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On fractional Laplacians – 2

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Dedicated to Louis Nirenberg on the occasion of his 90th birthday

Abstract

For $s > -1$ we compare two natural types of fractional Laplacians $(-\Delta)^s$, namely, the "Navier" and the "Dirichlet" ones.

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1. Introduction

Recall that the Sobolev space $H^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, is the space of distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ with finite norm

$$\|u\|_s^2 = \int\limits_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\mathcal{F}u(\xi)|^2 \, d\xi,$$

see for instance Section 2.3.3 of the monograph [8]. Here $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\, dx.$$

For arbitrary $s \in \mathbb{R}$ we define fractional Laplacian in $\mathbb{R}^n$ by the quadratic form

$$Q_s[u] = ((-\Delta)^s u, u) := \int\limits_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi,$$

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with domain

$$\mathrm{Dom}(Q_s) = \{u \in \mathcal{S}'(\mathbb{R}^n) : Q_s[u] < \infty\}.$$

Let Ω be a bounded and smooth domain in $\mathbb{R}^n$. We put

$$H^s(\Omega) = \left\{ u\big|_{\Omega} : u \in H^s(\mathbb{R}^n) \right\},$$

see [8, Sec. 4.2.1] and the extension theorem in [8, Sec. 4.2.3].

Also we introduce the space

$$\widetilde{H}^s(\Omega) = \{u \in H^s(\mathbb{R}^n) : \mathrm{supp}\, u \subset \overline{\Omega}\}.$$

By Theorem 4.3.2/1 of [8], for $s - \frac{1}{2} \notin \mathbb{Z}$ this space coincides with $H_0^s(\Omega)$, that is the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H^s(\Omega)$, while for $s - \frac{1}{2} \in \mathbb{Z}$ one has $\widetilde{H}^s(\Omega) \subsetneq H_0^s(\Omega)$. Moreover, $\mathcal{C}_0^\infty(\Omega)$ is dense in $\widetilde{H}^s(\Omega)$.

We introduce the "Dirichlet" fractional Laplacian in Ω (denoted by $(-\Delta_\Omega)_D^s$) as the restriction of $(-\Delta)^s$. The domain of its quadratic form is

$$\mathrm{Dom}(Q_{s,\Omega}^D) = \{u \in \mathrm{Dom}(Q_s) : \mathrm{supp}\, u \subset \overline{\Omega}\}.$$

Also we define the "Navier" fractional Laplacian as s-th power of the conventional Dirichlet Laplacian in the sense of spectral theory. Its quadratic form reads

$$Q_{s,\Omega}^N[u] = ((-\Delta_\Omega)_N^s u, u) := \sum\nolimits_j \lambda_j^s \cdot |(u, \varphi_j)|^2.$$

Here, λ_j and φ_j are eigenvalues and eigenfunctions of the Dirichlet Laplacian in Ω, respectively, and $\mathrm{Dom}(Q_{s,\Omega}^N)$ consists of distributions in Ω such that $Q_{s,\Omega}^N[u] < \infty$.

It is well known that for $s = 1$ these operators coincide: $(-\Delta_\Omega)_N = (-\Delta_\Omega)_D$. We emphasize that, in contrast to $(-\Delta_\Omega)_N^s$, the operator $(-\Delta_\Omega)_D^s$ is not the s-th power of the Dirichlet Laplacian for $s \neq 1$. In particular, $(-\Delta_\Omega)_D^{-s}$ is not inverse to $(-\Delta_\Omega)_D^s$.

The present paper is the natural evolution of [6], where we compared the operators $(-\Delta_\Omega)_D^s$ and $(-\Delta_\Omega)_N^s$ for $0 < s < 1$. In the first result we extend Theorem 2 of [6].

Theorem 1. *Let $s > -1$, $s \notin \mathbb{N}_0$. Then for $u \in \mathrm{Dom}(Q_{s,\Omega}^D)$, $u \not\equiv 0$, the following relations hold:*

$$Q_{s,\Omega}^N[u] > Q_{s,\Omega}^D[u], \quad \text{if} \quad 2k < s < 2k+1, \quad k \in \mathbb{N}_0; \tag{1}$$

$$Q_{s,\Omega}^N[u] < Q_{s,\Omega}^D[u], \quad \text{if} \quad 2k-1 < s < 2k, \quad k \in \mathbb{N}_0. \tag{2}$$

Next, we take into account the role of dilations in $\mathbb{R}^n$. We denote by $F(\Omega)$ the class of smooth and bounded domains containing Ω. If $\Omega' \in F(\Omega)$, then any $u \in \mathrm{Dom}(Q_{s,\Omega}^D)$ can be regarded as a function in $\mathrm{Dom}(Q_{s,\Omega'}^D)$, and the corresponding form $Q_{s,\Omega'}^D[u]$ does not change. In contrast, the form $Q_{s,\Omega'}^N[u]$ does depend on $\Omega' \supset \Omega$. However, roughly speaking, the difference between these quadratic forms disappears as $\Omega' \to \mathbb{R}^n$.

Theorem 2. *Let $s > -1$. Then for $u \in \mathrm{Dom}(Q_{s,\Omega}^D)$ the following facts hold:*

$$Q_{s,\Omega}^D[u] = \inf_{\Omega' \in F(\Omega)} Q_{s,\Omega'}^N[u], \quad \text{if} \quad 2k < s < 2k+1, \quad k \in \mathbb{N}_0; \tag{3}$$

$$Q_{s,\Omega}^D[u] = \sup_{\Omega' \in F(\Omega)} Q_{s,\Omega'}^N[u], \quad \text{if} \quad 2k-1 < s < 2k, \quad k \in \mathbb{N}_0. \tag{4}$$

For $-1 < s < 0$ we also obtain a pointwise comparison result reverse to the case $0 < s < 1$ (compare with [6, Theorem 1]).

Theorem 3. *Let $-1 < s < 0$, and let $f \in \mathrm{Dom}(Q_{s,\Omega}^D)$, $f \geq 0$ in the sense of distributions, $f \not\equiv 0$. Then the following relation holds:*

$$(-\Delta_\Omega)_N^s f < (-\Delta_\Omega)_D^s f. \tag{5}$$

Actually, fractional Laplacians of orders $s \in (-1, 0)$ play a crucial role in our arguments. In Section 2 we give a variational characterization of these operators, "dual" to variational characterization of fractional Laplacians of orders $s \in (0, 1)$ given in [4] and [7]. Theorems 1–3 are proved in Section 3.

Note that our statements hold in more general setting. Let Ω be a bounded and smooth domain in a complete smooth Riemannian manifold $\mathcal{M}$. Denote by $(-\Delta_\Omega)^s_N$ and $(-\Delta_\Omega)^s_D$, respectively, the s-th power of the Dirichlet Laplacian in Ω and the restriction of s-th power of the Dirichlet Laplacian in $\mathcal{M}$ to the set of functions supported in Ω. Then proofs of Theorems 1–3 (and of Theorem 1 in [6] as well) run with minimal changes.

2. Fractional Laplacians of negative orders

First, we recall some facts from the classical monograph [8] about the spaces $H^s(\Omega)$ and $\widetilde{H}^s(\Omega)$.

Proposition 1. *(A particular case of [8, Theorem 4.3.2/1].)*

1. *If* $0 < \sigma < \frac{1}{2}$ *then* $\widetilde{H}^\sigma(\Omega) = H^\sigma_0(\Omega) = H^\sigma(\Omega)$.
2. *If* $\sigma = \frac{1}{2}$ *then* $\widetilde{H}^\sigma(\Omega)$ *is dense in* $H^\sigma(\Omega) = H^\sigma_0(\Omega)$.
3. *If* $\frac{1}{2} < \sigma < 1$ *then* $\widetilde{H}^\sigma(\Omega) = H^\sigma_0(\Omega)$ *is a subspace of* $H^\sigma(\Omega)$.

Proposition 2. *(A particular case of [8, Theorem 2.10.5/1].) For any* $\sigma \in \mathbb{R}$ $(\widetilde{H}^\sigma(\Omega))' = H^{-\sigma}(\Omega)$.

As an immediate consequence we obtain

Corollary 1.

1. *If* $0 < \sigma < \frac{1}{2}$ *then* $\widetilde{H}^{-\sigma}(\Omega) = H^{-\sigma}(\Omega)$.
2. *If* $\sigma = \frac{1}{2}$ *then* $\widetilde{H}^{-\sigma}(\Omega)$ *is dense in* $H^{-\sigma}(\Omega)$.
3. *If* $\frac{1}{2} < \sigma < 1$ *then* $H^{-\sigma}(\Omega)$ *is a subspace of* $\widetilde{H}^{-\sigma}(\Omega)$.

Remark 1. In the one-dimensional case, for $\frac{1}{2} < \sigma < 1$ the codimension of $H^{-\sigma}(\Omega)$ in $\widetilde{H}^{-\sigma}(\Omega)$ equals 2 since the same is codimension of $\widetilde{H}^\sigma(\Omega)$ in $H^\sigma(\Omega)$.

The next statement gives explicit description of domains of quadratic forms under consideration.

Lemma 1. *Let* $0 < \sigma < 1$. *Then*

1. $\mathrm{Dom}(Q^N_{-\sigma,\Omega}) = H^{-\sigma}(\Omega)$.
2. $\mathrm{Dom}(Q^D_{-\sigma,\Omega}) = \widetilde{H}^{-\sigma}(\Omega)$ *if* $n \geq 2$ *or* $\sigma < \frac{1}{2}$.
3. $\mathrm{Dom}(Q^D_{\sigma,\Omega}) = \{u \in \widetilde{H}^{-\sigma}(\Omega) : \mathcal{F}u(0) = 0\}$ *if* $n = 1$ *and* $\sigma \geq \frac{1}{2}$.

Proof. The first statement follows from the relation $\mathrm{Dom}(Q^N_{\sigma,\Omega}) = \widetilde{H}^\sigma(\Omega)$, see, e.g., [8, Theorems 1.15.3 and 4.3.2/2], and from Proposition 2.

The second and the third statements follow directly from definition of $\widetilde{H}^{-\sigma}(\Omega)$, if we take into account that $\mathcal{F}u$ is a smooth function. □

By Lemma 1 and Corollary 1, for $0 < \sigma \leq \frac{1}{2}$ we have $\mathrm{Dom}(Q^D_{-\sigma,\Omega}) \subseteq \mathrm{Dom}(Q^N_{-\sigma,\Omega})$ (even $\mathrm{Dom}(Q^D_{-\sigma,\Omega}) = \mathrm{Dom}(Q^N_{-\sigma,\Omega})$ if $0 < \sigma < \frac{1}{2}$). In the case $\frac{1}{2} < \sigma < 1$, $\mathrm{Dom}(Q^N_{-\sigma,\Omega})$ is a subspace of $\mathrm{Dom}(Q^D_{-\sigma,\Omega})$ (for $n = 1$ this follows from Remark 1). However, for arbitrary $f \in \mathrm{Dom}(Q^D_{-\sigma,\Omega})$ we can consider f as a functional on $H^\sigma(\Omega)$, put $\widetilde{f} = f|_{\widetilde{H}^\sigma(\Omega)} \in \mathrm{Dom}(Q^N_{-\sigma,\Omega})$ and define $Q^N_{-\sigma,\Omega}[f] := Q^N_{-\sigma,\Omega}[\widetilde{f}]$.

Next, we recall that in the paper [4] the fractional Laplacian of order $\sigma \in (0, 1)$ in $\mathbb{R}^n$ was connected with the so-called *harmonic extension in* $n + 2 - 2\sigma$ *dimensions* and with generalized Dirichlet-to-Neumann map (see also

[3] for the case $\sigma = \frac{1}{2}$). In particular, for any $u \in \widetilde{H}^\sigma(\Omega)$ the function $w_\sigma^D(x,y)$ minimizing the weighted Dirichlet integral

$$\mathcal{E}_\sigma^D(w) = \int_0^\infty \int_{\mathbb{R}^n} y^{1-2\sigma} |\nabla w(x,y)|^2 \, dxdy$$

over the set

$$\mathcal{W}_\sigma^D(u) = \Big\{ w(x,y) \, : \, \mathcal{E}_\sigma^D(w) < \infty \, , \quad w\big|_{y=0} = u \Big\},$$

satisfies

$$Q_{\sigma,\Omega}^D[u] = \frac{C_\sigma}{2\sigma} \cdot \mathcal{E}_\sigma^D(w_\sigma^D), \tag{6}$$

where the constant C_σ is given by

$$C_\sigma := \frac{4^\sigma \Gamma(1+\sigma)}{\Gamma(1-\sigma)}.$$

Moreover, $w_\sigma^D(x,y)$ is the solution of the BVP

$$-\mathrm{div}(y^{1-2\sigma} \nabla w) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+; \qquad w\big|_{y=0} = u,$$

and for sufficiently smooth u

$$(-\Delta)^\sigma u(x) = -\frac{C_\sigma}{2\sigma} \cdot \lim_{y \to 0^+} y^{1-2\sigma} \partial_y w_\sigma^D(x,y), \qquad x \in \mathbb{R}^n \tag{7}$$

(we recall that $(-\Delta_\Omega)_D^\sigma u = (-\Delta)^\sigma u\big|_\Omega$).

In [7] this approach was developed in quite general situation. In particular, it was shown that for any $u \in \widetilde{H}^\sigma(\Omega)$ the BVP

$$-\mathrm{div}(y^{1-2\sigma} \nabla w) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+; \qquad w\big|_{y=0} = u; \qquad w\big|_{x \in \partial\Omega} = 0, \tag{8}$$

has a solution $w_\sigma^N(x,y)$, such that for sufficiently smooth u

$$(-\Delta_\Omega)_N^\sigma u(x) = -\frac{C_\sigma}{2\sigma} \cdot \lim_{y \to 0^+} y^{1-2\sigma} \partial_y w_\sigma^N(x,y). \tag{9}$$

Integrating by parts we see that the function $w_\sigma^N(x,y)$ minimizes the energy integral

$$\mathcal{E}_\sigma^N(w) = \int_0^\infty \int_\Omega y^{1-2\sigma} |\nabla w(x,y)|^2 \, dxdy$$

over the set

$$\mathcal{W}_{\sigma,\Omega}^N(u) = \{ w(x,y) \in \mathcal{W}_\sigma^D(u) \, : \, w\big|_{x \in \partial\Omega} = 0 \},$$

and

$$Q_{\sigma,\Omega}^N[u] = \frac{C_\sigma}{2\sigma} \cdot \mathcal{E}_\sigma^N(w_\sigma^N). \tag{10}$$

In a similar way, negative fractional Laplacians are connected with generalized Neumann-to-Dirichlet map. Namely, let $u \in \mathrm{Dom}(Q_{-\sigma,\Omega}^D)$. We consider the problem of minimizing the functional

$$\widetilde{\mathcal{E}}_{-\sigma}^D(w) = \mathcal{E}_\sigma^D(w) \, - \, 2\big\langle u, w\big|_{y=0}\big\rangle$$

over the set $\mathcal{W}_{-\sigma}^D$, that is closure of smooth functions on $\mathbb{R}^n \times \bar{\mathbb{R}}_+$ with bounded support, with respect to $\mathcal{E}_\sigma^D(\cdot)$. We recall that by Lemma 1 u can be considered as a compactly supported functional on $H^\sigma(\mathbb{R}^n)$, and thus the duality $\big\langle u, w\big|_{y=0}\big\rangle$ is well defined by the result of [4].

First, let $n > 2\sigma$ (this is a restriction only for $n = 1$). We claim that the Hardy type inequality

$$\mathcal{E}_\sigma^D(w) \ge \left(\frac{n-2\sigma}{2}\right)^2 \int\limits_0^\infty \int\limits_{\mathbb{R}^n} y^{1-2\sigma} \frac{w^2(x,y)}{r^2}\, dxdy \tag{11}$$

holds for $w \in \mathcal{W}_{-\sigma}^D$ (here $r^2 = |x|^2 + y^2$). Indeed, for a smooth function w with bounded support we consider the restriction of w to arbitrary ray in $\mathbb{R}^n \times \mathbb{R}_+$ and write down the classical Hardy inequality

$$\int\limits_0^\infty r^{n+1-2\sigma} w_r^2\, dr \ge \left(\frac{n-2\sigma}{2}\right)^2 \int\limits_0^\infty r^{n-1-2\sigma} w^2\, dr.$$

We multiply it by $\left(\frac{y}{r}\right)^{1-2\sigma}$, integrate over unit hemisphere in $\mathbb{R}^{n+1}$, and the claim follows.

By (11), a non-zero constant cannot be approximated by compactly supported functions. Thus, the minimizer of $\widetilde{\mathcal{E}}_{-\sigma}^D$ is determined uniquely. Denote it by $w_{-\sigma}^D(x,y)$. Then formulae (6) and (7) imply relations

$$Q_{-\sigma,\Omega}^D[u] = -\frac{2\sigma}{C_\sigma} \cdot \widetilde{\mathcal{E}}_{-\sigma}^D(w_{-\sigma}^D); \quad (-\Delta_\Omega)_D^{-\sigma} u(x) = \frac{2\sigma}{C_\sigma}\, w_{-\sigma}^D(x,0), \quad x \in \Omega, \tag{12}$$

that give the “dual” Caffarelli–Silvestre characterization of $(-\Delta_\Omega)_D^{-\sigma}$.

In case $n = 1 \le 2\sigma$ the above argument needs some modification. Namely, the minimizer $w_{-\sigma}^D(x,y)$ in this case is defined up to an additive constant. However, by Lemma 1 we have

$$\mathcal{F}u(0) \equiv \langle u, \mathbf{1}\rangle = 0.$$

Therefore, $\widetilde{\mathcal{E}}_{-\sigma}^D(w_{-\sigma}^D)$ does not depend on the choice of the constant, and the first relation in (12) holds. The second equality in (12) also holds if we choose the constant such that $w_{-\sigma}^D(x,0) \to 0$ as $|x| \to \infty$.

Remark 2. Note that for sufficiently smooth u the function $w_{-\sigma}^D$ solves the Neumann problem

$$-\mathrm{div}(y^{1-2\sigma}\nabla w) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+; \qquad \lim_{y\to 0^+} y^{1-2\sigma}\partial_y w = -u. \tag{13}$$

Analogously, formulae (10) and (9) imply the “dual” Stinga–Torrea characterization of $(-\Delta_\Omega)_N^{-\sigma}$. Namely, the function $w_{-\sigma}^N(x,y)$ minimizing the functional

$$\widetilde{\mathcal{E}}_{-\sigma}^N(w) = \mathcal{E}_\sigma^N(w) - 2\langle u, w\big|_{y=0}\rangle$$

over the set

$$\mathcal{W}_{-\sigma,\Omega}^N(u) = \{w(x,y) \in \mathcal{W}_{-\sigma}^D \,:\, w\big|_{x\notin\Omega} = 0\},$$

satisfies

$$Q_{-\sigma,\Omega}^N[u] = -\frac{2\sigma}{C_\sigma} \cdot \widetilde{\mathcal{E}}_{-\sigma}^N(w_{-\sigma}^N); \qquad (-\Delta_\Omega)_N^{-\sigma} u(x) = \frac{2\sigma}{C_\sigma}\, w_{-\sigma}^N(x,0). \tag{14}$$

Remark 3. Formula (14) shows that $w_{-\sigma}^N$ is the Stinga–Torrea extension of $\frac{C_\sigma}{2\sigma}(-\Delta_\Omega)_N^{-\sigma} u$. Similarly, from (12) we conclude that $w_{-\sigma}^D$ is the Caffarelli–Silvestre extension of $\frac{C_\sigma}{2\sigma}(-\Delta)^{-\sigma} u$ but not of $\frac{C_\sigma}{2\sigma}(-\Delta_\Omega)_D^{-\sigma} u$. This is due to the fact, already noticed in the introduction, that $(-\Delta_\Omega)_D^{-\sigma}$ is not the inverse of $(-\Delta_\Omega)_D^{\sigma}$.

Remark 4. The representation of $(-\Delta)^{-\sigma} u$ via solution of the problem (13) was used in [2]. Similar representation of $(-\Delta_\Omega)_N^{-\sigma} u$ via solution of corresponding mixed boundary value problem was used earlier in [5]. However, variational characterizations of negative fractional Laplacians (the first parts of formulae (12) and (14)) which play key role in what follows, are given for the first time.

3. Proofs of main theorems

We start by recalling an auxiliary result.

Lemma 2. *Let* $s > 1$. *Then*

$$\mathrm{Dom}(Q^D_{s,\Omega}) = \widetilde{H}^s(\Omega) = \mathrm{Dom}(Q^N_{s,\Omega}) \quad \textit{for} \quad s < 3/2;$$
$$\mathrm{Dom}(Q^D_{s,\Omega}) = \widetilde{H}^s(\Omega) \subsetneq \mathrm{Dom}(Q^N_{s,\Omega}) \quad \textit{for} \quad s \geq 3/2.$$

Proof. For $Q^D_{s,\Omega}$ the conclusion follows directly from its definition. For $Q^N_{s,\Omega}$ this fact is well known for $s \in \mathbb{N}$; in general case it follows immediately from [8, Theorem 1.17.1/1] and [8, Theorem 4.3.2/1]. □

Proof of Theorem 1. We split the proof in three parts.

1. Let $0 < s < 1$. Then the relation (1) is proved in [6, Theorem 2].

2. Let $-1 < s < 0$. We define $\sigma = -s \in (0, 1)$ and construct extensions $w^D_{-\sigma}$ and $w^N_{-\sigma}$ as described in Section 2. We evidently have $\mathcal{W}^N_{-\sigma,\Omega} \subset \mathcal{W}^D_{-\sigma}$ and $\widetilde{\mathcal{E}}^N_{-\sigma} = \widetilde{\mathcal{E}}^D_{-\sigma}\big|_{\mathcal{W}^N_{-\sigma,\Omega}}$. Therefore, (12) and (14) provide

$$Q^N_{s,\Omega}[u] = -\frac{2\sigma}{C_\sigma} \cdot \inf_{w \in \mathcal{W}^N_{-\sigma,\Omega}} \widetilde{\mathcal{E}}^N_{-\sigma}(w) \leq -\frac{2\sigma}{C_\sigma} \cdot \inf_{w \in \mathcal{W}^D_{-\sigma}} \widetilde{\mathcal{E}}^D_{-\sigma}(w) = Q^D_{s,\Omega}[u].$$

To complete the proof, we observe that for $u \not\equiv 0$ the function $w^N_{-\sigma}$ cannot be a solution of the homogeneous equation in (8) in the whole half-space, since such a solution is analytic in the half-space. Thus, it cannot provide $\inf\limits_{w \in \mathcal{W}^D_{-\sigma}} \widetilde{\mathcal{E}}^D_{-\sigma}(w)$, and (2) follows.

3. Now let $s > 1$, $s \notin \mathbb{N}$. We put $k = \lfloor \frac{s+1}{2} \rfloor$ and define for $u \in \widetilde{H}^s(\Omega)$

$$v = (-\Delta)^k u \in \widetilde{H}^{s-2k}(\Omega), \qquad s - 2k \in (-1, 0) \cup (0, 1).$$

Note that $v \not\equiv 0$ if $u \not\equiv 0$. Then we have

$$Q^N_{s,\Omega}[u] = Q^N_{s-2k,\Omega}[v], \qquad Q^D_{s,\Omega}[u] = Q^D_{s-2k,\Omega}[u],$$

and the conclusion follows from cases 1 and 2. □

Proof of Theorem 2. Here we again distinguish three cases.

1. Let $0 < s < 1$. Then the relation (3) is proved in [6, Theorem 3].

2. Let $-1 < s < 0$. We define $\sigma = -s \in (0, 1)$ and proceed similarly to the proof of [6, Theorem 3]. It is sufficient to prove the statement for $u \in \mathcal{C}^\infty_0(\Omega)$.

For $\Omega' \supset \Omega$ we have $\mathcal{W}^N_{-\sigma,\Omega} \subset \mathcal{W}^N_{-\sigma,\Omega'}$. By (14), the quadratic form $Q^N_{s,\Omega}[u]$ is monotone increasing with respect to the domain inclusion. Taking (2) into account, we obtain

$$Q^D_{s,\Omega}[u] > Q^N_{s,\Omega'}[u] \geq Q^N_{s,\Omega}[u]. \tag{15}$$

Denote by $w = w^D_{-\sigma}$ the Caffarelli–Silvestre extension of $\frac{C_\sigma}{2\sigma}(-\Delta)^{-\sigma}u$, described in Section 2. Next, for any $y \geq 0$ let $\phi_R(\cdot, y)$ be the harmonic extension of $w(\cdot, y)$ on the ball B_R, that is,

$$-\Delta\phi_R(\cdot, y) = 0 \quad \text{in } B_R; \qquad \phi_h(\cdot, y) = w(\cdot, y) \quad \text{on } \partial B_R\,.$$

Finally, for $x \in B_R$ and $y \geq 0$ we put

$$w_R(x, y) = w(x, y) - \phi_R(x, y)\,.$$

It is shown in the proof of [6, Theorem 3] that there exists a sequence $R_h \to \infty$ such that

$$\mathcal{E}^N_\sigma(w_{R_h}) \leq \mathcal{E}^D_\sigma(w) + o(1).$$

Further, since $(-\Delta_\Omega)^{-\sigma}_D u$ vanishes at infinity, for any multi-index β we evidently have $D^\beta\phi_{R_h}(\cdot, 0) \to 0$ locally uniformly as $R_h \to \infty$. This gives $\langle u, \phi_{R_h}(\cdot, 0)\rangle = o(1)$, and we obtain by (12) and (14)

$$Q^N_{s,B_{R_h}}[u] \ge -\frac{2\sigma}{C_\sigma} \cdot \widetilde{\mathcal{E}}^N_{-\sigma}(w_{R_h})$$
$$\ge -\frac{2\sigma}{C_\sigma} \cdot \widetilde{\mathcal{E}}^D_{\sigma}(w) - o(1) = Q^D_{s,\Omega}[u] - o(1)\,. \tag{16}$$

The relation (3) readily follows by comparing (15) and (16).

3. For $s > 1$, $s \notin \mathbb{N}$, the conclusion follows from cases 1 and 2 just as in the proof of Theorem 1. □

Remark 5. Assume that $0 \in \Omega$ and put $\alpha\Omega = \{\alpha x \,:\, x \in \Omega\}$. The above proof shows that

$$Q^D_{s,\Omega}[u] = \lim_{\alpha\to\infty} Q^N_{s,\alpha\Omega}[u] \quad \text{for any} \quad u \in \widetilde{H}^s(\Omega).$$

Now put $u_\alpha(x) = \alpha^{\frac{n-2s}{2}} u(\alpha x)$. Then the scaling gives

$$Q^D_{s,\Omega}[u_\alpha] \equiv Q^D_{s,\Omega}[u] = \lim_{\alpha\to\infty} Q^N_{s,\Omega}[u_\alpha] \quad \text{for any} \quad u \in \widetilde{H}^s(\Omega).$$

Proof of Theorem 3. First, let $f \in \mathcal{C}^\infty_0(\Omega)$. We define $\sigma = -s \in (0,1)$ and construct extensions $w^D_{-\sigma}$ and $w^N_{-\sigma}$ described in Section 2. Making the change of the variable $t = y^{2\sigma}$, we rewrite the BVP (13) for $w^D_{-\sigma}(x,t)$ as follows:

$$\Delta_x w^D_{-\sigma} + 4\sigma^2 t^{\frac{2\sigma-1}{\sigma}} \partial^2_{tt} w^D_{-\sigma} = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+; \qquad \partial_t w^D_{-\sigma}\big|_{t=0} = -\frac{f}{2\sigma}. \tag{17}$$

Since $w^D_{-\sigma}$ vanishes at infinity, $w^D_{-\sigma}(x,t) > 0$ for $t > 0$ by the maximum principle. Moreover, by [1, Theorem 1.4] (the boundary point lemma) we have $w^D_{-\sigma}(x,0) > 0$.

Further, the function $w^N_{-\sigma}$ satisfies the equalities (17) in $\Omega \times \mathbb{R}_+$. Since $w^N_{-\sigma}\big|_{x\notin\Omega} = 0$, we infer that the function

$$W(x,t) := w^D_{-\sigma}(x,t) - w^N_{-\sigma}(x,t)$$

meets the following relations:

$$\Delta_x W + 4\sigma^2 t^{\frac{2\sigma-1}{\sigma}} \partial^2_{tt} W = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+; \qquad \partial_t W\big|_{t=0} = 0; \qquad W\big|_{x\in\partial\Omega} > 0.$$

Again, [1, Theorem 1.4] gives $W(x,0) > 0$, which gives (5) in view of (12) and (14).

For $f \in \widetilde{H}^s(\Omega)$ the statement holds by approximation argument. □

Conflict of interest statement

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

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