Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 33 (2016) 1103–1130

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Parabolic limit with differential constraints of first-order quasilinear hyperbolic systems

Yue-Jun Peng [a,b,*], Victor Wasiolek [a,b]

[a] *Université Clermont Auvergne, Université Blaise Pascal, 63000 Clermont-Ferrand, France*
[b] *Laboratoire de Mathématiques, CNRS, UMR 6620, 63171 Aubière, France*

Received 24 April 2014; received in revised form 26 February 2015; accepted 30 March 2015

Available online 2 April 2015

Abstract

The goal of this work is to provide a general framework to study singular limits of initial-value problems for first-order quasilinear hyperbolic systems with stiff source terms in several space variables. We propose structural stability conditions of the problem and construct an approximate solution by a formal asymptotic expansion with initial layer corrections. In general, the equations defining the approximate solution may come together with differential constraints, and so far there are no results for the existence of solutions. Therefore, sufficient conditions are shown so that these equations are parabolic without differential constraint. We justify rigorously the validity of the asymptotic expansion on a time interval independent of the parameter, in the case of the existence of approximate solutions. Applications of the result include Euler equations with damping and an Euler–Maxwell system with relaxation. The latter system was considered in [27,9] which contain ideas used in the present paper.

MSC: 35C20; 35L60; 76M45

Keywords: First-order quasilinear hyperbolic system; Singular limit; Structural stability condition; Differential constraint; Parabolic limit equations

1. Introduction

This work is concerned with singular limits of first-order quasilinear hyperbolic equations with stiff source terms of the form

$$\partial_t U + \frac{1}{\varepsilon}\sum_{j=1}^{d} A_j(U)\partial_{x_j} U = \frac{Q(\varepsilon, U)}{\varepsilon^2}, \tag{1.1}$$

* Corresponding author at: Laboratoire de Mathématiques, CNRS, UMR 6620, 63171, Aubière cedex, France.
E-mail addresses: peng@math.univ-bpclermont.fr (Y.-J. Peng), wasiolek@math.univ-bpclermont.fr (V. Wasiolek).

http://dx.doi.org/10.1016/j.anihpc.2015.03.006

with initial conditions

$$U(0, x) = \bar{U}(x, \varepsilon). \tag{1.2}$$

Here $U : \mathbb{R}_t^+ \times \mathbb{R}_x^d \longrightarrow G \subset \mathbb{R}^n$ is the unknown variable with $x = (x_1, \cdots, x_d)$, $\varepsilon \in (0, 1]$ is a small parameter and $Q : [0, 1] \times G \longrightarrow \mathbb{R}^n$ is a smooth vector function. In physical models, ε often stands for a relaxation time. The set G is called the state space and A_j $(1 \le j \le d)$ are $n \times n$ smooth matrix functions defined on G. We suppose that (1.1) is symmetrizable hyperbolic (see [8]): i.e., there exists a symmetric positive definite matrix $A_0(U)$, called symmetrizer, such that for all $U \in G$,

(i) $A_0(U)\xi \cdot \xi \ge M_0|\xi|^2$, for all $\xi \in \mathbb{R}^n$;
(ii) $\tilde{A}_j(U) \stackrel{def}{=} A_0(U)A_j(U)$ is symmetric for all $1 \le j \le d$,

where $M_0 > 0$ is a constant, "$\cdot$" is the inner product of $\mathbb{R}^n$ and $|\cdot|$ is the Euclidean norm of $\mathbb{R}^n$. In general, Q only depends on U. The fact that it may also depend on ε is due to an Euler–Maxwell system with relaxation (see the last section).

In (1.1), the variable t should be understood as a slow time linked with the usual time t' by $t = \varepsilon t'$. Therefore, (1.1) is equivalent to

$$\partial_{t'} U + \sum_{j=1}^{d} A_j(U)\partial_{x_j} U = \frac{Q(\varepsilon, U)}{\varepsilon}. \tag{1.3}$$

System (1.3) is a general form of first-order quasilinear hyperbolic equations with stiff relaxation source terms. It was studied by many authors in the case where Q is a function of only U. Under stability conditions, the limit equations of (1.3) as $\varepsilon \to 0$ are of first-order hyperbolic type. For mathematical results and physical examples of (1.3), we refer to [32,19,6,12,3,25,26,33,29,34] and references therein.

The aim of the present work is to study the limit of smooth solutions of (1.1)–(1.2) as $\varepsilon \to 0$, in a d-dimensional torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$. Then $\bar{U}$ is supposed to be smooth and periodic with respect to x. As usual for first-order hyperbolic problems with relaxation, we assume

$$Q(0, U) = \begin{bmatrix} 0 \\ q(U) \end{bmatrix}, \tag{1.4}$$

where $q : G \longrightarrow \mathbb{R}^r$ is a smooth function, $1 \le r \le n$. With the same partition, we denote

$$U = \begin{bmatrix} u \\ v \end{bmatrix}, \quad u \in \mathbb{R}^{n-r}, \ v \in \mathbb{R}^r.$$

More generally, a vector $V \in \mathbb{R}^n$ and an $n \times n$ matrix M will be denoted by $\begin{bmatrix} V^I \\ V^{II} \end{bmatrix}$ and $\begin{bmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{bmatrix}$, respectively. In order to obtain a parabolic limit from (1.1), we further assume

$$q(U) = 0 \Longleftrightarrow v = 0, \text{ and } \partial_v q(u, 0) \text{ is invertible for all } u \in \mathbb{R}^{n-r}. \tag{1.5}$$

The singular limit problem $\varepsilon \to 0$ for (1.1) was considered in [22,21,23] in the case of special models. See also [4] for the approximation of parabolic equations by diffusive BGK models. Contrarily to (1.3), in general the limit equations of (1.1) are of parabolic type. In [16], Lattanzio and Yong considered a first-order symmetrizable hyperbolic system of the form

$$\partial_t U + \frac{1}{\varepsilon} \sum_{j=1}^{d} A_j(\varepsilon U)\partial_{x_j} U + \sum_{j=1}^{d} \bar{A}_j(U)\partial_{x_j} U = \frac{Q(U)}{\varepsilon^2}, \tag{1.6}$$

with smooth and periodic initial data $\bar{U}$ given in (1.2). Assuming appropriate stability conditions and the existence of approximate solutions, they proved the convergence of the system to parabolic type equations on a time interval independent of ε. In this problem, the singular limit arises from $Q(U)/\varepsilon^2$ and the term containing $A_j(\varepsilon U)/\varepsilon$, and there is no difficulty to treat the term containing $\bar{A}_j(U)$. The advantage of this system is that $\partial_{x_j} A_j(\varepsilon U)$ is always a term of order $O(\varepsilon)$. This is a very useful property in higher order energy estimates by using the Moser-type calculus inequalities.

When A_j $(1 \le j \le d)$ are constant matrices and $\bar{A}_j = 0$, system (1.6) is semilinear and was considered in [23] in one-dimensional case. The result in [23] can be applied to a linear wave equation of heat conduction and a generalized discrete two-velocity model. Now write (1.6) as

$$\partial_t U + \frac{1}{\varepsilon}\sum_{j=1}^d A_j(0)\partial_{x_j} U + \sum_{j=1}^d \Big(\bar{A}_j(U) + \frac{A_j(\varepsilon U) - A_j(0)}{\varepsilon}\Big)\partial_{x_j} U = \frac{Q(U)}{\varepsilon^2}.$$

Since $\big(A_j(\varepsilon U) - A_j(0)\big)/\varepsilon$ is of order $O(1)$, (1.6) is essentially an extension of semilinear problems in several space dimensions. Moreover, in the last section we will see that the result in [16] cannot be applied to the Euler equations with damping and the Euler–Maxwell system with relaxation, which are both quasilinear systems.

In order to study the singular limit $\varepsilon \to 0$ for the quasilinear system (1.1), we propose stability conditions on the system. As in previous works for singular perturbation problems, we construct an approximate solution by a formal series asymptotic expansion with initial layer corrections. The novelty here is that the limit equations defining the approximate solution are generally combined by differential constraints. So far no general results are available for the existence of solutions to such limit equations even (1.1) is semilinear. Then sufficient conditions are shown so that these equations are parabolic without differential constraint. Further sufficient conditions can be investigated for the local existence of smooth solutions to the limit equations, but this is beyond the goal of the present paper. We justify rigorously the validity of the asymptotic expansion on a time interval independent of the parameter, in the case of the existence of approximate solutions. Applications of this result include the Euler equations with damping and the Euler–Maxwell system with relaxation mentioned above. For the latter system, there are differential constraints in the limit equations.

Since $\bar{U}$ is smooth and periodic in x, according to Kato (see [13]), for all integer $s > d/2 + 1$, there exists a maximal time $T_\varepsilon > 0$ such that problem (1.1)–(1.2) admits a unique local-in-time smooth solution U^ε satisfying

$$U^\varepsilon \in C\big([0, T_\varepsilon), H^s(\mathbb{T}^d)\big) \cap C^1\big([0, T_\varepsilon), H^{s-1}(\mathbb{T}^d)\big). \tag{1.7}$$

The central problem of the study is to show that U^ε converges as $\varepsilon \to 0$ and $\inf_{0<\varepsilon\le 1} T_\varepsilon > 0$. More precisely, for all integer $m \in \mathbb{N}$ and a constant $T_m > 0$ being independent of ε, we denote by U_ε^m an approximate smooth solution to (1.1)–(1.2) defined on time interval $[0, T_m]$. The error of the approximation is defined by

$$R_m^\varepsilon = \partial_t U_\varepsilon^m + \frac{1}{\varepsilon}\sum_{j=1}^d A_j(U_\varepsilon^m)\partial_{x_j} U_\varepsilon^m - \frac{Q(\varepsilon, U_\varepsilon^m)}{\varepsilon^2}. \tag{1.8}$$

Then a necessary condition for U_ε^m to be an approximate solution to (1.1)–(1.2) is that $R_m^\varepsilon \to 0$ as $\varepsilon \to 0$.

In Section 4, we construct such an approximate solution by an asymptotic expansion with initial layer corrections of the form

$$U_\varepsilon^m(t,x) = \sum_{k=0}^m \varepsilon^k \big(U_k(t,x) + I_k(\tau, x)\big), \quad m \in \mathbb{N}, \tag{1.9}$$

where $\tau = t/\varepsilon^2$ is a fast time. The properties of the approximate solution strongly depend on its leading profile $U_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$, which is a formal limit of U^ε. From (1.1), (1.4)–(1.5) and (1.9), we have $v_0 = 0$,

$$\sum_{j=1}^d A_j^{11}(u_0, 0)\partial_{x_j} u_0 - \partial_\varepsilon Q^I(0, u_0, 0) = 0, \tag{1.10}$$

$$\partial_t u_0 + \sum_{j=1}^d A_j^{12}(U_0)\partial_{x_j} v_1 + \sum_{j=1}^d A_j^{11}(U_0)\partial_{x_j} u_1 + g_0(u_0, \nabla u_0, v_1) = 0, \tag{1.11}$$

and

$$v_1 = \partial_v q(u_0, 0)^{-1} \left[\sum_{j=1}^{d} A_j^{21}(u_0, 0)\partial_{x_j} u_0 - \partial_\varepsilon Q^{II}(0, u_0, 0) \right], \tag{1.12}$$

where

$$g_0(u_0, \nabla u_0, v_1) = \sum_{j=1}^{d} \partial_v A_j^{11}(U_0) v_1 \partial_{x_j} u_0 - \partial_v \partial_\varepsilon Q^I(0, u_0, 0) v_1 - \frac{1}{2}\partial_\varepsilon^2 Q^I(0, u_0, 0). \tag{1.13}$$

Thus, the system for u_0 is composed of (1.10)–(1.11). Comparing to the study in [16], Eq. (1.10) is quite new. It stands for a differential constraint for u_0. For system (1.6), since Q is independent of ε, the corresponding differential constraint for u_0 is

$$\sum_{j=1}^{d} A_j^{11}(0)\partial_{x_j} u_0 = 0.$$

It is trivially satisfied under assumption $A_j^{11}(0) = 0$ for all j, which was made in [16]. Thus, there is no differential constraint in the limit equation of (1.6).

Now we use a projection technique to eliminate u_1 in (1.11). Let D be a constant square matrix of order $n - r$ such that

$$DA_j^{11}(u, 0) = 0, \quad \forall u \in \mathbb{R}^{n-r}, \ \forall 1 \le j \le d. \tag{1.14}$$

Applying D to (1.11), we obtain

$$D\partial_t u_0 + D\sum_{j=1}^{d} A_j^{12}(U_0)\partial_{x_j} v_1 + Dg_0(u_0, \nabla u_0, v_1) = 0. \tag{1.15}$$

Then (1.11) can be written as

$$\sum_{j=1}^{d} A_j^{11}(U_0)\partial_{x_j} u_1 + (\mathbf{I}_{n-r} - D)\Big(\partial_t u_0 + \sum_{j=1}^{d} A_j^{12}(U_0)\partial_{x_j} v_1 + g_0(u_0, \nabla u_0, v_1)\Big) = 0. \tag{1.16}$$

When u_0 is solved, v_1 is given by (1.12). Hence, (1.16) is a constraint for u_1. Substituting (1.12) into (1.15) gives

$$D\partial_t u_0 + D\sum_{i,j=1}^{d} A_{ij}(u_0)\partial^2_{x_i x_j} u_0 + D\sum_{j=1}^{d} B_j(u_0)\partial_{x_j} u_0 + D\sum_{i,j=1}^{d} C_{ij}(u_0)\partial_{x_i} u_0 \, \partial_{x_j} u_0 + Df_0(u_0) = 0, \tag{1.17}$$

where

$$A_{ij}(u_0) = A_i^{12}(u_0, 0)\partial_v q(u_0, 0)^{-1} A_j^{21}(u_0, 0), \tag{1.18}$$

$$\begin{aligned} B_j(u_0) = {} & -A_j^{12}(u_0, 0)\partial_u\big[\partial_v q(u_0, 0)^{-1}\partial_\varepsilon Q^{II}(0, u_0, 0)\big] \\ & - \partial_v A_j^{11}(u_0, 0)\partial_v q(u_0, 0)^{-1}\partial_\varepsilon Q^{II}(0, u_0, 0) \\ & - \partial_v \partial_\varepsilon Q^I(0, u_0, 0)\partial_v q(u_0, 0)^{-1} A_j^{21}(u_0, 0), \end{aligned} \tag{1.19}$$

$$C_{ij}(u_0) = A_i^{12}(u_0, 0)\partial_u\big[\partial_v q(u_0, 0)^{-1} A_j^{21}(u_0, 0)\big] + \partial_v A_j^{11}(u_0, 0)\partial_v q(u_0, 0)^{-1} A_i^{21}(u_0, 0), \tag{1.20}$$

and

$$f_0(u_0) = -\frac{1}{2}\partial_\varepsilon^2 Q^I(0, u_0, 0) + \partial_v \partial_\varepsilon Q^I(0, u_0, 0)\partial_v q(u_0, 0)^{-1}\partial_\varepsilon Q^{II}(0, u_0, 0). \tag{1.21}$$

We point out that u_0 is a vector function and then (1.17) is a system of partial differential equations combined with the differential constraint (1.10). If $D = \mathbf{I}_{n-r}$, the unit matrix of order $n - r$, solving u_0 requires compatibility conditions between (1.17) and (1.10). In a simple case that

$$A_j^{11}(u,0)=0 \text{ and } \partial_\varepsilon Q^I(0,u,0)=0, \quad \forall u\in\mathbb{R}^{n-r},\ \forall 1\le j\le d,$$

the differential constraint disappears. Then the principal part of the system (1.17) is governed by the second-order partial differential operator of evolution-type:

$$\partial_t+\sum_{i,j=1}^{d}A_{ij}(u_0)\partial^2_{x_ix_j}.$$

If this operator is parabolic, it is possible to solve (1.17) locally in time (see [15]). In particular, when $n-r=1$, u_0 satisfies a scalar parabolic equation, which can be solved by standard techniques. This is the case of the semilinear examples and the Euler equations with damping given in the last section. Another interesting case is that the equations and the differential constraint are separated. For example, let $r_1, r_2\in\mathbb{N}$, with $r_1+r_2=n-r$. If the first r_1 lines of $A_j^{11}(U_0)$ and $\partial_\varepsilon Q^I(0,u_0,0)$ are zero, we take $D=\mathrm{diag}\big(\mathbf{I}_{r_1},\mathbf{0}_{r_2}\big)$, with $\mathbf{0}_{r_2}$ being the $r_2\times r_2$ zero matrix. Then (1.14) holds and (1.17) means that only the first r_1 components of u_0 satisfy a second-order evolution system of partial differential equations together with r_2 differential constraint conditions given by (1.10). A typical example of this situation is the Euler–Maxwell system with relaxation.

The main result of this paper is to prove that, for any fixed integer $m\ge 2$, we have $T_\varepsilon>T_m$ and

$$\sup_{0\le t\le T_m}\|U^\varepsilon(t,\cdot)-U^m_\varepsilon(t,\cdot)\|_s\le c\varepsilon^m,$$

where $\|\cdot\|_s$ stands for the norm of $H^s(\mathbb{T}^d)$ and $c>0$ is a constant independent of ε. It is stated in Theorem 2.1 in Section 2. The result implies that the convergence of system (1.3) is valid in $[0,T_m/\varepsilon]$. The proof of Theorem 2.1 is based on uniform energy estimates with respect to ε. However, usual energy estimates are not efficient for our problem. The main difficulty comes from the term $\partial_{x_j}A_j(U^\varepsilon)$ which is of order $O(1)$ instead of order $O(\varepsilon)$ for (1.6). To overcome this difficulty, we use a continuation argument as follows. Assume

$$\|U^\varepsilon(0,\cdot)-U^m_\varepsilon(0,\cdot)\|_s\le c\varepsilon^m.$$

For all $T^1_\varepsilon\in(0,T_\varepsilon)\cap(0,T_m]$, the function $t\longmapsto\|U^\varepsilon(t,\cdot)-U^m_\varepsilon(t,\cdot)\|_s$ is continuous on $[0,T^1_\varepsilon]$. It follows that, for any fixed integer $m\ge 2$, if ε is sufficiently small, there exists a maximal time $T^2_\varepsilon\in(0,T_\varepsilon)\cap(0,T_m]$, such that

$$\sup_{0\le t\le T^2_\varepsilon}\|U^\varepsilon(t,\cdot)-U^m_\varepsilon(t,\cdot)\|_s\le\varepsilon.$$

This result is shown in Lemma 3.2. Therefore, it remains to prove

$$\sup_{0\le t\le T^2_\varepsilon}\|U^\varepsilon(t,\cdot)-U^m_\varepsilon(t,\cdot)\|_s\le c\varepsilon^m.$$

Indeed, by a simple argument, the last inequality easily implies that $T^2_\varepsilon=T_m$. Hence, $T_\varepsilon>T_m$. Besides, the continuation argument allows to keep only the quadratic terms in energy estimates. Thus we avoid complicated calculus and the use of a nonlinear Gronwall-type inequality as in [33,16].

This paper is organized as follows. In the next section, we present the stability conditions for the singular limit of (1.1) and state Theorem 2.1. Section 3 is devoted to the proof of the theorem, which is achieved by a series of lemmas for energy estimates together with a Gronwall inequality with variable coefficients. In Section 4, we show the detailed derivation of the equations for U_k and I_k defined by (1.9). For small initial data, we prove that for all $0\le k\le m$, I_k exists globally in time and decays exponentially fast to zero as $\tau\to+\infty$. Finally, we give semilinear and quasilinear examples to which the approximate solutions can be rigorously constructed and thus the theorem can be applied.

2. Assumptions and main results

We first introduce the following notations. For a multi-index $\alpha=(\alpha_1,\cdots,\alpha_d)\in\mathbb{N}^d$, we denote

$$\partial^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}\quad\text{with}\quad|\alpha|=\alpha_1+\cdots+\alpha_d.$$

We denote by $\|\cdot\|_s$ the usual norm of the Sobolev space $H^s \overset{def}{=} H^s(\mathbb{T}^d)$, and by $\|\cdot\|$ and $\|\cdot\|_\infty$ the usual norms of $L^2 \overset{def}{=} L^2(\mathbb{T}^d)$ and $L^\infty \overset{def}{=} L^\infty(\mathbb{T}^d)$, respectively. We also make a convention that $\|\cdot\| = \|\cdot\|_0$. Finally, $\langle\cdot,\cdot\rangle$ stands for the inner product in $L^2(\mathbb{T}^d)$ and $Im(V)$ stands for the image of a function V. For two sets $\omega, \Omega \in \mathbb{R}^n$, $\omega \subset\subset \Omega$ means that ω is relatively compact in Ω.

Throughout this paper, $s > d/2 + 1$ is an integer and $c > 0$ stands for a generic constant independent of ε. We assume there exists an approximate solution U_ε^m to (1.1)–(1.2) defined on a time interval $[0, T_m]$, with $T_m > 0$ independent of ε. Here U_ε^m is not necessarily given by (1.9). Let $U_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$ with $v_0 = 0$ and $u_0 \in C([0, T_m], H^{s+1})$ being an arbitrary smooth solution of (1.17) and (1.10).

We make the following assumptions:

(H1) $A_j^{11}(u_0, 0)$ and $\tilde{A}_j^{11}(u_0, 0)$ are constant matrices and $\partial_u A_j^{11}(u_0, 0) = 0$ for all $1 \le j \le d$;
(H2) there is a constant $c_0 > 0$, depending only on G, such that

$$A_0(u,0)\partial_U Q(0,u,0)\xi \cdot \xi \le -c_0|\xi^{II}|^2, \quad \forall (u,0) \in G,\ \forall \xi \in \mathbb{R}^n;$$

(H3) $\partial_u \partial_\varepsilon Q^I(0, u, 0) = 0$ for all $u \in \mathbb{R}^{n-r}$;
(H4) $\bar{U} \in H^s$ and there exists a convex open set $G_0 \subset G$ such that $Im(\bar{U}) \subset\subset G_0$;
(H5) for sufficiently small $\varepsilon > 0$, U_ε^m satisfies $U_\varepsilon^m \in C([0, T_m], H^{s+1}) \cap C^1([0, T_m], H^s)$, $Im(U_\varepsilon^m) \subset\subset G_0$,

$$\|U_\varepsilon^m(0,\cdot) - \bar{U}(\cdot,\varepsilon)\|_s \le c\varepsilon^m, \tag{2.1}$$

and

$$\|\partial_t U_\varepsilon^m(t,\cdot)\|_s \le c + c\varepsilon^{-2} e^{-\frac{\mu t}{\varepsilon^2}}, \qquad \sup_{0\le t\le T_m} \|U_\varepsilon^m(t,\cdot) - U_0(t,\cdot)\|_s \le c\varepsilon + ce^{-\frac{\mu t}{\varepsilon^2}}, \tag{2.2}$$

where $\mu > 0$ is a constant independent of ε;
(H6) the error R_m^ε defined in (1.8) can be expressed as

$$R_m^\varepsilon = \varepsilon^{m-1} \begin{bmatrix} 0 \\ r_m \end{bmatrix} + \varepsilon^{m-1} F_m^\varepsilon,$$

with $r_m \in C([0, T_m], H^s)$, $F_m^\varepsilon \in C([0, T_m], H^s)$ and

$$\|F_m^\varepsilon(t)\|_s \le c\varepsilon + ce^{-\frac{\mu t}{\varepsilon^2}}.$$

Theorem 2.1. *Let $s > d/2 + 1$ be an integer and U^ε be the exact solution to (1.1)–(1.2) defined on the maximal time interval $[0, T_\varepsilon)$ satisfying (1.7). Let $m \ge 2$ be any fixed integer and U_ε^m be an approximate solution to (1.1)–(1.2) defined on $[0, T_m]$ with $T_m > 0$ being independent of ε. Assume (H1)–(H6) and (1.4)–(1.5) hold. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, we have $T_\varepsilon > T_m$ and*

$$\sup_{0\le t\le T_m} \|U^\varepsilon(t) - U_\varepsilon^m(t)\|_s \le c\varepsilon^m. \tag{2.3}$$

Moreover,

$$\int_0^{T_m} \|v^\varepsilon(t) - v_\varepsilon^m(t)\|_s^2\, dt \le c\varepsilon^{2(m+1)}. \tag{2.4}$$

Now the following remark is necessary.

Remark 2.1.

- (H1)–(H3) together with (1.4)–(1.5) are structural stability conditions of the quasilinear system (1.1) and they can be checked for a given system and a given u_0. In particular, they are satisfied with all examples given in the last section.

- (H1) is used in the proofs of Lemmas 3.3–3.4 and 3.7. It replaces condition $A_j^{11}(0) = 0$ for studying (1.6) in [16].
- (H1) is satisfied if $A_j^{11}(u, 0)$ and $\tilde{A}_j^{11}(u, 0)$ are constant matrices for all u and all $1 \le j \le d$. This is the case of all examples given in the last section. In particular, (H1) is satisfied when u_0 is a constant and $\partial_u A_j^{11}(u_0, 0) = 0$, or when (1.1) is semilinear, namely, A_j is a constant matrix for all $1 \le j \le d$.
- (H2) stands for the partial dissipation property of (1.1). Together with (1.4)–(1.5), it gives a property on the symmetrizer that we need later (see Lemma 3.1). When $r = n$ the dissipation is complete and $U = v$. This case is easier to treat comparing to the partial dissipation case $1 \le r \le n - 1$.
- (H3) is a technical assumption to treat the source term Q and it is trivially satisfied when Q is a function of only U.
- (H4) is necessary to apply the existence theorem of Kato, see [20].
- In (H5), (2.1) is a natural condition on the initial data. It stands for initial errors. Condition (2.2) and (H6) can be checked in the construction of U_ε^m in Section 4.

3. Justification of formal expansions

3.1. Preliminaries

Let $m \ge 2$ be an integer and U_ε^m be an approximate solution of (1.1)–(1.2) defined on $[0, T_m]$, with $T_m > 0$ being independent of ε. Then, for all $T_\varepsilon^1 \in (0, T_\varepsilon) \cap (0, T_m]$, both the exact solution U^ε and the approximate solution are defined on time interval $[0, T_\varepsilon^1]$, on which we define

$$W^\varepsilon = U^\varepsilon - U_\varepsilon^m.$$

From (1.1) and (1.8), we obtain

$$\partial_t W^\varepsilon + \frac{1}{\varepsilon} \sum_{j=1}^d A_j(U^\varepsilon)\partial_{x_j} W^\varepsilon = \frac{a^\varepsilon}{\varepsilon} + \frac{b^\varepsilon}{\varepsilon^2} - R_m^\varepsilon, \tag{3.1}$$

where

$$a^\varepsilon = \sum_{j=1}^d \left[A_j(U_\varepsilon^m) - A_j(U^\varepsilon)\right]\partial_{x_j} U_\varepsilon^m \tag{3.2}$$

and

$$b^\varepsilon = Q(\varepsilon, U^\varepsilon) - Q(\varepsilon, U_\varepsilon^m). \tag{3.3}$$

Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \le s$. Thanks to the symmetry of A_0 and $\tilde{A}_j$, we obtain an energy equality:

$$\begin{aligned}\frac{d}{dt}\langle A_0(U^\varepsilon)\partial^\alpha W^\varepsilon, \partial^\alpha W^\varepsilon\rangle = {} & \langle \operatorname{div}_\varepsilon A(U^\varepsilon)\partial^\alpha W^\varepsilon, \partial^\alpha W^\varepsilon\rangle + \frac{2}{\varepsilon}\langle A_0(U^\varepsilon)\partial^\alpha a^\varepsilon, \partial^\alpha W^\varepsilon\rangle \\ & + \frac{2}{\varepsilon^2}\langle A_0(U^\varepsilon)\partial^\alpha b^\varepsilon, \partial^\alpha W^\varepsilon\rangle - 2\langle A_0(U^\varepsilon)\partial^\alpha R_m^\varepsilon, \partial^\alpha W^\varepsilon\rangle \\ & + \frac{2}{\varepsilon}\langle A_0(U^\varepsilon) f_\alpha^\varepsilon, \partial^\alpha W^\varepsilon\rangle,\end{aligned}$$

where

$$\operatorname{div}_\varepsilon A(V) = \partial_t A_0(V) + \frac{1}{\varepsilon}\sum_{j=1}^d \partial_{x_j} \tilde{A}_j(V), \tag{3.4}$$

and

$$f_\alpha^\varepsilon = \sum_{j=1}^d f_{\alpha j}^\varepsilon, \quad f_{\alpha j}^\varepsilon = A_j(U^\varepsilon)\partial_{x_j}(\partial^\alpha W^\varepsilon) - \partial^\alpha\left(A_j(U^\varepsilon)\partial_{x_j} W^\varepsilon\right). \tag{3.5}$$

We write this equality as follows:

$$\frac{d}{dt}\langle A_0(U^\varepsilon)\partial^\alpha W^\varepsilon, \partial^\alpha W^\varepsilon\rangle = I^\alpha_{1,\varepsilon} + I^\alpha_{2,\varepsilon} + I^\alpha_{3,\varepsilon} + I^\alpha_{4,\varepsilon} + I^\alpha_{5,\varepsilon},$$

with the natural correspondence for $I^\alpha_{1,\varepsilon}, \cdots, I^\alpha_{5,\varepsilon}$.

We first give two preliminary results, which are useful in the proofs of results in Sections 3–4. The proof of Lemma 3.1 can be found in [33] with a minor variation.

Lemma 3.1. *For any $u \in \mathbb{R}^{n-r}$, (H2) together with (1.4)–(1.5) implies that $A_0^{12}(u, 0) = 0$.*

Lemma 3.2. *Assume (H5) holds and $m \ge 2$. If $\varepsilon > 0$ is sufficiently small, then there exists a maximal time $T^2_\varepsilon \in (0, T_\varepsilon) \cap (0, T_m]$, such that*

$$\|W^\varepsilon(t)\|_s \le \varepsilon, \quad \forall t \in [0, T^2_\varepsilon] \tag{3.6}$$

and

$$\text{either } \|W^\varepsilon(T^2_\varepsilon)\|_s = \varepsilon \text{ or } T^2_\varepsilon = T_m. \tag{3.7}$$

Moreover,

$$\|U^\varepsilon(t) - U_0(t)\|_s \le c\varepsilon + ce^{-\frac{\mu t}{\varepsilon^2}}, \quad \|U^\varepsilon(t)\|_s \le c, \quad \forall t \in [0, T^2_\varepsilon], \tag{3.8}$$

$$Im(U^\varepsilon(t, x)) \subset\subset G_0, \quad \forall (t, x) \in [0, T^2_\varepsilon] \times \mathbb{T}^d. \tag{3.9}$$

Proof. For all $T^1_\varepsilon \in (0, T_\varepsilon) \cap (0, T_m]$, since $W^\varepsilon \in C([0, T^1_\varepsilon], H^s)$, the function $t \longmapsto \|W^\varepsilon(t)\|_s$ is continuous on $[0, T^1_\varepsilon]$. Moreover, for any fixed integer $m \ge 2$, any fixed constant $c > 0$ and sufficiently small $\varepsilon > 0$, we always have $c\varepsilon^m < \varepsilon$.

If $T_m < T_\varepsilon$, then $[0, T_\varepsilon) \cap [0, T_m] = [0, T_m]$, which is a bounded closed interval. It follows from (2.1) that there exists a maximal time $T^2_\varepsilon \in (0, T_m]$, such that (3.6)–(3.7) hold. Otherwise, $T_m \ge T_\varepsilon$ and $[0, T_\varepsilon) \cap [0, T_m] = [0, T_\varepsilon)$. Since T_ε is the maximal existence time for U^ε, we have

$$\lim_{t \to T_\varepsilon^-} \|W^\varepsilon(t)\|_s = +\infty.$$

Hence, there still exists a maximal time $T^2_\varepsilon \in (0, T_\varepsilon)$, such that (3.6) holds and $\|W^\varepsilon(T^2_\varepsilon)\|_s = \varepsilon$. This proves (3.6)–(3.7). Finally, (3.8) follows from (3.6) and (2.2), and (3.9) follows from (3.6), $Im(U^m_\varepsilon) \subset\subset G_0$ and the continuous imbedding $H^s \hookrightarrow L^\infty$. □

3.2. Energy estimates

In general we start the energy estimates by an L^2-estimate for $\alpha = 0$ followed by higher order estimates for $|\alpha| \ge 1$. These estimates are indeed similar for the non-conservative system. In order to avoid repeated calculations, we consider a general estimate of order $|\alpha| \le s$ which includes the L^2-estimate as a particular case, by adopting a convention that $\|\cdot\|_{-1} = 0$.

In Lemmas 3.3–3.7 below, we establish the estimates for $I^\alpha_{1,\varepsilon}, \cdots, I^\alpha_{5,\varepsilon}$ on $[0, T^2_\varepsilon]$. For this purpose, we always assume that the conditions of Theorem 2.1 hold and we will repeatedly use (3.6), (3.8) and the continuous imbedding $H^s \hookrightarrow W^{1,\infty} \stackrel{def}{=} W^{1,\infty}(\mathbb{T}^d)$. For simplicity, in what follows we drop ε in W^ε and in $I^\alpha_{1,\varepsilon}, \cdots, I^\alpha_{5,\varepsilon}$, and we introduce

$$\nu_\varepsilon(t) = e^{-\frac{\mu t}{\varepsilon^2}}. \tag{3.10}$$

This function has already appeared in (H5)–(H6) and (3.8). We also write $W = \begin{bmatrix} W^I \\ W^{II} \end{bmatrix}$. From (1.4) and (1.5) we have

$$\partial_U Q(0, u, 0) = \begin{bmatrix} 0 & 0 \\ 0 & \partial_v q(u, 0) \end{bmatrix}, \quad \partial_U Q(0, u, 0)W = \begin{bmatrix} 0 \\ \partial_v q(u, 0)W^{II} \end{bmatrix}, \quad \forall u \in \mathbb{R}^{n-r}. \tag{3.11}$$

The strategy of the proof is to control each I_i^α $(i \neq 3)$ by

$$\frac{\delta}{\varepsilon^2}\|\partial^\alpha W^{II}(t)\|^2 + \frac{c}{\varepsilon^2}\|W^{II}(t)\|^2_{|\alpha|-1} + c\Big(1 + \frac{1}{\varepsilon}\nu_\varepsilon(t) + \frac{1}{\varepsilon^2}\|W^{II}(t)\|_s\Big)\|W(t)\|_s^2 + c\varepsilon^{2m}, \tag{3.12}$$

where $\delta > 0$ stands for an arbitrary small constant (independent of ε) to be chosen later and $c > 0$ may depend on δ. The first term in (3.12) will be absorbed by I_3^α due to the dissipation assumption (H2). Then the second term in (3.12) can be treated by an induction argument on $|\alpha|$. Finally, a Gronwall inequality yields the desired estimate.

Lemma 3.3. *It holds*

$$|I_1^\alpha(t)| \le \frac{\delta}{\varepsilon^2}\|\partial^\alpha W^{II}(t)\|^2 + c\Big(1 + \frac{1}{\varepsilon}\nu_\varepsilon(t) + \frac{1}{\varepsilon^2}\|W^{II}(t)\|_s\Big)\|W(t)\|_s^2, \quad \forall t \in [0, T_\varepsilon^2]. \tag{3.13}$$

Proof. Recall that

$$I_1^\alpha = \big\langle \operatorname{div}_\varepsilon A(U^\varepsilon)\partial^\alpha W, \partial^\alpha W\big\rangle,$$

where $\operatorname{div}_\varepsilon A(U^\varepsilon)$ is defined in (3.4). We first prove that

$$\big|\langle \partial_t A_0(U^\varepsilon)\partial^\alpha W, \partial^\alpha W\rangle\big| \le c\Big(1 + \frac{1}{\varepsilon}\nu_\varepsilon + \frac{1}{\varepsilon^2}\|W^{II}\|_s\Big)\|W\|_s^2. \tag{3.14}$$

Indeed, system (3.1) yields

$$\partial_t W = -\frac{1}{\varepsilon}\sum_{j=1}^d A_j(U^\varepsilon)\partial_{x_j} W + \frac{a^\varepsilon}{\varepsilon} + \frac{b^\varepsilon}{\varepsilon^2} - R_m^\varepsilon.$$

In view of the given expressions, we have obviously

$$\Big\|\sum_{j=1}^d A_j(U^\varepsilon)\partial_{x_j} W\Big\|_\infty \le c\|W\|_s \le c\varepsilon,$$

and

$$\|a^\varepsilon\|_\infty \le c\|W\|_s \le c\varepsilon.$$

From $m \ge 2$ and (H6), we also have

$$\|R_m^\varepsilon\|_\infty \le \varepsilon^{m-1}\|r_m\|_\infty + \varepsilon^{m-1}\|F_m^\varepsilon\|_\infty \le c.$$

Now we write b^ε as

$$\begin{aligned} b^\varepsilon &= Q(\varepsilon, U^\varepsilon) - Q(\varepsilon, U_\varepsilon^m) \\ &= Q(\varepsilon, U^\varepsilon) - Q(\varepsilon, U_\varepsilon^m) - \partial_U Q(\varepsilon, U_\varepsilon^m)W \\ &\quad + \big(\partial_U Q(\varepsilon, U_\varepsilon^m) - \partial_U Q(\varepsilon, U_0)\big)W + \big(\partial_U Q(\varepsilon, U_0) - \partial_U Q(0, U_0)\big)W + \partial_U Q(0, U_0)W. \end{aligned}$$

Noting (3.11), we obtain from (H5), (3.6) and (3.8) that

$$\|b^\varepsilon\|_\infty \le c\varepsilon^2 + c\varepsilon\nu_\varepsilon + c\|W^{II}\|_s,$$

which implies that

$$\|\partial_t W\|_\infty \le c + \frac{c}{\varepsilon}\nu_\varepsilon + \frac{c}{\varepsilon^2}\|W^{II}\|_s.$$

Therefore, (3.14) follows from (H5) and

$$\partial_t A_0(U^\varepsilon) = A_0'(U^\varepsilon)(\partial_t W + \partial_t U_\varepsilon^m).$$

Next, since $\tilde{A}_j(U^\varepsilon)$ is symmetric, we have

$$\begin{aligned}\langle \partial_{x_j}\tilde{A}_j(U^\varepsilon)\partial^\alpha W, \partial^\alpha W\rangle &= \langle \partial_{x_j}\tilde{A}_j^{11}(U^\varepsilon)\partial^\alpha W^I, \partial^\alpha W^I\rangle + 2\langle \partial_{x_j}\tilde{A}_j^{12}(U^\varepsilon)\partial^\alpha W^{II}, \partial^\alpha W^I\rangle \\ &\quad + \langle \partial_{x_j}\tilde{A}_j^{22}(U^\varepsilon)\partial^\alpha W^{II}, \partial^\alpha W^{II}\rangle.\end{aligned}$$

The last two terms on the right-hand side are bounded by

$$c\|W\|_s\|\partial^\alpha W^{II}\| \le \frac{\delta}{\varepsilon}\|\partial^\alpha W^{II}\|^2 + c\varepsilon\|W\|_s^2.$$

For the first term, we use (H1) to get

$$\begin{aligned}\left|\langle \partial_{x_j}\tilde{A}_j^{11}(U^\varepsilon)\partial^\alpha W^I, \partial^\alpha W^I\rangle\right| &= \left|\langle \partial_{x_j}(\tilde{A}_j^{11}(U^\varepsilon) - \tilde{A}_j^{11}(U_0))\partial^\alpha W^I, \partial^\alpha W^I\rangle\right| \\ &\le c\|\partial_{x_j}(U^\varepsilon - U_0)\|_\infty \|W\|_s^2 \\ &\le c(\varepsilon + \nu_\varepsilon)\|W\|_s^2.\end{aligned}$$

Hence,

$$\frac{1}{\varepsilon}\left|\left\langle \sum_{j=1}^d \partial_{x_j}\tilde{A}_j(U^\varepsilon)\partial^\alpha W, \partial^\alpha W\right\rangle\right| \le c\left(1 + \frac{1}{\varepsilon}\nu_\varepsilon\right)\|W\|_s^2.$$

Together with (3.14), this yields (3.13). □

Lemma 3.4. *It holds*

$$|I_2^\alpha(t)| \le \frac{\delta}{\varepsilon^2}\|\partial^\alpha W^{II}(t)\|^2 + \frac{c}{\varepsilon^2}\|W^{II}(t)\|_{|\alpha|-1}^2 + c\left(1 + \frac{1}{\varepsilon}\nu_\varepsilon(t)\right)\|W(t)\|_s^2, \quad \forall t \in [0, T_\varepsilon^2]. \tag{3.15}$$

Proof. Recall that

$$\begin{aligned}I_2^\alpha &= \frac{2}{\varepsilon}\langle A_0(U^\varepsilon)\partial^\alpha a^\varepsilon, \partial^\alpha W\rangle \\ &= \frac{2}{\varepsilon}\langle [A_0(U^\varepsilon) - A_0(U_0)]\partial^\alpha a^\varepsilon, \partial^\alpha W\rangle + \frac{2}{\varepsilon}\langle A_0(U_0)\partial^\alpha a^\varepsilon, \partial^\alpha W\rangle,\end{aligned} \tag{3.16}$$

where a^ε is defined in (3.2). From (H5) and (3.6), it is clear that

$$\|\partial^\alpha a^\varepsilon\| \le \|a^\varepsilon\|_s \le c\|W\|_s.$$

Similarly, (3.8) yields

$$\|A_0(U^\varepsilon) - A_0(U_0)\|_\infty \le c(\varepsilon + \nu_\varepsilon).$$

Therefore,

$$\frac{2}{\varepsilon}\left|\langle [A_0(U^\varepsilon) - A_0(U_0)]\partial^\alpha a^\varepsilon, \partial^\alpha W\rangle\right| \le c\left(1 + \frac{1}{\varepsilon}\nu_\varepsilon\right)\|W\|_s^2. \tag{3.17}$$

For the second term in (3.16), by Lemma 3.1 we have $A_0^{12}(U_0) = 0$. Then a straightforward calculation gives

$$\begin{aligned}\langle A_0(U_0)\partial^\alpha a^\varepsilon, \partial^\alpha W\rangle &= \sum_{j=1}^d \langle A_0(U_0)\partial^\alpha[(A_j(U_\varepsilon^m) - A_j(U^\varepsilon))\partial_{x_j}U_\varepsilon^m], \partial^\alpha W\rangle \\ &= \sum_{j=1}^d \langle A_0^{11}(U_0)\partial^\alpha[(A_j^{11}(U_\varepsilon^m) - A_j^{11}(U^\varepsilon))\partial_{x_j}u_\varepsilon^m], \partial^\alpha W^I\rangle \\ &\quad + \sum_{j=1}^d \langle A_0^{11}(U_0)\partial^\alpha\left[(A_j^{12}(U_\varepsilon^m) - A_j^{12}(U^\varepsilon))\partial_{x_j}v_\varepsilon^m\right], \partial^\alpha W^I\rangle\end{aligned}$$

$$+\sum_{j=1}^{d}\langle A_0^{22}(U_0)\partial^\alpha\big[(A_j^{21}(U_\varepsilon^m)-A_j^{21}(U^\varepsilon))\partial_{x_j}u_\varepsilon^m\big],\partial^\alpha W^{II}\rangle$$
$$+\sum_{j=1}^{d}\langle A_0^{22}(U_0)\partial^\alpha\Big[(A_j^{22}(U_\varepsilon^m)-A_j^{22}(U^\varepsilon))\partial_{x_j}v_\varepsilon^m\Big],\partial^\alpha W^{II}\rangle. \tag{3.18}$$

Obviously, the last two terms in (3.18) are bounded by

$$c\|W\|_s\|\partial^\alpha W^{II}\|\le\frac{\delta}{\varepsilon}\|\partial^\alpha W^{II}\|^2+c\varepsilon\|W\|_s^2.$$

Since $v_0=0$, (H5) yields $\|v_\varepsilon^m\|_s\le c\varepsilon+c\nu_\varepsilon$. Therefore,

$$\Big|\sum_{j=1}^{d}\langle A_0^{11}(U_0)\partial^\alpha\Big[(A_j^{12}(U_\varepsilon^m)-A_j^{12}(U^\varepsilon))\partial_{x_j}v_\varepsilon^m\Big],\partial^\alpha W^I\rangle\Big|\le c\,(\varepsilon+\nu_\varepsilon)\,\|W\|_s^2.$$

For the first term in (3.18), we have

$$\begin{aligned}A_j^{11}(U_\varepsilon^m)-A_j^{11}(U^\varepsilon)&=\big(A_j^{11}(u_\varepsilon^m,v_\varepsilon^m)-A_j^{11}(u^\varepsilon,v_\varepsilon^m)\big)+\big(A_j^{11}(u^\varepsilon,v_\varepsilon^m)-A_j^{11}(u^\varepsilon,v^\varepsilon)\big)\\&=-\int_0^1\partial_uA_j^{11}(u_\varepsilon^m+\theta(u^\varepsilon-u_\varepsilon^m),v_\varepsilon^m)W^I\,d\theta\\&\quad-\int_0^1\partial_vA_j^{11}(u^\varepsilon,v_\varepsilon^m+\theta(v^\varepsilon-v_\varepsilon^m))W^{II}\,d\theta.\end{aligned}$$

The second integral above is easily estimated due to the appearance of W^{II}. The first one can be treated due to condition $\partial_uA_j^{11}(u_0,0)=0$ in (H1). Precisely, we write

$$\begin{aligned}&\partial_uA_j^{11}\big(u_\varepsilon^m+\theta(u^\varepsilon-u_\varepsilon^m),v_\varepsilon^m\big)\\&=\big[\partial_uA_j^{11}\big(u_\varepsilon^m+\theta(u^\varepsilon-u_\varepsilon^m),v_\varepsilon^m\big)-\partial_uA_j^{11}(u_0,v_\varepsilon^m)\big]+\big[\partial_uA_j^{11}(u_0,v_\varepsilon^m)-\partial_uA_j^{11}(u_0,0)\big]\\&=\int_0^1\partial_{uu}^2A_j^{11}\big((1-\theta')u_0+\theta'(u_\varepsilon^m+\theta(u^\varepsilon-u_\varepsilon^m)),v_\varepsilon^m\big)\big(u_\varepsilon^m-u_0+\theta(u^\varepsilon-u_\varepsilon^m)\big)d\theta'\\&\quad+\int_0^1\partial_{uv}^2A_j^{11}(u_0,\theta'v_\varepsilon^m)v_\varepsilon^m\,d\theta'.\end{aligned}$$

Hence,

$$\begin{aligned}&-\int_0^1\partial_uA_j^{11}\big(u_\varepsilon^m+\theta(u^\varepsilon-u_\varepsilon^m),v_\varepsilon^m\big)W^I\,d\theta\\&=-\int_0^1\int_0^1\partial_{uu}^2A_j^{11}\big((1-\theta')u_0+\theta'((u_\varepsilon^m+\theta(u^\varepsilon-u_\varepsilon^m)),v_\varepsilon^m\big)\big(u_\varepsilon^m-u_0+\theta(u^\varepsilon-u_\varepsilon^m),W^I\big)\,d\theta d\theta'\\&\quad-\int_0^1\int_0^1\partial_{uv}^2A_j^{11}(u_0,\theta'v_\varepsilon^m)(W^I,v_\varepsilon^m)\,d\theta d\theta'.\end{aligned}$$

Since $\|U_\varepsilon^m\|_s\le c\varepsilon+c\nu_\varepsilon$, it follows that

$$\Big|\sum_{j=1}^{d}\langle A_0^{11}(U_0)\partial^\alpha\big[(A_j^{11}(U_\varepsilon^m)-A_j^{11}(U^\varepsilon))\partial_{x_j}u_\varepsilon^m\big],\partial^\alpha W^I\rangle\Big|$$
$$\le \frac{\delta}{\varepsilon}\|\partial^\alpha W^{II}(t)\|^2+\frac{c}{\varepsilon}\|W^{II}(t)\|^2_{|\alpha|-1}+c\,(\varepsilon+\nu_\varepsilon)\,\|W\|_s^2.$$

This proves

$$\frac{2}{\varepsilon}\big|\langle A_0(U_0)\partial^\alpha a^\varepsilon,\partial^\alpha W\rangle\big|\le\frac{\delta}{\varepsilon^2}\|\partial^\alpha W^{II}(t)\|^2+\frac{c}{\varepsilon^2}\|W^{II}(t)\|^2_{|\alpha|-1}+c\left(1+\frac{1}{\varepsilon}\nu_\varepsilon\right)\|W\|_s^2.$$

Together with (3.16)–(3.17) it yields (3.15). □

Lemma 3.5. *It holds*

$$I_3^\alpha(t)\le\frac{4\delta-2c_0}{\varepsilon^2}\|\partial^\alpha W^{II}(t)\|^2+\frac{c}{\varepsilon^2}\|W^{II}(t)\|^2_{|\alpha|-1}+c\Big(1+\frac{1}{\varepsilon^2}\nu_\varepsilon(t)\Big)\|W(t)\|_s^2,\ \ \forall t\in[0,T_\varepsilon^2],\tag{3.19}$$

where the positive constant c_0 is given in (H2).

Proof. Recall that

$$I_3^\alpha=\frac{2}{\varepsilon^2}\langle A_0(U^\varepsilon)\partial^\alpha b^\varepsilon,\partial^\alpha W\rangle,$$

where b^ε is defined in (3.3). We first write b^ε as

$$\begin{aligned}b^\varepsilon=\;&\partial_U Q(0,U_0)W\\&+\varepsilon\partial_U\partial_\varepsilon Q(0,U_0)W\\&+\big[Q(0,U^\varepsilon)-Q(0,U_\varepsilon^m)-\partial_U Q(0,U_0)W\big]\\&+\varepsilon\big[\partial_\varepsilon Q(0,U^\varepsilon)-\partial_\varepsilon Q(0,U_\varepsilon^m)-\partial_U\partial_\varepsilon Q(0,U_0)W\big]\\&+\big[Q(\varepsilon,U^\varepsilon)-Q(0,U^\varepsilon)-\varepsilon\partial_\varepsilon Q(0,U^\varepsilon)-\big(Q(\varepsilon,U_\varepsilon^m)-Q(0,U_\varepsilon^m)-\varepsilon\partial_\varepsilon Q(0,U_\varepsilon^m)\big)\big],\end{aligned}$$

which implies that

$$I_3^\alpha=I_{31}^\alpha+I_{32}^\alpha+I_{33}^\alpha+I_{34}^\alpha+I_{35}^\alpha,$$

with the natural correspondence for $I_{31}^\alpha,\cdots,I_{35}^\alpha$. Now we estimate each of these terms.

(i) For I_{31}^α we write

$$\begin{aligned}A_0(U^\varepsilon)\partial^\alpha\left[\partial_U Q(0,U_0)W\right]=\;&A_0(U_0)\partial_U Q(0,U_0)\partial^\alpha W\\&+A_0(U_0)\big[\partial^\alpha\left(\partial_U Q(0,U_0)W\right)-\partial_U Q(0,U_0)\partial^\alpha W\big]\\&+\big[A_0(U^\varepsilon)-A_0(U_0)\big]\partial^\alpha\left[\partial_U Q(0,U_0)W\right].\end{aligned}$$

Then (H2) implies that

$$\langle A_0(U_0)\partial_U Q(0,U_0)\partial^\alpha W,\partial^\alpha W\rangle\le -c_0\|\partial^\alpha W^{II}\|^2.$$

Noting (3.11) and $A_0^{12}(U_0)=0$, we obtain

$$\begin{aligned}&\langle A_0(U_0)[\partial^\alpha(\partial_U Q(0,U_0)W)-\partial_U Q(0,U_0)\partial^\alpha W],\partial^\alpha W\rangle\\=&\langle A_0^{22}(U_0)\big[\partial^\alpha(\partial_v q(U_0)W^{II})-\partial_v q(U_0)\partial^\alpha W^{II}\big],\partial^\alpha W^{II}\rangle.\end{aligned}$$

This term vanishes when $\alpha=0$. Hence, for all $|\alpha|\le s$, by the Moser-type calculus inequalities (see [14,20]), we always have

$$\begin{aligned}&\langle A_0(U_0)[\partial^\alpha(\partial_U Q(0,U_0)W)-\partial_U Q(0,U_0)\partial^\alpha W],\partial^\alpha W\rangle\\\le\;& c\|W^{II}\|_{|\alpha|-1}.\|\partial^\alpha W^{II}\|\\\le\;&\frac{\delta}{4}\|\partial^\alpha W^{II}\|^2+c\|W^{II}\|^2_{|\alpha|-1}.\end{aligned}$$

For the last term in I_{31}^α, a similar calculation yields

$$\begin{aligned}
&\big\langle [A_0(U^\varepsilon)-A_0(U_0)]\partial^\alpha[\partial_U Q(0,U_0)W],\partial^\alpha W\big\rangle\\
&=\big\langle\big[A_0^{12}(U^\varepsilon)-A_0^{12}(U_0)\big]\partial^\alpha\big[\partial_v q(U_0)W^{II}\big],\partial^\alpha W^I\big\rangle\\
&\quad+\big\langle\big[A_0^{22}(U^\varepsilon)-A_0^{22}(U_0)\big]\partial^\alpha\big[\partial_v q(U_0)W^{II}\big],\partial^\alpha W^{II}\big\rangle.
\end{aligned}$$

Since $\|A_0(U^\varepsilon)-A_0(U_0)\|_\infty\le c\varepsilon+c\nu_\varepsilon$ and $\nu_\varepsilon^2\le\nu_\varepsilon$, it is clear that

$$\begin{aligned}
&\big\langle\big[A_0(U^\varepsilon)-A_0(U_0)\big]\partial^\alpha\,[\partial_U Q(0,U_0)W]\,,\partial^\alpha W\big\rangle\\
&\le c\,(\varepsilon+\nu_\varepsilon)\,\|W^{II}\|_{|\alpha|}.\|W\|_{|\alpha|}\\
&\le\frac{\delta}{4}\|\partial^\alpha W^{II}\|^2+c\|W^{II}\|_{|\alpha|-1}^2+c(\varepsilon^2+\nu_\varepsilon)\|W\|_s^2.
\end{aligned}$$

This shows that

$$I_{31}^\alpha\le\frac{\delta-2c_0}{\varepsilon^2}\|\partial^\alpha W^{II}\|^2+\frac{c}{\varepsilon^2}\|W^{II}\|_{|\alpha|-1}^2+c\left(1+\frac{1}{\varepsilon^2}\nu_\varepsilon\right)\|W\|_s^2.$$

(ii) For I_{32}^α, (H3) yields

$$\partial_U\partial_\varepsilon Q(0,U_0)=\begin{bmatrix}0 & \partial_v\partial_\varepsilon Q^I(0,U_0)\\ \partial_u\partial_\varepsilon Q^{II}(0,U_0) & \partial_v\partial_\varepsilon Q^{II}(0,U_0)\end{bmatrix}.$$

Hence,

$$\begin{aligned}
I_{32}^\alpha=&\frac{2}{\varepsilon}\big\langle A_0^{11}(U^\varepsilon)\partial^\alpha\big(\partial_v\partial_\varepsilon Q^I(0,U_0)W^{II}\big),\partial^\alpha W^I\big\rangle\\
&+\frac{2}{\varepsilon}\big\langle\big[A_0^{12}(U^\varepsilon)-A_0^{12}(U_0)\big]\partial^\alpha\big(\partial_u\partial_\varepsilon Q^{II}(0,U_0)W^I\big),\partial^\alpha W^I\big\rangle\\
&+\frac{2}{\varepsilon}\big\langle A_0^{12}(U^\varepsilon)\partial^\alpha\big(\partial_v\partial_\varepsilon Q^{II}(0,U_0)W^{II}\big),\partial^\alpha W^I\big\rangle\\
&+\frac{2}{\varepsilon}\big\langle A_0^{21}(U^\varepsilon)\partial^\alpha\big[\partial_v\partial_\varepsilon Q^I(0,U_0)W^{II}\big],\partial^\alpha W^{II}\big\rangle\\
&+\frac{2}{\varepsilon}\big\langle A_0^{22}(U^\varepsilon)\partial^\alpha\big[\partial_u\partial_\varepsilon Q^{II}(0,U_0)W^I+\partial_v\partial_\varepsilon Q^{II}(0,U_0)W^{II}\big],\partial^\alpha W^{II}\big\rangle,
\end{aligned}$$

in which each term is quadratic containing W^{II}, except for the second one, which is obviously bounded by $c\left(1+\frac{1}{\varepsilon}\nu_\varepsilon\right)\|W\|_s^2$. Hence,

$$|I_{32}^\alpha|\le\frac{\delta}{\varepsilon^2}\|\partial^\alpha W^{II}\|^2+\frac{c}{\varepsilon^2}\|W^{II}\|_{|\alpha|-1}^2+c\left(1+\frac{1}{\varepsilon}\nu_\varepsilon\right)\|W\|_s^2.$$

(iii) For I_{33}^α, since $\partial_u q(U_0)=0$, we have (3.11) and $\partial_v q(U_0)W^{II}=\partial_U q(U_0)W$. Hence,

$$\begin{aligned}
&Q(0,U^\varepsilon)-Q(0,U_\varepsilon^m)-\partial_U Q(0,U_0)W\\
&=\begin{bmatrix}0\\ q(U^\varepsilon)-q(U_\varepsilon^m)-\partial_U q(U_\varepsilon^m)W\end{bmatrix}+\begin{bmatrix}0\\ (\partial_U q(U_\varepsilon^m)-\partial_U q(U_0))W\end{bmatrix}.
\end{aligned}$$

By the Taylor formula, it is clear that

$$\|\partial^\alpha\big(q(U^\varepsilon)-q(U_\varepsilon^m)-\partial_U q(U_\varepsilon^m)W\big)\|\le c\|W\|_s^2\le c\varepsilon\|W\|_s$$

and

$$\|\partial^\alpha\big(\partial_U q(U_\varepsilon^m)-\partial_U q(U_0))W\big)\|\le c(\varepsilon+\nu_\varepsilon)\|W\|_s.$$

Since $A_0^{12}(U_0)=A_0^{21}(U_0)=0$, we have

$$
\begin{aligned}
&\frac{2}{\varepsilon^2}\left|\left\langle A_0(U_0)\partial^\alpha\big(Q(0,U^\varepsilon)-Q(0,U_\varepsilon^m)-\partial_U Q(0,U_0)W\big),\partial^\alpha W\right\rangle\right|\\
&\le \frac{2}{\varepsilon^2}\left|\left\langle A_0^{22}(U_0)\partial^\alpha\big(q(U^\varepsilon)-q(U_\varepsilon^m)-\partial_U q(U_\varepsilon^m)W\big),\partial^\alpha W^{II}\right\rangle\right|\\
&\quad+\frac{2}{\varepsilon^2}\left|\left\langle A_0^{22}(U_0)\partial^\alpha\big(\partial_U q(U_\varepsilon^m)-\partial_U q(U_0)\big)W\big),\partial^\alpha W^{II}\right\rangle\right|\\
&\le \frac{c}{\varepsilon^2}(\varepsilon+\nu_\varepsilon)\|W\|_s\|\partial^\alpha W^{II}\|\\
&\le \frac{\delta}{2\varepsilon^2}\|\partial^\alpha W^{II}\|^2+c\left(1+\frac{1}{\varepsilon^2}\nu_\varepsilon\right)\|W\|_s^2.
\end{aligned}
$$

Now

$$
\begin{aligned}
I_{33}^\alpha&=\frac{2}{\varepsilon^2}\left\langle\big(A_0(U^\varepsilon)-A_0(U_0)\big)\partial^\alpha\big(Q(0,U^\varepsilon)-Q(0,U_\varepsilon^m)-\partial_U Q(0,U_0)W\big),\partial^\alpha W\right\rangle\\
&\quad+\frac{2}{\varepsilon^2}\left\langle A_0(U_0)\partial^\alpha\big(Q(0,U^\varepsilon)-Q(0,U_\varepsilon^m)-\partial_U Q(0,U_0)W\big),\partial^\alpha W\right\rangle.
\end{aligned}
$$

The first term can be estimated as above by using $\|U^\varepsilon-U_0\|_s\le c(\varepsilon+\nu_\varepsilon)$. Therefore,

$$
|I_{33}^\alpha|\le\frac{\delta}{\varepsilon^2}\|\partial^\alpha W^{II}\|^2+c\left(1+\frac{1}{\varepsilon^2}\nu_\varepsilon\right)\|W\|_s^2.
$$

(iv) Similarly, we obtain

$$
\begin{aligned}
|I_{34}^\alpha|&=\varepsilon\left|\left\langle A_0(U^\varepsilon)\partial^\alpha\big(\partial_\varepsilon Q(0,U^\varepsilon)-\partial_\varepsilon Q(0,U_\varepsilon^m)-\partial_U\partial_\varepsilon Q(0,U_0)W\big),\partial^\alpha W\right\rangle\right|\\
&\le\frac{\delta}{\varepsilon^2}\|\partial^\alpha W^{II}\|^2+\frac{c}{\varepsilon^2}\|W^{II}\|_{|\alpha|-1}^2+c\left(1+\frac{1}{\varepsilon}\nu_\varepsilon\right)\|W\|_s^2.
\end{aligned}
$$

(v) For this last term in I_3^α, we have

$$
\begin{aligned}
&Q(\varepsilon,U^\varepsilon)-Q(0,U^\varepsilon)-\varepsilon\partial_\varepsilon Q(0,U^\varepsilon)-\big(Q(\varepsilon,U_\varepsilon^m)-Q(0,U_\varepsilon^m)-\varepsilon\partial_\varepsilon Q(0,U_\varepsilon^m)\big)\\
&=\varepsilon^2\int_0^1\int_0^1\partial_U\partial_{\varepsilon\varepsilon}^2 Q\big(\theta\varepsilon,\tau U^\varepsilon+(1-\tau)U_\varepsilon^m\big)W\,d\tau\,d\theta,
\end{aligned}
$$

which implies that

$$
|I_{35}^\alpha|\le\frac{2}{\varepsilon^2}c\varepsilon^2\|W\|_s\|W\|_s\le c\|W\|_s^2.
$$

From the estimates in (i)–(v), we obtain (3.19). □

Lemma 3.6. *It holds*

$$
|I_4^\alpha(t)|\le\frac{\delta}{\varepsilon^2}\|\partial^\alpha W^{II}(t)\|^2+c\left(1+\frac{1}{\varepsilon^2}\nu_\varepsilon(t)\right)\|W(t)\|_s^2+c\varepsilon^{2m},\quad\forall t\in[0,T_\varepsilon^2].\tag{3.20}
$$

Proof. We have

$$
\begin{aligned}
I_4^\alpha&=-2\left\langle A_0(U^\varepsilon)\partial^\alpha R^\varepsilon,\partial^\alpha W\right\rangle\\
&=-2\left\langle\big[A_0(U^\varepsilon)-A_0(U_0)\big]\partial^\alpha R^\varepsilon,\partial^\alpha W\right\rangle-2\left\langle A_0(U_0)\partial^\alpha R^\varepsilon,\partial^\alpha W\right\rangle.
\end{aligned}
$$

Since

$$
\|A_0(U^\varepsilon)-A_0(U_0)\|_\infty\le c\|U^\varepsilon-U_0\|_\infty\le\|U^\varepsilon-U_0\|_s\le c(\varepsilon+\nu_\varepsilon),
$$

using (H6) and (3.6) we obtain

$$\begin{aligned}\left|\left\langle\left[A_0(U^\varepsilon)-A_0(U_0)\right]\partial^\alpha R^\varepsilon,\partial^\alpha W\right\rangle\right| &\le c\varepsilon^m\left(1+\frac{1}{\varepsilon}\nu_\varepsilon\right)(\|r_m\|_s+\|F_m^\varepsilon\|_s)\|W\|_s\\ &\le c\left(1+\frac{1}{\varepsilon^2}\nu_\varepsilon\right)\|W\|_s^2+c\varepsilon^{2m}.\end{aligned}$$

For the second term in I_4^α, we use $A_0^{12}(U_0)=A_0^{21}(U_0)=0$ to get

$$\begin{aligned}\left|\left\langle A_0(U_0)\partial^\alpha R^\varepsilon,\partial^\alpha W\right\rangle\right| &= \varepsilon^{m-1}\left|\left\langle A_0^{22}(U_0)\partial^\alpha r_m,\partial^\alpha W^{II}\right\rangle+\left\langle A_0(U_0)\partial^\alpha F_m^\varepsilon,\partial^\alpha W\right\rangle\right|\\ &\le \frac{\delta}{2\varepsilon^2}\|\partial^\alpha W^{II}\|^2+c\left(1+\frac{1}{\varepsilon^2}\nu_\varepsilon\right)\|W\|_s^2+c\varepsilon^{2m}.\end{aligned}$$

This proves (3.20). □

Lemma 3.7. *It holds*

$$|I_5^\alpha(t)|\le\frac{\delta}{\varepsilon^2}\|\partial^\alpha W^{II}(t)\|^2+\frac{c}{\varepsilon^2}\|W^{II}(t)\|_{|\alpha|-1}^2+c\left(1+\frac{1}{\varepsilon}\nu_\varepsilon(t)\right)\|W(t)\|_s^2,\quad\forall t\in[0,T_\varepsilon^2]. \tag{3.21}$$

Proof. Recall that

$$I_5^\alpha=\frac{2}{\varepsilon}\sum_{j=1}^d\left\langle A_0(U^\varepsilon)f_{\alpha j}^\varepsilon,\partial^\alpha W\right\rangle,$$

where $f_{\alpha j}^\varepsilon$ is defined in (3.5). Then

$$\begin{aligned}A_0(U^\varepsilon)f_{\alpha j}^\varepsilon &= A_0(U^\varepsilon)\left[\left(A_j(U^\varepsilon)-A_j(U_0)\right)\partial_{x_j}(\partial^\alpha W)-\partial^\alpha\left(\left(A_j(U^\varepsilon)-A_j(U_0)\right)\partial_{x_j}W\right)\right]\\ &\quad+\left(A_0(U^\varepsilon)-A_0(U_0)\right)\left[A_j(U_0)\partial_{x_j}(\partial^\alpha W)-\partial^\alpha\left(A_j(U_0)\partial_{x_j}W\right)\right]\\ &\quad+A_0(U_0)\left[A_j(U_0)\partial_{x_j}(\partial^\alpha W)-\partial^\alpha\left(A_j(U_0)\partial_{x_j}W\right)\right].\end{aligned}$$

Applying the Moser-type calculus inequalities together with

$$\|A_j(U^\varepsilon)-A_j(U_0)\|_s\le c(\varepsilon+\nu_\varepsilon),\qquad \|A_0(U^\varepsilon)-A_0(U_0)\|_s\le c(\varepsilon+\nu_\varepsilon),$$

the first two terms in $|I_5^\alpha|$ are bounded by $c\left(1+\frac{1}{\varepsilon}\nu_\varepsilon\right)\|W\|_s^2$.

For the last term in $|I_5^\alpha|$, we use again $A_0^{12}(U_0)=A_0^{21}(U_0)=0$. Since $A_j^{11}(U_0)$ is constant (thanks to (H1)), a straightforward calculation yields

$$\begin{aligned}&\left\langle A_0(U_0)\left[A_j(U_0)\partial_{x_j}(\partial^\alpha W)-\partial^\alpha\left(A_j(U_0)\partial_{x_j}W\right)\right],\partial^\alpha W\right\rangle\\ &=\left\langle A_0^{11}(U_0)\left[A_j^{12}(U_0)\partial_{x_j}(\partial^\alpha W^{II})-\partial^\alpha\left(A_j^{12}(U_0)\partial_{x_j}W^{II}\right)\right],\partial^\alpha W^I\right\rangle\\ &\quad+\left\langle A_0^{22}(U_0)\left[A_j^{21}(U_0)\partial_{x_j}(\partial^\alpha W^I)-\partial^\alpha\left(A_j^{21}(U_0)\partial_{x_j}W^I\right)\right],\partial^\alpha W^{II}\right\rangle\\ &\quad+\left\langle A_0^{22}(U_0)\left[A_j^{22}(U_0)\partial_{x_j}(\partial^\alpha W^{II})-\partial^\alpha\left(A_j^{22}(U_0)\partial_{x_j}W^{II}\right)\right],\partial^\alpha W^{II}\right\rangle,\end{aligned}$$

in which each term on the right-hand side contains W^{II}. By the Moser-type calculus inequalities, it is easy to see that

$$\begin{aligned}&\left|\left\langle A_0(U_0)\left[A_j(U_0)\partial_{x_j}(\partial^\alpha W)-\partial^\alpha\left(A_j(U_0)\partial_{x_j}W\right)\right],\partial^\alpha W\right\rangle\right|\\ &\le c\|W^{II}\|_{|\alpha|}\|W\|_s\\ &\le\frac{\delta}{2\varepsilon^2}\|\partial^\alpha W^{II}\|^2+\frac{c}{\varepsilon^2}\|W^{II}\|_{|\alpha|-1}^2+c\varepsilon^2\|W\|_s^2.\end{aligned}$$

This implies (3.21). □

3.3. Proof of Theorem 2.1

Adding the estimates in Lemmas 3.3–3.7 and taking δ to be sufficiently small, we conclude the following result.

Lemma 3.8. *There is a constant $c_1 \in (0, c_0]$, independent of ε, such that for all $t \in [0, T_\varepsilon^2]$ and all $\alpha \in \mathbb{N}^d$ with $|\alpha| \le s$, it holds*

$$\frac{d}{dt}\langle A_0(U^\varepsilon)\partial^\alpha W, \partial^\alpha W\rangle + \frac{c_1}{\varepsilon^2}\|\partial^\alpha W^{II}(t)\|^2 \le \frac{c}{\varepsilon^2}\|W^{II}(t)\|^2_{|\alpha|-1} + c\left(1+\frac{1}{\varepsilon^2}\nu_\varepsilon(t)\right)\|W(t)\|_s^2 + \frac{c}{\varepsilon^2}\|W^{II}(t)\|_s\,\|W(t)\|_s^2 + c\varepsilon^{2m}. \tag{3.22}$$

By an induction argument together with Lemma 3.8, we obtain the final energy estimate in H^s as follows.

Proposition 3.1. *Under the assumptions of Theorem 2.1, it holds*

$$\|W(t)\|_s^2 + \frac{1}{\varepsilon^2}\int_0^t \|W^{II}(t')\|_s^2\, dt' \le c\varepsilon^{2m}, \quad \forall t \in [0, T_\varepsilon^2]. \tag{3.23}$$

Proof. Recall $\|W^{II}\|_{-1} = 0$. Applying Lemma 3.8 with $|\alpha| = 1$, we see that $\frac{c}{\varepsilon^2}\|W^{II}\|^2$ on the right-hand side of (3.22) can be controlled by $\frac{c_1}{\varepsilon^2}\|W^{II}\|^2$ on the left-hand side of (3.22) with $|\alpha| = 0$. More generally, let $\eta \in (0, 1]$. Multiplying (3.22) by $\eta^{|\alpha|}$ and summing up the equalities for all index $\alpha \in \mathbb{N}^d$ with $|\alpha| \le s$ yields

$$\frac{d}{dt}\sum_{|\alpha|\le s}\eta^{|\alpha|}\langle A_0(U^\varepsilon)\partial^\alpha W, \partial^\alpha W\rangle + \frac{c_1}{\varepsilon^2}\sum_{|\alpha|\le s}\eta^{|\alpha|}\|\partial^\alpha W^{II}(t)\|^2$$
$$\le \frac{c}{\varepsilon^2}\sum_{|\alpha|\le s-1}\eta^{|\alpha|+1}\|W^{II}(t)\|^2_{|\alpha|} + c\left(1+\frac{1}{\varepsilon^2}\nu_\varepsilon(t)\right)\|W(t)\|_s^2 + \frac{c}{\varepsilon^2}\|W^{II}(t)\|_s\,\|W(t)\|_s^2 + c\varepsilon^{2m},$$

in which c is independent of η. Let η be suitably small. Then

$$\frac{c}{\varepsilon^2}\sum_{|\alpha|\le s-1}\eta^{|\alpha|+1}\|W^{II}(t)\|^2_{|\alpha|} \le \frac{c_1}{2\varepsilon^2}\sum_{|\alpha|\le s}\eta^{|\alpha|}\|\partial^\alpha W^{II}(t)\|^2$$

and

$$\frac{c_1\eta^s}{2\varepsilon^2}\|W^{II}(t)\|_s^2 \le \frac{c_1}{2\varepsilon^2}\sum_{|\alpha|\le s}\eta^{|\alpha|}\|\partial^\alpha W^{II}(t)\|^2.$$

Therefore,

$$\frac{d}{dt}\sum_{|\alpha|\le s}\eta^{|\alpha|}\langle A_0(U^\varepsilon)\partial^\alpha W, \partial^\alpha W\rangle + \frac{c_1\eta^s}{2\varepsilon^2}\|W^{II}(t)\|_s^2 \le c\left(1+\frac{1}{\varepsilon^2}\nu_\varepsilon(t)\right)\|W(t)\|_s^2 + \frac{c}{\varepsilon^2}\|W^{II}(t)\|_s\,\|W(t)\|_s^2 + c\varepsilon^{2m}.$$

By the Young inequality, we have

$$c\|W^{II}(t)\|_s\,\|W(t)\|_s^2 \le \frac{c_1\eta^s}{4}\|W^{II}(t)\|_s^2 + \frac{c^2}{c_1\eta^s}\|W(t)\|_s^4.$$

It follows from $\|W(t)\|_s \le c\varepsilon$ for all $t \in [0, T_\varepsilon^2]$ that

$$\frac{d}{dt}\sum_{|\alpha|\le s}\eta^{|\alpha|}\langle A_0(U^\varepsilon)\partial^\alpha W, \partial^\alpha W\rangle + \frac{c_1\eta^s}{4\varepsilon^2}\|W^{II}(t)\|_s^2 \le c\left(1+\frac{1}{\eta^s}+\frac{1}{\varepsilon^2}\nu_\varepsilon(t)\right)\|W(t)\|_s^2 + c\varepsilon^{2m}.$$

Now we fix $\eta > 0$. Integrating this inequality over $[0, t]$ with $t \le T_\varepsilon^2$ and noting that $\sum_{|\alpha|\le s} \eta^{|\alpha|}\langle A_0(U^\varepsilon)\partial^\alpha W, \partial^\alpha W\rangle$ is equivalent to $\|W\|_s^2$, we use (H5) to obtain

$$\|W(t)\|_s^2 + \frac{1}{\varepsilon^2}\int_0^t \|W^{II}(t')\|_s^2\,dt' \le c\int_0^t \left(1 + \frac{1}{\varepsilon^2}\nu_\varepsilon(t')\right)\|W(t')\|_s^2\,dt' + c\varepsilon^{2m}, \quad \forall t \in [0, T_\varepsilon^2].$$

Finally, noting $\int_0^t \left(1 + \frac{1}{\varepsilon^2}\nu_\varepsilon(t')\right)dt' \le const.$ for all $t \le T_\varepsilon^2 \le T_m$, the Gronwall inequality implies (3.23). □

Proof of Theorem 2.1. In view of the estimate established in Proposition 3.1, it remains to prove $T_\varepsilon^2 = T_m$, which implies that $T_\varepsilon > T_m$. Recall from Lemma 3.2 that $T_\varepsilon^2 \in (0, T_\varepsilon) \cap (0, T_m]$ and $[0, T_\varepsilon^2]$ is the maximal time interval on which (3.6)–(3.7) hold. On the other hand, by Proposition 3.1, we have

$$\|W(t)\|_s \le c\varepsilon^m, \quad \forall t \in [0, T_\varepsilon^2].$$

In particular, $\|W(T_\varepsilon^2)\|_s \le c\varepsilon^m$. When $m \ge 2$ and ε is sufficiently small, we always have $c\varepsilon^m < \varepsilon$ for any fixed constant $c > 0$. Thus, $T_\varepsilon^2 = T_m$ follows from (3.7). □

4. Formal asymptotic expansions

We are looking for an approximate solution to (1.1)–(1.2) of the form

$$\sum_{k=0}^{+\infty} \varepsilon^k\big(U_k(t, x) + I_k(\tau, x)\big), \quad \tau = t/\tau^2, \tag{4.1}$$

with profiles I_k that converge exponentially fast to zero when τ tends to infinity. In what follows, we present a detailed construction of U_k and I_k, and we show that U_ε^m defined by (1.9) satisfies conditions (H5)–(H6) together with the definition of R_m^ε in (1.8). Remark that for $V = \sum_{k=0}^{+\infty}\varepsilon^k V_k$ and a sufficiently smooth function H, we have formally

$$H(V) = H(V_0) + \sum_{k=1}^{+\infty}\varepsilon^k\big[\partial_V H(V_0)V_k + \mathcal{C}(H, k, \underline{V})\big], \tag{4.2}$$

where $\mathcal{C}(H, k, \underline{V})$ only depends on H and the first k elements of $\underline{V} = (V_0, V_1, V_2, \cdots)$, with $\mathcal{C}(H, 1, \underline{V}) = 0$. The derivation of U_k and I_k is based on the fact that both series $\sum_{k=0}^{+\infty}\varepsilon^k U_k(t, x)$ and (4.1) are formal solutions of (1.1).

4.1. The equations for U_k

Putting $\sum_{k=0}^{+\infty}\varepsilon^k U_k(t, x)$ into (1.1), the identification of the powers of ε yields

$$\varepsilon^{-2}: \quad Q(0, U_0) = 0, \tag{4.3}$$

$$\varepsilon^{-1}: \quad \sum_{j=1}^{d} A_j(U_0)\partial_{x_j}U_0 - \partial_\varepsilon Q(0, U_0) - \partial_U Q(0, U_0)U_1 = 0, \tag{4.4}$$

$$\varepsilon^k: \quad \partial_t U_k + \sum_{j=1}^{d} A_j(U_0)\partial_{x_j}U_{k+1} + \sum_{l=0}^{k}\sum_{j=1}^{d}\big[\partial_U A_j(U_0)U_{l+1} + \mathcal{C}(A_j, l+1, \underline{U})\big]\partial_{x_j}U_{k-l}$$

$$
\begin{aligned}
&- \sum_{l=0}^{k+1} \frac{1}{l!}\big[\partial_U \partial_\varepsilon^l Q(0,U_0)U_{k+2-l} + \mathcal{C}\big(\partial_\varepsilon^l Q(0,\cdot),k+2-l,\underline{U}\big)\big] \\
&- \frac{1}{(k+2)!}\partial_\varepsilon^{k+2} Q(0,U_0) = 0, \quad \forall k \in \mathbb{N}.
\end{aligned} \tag{4.5}
$$

Eq. (4.3) gives $v_0 = 0$ thanks to (1.5). Next, we separate (4.4) into two systems of $n-r$ and r equations:

$$
\sum_{j=1}^d A_j^{11}(u_0,0)\partial_{x_j} u_0 - \partial_\varepsilon Q^I(0,u_0,0) = 0, \tag{4.6}
$$

$$
\sum_{j=1}^d A_j^{21}(u_0,0)\partial_{x_j} u_0 - \partial_\varepsilon Q^{II}(0,u_0,0) - \partial_v q(u_0,0)v_1 = 0. \tag{4.7}
$$

System (4.6) is a differential constraint on u_0 which has been discussed in the introduction. From (4.7) and (1.5), we have

$$
v_1 = \partial_v q(u_0,0)^{-1}\left[\sum_{j=1}^d A_j^{21}(u_0,0)\partial_{x_j} u_0 - \partial_\varepsilon Q^{II}(0,u_0,0)\right]. \tag{4.8}
$$

Similarly, for $k \in \mathbb{N}$ we separate (4.5) into two systems of $n-r$ and r equations. Noting

$$
\partial_u A_j^{11}(U_0) = 0, \;\; \partial_U Q^I(0,U_0) = 0, \;\; \mathcal{C}\big(Q^I(0,\cdot),k+2,\underline{U}\big) = 0,
$$

we obtain

$$
\partial_t u_k + \sum_{j=1}^d A_j^{12}(U_0)\partial_{x_j} v_{k+1} + \sum_{j=1}^d A_j^{11}(U_0)\partial_{x_j} u_{k+1} + g_k\big((U_i,\nabla U_i)_{0\le i\le k}, v_{k+1}\big) = 0 \tag{4.9}
$$

and

$$
\begin{aligned}
&\partial_t v_k + \sum_{j=1}^d \big[A_j^{21}(U_0)\partial_{x_j} u_{k+1} + A_j^{22}(U_0)\partial_{x_j} v_{k+1}\big] - \frac{1}{(k+2)!}\partial_\varepsilon^{k+2} Q^{II}(0,U_0) \\
&+ \sum_{l=0}^k \sum_{j=1}^d \big[(\partial_u A_j^{21}(U_0)u_{l+1} + \partial_v A_j^{21}(U_0)v_{l+1}).\partial_{x_j} u_{k-l} \\
&+ \big(\partial_u A_j^{22}(U_0)u_{l+1} + \partial_v A_j^{22}(U_0)v_{l+1}\big).\partial_{x_j} v_{k-l}\big] \\
&+ \sum_{l=0}^k \sum_{j=1}^d \big[\mathcal{C}(A_j^{21},l+1,\underline{U})\partial_{x_j} u_{k-l} + \mathcal{C}(A_j^{22},l+1,\underline{U})\partial_{x_j} v_{k-l}\big] \\
&- \sum_{l=0}^{k+1} \frac{1}{l!}\big[\partial_u \partial_\varepsilon^l Q^{II}(0,U_0)u_{k+2-l} + \partial_v \partial_\varepsilon^l Q^{II}(0,U_0)v_{k+2-l} + \mathcal{C}\big(\partial_\varepsilon^l Q^{II}(0,\cdot),k+2-l,\underline{U}\big)\big] = 0,
\end{aligned} \tag{4.10}
$$

where

$$
\begin{aligned}
g_k\big((U_i,\nabla U_i)_{0\le i\le k}, v_{k+1}\big) &= \sum_{l=0}^k \sum_{j=1}^d \big[\mathcal{C}(A_j^{11},l+1,\underline{U})\partial_{x_j} u_{k-l} + \mathcal{C}(A_j^{12},l+1,\underline{U})\partial_{x_j} v_{k-l}\big] \\
&+ \sum_{l=0}^k \sum_{j=1}^d \big[\partial_v A_j^{11}(U_0)v_{l+1}.\partial_{x_j} u_{k-l} + \big(\partial_u A_j^{12}(U_0)u_{l+1} + \partial_v A_j^{12}(U_0)v_{l+1}\big).\partial_{x_j} v_{k-l}\big] \\
&- \sum_{l=1}^{k+1} \frac{1}{l!}\big[\partial_u \partial_\varepsilon^l Q^I(0,U_0)u_{k+2-l} + \partial_v \partial_\varepsilon^l Q^I(0,U_0)v_{k+2-l} + \mathcal{C}\big(\partial_\varepsilon^l Q^I(0,\cdot),k+2-l,\underline{U}\big)\big]
\end{aligned}
$$

$$- \frac{1}{(k+2)!}\partial_\varepsilon^{k+2} Q^I(0, U_0). \tag{4.11}$$

In the last summation of (4.10), from (1.5), we have

$$\partial_u \partial_\varepsilon^l Q^{II}(0, U_0) u_{k+2-l} + \partial_v \partial_\varepsilon^l Q^{II}(0, U_0) v_{k+2-l} = \partial_v q(U_0) v_{k+2}, \quad \text{for } l = 0.$$

Hence, (4.10) allows to express v_{k+2} as:

$$v_{k+2} = \partial_v q(u_0, 0)^{-1} \sum_{j=1}^{d} A_j^{21}(u_0, 0)\partial_{x_j} u_{k+1} + V_{k+2}^1 u_{k+1} + V_{k+2}^2, \quad k \in \mathbb{N}, \tag{4.12}$$

where V_{k+2}^1 and V_{k+2}^2 may depend on $U_0, U_1, \cdots, U_k, v_{k+1}$ and their first-order derivatives, but are independent of u_{k+1}. Remark that, due to (H3), u_{k+1} does not appear in (4.11).

Now let us make more details for these equations according to the value of k. For $k = 0$, system (4.9) becomes

$$\partial_t u_0 + \sum_{j=1}^{d} A_j^{12}(U_0)\partial_{x_j} v_1 + \sum_{j=1}^{d} A_j^{11}(U_0)\partial_{x_j} u_1 + g_0(u_0, \nabla u_0, v_1) = 0, \tag{4.13}$$

where g_0 is defined in (1.13). In (4.13), v_1 can be replaced by (4.8), but u_1 is an independent unknown. From (H1), $A_j^{11}(U_0)$ is a constant matrix for all $1 \le j \le d$. In order to eliminate u_1 in (4.13), we assume that there is a constant square matrix of order $n - r$, denoted by D, such that (1.14) holds, i.e., $DA_j^{11}(U_0) = 0$ for all $1 \le j \le d$. Applying D to (4.13) yields a nonlinear system of second-order partial differential equations for u_0:

$$D\partial_t u_0 + D\sum_{j=1}^{d} A_j^{12}(U_0)\partial_{x_j} v_1 + Dg_0(u_0, \nabla u_0, v_1) = 0. \tag{4.14}$$

Then (4.13) is a differential constraint for u_1, which can be rewritten as

$$\sum_{j=1}^{d} A_j^{11}(U_0)\partial_{x_j} u_1 + (\mathbf{I}_{n-r} - D)\Big(\partial_t u_0 + \sum_{j=1}^{d} A_j^{12}(U_0)\partial_{x_j} v_1 + g_0(u_0, \nabla u_0, v_1)\Big) = 0. \tag{4.15}$$

Putting (4.8) into (4.14) yields a closed system for u_0:

$$\begin{aligned} &D\partial_t u_0 + D\sum_{i,j=1}^{d} A_{ij}(u_0)\partial_{x_i x_j}^2 u_0 + D\sum_{j=1}^{d} B_j(u_0)\partial_{x_j} u_0 \\ &\qquad + D\sum_{i,j=1}^{d} C_{ij}(u_0)\partial_{x_i} u_0\, \partial_{x_j} u_0 + Df_0(u_0) = 0, \end{aligned} \tag{4.16}$$

where A_{ij}, B_j, C_{ij} and f_0 are defined in (1.18), (1.19), (1.20) and (1.21), respectively.

System (4.16) and its differential constraint (4.6) are just the limit equations (1.17) and (1.10) given in the introduction. Assume that (4.16) and (4.6) admit a local smooth solution u_0, defined on $[0, T_0]$ with $T_0 > 0$ being independent of ε. Then we have constructed $U_0 = \begin{bmatrix} u_0 \\ 0 \end{bmatrix}$ and v_1, which is given by (4.8). Moreover, we still have a constraint (4.15) on u_1.

Now let $k \ge 1$. By induction, assume that $U_0, U_1, \cdots, U_{k-1}$ and v_k are defined on $[0, T_{k-1}]$ with $T_{k-1} \in (0, T_0]$ being independent of ε, and we have a differential constraint on u_k of the same type as (4.15):

$$\begin{aligned} &\sum_{j=1}^{d} A_j^{11}(u_0, 0)\partial_{x_j} u_k + (\mathbf{I}_{n-r} - D)\Big(\partial_t u_{k-1} + \sum_{j=1}^{d} A_j^{12}(U_0)\partial_{x_j} v_k\Big) \\ &\qquad\qquad + (\mathbf{I}_{n-r} - D) g_{k-1}\big((U_i, \nabla U_i)_{0 \le i \le k-1}, v_k\big) = 0. \end{aligned} \tag{4.17}$$

Similarly to the case $k = 0$, applying D to (4.9) yields a linear system of partial differential equations for u_k:

$$D\partial_t u_k + D\sum_{j=1}^d A_j^{12}(U_0)\partial_{x_j} v_{k+1} + Dg_k\big((U_i, \nabla U_i)_{0\le i\le k}, v_{k+1}\big) = 0. \tag{4.18}$$

Then (4.9) becomes a linear differential constraint for u_{k+1}, which can be rewritten as

$$\sum_{j=1}^d A_j^{11}(u_0, 0)\partial_{x_j} u_{k+1} + (\mathbf{I}_{n-r} - D)\Big(\partial_t u_k + \sum_{j=1}^d A_j^{12}(U_0)\partial_{x_j} v_{k+1}\Big) \\ + (\mathbf{I}_{n-r} - D)g_k\big((U_i, \nabla U_i)_{0\le i\le k}, v_{k+1}\big) = 0. \tag{4.19}$$

Putting (4.12) with $k+1$ instead of $k+2$ into (4.18), we obtain the equations of u_k:

$$D\partial_t u_k + D\sum_{i,j=1}^d A_{ij}(u_0)\partial^2_{x_i x_j} u_k + D\sum_{j=1}^d B_j^k \partial_{x_j} u_k + D\sum_{j=1}^d C_j^k u_k + Df_k = 0, \tag{4.20}$$

where B_j^k, C_j^k and f_k may depend on $U_0, U_1, \cdots, U_{k-1}$ and their first-order derivatives, but are independent of u_k.

Assume that (4.20) and (4.17) admit a local smooth solution u_k defined on $[0, T_k]$ with $T_k \in (0, T_{k-1}]$ being independent of ε. Thus we have constructed U_k and v_{k+1}, which is given by (4.12) with $k+1$ instead of $k+2$. Finally, we still have a constraint (4.19) on u_{k+1}.

Remark that the second-order operator is the same in (4.16) and (4.20). If $\partial_\varepsilon Q^I(0, u_0, 0) = 0$ and $A_j^{11}(u_0, 0) = 0$ for all $1 \le j \le d$, then the constraints (4.6) and (4.17) are trivially satisfied with $D = \mathbf{I}_{n-r}$. Hence, u_0 and u_k are determined by (4.16) and (4.20). In this case, we give below a sufficient condition for (4.16) and (4.20) to be parabolic. Its proof is quite similar to those proved in [16,33]. A typical example of this situation is the Euler equations with damping given in the last section.

Proposition 4.1. *Let $\omega = (\omega_1, \cdots, \omega_d) \in \mathbb{R}^d \setminus \{0\}$ and $A^{21}(\omega, u_0) = \sum_{j=1}^d A_j^{21}(u_0, 0)\omega_j$. Assume $Ker\big(A^{21}(\omega, u_0)\big) = \{0\}$, i.e., $r \ge n - r$ and $A^{21}(\omega, u_0)$ is a full-rank matrix. Then, $\sum_{i,j=1}^d A_{ij}(u_0)\omega_i\omega_j$ is a negative matrix. Consequently, if $D = \mathbf{I}_{n-r}$, then both (4.16) and (4.20) are strictly parabolic.*

4.2. The determination of I_k and the initial date of u_k

Since $t = \varepsilon^2\tau$, we have formally

$$\sum_{k=0}^{+\infty} \varepsilon^k U_k(t, x) = \sum_{k=0}^{+\infty} \varepsilon^k P_k(\tau, x), \tag{4.21}$$

with

$$P_k(\tau, x) = \sum_{h=0}^{\lfloor k/2 \rfloor} \frac{\tau^h}{h!}\frac{\partial^h U_{k-2h}}{\partial t^h}(0, x).$$

Hence,

$$\sum_{k=0}^{+\infty} \varepsilon^k \big(U_k(t, x) + I_k(\tau, x)\big) = \sum_{k=0}^{+\infty} \varepsilon^k \big(P_k(\tau, x) + I_k(\tau, x)\big). \tag{4.22}$$

Now write (1.1) in variables (τ, x). Putting (4.22) into (1.1) and repeatedly using (4.2) and the same techniques as above, we obtain

$$\begin{cases} \partial_\tau(I_0+P_0) = Q(0, I_0+P_0), \\ \partial_\tau(I_1+P_1) = \partial_U Q(0, I_0+P_0)(I_1+P_1) + \partial_\varepsilon Q(0, I_0+P_0) - \sum_{j=1}^{d} A_j(I_0+P_0)\partial_{x_j}(I_0+P_0), \\ \partial_\tau(I_k+P_k) = \partial_U Q(0, I_0+P_0)(I_k+P_k) + \mathcal{F}(k, \underline{I+P}), \text{ for all } k \geq 2, \end{cases}$$

where

$$\mathcal{F}(k, \underline{I+P}) = \sum_{l=1}^{k-1} \frac{1}{l!}\big[\partial_U \partial_\varepsilon^l Q(0, I_0+P_0)(I_{k-l}+P_{k-l}) + \mathcal{C}\big(\partial_\varepsilon^l Q(0,\cdot), k-l, \underline{I+P}\big)\big]$$
$$+ \frac{1}{k!}\partial_\varepsilon^k Q(0, I_0+P_0) - \sum_{j=1}^{d} A_j(I_0+P_0)\partial_{x_j}(I_{k-1}+P_{k-1})$$
$$- \sum_{j=1}^{d}\sum_{l=0}^{k-2}\big[\partial_U A_j(I_0+P_0)(I_{l+1}+P_{l+1}) + \mathcal{C}(A_j, l+1, \underline{I+P})\big]\partial_{x_j}(I_{k-2-l}+P_{k-2-l})$$

depending only on the first k terms of $\underline{I+P} = (I_0+P_0, I_1+P_1, \cdots, I_{k-1}+P_{k-1}, \cdots)$.

On the other hand, due to (4.21), $\sum_{k=0}^{+\infty} \varepsilon^k P_k(\tau, x)$ is also a solution of (1.1). Hence, we obtain as above

$$\begin{cases} \partial_\tau P_0 = Q(0, P_0), \\ \partial_\tau P_1 = \partial_U Q(0, P_0) P_1 + \partial_\varepsilon Q(0, I_0) - \sum_{j=1}^{d} A_j(P_0)\partial_{x_j} P_0, \\ \partial_\tau P_k = \partial_U Q(0, P_0) P_k + \mathcal{F}(k, \underline{P}), \text{ for all } k \geq 2. \end{cases}$$

It follows from $P_0(\tau, x) = U_0(0, x)$ and $Q(0, P_0) = 0$ that

$$\begin{cases} \partial_\tau I_0 = Q(0, I_0+P_0), \\ \partial_\tau I_k = \partial_U Q(0, I_0+P_0) I_k + \big[\partial_U Q(0, I_0+P_0) - \partial_U Q(0, P_0)\big] P_k(\tau, x) + \mathcal{G}(k, \tau, x), \quad \forall k \geq 1, \end{cases} \tag{4.23}$$

where

$$\mathcal{G}(k, \tau, x) = \mathcal{F}(k, \underline{I+P}) - \mathcal{F}(k, \underline{P}), \quad \forall k \geq 1,$$

with

$$\mathcal{F}(1, \underline{I+P}) = \partial_\varepsilon Q(0, I_0+P_0) - \sum_{j=1}^{d} A_j(I_0+P_0)\partial_{x_j}(I_0+P_0).$$

Now we solve I_k and determine the initial conditions for u_k. Let $\bar{U}_k = \begin{bmatrix} \bar{u}_k \\ \bar{v}_k \end{bmatrix}$ be given smooth functions of x, obtained through a formal asymptotic expansion of the initial datum $\bar{U}$:

$$\bar{U}(x, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \bar{U}_k(x).$$

If $\sum_{k=0}^{+\infty} \varepsilon^k \big(U_k(t, x) + I_k(\tau, x)\big)$ is a solution of (1.1)–(1.2), we should have

$$U_k(0, x) + I_k(0, x) = \bar{U}_k(x),$$

or equivalently

$$\begin{cases} u_k(0, x) + I_k^I(0, x) = \bar{u}_k(x), \\ v_k(0, x) + I_k^{II}(0, x) = \bar{v}_k(x), \quad \forall k \geq 0. \end{cases} \tag{4.24}$$

From $Q^I(0,U)=0$ and the first equation of (4.23), we have $\partial_\tau I_0^I=0$, which means that there is no zero-th order initial layer for u. In this case, we may take $I_0^I=0$. Together with $v_0=0$, we obtain

$$u_0(0,x)=\bar{u}_0(x), \quad I_0^{II}(0,x)=\bar{v}_0(x),$$

which are the initial conditions for u_0 and I_0^{II}. Hence, the equation of I_0^{II} becomes

$$\partial_\tau I_0^{II}=q\big(\bar{u}_0(x), I_0^{II}\big), \quad x\in\mathbb{T}^d.$$

Lemma 4.1. *Let $(\bar{u}_0,\bar{v}_0)$ be sufficiently small and $\bar{v}_0$ be sufficiently close to zero. Then there exists a unique global smooth solution I_0 satisfying*

$$\|I_0(\tau,\cdot)\|_{s+m}\longrightarrow 0, \ \textit{exponentially as } \tau\to+\infty. \tag{4.25}$$

Proof. By Lemma 3.1, the condition in (H2) can be written in an equivalent way:

$$A_0^{22}(u,0)\partial_v q(u,0)\xi^{II}\cdot\xi^{II}\le -c_0|\xi^{II}|^2, \quad \forall u\in\mathbb{R}^{n-r},\ \xi^{II}\in\mathbb{R}^r.$$

Since A_0 is symmetric positive definite, so is A_0^{22}. It follows that each eigenvalue of $\partial_v q(u,0)$ is negative uniformly with respect to u. Therefore, for sufficiently small data $\bar{v}_0$, there is a unique global solution $I_0^{II}(\tau,x)$ which decays exponentially fast to zero as $\tau\to+\infty$ (see [1]). Next, by induction, for all $\alpha\in\mathbb{N}^d$ with $|\alpha|\le s+m$, $\partial_x^\alpha I_0^{II}$ satisfies a linear ordinary differential equation of the form

$$\partial_\tau Y=\partial_v q\big(\bar{u}_0(x), I_0^{II}\big)Y+g_\alpha(\tau,x), \quad x\in\mathbb{T}^d, \tag{4.26}$$

with exponential decay of g_α as $\tau\to+\infty$. This implies (4.25). We refer to [33] for solving the linear equation of Y. □

By induction, for $k\ge 1$ and for all $i\le k-1$, assume that I_i exists globally in time and $\|I_i(\tau,.)\|_{s+m-i}$ decays exponentially fast to zero as τ goes to infinity. Then so does $\|\mathcal{G}(k,\tau,x)\|_{s+m-k}$, since

$$\mathcal{G}(k,\tau,x)=\mathcal{F}(k,\underline{I+P})-\mathcal{F}(k,\underline{P}),$$

and $\mathcal{F}$ only depends on those of (I_i,P_i) for $i\le k-1$. The first $n-r$ equations in (4.23) are

$$\partial_\tau I_k^I=\mathcal{G}^I(k,\tau,x).$$

Hence,

$$I_k^I(\tau,x)=I_k^I(0,x)+\int_0^\tau \mathcal{G}^I(k,\tau',x)d\tau',$$

which admits a limit 0 as τ goes to infinity. Therefore,

$$I_k^I(\tau,x)=-\int_\tau^{+\infty}\mathcal{G}^I(k,\tau',x)d\tau'$$

and

$$\|I_k^I(\tau,\cdot)\|_{s+m-k}\longrightarrow 0, \ \text{exponentially as } \tau\to+\infty.$$

In particular,

$$I_k^I(0,x)=-\int_0^{+\infty}\mathcal{G}^I(k,\tau,x)d\tau.$$

Together with (4.24) it determines the initial value of u_k:

$$u_k(0,x) = \bar{u}_k^I(x) + \int_0^{+\infty} \mathcal{G}^I(k,\tau,x)d\tau.$$

Finally, the last r equations in (4.23) imply that I_k^{II} still satisfies a linear system of the form (4.26). Thus, I_k^{II} exists globally in time and

$$\|I_k^{II}(\tau,\cdot)\|_{s+m-k} \longrightarrow 0, \text{ exponentially as } \tau \to +\infty.$$

4.3. Error estimates

In the last two subsections we have constructed U_k and I_k on time interval $[0, T_k]$ for all $k \in \mathbb{N}$, with $0 < T_{k+1} \le T_k$. Now we show that, for any fixed $m \in \mathbb{N}$, the approximate solution U_ε^m defined by (1.9) satisfies (H5)–(H6). Indeed, since $I_0^I = 0$,

$$\partial_t I_0^{II}(t/\varepsilon^2,\cdot) = \varepsilon^{-2}\partial_\tau I_0^{II}(t/\varepsilon^2,\cdot) = \varepsilon^{-2}\partial_v q(\bar{u}_0, I_0^{II}),$$

and $I_0^{II}(\tau,\cdot)$ decays exponentially fast to zero as $\tau \to +\infty$, (H5) is obviously satisfied.

The following result (see [16]) implies that (H6) is also satisfied.

Proposition 4.2. *Let R_m^ε be defined by (1.8). Then*

$$R_m^\varepsilon = \varepsilon^{m-1}\begin{bmatrix} 0 \\ r_m \end{bmatrix} + \varepsilon^{m-1} F_m^\varepsilon,$$

where $r_m \in C([0,T_m], H^s)$ and $F_m^\varepsilon \in C([0,T_m], H^s)$ satisfying

$$\|F_m^\varepsilon(t)\|_s \le c\varepsilon + ce^{-\frac{\mu t}{\varepsilon^2}}, \quad \forall t \in [0, T_m].$$

5. Examples

5.1. Semilinear examples

We give two examples of semilinear equations of the form (1.1) with $n = 2$ and $d = 1$. Both were considered as applications of (1.6) in [16]. The first one concerns a wave equation of heat conduction and was studied by several authors (see [10,17] and references therein). It reads

$$\varepsilon^2 \partial_{tt}^2 w - \partial_{xx}^2 w + \partial_t w = 0, \quad t > 0, \ x \in \mathbb{R}.$$

Let

$$u = \partial_x w, \quad v = -\varepsilon \partial_t w.$$

Then the system is written as

$$\begin{cases} \partial_t u + \dfrac{1}{\varepsilon}\partial_x v = 0, \\ \partial_t v + \dfrac{1}{\varepsilon}\partial_x u = -\dfrac{v}{\varepsilon^2}. \end{cases}$$

It is of the form (1.1) with

$$U = \begin{bmatrix} u \\ v \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Q(U) = \begin{bmatrix} 0 \\ -v \end{bmatrix}.$$

Let $A_0 = \mathbf{I}_2$ and $r = 1$. It is easy to check that the system is symmetrizable hyperbolic and satisfies (1.4)–(1.5) and (H1)–(H3), with $A_0 \partial_U Q(u,0) = \mathrm{diag}(0,-1)$. The corresponding limit equations for U_0 are $v_0 = 0$ and the one-dimensional heat equation

$$\partial_t u_0 - \partial_x^2 u_0 = 0.$$

The second example concerns a generalized discrete two-velocity model in a slow time:

$$\begin{cases} \partial_t f + \varepsilon^{-1}\partial_x f = \varepsilon^{-2}(f+g)^\gamma (g-f), \\ \partial_t g - \varepsilon^{-1}\partial_x g = \varepsilon^{-2}(f+g)^\gamma (f-g), \quad t>0,\ x\in\mathbb{R}, \end{cases}$$

where γ is a real number and $f+g>0$. It was studied in [31,28,18]. With a change of variables $u=f+g$ and $v=f-g$, the system is written as

$$\begin{cases} \partial_t u + \dfrac{1}{\varepsilon}\partial_x v = 0, \\ \partial_t v + \dfrac{1}{\varepsilon}\partial_x u = -\dfrac{2u^\gamma v}{\varepsilon^2}. \end{cases}$$

Let $A_0 = \mathbf{I}_2$, $r=1$ and

$$U = \begin{bmatrix} u \\ v \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Q(U) = \begin{bmatrix} 0 \\ -2u^\gamma v \end{bmatrix}.$$

For $u \ge const. > 0$, the system is symmetrizable hyperbolic and satisfies (1.4)–(1.5) and (H1)–(H3), with $A_0\partial_U Q(u,0) = \mathrm{diag}(0, -2u^\gamma)$. The corresponding limit equations for U_0 are $v_0 = 0$ and

$$\partial_t u_0 - \frac{1}{2}\partial_x\big(u_0^{-\gamma}\partial_x u_0\big) = 0.$$

For both semilinear examples above, we have $A_1^{11}=0$ and Q only depends on U. Then the differential constraints disappear and $D=1$. It is easy to check that their approximate solutions U_ε^m can be constructed for all $m\in\mathbb{N}$ and thus Theorem 2.1 can be applied.

5.2. *Euler equations with damping*

The equations take the form (see [22,21,11,30,7] etc.)

$$\begin{cases} \partial_{t'}\rho + \mathrm{div}(\rho v) = 0, \\ \partial_{t'}(\rho v) + \mathrm{div}(\rho v\otimes v) + \nabla p(\rho) = -\dfrac{\rho v}{\varepsilon}, \quad t'>0,\ x\in\mathbb{R}^d, \end{cases}$$

where $\rho>0$, v, p and $\varepsilon>0$ stand for the fluid density, the velocity, the pressure and the relaxation time, respectively. As usual, we assume $p'(\rho)>0$ for all $\rho>0$. For smooth solutions, the system is equivalent to

$$\begin{cases} \partial_t \rho + \dfrac{1}{\varepsilon}\mathrm{div}(\rho v) = 0, \\ \partial_t v + \dfrac{1}{\varepsilon}[(v.\nabla)v + \nabla h(\rho)] = -\dfrac{v}{\varepsilon^2}, \quad t>0,\ x\in\mathbb{R}^d, \end{cases}$$

where $t=\varepsilon t'$ is the slow time and $h'(\rho) = \frac{p'(\rho)}{\rho}$. Let

$$u=\rho, \quad U = \begin{bmatrix} \rho \\ v \end{bmatrix}, \quad A_j(U) = \begin{bmatrix} v^j & \rho e_j^T \\ h'(\rho)e_j & v^j\mathbf{I}_d \end{bmatrix}, \quad j=1,2,\cdots,d,$$

and

$$Q(U) = \begin{bmatrix} 0 \\ -v \end{bmatrix}, \quad q(U) = -v,$$

where v^j is the j-th component of v, e_j is the j-th vector of the canonical basis in $\mathbb{R}^d$, and the superscript T stands for the transpose. Let $n=d+1$ and $r=d$. Since $\rho>0$ and $h'(\rho)>0$, with symmetrizer $A_0(U) = \mathrm{diag}\big(\rho^{-1}, h'(\rho)^{-1}\mathbf{I}_d\big)$, it is straightforward that the system is symmetrizable hyperbolic and satisfies (1.4)–(1.5) and (H1)–(H3).

It is important to point out that the system cannot be put in the form (1.6). Hence, the result in [16] cannot be applied.

The leading profile (ρ_0, v_0) satisfies $v_0 = 0$ and a porous medium equation

$$\partial_t \rho_0 - \Delta p(\rho_0) = 0,$$

which is strictly parabolic since p is a strictly increasing function. Hence, it admits a local smooth solution. It is easy to see that $v_1 = -\nabla h(\rho_0)$. The leading initial layer profile $I_0 = \begin{bmatrix} \tilde\rho_0 \\ \tilde v_0 \end{bmatrix}$ satisfies

$$\partial_\tau \tilde\rho_0 = 0, \quad \partial_\tau \tilde v_0 = -\tilde v_0.$$

Thus, it is clear that I_0 exists globally in time and decays exponentially fast to zero as $\tau \to +\infty$, even for large initial data.

Similarly, by induction we can construct higher order profiles ρ_k, $I_k = \begin{bmatrix} \tilde\rho_k \\ \tilde v_k \end{bmatrix}$ and v_{k+1} for $k \geq 1$. More precisely, the equations for ρ_k and I_k are

$$\partial_t \rho_k - p'(\rho_0)\Delta\rho_k + b_k = 0,$$
$$\partial_\tau \tilde\rho_k = g_k^I(\tau, x), \quad \partial_\tau \tilde v_k = -\tilde v_k + g_k^{II}(\tau, x),$$

where b_k only depends on (ρ_i, v_i) for $0 \leq i \leq k$ and their first-order derivatives, g_k^I and g_k^{II} decay exponentially fast to zero as $\tau \to +\infty$. Finally, v_{k+1} is given by expression (4.12). Thus, the approximate solution U_ε^m is constructed for all $m \in \mathbb{N}$ and Theorem 2.1 can be applied.

Finally, for this system, we have $A_j^{11}(\rho, v) = v^j$. Since $A_j^{11}(\rho, 0) = 0$ and Q is a function of only U, there is no differential constraint and thus $D = 1$ for all $k \in \mathbb{N}$.

5.3. *An Euler–Maxwell system with relaxation*

The system reads (see [2,5]):

$$\begin{cases} \partial_{t'}\rho + \operatorname{div}(\rho v) = 0, \\ \partial_{t'}(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla p(\rho) = -\rho(E + v \times B) - \dfrac{\rho v}{\varepsilon}, \\ \partial_{t'} E - \operatorname{rot} B = \rho v, \quad \operatorname{div} E = b(x) - \rho, \\ \partial_{t'} B + \operatorname{rot} E = 0, \quad \operatorname{div} B = 0, \quad t' > 0,\ x \in \mathbb{R}^3. \end{cases}$$

Here E and B are the electric field and the magnetic induction, b is a given time-independent function, ρ, v, h and ε have the same physical interpretations as in the previous example. The differential constraint equations

$$\operatorname{div} E = b(x) - \rho, \quad \operatorname{div} B = 0$$

are time invariant. This is a system of 10 equations. In the slow time $t = \varepsilon t'$, it becomes

$$\begin{cases} \partial_t \rho + \dfrac{1}{\varepsilon}\operatorname{div}(\rho v) = 0, \\ \partial_t B + \dfrac{1}{\varepsilon}\operatorname{rot} E = 0, \quad \operatorname{div} B = 0, \\ \partial_t E - \dfrac{1}{\varepsilon}\operatorname{rot} B = \dfrac{\rho v}{\varepsilon}, \quad \operatorname{div} E = b(x) - \rho, \\ \partial_t v + \dfrac{1}{\varepsilon}(v.\nabla)v + \dfrac{1}{\varepsilon}\nabla h(\rho) = -\dfrac{1}{\varepsilon^2}(\varepsilon E + \varepsilon v \times B + v). \end{cases}$$

Let

$$U = \begin{bmatrix} \rho \\ B \\ E \\ v \end{bmatrix}, \quad u = \begin{bmatrix} \rho \\ B \\ E \end{bmatrix}, \quad A_j(U) = \begin{bmatrix} v^j & 0 & 0 & \rho e_j^T \\ 0 & 0 & J_j & 0 \\ 0 & J_j^T & 0 & 0 \\ h'(\rho)e_j & 0 & 0 & v^j \mathbf{I}_3 \end{bmatrix}, \quad j = 1, 2, 3,$$

and

$$Q(\varepsilon, U) = \begin{bmatrix} 0 \\ 0 \\ \varepsilon \rho v \\ -v - \varepsilon E - \varepsilon v \times B \end{bmatrix}, \quad Q^I(\varepsilon, U) = \begin{bmatrix} 0 \\ 0 \\ \varepsilon \rho v \end{bmatrix}, \quad q(U) = -v,$$

where

$$J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the system is written in the form (1.1). By choosing a symmetrizer

$$A_0(U) = \mathrm{diag}\big(\rho^{-1}, \mathbf{I}_3, \mathbf{I}_3, h'(\rho)^{-1}\mathbf{I}_3\big),$$

we easily check that the system is symmetrizable hyperbolic and satisfies (1.4)–(1.5) and (H1)–(H3) with $n = 10$ and $r = 3$. See [27,9] for the derivation and the justification of the limit, of which the present paper is inspired.

Similarly as above, this system cannot be put in the form (1.6) and hence the result in [16] cannot be applied. Moreover, we have

$$A_j^{11}(u, 0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & J_j \\ 0 & J_j^T & 0 \end{bmatrix} \neq 0, \quad \partial_\varepsilon Q^I(0, u, 0) = 0.$$

Hence, there are differential constraints for the leading profile, which are given by

$$0 = \sum_{j=1}^{3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & J_j \\ 0 & J_j^T & 0 \end{bmatrix} \partial_{x_j} u_0 = \begin{bmatrix} 0 \\ \mathrm{rot}\, E_0 \\ -\mathrm{rot}\, B_0 \end{bmatrix},$$

namely,

$$\mathrm{rot}\, B_0 = \mathrm{rot}\, E_0 = 0.$$

Together with the constraints of the Maxwell equations:

$$\mathrm{div}\, B_0 = 0, \quad \mathrm{div}\, E_0 = b - \rho_0,$$

we deduce that B_0 is a constant and there is a potential function ϕ_0 such that $E_0 = \nabla \phi_0$. Finally, $v_0 = 0$ and the equation for ρ_0 is

$$\partial_t \rho_0 + \mathrm{div}(\rho_0 v_1) = 0, \quad v_1 = -\nabla\big(h(\rho_0) + \phi_0\big).$$

Therefore, (ρ_0, ϕ_0) satisfies the drift–diffusion system:

$$\begin{cases} \partial_t \rho_0 - \mathrm{div}\,\big(\rho_0 \nabla(h(\rho_0) + \phi_0)\big) = 0, \\ \Delta \phi_0 = b - \rho_0, \quad E_0 = \nabla \phi_0. \end{cases}$$

It is well-known that this system admits a local smooth solution (see [24]). Thus, we have constructed U_0 and v_1. It is easy to check that the differential constraints of u_1 are

$$\mathrm{rot}\, E_1 + \partial_t B_0 = 0, \quad -\mathrm{rot}\, B_1 + \partial_t E_0 - \rho_0 v_1 = 0.$$

The leading initial layer profile $I_0 = \begin{bmatrix} \tilde{\rho}_0 \\ \tilde{B}_0 \\ \tilde{E}_0 \\ \tilde{v}_0 \end{bmatrix}$ satisfies

$$\partial_\tau \tilde{\rho}_0 = 0, \quad \partial_\tau \tilde{B}_0 = \partial_\tau \tilde{E}_0 = 0, \quad \partial_\tau \tilde{v}_0 = -\tilde{v}_0.$$

Thus, as above I_0 exists globally in time and decays exponentially fast to zero as $\tau \to +\infty$.

Similarly, by induction and together with the differential constraints of u_k, we can construct higher order profiles u_k, I_k and v_{k+1} for $k \geq 1$. In particular, (ρ_k, ϕ_k) solves a linear drift–diffusion system:

$$\begin{cases} \partial_t \rho_k - \operatorname{div}\big(\rho_0 \nabla (h'(\rho_0)\rho_k + \phi_k)\big) + \operatorname{div}(\rho_k v_1) = \alpha_k, \\ \Delta \phi_k = -\rho_k + \beta_k, \end{cases}$$

B_k solves a linear div–rot system:

$$\operatorname{div} B_k = 0, \quad -\operatorname{rot} B_k + \partial_t E_{k-1} + \zeta_{k-1} = 0,$$

and

$$E_k = \nabla \phi_k - \partial_t \psi_{k-1}, \quad v_{k+1} = -\nabla (h'(\rho_0)\rho_k + \phi_k) + \partial_t \psi_{k-1} + \gamma_k,$$

where α_k, β_k and γ_k only depend on U_i and ψ_i for $0 \le i \le k-1$, and

$$B_{k-1} = \operatorname{rot} \psi_{k-1}, \quad \zeta_{k-1} = -\sum_{i=0}^{k-1} \rho_i v_{k-i}, \quad k \ge 1.$$

Moreover, we still have differential constraints for u_{k+1}:

$$\operatorname{rot} E_{k+1} + \partial_t B_k = 0, \quad -\operatorname{rot} B_{k+1} + \partial_t E_k + \zeta_k = 0.$$

The initial layer profile I_k satisfies a linear system of ordinary differential equations with the same principal part as I_0 and a source term decaying exponentially fast to zero. Thus, the approximate solution U_ε^m is constructed for all $m \in \mathbb{N}$ and Theorem 2.1 can be applied. In this example, the corresponding choice is

$$D = \operatorname{diag}\big(1, \mathbf{0}_6\big), \quad g_0(u_0, \nabla u_0, v_1) = \begin{bmatrix} v_1 . \nabla \rho_0 \\ 0 \\ -\rho_0 v_1 \end{bmatrix},$$

and

$$g_k\big((U_i, \nabla U_i)_{0 \le i \le k}, v_{k+1}\big) = \begin{bmatrix} v_{k+1} . \nabla \rho_0 + \operatorname{div}(\rho_k v_1) - \alpha_k \\ 0 \\ \zeta_k \end{bmatrix}, \ k \ge 1.$$

Conflict of interest statement

There is no conflict of interest.

Acknowledgements

The authors thank both referees for their useful comments and suggestions which allowed to improve the presentation of the paper.

References

[1] V. Arnold, Equations Différentielles Ordinaires, fourth ed., Editions MIR, Moscow, 1988 (in French), translated from the Russian by Djilali Embarek.
[2] C. Besse, P. Degond, F. Deluzet, J. Claudel, G. Gallice, C. Tessieras, A model hierarchy for ionospheric plasma modeling, Math. Models Methods Appl. Sci. 14 (2004) 393–415.
[3] G. Boillat, T. Ruggeri, Hyperbolic principal subsystems: entropy convexity and subcharacteristic conditions, Arch. Ration. Mech. Anal. 137 (1997) 305–320.
[4] F. Bouchut, F.R. Guarguaglini, R. Natalini, Diffusive BGK approximations for nonlinear multidimensional parabolic equations, Indiana Univ. Math. J. 49 (2000) 723–749.
[5] F. Chen, Introduction to Plasma Physics and Controlled Fusion, vol. 1, Plenum Press, New York, 1984.
[6] G.Q. Chen, C.D. Levermore, T.P. Liu, Hyperbolic conservation laws with stiff relaxation terms and entropy, Commun. Pure Appl. Math. 47 (1994) 787–830.
[7] J.F. Coulombel, T. Goudon, The strong relaxation limit of the multidimensional isothermal Euler equations, Trans. Am. Math. Soc. 359 (2007) 637–648.
[8] K.O. Friedrichs, Symmetric hyperbolic linear differential equations, Commun. Pure Appl. Math. 7 (1954) 345–392.
[9] M.L. Hajjej, Y.J. Peng, Initial layers and zero-relaxation limits of Euler–Maxwell equations, J. Differ. Equ. 252 (2012) 1441–1465.
[10] G.C. Hsiao, R.J. Weinacht, Singular perturbations for a semilinear hyperbolic equation, SIAM J. Math. Anal. 14 (1983) 1168–1179.

[11] L. Hsiao, T.P. Liu, Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping, Commun. Math. Phys. 143 (1992) 599–605.
[12] S. Jin, Z.P. Xin, The relaxation schemes for systems of conservation laws in arbitrary space dimensions, Commun. Pure Appl. Math. 48 (1995) 235–276.
[13] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, Arch. Ration. Mech. Anal. 58 (1975) 181–205.
[14] S. Klainerman, A. Majda, Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, Commun. Pure Appl. Math. 34 (1981) 481–524.
[15] O. Ladyženskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, Transl. Math. Monogr., vol. 23, Amer. Math. Soc., Providence, RI, 1968.
[16] C. Lattanzio, W.A. Yong, Hyperbolic–parabolic singular limits for first-order nonlinear systems, Commun. Partial Differ. Equ. 26 (2001) 939–964.
[17] T.T. Li, Nonlinear heat conduction with finite speed of propagation, in: China–Japan Symposium on Reaction–Diffusion Equations and Their Applications and Computational Aspects, Shanghai, 1994, World Sci. Publ., River Edge, NJ, 1997, pp. 81–91.
[18] P.L. Lions, G. Toscani, Diffusive limit for finite velocity Boltzmann kinetic models, Rev. Mat. Iberoam. 13 (1997) 473–513.
[19] T.P. Liu, Hyperbolic conservation laws with relaxation, Commun. Math. Phys. 108 (1987) 153–175.
[20] A. Majda, Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables, Springer-Verlag, New York, 1984.
[21] P. Marcati, A. Milani, The one-dimensional Darcy's law as the limit of a compressible Euler flow, J. Differ. Equ. 84 (1990) 129–147.
[22] P. Marcati, A. Milani, P. Secchi, Singular convergence of weak solutions for a quasilinear nonhomogeneous hyperbolic system, Manuscr. Math. 60 (1988) 49–69.
[23] P. Marcati, B. Rubino, Hyperbolic to parabolic relaxation theory for quasilinear first order systems, J. Differ. Equ. 162 (2000) 359–399.
[24] P.A. Markowich, C.A. Ringhofer, C. Schmeiser, Semiconductor Equations, Springer-Verlag, New York, 1990.
[25] I. Müller, T. Ruggeri, Rational Extended Thermodynamics, second ed., Springer-Verlag, New York, 1998, with supplementary chapters by H. Struchtrup and Wolf Weiss.
[26] R. Natalini, Recent results on hyperbolic relaxation problems, in: Analysis of Systems of Conservation Laws, Aachen, 1997, in: Monogr. Surv. Pure Appl. Math., vol. 99, Chapman & Hall/CRC, Boca Raton, FL, 1999, pp. 128–198.
[27] Y.J. Peng, S. Wang, Q.L. Gu, Relaxation limit and global existence of smooth solution of compressible Euler–Maxwell equations, SIAM J. Math. Anal. 43 (2011) 944–970.
[28] T. Platkowski, R. Illner, Discrete velocity models of the Boltzmann equation: a survey on the mathematical aspects of the theory, SIAM Rev. 30 (1988) 213–255.
[29] D. Serre, Relaxations semi-linéaire et cinétique des systèmes de lois de conservation, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 17 (2000) 169–192.
[30] T.C. Sideris, B. Thomases, D. Wang, Long time behavior of solutions to the 3D compressible Euler equations with damping, Commun. Partial Differ. Equ. 28 (2003) 795–816.
[31] L. Tartar, Solutions oscillantes des équations de Carleman, in: Séminaire Equations aux dérivées partielles (Ecole Polytechnique), Exp. n^0 12, 1980–1981, pp. 1–15.
[32] G.B. Whitham, Linear and Nonlinear Waves, Wiley, New York, 1974.
[33] W.A. Yong, Singular perturbations of first-order hyperbolic systems with stiff source terms, J. Differ. Equ. 155 (1999) 89–132.
[34] W.A. Yong, Basic aspects of hyperbolic relaxation systems, in: H. Freistüler, A. Szepessy (Eds.), Advances in the Theory of Shock Waves, in: Prog. Nonlinear Differ. Equ. Appl., vol. 47, Birkhäuser, Boston, 2001, pp. 259–305.