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Invariance of the Gibbs measure for the periodic quartic gKdV

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Abstract

We prove invariance of the Gibbs measure for the (gauge transformed) periodic quartic gKdV. The Gibbs measure is supported on $H^s(\mathbb{T})$ for $s < \frac{1}{2}$, and the quartic gKdV is analytically ill-posed in this range. In order to consider the flow in the support of the Gibbs measure, we combine a probabilistic argument with the second iteration and construct local-in-time solutions to the (gauge transformed) quartic gKdV almost surely in the support of the Gibbs measure. Then, we use Bourgain's idea to extend these local solutions to global solutions, and prove the invariance of the Gibbs measure under the flow. Finally, inverting the gauge, we construct almost sure global solutions to the (ungauged) quartic gKdV below $H^{\frac{1}{2}}(\mathbb{T})$.

Keywords: gKdV; Gibbs measure; Nonlinear smoothing; A.s. global well-posedness

1. Introduction

In this paper, we consider the periodic quartic generalized Korteweg–de Vries (gKdV) equation

$$\begin{cases} \partial_t u + \partial_x^3 u = \frac{1}{4}\partial_x\left(u^4\right), \quad x \in \mathbb{T},\ t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

Here u is a real-valued function on $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the one-dimensional torus. That is, $\mathbb{T} = [0, 2\pi]$ with the endpoints identified. We assume that the mean of u_0 is zero, and from conservation of the mean, it follows that the solution $u(t)$ of (1.1) (if it exists) has spatial mean zero for all $t \in \mathbb{R}$. Throughout this paper, we assume that the spatial mean $\hat{u}(0,t)$ is always zero for all $t \in \mathbb{R}$.

The system (1.1) is a special case of the gKdV equation

$$\begin{cases} \partial_t u + \partial_x^3 u = \pm\frac{1}{p}\partial_x\left(u^p\right), \quad x \in \mathbb{T},\ t \in \mathbb{R},\ p \geq 2 \text{ integer}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.2}$$

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The KdV ((1.2) with $p = 2$) is a canonical model for dispersive waves in physics. This equation has a rich history and the related literature is extensive. The modified KdV (mKdV, $p = 3$) has also appeared in physics, and it is closely related to KdV through the Miura transform [13]. Higher power gKdV equations ($p \geq 4$) have been studied mainly by mathematicians; there is interest in exploring the balance of a stronger nonlinearity with dispersion.

The system (1.2) has a conserved (if it is finite) Hamiltonian given by

$$H(u) := \frac{1}{2} \int_{\mathbb{T}} u_x^2 dx \pm \frac{1}{p(p+1)} \int_{\mathbb{T}} u^{p+1} dx.$$

Then (1.2) can be reformulated as

$$\partial_t u = \partial_x \frac{\partial H}{\partial u}, \tag{1.3}$$

where $\frac{\partial H}{\partial u}$ is the Fréchet derivative with respect to the $L^2(\mathbb{T})$-inner product.[2] This Hamiltonian structure leads to a natural question: is the Gibbs measure "$d\mu = e^{-H(u)} du$" invariant under the flow of (1.2)?

The Gibbs measure μ for (1.2), first constructed by Lebowitz, Rose and Speer [27], is supported on $H^{\frac{1}{2}-}(\mathbb{T}) = \bigcap_{s<\frac{1}{2}} H^s(\mathbb{T})$ (for $p \leq 5$ only, with appropriate restrictions). To ask the question of its invariance under the flow, one needs to prove that the evolution of (1.2) is well-defined (globally-in-time) for initial data in the support of μ.

Let us recall some well-posedness results for (1.1) and (1.2). In [1], Bourgain introduced a weighted space–time Sobolev space $X^{s,b}(\mathbb{T} \times \mathbb{R})$ whose norm is given by

$$\|u\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} = \left\| \langle n \rangle^s \langle \tau - n^3 \rangle^b \hat{u}(n,\tau) \right\|_{L^2_{n,\tau}(\mathbb{Z}\times\mathbb{R})}. \tag{1.4}$$

He used a fixed point argument to prove local well-posedness (LWP) of KdV in $L^2(\mathbb{T})$, and automatically obtained global well-posedness (GWP) by conservation of the $L^2(\mathbb{T})$-norm.

The study of well-posedness for the periodic quartic gKdV (1.1) was also initiated in [1]; a fixed point argument was used to establish LWP in $H^s(\mathbb{T})$, for $s > \frac{3}{2}$. This was improved to LWP in $H^s(\mathbb{T})$ for $s \geq 1$ by Staffilani [44], and then to $s \geq \frac{1}{2}$ by Colliander, Keel, Staffilani, Takaoka and Tao [12]. In [12], they also proved analytic ill-posedness of (1.1) below $H^{\frac{1}{2}}(\mathbb{T})$. That is, the data-to-solution map for (1.1) is not analytic in $H^s(\mathbb{T})$ for $s < \frac{1}{2}$. In fact, it is not C^4 (see also [2,11]).

Bourgain [3] rigorously proved the invariance of the Gibbs measure under the flow of KdV and mKdV, but to the knowledge of the author, this problem remains open for (1.2) with $p = 4$ and $p = 5$. For KdV and mKdV, he used a deterministic fixed point argument to establish well-posedness in the support of the Gibbs measure. Recall that the evolution of KdV is well-defined for all $u_0 \in L^2(\mathbb{T})$ [1] (see also [21]), so it is certainly well-defined for $u_0 \in H^{\frac{1}{2}-}(\mathbb{T})$ (globally-in-time). For mKdV, Bourgain proved LWP in a modified Besov-type space, slightly larger than $H^{\frac{1}{2}}(\mathbb{T})$ (where the Gibbs measure is also supported), but he could not use conservation of the $L^2(\mathbb{T})$-norm to extend solutions globally-in-time.

The main new idea implemented in [3] was to use the invariance of the Gibbs measure under the flow of the finite-dimensional system of ODEs obtained by the projecting mKdV[3] to the first $N > 0$ modes of the trigonometric basis (and an approximation argument) as a *substitute for a conservation law*, extending the local solutions of mKdV to global solutions (almost surely in the support of the Gibbs measure), and subsequently proving the invariance of the Gibbs measure μ under the flow.

We are interested in proving the invariance of the Gibbs measure under the flow of (1.1). Following the strategy developed in [3], the crucial ingredient is local well-posedness (and good approximation to the finite-dimensional ODEs) in the support of the Gibbs measure. Unfortunately, the C^4-failure of the data-to-solution map below $H^{\frac{1}{2}}(\mathbb{T})$ [12] indicates that one *cannot use the contraction mapping principle* to establish LWP of (1.1) in $H^s(\mathbb{T})$ for $s < \frac{1}{2}$, as this necessitates analyticity of the data-to-solution map. However, to establish local-in-time dynamics for (1.1) in

[2] This is at least formally correct, for the rigorous definition of gKdV as a Hamiltonian system see [25].

[3] In [3], Bourgain also proved the invariance of the Gibbs measure for periodic nonlinear Schrödinger equations, but we will focus on (1.2) in this discussion.

the support of the Gibbs measure, it suffices to prove something weaker: that (1.1) is locally well-posed *almost surely* with randomized initial data given by

$$u_{0,\omega}(x) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{g_n(\omega)}{|n|} e^{inx}, \tag{1.5}$$

where $\{g_n\}_{n=1}^{\infty}$ is a sequence of independent complex-valued Gaussian random variables of mean 0 and variance 1 on a probability space $(\Omega, \mathcal{F}, P)$, and $g_{-n} = \overline{g_n}$ (in order for $u_{0,\omega}$ to be real-valued). The expression (1.5) represents a typical element in the support of the Gaussian part of the Gibbs measure, also known as the Wiener measure (see (1.10) below).

The analysis of well-posedness for (1.1) is simplified by a gauge transformation. This transformation preserves the initial data, and it is invertible. A function u satisfies (1.1) if and only if its gauge transformation $v := \mathcal{G}(u)$ (see (1.14) below) satisfies

$$\begin{cases} \partial_t v + \partial_x^3 v = \mathbb{P}(v^3)\partial_x v, & x\in\mathbb{T},\ t\in\mathbb{R}, \\ v(x,0) = u_0(x), \end{cases} \tag{1.6}$$

where $\mathbb{P}(u) = u - \frac{1}{2\pi}\int_{\mathbb{T}} u\,dx$ is the projection to functions with mean zero. The analysis of well-posedness for (1.6) is simpler than for (1.1), but the data-to-solution map still fails to be C^4 below $H^{\frac{1}{2}}(\mathbb{T})$ [12].

To properly state our results we need a few more definitions. Consider the finite-dimensional Galerkin approximation of (1.6),

$$\begin{cases} \partial_t u^N + \partial_x^3 u^N = \mathbb{P}_N\big(\mathbb{P}\big((u^N)^3\big)\partial_x u^N\big), & x\in\mathbb{T},\ t\in\mathbb{R}, \\ u^N(x,0) = \mathbb{P}_N u_0(x) \in E_N, & u_0 \text{ mean zero.} \end{cases} \tag{1.7}$$

Here $\mathbb{P}_N$ denotes Dirichlet projection to $E_N = \operatorname{span}\{\sin(nx), \cos(nx) : 1 \le n \le N\}$. We also consider an extension of (1.7) to infinite dimensions, where the higher modes evolve according to linear dynamics. That is, we consider the system

$$\begin{cases} \partial_t v^N + \partial_x^3 v^N = \mathbb{P}_N\big(\mathbb{P}\big((\mathbb{P}_N v^N)^3\big)\partial_x \mathbb{P}_N v^N\big), & x\in\mathbb{T},\ t\in\mathbb{R}, \\ v^N(x,0) = v_0(x) \in H^{\frac{1}{2}-}, & v_0 \text{ mean zero,} \end{cases} \tag{1.8}$$

and let $\Phi^N(t)$ denote the flow map of (1.8).

In this paper, we exhibit nonlinear smoothing for (1.6) when the initial data are randomized according to (1.5). This is used to prove our first theorem: (1.6) is locally well-posed almost surely in $H^{\frac{1}{2}-}(\mathbb{T})$. In the statement below, $S(t) := e^{it\partial_x^3}$ denotes the evolution operator for the linear part of gKdV.

Theorem 1.1 *(Almost sure local well-posedness). The gauge-transformed periodic quartic gKdV (1.6) is locally well-posed almost surely with randomized data $u_{0,\omega}$ (given by (1.5)). More precisely, for all $0 < \delta_1 < \delta$, with δ sufficiently small, there exists $0 < \beta < \delta - \delta_1$, and $c > 0$ such that for each $0 < T \ll 1$, there is a set $\Omega_T \in \mathcal{F}$ with the following properties:*

(a) The complemental measure of Ω_T is small. More precisely, we have

$$P\big(\Omega_T^c\big) = \rho \circ u_0\big(\Omega_T^c\big) < e^{-\frac{c}{T^\beta}},$$

where ρ is the Wiener measure (see (1.10) below), and the initial data (given by (1.5)) is viewed as a map $u_0 : \Omega \to H^{1/2-}(\mathbb{T})$.

(b) For each $\omega \in \Omega_T$ there exists a solution u to (1.6) with data $u_{0,\omega}$ satisfying

(i) $u \in S(t)u_{0,\omega} + C([0,T]; H^{1/2+\delta}(\mathbb{T})) \subset C([0,T]; H^{1/2-}(\mathbb{T}))$.

(ii) The solution u is unique in $\{S(t)u_{0,\omega} + B_K\}$, for some $K > 0$, where B_K denotes a ball of radius K in the space $X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}$.

(iii) *u depends continuously on the initial data, in the sense that, for each $\omega \in \Omega_T$, the solution map*

$$\Phi : \left\{u_{0,\omega} + \left\{\|\cdot\|_{H^{\frac{1}{2}+\delta}} \le R\right\}\right\} \to \left\{S(t)u_{0,\omega} + \left\{\|\cdot\|_{C([0,T];H^{\frac{1}{2}+\delta})} \le \tilde{R}\right\}\right\}$$

is well-defined and Lipschitz, for some fixed $R, \tilde{R} \sim 1$.

(iv) *u is well-approximated by the solution of (1.8). More precisely,*

$$\left\|u - \Phi^N(t)u_{0,\omega}\right\|_{C([0,T];H^{\frac{1}{2}+\delta_1})} \lesssim N^{-\beta}. \tag{1.9}$$

For the definition of the $X_T^{s,b}$ space, see Section 2 below.

Following the method developed in [3], we use the invariance of finite-dimensional Gibbs measures under the flow of (1.7) and an approximation argument, to extend the local solutions of (1.6) (obtained from Theorem 1.1) to global solutions, almost surely, and to prove the invariance of the Gibbs measure under the flow.

Theorem 1.2 *(Invariance of the Gibbs measure). The gauge-transformed periodic quartic gKdV (1.6) is globally well-posed almost surely with randomized data $u_{0,\omega}$ (given by (1.5)). More precisely, for $\delta_2 > 0$ sufficiently small, it holds that given any $T > 0$, for almost every $\omega \in \Omega$, there is a (unique) solution u to (1.6) with data $u_{0,\omega}$ satisfying*

$$u \in S(t)u_{0,\omega} + C\left([0,T]; H^{1/2+\delta_2}(\mathbb{T})\right) \subset C\left([0,T]; H^{1/2-}(\mathbb{T})\right).$$

Furthermore, the Gibbs measure μ (given by (1.11) below) is invariant under the flow.

By inverting the gauge transformation, we obtain the following corollary.

Corollary 1.3 *(Almost sure global well-posedness). The periodic quartic gKdV (1.1) is globally well-posed almost surely in $H^{1/2-}(\mathbb{T})$. More precisely, given any $T > 0$, for almost every $\omega \in \Omega$, there exists a (unique) solution u to (1.1) for $t \in [0,T]$ with randomized data $u_{0,\omega}$ (given by (1.5)).*

Remark 1.4. In terms of global theory, GWP of (1.1) in $H^s(\mathbb{T})$ for $s > \frac{5}{6}$ was established in [12] using the I-method. This is mentioned to emphasize that, to the knowledge of the author, Corollary 1.3 is the first result to provide global-in-time solutions to (1.1) below $H^{\frac{5}{6}}(\mathbb{T})$. We further note that these solutions evolve from data at a spatial regularity where even local theory is unavailable at present (below $H^{\frac{1}{2}}(\mathbb{T})$).

Remark 1.5. The solutions of (1.6) and (1.1) produced by Theorem 1.2 and Corollary 1.3, respectively, are unique in a mild sense only. For a technical description of the uniqueness of solutions to (1.6) produced by Theorem 1.2, see Remark 5.9 in Section 5. This characterization applies to the gauge transformation (see (1.14) below) of the solutions to (1.1) produced by Corollary 1.3, which provides a mild form of uniqueness due to the invertibility of this transformation.

Remark 1.6. By composing with a modified and time-dependent gauge transformation $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}_t$, we can obtain a time-dependent measure $\nu_t := \mu \circ \widetilde{\mathcal{G}}_t$, supported on $H^s(\mathbb{T})$ for $s < \frac{1}{2}$, which (due to Theorem 1.2) satisfies $\Psi(t)^*\nu_t = \mu$ for each $t \ge 0$, where $\Psi(t)$ is the evolution operator for (1.1) (well-defined in the support of the Gibbs measure by Corollary 1.3). This leads to a natural question for future investigation: how is the time-dependent measure $\nu_t = \mu \circ \widetilde{\mathcal{G}}_t$ related to the Gibbs measure μ? Do we in fact have invariance of the Gibbs measure for the ungauged quartic gKdV (1.1)? This type of issue was recently explored for the periodic derivative NLS [30,32].

For the remainder of the introduction we provide more background on this problem, then outline the methods involved and the challenges confronted in the proofs of Theorem 1.1 and Theorem 1.2.

1.1. Background

The study of invariant Gibbs measures for Hamiltonian PDEs was initiated in [27]. They constructed the Gibbs measure for (1.2) as a weighted Wiener measure. Recall that the Wiener measure,[4] ρ, is the probability measure supported on $H^{\frac{1}{2}-}(\mathbb{T})$ with density

$$d\rho = Z_0^{-1} e^{-\frac{1}{2}\int u_x^2 dx} \prod_{x\in\mathbb{T}} du(x), \quad u \text{ mean zero.} \tag{1.10}$$

This is a purely formal expression, but it provides intuition. We can in fact define ρ as the push-forward of the probability measure P under the map from Ω to $H^{\frac{1}{2}-}(\mathbb{T})$ given by $\omega \mapsto u_{0,\omega}$ (as defined in (1.5), see Section 5 for details).

In [27], it was shown that the *Gibbs measure*,[5] μ, given by

$$\begin{aligned} d\mu &:= \chi_{\{\|u\|_2\le B\}} e^{\mp\frac{1}{p(p+1)}\int_{\mathbb{T}} u^{p+1}dx} d\rho \\ &= Z_0{}^{-1}\chi_{\{\|u\|_2\le B\}} e^{-H(u)} \prod_{x\in\mathbb{T}} du(x), \end{aligned} \tag{1.11}$$

is a finite Borel measure on $H^{\frac{1}{2}-}(\mathbb{T})$ (for integer $1 \le p \le 5$, and with restrictions on B for $p = 5$) that is absolutely continuous with respect to the Wiener measure ρ. That is, the Gibbs measure μ for (1.1) was defined in [27], see also [3].

1.2. Nonlinear smoothing for the second iteration

As discussed above, the Gibbs measure for (1.1) is supported below $H^{\frac{1}{2}}(\mathbb{T})$, and local well-posedness of the quartic gKdV (both gauged and ungauged) cannot be established in $H^s(\mathbb{T})$ for $s < \frac{1}{2}$ by applying the contraction principle to an equivalent integral equation, as the data-to-solution map is not C^4 [12]. In this paper we avoid this obstruction by exhibiting nonlinear smoothing under initial data randomization (according to (1.5)) on the second iteration of the integral formulation of (1.6). In this subsection we compare and contrast our approach with other related strategies.

Nonlinear smoothing induced by initial data randomization was also exploited by Bourgain [4]. He considered the Wick-ordered cubic NLS on $\mathbb{T}^2$, and proved invariance of the Gibbs measure under the flow. The Gibbs measure (in two dimensions) is supported on $H^s(\mathbb{T}^2)$ for $s < 0$, and the Wick-ordered cubic NLS is ill-posed below $L^2(\mathbb{T}^2)$ due to scaling. In order to discuss the flow in the support of the Gibbs measure, Bourgain considered randomized initial data given by

$$\tilde{u}_{0,\omega}(x) = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\sqrt{1+|n|^2}} e^{in\cdot x}, \tag{1.12}$$

where $\{g_n\}_{n\in\mathbb{Z}^2}$ is a collection of independent complex-valued Gaussian random variables of mean 0 and variance 1 on a probability space $(\Omega, \mathcal{F}, P)$, which represents a typical element in the support of the Wiener measure. He exhibited a nonlinear smoothing effect induced by this randomization, and used this to construct local solutions to the Wick-ordered cubic NLS almost surely in the support of the Gibbs measure.

Bourgain quantified the nonlinear smoothing effect by proving that, with high probability, the nonlinear part of the solution to the Wick-ordered cubic NLS with randomized data $\tilde{u}_{0,\omega}$ lies in a smoother space – $C([0,T]; H^s(\mathbb{T}^2))$ for some $s > 0$ – than the linear evolution. In contrast, the linear evolution almost surely stays below $L^2(\mathbb{T}^2)$ for all time. More precisely, for all $T > 0$ sufficiently small, he constructed a set $\Omega_T \subset \Omega$ (corresponding to "good" randomized data $\tilde{u}_{0,\omega}$), such that Ω_T is exponentially likely as a function of $T \searrow 0$, and such that for each $\omega \in \Omega_T$, he could prove local existence and uniqueness of the solution to the Wick-ordered cubic NLS with data $\tilde{u}_{0,\omega}$ for $t \in [0, T]$

[4] This is the mean zero Wiener measure, but we restrict attention to measures, data, and solutions with spatial mean zero throughout this paper, and will often omit the prefix "mean zero".

[5] The Gibbs measure was constructed for NLS in [27], but the same method applies to gKdV, see [3].

by performing a contraction argument in the space $\{e^{it\Delta}\tilde{u}_{0,\omega} + B\}$, where B is a ball in $Z_T^{s,\frac{1}{2}} \subset C([0,T]; H^s(\mathbb{T}^2))$ for some $s > 0$ (for the definition of the function space $Z_T^{s,\frac{1}{2}}$, consult Section 2 below). By taking an appropriate union over sets of this type (with $T \searrow 0$), he obtained local well-posedness almost surely for the Wick-ordered cubic NLS below $L^2(\mathbb{T}^2)$. For other works that have used nonlinear smoothing to establish local dynamics (and other properties) for dispersive PDEs in the support of measures on phase space, see (for example) Bourgain and Bulut [5,6], Burq, Thomann and Tzvetkov [7], Burq and Tzvetkov [8–10], Colliander and Oh [14], Deng [15,16], Luhrmann and Mendelson [28], Oh [34–37], Nahmod and Staffilani [33], Poiret [38], de Suzzoni [41,42], de Suzzoni and Tzvetkov [43] and Tzvetkov [46,47]. For related results on Navier–Stokes equations, see for example, Deng and Cui [17,18], Nahmod, Pavlović and Staffilani [31] and Zhang and Fang [51].

This paper considers (1.6) posed with randomized initial data of the form (1.5). To establish almost sure LWP, we found that (due in part to using the temporal regularity $b = \frac{1}{2}$) we could not follow the method of [4] directly, and perform a contraction argument for (1.6) (with exponential likelihood in T) in $\{S(t)u_{0,\omega} + B\}$, where B is a ball in the Banach space $Z_T^{s,\frac{1}{2}}$. Instead, we will establish a priori estimates on the *second iteration* of the Duhamel formulation of (1.6) in $X_T^{s,b}$, with $s > \frac{1}{2}$ and $b < \frac{1}{2}$.

More precisely, the local-in-time solution u to (1.6) will be constructed as the limit in $X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}$ (with $0 < \delta \ll 1$) of a sequence of smooth solutions u^N evolving from frequency truncated data $u_{0,\omega}^N = \mathbb{P}_N(u_{0,\omega})$. Each u^N will satisfy the Duhamel formulation

$$u^N(t) = S(t)u_{0,\omega}^N + \mathcal{D}\big(u^N\big)(t), \tag{1.13}$$

where

$$\mathcal{D}(v)(t) = \int_0^t S(t-s)\mathcal{N}\big(v(s)\big)ds$$

and $\mathcal{N}(u) = u_x(u^3 - \frac{1}{2\pi}\int_{\mathbb{T}} u^3 dx)$ is the gauge-transformed nonlinearity. We will simultaneously establish the convergence of u^N to u, and $\mathcal{D}(u^N)$ to $\mathcal{D}(u)$, in $X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}$ and $X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}$, respectively, as $N \to \infty$. Notice the increase in spatial regularity from $s = \frac{1}{2} - \delta$ to $s = \frac{1}{2} + \delta$; this is the nonlinear smoothing effect induced by initial data randomization. During the proof of nonlinear estimates required to establish these convergence results, there is a troublesome region of frequency space created by taking $b < \frac{1}{2}$. In this region we will substitute (1.13) into an appropriately chosen factor of the nonlinearity, and expand. Taking this approach will resolve the technical obstruction due to $b < \frac{1}{2}$, but by considering a second iteration of (1.13) into just one of the factors, the nonlinear analysis expands from quartic to septic. Indeed, we will establish probabilistic septilinear estimates on the second iteration of (1.13). This is the trade-off involved in proving a priori estimates on the second iteration in $X_T^{s,b}$ with $b < \frac{1}{2}$: *one can take $b < \frac{1}{2}$ at the cost of conducting a higher order multilinear analysis*.

This approach (using $b < \frac{1}{2}$ and the second iteration) was pioneered by Bourgain [2] in the analysis of KdV with measures as initial data. The argument was adapted to the setting of randomized initial data by Oh [37], who proved invariance of the white noise measure for the periodic KdV (see also [39]). Our approach is similar to [37], but we consider the quartic gKdV and the Gibbs measure (as opposed to the KdV and white noise). The additional source of difficulty for well-posedness of the quartic gKdV (including the C^4-failure below $H^{\frac{1}{2}}(\mathbb{T})$ established in [12]) is the existence of distinct frequencies $n, n_1, \cdots, n_4 \in \mathbb{Z}$, such that $n = n_1 + \cdots + n_4$, $|n| \sim |n_1| \sim N \gg 0$, but such that $|n^3 - n_1^3 - \cdots - n_4^3| \ll N^2$. This does not occur for KdV and mKdV, which have dispersion relations with cubic factorizations, and it makes the nonlinear analysis for quartic gKdV more labor intensive.[6] Indeed, the regions of frequency space where these conditions are satisfied require us to use $b < \frac{1}{2}$ (and thus to consider a second iteration of (1.13)). Furthermore, it is in these regions of frequency space where our nonlinear estimates will rely most heavily on probabilistic lemmata (involving, for example, hyper-contractivity properties of the Ornstein–Uhlenbeck semigroup, see Lemmas 6.4–6.6 in Section 6.2).

[6] Let us remark that KdV and mKdV are strongly ill-posed (discontinuous data-to-solution maps) below $H^{-1}(\mathbb{T})$ and $L^2(\mathbb{T})$, respectively [29].

1.3. Global-in-time solutions

To establish Theorem 1.2, we follow the scheme of [3]. Using the invariance of (finite-dimensional) Gibbs measures under the flow of (1.7), we extend local solutions of (1.6) (produced by Theorem 1.1) to global solutions, almost surely, and prove the invariance of the Gibbs measure under the flow. To be more precise, we follow the method of [7,49], and use the invariance of a sequence of truncated Gibbs measures for (1.8). All of these truncated measures are defined on a fixed infinite-dimensional space – they are Gaussian at high frequencies (see (5.2) in Section 5). This modification makes the limit properties of these truncated measures more flexible. We combine the invariance of these measures under (1.8) with the approximation (1.9) in order to extend local solutions of (1.6) globally-in-time, and to prove the invariance of the Gibbs measure under the flow.

To show that the system (1.7) preserves the finite-dimensional Gibbs measure, we need to prove that the L^2-norm of u^N, the Hamiltonian $H(u^N)$ and the Lebesgue measure on phase space (the space of Fourier coefficients) are all invariant under the flow. The invariance of the Lebesgue measure is trivial (by Liousville's Theorem) for a Hamiltonian system, but the Hamiltonian formulation of (1.1) is disrupted by the gauge transformation (and the same is true in finite dimensions). That is, the exact nature of the Hamiltonian formulation for (1.7) (if one exists) is not clear to the author. Instead, we verify invariance of the Lebesgue measure under the flow of (1.7) directly. To extend from invariance of the Gibbs measure for (1.7) to invariance of truncated Gibbs measures (Gaussian at high frequencies) for (1.8), we also need to use the invariance of complex Gaussians under rotation.

1.4. The gauge transformation

Following the standard reductions of [44,12], we consider the gauge transformation

$$v(x,t) = \mathcal{G}\big(u(x,t)\big) := u\left(x - \int_0^t \fint_{\mathbb{T}} u^3(x',t')dx'dt', t\right). \tag{1.14}$$

This transformation preserves the initial data u_0, and for fixed $t \in \mathbb{R}$ it is an isometry on $H^s(\mathbb{T})$. Also, it is invertible:

$$u(x,t) = \mathcal{G}^{-1}\big(v(x,t)\big) = v\left(x + \int_0^t \fint_{\mathbb{T}} v^3(x',t')dx'dt', t\right). \tag{1.15}$$

That is, u solves (1.1) if and only if[7] v solves (1.6). Eqs. (1.1) and (1.6) leave the same Hamiltonian

$$H(u) = \frac{1}{2}\int u_x^2 dx + \frac{1}{20}\int u^5 dx = \frac{1}{2}\int v_x^2 dx + \frac{1}{20}\int v^5 dx =: H(v), \tag{1.16}$$

invariant under the flow.

As a final remark, note that since $v^3 v_x = \frac{1}{4}\partial_x(v^4)$ and v_x both have mean zero, so does $\mathbb{P}(v^3)v_x$. We therefore have $\mathbb{P}(v^3)v_x = \mathbb{P}(\mathbb{P}(v^3)v_x)$. Then since $\fint_{\mathbb{T}} v^2 v_x = \frac{1}{3}\fint_{\mathbb{T}}(v^3)_x = 0$, and $\fint_{\mathbb{T}} v v_x = \frac{1}{2}\fint_{\mathbb{T}}(v^2)_x = 0$, we can subtract $3\mathbb{P}(v)\fint_{\mathbb{T}} v^2 v_x + 3\mathbb{P}(v^2)\fint_{\mathbb{T}} v v_x$ from the right hand side of (1.6), with no effect, and rewrite (1.6) as

$$\begin{cases} \partial_t v + \partial_x^3 v = \mathbb{P}\big(\mathbb{P}\big(v^3\big)v_x\big) - 3\mathbb{P}(v)\fint_{\mathbb{T}} v^2 v_x - 3\mathbb{P}\big(v^2\big)\fint_{\mathbb{T}} v v_x, \quad x \in \mathbb{T},\ t \in \mathbb{R}, \\ v(x,0) = u_0(x). \end{cases} \tag{1.17}$$

The reformulation (1.17) of (1.6) will be needed during the proof of certain nonlinear estimates. Indeed, after generalizing the right-hand side of (1.17) to a specific multilinear function (see (3.1)–(3.2) and (6.1)–(6.4) below), this reformulation removes resonant frequency interactions which would otherwise complicate our nonlinear analysis.

More precisely, the reformulation (1.17) is critical during the proof of Lemma 6.8 in Section 6. Lemma 6.8 is a nonlinear estimate which will eventually be used during the proof of Theorem 1.1 (through Proposition 3.2 and Proposition 6.1). The proof of Lemma 6.8 will involve nonlinear Fourier analysis, and the quartic nonlinearity in

[7] Note that (1.15) is well-defined for $v \in X^{1/2-,1/2-}$ by the embedding $X^{1/2-,1/2-} \subset L^3_{x,t}$, see Section 2.

(1.17) will produce a convolution over quintuples of integer frequencies (n, n_1, n_2, n_3, n_4) such that $n = n_1 + n_2 + n_3 + n_4$, where n_1 is the frequency corresponding to the derivative. The gauge transformation leading to (1.6) removes interactions in this convolution where $n = n_1$, and by replacing $\mathbb{P}(v^3)v_x$ with $\mathbb{P}(\mathbb{P}(v^3)v_x)$ we remove the interactions where $n = 0$. The term $\mathbb{P}(v) \fint_{\mathbb{T}} v^2 v_x$ is subtracted three times in (1.17) to remove interactions where $n = n_k$ for each $k = 2, 3, 4$, and the terms of the type $\mathbb{P}(v^2) \fint_{\mathbb{T}} v v_x$ are subtracted to remove interactions where $n_1 = -n_k$ for each $k = 2, 3, 4$. This procedure has redundancy (see the definition of $\zeta_2(n)$ in (6.3)), but we are able to control the terms created by this overlap.

Let us describe the benefit of removing these frequency interactions. During the proof of Lemma 6.8 we will order the frequencies $\{-n, n_1, n_2, n_3, n_4\}$ by magnitude using superscripts $|n^0| \geq |n^1| \geq |n^2| \geq |n^3| \geq |n^4|$. For the quartic gKdV, there is a region of frequency space where $|n^3 - n_1^3 - n_2^3 - n_3^3 - n_4^3| \ll (n^0)^2$ (recall that this does not occur for KdV and mKdV, which have cubic factorizations [1]). In fact, Lemma 6.8 is the nonlinear estimate which controls the contributions from precisely this region. By considering the reformulation (1.17) we can control the contributions from the subset of this region where $n^0 = -n^1$. This is described in Case 1 during the proof of Lemma 6.8 (see (6.17)). Without introducing (1.17), the regions of frequency space where $n = n_k$ for some $k = 1, 2, 3, 4$, or $n_1 = -n_k$ for some $k = 2, 3, 4$, would have produced the following subcases when $n^0 = -n^1$:

(i) $-n^0 = n = n_1 = n^1$, $|n^3 - n_1^3 - n_2^3 - n_3^3 - n_4^3| = 3|n_2 n_3 n_4| \ll (n^0)^2$,
(ii) $n^0 = n_1 = -n_2 = -n^1$, $|n^3 - n_1^3 - n_2^3 - n_3^3 - n_4^3| = 3|n n_3 n_4| \ll (n^0)^2$,

and similar cases when $n = n_k$ for $k = 2, 3, 4$, or $n_1 = -n_k$ for $k = 3, 4$. These contributions are all problematic, as the derivative produces a factor of n_1 in the nonlinearity, and the function spaces we will work with produce a positive power of n in the numerator, so that division by the cubic expression $|n^3 - n_1^3 - n_2^3 - n_3^3 - n_4^3|$ due to dispersion does not, in these cases, provide us with sufficient means for balancing these large factors. However, by introducing (1.17) we can avoid these interactions completely.

By controlling the contributions from the region where $|n^3 - n_1^3 - n_2^3 - n_3^3 - n_4^3| \ll (n^0)^2$ under the assumption that $n^0 = -n^1$, we can reduce to the case where $n^0 \neq -n^1$. This reduction produces additional restrictions under the condition $|n^3 - n_1^3 - n_2^3 - n_3^3 - n_4^3| \ll (n^0)^2$. Recall that $n^0 + \cdots + n^4 = 0$ is satisfied, so that

$$\begin{aligned}
|n^3 - n_1^3 - \cdots - n_4^3| &= \left|\left(n^1 + \cdots + n^4\right)^3 - \left(n^1\right)^3 - \cdots - \left(n^4\right)^3\right| \\
&= 3\left|\left(-n^0 n^1 + n^2\left(n^3 + n^4\right) + n^3 n^4\right)\left(n^2 + n^3 + n^4\right) - n^2 n^3 n^4\right| \\
&\gtrsim |n^0||n^1||n^0 + n^1|,
\end{aligned}$$

where the last inequality holds if both $|n^3| \ll |n^0|$ and $|n^2 n^3 n^4| \ll |n^0||n^1||n^0 + n^1|$. This is in contradiction with the condition $|n^3 - n_1^3 - n_2^3 - n_3^3 - n_4^3| \ll (n^0)^2$ (since $n^0 \neq -n^1$). Therefore we can assume that either $|n^3| \sim |n^0|$ or $|n^2 n^3 n^4| \gtrsim |n^0||n^1||n^0 + n^1|$ with $|n^3| \ll |n^0|$ (see Case 2 in the proof of Lemma 6.8), and this provides the analytical leverage required to complete the proof of Lemma 6.8.

1.5. Notation

We include some brief remarks on notation.

For simplicity, the appropriate factors of 2π will often be dropped when we use the Fourier transform.

Let $\eta \in C_c^\infty(\mathbb{R})$ denote a smooth bump function supported on $[-2, 2]$ such that $\eta \equiv 1$ on $[-1, 1]$, and write $\eta_\delta(t) := \eta(t/\delta)$. Also let $\chi = \chi_{[-1,1]}$ denote the characteristic function of the interval $[-1, 1]$ and $\chi_\delta(t) := \chi(t/\delta) = \chi_{[-\delta,\delta]}(t)$.

We write $A \lesssim B$ to denote an estimate of the form $A \leq CB$, where C is a general constant. Similarly, we write $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$ and use $A \ll B$ when there is no general constant C such that $B \leq CA$. Where appropriate, we will modify these notations by $A \lesssim_\varepsilon B$ (or $\sim_\varepsilon$, $\ll_\varepsilon$) to indicate that the implied constant depends on a parameter ε.

Lastly, we use $a+$ (and $a-$) to denote $a + \varepsilon$ (and $a - \varepsilon$), respectively, for arbitrarily small $\varepsilon \ll 1$.

1.6. Organization of paper

The remainder of this paper is organized as follows. In Section 2 we present the basic linear estimates related to the propagator $S(t) := e^{-\partial_x^3 t}$ of the linear part of gKdV. In Section 3 we present the nonlinear estimates to be used in the proof of local well-posedness (Theorem 1.1). In Section 4 we will prove Theorem 1.1. Section 5 contains the proof of Theorem 1.2. Section 6 is devoted to the proof of the crucial nonlinear estimates using certain technical lemmata. The proofs of these lemmata are included in Appendix A.

2. Linear estimates

In [1], Bourgain introduced a weighted space–time Sobolev space $X^{s,b}(\mathbb{T}\times\mathbb{R})$ whose norm is given by

$$\|u\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} = \big\|\langle n\rangle^s \big\langle \tau - n^3\big\rangle^b \hat{u}(n,\tau)\big\|_{L^2_{n,\tau}(\mathbb{Z}\times\mathbb{R})}.$$

Since the $X^{s,\frac{1}{2}}$ norm fails to control $L^\infty_t H^s_x$ norm, a smaller space $Z^{s,b}(\mathbb{T}\times\mathbb{R})$ was also introduced, whose norm is given by

$$\|u\|_{Z^{s,b}(\mathbb{T}\times\mathbb{R})} := \|u\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} + \|u\|_{Y^{s,b-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})}, \tag{2.1}$$

where $\langle\cdot\rangle = 1 + |\cdot|$ and $\|u\|_{Y^{s,b}(\mathbb{T}\times\mathbb{R})} = \|\langle n\rangle^s \langle \tau - n^3\rangle^b \hat{u}(n,\tau)\|_{l^2_n L^1_\tau(\mathbb{Z}\times\mathbb{R})}$. One also defines the local-in-time version $Z^{s,b}_T$ on $\mathbb{T}\times[0,T]$, by

$$\|u\|_{Z^{s,b}_T} = \inf\big\{\|\tilde{u}\|_{Z^{s,b}(\mathbb{T}\times\mathbb{R})} : \tilde{u}|_{[0,T]} = u\big\}.$$

The local-in-time versions of other function spaces are defined analogously.

In this section we present the basic linear estimates related to gKdV. Let $S(t) := e^{-\partial_x^3 t}$ and $T \le 1$ in the following. We first state the homogeneous and nonhomogeneous linear estimates. See [1,22] for details.

Lemma 2.1. *For any $s \in \mathbb{R}$ and $b < \frac{1}{2}$, we have $\|S(t)u_0\|_{X^{s,b}_T} \lesssim T^{\frac{1}{2}-b}\|u_0\|_{H^s}$.*

Lemma 2.2. *For any $s, b \in \mathbb{R}$, we have $\|\eta_T(t)S(t)u_0\|_{X^{s,b}} \lesssim_{T,b} \|u_0\|_{H^s}$.*

Lemma 2.3. *For any $s \in \mathbb{R}$ and $b \le \frac{1}{2}$, we have*

$$\left\|\int_0^t S(t-t')F(x,t')dt'\right\|_{X^{s,b}_T} \lesssim_{T,b} \|F\|_{Z^{s,b-1}_T}.$$

For $b \in (\frac{1}{2}, 1)$ we have

$$\left\|\int_0^t S(t-t')F(x,t')dt'\right\|_{X^{s,b}_T} \lesssim_{T,b} \|F\|_{X^{s,b-1}_T}.$$

Also, for any $b \in \mathbb{R}$, it holds that

$$\left\|\int_0^t S(t-t')F(x,t')dt'\right\|_{Y^{s,b}_T} \lesssim_{T,b} \|F\|_{Y^{s,b-1}_T}.$$

We will also require the following lemma concerning the $X^{s,b}_T$ spaces, which allows us to gain a small power of T by raising the temporal exponent b.

Lemma 2.4. *Let* $0 < b < \frac{1}{2}$, $s \in \mathbb{R}$, *then*

$$\|u\|_{X_T^{s,b}} \lesssim T^{\frac{1}{2}-b-} \|u\|_{X_T^{s,\frac{1}{2}}}.$$

The proof of Lemma 2.4 can be found in [14]; it is based on the following property of the $X_T^{s,b}$ spaces, which will be exploited throughout this paper. For any $b < \frac{1}{2}$, letting $\chi_{[0,T]}$ denote the characteristic function of the interval $[0,T]$, we have

$$\|u\|_{X_T^{s,b}} \sim \|\chi_{[0,T]} u\|_{X^{s,b}}. \tag{2.2}$$

Most of the probabilistic lemmata found in this paper will be presented in Section 6.2. They will be needed during the proofs of the crucial nonlinear estimates. Earlier in our analysis, however, we will require the following lemma regarding large deviations. This lemma is a special case of Fernique's Theorem [19] (for discussion see [26], specifically Remark 2 on page 104 and Theorem 3.1 on page 159), but it can also be proven by explicit computation.

Lemma 2.5. *Fix* $\gamma > 0$, *and let* $u_{0,\omega}$ *be given by* (1.5). *There exists* $c > 0$ *such that*

$$\mathbb{E}\big(\exp(c\|u_{0,\omega}\|^2_{H^{\frac{1}{2}-\gamma}})\big) < \infty.$$

Hence there exists $c' > 0$ *such that for each* $K > 0$,

$$P\big(\|u_{0,\omega}\|_{H^{\frac{1}{2}-\gamma}} \geq K\big) \leq e^{-c'K^2}.$$

Next we list some embeddings involving the $X^{s,b}$ spaces, to be used throughout this paper. We will use the trivial embedding

$$X^{s,b} \subset X^{s',b'} \tag{2.3}$$

for $s \geq s'$, $b \geq b'$. The spatial Sobolev embedding gives

$$X^{s,0} = L_t^2 H_x^s \subset L_t^2 L_x^p \tag{2.4}$$

where $0 \leq s < 1/2$ and $2 \leq p \leq \frac{2}{1-2s}$, or where $s > 1/2$ and $2 \leq p \leq \infty$. Also recall the energy estimate

$$X^{s,1/2+} \subset L_t^\infty H_x^s \subset L_t^\infty L_x^p \tag{2.5}$$

under the same conditions on s and p. This gives

$$X^{1/2+,1/2+} \subset L_{x,t}^\infty. \tag{2.6}$$

Interpolating (2.5) with (2.6), for $s > 1/2$, we have

$$X^{1/2+,1/2+} \subset L_t^q L_x^r \tag{2.7}$$

for all $2 \leq q, r \leq \infty$. Next we claim that by interpolating (2.7) with (2.4) (for $s = 0$ and $p = 2$), we find

$$X^{1/2-\delta,1/2-\delta} \subset L_t^{q'} L_x^{r'} \tag{2.8}$$

whenever $0 < \delta < 1/2$ and $2 \leq q', r' < 1/\delta$. Indeed, for each $0 < \delta < 1/2$, given any $2 \leq q' < 1/\delta$, we can select $\gamma > 0$ and $2 \leq q \leq \infty$ such that

$$\frac{1}{q'} = \frac{1}{2} + \frac{1-2\delta}{1+2\gamma}\left(\frac{1}{q} - \frac{1}{2}\right). \tag{2.9}$$

This should be clear to the reader by checking that, for fixed $0 < \delta < 1/2$, as a function of $(\gamma, q) \in (0,\infty) \times [2,\infty]$, the right-hand side of (2.9) maps onto $(\delta, 1/2]$. Letting $\theta = \frac{1-2\delta}{1+2\gamma}$, it follows that

$$\frac{1}{q'} = (1-\theta)\frac{1}{2} + \theta\frac{1}{q},$$

and

$$\frac{1}{2}-\delta=(1-\theta)0+\theta\left(\frac{1}{2}+\gamma\right).$$

The same computation applies with respect to the r' parameter, and the conclusion (2.8) follows by interpolating (2.7) with (2.4) (for $s=0$ and $p=2$) according to this algebraic scheme.

Recall the following Strichartz estimates from [1],

$$X^{0,1/3}\subset L^4_{x,t}, \tag{2.10}$$

and

$$X^{0+,1/2+}\subset L^6_{x,t}. \tag{2.11}$$

We can interpolate (2.10) with (2.11) to obtain

$$X^{0+,1/2-\sigma}\subset L^q_{x,t}, \tag{2.12}$$

whenever $4<q<6$ and $\sigma<2(\frac{1}{q}-\frac{1}{6})$.

Lastly we recall the following embeddings for the $Y^{s,b}$ space, which are easily established from the definitions: for $s\in\mathbb{R}$, we have

$$X_T^{s,\frac{1}{2}+}\subset Y_T^{s,0}\subset C\big([0,T];H^s(\mathbb{T})\big). \tag{2.13}$$

3. Nonlinear estimates

In this section we will formulate and state two key propositions (see Proposition 3.2 and Proposition 3.3 below). These propositions provide multilinear estimates to be used in the proof of Theorem 1.1 (which can be found in the next section). The proofs of Proposition 3.2 and Proposition 3.3 will be postponed to Section 6.

We begin by defining the multilinear functions which will appear in Proposition 3.2 and Proposition 3.3. In this paper, we solve the integral formulation of (1.6) with data $u_{0,\omega}$ (given by (1.5)),

$$u=S(t)u_{0,\omega}+\mathcal{D}(u). \tag{3.1}$$

Here $\mathcal{D}(u):=\mathcal{D}(u,u,u,u)$ and

$$\mathcal{D}(u_1,u_2,u_3,u_4):=\int_0^t S(t-t')\mathcal{N}(u_1,u_2,u_3,u_4)(t')dt', \tag{3.2}$$

with $\mathcal{N}(u_1,u_2,u_3,u_4)$ defined by its Fourier transform in space:

$$\big(\mathcal{N}(u_1,u_2,u_3,u_4)\big)^\wedge(n,t)=\sum_{(n_1,n_2,n_3,n_4)\in\zeta(n)}(in_1)\widehat{u_1}(n_1,t)\widehat{u_2}(n_2,t)\widehat{u_3}(n_3,t)\widehat{u_4}(n_4,t),$$

where $\zeta(n)$ is a set of frequencies (n_1,n_2,n_3,n_4) satisfying certain restrictions (dictated by the nonlinearity of (1.17)). The definition of $\zeta(n)$ is slightly cumbersome, and we avoid it here. See (6.2)–(6.4) in Section 6.1 for details.

Taking the Fourier transform in time, we have

$$\big(\mathcal{N}(u_1,u_2,u_3,u_4)\big)^\wedge(n,\tau)=\sum_{\zeta(n)}\int_{\tau=\tau_1+\cdots+\tau_4}(in_1)\hat{u_1}(n_1,\tau_1)\cdots\hat{u_4}(n_4,\tau_4). \tag{3.3}$$

Let A be the domain of integration in (3.3), given by

$$A:=\big\{(n,n_1,\dots,n_4,\tau,\tau_1,\dots,\tau_4)\in\mathbb{Z}^5\times\mathbb{R}^5:(n_1,n_2,n_3,n_4)\in\zeta(n),\ \tau=\tau_1+\cdots+\tau_4\big\}.$$

We will decompose A depending on the relative sizes of the dispersive weights $\sigma:=\tau-n^3$, $\sigma_k:=\tau_k-n_k^3$, and the spatial frequencies n, n_k, for $k=1,\dots,4$. Specifically, letting $|\sigma_{\max}|:=\max(|\sigma|,|\sigma_1|,|\sigma_2|,|\sigma_3|,|\sigma_4|)$ and $|n_{\max}|:=\max(|n|,|n_1|,|n_2|,|n_3|,|n_4|)$, we express $A=A_{-1}\cup A_0\cup\cdots\cup A_4$ by letting

$$A_{-1} := A \cap \{|\sigma_{\max}| \ll |n_{\max}|^2\},$$
$$A_0 := A \cap \{|\sigma| \gtrsim |n_{\max}|^2\},$$
$$A_k := A \cap \{|\sigma_k| \gtrsim |n_{\max}|^2\}, \tag{3.4}$$

for $k = 1, 2, 3, 4$. We will use $\mathcal{N}_j(u_1, u_2, u_3, u_4)$ to denote the contribution to $\mathcal{N}(u_1, u_2, u_3, u_4)$ coming from A_j, for $j = -1, 0, 1, 2, 3, 4$. Similarly we will use $\mathcal{D}_j(u_1, u_2, u_3, u_4)$ to denote each contribution to $\mathcal{D}(u_1, u_2, u_3, u_4)$.

The partition of type (3.4) is standard; see for example [23] in the context of KdV. In the analysis of KdV (and mKdV), the region A_{-1} is empty. However, in the analysis of (1.6) (with quartic nonlinearity), there are nontrivial contributions from the region A_{-1}. In fact, the counterexample which produces C^4-failure of the data-to-solution map for (1.6) in $H^s(\mathbb{T})$ for $s < \frac{1}{2}$ is based on this fact (see [12]).

The multilinear estimates we establish in the region A_{-1} will use a probabilistic analysis. These will be estimates on the first iteration of the integral formulation of (1.6) (i.e. we do not require the second iteration in the region A_{-1}). That is, in Proposition 3.2 below, we will establish *probabilistic quadrilinear estimates* on $\mathcal{D}_{-1}(u_1, \dots, u_4)$. In the regions $A_1, \dots, A_4$, we require both the second iteration and a probabilistic analysis. That is, we establish *probabilistic septilinear estimates* on $\mathcal{D}_1(\mathcal{D}(u_5, u_6, u_7, u_8), u_2, u_3, u_4)$, and analogous septilinear estimates on contributions from the regions A_2, A_3 and A_4 (see Proposition 3.2 below). In the region A_0, however, we can use a deterministic analysis. Indeed, we establish *deterministic quadrilinear estimates* on $\mathcal{D}_0(u_1, \dots, u_4)$ (see Proposition 3.3 below).

Remark 3.1. There is a condition which we will implicitly impose in the statements of Proposition 3.2 and Proposition 3.3 below: in all cases, the input factors u_j have spatial mean zero for all time. All of the factors we will consider in application of these estimates will be solutions (or differences of solutions) to (1.6) (equivalently (1.17)), or the truncation of this system to finite dimensions, evolving from initial data with spatial mean zero. That is, this mean zero condition will always be satisfied when these estimates are applied.

The probabilistic estimates established in the regions $A_{-1}, A_1, \dots, A_4$ are grouped into the following proposition.

Proposition 3.2 *(Probabilistic nonlinear estimates). For $\delta \gg \beta > 0$ sufficiently small, any $\delta_0 \geq 0$ such that $\delta > \delta_0$, and any $0 < T \ll 1$, there exist $c, C > 0$ and a measurable set $\Omega_T \subset \Omega$ satisfying $P(\Omega_T^c) < e^{-\frac{c}{T^\beta}}$ and the following conditions:*

(i) If $\omega \in \Omega_T$, then for every quadruple of Fourier multipliers $\Lambda_1, \dots, \Lambda_4$ defined by

$$\widehat{\Lambda_j f}(n) = \chi_{N_j \leq |n| \leq M_j} \hat{f}(n), \tag{3.5}$$

for some dyadic numbers $N_j < M_j \leq \infty$, and for every quadruple of real numbers

$$(\delta_1, \dots, \delta_4) \in \{(\delta_0, 0, 0, 0), \dots, (0, 0, 0, \delta_0)\},$$

we have the estimate

$$\|\mathcal{D}_{-1}(u_1, u_2, u_3, u_4)\|_{X_T^{\frac{1}{2}+\delta-\delta_0, \frac{1}{2}+\delta}} \leq CT^{-\beta} \prod_{j=1}^{4} \big(N_j^{-\beta} + \|u_j\|_{X_T^{\frac{1}{2}-\delta-\delta_j, \frac{1}{2}-\delta}} + \|u_j - S(t)\Lambda_j(u_{0,\omega})\|_{X_T^{\frac{1}{2}+\delta-\delta_j, \frac{1}{2}-\delta}}\big). \tag{3.6}$$

(ii) If $\omega \in \Omega_T$, then for every heptuple of Fourier multipliers $\Lambda_2, \dots, \Lambda_8$ defined by (3.5) for some dyadic numbers $N_j < M_j < \infty$, and for every heptuple of real numbers

$$(\delta_2, \dots, \delta_8) \in \{(\delta_0, 0, \dots, 0), \dots, (0, \dots, 0, \delta_0)\},$$

we have the estimates

$$\big\|\mathcal{D}_1\big(\mathcal{D}(u_5,u_6,u_7,u_8),u_2,u_3,u_4\big)\big\|_{X_T^{\frac{1}{2}+\delta-\delta_0,\frac{1}{2}+\delta}}$$
$$\leq CT^{-\beta}\prod_{j=2}^{8}\big(N_j^{-\beta}+\|u_j\|_{X_T^{\frac{1}{2}-\delta-\delta_j,\frac{1}{2}-\delta}}+\big\|u_j-S(t)\Lambda_j(u_{0,\omega})\big\|_{X_T^{\frac{1}{2}+\delta-\delta_j,\frac{1}{2}-\delta}}\big). \tag{3.7}$$

(iii) Estimates analogous to (3.7) on $\mathcal{D}_2(u_1,\mathcal{D}(u_5,u_6,u_7,u_8),u_3,u_4),\dots,\mathcal{D}_4(u_1,u_2,u_3,\mathcal{D}(u_5,u_6,u_7,u_8))$ *also hold for* $\omega\in\Omega_T$*.*

The estimates of Proposition 3.2 are based on nonlinear smoothing due to initial data randomization. However, in some regions (e.g. A_0), the choice of $b=\frac{1}{2}-\delta<\frac{1}{2}$ allows us to establish deterministic estimates.

Proposition 3.3 *(Deterministic nonlinear estimates). For* $\delta>0$ *sufficiently small, any* $\delta_0\geq 0$ *such that* $\delta>\delta_0$*, and any* $T>0$*, there exist* $\theta, C>0$ *such that*

$$\big\|\mathcal{D}_0(u_1,u_2,u_3,u_4)\big\|_{X_T^{\frac{1}{2}+\delta-\delta_0,\frac{1}{2}-\delta}}\leq CT^{\theta}\prod_{j=1}^{4}\|u_j\|_{X_T^{\frac{1}{2}-\delta-\delta_j,\frac{1}{2}-\delta}}, \tag{3.8}$$

$$\big\|\mathcal{D}_0(u_1,u_2,u_3,u_4)\big\|_{Y_T^{\frac{1}{2}+\delta-\delta_0,0}}\leq CT^{\theta}\prod_{j=1}^{4}\|u_j\|_{X_T^{\frac{1}{2}-\delta-\delta_j,\frac{1}{2}-\delta}}, \tag{3.9}$$

and

$$\big\|\mathcal{D}_k(u_1,u_2,u_3,u_4)\big\|_{X_T^{\frac{1}{2}+\delta-\delta_0,\frac{1}{2}+\delta}}\leq CT^{\theta}\|u_k\|_{X_T^{\frac{1}{2}-\delta-\delta_k,\frac{1}{2}+2\delta}}\prod_{j=1,\,j\neq k}^{4}\|u_j\|_{X_T^{\frac{1}{2}-\delta-\delta_j,\frac{1}{2}-\delta}}, \tag{3.10}$$

for each $k=1,2,3,4$*, and for all quadruples*

$$(\delta_1,\delta_2,\delta_3,\delta_4)\in\big\{(\delta_0,0,0,0),(0,\delta_0,0,0),(0,0,\delta_0,0),(0,0,0,\delta_0)\big\}.$$

We also have

$$\big\|\mathcal{D}_k(u_1,u_2,u_3,u_4)\big\|_{X_T^{\frac{1}{2}-4\delta,\frac{1}{2}+\delta}}\leq C\prod_{j=1}^{4}\|u_j\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}, \tag{3.11}$$

for each $k=1,2,3,4$*.*

Remark 3.4. In order to establish the crucial approximation (1.9) in the proof of Theorem 1.1, we will exploit flexibility of the estimates stated above with respect to frequency truncation of the nonlinearity. For the most part, this flexibility is implicit to the definition of the $X_T^{s,b}$-norm (and $Y_T^{s,b}$-norm), but this is less obvious when we consider a second iteration of the integral formulation of (1.7), and attempt to use (3.7). In particular, it should be mentioned that (for $\omega\in\Omega_T$) we also have an estimate analogous to (3.7) on the multilinear expression $\mathcal{D}_1(\mathbb{P}_N\mathcal{D}(u_5,u_6,u_7,u_8),u_2,u_3,u_4)$, and similarly for the contributions from A_2, A_3 and A_4. For an explicit reference to our use of this flexibility, see line (4.52) in Section 4.

Remark 3.5. There is another flexibility implicit to the nonlinear estimates of Proposition 3.2. The time interval $[0,T]$ can be replaced with an interval I of length T, and we do not need the randomized data $S(t)u_{0,\omega}$ to evolve from the left end-point of the interval I. In particular, we can take $I=[t_0,t_0+T]$, and prove Proposition 3.2 by replacing $S(t)u_{0,\omega}$ with $S(t+t_0)u_{0,\omega}$, for any $t_0\in\mathbb{R}$, since the linear gKdV evolution $S(t_0)$ preserves the Gaussian probability densities of the (independent) randomized Fourier coefficients in (1.5). However, by varying t_0 the probabilistic set $\Omega_T=\Omega_T(t_0)$ varies as well. That is, we can use this flexibility (varying $t_0\in\mathbb{R}$), but the measurable set of good data produced by Proposition 3.2 changes. We will stick to the following notation: $\Omega_T(t_0)$ is the set satisfying the conclusions of Proposition 3.2 on the time interval $I=[t_0,t_0+T]$ (instead of $[0,T]$) with initial data $u_{0,\omega}$ posed at time $t=0$.

Remark 3.6. By using nonlinear estimates in $X_T^{s,b}$ with $b = \frac{1}{2} + \delta > \frac{1}{2}$ (as in (3.6)), and applying the embedding (2.13), we will find (during the proof of Theorem 1.1) that the contributions to the nonlinear part of the solution from the regions $A_{-1}, A_1, \dots, A_4$ are automatically continuous in time with values in a Sobolev space of higher regularity than the data (condition (i) in the statement of Theorem 1.1). However, for the contribution from the region A_0, we will use the $X_T^{s,b}$ estimate (3.8) with $b = \frac{1}{2} - \delta < \frac{1}{2}$, and the proof of continuity will require a modified argument. This is where the $Y_T^{s,b}$ estimate (3.9) will be needed.

Remark 3.7. There is one region of frequency space, produced by using the second iteration, which appears lethal, at first glance, to the proof of (3.7). Luckily there is a cancellation in this region that saves the analysis. A technicality emerges, due to this cancellation, that needs to be addressed in this section. In particular, the estimate (3.7) has potentially different input functions $u_2, u_3, \dots, u_8$, but the cancellation that we need to invoke in the troublesome region of frequency space requires that all input functions are the same. This is not problematic, however, as we only need multilinear estimates with different input functions in order to bound the difference of two expressions, each given by $\mathcal{D}_1(\mathcal{D}(\cdot,\cdot,\cdot,\cdot),\cdot,\cdot,\cdot)$ evaluated with all input functions equivalent. We bound this difference using multilinearity and a telescoping sum. Therefore, to incorporate the cancellation with different input functions, we simply define $\mathcal{D}_1(\mathcal{D}(u_5, u_6, u_7, u_8), u_2, u_3, u_4)$ with the cancellation *imposed* in the troublesome region. For a more precise discussion of this cancellation (and the proper definition of the multilinear functions appearing in Proposition 3.2 above) see Section 6.3 (and, more precisely, Case 2.b.ii of the proof of Proposition 6.2).

The proofs of Proposition 3.2 and Proposition 3.3 are postponed to Section 6.

4. Almost sure local well-posedness

In this section we present the proof of Theorem 1.1. The key inputs for this proof are Proposition 3.2 and Proposition 3.3.

Proof of Theorem 1.1. We will construct the local solution to (1.6) as the limit of a sequence of solutions u^N which evolve from frequency truncated data. Consider initial data of the form

$$u_{0,\omega}^N(x) = \mathbb{P}_N\big(u_{0,\omega}(x)\big),$$

where $u_{0,\omega}$ is given by (1.5), and $\mathbb{P}_N$ is the Dirichlet projection to E_N. Notice that $u_{0,\omega}^N \in H^s(\mathbb{T})$ almost surely, for every $s \in \mathbb{R}$. By Theorem 2 in [12], for each $N > 0$, almost surely, there exists a unique global-in-time solution u^N to (1.6) with data $u_{0,\omega}^N$. Then u^N satisfies

$$u^N = S(t)u_{0,\omega}^N + \mathcal{D}^N, \tag{4.1}$$

where $\mathcal{D}^N := \mathcal{D}(u^N)$, $\mathcal{D}(u) = \mathcal{D}(u,u,u,u)$ and $\mathcal{D}(u_1,u_2,u_3,u_4)$ is defined by (3.2).

Here is an outline of the proof of Theorem 1.1: we show that for $0 < \delta_1 < \delta$ with δ sufficiently small, $\exists\, 0 < \beta < \delta - \delta_1$ and $c > 0$ such that for each $0 < T \ll 1$ there exists $\Omega_T \subset \Omega$ with $P(\Omega_T^c) < e^{-\frac{c}{T^\beta}}$ such that for $\omega \in \Omega_T$, we have the following: for every $N > M > 0$,

$$\big\| u^N - u^M \big\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} \lesssim M^{-\beta}, \tag{4.2}$$

$$\big\| \mathcal{D}^N - \mathcal{D}^M \big\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \lesssim M^{-\beta}. \tag{4.3}$$

These estimates show that u^N and $\mathcal{D}^N$ are Cauchy in $X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}$ and $X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}$, respectively. Then we show that the convergent u (of u^N) is a solution to (1.6), and proceed to prove continuity, uniqueness and stability properties of this solution (properties (i)–(iv) in the statement of Theorem 1.1).

We begin by constructing $\Omega_T \subset \Omega$ such that if $\omega \in \Omega_T$, we have

$$\|u^N\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} \leq R, \tag{4.4}$$

$$\|\mathcal{D}^N\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \leq \tilde{R}, \tag{4.5}$$

for some constants $R, \tilde{R} \sim 1$ (independent of N). Then using the estimates (4.4) and (4.5), and imposing additional constraints on T, we will show that if $\omega \in \Omega_T$, then (4.2) and (4.3) hold true as well. By (4.1) and Lemma 2.1, we find

$$\begin{aligned}\|u^N\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} &\leq \|S(t)u^N_{0,\omega}\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + \|\mathcal{D}^N\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} \\ &\leq C_0 T^\delta \|u^N_{0,\omega}\|_{H^{\frac{1}{2}-\delta}} + \|\mathcal{D}^N\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}.\end{aligned} \tag{4.6}$$

Next we use the triangle inequality and Lemma 2.4 to find

$$\begin{aligned}\|\mathcal{D}^N\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} &\leq \|\mathcal{D}_{-1}(u^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} + \cdots + \|\mathcal{D}_4(u^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \\ &\leq \|\mathcal{D}_0(u^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} + T^{\delta-} \sum_{k=-1,k\neq 0}^{4} \|\mathcal{D}_k(u^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}}.\end{aligned} \tag{4.7}$$

By Proposition 3.2, for $\delta \gg \beta > 0$ sufficiently small, choosing $\delta_0 = 0$ (for now), for any $0 < T \ll 1$, there exist $c, C > 0$ and a measurable set $\Omega_T^0 \subset \Omega$ satisfying $P((\Omega_T^0)^c) < e^{-\frac{c}{T^\beta}}$ such that if $\omega \in \Omega_T^0$ the estimates (3.6)–(3.7) hold true. In particular using (3.8) (with $\delta_0 = 0$) and (3.6) with each $u_j = u^N$, and each $\Lambda_j = \mathbb{P}_N$, we have that if $\omega \in \Omega_T^0$ then

$$\begin{aligned}&\|\mathcal{D}_0(u^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} + T^{\delta-}\|\mathcal{D}_{-1}(u^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}} \\ &\lesssim T^\kappa\big(1 + \|u^N\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + \|u^N - S(t)u^N_{0,\omega}\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\big)^4 \\ &= T^\kappa\big(1 + \|u^N\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + \|\mathcal{D}^N\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\big)^4.\end{aligned} \tag{4.8}$$

We have used (4.1) in the last line, and taken $\kappa \geq \theta \vee (\delta - \beta)$.

When we estimate $\mathcal{D}_k(u^N)$, for $k = 1, 2, 3, 4$, we consider a second iteration of (4.1) in the kth slot. For example, in the region A_1 we substitute (4.1) into the first slot of $\mathcal{D}_1(u^N)$, and estimate the contributions from the linear and nonlinear parts of (4.1) separately. To estimate the linear contribution, we find by (3.10) (with $\delta_0 = 0$) and Lemma 2.2,

$$\begin{aligned}T^{\delta-}\|\mathcal{D}_1(S(t)u^N_{0,\omega}, u^N, u^N, u^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}} &\lesssim T^{\delta-}\|\eta_T(t)S(t)u^N_{0,\omega}\|_{X^{\frac{1}{2}-\delta,\frac{1}{2}+2\delta}} \|u^N\|^3_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} \\ &\lesssim T^{\delta-}\|u^N_{0,\omega}\|_{H^{\frac{1}{2}-\delta}} \|u^N\|^3_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}.\end{aligned} \tag{4.9}$$

Next we estimate $\mathcal{D}_1(\mathcal{D}^N, u^N, u^N, u^N)$. Using (3.7) with each $u_j = u^N$ and each $\Lambda_j = \mathbb{P}_N$, and (4.1), we have for $\omega \in \Omega_T^0$,

$$T^{\delta-}\|\mathcal{D}_1(\mathcal{D}^N, u^N, u^N, u^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}} \lesssim T^\kappa\big(1 + \|u^N\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + \|\mathcal{D}^N\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\big)^7. \tag{4.10}$$

Using the estimates analogous to (3.7) for $k = 2, 3, 4$ (also produced by Proposition 3.2), we have statements similar to (4.9) and (4.10) (for $\omega \in \Omega_T^0$) to bound $\mathcal{D}_2(u^N), \mathcal{D}_3(u^N)$ and $\mathcal{D}_4(u^N)$. Combining these estimates with (4.7) and (4.8) we have

$$\|\mathcal{D}^N\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \leq C_1 T^\kappa Z\big(\|u^N_{0,\omega}\|_{H^{\frac{1}{2}-\delta}}, \|u^N\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}, \|\mathcal{D}^N\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\big), \tag{4.11}$$

for some $\kappa > 0$, where $Z(x, y, z)$ is a polynomial of degree 7 with positive coefficients.

Now, fix $C > 0$, and let

$$\Omega_T := \Omega_T^0 \cap \left\{\omega \in \Omega \;\middle|\; \left\| \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{g_n(\omega)e^{inx}}{|n|^{1-\beta}} \right\|_{H^{\frac{1}{2}-\delta}} \le \frac{C}{T^{\frac{\beta}{2}}} \right\}.$$

We have by Lemma 2.5 (with $\gamma = \delta - \beta > 0$) that for $T > 0$ sufficiently small $P((\Omega_T)^c) \le P((\Omega_T^0)^c) + e^{-c(K(T))^2} \le e^{-\frac{c}{T^\beta}}$. Combining (4.6) and (4.11) with a standard continuity argument, if $\omega \in \Omega_T$, then (4.4) and (4.5) are satisfied for each $N > 0$. See Section 3.4 of [40] for details.

To establish (4.2) and (4.3) we impose further restrictions on T. For $\omega \in \Omega_T$, consider

$$\begin{aligned} \|u^N - u^M\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} &\le \|S(t)(u_0^N - u_0^M)\|_{X^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + \|\mathcal{D}^N - \mathcal{D}^M\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} \\ &\le M^{-\beta}\frac{\tilde{C}}{T^{\frac{\beta}{2}}} + \|\mathcal{D}^N - \mathcal{D}^M\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}. \end{aligned} \tag{4.12}$$

Then, using the multilinearity of $\mathcal{D}$,

$$\begin{aligned} \|\mathcal{D}^N - \mathcal{D}^M\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} &\le \|\mathcal{D}(u^N - u^M, u^N, u^N, u^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} + \|\mathcal{D}(u^M, u^N - u^M, u^N, u^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \\ &\quad + \|\mathcal{D}(u^M, u^M, u^N - u^M, u^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \\ &\quad + \|\mathcal{D}(u^M, u^M, u^M, u^N - u^M)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}. \end{aligned} \tag{4.13}$$

Each of the terms in (4.13) will be bounded in a similar way; we proceed to estimate the first term explicitly. As above, we will bound the contributions from each region A_k, $k = -1, 0, 1, 2, 3, 4$, using Propositions 3.2 and 3.3. Consider (3.6) and (3.8) applied with $\delta_0 = 0$, $u_1 = u^N - u^M$, $\Lambda_1 = \mathbb{P}_N - \mathbb{P}_M$, and $u_j = u^N$, $\Lambda_j = \mathbb{P}_N$ for $j = 2, 3, 4$. This gives, for $\omega \in \Omega_T$,

$$\begin{aligned} &\|\mathcal{D}_0(u^N - u^M, u^N, u^N, u^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} + T^{\delta-}\|\mathcal{D}_{-1}(u^N - u^M, u^N, u^N, u^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}} \\ &\quad\lesssim T^\kappa\big(M^{-\beta} + \|u^N - u^M\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + \|\mathcal{D}^N - \mathcal{D}^M\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\big) \\ &\qquad \cdot \big(1 + \|u^N\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + \|\mathcal{D}^N\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\big)^3. \end{aligned} \tag{4.14}$$

To estimate $\mathcal{D}_k(u^N - u^M, u^N, u^N, u^N)$, for $k = 1, 2, 3, 4$, we again consider a second iteration of (4.1) in the kth factor. This argument requires minor modification when we consider $\mathcal{D}_1(u^N - u^M, u^N, u^N, u^N)$. We substitute

$$u^N - u^M = S(t)\big(u_{0,\omega}^N - u_{0,\omega}^M\big) + \mathcal{D}\big(u^N\big) - \mathcal{D}\big(u^M\big). \tag{4.15}$$

Then to estimate $\mathcal{D}_1(S(t)(u_{0,\omega}^N - u_{0,\omega}^M), u^N, u^N, u^N)$, we proceed as in (4.9) above. To be precise, by (3.10), Lemma 2.2, and the definition of Ω_T, we have

$$\begin{aligned} &T^{\delta-}\|\mathcal{D}_1(S(t)(u_{0,\omega}^N - u_{0,\omega}^M), u^N, u^N, u^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}} \\ &\quad\lesssim T^{\delta-}\|\eta(t)S(t)(u_{0,\omega}^N - u_{0,\omega}^M)\|_{X^{\frac{1}{2}-\delta,\frac{1}{2}+\delta}}\|u_j\|^3_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} \\ &\quad\lesssim T^\kappa M^{-\beta}\|u_j\|^3_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}. \end{aligned} \tag{4.16}$$

Next we estimate $\mathcal{D}_1(\mathcal{D}^N - \mathcal{D}^M, u^N, u^N, u^N)$. We find

$$\begin{aligned} &\|\mathcal{D}_1(\mathcal{D}^N - \mathcal{D}^M, u^N, u^N, u^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}} \\ &\quad\lesssim \|\mathcal{D}_1(\mathcal{D}(u^N - u^M, u^N, u^N, u^N), u^N, u^N, u^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}} \end{aligned}$$

$$
\begin{aligned}
&+\big\|\mathcal{D}_1\big(\mathcal{D}\big(u^M,u^N-u^M,u^N,u^N\big),u^N,u^N,u^N\big)\big\|_{X_T^{\frac12+\delta,\frac12+\delta}}\\
&+\big\|\mathcal{D}_1\big(\mathcal{D}\big(u^M,u^M,u^N-u^M,u^N\big),u^N,u^N,u^N\big)\big\|_{X_T^{\frac12+\delta,\frac12+\delta}}\\
&+\big\|\mathcal{D}_1\big(\mathcal{D}\big(u^M,u^M,u^M,u^N-u^M\big),u^N,u^N,u^N\big)\big\|_{X_T^{\frac12+\delta,\frac12+\delta}}. \qquad (4.17)
\end{aligned}
$$

Each of the terms in (4.17) will be bounded in a similar way; we bound the first term explicitly. Applying (3.7) with $\delta_0=0$, $u_5=u^N-u^M$, $\Lambda_5=\mathbb{P}_N-\mathbb{P}_M$ and $u_j=u^N$, $\Lambda_j=\mathbb{P}_N$ for $j=2,3,4,6,7,8$, we find

$$
\begin{aligned}
&T^{\delta-}\big\|\mathcal{D}_1\big(\mathcal{D}\big(u^N-u^M,u^N,u^N,u^N\big),u^N,u^N,u^N\big)\big\|_{X_T^{\frac12+\delta,\frac12+\delta}}\\
&\lesssim T^{\kappa}\big(M^{-\beta}+\|u^N-u^M\|_{X_T^{\frac12-\delta,\frac12-\delta}}+\|\mathcal{D}^N-\mathcal{D}^M\|_{X_T^{\frac12+\delta,\frac12-\delta}}\big)\cdot\big(1+\|u^N\|_{X_T^{\frac12-\delta,\frac12-\delta}}+\|\mathcal{D}^N\|_{X_T^{\frac12+\delta,\frac12-\delta}}\big)^6.
\end{aligned}
$$

Using (4.17) this leads to the bound

$$
\begin{aligned}
&T^{\delta-}\big\|\mathcal{D}_1\big(u^N-u^M,u^N,u^N,u^N\big)\big\|_{X_T^{\frac12+\delta,\frac12+\delta}}\\
&\lesssim T^{\kappa}\big(M^{-\beta}+\|u^N-u^M\|_{X_T^{\frac12-\delta,\frac12-\delta}}+\|\mathcal{D}^N-\mathcal{D}^M\|_{X_T^{\frac12+\delta,\frac12-\delta}}\big)\\
&\quad\cdot\big(1+\|u^N\|_{X_T^{\frac12-\delta,\frac12-\delta}}+\|u^M\|_{X_T^{\frac12-\delta,\frac12-\delta}}+\|\mathcal{D}^N\|_{X_T^{\frac12+\delta,\frac12-\delta}}+\|\mathcal{D}^M\|_{X_T^{\frac12+\delta,\frac12-\delta}}\big)^6.
\end{aligned}
$$

With similar arguments, using the inequalities of Proposition 3.2, we find

$$
\begin{aligned}
&T^{\delta-}\big\|\mathcal{D}_k\big(u^N-u^M,u^N,u^N,u^N\big)\big\|_{X_T^{\frac12+\delta,\frac12+\delta}}\\
&\lesssim T^{\kappa}\big(M^{-\beta}+\|u^N-u^M\|_{X_T^{\frac12-\delta,\frac12-\delta}}+\|\mathcal{D}^N-\mathcal{D}^M\|_{X_T^{\frac12+\delta,\frac12-\delta}}\big)\\
&\quad\cdot\big(1+\|u^N\|_{X_T^{\frac12-\delta,\frac12-\delta}}+\|u^M\|_{X_T^{\frac12-\delta,\frac12-\delta}}+\|\mathcal{D}^N\|_{X_T^{\frac12+\delta,\frac12-\delta}}+\|\mathcal{D}^M\|_{X_T^{\frac12+\delta,\frac12-\delta}}\big)^6, \qquad (4.18)
\end{aligned}
$$

for all $k=1,2,3,4$. Combining (4.14) and (4.18), preceded by Lemma 2.4, we have

$$
\begin{aligned}
&\big\|\mathcal{D}\big(u^N-u^M,u^N,u^N,u^N\big)\big\|_{X_T^{\frac12+\delta,\frac12-\delta}}\\
&\lesssim T^{\kappa}\big(M^{-\beta}+\|u^N-u^M\|_{X_T^{\frac12-\delta,\frac12-\delta}}+\|\mathcal{D}^N-\mathcal{D}^M\|_{X_T^{\frac12+\delta,\frac12-\delta}}\big)\\
&\quad\cdot Z_0\big(\|u^N\|_{X_T^{\frac12-\delta,\frac12-\delta}},\|u^M\|_{X_T^{\frac12-\delta,\frac12-\delta}},\|\mathcal{D}^N\|_{X_T^{\frac12+\delta,\frac12-\delta}},\|\mathcal{D}^M\|_{X_T^{\frac12+\delta,\frac12-\delta}}\big), \qquad (4.19)
\end{aligned}
$$

where $Z_0(x,y,z,w)$ is a polynomial of degree 6 with positive coefficients. Each of the terms in (4.13) can be bounded with similar arguments. This leads to

$$
\begin{aligned}
\|\mathcal{D}^N-\mathcal{D}^M\|_{X_T^{\frac12+\delta,\frac12-\delta}}&\le C_1T^{\kappa}\big(M^{-\beta}+\|u^N-u^M\|_{X_T^{\frac12-\delta,\frac12-\delta}}+\|\mathcal{D}^N-\mathcal{D}^M\|_{X_T^{\frac12+\delta,\frac12-\delta}}\big)\\
&\quad\cdot Z_1\big(\|u^N\|_{X_T^{\frac12-\delta,\frac12-\delta}},\|u^M\|_{X_T^{\frac12-\delta,\frac12-\delta}},\|\mathcal{D}^N\|_{X_T^{\frac12+\delta,\frac12-\delta}},\|\mathcal{D}^M\|_{X_T^{\frac12+\delta,\frac12-\delta}}\big), \qquad (4.20)
\end{aligned}
$$

where $Z_1(x,y,z,w)$ is a polynomial of degree 6 with positive coefficients. If we choose $T>0$ sufficiently small such that

$$
C_1T^{\kappa}Z_1(R,R,\tilde R,\tilde R)\le\frac14, \qquad (4.21)
$$

we find from (4.20), (4.4) (4.5) and (4.21),

$$
\|\mathcal{D}^N-\mathcal{D}^M\|_{X_T^{\frac12+\delta,\frac12-\delta}}\le\frac12M^{-\beta}+\frac12\|u^N-u^M\|_{X_T^{\frac12-\delta,\frac12-\delta}}. \qquad (4.22)
$$

Then combining (4.12) and (4.22), we have

$$\|u^N - u^M\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} \le \big(2\tilde{C}T^{\delta-\frac{\beta}{2}}+1\big)M^{-\beta} \lesssim M^{-\beta},$$

by taking $T<1$. With (4.22) this gives

$$\|\mathcal{D}^N - \mathcal{D}^M\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \lesssim M^{-\beta},$$

and we conclude that (4.2) and (4.3) hold for $\omega \in \Omega_T$.

By (4.2) and (4.3), u^N and $\mathcal{D}^N$ converge in $X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}$ and $X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}$, respectively, for $\omega \in \Omega_T$. It remains to be shown that, for $\omega \in \Omega_T$,

(i) The convergent u of u^N is indeed a solution to (1.6) with initial data $u_{0,\omega}$.
(ii) $u - S(t)u_{0,\omega} \in C([0,T]; H^{\frac{1}{2}+\delta}(\mathbb{T}))$.
(iii) u is unique in $\{S(t)u_{0,\omega} + \{\|\cdot\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \le \tilde{R}\}\}$.
(iv) u depends continuously on the initial data, in the sense that the solution map $\Phi : \{u_{0,\omega} + \{\|\cdot\|_{H^{\frac{1}{2}+\delta}} \le K\}\} \to \{S(t)u_{0,\omega} + \{\|\cdot\|_{C([0,T];H^{\frac{1}{2}+\delta})} \le \tilde{K}\}\}$ is Lipschitz.
(v) The solution u is well-approximated by the solution of (1.8). More precisely,

$$\|u - \Phi^N(t)u_{0,\omega}\|_{C([0,T];H^{\frac{1}{2}+\delta_1})} \lesssim N^{-\beta}.$$

To establish (i), we need to prove that $u = \lim_{N\to\infty} u^N$ satisfies

$$u = S(t)u_{0,\omega} + \mathcal{D}(u), \tag{4.23}$$

where equality holds in the sense of distributions. Clearly, for $\omega \in \Omega_T$, we have $S(t)u^N_{0,\omega} \to S(t)u_{0,\omega}$ in $X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}$. Using a telescoping sum and the deterministic estimate (3.8), we have

$$\mathcal{D}_0\big(u^N\big) \to \mathcal{D}_0(u) \quad \text{in } X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}. \tag{4.24}$$

Also, by modifying the technique used to prove (4.3) (and invoking both (4.2) and (4.3)), we conclude that $\mathcal{D}_k(u^N)$ is Cauchy in $X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}$, for each $k=-1,1,2,3,4$. That is,

$$\mathcal{D}_k\big(u^N\big) \to v_k \quad \text{in } X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta} \tag{4.25}$$

for some $v_k \in X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}$, $k=-1,1,2,3,4$. We can therefore express

$$u = S(t)u_{0,\omega} + \mathcal{D}_0(u) + v_{-1} + v_1 + \cdots + v_4. \tag{4.26}$$

It remains to be shown that $v_k = \mathcal{D}_k(u)$ for each $k=-1,1,2,3,4$.

First observe that by (4.2), (4.24) and (4.25), given the uniform bounds of (4.4) and (4.5), we have

$$\|u\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} = \lim_{N\to\infty}\|u^N\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} \le R, \tag{4.27}$$

$$\Bigg\|\mathcal{D}_0(u) + \sum_{\substack{k=-1\\k\neq 0}}^{4} v_k\Bigg\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} = \lim_{N\to\infty}\|\mathcal{D}^N\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \le \tilde{R}. \tag{4.28}$$

Then to show that $v_{-1} = \mathcal{D}_{-1}(u)$, we apply (3.6) with $\delta_0 = 0$, $u_1 = u - u^N$, $\Lambda_1 = \mathrm{Id} - \mathbb{P}_N$ and $u_j = u$, $\Lambda_j = \mathrm{Id}$, for $j = 2,3,4$, we find (by substituting (4.1) and (4.26)), for $\omega \in \Omega_T$, we have

$$
\begin{aligned}
&\big\|\mathcal{D}_{-1}\big(u-u^N,u,u,u\big)\big\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}} \\
&\lesssim T^{-\beta}\Bigg(N^{-\beta}+\big\|u-u^N\big\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}+\Bigg\|\mathcal{D}_0(u)+\sum_{\substack{k=-1\\k\neq 0}}^{4} v_k-\mathcal{D}\big(u^N\big)\Bigg\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\Bigg) \\
&\quad\cdot\Bigg(1+\|u\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}+\Bigg\|\mathcal{D}_0(u)+\sum_{\substack{k=-1\\k\neq 0}}^{4} v_k\Bigg\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\Bigg)^3 \\
&\lesssim T^{-\beta}\Bigg(N^{-\beta}+\big\|u-u^N\big\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}+\sum_{\substack{k=-1\\k\neq 0}}^{4}\big\|v_k-\mathcal{D}_k\big(u^N\big)\big\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \\
&\quad+\big\|\mathcal{D}_0(u)-\mathcal{D}_0\big(u^N\big)\big\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\Bigg)(1+R+\tilde{R})^3 \\
&\to 0,\quad \text{as } N\to\infty.
\end{aligned}
$$

Note that we have used (4.27) and (4.28) in the second last line, and the convergence to zero follows from (4.24) and (4.25). Using a telescoping sum, we can apply similar arguments to conclude that $v_{-1}=\lim_{N\to\infty}\mathcal{D}_{-1}(u^N)=\mathcal{D}_{-1}(u)$.

It remains to show that $v_k=\mathcal{D}_k(u)$, for each $k=1,2,3,4$. The justification is similar for each such k; we focus on $k=1$. We demonstrate this equivalence using a weaker norm. In particular, it is clear from (3.11) that $\mathcal{D}_1(u^N)$ converges to $\mathcal{D}_1(u)$ in $X_T^{\frac{1}{2}-4\delta,\frac{1}{2}-\delta}$, and therefore $v_1=\mathcal{D}_1(u)$. In fact, by (4.25), it follows that $\mathcal{D}_1(u^N)$ converges to $\mathcal{D}_1(u)$ in $X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}$. The same type of argument applies for $k=2,3,4$, and (4.23) follows. We conclude that u is indeed a (mild) solution to (1.6) with data $u_{0,\omega}$ for $t\in[0,T]$. The discussion of point (i) is complete.

To address point (ii), we remark that by (4.25) and (2.13), since each $v_k=\mathcal{D}_k(u)$, if $\omega\in\Omega_T$, then $\mathcal{D}_k(u)\in C([0,T];H^{\frac{1}{2}+\delta}(\mathbb{T}))$ for all $k\in\{-1,1,2,3,4\}$. For $k=0$, we have by (2.13), followed by a telescoping sum and application of (3.9), that

$$
\begin{aligned}
\big\|\mathcal{D}_0(u)-\mathcal{D}_0\big(u^N\big)\big\|_{C([0,T];H^{\frac{1}{2}+\delta}(\mathbb{T}))}&\lesssim\big\|\mathcal{D}_0(u)-\mathcal{D}_0\big(u^N\big)\big\|_{Y_T^{\frac{1}{2}+\delta,0}} \\
&\lesssim T^{\theta}\big\|u-u^N\big\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}R^3.
\end{aligned}
$$

With (4.2) we conclude that $\mathcal{D}_0(u)\in C([0,T];H^{\frac{1}{2}+\delta}(\mathbb{T}))$. Therefore, if $\omega\in\Omega_T$, we have

$$
u-S(t)u_{0,\omega}=\mathcal{D}(u)\in C\big([0,T];H^{\frac{1}{2}+\delta}(\mathbb{T})\big).
$$

Turning to point (iii) (uniqueness), we establish that, for $\omega\in\Omega_T$, the solution u to (1.6) with data $u_{0,\omega}$ (obtained as the limit of u^N given by (4.1)) is unique in $\{S(t)u_{0,\omega}+\{\|\cdot\|_{\frac{1}{2}+\delta,\frac{1}{2}-\delta,T}\leq R\}\}$. Suppose $\tilde{u}$ is another solution to (1.6) with data $u_{0,\omega}$ that belongs to this function space. Following the methods used above, if $\omega\in\Omega_T$, then we have

$$
\begin{aligned}
\big\|\mathcal{D}(\tilde{u})-\mathcal{D}(u)\big\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}&\leq T^{\kappa}\big(\|\tilde{u}-u\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}+\big\|\mathcal{D}(\tilde{u})-\mathcal{D}(u)\big\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\big) \\
&\quad\cdot Z_2\big(\|\tilde{u}\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}},\|u\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}},\big\|\mathcal{D}(\tilde{u})\big\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}},\big\|\mathcal{D}(u)\big\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\big), \qquad (4.29)
\end{aligned}
$$

where $Z_2(x,y,z,w)$ is a polynomial of degree 6 with positive coefficients. With the definition of Ω_T, we have

$$
\begin{aligned}
\big\|\mathcal{D}(\tilde{u})-\mathcal{D}(u)\big\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}&\leq T^{\kappa}\big\|\mathcal{D}(\tilde{u})-\mathcal{D}(u)\big\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\cdot Z_2\big(CT^{\kappa}+R,CT^{\kappa}+R,R,R\big) \\
&\leq\frac{1}{2}\big\|\mathcal{D}(\tilde{u})-\mathcal{D}(u)\big\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}, \qquad (4.30)
\end{aligned}
$$

for $T>0$ sufficiently small. We conclude that $\mathcal{D}(\tilde{u})=\mathcal{D}(u)$ in $X_T^{\frac12+\delta,\frac12-\delta}$, and thus $u=\tilde{u}$ in $X_T^{\frac12-\delta,\frac12-\delta}$, for $\omega\in\Omega_T$. The proof of uniqueness is complete.

Next we discuss point (iv). We will show that, for $\omega\in\Omega_T$, the solution map $\Phi:\{u_{0,\omega}+\{\|\cdot\|_{H^{\frac12+\delta}}\le R\}\}\to\{S(t)u_{0,\omega}+\{\|\cdot\|_{C([0,T];H^{\frac12+\delta})}\le\tilde{R}\}\}$ for (1.6) is well-defined and Lipschitz. That is, given v_0 such that $\|u_{0,\omega}-v_0\|_{H^{\frac12+\delta}}\le R$, we will demonstrate that:

(a) The solution to (1.6) with data v_0 exists, is unique in the sense described above, and satisfies

$$\|v\|_{X_T^{\frac12-\delta,\frac12-\delta}}\le R,\qquad \big\|\mathcal{D}(v)\big\|_{X_T^{\frac12+\delta,\frac12-\delta}}\le\tilde{R}.$$

(b) The map Φ is Lipschitz.

To establish point (a), for $N>0$ we let $v_0^N:=\mathbb{P}_N v_0$. By Theorem 2 in [12] the solution v^N to (1.6) with data v_0^N exists for all $t\in\mathbb{R}$. We will show that, if $\omega\in\Omega_T$, then for all $N>M>0$,

$$\big\|v^N\big\|_{X_T^{\frac12-\delta,\frac12-\delta}}\le R, \tag{4.31}$$

$$\big\|\mathcal{D}\big(v^N\big)\big\|_{X_T^{\frac12+\delta,\frac12-\delta}}\le\tilde{R}, \tag{4.32}$$

and

$$\big\|v^N-v^M\big\|_{X_T^{\frac12-\delta,\frac12-\delta}},\ \big\|\mathcal{D}\big(v^N\big)-\mathcal{D}\big(v^M\big)\big\|_{X_T^{\frac12+\delta,\frac12-\delta}}\to0,\quad\text{as } M\to\infty. \tag{4.33}$$

The existence of a convergent $v\in X_T^{\frac12-\delta,\frac12-\delta}$ of the v^N will follow from (4.31)–(4.33). Afterwards, the justification of points (i)–(iii) for the convergent v can follow the discussion above (for u with data $u_{0,\omega}$) very closely, and we omit details. We proceed to justify (4.31)–(4.33).

The solution v^N to (1.6) with data v_0^N satisfies

$$\begin{aligned} v^N&=S(t)v_0^N+\mathcal{D}\big(v^N\big)\\ &=S(t)u_{0,\omega}^N+S(t)\big(v_0^N-u_{0,\omega}^N\big)+\mathcal{D}\big(v^N\big), \end{aligned} \tag{4.34}$$

and

$$v^N-v^M=S(t)(\mathbb{P}_N-\mathbb{P}_M)u_{0,\omega}+S(t)(\mathbb{P}_N-\mathbb{P}_M)(v_0-u_{0,\omega})+\mathcal{D}\big(v^N\big)-\mathcal{D}\big(v^M\big). \tag{4.35}$$

Using Lemma 2.2, the new contributions to (4.34) and (4.35) (i.e. contributions which were absent in the analysis of the sequence u^N above) satisfy, for any $b\in\mathbb{R}$,

$$\big\|S(t)\big(v_0^N-u_{0,\omega}^N\big)\big\|_{X_T^{\frac12+\delta,b}}\lesssim\|v_0-u_{0,\omega}\|_{H^{\frac12+\delta}(\mathbb{T})}, \tag{4.36}$$

and

$$\big\|S(t)(\mathbb{P}_N-\mathbb{P}_M)(v_0-u_{0,\omega})\big\|_{X_T^{\frac12+\delta,b}}\lesssim\big\|(\mathrm{Id}-\mathbb{P}_M)(v_0-u_{0,\omega})\big\|_{H^{\frac12+\delta}(\mathbb{T})}\le C_M\to0, \tag{4.37}$$

as $M\to\infty$. The point is that, thanks to (4.36) and (4.37), we can estimate the contributions from $(v_0-u_{0,\omega})$ in (4.34)–(4.35) using the $X_T^{\frac12+\delta,\frac12+\delta}$ norm (with spatial and temporal regularity $s=b=\frac12+\delta$). In the proof of inequalities (4.31)–(4.33), this has two benefits. Firstly, we can estimate contributions from these terms as we did the nonlinear terms (with spatial regularity $s=\frac12+\delta$) in our estimates above. Secondly, when we consider a second iteration of (4.34) in some regions of frequency space (as in line (4.15) above for u^N), we cannot expand the contributions from $(v_0-u_{0,\omega})$ in (4.34) and (4.35) into septilinear expressions (as we can for $\mathcal{D}(v^N)$); instead these contributions can be bounded using (4.36) and (4.37) with temporal regularity $b=\frac12+\delta$, by applying the deterministic estimates (3.10).

Using this approach, we can establish

$$\|v^N\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} \le C_0 T^{\delta-}\|u_{0,\omega}^N\|_{H^{\frac{1}{2}-\delta}} + \tilde{C}T^{\delta-}\|v_0^N - u_{0,\omega}^N\|_{H^{\frac{1}{2}+\delta}} + \|\mathcal{D}(v^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}},$$

and if $\omega \in \Omega_T$, then

$$\|\mathcal{D}(v^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \le C_1 T^\kappa Z_2\big(T^{\delta-}\|u_{0,\omega}^N\|_{H^{\frac{1}{2}-\delta}}, \|v_0^N - u_{0,\omega}^N\|_{H^{\frac{1}{2}+\delta}}, \|v^N\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}, \|\mathcal{D}(v^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\big), \tag{4.38}$$

where $Z_2(x,y,z,w)$ is a polynomial of degree 7 with positive coefficients. Under the assumption $\|v_0^N - u_{0,\omega}^N\|_{H^{\frac{1}{2}+\delta}} \le R$, we can repeat the analysis done for u^N, and (4.31)–(4.32) follows for $T > 0$ sufficiently small. To prove (4.33), we proceed as above, using (4.34)–(4.37), Proposition 3.2 and Proposition 3.3 to establish, if $\omega \in \Omega_T$, then for all $N > M > 0$ we have

$$\begin{aligned}
&\|\mathcal{D}(v^N) - \mathcal{D}(v^M)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \\
&\le C_1 T^\kappa \big(M^{-\beta} + \|S(t)(\mathbb{P}^N - \mathbb{P}^M)(v_0 - u_{0,\omega})\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}} \\
&\quad + \|v^N - v^M\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + \|\mathcal{D}(v^N) - \mathcal{D}(v^M)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\big) \\
&\quad \cdot Z_3\big(T^{\delta-}\|u_{0,\omega}^N\|_{H^{\frac{1}{2}-\delta}}, T^{\delta-}\|u_{0,\omega}^M\|_{H^{\frac{1}{2}-\delta}}, \|v_0^N - u_{0,\omega}^N\|_{H^{\frac{1}{2}+\delta}}, \|v_0^M - u_{0,\omega}^M\|_{H^{\frac{1}{2}+\delta}}, \\
&\qquad \|v^N\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}, \|v^M\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}, \|\mathcal{D}(v^N)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}, \|\mathcal{D}(v^M)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\big),
\end{aligned}$$

where $Z_3(s,t,u,v,w,x,y,z)$ is a polynomial of degree 6 with positive coefficients. Then using (4.31), (4.32) and the definition of Ω_T, we find

$$\|\mathcal{D}(v^N) - \mathcal{D}(v^M)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \le C_1 T^\kappa \big(M^{-\beta} + C_M + \|\mathcal{D}(v^N) - \mathcal{D}(v^M)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\big) \cdot Z_3\big(CT^\kappa, CT^\kappa, R, R, R, R, \tilde{R}, \tilde{R}\big).$$

By taking $T > 0$ sufficiently small (followed by rearrangement of the last inequality), and using $C_M \to 0$ as $M \to \infty$, we conclude that (4.33) holds true for $\omega \in \Omega_T$. This completes the justification of point (a): for $\omega \in \Omega_T$, the local solution v to (1.6) with data $v_0 \in \{u_{0,\omega} + \{\|\cdot\|_{H^{\frac{1}{2}+\delta}} \le R\}\}$ exists and it is continuous and unique (in the sense described above).

We proceed to establish point (b). That is, we show that the solution map

$$\Phi : \big\{u_{0,\omega} + \big\{\|\cdot\|_{H^{\frac{1}{2}+\delta}} \le R\big\}\big\} \to \big\{S(t)u_{0,\omega} + \big\{\|\cdot\|_{C([0,T];H^{\frac{1}{2}+\delta})} \le \tilde{R}\big\}\big\}$$

for (1.6) is Lipschitz. Suppose $u_0, v_0 \in \{u_{0,\omega} + \{\|\cdot\|_{H^{\frac{1}{2}+\delta}} \le R\}\}$ and let $u, v \in \{S(t)u_{0,\omega} + \{\|\cdot\|_{C([0,T];H^{\frac{1}{2}+\delta})} \le \tilde{R}\}\}$ denote the corresponding solutions to (1.6). As above, we have

$$\begin{aligned}
\|u - v\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} &\le \|S(t)(u_0 - v_0)\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + \|\mathcal{D}(u) - \mathcal{D}(v)\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} \\
&\lesssim T^{\delta-}\|u_0 - v_0\|_{H^{\frac{1}{2}+\delta}} + \|\mathcal{D}(u) - \mathcal{D}(v)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}},
\end{aligned}$$

and if $\omega \in \Omega_T$, then

$$\begin{aligned}
&\|\mathcal{D}(u) - \mathcal{D}(v)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \\
&\le C_1 T^\kappa \big(\|u_0 - v_0\|_{H^{\frac{1}{2}+\delta}} + \|u - v\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + \|\mathcal{D}(u) - \mathcal{D}(v)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\big) \\
&\quad \cdot Z_4\big(\|u_0 - v_0\|_{H^{\frac{1}{2}+\delta}}, \|u\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}, \|v\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}, \|\mathcal{D}(u)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}, \|\mathcal{D}(v)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\big),
\end{aligned}$$

where $Z_4(v, w, x, y, z)$ is a polynomial of degree 6 with positive coefficients. Repeating previous arguments, we conclude that, if $T > 0$ is sufficiently small, then for $\omega \in \Omega_T$, we have

$$\|u - v\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} \lesssim T^\kappa \|u_0 - v_0\|_{H^{\frac{1}{2}+\delta}},$$
$$\|\mathcal{D}(u) - \mathcal{D}(v)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \lesssim T^\kappa \|u_0 - v_0\|_{H^{\frac{1}{2}+\delta}}. \tag{4.39}$$

Then we find

$$\begin{aligned}\|u - v\|_{C([0,T];H^{\frac{1}{2}+\delta})} &\le \|S(t)(u_0 - v_0)\|_{C([0,T];H^{\frac{1}{2}+\delta})} + \|\mathcal{D}(u) - \mathcal{D}(v)\|_{C([0,T];H^{\frac{1}{2}+\delta})} \\ &= \|u_0 - v_0\|_{H^{\frac{1}{2}+\delta}} + \|\mathcal{D}(u) - \mathcal{D}(v)\|_{C([0,T];H^{\frac{1}{2}+\delta})},\end{aligned} \tag{4.40}$$

and using (2.13), Proposition 3.2, Proposition 3.3 and (4.40),

$$\begin{aligned}&\|\mathcal{D}(u) - \mathcal{D}(v)\|_{C([0,T];H^{\frac{1}{2}+\delta})} \\ &\quad\lesssim \|\mathcal{D}(u) - \mathcal{D}(v)\|_{Y_T^{\frac{1}{2}+\delta,0}} \\ &\quad\le \sum_{j=-1}^{4} \|\mathcal{D}_j(u) - \mathcal{D}_j(v)\|_{Y_T^{\frac{1}{2}+\delta,0}} \\ &\quad\le \|\mathcal{D}_0(u) - \mathcal{D}_0(v)\|_{Y_T^{\frac{1}{2}+\delta,0}} + \sum_{j=-1, j\neq 0}^{4} \|\mathcal{D}_j(u) - \mathcal{D}_j(v)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}} \\ &\quad\lesssim \Big(T^\kappa \|u_0 - v_0\|_{H^{\frac{1}{2}+\delta}} + T^{-\beta}\|u - v\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + T^{-\beta}\|\mathcal{D}(u) - \mathcal{D}(v)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\Big) \\ &\qquad\cdot Z_4\Big(\|u_0 - v_0\|_{H^{\frac{1}{2}+\delta}}, \|u\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}, \|v\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}, \|\mathcal{D}(u)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}, \|\mathcal{D}(v)\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\Big) \\ &\quad\lesssim T^{\frac{\kappa}{2}} \|u_0 - v_0\|_{H^{\frac{1}{2}+\delta}} Z_4(2R, R, R, \tilde{R}, \tilde{R}) \\ &\quad\lesssim \|u_0 - v_0\|_{H^{\frac{1}{2}+\delta}}.\end{aligned} \tag{4.41}$$

From (4.40) and (4.41) we conclude that the solution map Φ for (1.6) is Lipschitz. This completes the discussion of point (iv).

Lastly, we need to address point (v). To do this, we will compare solutions $\tilde{u}^N$ of the truncated system (1.7) to the local solution u of (1.6) constructed above. Let us be clear that we will use $\tilde{u}^N$ to denote the solution to the frequency truncated PDE (1.7), not to be confused with the solution u^N of (1.6) with frequency truncated data. Avoiding frequency truncation of the nonlinearity was useful above (to establish the existence of a convergent u), but it remains to justify (1.9). We will use the notation $\widetilde{\mathcal{D}}^N := \mathbb{P}_N \mathcal{D}(\tilde{u}^N, \dots, \tilde{u}^N)$ and $\mathcal{D} := \mathcal{D}(u, u, u, u)$.

Let us remark that the analysis applied to the sequence u^N above, for *fixed* $N > 0$, applies to the frequency truncated sequence $\tilde{u}^N$ as well (the $X^{s,b}$-norm "behaves nicely" with respect to frequency truncation, since it is defined in terms of Fourier coefficients, see also Remark 3.4). In particular, by proving (4.4)–(4.5) with $\tilde{u}^N$ (instead of u^N) imply that for $\omega \in \Omega_T$ we have

$$\|\tilde{u}^N\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} \le R,$$
$$\|\widetilde{\mathcal{D}}^N\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \le \tilde{R}, \tag{4.42}$$

for each $N > 0$. It should also be mentioned that the argument used above to justify point (iv) (Lipschitz continuity of the data-to-solution map Φ) applies to the finite-dimensional data-to-solution map Φ^N of (1.7) (for the same reasons).

Given $0 < \delta_1 < \delta$ as in the statement of Theorem 1.1, we impose the smallness condition $0 < \beta < \delta - \delta_1$ on the small constant β used above, and this has no effect on our prior analysis. We claim that the following estimates hold for $\omega \in \Omega_T$:

$$\|\tilde{u}^N - u\|_{X_T^{\frac{1}{2}-2\delta+\delta_1,\frac{1}{2}-\delta}} \lesssim N^{-\beta}, \tag{4.43}$$

$$\|\widetilde{\mathcal{D}}^N - \mathcal{D}\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}-\delta}} \lesssim N^{-\beta}. \tag{4.44}$$

We proceed to justify (4.43) and (4.44). In fact, we will establish, for each $M > N > 0$,

$$\|\tilde{u}^N - u^M\|_{X_T^{\frac{1}{2}-2\delta+\delta_1,\frac{1}{2}-\delta}} \lesssim N^{-\beta}, \tag{4.45}$$

$$\|\widetilde{\mathcal{D}}^N - \mathcal{D}^M\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}-\delta}} \lesssim N^{-\beta}. \tag{4.46}$$

Then (4.43)–(4.44) follows by taking the limit $M \to \infty$, using (4.2), (4.3) and (4.23). The point here is that, to justify (4.45)–(4.46), we can proceed with applications of Proposition 3.2 with u^M as input functions (instead of u) with corresponding $\Lambda_j = \mathbb{P}_M$, because then $M_j = M < \infty$ (as required to apply the estimates from Proposition 3.2 of the form (3.7)).

We proceed to justify (4.45)–(4.46). Using the integral formulations of (1.6) and (1.7), the condition $0 < \beta < \delta - \delta_1$, and the definition of Ω_T, we have

$$\begin{aligned}
&\|\tilde{u}^N - u^M\|_{X_T^{\frac{1}{2}-2\delta+\delta_1,\frac{1}{2}-\delta}} \\
&\quad \le \|S(t)(\mathbb{P}_M - \mathbb{P}_N)u_{0,\omega}\|_{X_T^{\frac{1}{2}-2\delta+\delta_1,\frac{1}{2}-\delta}} + \|\widetilde{\mathcal{D}}^N - \mathcal{D}^M\|_{X_T^{\frac{1}{2}-2\delta+\delta_1,\frac{1}{2}-\delta}} \\
&\quad \le N^{-(\delta-\delta_1)}\|S(t)(\mathbb{P}_M - \mathbb{P}_N)u_{0,\omega}\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + \|\widetilde{\mathcal{D}}^N - \mathcal{D}^M\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}-\delta}} \\
&\quad \le \frac{N^{-\beta}}{T^{\frac{\beta}{2}}} + \|\widetilde{\mathcal{D}}^N - \mathcal{D}^M\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}-\delta}}.
\end{aligned} \tag{4.47}$$

Then

$$\|\widetilde{\mathcal{D}}^N - \mathcal{D}^M\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}-\delta}} \le \|(\mathrm{Id} - \mathbb{P}_N)\mathcal{D}^M\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}-\delta}} + \|\mathbb{P}_N(\widetilde{\mathcal{D}}^N - \mathcal{D}^M)\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}-\delta}}.$$

We find

$$\|(\mathrm{Id} - \mathbb{P}_N)\mathcal{D}^M\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}-\delta}} \le N^{-(\delta-\delta_1)}\|\mathcal{D}^M\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \le N^{-\beta}\tilde{R}. \tag{4.48}$$

The estimate (4.48) illustrates how the slack in spatial derivatives due to the parameter $\delta_1 > 0$ can be used to establish the decay in N on the right-hand sides (4.45)–(4.46) (and thus (4.43)–(4.44)). Next, in order to estimate $\|\mathbb{P}_N(\widetilde{\mathcal{D}}^N - \mathcal{D}^M)\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}-\delta}}$, by using a telescoping series it suffices to consider expressions of the type $\|\mathbb{P}_N\mathcal{D}(\tilde{u}^N - u^M, \tilde{u}^N, \tilde{u}^N, \tilde{u}^N)\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}-\delta}}$. Furthermore, with the decomposition of frequency space from Section 3, it is enough to consider expressions of the type $\|\mathbb{P}_N\mathcal{D}_k(\tilde{u}^N - u^M, \tilde{u}^N, \tilde{u}^N, \tilde{u}^N)\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}-\delta}}$, for each $k \in \{-1, 0, \dots, 4\}$. By Proposition 3.2, for $\delta \gg \beta > 0$ sufficiently small, choosing $\delta_0 = \delta - \delta_1$, for any $0 < T \ll 1$, there exist $c, C > 0$ and a measurable set $\Omega_T^1 \subset \Omega$ satisfying $P((\Omega_T^1)^c) < e^{-\frac{c}{T^\beta}}$ such that if $\omega \in \Omega_T^1$ the estimates (3.6)–(3.7) (with $\delta_0 = \delta - \delta_1$) hold true. From here we redefine the set Ω_T above by intersecting with the set Ω_T^1.

For $k = -1$ and $k = 0$ we have by (3.6) and (3.8) (with $\delta_0 = \delta - \delta_1 > 0$),

$$\begin{aligned}
&\|\mathbb{P}_N\mathcal{D}_0(\tilde{u}^N - u^M, \tilde{u}^N, \tilde{u}^N, \tilde{u}^N)\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}-\delta}} + T^{\delta-}\|\mathbb{P}_N\mathcal{D}_{-1}(\tilde{u}^N - u^M, \tilde{u}^N, \tilde{u}^N, \tilde{u}^N)\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}+\delta}} \\
&\quad \lesssim T^\kappa\big(N^{-\beta} + \|\tilde{u}^N - u^M\|_{X_T^{\frac{1}{2}-2\delta+\delta_1,\frac{1}{2}-\delta}} + \|\widetilde{\mathcal{D}}^N - \mathcal{D}^M\|_{X_T^{\frac{1}{2}+\delta_1,\frac{1}{2}-\delta}}\big)(1 + R + \tilde{R})^3.
\end{aligned} \tag{4.49}$$

Then for $k = 1$ we find

$$\begin{aligned}&\big\|\mathbb{P}_N\mathcal{D}_1\big(\tilde u^N-u^M,\tilde u^N,\tilde u^N,\tilde u^N\big)\big\|_{X_T^{\frac12+\delta_1,\frac12-\delta}}\\&\quad\le\big\|\mathbb{P}_N\mathcal{D}_1\big((\mathrm{Id}-\mathbb{P}_N)u^M,\tilde u^N,\tilde u^N,\tilde u^N\big)\big\|_{X_T^{\frac12+\delta_1,\frac12-\delta}}+\big\|\mathbb{P}_N\mathcal{D}_1\big(\mathbb{P}_N\big(\tilde u^N-u^M\big),\tilde u^N,\tilde u^N,\tilde u^N\big)\big\|_{X_T^{\frac12+\delta_1,\frac12-\delta}},\end{aligned}$$

and

$$\begin{aligned}&\big\|\mathbb{P}_N\mathcal{D}_1\big((\mathrm{Id}-\mathbb{P}_N)u^M,\tilde u^N,\tilde u^N,\tilde u^N\big)\big\|_{X_T^{\frac12+\delta_1,\frac12-\delta}}\\&\quad\sim\big\|\mathbb{P}_N\mathcal{D}_1\big((\mathrm{Id}-\mathbb{P}_N)\mathcal{D}^M,\tilde u^N,\tilde u^N,\tilde u^N\big)\big\|_{X_T^{\frac12+\delta_1,\frac12-\delta}}\\&\quad\le\big\|\mathbb{P}_N\mathcal{D}_1\big(\mathcal{D}\big((\mathrm{Id}-\mathbb{P}_{N/4})u^M,u^M,u^M,u^M\big),\tilde u^N,\tilde u^N,\tilde u^N\big)\big\|_{X_T^{\frac12+\delta_1,\frac12-\delta}}\\&\qquad+\big\|\mathbb{P}_N\mathcal{D}_1\big(\mathcal{D}\big(u^M,(\mathrm{Id}-\mathbb{P}_{N/4})u^M,u^M,u^M\big),\tilde u^N,\tilde u^N,\tilde u^N\big)\big\|_{X_T^{\frac12+\delta_1,\frac12-\delta}}\\&\qquad+\big\|\mathbb{P}_N\mathcal{D}_1\big(\mathcal{D}\big(u^M,u^M,(\mathrm{Id}-\mathbb{P}_{N/4})u^M,u^M\big),\tilde u^N,\tilde u^N,\tilde u^N\big)\big\|_{X_T^{\frac12+\delta_1,\frac12-\delta}}\\&\qquad+\big\|\mathbb{P}_N\mathcal{D}_1\big(\mathcal{D}\big(u^M,u^M,u^M,(\mathrm{Id}-\mathbb{P}_{N/4})u^M\big),\tilde u^N,\tilde u^N,\tilde u^N\big)\big\|_{X_T^{\frac12+\delta_1,\frac12-\delta}}.\end{aligned}\tag{4.50}$$

Once again, each term on the right-hand side of (4.50) will be bounded in a similar way, and we proceed to bound the first term explicitly. Using (3.7) (with $\delta_0=\delta-\delta_1$), we find

$$\begin{aligned}&\big\|\mathbb{P}_N\mathcal{D}_1\big(\mathcal{D}\big((\mathrm{Id}-\mathbb{P}_{N/4})u^M,u^M,u^M,u^M\big),\tilde u^N,\tilde u^N,\tilde u^N\big)\big\|_{X_T^{\frac12+\delta_1,\frac12-\delta}}\\&\quad\lesssim T^\kappa\big(N^{-\beta}+\big\|(\mathrm{Id}-\mathbb{P}_{N/4})u^M\big\|_{X_T^{\frac12-2\delta+\delta_1,\frac12-\delta}}+\big\|(\mathrm{Id}-\mathbb{P}_{N/4})\mathcal{D}^M\big\|_{X_T^{\frac12+\delta_1,\frac12-\delta}}\big)\\&\qquad\cdot\big(1+\big\|u^M\big\|_{X_T^{\frac12-\delta,\frac12-\delta}}+\big\|\mathcal{D}^M\big\|_{X_T^{\frac12+\delta,\frac12-\delta}}\big)^3\big(1+\big\|\tilde u^N\big\|_{X_T^{\frac12-\delta,\frac12-\delta}}+\big\|\widetilde{\mathcal{D}}^N\big\|_{X_T^{\frac12+\delta,\frac12-\delta}}\big)^3\\&\quad\lesssim T^\kappa\big(N^{-\beta}+N^{-(\delta-\delta_1)}\big(\big\|u^M\big\|_{X_T^{\frac12-\delta,\frac12-\delta}}+\big\|\mathcal{D}^M\big\|_{X_T^{\frac12+\delta,\frac12-\delta}}\big)\big)(1+R+\tilde R)^6\\&\quad\lesssim T^\kappa N^{-\beta}(1+R+\tilde R)^7.\end{aligned}\tag{4.51}$$

Next we find

$$\big\|\mathbb{P}_N\mathcal{D}_1\big(\mathbb{P}_N\big(\tilde u^N-u^M\big),\tilde u^N,\tilde u^N,\tilde u^N\big)\big\|_{X_T^{\frac12+\delta_1,\frac12-\delta}}=\big\|\mathbb{P}_N\mathcal{D}_1\big(\mathbb{P}_N\big(\widetilde{\mathcal{D}}^N-\mathcal{D}^M\big),\tilde u^N,\tilde u^N,\tilde u^N\big)\big\|_{X_T^{\frac12+\delta_1,\frac12-\delta}},$$

and by using a telescoping series, in order to estimate this expression it suffices to consider terms of the type $\|\mathbb{P}_N\mathcal{D}_1(\mathbb{P}_N(\mathcal{D}(\tilde u^N-u^M,\tilde u^N,\tilde u^N,\tilde u^N)),\tilde u^N,\tilde u^N,\tilde u^N)\|_{X_T^{\frac12+\delta_1,\frac12-\delta}}$. By the modification of (3.7) described in Remark 3.4 (with $\delta_0=\delta-\delta_1$), we have

$$\begin{aligned}&\big\|\mathbb{P}_N\mathcal{D}_1\big(\mathbb{P}_N\mathcal{D}\big(\tilde u^N-u^M,\tilde u^N,\tilde u^N,\tilde u^N\big),\tilde u^N,\tilde u^N,\tilde u^N\big)\big\|_{X_T^{\frac12+\delta_1,\frac12-\delta}}\\&\quad\le T^\kappa\big(N^{-\beta}+\big\|\tilde u^N-u^M\big\|_{X_T^{\frac12-2\delta+\delta_1,\frac12-\delta}}+\big\|\widetilde{\mathcal{D}}^N-\mathcal{D}^M\big\|_{X_T^{\frac12+\delta_1,\frac12-\delta}}\big)(1+R+\tilde R)^6.\end{aligned}\tag{4.52}$$

By (4.47)–(4.52), with a continuity argument, as in the proof of (4.2)–(4.3) above, we arrive at (4.45)–(4.46).

Given (4.43)–(4.44), by following the approach taken in the discussion of point (ii) above (using an estimate of the type (4.48) to control high frequencies), we can establish that for $\omega\in\Omega_T$ we have

$$\big\|\mathcal{D}-\widetilde{\mathcal{D}}^N\big\|_{C([0,T];H^{\frac12+\delta_1})}\lesssim N^{-\beta}.$$

Notice that, by definition, we have $\Phi^N(t)u_{0,\omega}=\tilde u^N+S(t)(\mathrm{Id}-\mathbb{P}_N)u_{0,\omega}$, and therefore

$$\begin{aligned}\big\|u-\Phi^N(t)u_{0,\omega}\big\|_{C([0,T];H^{\frac12+\delta_1})}&=\big\|u-S(t)u_{0,\omega}-\big(\tilde u^N-S(t)\mathbb{P}_Nu_{0,\omega}\big)\big\|_{C([0,T];H^{\frac12+\delta_1})}\\&=\big\|\mathcal{D}-\widetilde{\mathcal{D}}^N\big\|_{C([0,T];H^{\frac12+\delta_1})}\lesssim N^{-\beta}.\end{aligned}\tag{4.53}$$

This completes the discussion of point (v), and the proof of Theorem 1.1 is complete. □

5. Global well-posedness and invariance of the Gibbs measure

In this section we will extend the local solutions produced by Theorem 1.1 to global solutions, and prove Theorem 1.2 (invariance of the Gibbs measure under the flow of (1.6)). This section is divided into parts: 5.1. Construction of the Gibbs measure, 5.2. Invariance of the truncated Gibbs measure under the flow of (1.8), 5.3. Global well-posedness (almost surely) for (1.6), and 5.4. Invariance of the Gibbs measure for (1.6).

5.1. Construction of the Gibbs measure

We begin by defining the Wiener measure in finite and infinite dimensions. The finite-dimensional Wiener measure ρ_N on $E_N = \text{span}\{\sin(nx), \cos(nx) : 1 \le n \le N\}$ is the push-forward of P under the map from $(\Omega, \mathcal{F}, P)$ to E_N (equipped with the Borel sigma algebra) given by $\omega \mapsto \mathbb{P}_N u_{0,\omega}$, where $u_{0,\omega}$ is defined in (1.5). Fix $\delta > 0$, the Wiener measure ρ[8] on $H^{\frac{1}{2}-\delta}(\mathbb{T})$ is the push-forward of P under the map from $(\Omega, \mathcal{F}, P)$ to $H^{\frac{1}{2}-\delta}(\mathbb{T})$ (equipped with the Borel sigma algebra) given by

$$\omega \longmapsto u_{0,\omega}. \tag{5.1}$$

Next we define the Gibbs measure μ for (1.1). We consider the truncated Gibbs measure μ_N on $H^{\frac{1}{2}-\delta}(\mathbb{T})$ defined as the push-forward under the map (5.1) of the weighted measure

$$e^{-\frac{1}{20}\int_{\mathbb{T}}(\mathbb{P}_N u_{0,\omega}(x))^5 dx}\chi_{\{\|\mathbb{P}_N u_{0,\omega}\|_2 \le B\}} dP(\omega). \tag{5.2}$$

We recall a crucial result from [27,3].

Proposition 5.1. *(See [27,3].) Let $B < \infty$, then for each $r \ge 1$, we have*

$$e^{-\frac{1}{20}\int_{\mathbb{T}}(u_{0,\omega}(x))^5 dx}\chi_{\{\|u_{0,\omega}\|_2 \le B\}} \in L^r(\Omega).$$

*In particular the **Gibbs measure** μ, defined as the push-forward under (5.1) of the weighted measure*

$$e^{-\frac{1}{20}\int_{\mathbb{T}}(u_{0,\omega}(x))^5 dx}\chi_{\{\|u_{0,\omega}\|_2 \le B\}} dP(\omega),$$

is absolutely continuous with respect to the Wiener measure ρ.

The proof of Proposition 5.1 first appeared in [27]. A different proof can be found in [3].

Remark 5.2. It is easily verified from the proof of Proposition 5.1 in [3] that we also have the following conclusion: there exists $0 < C < \infty$ such that for all $N > 0$,

$$\left\| e^{-\frac{1}{20}\int_{\mathbb{T}}(\mathbb{P}_N u_{0,\omega}(x))^5 dx}\chi_{\{\|\mathbb{P}_N u_{0,\omega}\|_2 \le B\}} \right\|_{L^r(\Omega)} \le C < \infty.$$

Having defined these measures, we establish a convergence property to be used in the proof of Theorem 1.2. The application of this proposition (and its proof) are inspired by similar arguments appearing in [10].

Proposition 5.3. *Set*

$$f(u) = e^{-\frac{1}{20}\int_{\mathbb{T}} u^5 dx}\chi_{\{\|u\|_2 \le B\}} \quad \text{and} \quad f_N(u) = e^{-\frac{1}{20}\int_{\mathbb{T}}(\mathbb{P}_N u(x))^5 dx}\chi_{\{\|\mathbb{P}_N(u)\|_2 \le B\}}.$$

Then

$$\lim_{N\to\infty} \int\limits_{H^{\frac{1}{2}-\delta}} \left| f_N(u) - f(u) \right| d\rho(u) = 0.$$

[8] We will state the results required for the proof of Theorem 1.2. For more details about the Wiener measure, and Gaussian measures on Banach spaces, see [26,50].

Proof. We first claim that $f_N(u) \to f(u)$ in measure with respect to ρ, and this follows from showing that $f_N(u) \to f(u)$ ρ-almost surely (by Egorov's Theorem). Clearly we have

$$\chi_{\{\|\mathbb{P}_N u\|_2 \le B\}} \longrightarrow \chi_{\{\|u\|_2 \le B\}}$$

ρ-almost surely. By continuity of the exponential function, we need only verify that $\mathbb{P}_N u \to u$ in $L^5(\mathbb{T})$, ρ-almost surely, and this follows easily from the Sobolev embedding $L^5(\mathbb{T}) \hookrightarrow H^{\frac{1}{2}-\delta}(\mathbb{T})$ (for $\delta > 0$ sufficiently small).

Next fix any $\varepsilon > 0$, and let $A_{N,\varepsilon} := \{u \in H^{\frac{1}{2}-\delta}(\mathbb{T}) : |f_N(u) - f(u)| \le \varepsilon\}$. We apply Cauchy–Schwarz followed by Proposition 5.1,

$$\begin{aligned}\int_{H^{\frac{1}{2}-\delta}(\mathbb{T})} |f_N(u) - f(u)| d\rho(u) &\le \Big(\int_{A_{N,\varepsilon}} + \int_{A^c_{N,\varepsilon}} \Big) |f_N(u) - f(u)| d\rho(u) \\ &\le \int_{A_{N,\varepsilon}} |f_N(u) - f(u)| d\rho(u) + \|f_N - f\|_{L^2(d\rho)} \big(\rho\big(A^c_{N,\varepsilon}\big)\big)^{\frac{1}{2}} \\ &\le \varepsilon + 2C\big(\rho\big(A^c_{N,\varepsilon}\big)\big)^{\frac{1}{2}}.\end{aligned}$$

Then since $f_N(u) \to f(u)$ in measure with respect to ρ, we have $\rho(A^c_{N,\varepsilon}) \to 0$ as $N \to \infty$, and the proof of Proposition 5.3 is complete. □

We have the following useful corollary of Proposition 5.3.

Corollary 5.4. *For any Borel set $A \subset H^{\frac{1}{2}-\delta}(\mathbb{T})$, we have*

$$\mu(A) = \lim_{N\to\infty} \mu_N(A). \tag{5.3}$$

5.2. Invariance of the finite-dimensional Gibbs measure

Consider the frequency truncated and gauge-transformed quartic gKdV (1.7). We can write (1.7) in coordinates as a system of N complex ODEs (for the Fourier coefficients) $c_n := \widehat{u^N}(n)$, $1 \le n \le N$. This system is locally well-posed by the Cauchy–Lipschitz Theorem, and it is easily verified that the L^2-norm of the solution u^N to (1.7) is preserved under the flow. This provides an a priori bound on the ℓ^∞_n-norm of the Fourier coefficients $\{c_n\}_{1\le n\le N}$, and it follows that the solution u^N to (1.7) is global-in-time.

Recall that $\Phi^N(t)$ is defined as the flow map of (1.8), and let $\widetilde{\Phi}^N(t)$ denote the flow-map of (1.7). Let $E_N^\perp$ denote the orthogonal complement of E_N in $H^{\frac{1}{2}-\delta}$. Then $\Phi^N(t) = (\widetilde{\Phi}^N(t)\mathbb{P}_N, S(t)(\mathrm{Id} - \mathbb{P}_N))$ is defined as the flow of (1.7) on E_N and the linear flow (i.e. the solution to linear KdV) on $E_N^\perp$. In this subsection we establish the following proposition.

Proposition 5.5. *For each $N > 0$, $t \in \mathbb{R}$, the map $\Phi^N(t)$ is measure preserving on $H^{\frac{1}{2}-\delta}(\mathbb{T})$ equipped with the Gibbs measure μ_N (as defined by (5.2)).*

We define the (truly) finite-dimensional Gibbs measure $\widetilde{\mu}_N$ on E_N by the density $d\widetilde{\mu}_N = f_N d\rho_N$. Notice that we can write $d\mu_N = d\widetilde{\mu}_N \times d\rho_N^\perp$, where $\rho_N^\perp$ denotes the Wiener measure on $E_N^\perp$.

Lemma 5.6. *For each $N > 0$, $t \in \mathbb{R}$, the map $\widetilde{\Phi}^N(t)$ is measure preserving on E_N equipped with the finite-dimensional Gibbs measure $\widetilde{\mu}_N$.*

Lemma 5.7. *For each $t \in \mathbb{R}$, the linear propagator $S(t) = e^{-it\partial_x^3}$ is measure preserving on $E_N^\perp$ equipped with the Wiener measure $\rho_N^\perp$.*

Then, since $d\mu_N = d\widetilde{\mu}_N \times d\rho_N^\perp$ and $\Phi^N(t) = (\widetilde{\Phi}^N(t)\mathbb{P}_N, S(t)(\mathrm{Id} - \mathbb{P}_N))$, Proposition 5.5 is immediate from Lemma 5.6 and Lemma 5.7. We proceed with the proofs of these lemmata.

Proof of Lemma 5.6. Suppose u^N solves (1.7). We can write (1.7) in coordinates as a system of N complex ODEs (for the Fourier coefficients) $c_n := \widehat{u^N}(n)$, $1 \le n \le N$ (recall that $c_{-n} = \overline{c_n}$). We verify the invariance of $\widetilde{\mu}_N$ in the coordinate space $\mathbb{C}^N$, and this extends to E_N by its definition. Straightforward computations verify that the Hamiltonian $H(u^N)$, the L^2-norm of u^N, and the Lebesgue measure $\prod_{1\le n\le N} da_n db_n$ on $\mathbb{C}^N$ are invariant under the flow of (1.7). See Section 3.5.2 of [40] for details. The proof of Lemma 5.6 is complete. □

Proof of Lemma 5.7. This proof follows the argument of [49] (see also [41,47]), and it is essentially a consequence of the invariance of complex Gaussians under rotation. For $M > N > 0$, we consider the measure ρ_N^M on

$$E_N^M := \mathrm{span}\{\cos(nx), \sin(nx) : N < n \le M\},$$

defined as the push-forward of P under the map $\omega \mapsto (\mathbb{P}_M - \mathbb{P}_N)u_{0,\omega}$, where $u_{0,\omega}$ is given by (1.5). It holds that, for any closed $F \subset E_N^\perp$,

$$\rho_N^\perp(F) \ge \limsup_{M\to\infty} \rho_N^M(F \cap E_N^M). \tag{5.4}$$

The inequality (5.4) follows from the complementary statement: for any open $U \subset E_N^\perp$,

$$\rho_N^\perp(U) \le \liminf_{M\to\infty} \rho_N^M(U \cap E_N^M). \tag{5.5}$$

To justify (5.5), recall that for any open $U \subset E_N^\perp$,

$$U \subset \liminf_{M\to\infty}\{u \in E_N^\perp : \mathbb{P}_M u \in U\} = \bigcup_{M=1}^{\infty} \bigcap_{M_1=M}^{\infty} \{u \in E_N^\perp : \mathbb{P}_M u \in U\},$$

so that $\chi_U \le \liminf_{M\to\infty} \chi_{\{u\in E_N^\perp : \mathbb{P}_M u\in U\}}$. Then by using the definition of ρ_N^M, and applying Fatou's lemma, we have

$$\liminf_{M\to\infty} \rho_N^M(U \cap E_N^M) = \liminf_{M\to\infty} \int_{E_N^\perp} \chi_{\{u\in E_N^\perp:\mathbb{P}_M u\in U\}} d\rho_N^\perp \tag{5.6}$$

$$\ge \int_{E_N^\perp} \liminf_{M\to\infty} \chi_{\{u\in E_N^\perp:\mathbb{P}_M u\in U\}} d\rho_N^\perp \tag{5.7}$$

$$\ge \int_{E_N^\perp} \chi_U \, d\rho_N^\perp = \rho_N^\perp(U). \tag{5.8}$$

Note that for each fixed $n \in \mathbb{Z}$, $t \in \mathbb{R}$, the KdV propagator $S(t) = e^{-it\partial_x^3}$ acts as a rotation on the 2-dimensional real vector space $V_n = \mathrm{span}\{\cos(nx), \sin(nx)\}$. Thus any centered Gaussian measure on V_n is invariant under $S(t)$, and it follows that $S(t)$ is measure preserving on E_N^M equipped with ρ_N^M, since ρ_N^M is defined as a product of centered Gaussian measures.

Let B_ε denote the ball of radius ε in $E_N^\perp$ equipped with the $H^{\frac{1}{2}-}(\mathbb{T})$ topology. Then, since $S(t)$ is a linear isometry on $H^{\frac{1}{2}-}(\mathbb{T})$ which leaves $E_N^\perp$ invariant, we have

$$S(t)\big((F + B_\varepsilon) \cap E_N^M\big) \subset \big(S(t)F + \overline{B_\epsilon}\big) \cap E_N^M. \tag{5.9}$$

Then using (5.4), (5.9) and the invariance of ρ_N^M under $S(t)$, we have

$$\begin{aligned}
\rho_N^\perp\big(S(t)F + \overline{B_\epsilon}\big) &\ge \limsup_{M\to\infty} \rho_N^M\big(\big(S(t)F + \overline{B_\epsilon}\big) \cap E_N^M\big) \\
&\ge \limsup_{M\to\infty} \rho_N^M\big(S(t)\big((F + B_\epsilon) \cap E_N^M\big)\big) \\
&= \limsup_{M\to\infty} \rho_N^M\big((F + B_\epsilon) \cap E_N^M\big).
\end{aligned} \tag{5.10}$$

Note that $F + B_\varepsilon$ is open in $E_N^\perp$, so that by (5.5) we have

$$\liminf_{M\to\infty} \rho_N^M\big((F + B_\epsilon) \cap E_N^M\big) \geq \rho_N^\perp(F + B_\epsilon) \geq \rho_N^\perp(F). \tag{5.11}$$

By combining (5.10) and (5.11) we have $\rho_N^\perp(S(t)F + \overline{B_\epsilon}) \geq \rho_N^\perp(F)$ for every closed $F \subset E_N^\perp$. By taking $\varepsilon \to 0$, it follows from the dominated convergence theorem that $\rho_N^\perp(S(t)F) \geq \rho_N^\perp(F)$, and by time reversibility of $S(t)$ we conclude that $\rho_N^\perp(S(t)F) = \rho_N^\perp(F)$ for every closed $F \subset E_N^\perp$. By standard approximation arguments this passes to measurable sets, and the proof is complete. □

5.3. Extending to global-in-time solutions

In this subsection we establish a proposition that uses the truncated Gibbs measure invariance (Proposition 5.5), and an approximation argument, to extend the local solutions of (1.6) (produced by Theorem 1.1) to global solutions. Recall that $\Phi^N(t)$ denotes the data-to-solution map for (1.8).

Proposition 5.8. *$\forall\, 0 < \varepsilon < 1$ and $T^* > 0, \exists\, \delta_1 > \delta_2 > \beta > 0$ sufficiently small and a measurable set $\Lambda_{\varepsilon,T^*} \subset H^{\frac{1}{2}-\delta_1}(\mathbb{T})$ such that $\mu(\Lambda^c_{\varepsilon,T^*}) < \varepsilon$ and $\forall\, u_0 \in \Lambda_{\varepsilon,T^*}$ there exists a (unique) solution $u \in S(t)u_0 + C([0,T^*]; H^{\frac{1}{2}+\delta_2}(\mathbb{T})) \subset C([0,T^*]; H^{\frac{1}{2}-\delta_1}(\mathbb{T}))$ to (1.6) with initial data u_0. Furthermore, for all $N \gg 0$, we have*

$$\big\|u - \Phi^N(t)u_0\big\|_{C([0,T^*]; H^{\frac{1}{2}+\delta_2}(\mathbb{T}))} \lesssim C(\varepsilon)N^{-\beta}. \tag{5.12}$$

Remark 5.9. Regarding the uniqueness of the solution in Proposition 5.8, recall that, for $\omega \in \Omega_T$, the local solution produced by Theorem 1.1 (with $\delta = \delta_1$ and $T = T_0$) is unique in a ball in $X^{\frac{1}{2}+\delta_1,\frac{1}{2}-\delta_1}_{[0,T_0]}$ centered at the randomized data $S(t)u_{0,\omega}$. For the solution produced by Proposition 5.8, this characterization is extended to $\frac{T^*}{T_0}$ intervals of size T_0 for some $T_0 > 0$ sufficiently small. That is, for each $j = 1, \ldots, \frac{T^*}{T_0}$, u is the unique solution to (1.6) for $t \in [jT_0, (j+1)T_0]$ (with data $u(jT_0)$) in a ball in $X^{\frac{1}{2}+\delta_1,\frac{1}{2}-\delta_1}_{[jT_0,(j+1)T_0]}$ centered at $S(t - jT_0)u(jT_0)$.

The proof of Proposition 5.8 (using Theorem 1.1, Proposition 5.3 and Proposition 5.5) follows the method of [4]. See also [8] and [10]. For the details consult Section 3.5.3 of [40].

5.4. Invariance of the Gibbs measure

In this subsection we prove Theorem 1.2.

Proof of Theorem 1.2. Given $n, j \in \mathbb{N}$, let $T_j = 2^j$ and $\varepsilon_{n,j} = \frac{1}{n2^j}$. Also let $\Lambda_{\varepsilon_{n,j},T_j}$ be the subset of $H^{\frac{1}{2}-\delta_1}(\mathbb{T})$ produced by Proposition 5.8 with $\varepsilon = \varepsilon_{n,j}$ and $T^* = T_j$. Define $\Sigma_n := \bigcap_{j=1}^\infty \Lambda_{\varepsilon_{n,j},T_j}$, so that $\mu(\Sigma_n^c) < \frac{1}{n}$. By taking $\Sigma := \bigcup_{n=1}^\infty \Sigma_n$, it follows that $\mu(\Sigma^c) = 0$. Moreover if $u_0 \in \Sigma$, we have $u_0 \in \bigcap_{j=1}^\infty \Lambda_{\varepsilon_{n,j},T_j}$ for some n, and (1.6) is globally well-posed with data u_0 by the conclusions of Proposition 5.8.

Next we prove that the Gibbs measure μ is invariant under the flow. For $u_0 \in \Sigma$, let $\Phi(t)$ denote the data-to-solution map of (1.6). One formulation of invariance is the following: for all $F \in L^1(H^{\frac{1}{2}-\delta_1}(\mathbb{T}), d\mu)$, we have

$$\int_\Sigma F\big(\Phi(t)(u)\big)d\mu(u) = \int_\Sigma F(u)d\mu(u) \tag{5.13}$$

for all $t \geq 0$. It suffices to establish (5.13) on a dense set in $L^1(H^{\frac{1}{2}-\delta_1}(\mathbb{T}), d\mu)$, in particular we choose $\mathcal{H} \subset L^1(H^{\frac{1}{2}-\delta_1}(\mathbb{T}), d\mu)$ given by the subset of bounded and continuous functions. Fix $F \in \mathcal{H}$, and $\kappa > 0$. Recall that $d\mu_N = f_N d\rho$ and $d\mu = f d\rho$, where f_N and f are defined in Proposition 5.3. Boundedness of F combined with Proposition 5.3 implies that for $N > 0$ sufficiently large we have

$$\left|\int_{\Sigma} F(u)d\mu(u) - \int_{\Sigma} F(u)d\mu_N(u)\right| + \left|\int_{\Sigma} F\big(\Phi(t)(u)\big)d\mu(u) - \int_{\Sigma} F\big(\Phi(t)(u)\big)d\mu_N(u)\right| < \frac{\kappa}{2}. \tag{5.14}$$

Let $n > 0$ be such that $\frac{1}{n} < \frac{\kappa}{32\|F\|_{L^\infty}}$. Then we have

$$\left|\int_{\Sigma\setminus\Sigma_n} F\big(\Phi(t)(u)\big)d\mu_N(u) - \int_{\Sigma\setminus\Sigma_n} F\big(\Phi^N(t)(u)\big)d\mu_N(u)\right| \leq 2\|F\|_{L^\infty}\mu_N(\Sigma\setminus\Sigma_n) \leq 2\|F\|_{L^\infty}\mu_N\big(\Sigma_n^c\big) < \frac{\kappa}{4}, \tag{5.15}$$

for $N > 0$ sufficiently large, where we have used Corollary 5.4 in the last line.

By continuity of F, there exists $\gamma > 0$ such that if $\|\Phi(t)u_0 - \Phi^N(t)u_0\|_{H^{\frac{1}{2}-\delta_1}(\mathbb{T})} < \gamma$, then $|F(\Phi(t)u_0) - F(\Phi^N(t)u_0)| < \frac{\kappa}{8\mu(H^{\frac{1}{2}-\delta_1}(\mathbb{T}))}$. For $u_0 \in \Sigma_n$, we fix $t \geq 0$ and from (5.12) it follows that for all $N > 0$,

$$\big\|\Phi(t)u_0 - \Phi^N(t)u_0\big\|_{H^{\frac{1}{2}+\delta_2}(\mathbb{T})} \leq C(n)N^{-\beta}.$$

Taking $N > 0$ sufficiently large, we have $\|\Phi(t)u_0 - \Phi^N(t)u_0\|_{H^{\frac{1}{2}-\delta_1}(\mathbb{T})} < \gamma$, and $|F(\Phi(t)u_0) - F(\Phi^N(t)u_0)| \leq \frac{\kappa}{8\mu(H^{\frac{1}{2}-\delta_1}(\mathbb{T}))}$ is satisfied. This gives

$$\left|\int_{\Sigma_n} F\big(\Phi(t)(u)\big)d\mu_N(u) - \int_{\Sigma_n} F\big(\Phi^N(t)(u)\big)d\mu_N(u)\right| < \frac{\kappa}{4}, \tag{5.16}$$

for $N > 0$ sufficiently large, where we have applied Corollary 5.4 once more.

We also have, by Proposition 5.5,

$$\int_{\Sigma} F\big(\Phi^N(t)(u)\big)d\mu_N(u) = \int_{\Sigma} F(u)d\mu_N(u). \tag{5.17}$$

By combining (5.14)–(5.17), we conclude that for $N > 0$ sufficiently large, we have

$$\left|\int_{\Sigma} F\big(\Phi(t)(u)\big)d\mu(u) - \int_{\Sigma} F(u)d\mu(u)\right| < \kappa.$$

Since κ was arbitrary, we conclude that $\int_\Sigma F(\Phi(t)(u))d\mu(u) = \int_\Sigma F(u)d\mu(u)$, and the Gibbs measure μ is invariant under the flow of (1.6).

We have now established global well-posedness of (1.6) on a set $\Sigma \subset H^{\frac{1}{2}-\delta_1}(\mathbb{T})$ of full μ-measure, and invariance of the Gibbs measure under the flow. This is easily extended to global well-posedness almost surely, with randomized initial data given by (1.5) (see [40]). The proof of Theorem 1.2 is complete. □

6. Proof of nonlinear estimates

In this section we prove the crucial nonlinear estimates (Proposition 3.2 and Proposition 3.3). In Subsections 6.1–6.3 we establish Proposition 3.2. In Subsection 6.4 we present the proof of Proposition 3.3.

6.1. Setup

Here we will prove Proposition 3.2 using two sets of estimates: quadrilinear estimates (see Proposition 6.1 below) and septilinear estimates (see Proposition 6.2). The proof of Proposition 6.1 can be found in Subsection 6.2, and the proof of Proposition 6.2 is in Subsection 6.3.

We begin by identifying the exact form of the expressions appearing in the estimates of Proposition 3.2. Following the reformulation (1.17) of (1.6), we consider the multilinear function

$$\begin{aligned}\mathcal{N}(u_1,u_2,u_3,u_4) :=\ & \mathbb{P}\big[(u_1)_x\mathbb{P}(u_2u_3u_4)\big] - \mathbb{P}(u_2)\fint_{\mathbb{T}}(u_1)_xu_3u_4 - \mathbb{P}(u_3)\fint_{\mathbb{T}}(u_1)_xu_2u_4\\ & - \mathbb{P}(u_4)\fint_{\mathbb{T}}(u_1)_xu_2u_3 - \mathbb{P}(u_3u_4)\fint_{\mathbb{T}}(u_1)_xu_2 - \mathbb{P}(u_2u_4)\fint_{\mathbb{T}}(u_1)_xu_3\\ & - \mathbb{P}(u_2u_4)\fint_{\mathbb{T}}(u_1)_xu_4. \end{aligned} \tag{6.1}$$

With this definition, the integral formulation of (1.6) with data $u_{0,\omega}$ is given by

$$u = S(t)u_{0,\omega} + \mathcal{D}(u), \tag{6.2}$$

where $\mathcal{D}(u)$ is defined in (3.2). For fixed $n \in \mathbb{Z}\setminus\{0\}$, $t \in \mathbb{R}$, consider the nth Fourier coefficient of $\mathcal{N}(u_1,u_2,u_3,u_4)(t)$ (we suppress the dependence on time)

$$\begin{aligned}&\big(\mathcal{N}(u_1,u_2,u_3,u_4)\big)^\wedge(n)\\ &\quad= \sum_{n=n_1+m_1}(in_1)\widehat{u_1}(n_1)\big(\mathbb{P}(u_2u_3u_4)\big)^\wedge(m_1) - \big(\mathbb{P}(u_2)\big)^\wedge(n)\fint_{\mathbb{T}}(u_1)_xu_3u_4\\ &\qquad - \big(\mathbb{P}(u_3)\big)^\wedge(n)\fint_{\mathbb{T}}(u_1)_xu_2u_4 - \big(\mathbb{P}(u_4)\big)^\wedge(n)\fint_{\mathbb{T}}(u_1)_xu_2u_3 - \big(\mathbb{P}(u_3u_4)\big)^\wedge(n)\fint_{\mathbb{T}}(u_1)_xu_2\\ &\qquad - \big(\mathbb{P}(u_2u_4)\big)^\wedge(n)\fint_{\mathbb{T}}(u_1)_xu_3 - \big(\mathbb{P}(u_2u_3)\big)^\wedge(n)\fint_{\mathbb{T}}(u_1)_xu_4\\ &\quad= \Bigg(\sum_{\substack{n=n_1+\cdots+n_4\\ n\neq 0}} - \sum_{k=1}^{4}\sum_{\substack{n=n_1+\cdots+n_4\\ 0\neq n=n_k}} - \sum_{k=2}^{4}\sum_{\substack{n=n_1+\cdots+n_4\\ n\neq 0, n_1=-n_k}}\Bigg) n_1\widehat{u_1}(n_1)\widehat{u_2}(n_2)\widehat{u_3}(n_3)\widehat{u_4}(n_4)\\ &\quad= \Bigg(\sum_{\zeta_1(n)} - \sum_{\zeta_2(n)}\Bigg) n_1\widehat{u_1}(n_1)\widehat{u_2}(n_2)\widehat{u_3}(n_3)\widehat{u_4}(n_4). \end{aligned} \tag{6.3}$$

To obtain (6.3) we have used the identity $\fint_{\mathbb{T}} w = \hat{w}(0)$, and have taken

$$\begin{aligned}\zeta_1(n) := \big\{&(n_1,n_2,n_3,n_4) \in \mathbb{Z}^4 : n = n_1+n_2+n_3+n_4,\ n \neq n_k \text{ for each } k \in \{1,2,3,4\},\\ &\text{and } n_1 \neq -n_j \text{ for each } j \in \{2,3,4\}\big\},\end{aligned}$$

and

$$\begin{aligned}\zeta_2(n) := \big\{&(n_1,n_2,n_3,n_4) \in \mathbb{Z}^4 : n = n_1+n_2+n_3+n_4, \text{with } n_k, n_j \in \{n,-n_1\},\\ &\text{for some } k \neq j,\ k,j \in \{1,2,3,4\},\ \text{where } n_k = n \text{ if } k=1 \text{ (and } n_j = n \text{ if } j=1)\big\}.\end{aligned}$$

We define $\zeta(n) := \zeta_1(n) \cup \zeta_2(n)$, and abuse notation by writing $\sum_{\zeta(n)} := \sum_{\zeta_1(n)} - \sum_{\zeta_2(n)}$. Reinserting the dependence on time, this gives

$$\big(\mathcal{N}(u_1,u_2,u_3,u_4)\big)^\wedge(n,t) = \sum_{\zeta(n)}(in_1)\widehat{u_1}(n_1,t)\widehat{u_2}(n_2,t)\widehat{u_3}(n_3,t)\widehat{u_4}(n_4,t). \tag{6.4}$$

Let us also recall the mean zero condition on each factor in our nonlinear analysis, as described in Remark 3.1 (prior to the statements of Propositions 3.2 and 3.3). This condition allows us to assume that:

$$\text{each } n_k \neq 0, \quad \text{for } k = 1,2,3,4. \tag{6.5}$$

To avoid cumbersome notation, we will not carry this restriction with us explicitly.

Proposition 3.2 will follow from standard linear estimates (Lemmas 2.1–2.4) and the probabilistic multilinear estimates given by the following propositions.

Proposition 6.1. *For $\delta > 0$ sufficiently small, and any $0 < T \ll 1$, there exist $\beta, C, c > 0$ and a measurable set $\Omega_T \subset \Omega$ satisfying $P(\Omega_T^c) < e^{-\frac{c}{T^\beta}}$ and the following conditions: if $\omega \in \Omega_T$, then for every quadruple of Fourier multipliers $\Lambda_1, \dots, \Lambda_4$ defined by*

$$\widehat{\Lambda_i f}(n) = \chi_{N_i \le |n| \le M_i} \hat{f}(n),$$

for some dyadic numbers $N_i < M_i \le \infty$, we have the following estimate:

$$\begin{aligned}&\left\|\mathcal{N}_{-1}(\chi_{[0,T]}u_1, \chi_{[0,T]}u_2, \chi_{[0,T]}u_3, \chi_{[0,T]}u_4)\right\|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}+\delta}}\\ &\quad\le CT^{-\beta}\prod_{j=1}^{4}\left(N_j^{-\beta} + \|u_j\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + \left\|u_j - S(t)\Lambda_j(u_{0,\omega})\right\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\right).\end{aligned} \tag{6.6}$$

Proposition 6.2. *For $\delta > 0$ sufficiently small, and any $0 < T \ll 1$, there exist $\beta, C, c > 0$ and a measurable set $\Omega_T \subset \Omega$ satisfying $P(\Omega_T^c) < e^{-\frac{c}{T^\beta}}$ and the following conditions: if $\omega \in \Omega_T$, then for every heptuple of Fourier multipliers $\Lambda_2, \dots, \Lambda_8$ defined by*

$$\widehat{\Lambda_i f}(n) = \chi_{N_i \le |n| \le M_i} \hat{f}(n),$$

for some dyadic numbers $N_i < M_i < \infty$, we have the following estimates:

$$\begin{aligned}&\left\|\mathcal{N}_1\left(\mathcal{D}(\chi_{[0,T]}u_5, \chi_{[0,T]}u_6, \chi_{[0,T]}u_7, \chi_{[0,T]}u_8), \chi_{[0,T]}u_2, \chi_{[0,T]}u_3, \chi_{[0,T]}u_4\right)\right\|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}+\delta}}\\ &\quad\le CT^{-\beta}\prod_{j=2}^{8}\left(N_j^{-\beta} + \|u_j\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + \left\|u_j - S(t)\Lambda_j(u_{0,\omega})\right\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\right).\end{aligned} \tag{6.7}$$

Analogous septilinear estimates also hold in the regions A_2, A_3 and A_4.

Remark 6.3. Notice that in Propositions 6.1 and 6.2 we have taken $\delta_0 = 0$ (compared with the estimates of Proposition 3.2). This is done to avoid cumbersome notation throughout this section. It is not hard to verify that, for the set Ω_T produced by these propositions, if $\omega \in \Omega_T$, then the inequalities (6.6) and (6.7) hold for any fixed $0 \le \delta_0 < \delta$ (where these estimates are modified to incorporate δ_0 as it appears in the estimates of Proposition 3.2). Indeed, the proofs of Proposition 6.1 (or rather, of Lemma 6.8 found below) and Proposition 6.2 will be flexible with respect to this particular manipulation.

Let us expand on this claim. If we wish to prove the statement analogous to (6.6) with $0 < \delta_0 < \delta$, then on the left-hand side of the inequality, we will have lowered the spatial Sobolev regularity to $s = \frac{1}{2} + \delta - \delta_0$ from $s = \frac{1}{2} + \delta$ (e.g. in the line (6.23)). This amounts to having the factor $|n|^{\frac{1}{2}+\delta-\delta_0}$ in the numerator of the nonlinear estimates below, instead of $|n|^{\frac{1}{2}+\delta}$.

In every case of the proofs (found below) of Lemma 6.8 and Proposition 6.2 (excluding Case 1 in the proof of Lemma 6.8, and Case 2.b.ii in the proof of Proposition 6.2, which we discuss in the next paragraph), we control the factor $|n|^{\frac{1}{2}+\delta}$ using the estimate $|n| \le N^0$. That is, we control this factor using terms in the denominator that are known to be the size of the largest frequency N^0 (see, for example, (6.37)). This means that, for each estimate we establish in these proofs, we can replace $|n|^{\frac{1}{2}+\delta}$ with $|n|^{\frac{1}{2}+\delta-\delta_0}|n_k|^{\delta_0}$, for any $k = 1, 2, 3, 4$ (or $k = 2, 3, 4, 5, 6, 7, 8$ for Proposition 6.2), and the proof we present still applies. Establishing an estimate with an extra factor of $|n_k|^{\delta_0}$ in the numerator of the left-hand side corresponds to lowering the spatial Sobolev regularity of the u_k factor on the right-hand side by the same amount $\delta_0 > 0$. That is, we can establish the same nonlinear estimate with $0 < \delta_0 < \delta$ included as in the statement of Proposition 3.2.

We should comment that, in Case 1.b during the proof of Lemma 6.8, and Case 2.b.ii in the proof of Proposition 6.2, the reasoning of the last paragraph does not apply. However, it is easily verified that we can still lower the spatial regularity of any one of the factors on the right-hand side by a small amount $\delta_0 > 0$, as the estimates in these cases have some room to spare in each factor.

Before we prove Proposition 6.1 and Proposition 6.2, let us use them to establish Proposition 3.2.

Proof of Proposition 3.2. Apply Proposition 6.2, and suppose $\omega \in \Omega_T$ so that the estimate (6.6) holds true. Note that by the equivalence

$$\chi_{[0,T]}\mathcal{D}(u_1, \dots, u_4) = \chi_{[0,T]}\mathcal{D}(\chi_{[0,T]}u_1, \dots, \chi_{[0,T]}u_4),$$

we have

$$\left\|\mathcal{D}_{-1}(u_1, u_2, u_3, u_4)\right\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}} \le \left\|\mathcal{D}_{-1}(\chi_{[0,T]}u_1, \chi_{[0,T]}u_2, \chi_{[0,T]}u_3, \chi_{[0,T]}u_4)\right\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}}.$$

Applying Lemma 2.4, Lemma 2.3, and (6.6), we find

$$\begin{aligned}
&\left\|\mathcal{D}_{-1}(\chi_{[0,T]}u_1, \dots, \chi_{[0,T]}u_4)\right\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}} \\
&\quad \lesssim \left\|\mathcal{D}_{-1}(\chi_{[0,T]}u_1, \dots, \chi_{[0,T]}u_4)\right\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}+\delta}} \\
&\quad \lesssim \left\|\mathcal{N}_{-1}(\chi_{[0,T]}u_1, \dots, \chi_{[0,T]}u_4)\right\|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}+\delta}} \\
&\quad \lesssim T^{-\beta}\prod_{j=1}^{4}\left(M_j^{-\beta} + \|u_j\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + \left\|u_j - S(t)\Lambda_j(u_{0,\omega})\right\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\right).
\end{aligned}$$

The proof of (3.6) is complete. The justification of (3.7) follows from (6.7) using the same type of argument. This completes the proof of Proposition 3.2. □

6.2. Probabilistic quadrilinear estimates

In this subsection we present the proof of Proposition 6.1. We begin by presenting some probabilistic lemmata to be used in the proof. In each lemma, we are considering the probability space $(\Omega, \mathcal{F}, P)$ with $P = \rho \circ u_{0,\omega}$, where ρ is the Wiener measure defined in (1.10), and the initial data (given by (1.5)) is viewed as a map $u_{0,\omega} : \Omega \to H^{1/2-}(\mathbb{T})$.

Lemma 6.4. *Let $\varepsilon, \beta > 0$, $T \ll 1$, and $\{g_n(\omega)\}_{n=1}^{\infty}$ be a sequence of independent $\mathbb{C}$-valued standard Gaussian random variables. Then there exists $\widetilde{\Omega}_T \subset \Omega$ with $P(\widetilde{\Omega}_T^c) < e^{-\frac{1}{T^\beta}}$, such that for $\omega \in \widetilde{\Omega}_T$, we have*

$$\left|g_n(\omega)\right| \le CT^{-\frac{\beta}{2}}\langle n\rangle^{\varepsilon}$$

for all $n \in \mathbb{N}$.

Proof. Recall from [34] that

$$P\left(\sup_{n\in\mathbb{N}}\langle n\rangle^{-\varepsilon}\left|g_n(\omega)\right| > K\right) \le e^{-cK^2}$$

for K sufficiently large. Lemma 6.4 follows by taking $K \sim T^{-\frac{\beta}{2}}$. □

Lemma 6.5. *(See Thomann and Tzvetkov [45], Tzvetkov [48].) Let $d \ge 1$ and $c(n_1, \dots, n_k) \in \mathbb{C}$. Let $\{\gamma_n(\omega)\}_{1\le n\le d}$ be a sequence of independent $\mathbb{R}$-valued standard Gaussian random variables. For $k \ge 1$, denote by $A(k,d) = \{(n_1, \dots, n_k) \in \{1, \dots, d\}^k : n_1 \le \dots \le n_k\}$, and*

$$S_k(\omega) = \sum_{A(k,d)} c(n_1, \dots, n_k)\gamma_{n_1}(\omega)\cdots\gamma_{n_k}(\omega).$$

Then, for each $p \geq 1$, we have

$$\|S_k\|_{L^p(\Omega)} \leq \sqrt{k+1}(p-1)^{\frac{k}{2}} \|S_k\|_{L^2(\Omega)}.$$

The proof of Lemma 6.5 can be found in [45]; it relies on hypercontractivity of the Ornstein–Uhlenbeck semigroup.

Lemma 6.6. *(See Tzvetkov [48].) Let $F : H^{\frac{1}{2}-}(\mathbb{T}) \to \mathbb{R}$ be measurable. Assume there exist $\alpha > 0$, $\tilde{N} > 0$, $k \geq 1$ and $C > 0$ such that for all $p \geq 2$,*

$$\|F\|_{L^p(d\rho)} \leq C\tilde{N}^{-\alpha} p^{\frac{k}{2}}. \tag{6.8}$$

Then there exist $\delta > 0$, c_1 independent of N and α such that

$$\int\limits_{H^{\frac{1}{2}-}(\mathbb{T})} e^{\delta \tilde{N}^{\frac{2\alpha}{k}} |F(u)|^{\frac{2}{k}}} d\rho(u) \leq c_1.$$

As a consequence, for all $\lambda > 0$,

$$P\big(\omega \in \Omega : \big|F(u_{0,\omega})\big| > \lambda\big) \leq c_1 e^{-\delta \tilde{N}^{\frac{2\alpha}{k}} \lambda^{\frac{2}{k}}}. \tag{6.9}$$

The proof of Lemma 6.6 can be found in [48]. We will also need the following basic observation from linear algebra.

Lemma 6.7. *Let $A = \{a_{i,j}\}_{1 \leq i,j \leq N}$ be a square $(N \times N)$ matrix with complex entries. Then*

$$\|A\| \leq \sup_{1 \leq n \leq N} |a_{n,n}| + \bigg(\sum_{n \neq n'} |a_{n,n'}|^2\bigg)^{\frac{1}{2}},$$

where $\|\cdot\|$ denotes the matrix 2-norm.

Our application of this lemma will follow the analysis found in [4]. The proof of Lemma 6.7 is omitted (the analysis required is straightforward).

For the proof of Proposition 6.1, we will use a dyadically localized estimate. That is, we will establish probabilistic quadrilinear estimates which are independent of the Fourier multipliers $\Lambda_1, \ldots, \Lambda_4$ appearing in the statement of Proposition 6.1. In the following, subscripts with capital letters denote dyadic localization; i.e. $u_{N_j} = (\chi_{|n_j| \sim N_j} \widehat{u_j})^\vee$ for N_j dyadic. Let

$$f_{0,j} := \|u_{N_j}\|_{X_T^{\frac{1}{2}+\delta, \frac{1}{2}-\delta}},$$

$$f_{1,j} := N_j^{-\beta} + \|u_{N_j}\|_{X_T^{\frac{1}{2}-\delta, \frac{1}{2}-\delta}} + \big\|u_{N_j} - \big(S(t)u_{0,\omega}\big)_{N_j}\big\|_{X_T^{\frac{1}{2}+\delta, \frac{1}{2}-\delta}}.$$

Here is the dyadically localized probabilistic quadrilinear estimate.

Lemma 6.8. *For $\delta > 0$ sufficiently small, and any $0 < T \ll 1$, there exist $\alpha, \beta, \kappa, C, c > 0$ with $\alpha, \beta, \kappa \ll \delta$ such that for every quintuple of dyadic frequencies $N, N_1, \ldots, N_4$, $\exists\, \Omega_{N,N_1,\ldots,N_4,T} \subset \Omega$ with $P(\Omega^c_{N,N_1,\ldots,N_4,T}) < \frac{1}{(NN_1 \cdots N_4)^\kappa} e^{-\frac{\tilde{c}}{T^\beta}}$ such that for all $\omega \in \widetilde{\Omega}_T \cap \Omega_{N,N_1,\ldots,N_4,T}$ we have*

$$\big\|\mathcal{N}_{-1}|_{|n| \sim N}(u_{N_1}, u_{N_2}, u_{N_3}, u_{N_4})\big\|_{X_T^{\frac{1}{2}+\delta, -\frac{1}{2}+\delta}} \leq \frac{CT^{-\beta}}{(NN_1 \cdots N_4)^\alpha} \prod_{j=1}^4 \min(f_{0,j}, f_{1,j}), \tag{6.10}$$

where $\widetilde{\Omega}_T$ is the set obtained from Lemma 6.4.

Remark 6.9. For simplicity, we have dropped implicit factors of $\chi_{[0,T]}$ from the left-hand side of (6.6) and will reintroduce them when needed.

We proceed to prove Proposition 6.1 using Lemma 6.8. Then we present the proof of Lemma 6.8, followed by the proof of Proposition 6.2.

Proof of Proposition 6.1. Fix any dyadic N_j for $j \in \{1,2,3,4\}$. Observe that

$$\min(f_{0,j}, f_{1,j}) \le M_j^{-\beta} + \|u_j\|_{X_T^{\frac12-\delta,\frac12-\delta}} + \|u_j - S(t)\Lambda_j(u_{0,\omega})\|_{X_T^{\frac12+\delta,\frac12-\delta}}. \tag{6.11}$$

Indeed, suppose $N_j \in [M_j, K_j] = \operatorname{supp}(\Lambda_j)$, then we have

$$\begin{aligned} f_{1,j} &= N_j^{-\beta} + \|u_{N_j}\|_{X_T^{\frac12-\delta,\frac12-\delta}} + \big\|u_{N_j} - \big(S(t)u_{0,\omega}\big)_{N_j}\big\|_{X_T^{\frac12+\delta,\frac12-\delta}} \\ &\le M_j^{-\beta} + \|u_j\|_{X_T^{\frac12-\delta,\frac12-\delta}} + \|u_j - S(t)\Lambda_j(u_{0,\omega})\|_{X_T^{\frac12+\delta,\frac12-\delta}}. \end{aligned}$$

On the other hand, if $N_j \notin [M_j, K_j]$, we have

$$\begin{aligned} f_{0,j} &= \|u_{N_j}\|_{X_T^{\frac12+\delta,\frac12-\delta}} \\ &= \big\|u_{N_j} - \big(S(t)\Lambda_j(u_{0,\omega})\big)_{N_j}\big\|_{X_T^{\frac12+\delta,\frac12-\delta}} \\ &\le M_j^{-\beta} + \|u_j\|_{X_T^{\frac12-\delta,\frac12-\delta}} + \|u_j - S(t)\Lambda_j(u_{0,\omega})\|_{X_T^{\frac12+\delta,\frac12-\delta}}, \end{aligned}$$

and the inequality (6.11) holds true.

We proceed to build a set $\Omega_T \subset \Omega$ (satisfying the necessary conditions) where the estimate (6.6) is satisfied. Consider a dyadic decomposition of the nonlinearity,

$$\begin{aligned} &\|\mathcal{N}_{-1}(u_1,u_2,u_3,u_4)\|_{X_T^{\frac12+\delta,-\frac12+\delta}} \\ &\quad\le \sum_{N,N_1,\dots,N_4} \|\mathcal{N}_{-1}|_{|n|\sim N}(u_{N_1},u_{N_2},u_{N_3},u_{N_4})\|_{X_T^{\frac12+\delta,-\frac12+\delta}}. \end{aligned} \tag{6.12}$$

Now let $\Omega_T := \widetilde{\Omega}_T \bigcap_{\text{dyadic } N,N_1,\dots,N_4} \Omega_{N,N_1,\dots,N_4,T}$. Then

$$P(\Omega_T^c) \le \sum_{N,N_1,\dots,N_4} P(\Omega^c_{N,N_1,\dots,N_4,T}) < \sum_{N,N_1,\dots,N_4} \frac{1}{(NN_1\cdots N_4)^\kappa} e^{-\frac{\tilde c}{T^\beta}} \le e^{-\frac{c}{T^\beta}},$$

where $c = c(\tilde c, \kappa) > 0$. Furthermore, if $\omega \in \Omega_T$, then for every combination of dyadic scales $N, N_1, \dots, N_4$, the conclusion (6.10) holds true. With (6.11), this gives

$$\begin{aligned} (6.12) &\lesssim \sum_{N,N_1,\dots,N_4} \frac{T^{-\beta}}{(NN_1\cdots N_4)^\alpha} \prod_{j=1}^4 \min(f_{0,j}, f_{1,j}) \\ &\le \sum_{N,N_1,\dots,N_4} \frac{T^{-\beta}}{(NN_1\cdots N_4)^\alpha} \prod_{j=1}^4 \Big(M_j^{-\beta} + \|u_j\|_{X_T^{\frac12-\delta,\frac12-\delta}} + \|u_j - S(t)\Lambda_j(u_{0,\omega})\|_{X_T^{\frac12+\delta,\frac12-\delta}}\Big) \\ &\lesssim T^{-\beta} \prod_{j=1}^4 \Big(M_j^{-\beta} + \|u_j\|_{X_T^{\frac12-\delta,\frac12-\delta}} + \|u_j - S(t)\Lambda_j(u_{0,\omega})\|_{X_T^{\frac12+\delta,\frac12-\delta}}\Big). \qquad \square \end{aligned} \tag{6.13}$$

Before we proceed with the proof of Lemma 6.8, let us highlight an important property of our frequency space restrictions:

$$\text{If } (n, n_1, n_2, n_3, n_4, \tau, \tau_1, \tau_2, \tau_3, \tau_4) \in A_{-1}, \text{ then } (n_1, n_2, n_3, n_4) \in \zeta_1(n). \tag{6.14}$$

To justify (6.14) recall that in the region A_{-1} of frequency space (defined in (3.4)), the condition $|n^3 - n_1^3 - \cdots - n_4^3| \ll |n_{\max}|^2$ is satisfied. To establish (6.14) we show that this condition necessitates $(n_1, n_2, n_3, n_4) \in \zeta_1(n)$ (see (6.3) for the definition of $\zeta_1(n)$). In fact, we show the contrapositive; that $(n_1, n_2, n_3, n_4) \notin \zeta_1(n)$ implies $|n^3 - n_1^3 - \cdots - n_4^3| \gtrsim |n_{\max}|^2$. Recall from (3.3) that in the domain of integration we have $(n_1, n_2, n_3, n_4) \in \zeta(n) = \zeta_1(n) \cup \zeta_2(n)$, and $(n_1, n_2, n_3, n_4) \notin \zeta_1(n)$ is therefore equivalent to $(n_1, n_2, n_3, n_4) \in \zeta_2(n)$. Suppose $(n_1, n_2, n_3, n_4) \in \zeta_2(n)$, then there are six possibilities (up to permutations of (n_2, n_3, n_4)):

(i) $n = n_1 = n_2$
(ii) $n = n_2 = n_3$
(iii) $n = n_1 = -n_2$
(iv) $n_1 = -n_2 = -n_3$
(v) $n = -n_1 = n_2$
(vi) $n = n_2, n_1 = -n_3$

We proceed to show $|n^3 - n_1^3 - \cdots - n_4^3| \gtrsim |n_{\max}|^2$ in each circumstance. Suppose possibility (i) holds, and we have $n = n_1 = n_2$. Then $n = n_1 + \cdots + n_4$ gives $n_2 + n_3 + n_4 = 0$, and we find

$$n^3 - n_1^3 - \cdots - n_4^3 = -n_2^3 - n_3^3 - n_4^3 = -3n_2n_3n_4. \tag{6.15}$$

Recall that each $n_i \neq 0$ by the mean zero condition (6.5). If $|n_3| \sim |n_4| \sim |n_{\max}|$, then by (6.15), and the mean zero condition, we have $|n^3 - n_1^3 - \cdots - n_4^3| \gtrsim |n_{\max}|^2$, which is impossible in the region A_{-1}. Therefore, we must have $|n| = |n_1| = |n_2| = |n_{\max}|$. Then $n_2 + n_3 + n_4 = 0$ gives (without loss of generality) that $|n_3| \sim |n_2| = |n_{\max}|$, and again we arrive at $|n^3 - n_1^3 - \cdots - n_4^3| \gtrsim |n_{\max}|^2$. We conclude that possibility (i) cannot occur in the region A_{-1}. It is straightforward to verify that the same argument rules out (ii)–(v); only (vi) remains to be considered. Suppose (vi) holds, and we have $n = n_2$, $n_1 = -n_3$. Then $n_4 = n - n_1 - n_2 - n_3 = 0$, which is impossible by the mean zero condition (6.5). Therefore, in the region A_{-1}, we cannot have $(n_1, n_2, n_3, n_4) \in \zeta_2(n)$, and we conclude that (6.14) holds true.

Proof of Lemma 6.8. Throughout this proof, all factors u_{N_j} are dyadically localized, and we simplify notation by taking $u_j = u_{N_j}$. Also, we have dropped the $\chi_{[0,T]}$ from in front of each factor u_j, but may reintroduce them as needed.

This proof is based on multiple decompositions of frequency space. In each region of frequency space we impose one of the following two conditions: for each $j \in \{1, 2, 3, 4\}$, either

(i) $u_j \in X_T^{\frac{1}{2}-\delta, \frac{1}{2}-\delta}$,

or

(ii) $u_j - \xi_j (S(t)u_{0,\omega})_{N_j} \in X_T^{\frac{1}{2}+\delta, \frac{1}{2}-\delta}$, for each $\xi_j \in \{0, 1\}$.

The additional parameters $\xi_j \in \{0, 1\}$ are introduced in order to establish a single result for variable ξ_j, which produces factors of $f_{0,j}$ with $\xi_j = 0$ and factors of $f_{1,j}$ with $\xi_j = 1$. That is, by keeping each ξ_j variable, we will produce the right hand side of (6.10).

Contributions to the left-hand side of (6.10) from a region where u_j satisfies condition (i) produce a corresponding factor of $\|u_j\|_{X_T^{\frac{1}{2}-\delta, \frac{1}{2}-\delta}}$ on the right-hand side of the inequality. For contributions from regions where u_j satisfies condition (ii), we establish probabilistic bounds, using what will be referred to as a type (I)–type (II) analysis (see [4,14]) by writing

$$u_j = \underbrace{\xi_j \big(S(t)u_{0,\omega}\big)_{N_j}}_{\text{type (I)}} + \underbrace{u_j - \xi_j \big(S(t)u_{0,\omega}\big)_{N_j}}_{\text{type (II)}}.$$

We show that each type (I) contribution produces a factor of $\xi_j N_j^{-\beta}$ on the right-hand side of the inequality, for $\omega \in \Omega_{N,\dots,N_4,T}$. The type (II) contribution will produce a factor of $\|u_j - \xi_j(S(t)u_{0,\omega})_{N_j}\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}$ on the right-hand side. Combining the contributions from (i) and (ii), each u_j will produce a factor of

$$\xi_j N_j^{-\beta} + \|u_j\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + \left\|u_j - \xi_j\big(S(t)u_{0,\omega}\big)_{N_j}\right\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}. \tag{6.16}$$

Notice that (6.16) $\lesssim f_{0,j}$ for $\xi_j = 0$, and (6.16) $= f_{1,j}$ for $\xi_j = 1$. By establishing these estimates for all combinations of $\xi_j \in \{0,1\}$, $j = 1,2,3,4$, we can always choose the smaller of the two bounds, and each u_j contributes a factor of $\min(f_{0,j}, f_{1,j})$ to the right-hand side of our inequality.

Summarizing the previous paragraphs, we prove Lemma 6.8 by constructing $\Omega_{N,\dots,N_4,T} \subset \Omega_T$ with $P(\Omega^c_{N,N_1,\dots,N_4,T}) < \frac{1}{(NN_1\cdots N_4)^\kappa} e^{-\frac{\tilde{c}}{T^\beta}}$ such that for all $\omega \in \widetilde{\Omega}_T \cap \Omega_{N,N_1,\dots,N_4,T}$ we can, throughout frequency space, either bound each u_j deterministically, using condition (i), or probabilistically, using condition (ii) and Lemmas 6.4–6.6 (the type (I)–type (II) decomposition).

In the break down of cases that follows, as we estimate the left-hand side of (6.10) using the method just described, each factor u_j may be declared to be of the following types

- type (I) (rough but random): $u_j = (S(t)u_{0,\omega})_{N_j}$,
- type (II) (smooth and deterministic): $u_j \in X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}$.

In a given case, if u_j is declared to be of type (I) or type (II), this means that we are choosing to use condition (ii) in this factor, and according to the decomposition above, we must consider each case of u_j type (I) and u_j type (II). If we make no declaration about a particular factor u_j in a given case, it means that we are imposing condition (i) in that factor: $u_j \in X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}$.

We will use superscripts n^k (N^k), $k = 0,1,\dots,4$, to indicate frequencies (and corresponding dyadic blocks) which have been ordered from largest to smallest. That is, $|n^0| \geq |n^1| \geq \cdots \geq |n^4|$ (and $N^0 \geq N^1 \geq \cdots \geq N^4$). Note that we order the frequency n as $-n$ (e.g. if n is the frequency of largest magnitude, then $n^0 = -n$). Also, by symmetry of $\mathcal{N}(u_1,u_2,u_3,u_4)$ in (u_2,u_3,u_4), we can assume that $|n_2| \geq |n_3| \geq |n_4|$.

We begin with an overview of each case to be considered in the proof.

- **CASE 1.** $n^0 = -n^1$.
- **CASE 2.** $n^0 \neq -n^1$.
  - **CASE 2.a.** $N^3 \ll N^0$ and $N^2N^3N^4 \ll N^0N^1|n^0+n^1|$.
    We will find that there is no contribution from this case.
  - **CASE 2.b.** $N^3 \sim N^0$.
    We will use a type (I)–type (II) decomposition in the u_k factor for $k = 1,2,3$.
    - **CASE 2.b.i.** At least two u_i of type (I), $i = 1,2,3$.
      That is, u_1,u_2,u_3 of types (I)(I)(I), (I)(I)(II), (I)(II)(I) and (II)(I)(I).
    - **CASE 2.b.ii.** One of u_i of type (I), $i = 1,2,3$, others type (II).
      That is, u_1,u_2,u_3 of types (I)(II)(II), (II)(I)(II) and (II)(II)(I).
    - **CASE 2.b.iii.** u_1,u_2,u_3 all type (II).
  - **CASE 2.c.** $N^3 \ll N^0$ and $N^2N^3N^4 \gtrsim N^0N^1|n^0+n^1|$.
    We will use a type (I)–type (II) decomposition for each $k = 1,2,3,4$.
    - **CASE 2.c.i.** u_1 type (I) and at least two of u_2,u_3,u_4 type (I).
      That is, u_1,u_2,u_3,u_4 of types (I)(I)(I)(I), (I)(I)(I)(II), (I)(I)(II)(I) and (I)(II)(I)(I).
    - **CASE 2.c.ii.** u_1 type (II) and u_2,u_3,u_4 type (I).
    - **CASE 2.c.iii.** Two of u_1,u_2,u_3,u_4 type (I) and two type (II).
    - **CASE 2.c.iv.** At least three of u_1,u_2,u_3,u_4 type (II).

We proceed with the analysis of each case.

- **CASE 1.** $n^0 = -n^1$.

By (6.14) we have $(n_1, n_2, n_3, n_4) \in \zeta_1(n)$, and therefore $n \neq n_i$ for all $i \in \{1, 2, 3, 4\}$, and $n_1 \neq -n_k$ for all $k \in \{2, 3, 4\}$. It follows that if $n^0 = -n^1$, we must have $n^0 = n_k = -n_j = -n^1$ for some $k, j \in \{2, 3, 4\}$ (recall that we have ordered the frequency n as $-n$). By the condition $|n_2| \geq |n_3| \geq |n_4|$ it follows that $n^0 = n_2 = -n_3 = -n^1$. With $n_2 = -n_3$, we have $n = n_1 + n_4$ and

$$\begin{aligned}\max\big(|\sigma|, |\sigma_1|, |\sigma_2|, |\sigma_3|, |\sigma_4|\big) &\gtrsim |\sigma - \sigma_1 - \sigma_2 - \sigma_3 - \sigma_4| = \big|n^3 - n_1^3 - n_2^3 - n_3^3 - n_4^3\big| \\ &= \big|n^3 - n_1^3 - n_4^3\big| = 3|nn_1n_4|.\end{aligned}$$

In this case we establish:

$$\big\|\mathcal{N}_{-1}|_{\text{CASE 1}}(u_1, u_2, u_3, u_4)\big\|_{X_T^{\frac{1}{2}+\delta, -\frac{1}{2}+\delta}} \lesssim \frac{1}{(NN_1 \cdots N_4)^\alpha} \prod_{j=1}^{4} \|u_j\|_{X_T^{\frac{1}{2}-\delta, \frac{1}{2}-\delta}}. \tag{6.17}$$

We consider various subcases.

- **CASE 1.a.** $|\sigma| \gtrsim |nn_1n_4|$.

In this case we find

$$\frac{|n|^{\frac{1}{2}+\delta}|n_1|}{|\sigma|^{\frac{1}{2}-\delta}|n_1|^{\frac{1}{2}-\delta}|n_2|^{5\delta}} \lesssim \frac{1}{(NN_1N_2N_3N_4)^\alpha}.$$

Using this estimate, (6.17) follows from

$$\|f_1 f_2 u_3 u_4\|_{L^2_{x,t}} \leq \|f_1\|_{X^{0,\frac{1}{2}-\delta}} \|f_2\|_{X^{\frac{1}{2}-6\delta, \frac{1}{2}-\delta}} \|u_3\|_{X^{\frac{1}{2}-\delta, \frac{1}{2}-\delta}} \|u_4\|_{X^{\frac{1}{2}-\delta, \frac{1}{2}-\delta}}. \tag{6.18}$$

We can establish (6.18) using Hölder, (2.8) and (2.10),

$$\begin{aligned}\|f_1 f_2 u_3 u_4\|_{L^2_{x,t}} &\lesssim \|f_1\|_{L^4_{x,t}} \|f_2\|_{L^{12}_{x,t}} \|u_3\|_{L^{12}_{x,t}} \|u_4\|_{L^{12}_{x,t}} \\ &\lesssim \|f_1\|_{X^{0,\frac{1}{2}-\delta}} \|f_2\|_{X^{\frac{1}{2}-6\delta, \frac{1}{2}-\delta}} \|u_3\|_{X^{\frac{1}{2}-\delta, \frac{1}{2}-\delta}} \|u_4\|_{X^{\frac{1}{2}-\delta, \frac{1}{2}-\delta}},\end{aligned} \tag{6.19}$$

for $\delta > 0$ sufficiently small.

- **CASE 1.b.** $|\sigma_k| \gtrsim |nn_1n_4|$ for some $k = 1, 2, 3, 4$.

The analysis of this case is similar to the previous one. For details see [40].

- **CASE 2.** $n^0 \neq -n^1$.

Before we proceed with each subcase, let us identify a useful restriction which holds throughout Case 2:

$$\text{No two integers in the set } \{-n, n_1, n_2, n_3, n_4\} \text{ sum to zero.} \tag{6.20}$$

Indeed, recall from (6.14) that in the region A_{-1}, we have $(n_1, n_2, n_3, n_4) \in \zeta_1(n)$, and it follows that $n \neq n_k$ for all $k = 1, 2, 3, 4$ and $n_1 \neq -n_k$ for all $k = 2, 3, 4$. The only pairs of integers that could sum to zero are within the set $\{n_2, n_3, n_4\}$. Suppose, for example, that $n_2 = -n_3$, then by the restriction $n^0 \neq -n^1$ we must have $N_2, N_3 \ll N^0$. Then $n = n_1 + n_4$ and we have

$$\big|n^3 - n_1^3 - \cdots - n_4^3\big| = \big|n^3 - n_1^3 - n_4^3\big| = 3|nn_1n_4| \gtrsim \big(N^0\big)^2,$$

in contradiction with restriction to the region A_{-1} in this case. The same argument applies if $n_2 = -n_4$ or $n_3 = -n_4$, and (6.20) follows.

- **CASE 2.a.** $N^3 \ll N^0$ and $N^2N^3N^4 \ll N^0N^1|n^0 + n^1|$.

Recall that we have taken $n = -n^k$ for some $k \in \{0, 1, \dots, 4\}$, so that $n^0 + \cdots + n^4 = 0$ is satisfied. Then

$$\begin{aligned}\left|n^3 - n_1^3 - \cdots - n_4^3\right| &= \left|\left(n^1 + \cdots + n^4\right)^3 - \left(n^1\right)^3 - \cdots - \left(n^4\right)^3\right| \\ &= 3\left|\left(-n^0 n^1 + n^2\left(n^3 + n^4\right) + n^3 n^4\right)\left(n^2 + n^3 + n^4\right) - n^2 n^3 n^4\right| \\ &\gtrsim N^0 N^1 \left|n^0 + n^1\right|,\end{aligned}$$

since $N^3 \ll N^0$, $N^2 N^3 N^4 \ll N^0 N^1 |n^0 + n^1|$ (recall $n^0 \neq -n^1$). Then

$$\max\left(|\sigma|, |\sigma_1|, \dots, |\sigma_4|\right) \gtrsim \left|n^3 - n_1^3 - \cdots - n_4^3\right| \gtrsim N^0 N^1 \left|n^0 + n^1\right| \geq |n_{\max}|^2,$$

and we cannot be in the region A_{-1}. That is, there is no contribution to $\mathcal{N}_{-1}$ from this case, and we proceed to the next one.

- **CASE 2.b.** $N^3 \sim N^0$.

We consider a type (I)–type (II) decomposition in the u_k factor for $k = 1, 2, 3$. With $N_2 \geq N_3 \geq N_4$, the restriction $N^3 \sim N^0$ implies, in particular, that

$$N_3 \sim N^0. \tag{6.21}$$

- **CASE 2.b.i.** At least two u_i of type (I), $i = 1, 2, 3$. That is, u_1, u_2, u_3 of types (I)(I)(I), (I)(I)(II), (I)(II)(I) and (II)(I)(I).

Suppose u_1, u_2 are type (I). We will comment on adapting these arguments to the other cases afterward. We will use $\mathcal{N}_{-1}|_{2.\mathrm{b.i}}$ to denote the contribution to the nonlinearity from this case. We establish the estimate:

$$\begin{aligned}&\left\|\mathcal{N}_{-1}|_{2.\mathrm{b.i}}(u_1, u_2, u_3, u_4)\right\|_{X_T^{\frac{1}{2}+\delta, -\frac{1}{2}+\delta}} \\ &\quad\lesssim \frac{T^{-\beta}}{(N N_1 \cdots N_4)^{\alpha}} (N_1 N_2)^{-\beta} \|u_3\|_{X_T^{\frac{1}{2}-\delta, \frac{1}{2}-\delta}} \|u_4\|_{X_T^{\frac{1}{2}-\delta, \frac{1}{2}-\delta}}.\end{aligned} \tag{6.22}$$

By changing variables and taking out a supremum, we find

$$\begin{aligned}&\left\|\mathcal{N}_{-1}|_{2.\mathrm{b.i}}(u_1, u_2, u_3, u_4)\right\|_{X_T^{\frac{1}{2}+\delta, -\frac{1}{2}+\delta}} \\ &\quad= \left\|\frac{\langle n\rangle^{\frac{1}{2}+\delta}}{\langle \sigma\rangle^{\frac{1}{2}-\delta}} \widehat{\mathcal{N}}_{-1}|_{2.\mathrm{b.i}}(u_1, u_2, u_3, u_4)(n, \tau)\right\|_{L^2_{\{|n|\sim N\}, \tau}} \\ &\quad= \left\|\chi_{\{|\lambda|<(N^0)^2\}} \frac{\langle n\rangle^{\frac{1}{2}+\delta}}{\langle \lambda\rangle^{\frac{1}{2}-\delta}} \widehat{\mathcal{N}}_{-1}|_{2.\mathrm{b.i}}(u_1, u_2, u_3, u_4)\left(n, \lambda + n^3\right)\right\|_{L^2_{\{|n|\sim N\}, \lambda}} \\ &\quad\leq \left(N^0\right)^{\delta} \sup_{|\lambda|<(N^0)^2} \left\|\widehat{\mathcal{N}}_{-1}|_{2.\mathrm{b.i}}(u_1, u_2, u_3, u_4)\left(n, \lambda + n^3\right)\right\|_{H^{\frac{1}{2}+\delta}_{|n|\sim N}}.\end{aligned} \tag{6.23}$$

For the factors $u_3, u_4 \in X_T^{\frac{1}{2}-\delta, \frac{1}{2}-\delta}$, we will use the following standard representation for functions in $X^{s,b}$ (see [24], for example). Given a function $v(x, t)$, we can write v as

$$v(x, t) = \int \langle\lambda\rangle^{-b} \left(\sum_n \langle n\rangle^{2s} \langle\lambda\rangle^{2b} \left|\hat{v}\left(n, n^3 + \lambda\right)\right|^2\right)^{\frac{1}{2}} \left\{e^{i\lambda t} \sum_n a_\lambda(n) e^{i(nx + n^3 t)}\right\} d\lambda \tag{6.24}$$

where $a_\lambda(n) = \frac{\hat{v}(n, n^3+\lambda)}{(\sum_n \langle n\rangle^{2s} |\hat{v}(n, n^3+\lambda)|^2)^{\frac{1}{2}}}$. Notice that $\sum_n \langle n\rangle^{2s} |a_\lambda(n)|^2 = 1$. For $v \in X^{s,b}$, with $b < \frac{1}{2}$, we have

$$\int\limits_{|\lambda|<K} \langle\lambda\rangle^{-b} \left(\sum_n \langle n\rangle^{2s} \langle\lambda\rangle^{2b} \left|\hat{v}\left(n, n^3 + \lambda\right)\right|^2\right)^{\frac{1}{2}} d\lambda \lesssim K^{\frac{1}{2}-b} \|v\|_{X^{s,b}}, \tag{6.25}$$

by Cauchy–Schwarz. In our context, for each $j = 3, 4$, we have $u_j = \chi_{[0,T]} u_j \in X^{\frac{1}{2}-\delta, \frac{1}{2}-\delta}$, and $|\tau_j - n_j^3| < (N^0)^2$. Using (6.24) we can write

$$\hat{u}_j(n_j, \tau_j) = \int\limits_{|\lambda_j| < (N^0)^2} \langle \lambda_j \rangle^{-\frac{1}{2}+\delta} c_j(\lambda_j) a_{\lambda_j}(n_j) \delta\big(\tau_j - n_j^3 - \lambda_j\big) d\lambda_j,$$

with $\sum_n \langle n \rangle^{2s} |a_\lambda(n)|^2 = 1$ and $c_j(\lambda_j) = (\sum_n \langle n \rangle^{1-2\delta} \langle \lambda_j \rangle^{1-2\delta} |\hat{u_j}(n, n^3 + \lambda)|^2)^{\frac{1}{2}}$. Inserting this representation for u_3, u_4, and the assumption that u_1 and u_2 are type (I), we have

$$\begin{aligned}(6.23) = N^{\frac{1}{2}+\delta} \big(N^0\big)^{\delta} \sup_{|\lambda| < (N^0)^2} \Bigg\| \sum_{\{|n_j| \sim N_j\} \cap (6.20)} (in_1) \prod_{i=1}^{2} \frac{g_{n_i}(\omega) \delta(\tau_i - n_i^3)}{|n_i|} \\ \cdot \iint\limits_{|\lambda_3|, |\lambda_4| < (N^0)^2} \prod_{j=3}^{4} \langle \lambda_j \rangle^{-\frac{1}{2}+\delta} c_j(\lambda_j) a_{\lambda_j}(n_j) \delta\big(\tau_j - n_j^3 - \lambda_j\big) d\lambda_j \Bigg\|_{L^2_{|n| \in N}}. \qquad (6.26)\end{aligned}$$

By Minkowski in λ_3, λ_4, we find

$$\begin{aligned}(6.26) \le N^{\frac{1}{2}+\delta} \big(N^0\big)^{\delta} \iint\limits_{|\lambda_3|, |\lambda_4| < (N^0)^2} \prod_{j=3}^{4} \langle \lambda_j \rangle^{-\frac{1}{2}+\delta} |c_j(\lambda_j)| d\lambda_j \\ \cdot \sup_{|\lambda|, |\lambda_3|, |\lambda_4| < (N^0)^2} \Bigg\| \sum_{\{|n_j| \sim N_j\} \cap (6.20)} (in_1) \frac{g_{n_1}(\omega) g_{n_2}(\omega)}{|n_1||n_2|} a_{\lambda_3}(n_3) a_{\lambda_4}(n_4) \\ \cdot \iiint\limits_{\tau_1, \tau_2, \tau_3} \delta\big(\tau_1 - n_1^3\big) \delta\big(\tau_2 - n_2^3\big) \delta\big(\tau_3 - n_3^3 - \lambda_3\big) \\ \cdot \delta\big(\lambda + n^3 - \tau_1 - \tau_2 - \tau_3 - n_4^3 - \lambda_4\big) d\tau_1 d\tau_2 d\tau_3 \Bigg\|_{L^2_{|n| \in N}}. \qquad (6.27)\end{aligned}$$

For fixed $n, n_1, n_2, n_3, \lambda, \lambda_3, \lambda_4$, we find

$$\begin{aligned}&\iiint\limits_{\tau_1, \tau_2, \tau_3} \delta\big(\tau_1 - n_1^3\big) \delta\big(\tau_2 - n_2^3\big) \delta\big(\tau_3 - n_3^3 - \lambda_3\big) \delta\big(\lambda + n^3 - \tau_1 - \tau_2 - \tau_3 - n_4^3 - \lambda_4\big) d\tau_1 d\tau_2 d\tau_3 \\ &= \begin{cases} 1, & \text{if } \lambda - \lambda_3 - \lambda_4 + n^3 - n_1^3 - \cdots - n_4^3 = 0, \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

Then we have

$$\begin{aligned}(6.27) &\le N^{\frac{1}{2}+\delta} \big(N^0\big)^{\delta} \iint\limits_{|\lambda_3|, |\lambda_4| < (N^0)^2} \prod_{j=3}^{4} \langle \lambda_j \rangle^{-\frac{1}{2}+\delta} |c_j(\lambda_j)| d\lambda_j \\ &\quad \cdot \sup_{|\lambda|, |\lambda_3|, |\lambda_4| < (N^0)^2} \Bigg\| \sum_{*(n, \lambda+\lambda_3+\lambda_4)} (in_1) \frac{g_{n_1}(\omega) g_{n_2}(\omega)}{|n_1||n_2|} a_{\lambda_3}(n_3) a_{\lambda_4}(n_4) \Bigg\|_{L^2_{|n| \in N}} \\ &\le N^{\frac{1}{2}+\delta} \big(N^0\big)^{3\delta} \prod_{j=3}^{4} \|u_j\|_{X_T^{\frac{1}{2}-\delta, \frac{1}{2}-\delta}} \sup_{|\mu| < 3(N^0)^2} \Bigg\| \sum_{*(n,\mu)} (in_1) \frac{g_{n_1}(\omega) g_{n_2}(\omega)}{|n_1||n_2|} a_{n_3} a_{n_4} \Bigg\|_{L^2_{|n| \in N}}, \qquad (6.28)\end{aligned}$$

by (6.25), where $\sum_{n_i} |n_i|^{1-2\delta} |a_{n_i}|^2 = 1$, for $i = 3, 4$, and

$$*(n, \mu) := \big\{(n_1, n_2, n_3, n_4) \in \mathbb{Z}^4 \;\big|\; \text{Each } |n_i| \sim N_i,\ (6.20) \text{ is satisfied, and } \mu = n^3 - n_1^3 - \cdots - n_4^3\big\}. \qquad (6.29)$$

When we fix numbers other than n, μ, for example n_1, we let

$$*(n,\mu,n_1) := \left\{(n_2,n_3,n_4)\in\mathbb{Z}^3 \;\middle|\; \text{Each } |n_i|\sim N_i,\ \text{(6.20) is satisfied, and } \mu = n^3 - n_1^3 - \cdots - n_4^3\right\}, \tag{6.30}$$

and define $*(n,\mu,n_2,n_3)\subset\mathbb{Z}^2$, $*(n,\mu,n_1,n_2,n_3)\subset\mathbb{Z},\ldots$, etc., similarly. Notice that we have dropped the dependence on λ_3,λ_4 in (6.28); this is justified a posteriori by using estimates which are independent of λ_3,λ_4.

Now for each fixed $|n|\in N$, $|\mu|<3(N^0)^2$, we write

$$\begin{aligned}
&\left|\sum_{*(n,\mu)}(in_1)\frac{g_{n_1}(\omega)g_{n_2}(\omega)}{\langle n_1\rangle\langle n_2\rangle}a_{n_3}a_{n_4}\right|^2\\
&\quad=\left|\sum_{|n_4|\sim N_4}|n_4|^{\frac12-\delta}a_{n_4}\cdot\frac{1}{|n_4|^{\frac12-\delta}}\left(\sum_{*(n,\mu,n_4)}(in_1)\frac{g_{n_1}(\omega)g_{n_2}(\omega)}{\langle n_1\rangle\langle n_2\rangle}a_{n_3}\right)\right|^2\\
&\quad\lesssim\sum_{|n_4|\sim N_4}\frac{1}{|n_4|^{1-2\delta}}\left|\sum_{*(n,\mu,n_4)}(in_1)\frac{g_{n_1}(\omega)g_{n_2}(\omega)}{\langle n_1\rangle\langle n_2\rangle}a_{n_3}\right|^2,
\end{aligned}$$

by Cauchy–Schwarz in n_4. For each fixed $|n_4|\sim N_4$, $\mu<3(N^0)^2$, we write

$$\sum_{|n|\sim N}\left|\sum_{*(n,\mu,n_4)}(in_1)\frac{g_{n_1}(\omega)g_{n_2}(\omega)}{\langle n_1\rangle\langle n_2\rangle}a_{n_3}\right|^2=\sum_{|n|\sim N}\left|\sum_{|n_3|\sim N_3}\sigma_{n,n_3}^{n_4,\mu}|n_3|^{\frac12-\delta}a_{n_3}\right|^2 \tag{6.31}$$

where $\sigma_{n,n_3}^{n_4,\mu}$ is the (n,n_3)rd entry of a matrix $\sigma^{n_4,\mu}$ (for n_4,μ fixed) with columns indexed by $|n_3|\sim N_3$, and rows indexed by $|n|\sim N$. These entries are given by

$$\sigma_{n,n_3}^{n_4,\mu}=\sum_{(n_1,n_2)\in*(n,n_3,n_4,\mu)}(in_1)\frac{g_{n_1}(\omega)g_{n_2}(\omega)}{\langle n_1\rangle\langle n_2\rangle|n_3|^{\frac12-\delta}}. \tag{6.32}$$

Recall the following property of matrix norms: $\|A^*A\|=\|AA^*\|$. Using Cauchy–Schwarz, the condition $\sum_{n_3}|n_3|^{1-2\delta}|a_3(n_3)|^2=1$, and applying Lemma 6.7, we find

$$\begin{aligned}
\text{(6.31)}&\lesssim\left\|\left(\sigma^{n_4,\mu}\right)^*\sigma^{n_4,\mu}\right\|=\left\|\sigma^{n_4,\mu}\left(\sigma^{n_4,\mu}\right)^*\right\|\\
&\le\sup_{|n|\sim N}\sum_{|n_3|\sim N_3}\left|\sigma_{n,n_3}^{n_4,\mu}\right|^2+\left(\sum_{\substack{n\neq n'\\|n|,|n'|\sim N}}\left|\sum_{|n_3|\sim N_3}\sigma_{n,n_3}^{n_4,\mu}\overline{\sigma_{n',n_3}^{n_4,\mu}}\right|^2\right)^{\frac12}\\
&=:I_1(n_4,\mu)+I_2(n_4,\mu).
\end{aligned} \tag{6.33}$$

To recap, combining (6.23), (6.26), (6.27) and (6.28), we now have

$$\begin{aligned}
&\left\|\mathcal{N}_{-1}|_{2.\text{b.i}}(u_1,u_2,u_3,u_4)\right\|_{X_T^{\frac12+\delta,-\frac12+\delta}}\\
&\quad\lesssim\left(N^0\right)^{3\delta}N^{\frac12+\delta}\prod_{j=3}^4\|u_j\|_{X_T^{\frac12-\delta,\frac12-\delta}}\sup_{|\mu|<3(N^0)^2}\left(\sum_{|n_4|\sim N_4}\frac{1}{|n_4|^{1-2\delta}}\left(I_1(n_4,\mu)+I_2(n_4,\mu)\right)\right)^{\frac12},
\end{aligned} \tag{6.34}$$

and we estimate the contributions from $I_1(n_4,\mu)$ and $I_2(n_4,\mu)$ separately.

We remark that the sum in (6.32) has at most two terms. Indeed, for n, n_3, n_4, and μ fixed, if $(n_1,n_2)\in*(n,n_3,n_4,\mu)$, then n_2 is determined by n_1 through the condition $n=n_1+\cdots+n_4$, and n_1 satisfies the equation $\mu=n^3-n_1^3-\cdots-n_4^3$. Since $n_1\neq-n_2$ (recall (6.20)), this is a non-degenerate quadratic equation in n_1:

$$\begin{aligned}
\mu&=n^3-n_1^3-\cdots-n_4^3=n^3-n_1^3-(n-n_1-n_3-n_4)^3-\cdots-n_4^3\\
&=-3(n-n_3-n_4)n_1^2-3(n-n_3-n_4)^2n_1+n^3-(n-n_3-n_4)^3-n_3^3-n_4^3,
\end{aligned} \tag{6.35}$$

with $n-n_3-n_4=n_1+n_2\neq0$, and this equation has at most two roots n_1.

Then to estimate $I_1(n_4,\mu)$, for n,n_3,n_4,μ fixed, we bring the absolute value inside the sum of (at most) two terms in (6.32) and apply Lemma 6.4 (with $\varepsilon=\beta$) to obtain, for $\omega\in\widetilde{\Omega}_T$:

$$I_1(n_4,\mu)\le \sup_{|n|\sim N}\sum_{\substack{|n_3|\sim N_3\\(n_1,n_2)\in*(n,n_3,n_4,\mu)}}\frac{|n_1|^2|g_{n_1}(\omega)||g_{n_2}(\omega)|}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^{1-2\delta}}$$
$$\lesssim T^{-\beta}\sup_{|n|\sim N}\sum_{\substack{|n_3|\sim N_3\\(n_1,n_2)\in*(n,n_3,n_4,\mu)}}\frac{|n_1|^2}{\langle n_1\rangle^{2-\beta}\langle n_2\rangle^{2-\beta}\langle n_3\rangle^{1-2\delta}}$$
$$\lesssim \frac{T^{-\beta}}{(N^0)^{2-2\beta-2\delta-\gamma}}\sum_{|n_3|\sim N_3}\frac{1}{\langle n_3\rangle^{1+\gamma}}\lesssim\frac{T^{-\beta}}{(N^0)^{2-2\beta-2\delta-\gamma}},\tag{6.36}$$

where we have used $N_3\sim N^0$ and $N_2\ge N_3\ge N_4$ in the second last line. Then we can estimate the contribution to (6.34) coming from $I_1(n_4,\mu)$ by

$$\frac{T^{-\frac{\beta}{2}}N^{\frac12+\delta}}{(N^0)^{1-\beta-5\delta-\frac{\gamma}{2}}}\prod_{j=3}^{4}\|u_j\|_{X_T^{\frac12-\delta,\frac12-\delta}}\Big(\sum_{|n_4|\sim N_4}\frac{1}{\langle n_4\rangle^{1-2\delta}}\Big)^{\frac12}$$
$$\lesssim\frac{T^{-\frac{\beta}{2}}N^{\frac12+\delta}}{(N^0)^{1-\beta-6\delta-\gamma}}\prod_{j=3}^{4}\|u_j\|_{X_T^{\frac12-\delta,\frac12-\delta}}\Big(\sum_{|n_4|\sim N_4}\frac{1}{\langle n_4\rangle^{1+\gamma}}\Big)^{\frac12}$$
$$\lesssim\frac{T^{-\beta}}{(NN_1\cdots N_4)^{\alpha}}(N_1N_2)^{-\beta}\|u_3\|_{X_T^{\frac12-\delta,\frac12-\delta}}\|u_4\|_{X_T^{\frac12-\delta,\frac12-\delta}}.\tag{6.37}$$

To estimate $I_2(n_4,\mu)$, note that

$$I_2(n_4,\mu)=\Bigg(\sum_{\substack{n\neq n'\\|n|,|n'|\sim N}}\Bigg|\sum_{|n_3|\sim N_3}\Big(\sum_{*(n,n_3,n_4,\mu)}\frac{(in_1)g_{n_1}(\omega)g_{n_2}(\omega)}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle^{\frac12-\delta}}\Big)$$
$$\cdot\Big(\sum_{*(n',n_3,n_4,\mu)}\frac{(-in_1')\overline{g_{n_1'}}(\omega)\overline{g_{n_2'}}(\omega)}{\langle n_1'\rangle\langle n_2'\rangle\langle n_3'\rangle^{\frac12-\delta}}\Big)\Bigg|^2\Bigg)^{\frac12}.\tag{6.38}$$

For each fixed n,n',n_4,μ, let

$$F_{n,n',n_4,\mu}(\omega):=\sum_{\substack{|n_3|\sim N_3\\n_1,n_2\in*(n,\mu,n_3,n_4)\\n_1',n_2'\in*(n',\mu,n_3,n_4)}}\frac{n_1n_1'g_{n_1}(\omega)g_{n_2}(\omega)\overline{g_{n_1'}}(\omega)\overline{g_{n_2'}}(\omega)}{\langle n_1\rangle\langle n_2\rangle\langle n_1'\rangle\langle n_2'\rangle\langle n_3\rangle^{1-2\delta}}.$$

Notice that $F_{n,n',n_4,\mu}(\omega):=F_{n,n',n_4,\mu}(u_{0,\omega})$ is ρ-measurable (it is a polynomial function of the randomized Fourier coefficients). By Lemma 6.5,

$$\|F_{n,n',n_4,\mu}\|_{L^p(\Omega)}\le\sqrt{5}(p-1)^2\|F_{n,n',n_4,\mu}\|_{L^2(\Omega)},$$

for each $2<p<\infty$. Then by Lemma 6.6 (applied with $\tilde N=(\|F_{n,n',n_4,\mu}\|_{L^2(\Omega)})^{-1}$, $\alpha=1$ and $k=4$) it follows that

$$P\big(\big|F_{n,n',n_4,\mu}(\omega)\big|\ge\lambda\big)\le e^{-c\|F_{n,n',n_4,\mu}\|_{L^2(\Omega)}^{-\frac12}\lambda^{\frac12}}.$$

Taking $\lambda=\|F_{n,n',n_4,\mu}\|_{L^2(\Omega)}(N^0)^{2\beta}T^{-2\beta}$, we have

$$P\big(\big|F_{n,n',n_4,\mu}(\omega)\big|\ge\|F_{n,n',n_4,\mu}\|_{L^2(\Omega)}\big(N^0\big)^{2\beta}T^{-2\beta}\big)\le e^{-c\frac{(N^0)^\beta}{T^\beta}}.$$

Let

$$\Omega_{N,N_1,\dots,N_4,T}:=\bigcap_{\substack{|n|\sim N,|n'|\sim N\\|n_4|\sim N_4,|\mu|<3(N^0)^2}}\big\{\big|F_{n,n',n_4,\mu}(\omega)\big|<\|F_{n,n',n_4,\mu}\|_{L^2(\Omega)}\big(N^0\big)^{2\beta}T^{-2\beta}\big\}.$$

Then

$$P\big((\Omega_{N,N_1,\dots,N_4,T})^c\big) \le \sum_{\substack{|n|\sim N, |n'|\sim N\\ |n_4|\sim N_4, |\mu|<3(N^0)^2}} e^{-c\frac{(N^0)^\beta}{T^\beta}} \le \big(N^0\big)^5 e^{-c\frac{(N^0)^\beta}{T^\beta}}$$

$$\le \big(N^0\big)^{-5\alpha} e^{-\frac{\tilde{c}}{T^\beta}} \le (NN_1\cdots N_4)^{-\alpha} e^{-\frac{\tilde{c}}{T^\beta}},$$

for some $\tilde{c}(\beta), \kappa(\beta) > 0$. Furthermore, if $\omega \in \Omega_{N,N_1,\dots,N_4,T}$, then for each $|n_4| \sim N_4$, $|\mu| < 3(N^0)^2$, we have

$$I_2(n_4,\mu) = \Bigg(\sum_{\substack{n\neq n'\\ |n|,|n'|\sim N}} \big|F_{n,n',n_4,\mu}(\omega)\big|^2\Bigg)^{\frac{1}{2}}$$

$$< \Bigg(\sum_{\substack{n\neq n'\\ |n|,|n'|\sim N}} \|F_{n,n',n_4,\mu}\|^2_{L^2(\Omega)} \big(N^0\big)^{4\beta} T^{-4\beta}\Bigg)^{\frac{1}{2}}. \tag{6.39}$$

Next we compute

$$\|F_{n,n',n_4,\mu}\|^2_{L^2(\Omega)}$$

$$= \sum_{\substack{|n_3|,|m_3|\sim N_3\\ (n_1,n_2)\in *(n,\mu,n_3,n_4),\ (n_1',n_2')\in *(n',\mu,n_3,n_4)\\ (m_1,m_2)\in *(n,\mu,m_3,n_4),\ (m_1',m_2')\in *(n',\mu,m_3,n_4)}} \frac{(-n_1n_1')(-m_1m_1')}{\langle n_1\rangle\langle n_2\rangle\langle n_1'\rangle\langle n_2'\rangle\langle n_3\rangle^{1-2\delta}\langle m_1\rangle\langle m_2\rangle\langle m_1'\rangle\langle m_2'\rangle\langle m_3\rangle^{1-2\delta}}$$

$$\mathbb{E}\big(g_{n_1}(\omega)g_{n_2}(\omega)\overline{g_{n_1'}}(\omega)\overline{g_{n_2'}}(\omega)\overline{g_{m_1}}(\omega)\overline{g_{m_2}}(\omega)g_{m_1'}(\omega)g_{m_2'}(\omega)\big)$$

$$\lesssim \frac{1}{(N^0)^{6-4\delta}} \sum_{\substack{|n_3|,|m_3|\sim N_3\\ (n_1,n_2)\in *(n,\mu,n_3,n_4),\ (n_1',n_2')\in *(n',\mu,n_3,n_4)\\ (m_1,m_2)\in *(n,\mu,m_3,n_4),\ (m_1',m_2')\in *(n',\mu,m_3,n_4)}}$$

$$\big|\mathbb{E}\big(g_{n_1}(\omega)g_{n_2}(\omega)\overline{g_{n_1'}}(\omega)\overline{g_{n_2'}}(\omega)\overline{g_{m_1}}(\omega)\overline{g_{m_2}}(\omega)g_{m_1'}(\omega)g_{m_2'}(\omega)\big)\big|. \tag{6.40}$$

Then combining (6.39) and (6.40) we have

$$I_2(n_4,\mu) = \Bigg(\sum_{\substack{n\neq n'\\ |n|,|n'|\sim N}} \big|F_{n,n',n_4,\mu}(\omega)\big|^2\Bigg)^{\frac{1}{2}}$$

$$< \frac{T^{-2\beta}}{(N^0)^{3-2\delta-2\beta}} \Bigg(\sum_{\substack{n\neq n',\ |n|,|n'|\sim N,\ |n_3|,|m_3|\sim N_3\\ (n_1,n_2)\in *(n,\mu,n_3,n_4)\ (n_1',n_2')\in *(n',\mu,n_3,n_4)\\ (m_1,n_2)\in *(n,\mu,m_3,n_4)\ (m_1',m_2')\in *(n',\mu,m_3,n_4)}}$$

$$\big|\mathbb{E}\big(g_{n_1}(\omega)g_{n_2}(\omega)\overline{g_{n_1'}}(\omega)\overline{g_{n_2'}}(\omega)g_{m_1}(\omega)g_{m_2}(\omega)\overline{g_{m_1'}}(\omega)\overline{g_{m_2'}}(\omega)\big)\big|\Bigg)^{\frac{1}{2}}$$

$$< \frac{T^{-2\beta}}{(N^0)^{\frac{3}{2}-2\delta-2\beta}}, \tag{6.41}$$

by the following lemma.

Lemma 6.10. *Let*

$$S(n_4,\mu):=\Big\{\big(n,n_1,n_2,n_3,n',n_1',n_2',m_1,m_2,m_3,m_1',m_2'\big)\ \Big|\ n\neq n',\ |n|,\big|n'\big|\sim N,\ |n_3|,|m_3|\sim N_3,\\ (n_1,n_2)\in *(n,\mu,n_3,n_4),\ \big(n_1',n_2'\big)\in *\big(n',\mu,n_3,n_4\big),\\ (m_1,m_2)\in *(n,\mu,m_3,n_4),\ \big(m_1',m_2'\big)\in *\big(n',\mu,m_3,n_4\big),\\ \text{and } \mathbb{E}\big(g_{n_1}(\omega)g_{n_2}(\omega)\overline{g_{n_1'}}(\omega)\overline{g_{n_2'}}(\omega)\overline{g_{m_1}}(\omega)\overline{g_{m_2}}(\omega)g_{m_1'}(\omega)g_{m_2'}(\omega)\big)\neq 0\Big\}.$$

Then $\#\{S(n_4,\mu)\}<(N^0)^3$.

The proof of Lemma 6.10 can be found in Appendix A. Using (6.41) (which was established with Lemma 6.10), if $\omega\in\Omega_{N,N_1,\dots,N_4,T}$, we can estimate the contribution to (6.34) coming from $I_2(n_4,\mu)$ by

$$\begin{aligned}&\frac{T^{-\beta}N^{\frac12+\delta}}{(N^0)^{\frac34-5\delta-\beta}}\prod_{j=3}^4\|u_j\|_{X_T^{\frac12-\delta,\frac12-\delta}}\Big(\sum_{|n_4|\sim N_4}\frac{1}{\langle n_4\rangle^{1-2\delta}}\Big)^{\frac12}\\ &\lesssim\frac{T^{-\beta}N^{\frac12+\delta}}{(N^0)^{\frac34-6\delta-\beta-\gamma}}\prod_{j=3}^4\|u_j\|_{X_T^{\frac12-\delta,\frac12-\delta}}\Big(\sum_{|n_4|\sim N_4}\frac{1}{\langle n_4\rangle^{1+2\gamma}}\Big)^{\frac12}\\ &\lesssim\frac{T^{-\beta}}{(NN_1\cdots N_4)^{\alpha}}(N_1N_2)^{-\beta}\|u_3\|_{X_T^{\frac12-\delta,\frac12-\delta}}\|u_4\|_{X_T^{\frac12-\delta,\frac12-\delta}}.\end{aligned}\tag{6.42}$$

Combining (6.34), (6.37) and (6.42), if $\omega\in\widetilde{\Omega}_T\cap\Omega_{N,N_1,\dots,N_4,T}$, then the estimate (6.22) holds true.

It is straightforward to check that the crucial inequalities in lines (6.36) and (6.41) remain true (using (6.20) and (6.21)) under permutations of the roles of (n_1,n_2,n_3) in the preceding analysis. The analysis of Case 2.b.i is complete.

- **CASE 2.b.ii.** One of u_i of type (I), $i=1,2,3$, others type (II).

We will begin by assuming u_1 is type (I), and u_2,u_3 are type (II). We will discuss modifications for other possibilities afterwards. In this case we establish the estimate

$$\begin{aligned}&\big\|\mathcal{N}_{-1}|_{2.\mathrm{b.ii}}(u_1,u_2,u_3,u_4)\big\|_{X_T^{\frac12+\delta,-\frac12+\delta}}\\ &\lesssim T^{-\beta}\frac{1}{(NN_1\cdots N_4)^{\alpha}N_1^{\beta}}\|u_2\|_{X_T^{\frac12+\delta,\frac12-\delta}}\|u_3\|_{X_T^{\frac12+\delta,\frac12-\delta}}\|u_4\|_{X_T^{\frac12-\delta,\frac12-\delta}}.\end{aligned}\tag{6.43}$$

With the condition (6.21) and $N_2\ge N_3\ge N_4$, we have $N_2\sim N_3\sim N^0$, and for $\gamma,\alpha,\beta\ll\delta$ this gives

$$\begin{aligned}\frac{|n|^{\frac12+\delta}|n_1|}{|n_1|^{\frac12-\gamma}|n_2|^{\frac12+\delta-\gamma}|n_3|^{\frac12+\delta-\gamma}}&\lesssim\frac{|n|^{\frac12+\delta}|n_1|^{\frac12+\gamma}}{|n_2|^{\frac12+\delta-\gamma}|n_3|^{\frac12+\delta-\gamma}}\lesssim\frac{1}{(N^0)^{\delta-3\gamma}}\\ &\lesssim\frac{1}{|n|^{\gamma}}\frac{1}{(NN_1\cdots N_4)^{\alpha}(N_1)^{\beta}}.\end{aligned}\tag{6.44}$$

Using (6.44), (6.43) follows from

$$\Big\|\Big(\sum_{|n_1|\sim N_1}\frac{|g_{n_1}(\omega)|e^{in_1x+in_1^3t}}{|n_1|^{\frac12+\gamma}}\Big)f_2f_3u_4\Big\|_{X^{-\gamma,-\frac12+\delta}}\lesssim T^{-\beta}\|f_2\|_{X^{\gamma,\frac12-\delta}}\|f_3\|_{X^{\gamma,\frac12-\delta}}\|u_4\|_{X^{\frac12-\delta,\frac12-\delta}}.\tag{6.45}$$

To establish (6.45), notice that by Lemma 2.1 and Lemma 6.4, if $\omega\in\widetilde{\Omega}_T$, then

$$\Big\|\sum_{|n_1|\sim N_1}\frac{|g_{n_1}(\omega)|e^{in_1x+in_1^3t}}{|n_1|^{\frac12+\gamma}}\Big\|_{X^{\frac{\gamma}{2},\frac12-\delta}}\lesssim T^{-\beta}\Big(\sum_{|n_1|\sim N_1}\frac{1}{|n_1|^{1+\gamma-2\beta}}\Big)^{\frac12}\lesssim T^{-\beta},\tag{6.46}$$

by taking $\beta=\beta(\gamma)$ sufficiently small. Then using duality, Hölder's inequality, (2.12) and (6.46),

$$\begin{aligned}
&\left|\int v\cdot\Big(\sum_{|n_1|\sim N_1}\frac{|g_{n_1}(\omega)|e^{in_1x+in_1^3t}}{|n_1|^{\frac12+\gamma}}\Big)f_2f_3u_4\,dxdt\right|\\
&\quad\le \|v\|_{L^5_{x,t}}\Big\|\sum_{|n_1|\sim N_1}\frac{|g_{n_1}(\omega)|e^{in_1x+in_1^3t}}{|n_1|^{\frac12+\gamma}}\Big\|_{L^5_{x,t}}\prod_{j=2,3}\|f_j\|_{L^5_{x,t}}\|u_4\|_{L^5_{x,t}}\\
&\quad\lesssim \|v\|_{X^{\frac{\gamma}{2},\frac12-\delta}}\Big\|\sum_{|n_1|\sim N_1}\frac{|g_{n_1}(\omega)|e^{in_1x+in_1^3t}}{|n_1|^{\frac12+\gamma}}\Big\|_{X^{\frac{\gamma}{2},\frac12-\delta}}\prod_{j=2,3}\|f_j\|_{X^{\frac{\gamma}{2},\frac12-\delta}}\|u_4\|_{X^{\frac{\gamma}{2},\frac12-\delta}}\\
&\quad\lesssim T^{-\beta}\|v\|_{X^{\gamma,\frac12-\delta}}\|f_2\|_{X^{\gamma,\frac12-\delta}}\|f_3\|_{X^{\gamma,\frac12-\delta}}\|u_4\|_{X^{\frac12-\delta,\frac12-\delta}},
\end{aligned}$$

and (6.45) holds for $\omega\in\widetilde{\Omega}_T$.

It is easy to verify that the crucial inequality, (6.44), remains true (by (6.21)) if we permute the roles of (n_1,n_2,n_3) in the preceding analysis. The analysis of Case 2.b.ii is complete.

- **CASE 2.b.iii.** u_1,u_2,u_3 all type (II).

In this subcase we establish the deterministic estimate

$$\|\mathcal{N}_{-1}|_{2.\mathrm{b.iii}}(u_1,u_2,u_3,u_4)\|_{X_T^{\frac12+\delta,-\frac12+\delta}}\lesssim\frac{1}{(NN_1\cdots N_4)^{\alpha}}\prod_{j=1}^{3}\|u_j\|_{X_T^{\frac12+\delta,\frac12-\delta}}\|u_4\|_{X_T^{\frac12-\delta,\frac12-\delta}}. \tag{6.47}$$

Using (6.21) we find

$$\frac{|n|^{\frac12+\delta}|n_1|}{|n_1|^{\frac12+\delta-\gamma}|n_2|^{\frac12+\delta-\gamma}|n_3|^{\frac12+\delta-\gamma}}\lesssim\frac{1}{(N^0)^{2\delta-3\gamma}}\lesssim\frac{1}{|n|^{\gamma}(NN_1\cdots N_4)^{\alpha}}.$$

Then (6.47) follows from

$$\|f_1f_2f_3u_4\|_{X^{-\gamma,-\frac12+\delta}}\lesssim\prod_{j=1}^{3}\|f_j\|_{X^{\gamma,\frac12-\delta}}\|u_4\|_{X^{\frac12-\delta,\frac12-\delta}}.$$

By duality, the last estimate is equivalent to

$$\left|\int v\cdot f_1f_2f_3u_4\,dxdt\right|\lesssim\|v\|_{X^{\gamma,\frac12-\delta}}\prod_{j=1}^{3}\|f_j\|_{X^{\gamma,\frac12-\delta}}\|u_4\|_{X^{\frac12-\delta,\frac12-\delta}}. \tag{6.48}$$

We obtain (6.48) with Hölder's inequality and (2.12)

$$\begin{aligned}
\left|\int v\cdot f_1f_2f_3u_4\,dxdt\right|&\le\|v\|_{L^5_{x,t}}\prod_{j=1}^{3}\|f_j\|_{L^5_{x,t}}\|u_4\|_{L^5_{x,t}}\\
&\lesssim\|v\|_{X^{\gamma,\frac12-\delta}}\prod_{j=1}^{3}\|f_j\|_{X^{\gamma,\frac12-\delta}}\|u_4\|_{X^{\frac12-\delta,\frac12-\delta}}.
\end{aligned}$$

This concludes the justification of (6.47), and case 2.b. is complete.

- **CASE 2.c.** $N^3\ll N^0$ and $N^2N^3N^4\gtrsim N^0N^1|n^0+n^1|$.

We perform a type (I)–type (II) decomposition in each factor. Observe that the assumptions of this case provide the additional condition

$$N^3N^4\gtrsim N^0, \tag{6.49}$$

otherwise we would find $(N^0)^2 \lesssim N^0 N^1 |n^0+n^1| \lesssim N^2 N^3 N^4 \le N^0 N^3 N^4 \ll (N^0)^2$, a contradiction. In this region, we also have

$$N_2 N_3 N_4 \gtrsim N^2 N^3 N^4 \gtrsim N^0 N^1 |n^0+n^1| \sim (N^0)^2 |n^0+n^1|. \tag{6.50}$$

Then we find, by (6.50),

$$\frac{|n|^{\frac{1}{2}+\delta}|n_1|}{|n_1|^{\frac{1}{2}-2\delta}|n_2 n_3 n_4|^{\frac{1}{2}-2\delta}} \lesssim \frac{(N^0)^{7\delta}}{|n^0+n^1|^{\frac{1}{2}-2\delta}} \lesssim \frac{1}{(N^0)^{\delta}}, \tag{6.51}$$

unless $|n^0+n^1| \ll (N^0)^{\frac{16\delta}{1-4\delta}}$. If (6.51) holds, we can proceed with (a modification of) the method used in case 2.b.iii to establish

$$\left\|\mathcal{N}_{-1}|_{2.c.}(u_1,u_2,u_3,u_4)\right\|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}+\delta}} \lesssim \frac{1}{(NN_1\cdots N_4)^{\alpha}} \prod_{j=1}^{4} \|u_j\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}.$$

We therefore assume that

$$\left|n^0+n^1\right| \ll \left(N^0\right)^{\frac{16\delta}{1-4\delta}}, \tag{6.52}$$

for the remainder of Case 2.c.

- **CASE 2.c.i.** u_1 type (I) and two of u_2,u_3,u_4 type (I). That is, u_1,u_2,u_3,u_4 of types (I)(I)(I)(I), (I)(I)(I)(II), (I)(I)(II)(I) and (I)(II)(I)(I).

Let us assume that u_1, u_2 and u_3 are all type (I). We will discuss the other possibilities afterwards. In this case we establish

$$\begin{aligned}&\left\|\mathcal{N}_{-1}|_{2.c.i}(u_1,u_2,u_3,u_4)\right\|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}+\delta}} \\ &\quad \lesssim T^{-3\beta} \frac{1}{(NN_1\cdots N_4)^{\alpha}} \frac{1}{(N_1N_2N_3)^{\beta}} \|u_4\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}.\end{aligned} \tag{6.53}$$

Using the representation (6.24) for u_4, we apply the Minkowski inequality in λ_4 to find

$$\begin{aligned}&\left\|\mathcal{N}_{-1}|_{2.c.i}(u_1,u_2,u_3,u_4)\right\|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}+\delta}} \\ &\quad \lesssim \|u_4\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \left(N^0\right)^{\delta} \\ &\quad\quad \cdot \sup_{\lambda_4,\mu \ll (N^0)^2} \left| \sum_{|n|\sim N} |n|^{1+2\delta} \left| \sum_{*(n,\mu+\lambda_4)\cap \text{Case 2.c}} (in_1) \frac{g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle} a_{\lambda_4}(n_4)\right|^2\right|^{\frac{1}{2}} \\ &\quad \lesssim \|u_4\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \left(N^0\right)^{\delta} N^{\frac{1}{2}+\delta} \\ &\quad\quad \cdot \sup_{\mu \ll (N^0)^2} \left| \sum_{|n|\sim N} \left| \sum_{*(n,\mu)\cap \text{Case 2.c}} (in_1) \frac{g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle} a_{n_4}\right|^2\right|^{\frac{1}{2}},\end{aligned} \tag{6.54}$$

where $\sum_{n_4} |n_4|^{1-2\delta}|a_{n_4}|^2 = 1$. We have dropped the dependence on λ_4 in the previous expression; this is justified a posteriori by obtaining estimates which are uniform in λ_4. For each fixed μ, we consider

$$\sum_{|n|\sim N} \left| \sum_{*(n,\mu)\cap \text{Case 2.c}} (in_1)\, \frac{g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)}{\langle n_1\rangle\langle n_2\rangle} a_{n_4}\right|^2 = \sum_{|n|\sim N} \left| \sum_{n_4} \sigma^{\mu}_{n,n_4} |n_4|^{\frac{1}{2}-\delta} a_{n_4}\right|^2 \tag{6.55}$$

where σ^{μ}_{n,n_4} is the (n,n_4) entry of a matrix σ_μ (for μ fixed) with columns indexed by $|n_4| \sim N_4$, and rows indexed by $|n| \sim N$. That is, the entries of this matrix are given by

$$\sigma^{\mu}_{n,n_4} = \sum_{*(n,n_4,\mu)\cap \text{Case 2.c}} \frac{in_1 g_{n_1}(\omega) g_{n_2}(\omega) g_{n_3}(\omega)}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle |n_4|^{\frac{1}{2}-\delta}}.$$

Then by Lemma 6.7

$$\begin{aligned}(6.55) &\lesssim \left\|\left(\sigma^{\mu}_{n,n_4}\right)^* \sigma^{\mu}_{n,n_4}\right\| = \left\|\sigma^{\mu}_{n,n_4}\left(\sigma^{\mu}_{n,n_4}\right)^*\right\| \\ &\le \sup_{|n|\sim N} \sum_{n_4} \left|\sigma^{\mu}_{n,n_4}\right|^2 + \left(\sum_{n\neq n'} \left|\sum_{n_4} \sigma^{\mu}_{n,n_4}\overline{\sigma^{\mu}_{n',n_4}}\right|^2\right)^{\frac{1}{2}} \\ &= I_1(\mu) + I_2(\mu). \end{aligned} \tag{6.56}$$

To recap we have

$$\left\|\mathcal{N}_{-1}|_{2.c.}(u_1,u_2,u_3,u_4)\right\|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}+\delta}} \lesssim \|u_4\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \left(N^0\right)^{\delta} N^{\frac{1}{2}+\delta} \cdot \sup_{|\mu|<(N^0)^2} \left(I_1(\mu)+I_2(\mu)\right)^{\frac{1}{2}}. \tag{6.57}$$

To estimate $I_1(\mu) = \sup_{|n|\sim N}\sum_{n_4}|\sigma^{\mu}_{n,n_4}|^2$, we consider $F_{n,n_4,\mu}(\omega) := \sigma^{\mu}_{n,n_4}(\omega)$, then by Lemma 6.5,

$$\|F_{n,n_4,\mu}\|_{L^p(\Omega)} \le p^{\frac{3}{2}} \|F_{n,n_4,\mu}\|_{L^2(\Omega)},$$

for each $2 < p < \infty$. Applying Lemma 6.6 it follows that

$$P\left(\left|F_{n,n_4,\mu}(\omega)\right| \ge \lambda\right) \le e^{-c\|F_{n,n_4,\mu}\|_{L^2(\Omega)}^{-\frac{2}{3}}\lambda^{\frac{2}{3}}}.$$

Taking $\lambda = \|F_{n,n_4,\mu}\|_{L^2(\Omega)}(N^0)^{\frac{3\beta}{2}}T^{-\frac{3\beta}{2}}$, we have

$$P\left(\left|F_{n,n_4,\mu}(\omega)\right| \ge \|F_{n,n_4,\mu}\|_{L^2(\Omega)}\left(N^0\right)^{\frac{3\beta}{2}}T^{-\frac{3\beta}{2}}\right) \le e^{-c\frac{(N^0)^{\beta}}{T^{\beta}}}.$$

Then letting $\Omega_{n,N_1,N_2,N_3,n_4,\mu,T} := \{|F_{n,n_4,\mu}(\omega)| \ge \|F_{n,n_4,\mu}\|_{L^2(\Omega)}(N^0)^{\frac{3\beta}{2}}T^{-\frac{3\beta}{2}}\}$ and

$$\Omega_{N,N_1,N_2,N_3,N_4,T} := \bigcap_{|n|\sim N,|n_4|\sim N_4,|\mu|<(N^0)^2} \Omega_{n,N_1,N_2,N_3,n_4,\mu,T},$$

we have

$$P\left(\Omega^c_{N,N_1,N_2,N_3,N_4,T}\right) \le \sum_{|n|\sim N,|n_4|\sim N_4,|\mu|<(N^0)^2} e^{-c\frac{(N^0)^{\beta}}{T^{\beta}}} \lesssim \left(N^0\right)^4 e^{-c\frac{(N^0)^{\beta}}{T^{\beta}}} \lesssim \left(N^0\right)^{0-} e^{-\frac{c'}{T^{\beta}}}.$$

Then for each $|\mu| \ll (N^0)^2$, if $\omega \in \Omega_{N,N_1,N_2,N_3,N_4,T}$,

$$I_1(\mu) = \sup_{|n|\sim N} \sum_{|n_4|\sim N_4} \left|F_{n,n_4,\mu}(\omega)\right|^2 \lesssim \left(N^0\right)^{3\beta} T^{-3\beta} \sup_{|n|\sim N} \sum_{|n_4|\sim N_4} \|F_{n,n_4,\mu}\|^2_{L^2(\Omega)}. \tag{6.58}$$

We compute that

$$\begin{aligned}\|F_{n,n_4,\mu}\|^2_{L^2(\Omega)} &= \mathbb{E}\left(\left|\sum_{(n_1,n_2,n_3)\in *(n,n_4,\mu)\cap \text{case 2.c.}} \frac{(in_1)g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)}{\langle n_1\rangle\langle n_1\rangle\langle n_1\rangle|n_4|^{\frac{1}{2}-\delta}}\right|^2\right) \\ &\lesssim \sum_{\substack{(n_1,n_2,n_3)\in *(n,n_4,\mu)\cap \text{case 2.c.} \\ (m_1,m_2,m_3)\in *(n,n_4,\mu)\cap \text{case 2.c.}}} \frac{1}{(N_2N_3)^2 N_4^{1-2\delta}} \\ &\quad \cdot \left|\mathbb{E}\left(g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)\overline{g_{m_1}}(\omega)\overline{g_{m_2}}(\omega)\overline{g_{m_3}}(\omega)\right)\right|. \end{aligned} \tag{6.59}$$

To bound this sum we use the following lemma.

Lemma 6.11. *Let*

$$S(n,\mu) := \big\{(n_1,n_2,n_3,n_4,m_1,m_2,m_3)\ \big|\ |n_4|\sim N_4,\\ (n_1,n_2,n_3)\in *(n,n_4,\mu),\ (m_1,m_2,m_3)\in *(n,n_4,\mu),\\ \mathbb{E}\big(g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)\overline{g_{m_1}}(\omega)\overline{g_{m_2}}(\omega)\overline{g_{m_3}}(\omega)\big)\neq 0\big\}.$$

Then $\#\{S(n,\mu)\} < \min(N_1N_2, N_1N_3, N_2N_3)$.

The proof of Lemma 6.11 can be found in Appendix A. By combining (6.55)–(6.59) and Lemma 6.11, the contribution to (6.57) from $I_1(\mu)$ is bounded by

$$\begin{aligned}\|u_4\|_{X_T^{\frac12-\delta,\frac12-\delta}}(N^0)^\delta N^{\frac12+\delta}\sup_{|\mu|<(N^0)^2}\big(I_1(\mu)\big)^{\frac12} &\lesssim \frac{T^{-\frac{3\beta}{2}}(N^0)^{\delta+\frac{\beta}{2}}N^{\frac12+\delta}}{(N_2N_3)^{\frac12}N_4^{\frac12-\delta}}\|u_4\|_{X_T^{\frac12-\delta,\frac12-\delta}}\\ &\lesssim \frac{T^{-\frac{3\beta}{2}}(N^0)^{2\delta+\frac{\beta}{2}}N^{\frac12+\delta}}{N^0}\|u_4\|_{X_T^{\frac12-\delta,\frac12-\delta}}\\ &\lesssim \frac{T^{-\frac{3\beta}{2}}}{(NN_1\cdots N_4)^\alpha}(N_1N_2N_3)^{-\beta}\|u_4\|_{X_T^{\frac12-\delta,\frac12-\delta}}.\end{aligned}$$

It remains to control the contribution to (6.54) from $I_2(\mu)$. Consider

$$I_2(\mu) = \Bigg(\sum_{n\neq n'}\Bigg|\sum_{|n_4|\sim N_4}\sigma^\mu_{n,n_4}\overline{\sigma^\mu_{n',n_4}}\Bigg|^2\Bigg)^{\frac12} = \Bigg(\sum_{n\neq n'}\big|G_{n,n',\mu}(\omega)\big|^2\Bigg)^{\frac12}, \tag{6.60}$$

where, for each fixed n, n', μ, we have taken

$$G_{n,n',\mu}(\omega) := \sum_{\substack{|n_4|\sim N_4\\ (n_1,n_2,n_3)\in*(n,n_4,\mu)\cap\text{Case 2.c}\\ (n_1',n_2',n_3')\in*(n',n_4,\mu)\cap\text{Case 2.c}}}\frac{-n_1n_1' g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)\overline{g_{n_1'}}(\omega)\overline{g_{n_2'}}(\omega)\overline{g_{n_3'}}(\omega)}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle\langle n_1'\rangle\langle n_2'\rangle\langle n_3'\rangle\langle n_4\rangle^{1-2\delta}}.$$

By Lemma 6.5 we have

$$\|G_{n,n',\mu}\|_{L^p(\Omega)} \le p^3\|G_{n,n',\mu}\|_{L^2(\Omega)},$$

for each $2<p<\infty$. With Lemma 6.6 it follows that

$$P\big(\big|G_{n,n',\mu}(\omega)\big|\ge\lambda\big)\le e^{-c\|G_{n,n',\mu}\|_{L^2(\Omega)}^{-\frac13}\lambda^{\frac13}}.$$

Taking $\lambda = \|G_{n,n',\mu}\|_{L^2(\Omega)}(N^0)^{3\beta}T^{-3\beta}$, we have

$$P\big(\big|G_{n,n',\mu}(\omega)\big|\ge \|G_{n,n',\mu}\|_{L^2(\Omega)}\big(N^0\big)^{3\beta}T^{-3\beta}\big)\le e^{-c\frac{(N^0)^\beta}{T^\beta}}.$$

Then letting $\Omega_{n,n',N_1,N_2,N_3,N_4,\mu,T} := \{|G_{n,n',\mu}(\omega)|\ge\|G_{n,n',\mu}\|_{L^2(\Omega)}(N^0)^{3\beta}T^{-3\beta}\}$ and

$$\Omega_{N,N_1,N_2,N_3,N_4,T} := \bigcap_{|n|,|n'|\sim N,|\mu|<(N^0)^2}\Omega_{n,n',N_1,N_2,N_3,N_4,\mu,T},$$

we have

$$P\big(\Omega^c_{N,N_1,N_2,N_3,N_4,T}\big)\le\sum_{|n|,|n'|\sim N,|\mu|<(N^0)^2}e^{-c\frac{(N^0)^\beta}{T^\beta}}\lesssim\big(N^0\big)^4e^{-c\frac{(N^0)^\beta}{T^\beta}}\lesssim\big(N^0\big)^{0-}e^{-\frac{c'}{T^\beta}},$$

for some $c'>0$. Then for each $|\mu|\ll(N^0)^2$, if $\omega\in\Omega_{N,N_1,N_2,N_3,N_4,T}$,

$$I_2(\mu) \lesssim \Big(\sum_{n\neq n'} \big|G_{n,n',\mu}(\omega)\big|^2\Big)^{\frac{1}{2}} \leq T^{-3\beta}\big(N^0\big)^{3\beta}\Big(\sum_{n\neq n'} \|G_{n,n',\mu}\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}.$$

We compute that

$$\sum_{n\neq n'} \|G_{n,n',\mu}\|^2_{L^2(\Omega)}$$
$$= \sum_{n\neq n'} \mathbb{E}\Bigg(\Bigg|\sum_{\substack{|n_4|\sim N_4\\ (n_1,n_2,n_3)\in *(n,n_4,\mu)\cap\text{Case 2.c}\\ (n_1',n_2',n_3')\in *(n',n_4,\mu)\cap\text{Case 2.c}}} \frac{-n_1n_1' g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)\overline{g_{n_1'}}(\omega)\overline{g_{n_2'}}(\omega)\overline{g_{n_3'}}(\omega)}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle\langle n_1'\rangle\langle n_2'\rangle\langle n_3'\rangle\langle n_4\rangle^{1-2\delta}}\Bigg|^2\Bigg)$$
$$\lesssim \sum_{\substack{|n|,|n'|\sim N,|n_4|,|m_4|\sim N_4\\ (n_1,n_2,n_3)\in *(n,n_4,\mu)\\ (n_1',n_2',n_3')\in *(n',n_4,\mu)\\ (m_1,m_2,m_3)\in *(n,m_4,\mu)\\ (m_1',m_2',m_3')\in *(n',m_4,\mu)}} \frac{1}{(N_2N_3)^4N_4^{2-4\delta}}\mathbb{E}\big(g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)\overline{g_{n_1'}}(\omega)\overline{g_{n_2'}}(\omega)\overline{g_{n_3'}}(\omega)$$
$$\cdot\,\overline{g_{m_1}}(\omega)\overline{g_{m_2}}(\omega)\overline{g_{m_3}}(\omega)g_{m_1'}(\omega)g_{m_2'}(\omega)g_{m_3'}(\omega)\big). \tag{6.61}$$

Using (6.21) we have

$$\frac{1}{(N_2N_3)^4N_4^{2-4\delta}} = \frac{N_4^{2+4\delta}}{(N_2N_3N_4)^4} \lesssim \frac{N_4^{2+4\delta}}{(N^0)^8}. \lesssim \frac{1}{(N^0)^{6-4\delta}}. \tag{6.62}$$

To further control (6.61) we establish the following lemma.

Lemma 6.12. *Let*

$$\begin{aligned}S(\mu) := \big\{&\big(n,n',n_1,n_2,n_3,n_1',n_2',n_3',m_1,m_2,m_3,m_1',m_2',m_3'\big)\ \big|\\ &|n|,\big|n'\big|\sim N,\ |n_4|,|m_4|\sim N_4,\ (n_1,n_2,n_3)\in *(n,n_4,\mu),\\ &\big(n_1',n_2',n_3'\big)\in *\big(n',n_4,\mu\big),\ (m_1,m_2,m_3)\in *(n,m_4,\mu),\\ &\big(m_1',m_2',m_3'\big)\in *\big(n',m_4,\mu\big),\ \text{with } \big|n^0+n^1\big| \ll \big(N^0\big)^{\frac{16\delta}{1-4\delta}} \text{ in all quintuples},\\ &\mathbb{E}\big(g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)\overline{g_{n_1'}}(\omega)\overline{g_{n_2'}}(\omega)\overline{g_{n_3'}}(\omega)\overline{g_{m_1}}(\omega)\overline{g_{m_2}}(\omega)\overline{g_{m_3}}(\omega)g_{m_1'}(\omega)g_{m_2'}(\omega)g_{m_3'}(\omega)\big)\neq 0\big\}.\end{aligned}$$

It holds that $\#\{S(\mu)\} \lesssim (N^0)^{3+\frac{32\delta}{1-4\delta}}$.

The proof of Lemma 6.12 can be found in Appendix A. By combining (6.62) with Lemma 6.12 we have

$$(6.61) \lesssim \frac{1}{(N^0)^{3-4\delta-\frac{32\delta}{1-4\delta}}}. \tag{6.63}$$

Then from (6.63) we can estimate the contribution to (6.57) coming from $I_2(\mu)$ by

$$\begin{aligned}\|u_4\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}\big(N^0\big)^{\delta}N^{\frac{1}{2}+\delta}\sup_{|\mu|<(N^0)^2}\big(I_2(\mu)\big)^{\frac{1}{2}} &\lesssim \|u_4\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}\frac{T^{-\frac{3\beta}{2}}(N^0)^{\delta+\frac{3\beta}{2}}N^{\frac{1}{2}+\delta}}{(N^0)^{\frac{3}{4}-\delta-\frac{8\delta}{1-4\delta}}}\\ &\lesssim \frac{T^{-\frac{3\beta}{2}}}{(NN_1\cdots N_4)^{\alpha}}(N_1N_2N_3)^{-\beta}\|u_4\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}},\end{aligned}$$

for $\delta,\beta,\alpha>0$ sufficiently small. It is clear that the previous analysis applies upon permutation of the variables n_2, n_3 and n_4, as we did not use the ordering $N_2\geq N_3\geq N_4$ in this case (see Remark 5.4 in [40]). The analysis of Case 2.c.i is complete.

- **CASE 2.c.ii.** u_1 type (II), and u_2, u_3, u_4 type (I).

In this case we proceed precisely as in Case 2.c.ii, swapping the roles of n_1 and n_4. The analysis requires modification in the lines (6.59) and (6.61), where we need to include the factor $\frac{N_1^{1-2\delta}}{(N_2N_3N_4)^2}$ instead of $\frac{1}{(N_2N_3)^2N_4^{1-2\delta}}$. In order to estimate (6.59), by $N_2 \geq N_3 \geq N_4$ and (6.50), we find

$$\frac{1}{N_2^2N_3N_4} \leq \frac{1}{(N_2N_3N_4)^{\frac{4}{3}}} \lesssim \frac{1}{(N^0)^{\frac{8}{3}}},$$

and we have

$$\frac{(N^0)^{2\delta}N^{1+2\delta}N_1^{1-2\delta}}{N_2^2N_3N_4} \lesssim \frac{1}{(N^0)^{\frac{2}{3}-2\delta}}.$$

By combining this inequality with Lemma 6.12 (with n_1 and n_4 swapped) we can estimate the contribution to (6.57) from $I_1(\mu)$ as we did in Case 2.c.i.

In the modification of (6.61), we consider

$$\frac{(N_1)^{2-4\delta}}{(N_2N_3N_4)^4} \lesssim \frac{(N_1)^{2-4\delta}}{(N^0)^8} \lesssim \frac{1}{(N^0)^{6+4\delta}},$$

which is precisely the conclusion we reached in Case 2.c.i. These are the only modifications required, and the analysis of Case 2.c.ii is complete.

- **CASE 2.c.iii:** Two type (I), two type (II).

We will consider four subcases, and begin with a description each of them.

- **CASE 2.c.iii.a:** u_1 type (II) and u_4 type (I).
  That is, u_1, u_2, u_3, u_4 types (II)(I)(II)(I) and (II)(II)(I)(I).
- **CASE 2.c.iii.b:** u_1, u_2, u_3, u_4 of types (II)(I)(I)(II).
- **CASE 2.c.iii.c:** u_1, u_2, u_3, u_4 of types (I)(II)(II)(I).
- **CASE 2.c.iii.d:** u_1 type (I) and u_4 type (II).
  That is, u_1, u_2, u_3, u_4 types (I)(II)(I)(II) and (I)(I)(II)(II).

We proceed with the analysis of each subcase.

- **CASE 2.c.iii.a:** u_1 type (II) and u_4 type (I). That is, u_1, u_2, u_3 and u_4 types (II)(I)(II)(I) and (II)(II)(I)(I).

Let us assume u_1, u_2, u_3 and u_4 are types (II)(II)(I)(I), respectively. It is easily verified (a posteriori) that the analysis of this subcase is symmetric with respect to the functions u_2 and u_3, and the preceding assumption holds without loss of generality.

In this case we exploit one more condition which restricts the size of N_4. Specifically, we notice that if $N_4 \geq (N^0)^{\frac{2}{3}+5\delta}$, then we have, using $N_2 \geq N_3 \geq N_4$,

$$\frac{|n|^{\frac{1}{2}+\delta}|n_1|}{|n_1|^{\frac{1}{2}-2\delta}|n_2n_3n_4|^{\frac{1}{2}-2\delta}} \lesssim \frac{(N^0)^{1+3\delta}}{(N^0)^{3(\frac{1}{2}-2\delta)(\frac{2}{3}+5\delta)}} \lesssim \frac{1}{(N^0)^{\frac{\delta}{2}-30\delta^2}}. \tag{6.64}$$

Once again, if (6.64) holds, we can proceed with (a modification of) the method used in Case 2.b.iii to establish

$$\|\mathcal{N}_{-1}(u_1,u_2,u_3,u_4)\|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}+\delta}} \lesssim \frac{1}{(NN_1\cdots N_4)^{\alpha}} \prod_{j=1}^{4} \|u_j\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}.$$

We therefore assume for the remainder of this proof that

$$N_4 \ll \left(N^0\right)^{\frac{2}{3}+5\delta}. \tag{6.65}$$

Using (6.50), we have

$$\begin{aligned}\frac{N^{\frac{1}{2}+\delta}N_1}{(N_1N_2)^{\frac{1}{2}+\frac{11\delta}{12}}(N_3N_4)^{\frac{1}{2}-\frac{\delta}{12}}} &\le \frac{N^{\frac{1}{2}+\delta}N_1(N_3N_4)^{\delta}}{(N_1N_2N_3N_4)^{\frac{1}{2}+\frac{11\delta}{12}}} \le \frac{(N^0)^{1+\frac{13\delta}{12}}N_4^{\delta}}{(N_2N_3N_4)^{\frac{1}{2}+\frac{11\delta}{12}}}\\ &\le \frac{(N^0)^{1+\frac{13\delta}{12}}(N^0)^{(\frac{2}{3}+5\delta)\delta}}{(N^0)^{1+\frac{11\delta}{6}}} \le \frac{1}{(N^0)^{\frac{\delta}{12}-5\delta^2}}.\end{aligned} \tag{6.66}$$

By using (6.66) and (a straightforward modification of) the methods of Case 2.b.iii we establish

$$\begin{aligned}&\left\|\mathcal{N}_{-1}|_{2.\text{c.iii.a}}(u_1,u_2,u_3,u_4)\right\|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}+\delta}}\\ &\quad\lesssim \frac{T^{-\beta}}{(NN_1\cdots N_4)^{\alpha}}(N_3N_4)^{-\beta}\|u_1\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}}\|u_2\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}},\end{aligned} \tag{6.67}$$

and the analysis of Case 2.c.iii.a is complete.

- **CASE 2.c.iii.b:** u_1 and u_4 type (II). That is, u_1, u_2, u_3 and u_4 types (II)(I)(I)(II).

In this case we can obtain a stronger restriction on N_4. More precisely, we have

$$\frac{|n|^{\frac{1}{2}+\delta}|n_1|}{|n_1|^{\frac{1}{2}+\delta-\gamma}|n_2|^{\frac{1}{2}-\gamma}|n_3|^{\frac{1}{2}-\gamma}|n_4|^{\frac{1}{2}+\delta-\gamma}} = \frac{|n|^{\frac{1}{2}+\delta}|n_1|^{\frac{1}{2}-\delta+\gamma}}{|n_2n_3n_4|^{\frac{1}{2}-\gamma}|n_4|^{\delta}} \lesssim \frac{(N^0)^{3\gamma}}{N_4^{\delta}} \lesssim \frac{1}{(N^0)^{\gamma}}, \tag{6.68}$$

unless $N_4^{\delta} \ll (N^0)^{4\gamma}$. If (6.68) holds, we can proceed with a straightforward modification of the method in Case 1.b.ii. Therefore, by taking $\gamma = \gamma(\delta) > 0$ sufficiently small, we may assume that

$$N_4 \ll \left(N^0\right)^{\delta}, \tag{6.69}$$

for the remainder of this case.

The analysis of this case closely follows the method of Case 2.b.i, with the roles of u_1 and u_3 swapped. Indeed, the analysis is identical until the line (6.34), where, due to the assumption $u_1, u_4 \in X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}$ (instead of $u_3, u_4 \in X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}$), we obtain

$$\begin{aligned}&\left\|\mathcal{N}_{-1}|_{2.\text{c.iii.b}}(u_1,u_2,u_3,u_4)\right\|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}+\delta}}\\ &\quad\lesssim N^{\frac{1}{2}+\delta}\prod_{j=1,4}\|u_j\|_{\frac{1}{2}+\delta,\frac{1}{2}-\delta,T}\cdot\sup_{|\mu|<3(N^0)^2}\left(\sum_{|n_4|\sim N_4}\frac{1}{|n_4|}\big(I_1(n_4,\mu)+I_2(n_4,\mu)\big)\right)^{\frac{1}{2}},\end{aligned} \tag{6.70}$$

where

$$I_1(n_4,\mu) = \sup_{|n|\sim N}\sum_{|n_1|\sim N_1}\left|\sigma_{n,n_1}^{n_4,\mu}\right|^2,$$

and

$$I_2(n_4,\mu) = \left(\sum_{\substack{n\neq n'\\ |n|,|n'|\sim N}}\left|\sum_{|n_1|\sim N_1}\sigma_{n,n_1}^{n_4,\mu}\overline{\sigma_{n',n_1}^{n_4,\mu}}\right|\right)^{\frac{1}{2}},$$

with

$$\sigma_{n,n_1}^{n_4,\mu} = \sum_{(n_2,n_3)\in *(n,n_1,n_4,\mu)} (in_1) \frac{g_{n_2}(\omega) g_{n_3}(\omega)}{|n_1|^{\frac{1}{2}+\delta} \langle n_2\rangle \langle n_3\rangle}.$$

Next when we estimate $I_1(n_4,\mu)$ as in line (6.36), we apply Lemma 6.4 to obtain, for $\omega \in \widetilde{\Omega}_T$:

$$\begin{aligned} I_1(n_4,\mu) &\le \sup_{|n|\sim N} \sum_{\substack{|n_1|\sim N_1 \\ (n_2,n_3)\in *(n,n_1,n_4,\mu)\cap \text{Case 2.c}}} \frac{|n_1|^2 |g_{n_2}(\omega)||g_{n_3}(\omega)|}{|n_1|^{1+2\delta}|n_2|^2|n_3|^2} \\ &\le T^{-\beta} \sup_{|n|\sim N} \sum_{(n_1,n_2,n_3)\in *(n,n_4,\mu)\cap \text{Case 2.c}} \frac{|n_1|^{1-2\delta}}{|n_2|^{2-\beta}|n_3|^{2-\beta}}. \end{aligned} \tag{6.71}$$

We will need the following lemma.

Lemma 6.13. *Let*

$$S(n,n_4,\mu) := \big\{(n_1,n_2,n_3) : (n_1,n_2,n_3) \in *(n,n_4,\mu) \text{ and (6.52) holds}\big\}.$$

Then we have $|S(n,n_4,\mu)| < (N^0)^{\frac{16\delta}{1-4\delta}}$.

The proof of Lemma 6.13 can be found in Appendix A. Combining (6.71) and Lemma 6.13 we have

$$\begin{aligned} I_1(n_4,\mu) &\le T^{-\beta} \frac{(N^0)^{\frac{16\delta}{1-4\delta}} N_1^{1-2\delta}}{N_2^{2-\beta} N_3^{2-\beta}} \le T^{-\beta} \frac{(N^0)^{\frac{16\delta}{1-4\delta}} N_1^{1-2\delta} N_4^{2-\beta}}{N_2^{2-\beta} N_3^{2-\beta} N_4^{2-\beta}} \\ &\le T^{-\beta} \frac{N_1^{1-2\delta} N_4^{2-\beta}}{(N^0)^{4-2\beta-\frac{16\delta}{1-4\delta}}} \lesssim T^{-\beta} \frac{N_1^{1-2\delta}(N^0)^{2\delta-\beta\delta}}{(N^0)^{4-2\beta-\frac{16\delta}{1-4\delta}}} \lesssim \frac{T^{-\beta}}{(N^0)^{3-(2+\delta)\beta-\frac{16\delta}{1-4\delta}}}. \end{aligned} \tag{6.72}$$

Notice that we have applied (6.50) and (6.69) in the previous lines. From here the estimates on the contribution to (6.70) from $I_1(n_4,\mu)$ proceed as in Case 2.b.i.

In the analysis of the contribution to (6.70) from $I_2(n_4,\mu)$, we have to modify our analysis once again. In particular, in the line of inequalities in (6.40), we need to obtain the same prefactor of $(N^0)^{-(6-)}$. This is done quite easily by following the approach used in (6.72) above. We find

$$\begin{aligned} \frac{n_1^2 m_1^2}{\langle n_1\rangle^{1+2\delta}\langle n_2\rangle\langle n_3\rangle\langle n_2'\rangle\langle n_3'\rangle\langle m_1\rangle^{1+2\delta}\langle m_2\rangle\langle m_3\rangle\langle m_2'\rangle\langle m_3'\rangle} &\lesssim \frac{N_1^{2-4\delta}N_4^4}{(N_2N_3N_4)^4} \\ &\lesssim \frac{N_1^{2-4\delta}(N^0)^{4\delta}}{(N^0)^8} \lesssim \frac{1}{(N^0)^6}, \end{aligned}$$

and from here the analysis proceeds as in Case 2.b.i. This completes the analysis of Case 2.c.iii.b.

- **CASE 2.c.iii.c:** u_1 and u_4 type (I). That is, u_1, u_2, u_3 and u_4 types (I)(II)(II)(I).

We find

$$\begin{aligned} \frac{|n|^{\frac{1}{2}+\delta}|n_1|}{|n_1|^{\frac{1}{2}-\gamma}|n_2n_3|^{\frac{1}{2}+\delta-\gamma}|n_4|^{\frac{1}{2}-\gamma}} &\lesssim \frac{|n|^{\frac{1}{2}+\delta}|n_1|^{\frac{1}{2}+\gamma}|n_4|^{\delta}}{|n_2n_3n_4|^{\frac{1}{2}+\delta-\gamma}} \\ &\lesssim \frac{|n|^{\frac{1}{2}+\delta}|n_1|^{\frac{1}{2}+\gamma}|n_4|^{\delta}}{(N^0)^{1+2\delta-2\gamma}} \lesssim \frac{{N_4}^{\delta}}{(N^0)^{\delta-2\gamma}} \lesssim \frac{1}{(N^0)^{\frac{\delta}{3}-3\gamma-5\delta^2}}. \end{aligned} \tag{6.73}$$

In the preceding inequalities, we have applied both (6.50) and (6.65). Using (6.73), we can proceed with a straightforward modification the method from Case 2.b.ii.

- **CASE 2.c.iii.d:** u_1 type (I) and u_4 type (II). That is, u_1, u_2, u_3 and u_4 types (I)(I)(II)(II) or (I)(II)(I)(II).

Let us consider the case (I)(I)(II)(II), and briefly describe the adaptation to (I)(II)(I)(II) throughout. Once again, the analysis of this case closely follows the method of Case 2.b.i. Indeed, the analysis is identical until we estimate $I_1(n_4,\mu)$ as in line (6.36), and apply Lemmas 6.4 and 6.13 to obtain, for $\omega \in \widetilde{\Omega}_T$:

$$\begin{aligned} I_1(n_4,\mu) &\le \sup_{|n|\sim N} \sum_{\substack{|n_1|\sim N_1 \\ (n_2,n_3)\in *(n,n_1,n_4,\mu)\cap \text{Case 2.c}}} \frac{|n_1|^2 |g_{n_1}(\omega)||g_{n_2}(\omega)|}{|n_1|^2|n_2|^2|n_3|^{1+2\delta}} \\ &\le T^{-\beta} \sup_{|n|\sim N} \sum_{(n_1,n_2,n_3)\in *(n,n_4,\mu)\cap \text{Case 2.c}} \frac{|n_1|^{\beta}}{|n_2|^{2-\beta}|n_3|^{1+2\delta}} \\ &\lesssim T^{-\beta} \frac{|N_1|^{\beta}(N^0)^{\frac{16\delta}{1-4\delta}}}{|N_2|^{2-\beta}|N_3|^{1+2\delta}} \lesssim T^{-\beta} \frac{(N^0)^{\beta+\frac{16\delta}{1-4\delta}} N_4^{1+2\delta}}{|N_2|^{1-\beta-2\delta}(N^0)^{2+4\delta}} \\ &\lesssim \frac{T^{-\beta}}{(N^0)^{\frac{4}{3}-\frac{7\delta}{3}-10\delta^2-\beta-\frac{16\delta}{1-4\delta}}}. \end{aligned} \tag{6.74}$$

In the previous lines, we have used (6.50) and (6.65). The inequality (6.74) is enough to estimate the contribution from $I(n_4,\mu)$ as in Case 1.b.i. Indeed, it is easily verified that, from the inequality (6.74), we only require a negative power of N^0 with magnitude greater than 1. Note that, by taking $\delta > \beta > 0$ sufficiently small, this is exactly what we have accomplished. Let us pause to remark that the analysis above is easily accomplished with types (I)(II)(I)(II) as well.

Before we estimate the contribution from $I_2(n_4,\mu)$, let us first observe that, in the case of types (I)(I)(II)(II), we can obtain a stronger restriction on the size of N_2. More precisely, we have

$$\begin{aligned} \frac{|n|^{\frac{1}{2}+\delta}|n_1|}{|n_1|^{\frac{1}{2}-\gamma}|n_2|^{\frac{1}{2}-\gamma}|n_3n_4|^{\frac{1}{2}+\delta-\gamma}} &\lesssim \frac{|n|^{\frac{1}{2}+\delta}|n_1|^{\frac{1}{2}+\gamma}|n_2|^{\delta}}{|n_2n_3n_4|^{\frac{1}{2}+\delta-\gamma}} \\ &\lesssim \frac{|n|^{\frac{1}{2}+\delta}|n_1|^{\frac{1}{2}+\gamma}|n_2|^{\delta}}{(N^0)^{1+2\delta-2\gamma}} \lesssim \frac{{N_2}^{\delta}}{(N^0)^{\delta-3\gamma}} \lesssim \frac{1}{(N^0)^{\frac{\delta}{5}-3\gamma}}, \end{aligned} \tag{6.75}$$

unless $N_2 \gtrsim (N^0)^{\frac{4}{5}}$. If (6.75) holds, we can proceed with a modification of the analysis in Case 2.b.ii. We will therefore assume, for the remainder of this case, that

$$N_2 \gtrsim (N^0)^{\frac{4}{5}}. \tag{6.76}$$

Turning to the contribution from $I_2(n_4,\mu)$, we proceed with the approach of Case 2.b.i until (6.40), where we find, using (6.76),

$$\begin{aligned} &\|F_{n,n',n_4,\mu}\|^2_{L^2(\Omega)} \\ &= \mathbb{E}\left(\left| \sum_{\substack{|n_3|\sim N_3 \\ (n_1,n_2)\in *(n,\mu,n_3,n_4) \\ (n_1',n_2')\in *(n',\mu,n_3,n_4)}} \frac{-n_1n_1' g_{n_1}(\omega)g_{n_2}(\omega)\overline{g_{n_1'}}(\omega)\overline{g_{n_2'}}(\omega)}{\langle n_1\rangle\langle n_2\rangle\langle n_1'\rangle\langle n_2'\rangle\langle n_3\rangle^{1+2\delta}}\right|^2\right) \\ &= \sum_{\substack{|n_3|,|m_3|\sim N_3 \\ (n_1,n_2)\in *(n,\mu,n_3,n_4),\ (n_1',n_2')\in *(n',\mu,n_3,n_4) \\ (m_1,m_2)\in *(n,\mu,m_3,n_4),\ (m_1',m_2')\in *(n',\mu,m_3,n_4)}} \frac{(-n_1n_1')(-m_1m_1')}{\langle n_1\rangle\langle n_2\rangle\langle n_1'\rangle\langle n_2'\rangle\langle n_3\rangle^{1+2\delta}\langle m_1\rangle\langle m_2\rangle\langle m_1'\rangle\langle m_2'\rangle\langle m_3\rangle^{1+2\delta}} \\ &\quad \mathbb{E}\left(g_{n_1}(\omega)g_{n_2}(\omega)\overline{g_{n_1'}}(\omega)\overline{g_{n_2'}}(\omega)\overline{g_{m_1}}(\omega)\overline{g_{m_2}}(\omega)g_{m_1'}(\omega)g_{m_2'}(\omega)\right) \\ &\lesssim \frac{1}{(N^0)^{\frac{16}{5}}N_3^{2+4\delta}} \sum_{\substack{|n_3|,|m_3|\sim N_3 \\ (n_1,n_2)\in *(n,\mu,n_3,n_4),\ (n_1',n_2')\in *(n',\mu,n_3,n_4) \\ (m_1,m_2)\in *(n,\mu,m_3,n_4),\ (m_1',m_2')\in *(n',\mu,m_3,n_4)}} \\ &\quad \left|\mathbb{E}\left(g_{n_1}(\omega)g_{n_2}(\omega)g_{n_1'}(\omega)g_{n_2'}(\omega)\overline{g_{m_1}}(\omega)\overline{g_{m_2}}(\omega)\overline{g_{m_1'}}(\omega)\overline{g_{m_2'}}(\omega)\right)\right|. \end{aligned} \tag{6.77}$$

Then combining (6.39) and (6.77), we have

$$\begin{aligned}
I_2(n_4,\mu) &= \Bigg(\sum_{\substack{n\neq n' \\ |n|,|n'|\sim N}} \big|F_{n,n',n_4,\mu}(\omega)\big|^2 \Bigg)^{\frac{1}{2}} \\
&< \frac{T^{-2\beta}}{(N^0)^{\frac{8}{5}} N_3^{1+2\delta}} \Bigg(\sum_{\substack{n\neq n',\ |n|,|n'|\sim N,\ |n_3|,|m_3|\sim N_3 \\ (n_1,n_2)\in *(n,\mu,n_3,n_4),\ (n_1',n_2')\in *(n',\mu,n_3,n_4) \\ (m_1,n_2)\in *(n,\mu,m_3,n_4),\ (m_1',m_2')\in *(n',\mu,m_3,n_4)}} \\
&\qquad \big|\mathbb{E}\big(g_{n_1}(\omega)g_{n_2}(\omega)g_{n_1'}(\omega)g_{n_2'}(\omega)\overline{g_{m_1}}(\omega)\overline{g_{m_2}}(\omega)\overline{g_{m_1'}}(\omega)\overline{g_{m_2'}}(\omega)\big)\big| \Bigg)^{\frac{1}{2}} \\
&\lesssim \frac{T^{-2\beta}}{(N^0)^{\frac{8}{5}}} \sup_{|n_3|,|m_3|\sim N_3} \Bigg(\sum_{\substack{(n,n_1,n_2)\in *(\mu,n_3,n_4),\ (n',n_1',n_2')\in *(\mu,n_3,n_4) \\ (n,m_1,m_2)\in *(\mu,m_3,n_4),\ (n',m_1',m_2')\in *(\mu,m_3,n_4)}} \Bigg)^{\frac{1}{2}} \\
&\lesssim \frac{T^{-2\beta}}{(N^0)^{\frac{8}{5}-\frac{32\delta}{1-4\delta}}} \lesssim \frac{T^{-2\beta}}{(N^0)^{\frac{3}{2}}},
\end{aligned} \tag{6.78}$$

by taking $\delta > 0$ sufficiently small. Let us remark that, to obtain (6.78) above, we have applied Lemma 6.13 and (6.76). With (6.78), we have established an estimate superior to (6.41), and the remaining analysis of this case follows Case 2.b.i.

With the combination of types (I)(II)(I)(II), we can follow the same scheme to estimate the contribution from $I_2(n_4,\mu)$, but the roles of n_2 and n_3 are swapped (including (6.76), which in this case restricts the size of N_3). This completes the analysis of Case 2.c.iii.

- **CASE 2.c.iv:** At least 3 of u_1, u_2, u_3, u_4 of type (II).

We will consider three subcases.

- **CASE 2.c.iv.a:** u_1, u_2, u_3, u_4 of types (I)(II)(II)(II).

We use (6.50) to find

$$\frac{|n|^{\frac{1}{2}+\delta}|n_1|}{|n_1|^{\frac{1}{2}-\gamma}|n_2n_3n_4|^{\frac{1}{2}+\delta-\gamma}} \lesssim \frac{1}{(N^0)^{\delta-3\gamma}}. \tag{6.79}$$

With (6.79), we may proceed as in Case 2.b.ii.

- **CASE 2.c.iv.b:** u_1, u_2, u_3, u_4 of types (II)(I)(II)(II), (II)(II)(I)(II) and (II)(II)(II)(I).

Suppose u_1, u_2, u_3, u_4 are of types (II)(I)(II)(II). We find

$$\begin{aligned}
\frac{|n|^{\frac{1}{2}+\delta}|n_1|}{|n_1|^{\frac{1}{2}+\delta-\gamma}|n_2|^{\frac{1}{2}-\gamma}|n_3n_4|^{\frac{1}{2}+\delta-\gamma}} &\lesssim \frac{|n|^{\frac{1}{2}+\delta}|n_1|^{\frac{1}{2}-\delta+\gamma}|n_2|^{\delta}}{|n_2n_3n_4|^{\frac{1}{2}+\delta-\gamma}} \\
&\lesssim \frac{|n_2|^{\delta}}{(N^0)^{2\delta-3\gamma}} \lesssim \frac{1}{(N^0)^{\delta-3\gamma}}.
\end{aligned} \tag{6.80}$$

Again, using (6.80), we may proceed as in Case 2.b.ii. It is trivial to verify that this approach applies with types (II)(II)(I)(II) and (II)(II)(II)(I) as well, and this case is complete.

- **CASE 2.c.iv.c:** u_1, u_2, u_3, u_4 all type (II).

It is obvious that the analysis of the last two cases applies here as well. This completes the analysis of Case 2.c.iv, our final case, and the proof of Lemma 6.8 is complete. □

6.3. Septilinear estimates

In this subsection we prove Proposition 6.2. This proof will involve some probability theory, but the analysis will be simpler than what was used in the proof of Lemma 6.8. In particular, we will invoke Lemma 6.4, but will not need Lemma 6.5.

Proof of Proposition 6.2. We split into cases depending on the relative sizes of the spatial frequencies $n, n_2, \dots, n_8$, where $n = n_2 + \cdots + n_8$. Recall the defining condition of the region A_1: $|\sigma_1| \gtrsim |n_{\max}|^2$, where $|n_{\max}| = \max(|n|, |n_1|, |n_2|, |n_3|, |n_4|)$. It should be emphasized that we cannot assume that $|n_{\max}| \geq |n_5|, |n_6|, |n_7|, |n_8|$. Here is a list of the cases to be considered.

We use a type (I)–type (II) decomposition in the u_5 factor.

- **CASE 1.** u_5 type (II).
  - **CASE 1.a.** $|n_{\max}| \gtrsim |n_5|$.
  - **CASE 1.b.** $|n_{\max}| \ll |n_5|$.
- **CASE 2.** u_5 type (I).
  - **CASE 2.a.** $n \neq n_5$.
    - **CASE 2.a.i.** $|n_{\max}| \gtrsim |n_k|$ for all $k \in \{5, 6, 7, 8\}$.
    - **CASE 2.a.ii.** $|n_k|, |n_j| \gtrsim |n_5|$ for some distinct $k, j \in \{6, 7, 8\}$.
    - **CASE 2.a.iii.** $|n_6| \sim |n_5| \gg |n_{\max}|$ (equivalent WLOG to $|n_k| \sim |n_5| \gg |n_{\max}|$ for some $k \in \{6, 7, 8\}$).
      - **CASE 2.a.iii.a:** u_6 type (II).
      - **CASE 2.a.iii.b:** u_6 type (I).
  - **CASE 2.b.** $n = n_5$.
    - **CASE 2.b.i.** $|\sigma| \gtrsim |n|^{\sqrt{2\delta}}$, $|\sigma_k| \gtrsim |n|^{\sqrt{2\delta}}$ or $|n_k| \gtrsim |n|^{\sqrt{2\delta}}$ for some $k \in \{2, 3, 4, 6, 7, 8\}$.
    - **CASE 2.b.ii.** $|\sigma| \ll |n|^{\sqrt{2\delta}}$, $|\sigma_k| \ll |n|^{\sqrt{2\delta}}$ and $|n_k| \ll |n|^{\sqrt{2\delta}}$ for all $k \in \{2, 3, 4, 6, 7, 8\}$.

We proceed with the analysis of each case.

- **CASE 1.** u_5 type (II).

In this case we establish

$$\left\| \mathcal{N}_1\big(\mathcal{D}(u_5, u_6, u_7, u_8), u_2, u_3, u_4\big)|_{\text{Case 1}} \right\|_{X_T^{\frac{1}{2}+\delta, -\frac{1}{2}+\delta}} \lesssim \|u_5\|_{X_T^{\frac{1}{2}+\delta, \frac{1}{2}-\delta}} \prod_{\substack{j=2 \\ j\neq 5}}^{8} \|u_j\|_{X_T^{\frac{1}{2}-\delta, \frac{1}{2}-\delta}}. \tag{6.81}$$

- **CASE 1.a.** $|n_{\max}| \gtrsim |n_5|$.

We use $|\sigma_1| \gtrsim |n_{\max}|^2 \gtrsim |n_5|^2$ to estimate

$$\frac{|n|^{\frac{1}{2}+\delta} |n_1| |n_5|}{|\sigma_1| |n_5|^{\frac{1}{2}+\delta}} \lesssim 1. \tag{6.82}$$

Using (6.82), the inequality (6.81) follows from

$$\left\| f_5 \prod_{j=2}^{8} u_j \right\|_{L^2_{x,t}} \lesssim \|f_5\|_{X^{0, \frac{1}{2}-\delta}} \prod_{\substack{j=2 \\ j\neq 5}}^{8} \|u_j\|_{X^{\frac{1}{2}-\delta, \frac{1}{2}-\delta}}. \tag{6.83}$$

Then (6.83) is obtained with Hölder, (2.8) and (2.10),

$$\Big\| f_5 \prod_{j=2}^{8} u_j \Big\|_{L^2_{x,t}} \lesssim \|f_5\|_{L^4_{x,t}} \prod_{\substack{j=2\\ j\neq 5}}^{8} \|u_j\|_{L^{24}_{x,t}} \lesssim \|f_5\|_{X^{0,\frac{1}{2}-\delta}} \prod_{\substack{j=2\\ j\neq 5}}^{8} \|u_j\|_{X^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}.$$

- **CASE 1.b.** $|n_{\max}| \ll |n_5|$.

Since $|n_{\max}| \ll |n_5|$ and $n_1 = n_5 + \cdots + n_8$ we must have (without loss of generality) that

$$|n_6| \gtrsim |n_5| \gg |n_{\max}|. \tag{6.84}$$

With (6.84) we find

$$\frac{|n|^{\frac{1}{2}+\delta}|n_1||n_5|}{|\sigma_1||n_5|^{\frac{1}{2}+\delta}|n_6|^{\frac{1}{2}-\delta}} \lesssim \frac{1}{|n|^{\gamma}}. \tag{6.85}$$

Using (6.85) and duality, (6.81) follows from

$$\begin{aligned} &\Big| \int v \cdot f_5 f_6 \prod_{\substack{j=2\\ j\neq 5,6}}^{8} u_j dx dt \Big| \\ &\lesssim \|v\|_{X^{\gamma,\frac{1}{2}-\delta}} \|f_5\|_{X_T^{0,\frac{1}{2}-\delta}} \|f_6\|_{X^{0,\frac{1}{2}-\delta}} \prod_{\substack{j=2\\ j\neq 5,6}}^{8} \|u_j\|_{X^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}. \end{aligned} \tag{6.86}$$

Then (6.86) is easily established using Hölder, (2.8), (2.10) and (2.12).

- **CASE 2.** u_5 type (I).

From here we proceed with a dyadic decomposition in all factors. That is, we assume that $|n| \sim N$, $|n_i| \sim N_i$, and as in the proof of Lemma 6.8, we will order the frequencies (and corresponding dyadic shells) from largest to smallest using superscripts.

We remark that it can be assumed that

$$|\sigma_k| \ll \left(N^0\right)^2. \tag{6.87}$$

Otherwise we have $|\sigma_k| \gtrsim (N^0)^2$, and this gives

$$\frac{|n|^{\frac{1}{2}+\delta}|n_1||n_5|}{|\sigma_1||\sigma_k|^{2\delta}|n_5|^{\frac{1}{2}-\delta}} \lesssim \frac{1}{(N^0)^{2\delta}}. \tag{6.88}$$

With (6.88) (and dyadic summation) we can establish

$$\big\|\mathcal{N}_1\big(\mathcal{D}(u_5,u_6,u_7,u_8),u_2,u_3,u_4\big)\big|_{2.\mathrm{a}}\big\|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}+\delta}} \lesssim \prod_{j=2}^{8} \|u_j\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} \tag{6.89}$$

using the methods of previous cases.

- **CASE 2.a.** $n \neq n_5$.
- **CASE 2.a.i.** $|n_{\max}| \gtrsim |n_k|$ for all $k \in \{5,6,7,8\}$.

In this subcase we show there exists $\beta > 0$ and $\Omega_T \subset \Omega$, with $P(\Omega_T^c) < e^{-\frac{1}{T^{\beta}}}$, such that if $\omega \in \Omega_T$, then we have

$$\left\|\mathcal{N}_1\big(\mathcal{D}\big(S(t)\Lambda_5 u_{0,\omega}, u_6, u_7, u_8\big), u_2, u_3, u_4\big)\big|_{2.\mathrm{a.i}}\right\|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}+\delta}}$$

$$\lesssim T^{-\beta} N_5^{-\beta} \prod_{\substack{j=2\\ j\neq 5}}^{8} \|u_j\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}. \tag{6.90}$$

Recall that Λ_5 is the Fourier multiplier corresponding to the characteristic function of the interval $[N_5, M_5]$ in frequency space, for some dyadic integers $N_5 < M_5 < \infty$.

Using the representation (6.24) for u_j and Minkowski in λ_j, for each $j = 2, 3, 4, 6, 7, 8$, we find (letting $\mu = \sum_{j=2, j\neq 5}^{8} \lambda_j - \lambda$, and invoking (6.87)),

$$\left\|\mathcal{N}_1\big(\mathcal{D}\big(S(t)\Lambda_5 u_{0,\omega}, u_6, u_7, u_8\big), u_2, u_3, u_4\big)\right\|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}+\delta}}$$

$$\lesssim \big(N^0\big)^{6\delta} \prod_{\substack{j=2\\ j\neq 5}}^{8} \|u_j\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} \cdot \sup_{\mu < C(N^0)^{\sqrt{2\delta}}} \left\| |n|^{\frac{1}{2}+\delta} \sum_{\substack{*(n,\mu)\\ n\neq n_5}} \frac{-n_1 n_5 g_{n_5}(\omega)}{\sigma_1 |n_5|} \prod_{\substack{j=2\\ j\neq 5}}^{8} a_{n_j} \right\|_{l_n^2}, \tag{6.91}$$

where $a_{n_j} := a_{\lambda_j}(n_j)$ (we have removed the dependence on λ_j because our estimates will hold uniformly with respect to these parameters), and

$$*(n,\mu) = \Big\{ (n_2, n_3, n_4, n_5, n_6, n_7, n_8) \in \mathbb{Z}^7 : (n_1, n_2, n_3, n_4) \in \zeta(n),$$

$$(n_5, n_6, n_7, n_8) \in \zeta(n_1) \text{ and } n^3 - \sum_{j=2}^{8} n_j^3 = \mu \Big\}.$$

By bringing the absolute value inside and applying Lemma 6.4, for each $\omega \in \tilde{\Omega}_T$ we have

$$\left\| |n|^{\frac{1}{2}+\delta} \sum_{\substack{*(n,\mu)\\ n_5\neq n}} \frac{-n_1 n_5 g_{n_5}(\omega)}{\sigma_1 |n_5|} \prod_{\substack{j=2\\ j\neq 5}}^{8} a_{n_j} \right\|_{l_n^2}$$

$$\lesssim \frac{T^{-\beta/2}}{(N^0)^{\frac{1}{2}-\delta-\beta}} \sup_{|\mu|<C(N^0)^2} \left(\sum_{|n|\sim N} \left| \sum_{\substack{*(n,\mu)\\ n_5\neq n}} \prod_{\substack{j=2\\ j\neq 5}}^{8} |a_{n_j}| \right|^2 \right)^{\frac{1}{2}}, \tag{6.92}$$

where we have used the condition $|\sigma_1| \gtrsim (N^0)^2$. With repeated applications of Cauchy–Schwarz, we find

$$(6.92) \leq \frac{T^{-\beta/2}}{(N^0)^{\frac{1}{2}-\delta-\beta}} \sup_{|\mu|<C(N^0)^2} \left(\sum_{\substack{|n|\sim N, |n_k|\sim N_k, k=2,3,4,6,7\\ \{(n_5,n_8):(n_2,\dots,n_8)\in *(n,\mu), n_5\neq n\}}} \prod_{\substack{j=2\\ j\neq 5}}^{8} \frac{1}{|n_j|^{1-2\delta}} \right)^{\frac{1}{2}}$$

$$= \frac{T^{-\beta/2}}{(N^0)^{\frac{1}{2}-\delta-\beta}} \sup_{|\mu|<C(N^0)^2} \left(\sum_{\substack{|n_k|\sim N_k,\, k=2,3,4,6,7,8\\ \{(n,n_5):(n_2,\dots,n_8)\in *(n,\mu), n\neq n_5\}}} \prod_{\substack{j=2\\ j\neq 5}}^{8} \frac{1}{|n_j|^{1-2\delta}} \right)^{\frac{1}{2}}$$

$$\lesssim \frac{T^{-\beta/2}}{(N^0)^{\frac{1}{2}-7\delta-\beta}}. \tag{6.93}$$

Notice that, in the first line of (6.93), we have used the condition $n_5 \neq -n_8$. This condition holds without loss of generality, since at least one of the integers $n_2, n_3, n_4, n_6, n_7, n_8$ is not equal to $-n_5$, otherwise the inequality (6.93) holds trivially. Then, for fixed $n, n_2, n_3, n_4, n_6, n_7$ and μ, n_5 is determined by n_8, and n_8 satisfies a non-degenerate (since $n_5 \neq -n_8$) quadratic equation with at most two roots. We have used the same argument (with $n \neq n_5$) to avoid summation with respect to n and n_5 in the last line of (6.93).

Combining (6.91)–(6.93), the inequality (6.90) follows by dyadic summation, and Case 2.a.i is complete.

- **CASE 2.a.ii.** $|n_k|, |n_j| \gtrsim |n_5|$ for some distinct $k, j \in \{6, 7, 8\}$.

In this subcase we find

$$\frac{|n|^{\frac{1}{2}+\delta}|n_1||n_5|}{|\sigma_1||n_5|^{\frac{1}{2}-\delta}|n_k|^{\frac{1}{4}+\frac{\delta}{2}}|n_j|^{\frac{1}{4}+\frac{\delta}{2}}} \lesssim 1, \tag{6.94}$$

and by using (6.94) we can establish an estimate of the type (6.89) with the methods of previous cases.

- **CASE 2.a.iii.** $|n_6| \sim |n_5| \gg |n_{\max}|$.
- **CASE 2.a.iii.a.** u_6 type (II).

In this subcase we establish

$$\begin{aligned} &\left\|\mathcal{N}_1\big(\mathcal{D}(u_5, u_6, u_7, u_8), u_2, u_3, u_4\big)\big|_{2.\text{a.iii.a}}\right\|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}+\delta}} \\ &\quad \lesssim \|u_6\|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} \prod_{\substack{j=2\\ j\neq 6}}^{8} \|u_j\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}. \end{aligned} \tag{6.95}$$

Notice that we have

$$\frac{|n|^{\frac{1}{2}+\delta}|n_1||n_5|}{|\sigma_1||n_5|^{\frac{1}{2}-\delta}|n_6|^{\frac{1}{2}+\delta}} \lesssim \frac{1}{|n|^{\gamma}},$$

and from here we can establish (6.95) using the method of Case 1.b.

- **CASE 2.a.iii.b.** u_6 type (I).

In this subcase we establish the estimate

$$\begin{aligned} &\left\|\mathcal{N}_1\big(\mathcal{D}\big(S(t)\Lambda_5 u_{0,\omega}, S(t)\Lambda_6 u_{0,\omega}, u_7, u_8\big), u_2, u_3, u_4\big)\big|_{2.\text{a.iii.b}}\right\|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}+\delta}} \\ &\quad \lesssim T^{-\beta}(M_5M_6)^{-\beta} \prod_{\substack{j=2\\ j\neq 5,6}}^{8} \|u_j\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}. \end{aligned} \tag{6.96}$$

Using the representation (6.24) for u_j and Minkowski in λ_j, for each $j = 2, 3, 4, 7, 8$, we find (letting $\mu = \sum_{j=2, j\neq 5,6}^{8}\lambda_j - \lambda$, and invoking (6.87)),

$$\begin{aligned} &\left\|\mathcal{N}_1\big(\mathcal{D}\big(S(t)\Lambda_5 u_{0,\omega}, S(t)\Lambda_6 u_{0,\omega}, u_7, u_8\big), u_2, u_3, u_4\big)\right\|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}+\delta}} \\ &\quad \lesssim \big(N^0\big)^{6\delta} \prod_{\substack{j=2\\ j\neq 5,6}}^{8} \|u_j\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}} \cdot \sup_{\mu < C(N^0)^{\sqrt{2\delta}}} \left\| |n|^{\frac{1}{2}+\delta} \sum_{\substack{*(n,\mu)\\ n\neq n_5}} \frac{-n_1 n_5 g_{n_5}(\omega) g_{n_6}(\omega)}{\sigma_1 |n_5||n_6|} \prod_{\substack{j=2\\ j\neq 5,6}}^{8} a_{n_j} \right\|_{l_n^2}. \end{aligned} \tag{6.97}$$

By bringing the absolute value inside and applying Lemma 6.4, for each $\omega \in \tilde{\Omega}_T$ we have

$$\begin{aligned} &\left\| |n|^{\frac{1}{2}+\delta} \sum_{\substack{*(n,\mu)\\ n_5\neq n}} \frac{-n_1 n_5 g_{n_5}(\omega) g_{n_6}(\omega)}{\sigma_1 |n_5||n_6|} \prod_{\substack{j=2\\ j\neq 5,6}}^{8} a_{n_j} \right\|_{l_n^2} \\ &\quad \lesssim \frac{T^{-\beta/2}}{(N_{\max})^{\frac{1}{2}-\delta}(N^0)^{1-2\beta}} \sup_{|\mu| < C(N^0)^2} \left(\sum_{|n|\sim N} \left| \sum_{\substack{*(n,\mu)\\ n_5\neq n}} \prod_{\substack{j=2\\ j\neq 5,6}}^{8} |a_{n_j}| \right|^2 \right)^{\frac{1}{2}}, \end{aligned} \tag{6.98}$$

where we have used the conditions $|\sigma_1| \gtrsim (N_{\max})^2$ and $|n_6| \sim |n_5| \gg |n_{\max}|, |n_7|, |n_8|$. With repeated applications of Cauchy–Schwarz, we find

$$(6.98) \leq \frac{T^{-\beta/2}}{(N_{\max})^{\frac{1}{2}-\delta}(N^0)^{1-2\beta}} \cdot \sup_{|\mu|<C(N^0)^2} \left(\sum_{\substack{|n|\sim N,\, |n_k|\sim N_k,\, k=2,3,4,7,8 \\ \{(n_5,n_6):(n_2,\dots,n_8)\in *(n,\mu), n_5\neq n\}}} \prod_{\substack{j=2 \\ j\neq 5,6}}^{8} \frac{1}{|n_j|^{1-2\delta}} \right)^{\frac{1}{2}}$$
$$\lesssim \frac{T^{-\beta/2}}{(N^0)^{1-2\beta-6\delta}}. \tag{6.99}$$

Notice that we have used the condition $n_5 \neq -n_6$ (in order to avoid summation in these variables). This holds because if $(n_5, n_6, n_7, n_8) \in \zeta(n_1)$, then $n_5 \neq -n_6$ unless we also have $n_1 = n_5$, $n_5 = -n_7$ or $n_5 = -n_8$, but this is impossible given the assumption $|n_6| \sim |n_5| \gg |n_{\max}|, |n_7|, |n_8|$. Combining (6.97)–(6.99), the inequality (6.96) follows by dyadic summation, and Case 2.a.iii.b is complete.

- **CASE 2.b.** $n = n_5$.
- **CASE 2.b.i.** $|\sigma| \gtrsim |n|^{\sqrt{2\delta}}$, $|\sigma_k| \gtrsim |n|^{\sqrt{2\delta}}$ or $|n_k| \gtrsim |n|^{\sqrt{2\delta}}$ for some $k \in \{2, 3, 4, 6, 7, 8\}$.

In this subcase we will not use the assumption that u_5 is type (I). Instead we establish a deterministic estimate of the type (6.89). Suppose $|n_k| \gtrsim |n|^{\sqrt{2\delta}}$ for some $k \in \{2, 3, 4, 6, 7, 8\}$. Since $n = n_5$ in this case, we have $|\sigma_1| \gtrsim |n_{\max}|^2 \geq |n_5|^2$ and find

$$\frac{|n|^{\frac{1}{2}+\delta}|n_1||n_5|}{|\sigma_1||n_5|^{\frac{1}{2}-\delta}|n_k|^{\sqrt{2\delta}}} \lesssim 1. \tag{6.100}$$

Using (6.100), an inequality of type (6.89) follows from

$$\left\| \prod_{j=2}^{8} f_j \right\|_{L^2_{x,t\in[0,T]}} \lesssim \|f_5\|_{X_T^{0,\frac{1}{2}-\delta}} \|f_k\|_{X_T^{\frac{1}{2}-\delta-\sqrt{2\delta},\frac{1}{2}-\delta}} \prod_{\substack{j=2 \\ j\neq 5,k}}^{8} \|f_j\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}. \tag{6.101}$$

Then (6.101) is obtained with Hölder, (2.8) and (2.10), as in previous cases. If $|\sigma_k| \gtrsim |n|^{\sqrt{2\delta}}$ or $|\sigma| \gtrsim |n|^{\sqrt{2\delta}}$, the justification of (6.89) follows the same method.

- **CASE 2.b.ii.** $|\sigma| \ll |n|^{\sqrt{2\delta}}$, $|n_k| \ll |n|^{\sqrt{2\delta}}$ and $|\sigma_k| \ll |n|^{\sqrt{2\delta}}$ for each $k \in \{2, 3, 4, 6, 7, 8\}$.

In this subcase we establish an estimate of the type (6.90). This is the region where we will exploit a deterministic cancellation (see Remark 3.7). We proceed to identify the cancellation, and to properly define $\mathcal{N}(u_1, u_2, u_3, u_4)$ and $\mathcal{N}_1(\mathcal{D}(u_5, u_6, u_7, u_8), u_2, u_3, u_4)$ with different input functions. First suppose all factors are the same, and consider

$$\mathcal{N}_1\big(\mathcal{D}(u,u,u,u),u,u,u\big)^\wedge(n,\tau) = \sum_{\substack{(n_1,n_2,n_3,n_4)\in\zeta(n) \\ (n_5,n_6,n_7,n_8)\in\zeta(n_1)}} \int_{\tau=\tau_2+\dots+\tau_8} \chi_{A_1} \frac{-n_1 n_5}{\sigma_1} \prod_{j=2}^{8} \hat{u}(n_j,\tau_j). \tag{6.102}$$

We will induce cancellation in the contribution to (6.102) from when $n_5 = n$ and the remaining frequencies satisfy certain smallness conditions. Consider

$$\begin{aligned} A_{1,c} = \Big\{ &(n, n_2, n_3, n_4, n_5, n_6, n_7, n_8, \tau, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8) \in \big(\mathbb{Z}\setminus\{0\}\big)^8 \times \mathbb{R}^8 : n = n_5, \\ &\tau = \tau_2 + \dots + \tau_8,\ n_2 + n_3 + n_4 + n_6 + n_7 + n_8 = 0,\ n_2 + n_3 + n_4 \neq 0, \\ &|\sigma| < |n|^{\sqrt{2\delta}},\ |\sigma_k| < |n|^{\sqrt{2\delta}},\ |n_k| < |n|^{\sqrt{2\delta}} \text{ for } k = 2, 3, 4, 6, 7, 8 \Big\}. \end{aligned} \tag{6.103}$$

Notice that, if $(n, n_2, \dots, n_8, \tau, \tau_2, \dots, \tau_8) \in A_{1,c}$, then $(n_1, n_2, n_3, n_4) \in \zeta_1(n)$, $(n, n_6, n_7, n_8) \in \zeta_1(n_1)$ and $(n, n_1, n_2, n_3, n_4, \tau, \tau_1, \tau_2, \tau_3, \tau_4) \in A_1$. Indeed, the restrictions $|n_k| < |n|^{\sqrt{2\delta}}$ for $k = 2, 3, 4$ and $n_2 + n_3 + n_4 \neq 0$ guarantee that $n \neq n_k$ for $k = 1, 2, 3, 4$ and $n_1 \neq -n_k$ for $k = 2, 3, 4$, thus $(n_1, n_2, n_3, n_4) \in \zeta(n)$. Similarly $|n_k| < |n|^{\sqrt{2\delta}}$ for $k = 6, 7, 8$ and $n_6 + n_7 + n_8 \neq 0$ guarantees $n_1 \neq n, n_k$ and $n \neq -n_k$ for $k = 6, 7, 8$, and thus $(n, n_6, n_7, n_8) \in \zeta_1(n_1)$. Lastly using the restrictions $|\sigma|, |\sigma_k|, |n_k| < |n|^{\sqrt{2\delta}}$ for $k = 2, 3, 4, 6, 7, 8$ and $n = n_5$, we can easily show that $|\sigma_1| \gtrsim |n_{\max}|^2$, and therefore $(n, n_1, n_2, n_3, n_4, \tau, \tau_1, \tau_2, \tau_3, \tau_4) \in A_1$.

Because of this, we can consider the following contribution to (6.102),

$$\begin{aligned}
&\mathcal{N}_1\big(\mathcal{D}(u,u,u,u),u,u,u\big)^{\wedge}(n,\tau)|_{A_{1,c}} \\
&:= -n^2 \sum_{n_2+n_3+n_4+n_6+n_7+n_8=0} \int_{\tau=\tau_2+\tau_3+\tau_4+\tau_5+\tau_6+\tau_7+\tau_8} \chi_{A_{1,c}} \prod_{j=2}^{8} \hat{u}(n_j,\tau_j) \\
&\quad \cdot \left(\frac{n-n_2-n_3-n_4}{\tau-\tau_2-\tau_3-\tau_4-(n-n_2-n_3-n_4)^3} \right).
\end{aligned}$$

Inserting the assumption that u_5 is type (I), we have

$$\begin{aligned}
&\mathcal{N}_1\big(\mathcal{D}\big(S(t)\Lambda_5 u_{0,\omega},u,u,u\big),u,u,u\big)^{\wedge}(n,\tau)|_{A_{1,c}} \\
&= -n g_n(\omega) \sum_{n_2+n_3+n_4+n_6+n_7+n_8=0} \int_{\tau=\tau_2+\tau_3+\tau_4+n^3+\tau_6+\tau_7+\tau_8} \chi_{A_{1,c}} \prod_{j=2, j\neq 5}^{8} \hat{u}(n_j,\tau_j) \\
&\quad \cdot \left(\frac{n}{\tau-\tau_2-\tau_3-\tau_4-(n-n_2-n_3-n_4)^3} - \frac{n_2+n_3+n_4}{\tau-\tau_2-\tau_3-\tau_4-(n-n_2-n_3-n_4)^3} \right) \\
&=: K_1(u,\dots,u)(n,\tau) + K_2(u,\dots,u)(n,\tau),
\end{aligned} \tag{6.104}$$

where we have defined K_1 and K_2 by expanding the parentheses in the second last line. We will only need cancellation to control K_1 (K_2 will be estimated directly). Let us now describe this cancellation. We swap the variable names $(n_2, n_3, n_4, \tau_2, \tau_3, \tau_4)$ with $(n_6, n_7, n_8, \tau_6, \tau_7, \tau_8)$ and use the invariance of $A_{1,c}$ under this modification to obtain

$$\begin{aligned}
K_1(u,\dots,u)(n,\tau) &= -\frac{n g_n(\omega)}{2} \sum_{n_2+n_3+n_4+n_6+n_7+n_8=0} \int_{\tau=\tau_2+\tau_3+\tau_4+n^3+\tau_6+\tau_7+\tau_8} \chi_{A_{1,c}} \\
&\quad \cdot \prod_{j=2, j\neq 5}^{8} \hat{u}(n_j,\tau_j) \left(\frac{1}{\tau-\tau_6-\tau_7-\tau_8-(n-n_6-n_7-n_8)^3} \right. \\
&\quad \left. + \frac{1}{\tau-\tau_2-\tau_3-\tau_4-(n-n_2-n_3-n_4)^3} \right).
\end{aligned} \tag{6.105}$$

Using $n - n_1 = n_2 + n_3 + n_4 = -n_6 - n_7 - n_8$, we find

$$\begin{aligned}
&\frac{1}{\tau-\tau_6-\tau_7-\tau_8-(n-n_6-n_7-n_8)^3} + \frac{1}{\tau-\tau_2-\tau_3-\tau_4-(n-n_2-n_3-n_4)^3} \\
&= \frac{-6n(n-n_1)^2+\sigma}{\tau-\tau_6-\tau_7-\tau_8-(n-n_6-n_7-n_8)^3} \cdot \frac{1}{\tau-\tau_2-\tau_3-\tau_4-(n-n_2-n_3-n_4)^3}.
\end{aligned} \tag{6.106}$$

This gives

$$\begin{aligned}
K_1(u,\dots,u)(n,\tau) &= -\frac{n g_n(\omega)}{2} \sum_{n_2+n_3+n_4+n_6+n_7+n_8=0} \int_{\tau=\tau_2+\tau_3+\tau_4+n^3+\tau_6+\tau_7+\tau_8} \chi_{A_{1,c}} \\
&\quad \cdot \prod_{j=2, j\neq 5}^{8} \hat{u}(n_j,\tau_j) \frac{-6n(n-n_1)^2+\sigma}{\tau-\tau_6-\tau_7-\tau_8-(n-n_6-n_7-n_8)^3} \\
&\quad \cdot \frac{1}{\tau-\tau_2-\tau_3-\tau_4-(n-n_2-n_3-n_4)^3}.
\end{aligned} \tag{6.107}$$

Now, as anticipated, we will *define* $\mathcal{N}(u_1, u_2, u_3, u_4)$ and $\mathcal{N}_1(\mathcal{D}(u_5, u_6, u_7, u_8), u_2, u_3, u_4)$ with (non-equivalent) input functions by extending the definition of $K_1(u, \dots, u)(n, \tau)$ according to (6.107). The expression $\mathcal{N}(u_1, u_2, u_3, u_4)$ is defined piecewise through a decomposition in frequency space. The region of integration A is divided into A_{-1}, A_0, A_1, A_2, A_3 and A_4. In the regions A_k for $k = -1, 0, 2, 3, 4$, we interpret $\mathcal{N}_k(u_1, u_2, u_3, u_4)$ directly. In the region A_1, we insert an equation satisfied by u_1 (the second iteration). When the inputs are equivalent, we interpret the expression directly, and exploit a cancellation in the region $A_{1,c}$ (it is straightforward to verify that the cancellation is not required in A_2, A_3 and A_4). When the inputs are not equivalent, the definition of $\mathcal{N}_1(u_1, u_2, u_3, u_4)$ will vary with the equation satisfied by u_1.

The algorithm for determining this definition is straightforward. During the proof of Theorem 1.1, the factor u_1 will satisfy an equation of the form (4.1), or one of its variants. The important point is that the equation satisfied by u_1 will always be decomposed into contributions of type (I) (linear part, rough but random) and type (II) (nonlinear part, smooth and deterministic). The contributions from the type (I) part of u_1 are always interpreted directly. For the contributions from the type (II) factor, we either (i) bound this factor using the higher temporal regularity $b = \frac{1}{2} + 2\delta$, via the estimate (3.10), in which case the nonlinearity is interpreted directly, or (ii) we expand the type (II) contribution into a septilinear expression. Thus in situation (ii) we are considering a nonlinearity as in Proposition 6.2, and in all of the prior subcases of this proof (in particular in the complement of $A_{1,c}$), we interpret K_1 directly, as in (6.102). For the contribution to situation (ii) from Case 2.b.ii, we will force the cancellation (6.106). That is, for each $n > 0$ and $\tau \in \mathbb{R}$, with $|\tau - n^3| < |n|^{\sqrt{2\delta}}$, we define

$$\begin{aligned} &\mathcal{N}_1\big(\mathcal{D}\big(S(t)\Lambda_5 u_{0,\omega}, u_6, u_7, u_8\big), u_2, u_3, u_4\big)^{\wedge}(n, \tau)|_{A_{1,c}} \\ &\quad := K_1(u_2, u_3, u_4, u_6, u_7, u_8)(n, \tau) + K_2(u_2, u_3, u_4, u_6, u_7, u_8)(n, \tau) \end{aligned} \tag{6.108}$$

where K_2 is given as in (6.104) (but with potentially non-equivalent factors u_j), and

$$\begin{aligned} &K_1(u_2, u_3, u_4, u_6, u_7, u_8)(n, \tau) \\ &\quad := -\frac{n g_n(\omega)}{2} \sum_{n_2+n_3+n_4+n_6+n_7+n_8=0} \int_{\tau=\tau_2+\tau_3+\tau_4+n^3+\tau_6+\tau_7+\tau_8} \chi_{A_{1,c}} \\ &\qquad \cdot \prod_{j=2, j\neq 5}^{8} \hat{u}_j(n_j, \tau_j) \frac{-6n(n-n_1)^2 + \sigma}{\tau - \tau_6 - \tau_7 - \tau_8 - (n - n_6 - n_7 - n_8)^3} \\ &\qquad \cdot \frac{1}{\tau - \tau_2 - \tau_3 - \tau_4 - (n - n_2 - n_3 - n_4)^3}. \end{aligned} \tag{6.109}$$

Let us comment on properly justifying that, with equivalent inputs $u_2 = \cdots = u_8$, the cancellation (6.107) can be used to control the entire contribution to the left-hand side of (6.90) from the region $A_{1,c}$ in Case 2.b.ii. Inverting the Fourier transform, the cancellation (6.107) certainly holds over a sum of finitely many n. Luckily, in this subcase, we have $n = n_5$, with u_5 a type (I) factor, and thus $|n| \le M_5 < \infty$. That is, we are justified in using the representation (6.107) for the entire contribution to the left-hand side of (6.90) from the region $A_{1,c}$ in Case 2.b.ii because $M_5 < \infty$ (by assumption in the statement of Proposition 3.2).

We proceed to justify an estimate of the type (6.90). The restriction of this subcase indicates that

$$(n, n_2, \dots, n_8, \tau, \tau_2, \dots, \tau_8) \in A_{1,c},$$

as given in (6.103), and therefore, according to our definitions above, the nonlinearity is defined through the expressions (6.108)–(6.109).

Notice that, using $|\sigma|, |n_k| \ll |n|^{\sqrt{2\delta}}$ and $|\sigma_1| \gtrsim (N^0)^2$, we have

$$\begin{aligned} &\frac{|n||-6n(n-n_1)^2 + \sigma|}{|\tau - \tau_6 - \tau_7 - \tau_8 - (n - n_6 - n_7 - n_8)^3|} \cdot \frac{1}{|\tau - \tau_2 - \tau_3 - \tau_4 - (n - n_2 - n_3 - n_4)^3|} \\ &\quad \lesssim \frac{1}{(N^0)^{2-\gamma}} \end{aligned} \tag{6.110}$$

and

$$\frac{|n_2+n_3+n_4|}{|\sigma_1|} \lesssim \frac{1}{(N^0)^{2-\gamma}}, \tag{6.111}$$

for some small $0<\gamma=\gamma(\delta)\ll 1$.

With (6.108)–(6.109) this gives, for each fixed n,τ,

$$\begin{aligned}&\big|\mathcal{N}_1\big(\mathcal{D}\big((S(t)u_{0,\omega})_{N_5},u_6,u_7,u_8\big),u_2,u_3,u_4\big)(n,\tau)|_{2.a.ii}\big|\\ &\lesssim \frac{1}{(N^0)^{2-2\gamma}}\big|g_n(\omega)\big|\cdot\Bigg|\sum_{\substack{n_2+n_3+n_4+n_6+n_7+n_8=0\\ |n_j|<(N^0)^{\sqrt{2\delta}}}}\int_{\substack{\tau-n^3=\tau_2+\tau_3+\tau_4+\tau_6+\tau_7+\tau_8\\ |\sigma_j|<(N^0)^{\sqrt{2\delta}}}}\prod_{\substack{j=2\\ j\neq 5}}^{8}\hat{u}_j(n_j,\tau_j)\Bigg|.\end{aligned}$$

Next we use the representation (6.24) for u_j, and apply Minkowski in λ_j, for each $j\in\{2,3,4,6,7,8\}$, and this gives

$$\begin{aligned}&\big\|\mathcal{N}_1\big(\mathcal{D}\big((S(t)u_{0,\omega})_{N_5},u_6,u_7,u_8\big),u_2,u_3,u_4\big)|_{2.a.ii}\big\|_{X_T^{\frac12+\delta,-\frac12+\delta}}\\ &\lesssim \frac{1}{(N^0)^{2-\gamma-6\sqrt{2}\delta^{3/2}}}\prod_{\substack{j=2\\ j\neq 5}}^{8}\|u_j\|_{X_T^{\frac12-\delta,\frac12-\delta}}\cdot\sup_{|\mu|<C(N^0)^{\sqrt{2\delta}}}\Bigg\||n|^{\frac12+\delta}\big|g_n(\omega)\big|\sum_{\substack{*(n,\mu)\cap\{n=n_5\}\\ |n_j|<(N^0)^{\alpha}}}\prod_{\substack{j=2\\ j\neq 5}}^{8}a_{n_j}\Bigg\|_{l_n^2},\end{aligned}$$

where $a_{n_j}:=a_{\lambda_j}(n_j)$ (we have removed the dependence on λ_j because our estimates will hold uniformly with respect to these parameters). Then by Lemma 6.4 (with $\varepsilon=\beta$) and repeated applications of Cauchy–Schwarz, we have for $\omega\in\tilde{\Omega}_T$,

$$\begin{aligned}&\big\|\mathcal{N}_1\big(\mathcal{D}\big((S(t)u_{0,\omega})_{N_5},u_6,u_7,u_8\big),u_2,u_3,u_4\big)\big\|_{X_T^{\frac12+\delta,-\frac12+\delta}}\\ &\lesssim \frac{T^{-\frac{\beta}{2}}}{(N^0)^{\frac32-\gamma-6\sqrt{2}\delta^{3/2}-\delta-\beta}}\prod_{\substack{j=2\\ j\neq 5}}^{8}\|u_j\|_{X_T^{\frac12-\delta,\frac12-\delta}}\sup_{|\mu|<C(N^0)^{\sqrt{2\delta}}}\Bigg\|\sum_{\substack{*(n,\mu)\cap\{n=n_5\}\\ |n_j|<(N^0)^{\alpha}}}\prod_{\substack{j=2\\ j\neq 5}}^{8}a_{n_j}\Bigg\|_{l_n^2}\\ &\lesssim \frac{T^{-\frac{\beta}{2}}}{(N^0)^{1-\gamma-6\sqrt{2}\delta^{3/2}-\delta-\beta}}\prod_{\substack{j=2\\ j\neq 5}}^{8}\|u_j\|_{X_T^{\frac12-\delta,\frac12-\delta}}\sup_{|n|\sim N,|\mu|<C(N^0)^{\sqrt{2\delta}}}\Bigg|\sum_{\substack{*(n,\mu)\cap\{n=n_5\}\\ |n_j|<(N^0)^{\alpha}}}\prod_{\substack{j=2\\ j\neq 5}}^{8}a_{n_j}\Bigg|\\ &\lesssim \frac{T^{-\frac{\beta}{2}}}{(N^0)^{1-\gamma-6\sqrt{2}\delta^{3/2}-\delta-\beta}}\prod_{\substack{j=2\\ j\neq 5}}^{8}\|u_j\|_{X_T^{\frac12-\delta,\frac12-\delta}}\sup_{|n|\sim N}\Bigg|\sum_{\substack{n_2+n_3+n_4+n_6+n_7+n_8=0\\ |n_j|<(N^0)^{\alpha}}}\prod_{\substack{j=2\\ j\neq 5}}^{8}|a_{n_j}|\Bigg|\\ &\lesssim \frac{T^{-\frac{\beta}{2}}}{(N^0)^{1-2\gamma-6\sqrt{2}\delta^{3/2}-7\delta-\beta}}\prod_{\substack{j=2\\ j\neq 5}}^{8}\|u_j\|_{X_T^{\frac12-\delta,\frac12-\delta}},\end{aligned} \tag{6.112}$$

and (6.90) follows by dyadic summation. This completes the analysis of Case 2.b.ii. The proof of Proposition 6.2 is complete. □

6.4. Deterministic nonlinear estimates

In this subsection we present the proof of Proposition 3.3. We require the following calculus inequality:

Lemma 6.14. *Let $0<\delta_1\le\delta_2$ satisfy $\delta_1+\delta_2>1$, and let $a\in\mathbb{R}$, then*

$$\int_{-\infty}^{\infty}\frac{d\theta}{\langle\theta\rangle^{\delta_1}\langle a-\theta\rangle^{\delta_2}}\lesssim\frac{1}{\langle a\rangle^{\alpha}},$$

where $\alpha=\delta_1-(1-\delta_2)_+$. Here $(\lambda)_+:=\lambda$ if $\lambda>0$, $=\varepsilon>0$ if $\lambda=0$, and $=0$ if $\lambda<0$.

The proof of Lemma 6.14 can be found in [20].

Proof of Proposition 3.3. We establish Proposition 3.3 with $\delta_0 = 0$ (see Remark 6.3). We begin with the proof of (3.9). In fact, we establish

$$\left\|\mathcal{N}_0\left(u_1, u_2, u_3, u_4\right)\right\|_{Y_T^{\frac{1}{2}+\delta,-1}} \lesssim \prod_{j=1}^{4} \|u_j\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\frac{3\delta}{2}}}. \tag{6.113}$$

Then (3.9) follows easily from (6.113), Lemma 2.3 and Lemma 2.4 with $\theta = 2\delta-$.

Using $\|\cdot\|_{Y_T^{s,b}} \leq \|\chi_{[0,T]}(t)\cdot\|_{Y^{s,b}}$ it suffices to establish

$$\left\|\mathcal{N}_0(u_1, u_2, u_3, u_4)\right\|_{Y^{\frac{1}{2}+\delta,-1}} \lesssim T^{\theta} \prod_{j=1}^{4} \|u_j\|_{X^{\frac{1}{2}-\delta,\frac{1}{2}-\frac{3\delta}{2}}}, \tag{6.114}$$

where each u_j satisfies $u_j = \chi_{[0,T]}(t)u_j$. We proceed to prove (6.114).

Let

$$f_j(n_j, \tau_j) := \langle n_j\rangle^{\frac{1}{2}-\delta}\langle \tau_j - n_j^3\rangle^{\frac{1}{2}-\frac{3\delta}{2}}\, \widehat{u_j}(n_j, \tau_j)$$

for each $j = 1, 2, 3, 4$. To prove (3.9) it is enough to show that

$$\left\| \frac{|n|^{\frac{1}{2}+\delta}}{\langle\sigma\rangle} \sum_{n=n_1+\cdots+n_4} \int_{\tau=\tau_1+\cdots+\tau_4} \chi_{A_0}\cdot|n_1| \prod_{j=1}^{4} \frac{|f_j(n_j,\tau_j)|}{|n_j|^{\frac{1}{2}-\delta}\langle\sigma_j\rangle^{\frac{1}{2}-\frac{3\delta}{2}}} \right\|_{l_n^2 L_\tau^1} \lesssim \prod_{j=1}^{4} \|f_j\|_{L^2_{n,\tau}}. \tag{6.115}$$

Using the condition $|\sigma| \gtrsim |n_{\max}|^2$, we have

$$\frac{|n|^{\frac{1}{2}+\delta}|n_1|}{\langle\sigma\rangle^{1-6\delta-\gamma}|n_1|^{\frac{1}{2}-\delta}} \lesssim \frac{1}{|n|^{1-17\delta-5\gamma}\prod_{j=2}^{4}|n_j|^{\delta+\gamma}}. \tag{6.116}$$

Applying (6.116), and subsequently removing all restrictions in frequency space (notice we have brought absolute values inside), we have

$$\begin{aligned}
\text{LHS of (6.115)} &\lesssim \left\| \frac{1}{\langle\sigma\rangle^{6\delta+\gamma}|n|^{1-17\delta-5\gamma}} \sum_{n=n_1+\cdots+n_4} \int_{\tau=\tau_1+\cdots+\tau_4} \frac{|f_1(n_1,\tau_1)|}{\langle\sigma_1\rangle^{\frac{1}{2}-\frac{3\delta}{2}}} \prod_{j=2}^{4} \frac{|f_j(n_j,\tau_j)|}{|n_j|^{\frac{1}{2}+\gamma}\langle\sigma_j\rangle^{\frac{1}{2}-\frac{3\delta}{2}}} \right\|_{l_n^2 L_\tau^1} \\
&\leq \left\| \frac{1}{\langle\sigma\rangle^{6\delta+\gamma}|n|^{1-17\delta-5\gamma}} \left(\sum_{n=n_1+\cdots+n_4} \int_{\tau=\tau_1+\cdots+\tau_4} \prod_{j=1}^{4} |f_j(n_j,\tau_j)|^2 \right)^{\frac{1}{2}} \right. \\
&\quad \left. \cdot \left(\sum_{n=n_1+\cdots+n_4} \int_{\tau=\tau_1+\cdots+\tau_4} \frac{1}{\langle\sigma_1\rangle^{1-3\delta}} \prod_{j=2}^{4} \frac{1}{|n_j|^{1+2\gamma}\langle\sigma_j\rangle^{1-3\delta}} \right)^{\frac{1}{2}} \right\|_{l_n^2 L_\tau^1}
\end{aligned} \tag{6.117}$$

by Cauchy–Schwarz in $n_1, n_2, n_3, \tau_1, \tau_2, \tau_3$, for fixed n, τ. Next we fix τ, n, n_1, n_2, n_3, and repeatedly apply Lemma 6.14 to obtain

$$\begin{aligned}
&\int_{\tau_1,\tau_2,\tau_3} \frac{d\tau_1 d\tau_2 d\tau_3}{\langle\tau_1 - n_1^3\rangle^{1-3\delta}\langle\tau_2 - n_2^3\rangle^{1-3\delta}\langle\tau_3 - n_3^3\rangle^{1-3\delta}\langle\tau - \tau_1 - \tau_2 - \tau_3 - n_4^3\rangle^{1-3\delta}} \\
&\quad\lesssim \frac{1}{\langle\tau - n_1^3 - n_2^3 - n_3^3 - n_4^3\rangle^{1-12\delta}}.
\end{aligned} \tag{6.118}$$

Using (6.118) we have

$$
\begin{aligned}
(6.117) &\lesssim \left\| \frac{1}{\langle\sigma\rangle^{6\delta+\gamma}|n|^{1-17\delta-5\gamma}} \left(\sum_{n=n_1+\cdots+n_4} \int_{\tau=\tau_1+\cdots+\tau_4} \prod_{j=1}^{4} \left| f_j(n_j,\tau_j) \right|^2 \right)^{\frac{1}{2}} \right. \\
&\quad \left. \cdot \left(\sum_{n=n_1+\cdots+n_4} \frac{1}{\langle \tau - n_1^3 - \cdots - n_4^3 \rangle^{1-12\delta}} \prod_{j=2}^{4} \frac{1}{\langle n_j \rangle^{1+2\gamma}} \right)^{\frac{1}{2}} \right\|_{l_n^2 L_\tau^1} \\
&\lesssim \left\| \left(\sum_{n=n_1+\cdots+n_4} \int_{\tau=\tau_1+\cdots+\tau_4} \prod_{j=1}^{4} \left| f_j(n_j,\tau_j) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2_{n,\tau}} \sup_{n\neq 0} \frac{1}{|n|^{1-17\delta-5\gamma}} \\
&\quad \cdot \left(\int_\tau \sum_{n=n_1+\cdots+n_4} \frac{1}{\langle\sigma\rangle^{12\delta+2\gamma} \langle \tau - n_1^3 - \cdots - n_4^3 \rangle^{1-12\delta}} \prod_{j=2}^{4} \frac{1}{|n_j|^{1+2\gamma}} \right)^{\frac{1}{2}}.
\end{aligned}
\tag{6.119}
$$

In the last line we applied Cauchy–Schwarz in τ, and took out the supremum in n afterward. Applying Fubini we compute that

$$
\left\| \left(\sum_{n=n_1+\cdots+n_4} \int_{\tau=\tau_1+\cdots+\tau_4} \prod_{j=1}^{4} \left| f_j(n_j,\tau_j) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2_{n,\tau}} = \prod_{j=1}^{4} \| f_j \|_{L^2_{n,\tau}}. \tag{6.120}
$$

It remains to estimate the second factor in (6.119). We change the order of integration and summation, and integrate in τ for fixed n, n_1, n_2, n_3. By Lemma 6.14 we have

$$
\int_\tau \frac{1}{\langle \tau - n^3 \rangle^{12\delta+2\gamma} \langle \tau - n_1^3 - \cdots - n_4^3 \rangle^{1-12\delta}} \lesssim \frac{1}{\langle n^3 - n_1^3 - \cdots - n_4^3 \rangle^{2\gamma}} \leq 1.
$$

This gives

$$
\begin{aligned}
&\sup_{n\neq 0} \frac{1}{|n|^{1-17\delta-5\gamma}} \left(\int_\tau \sum_{n=n_1+\cdots+n_4} \frac{1}{\langle\sigma\rangle^{12\delta+2\gamma} \langle \tau - n_1^3 - \cdots - n_4^3 \rangle^{1-12\delta}} \prod_{j=2}^{4} \frac{1}{|n_j|^{1+2\gamma}} \right)^{\frac{1}{2}} \\
&\quad \lesssim \sup_{n\neq 0} \frac{1}{|n|^{1-17\delta-5\gamma}} \left(\sum_{n=n_1+\cdots+n_4} \prod_{j=2}^{4} \frac{1}{|n_j|^{1+2\gamma}} \right)^{\frac{1}{2}} \leq 1.
\end{aligned}
\tag{6.121}
$$

Combining (6.117), (6.119), (6.120) and (6.121), we obtain the estimate (6.115). The proof of (3.9) is complete.

Next we establish (3.8). First we prove that

$$
\| \mathcal{N}_0(u_1,u_2,u_3,u_4) \|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}-\delta}} \lesssim \prod_{j=1}^{4} \| u_j \|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\frac{3\delta}{2}}}. \tag{6.122}
$$

Then (3.8) follows from Lemma 2.3, (6.122), (3.9) and Lemma 2.4,

$$
\begin{aligned}
\| \mathcal{D}_0(u_1,u_2,u_3,u_4) \|_{X_T^{\frac{1}{2}+\delta,\frac{1}{2}-\delta}} &\lesssim \| \mathcal{N}_0(u_1,u_2,u_3,u_4) \|_{X_T^{\frac{1}{2}+\delta,-\frac{1}{2}-\delta}} + \| \mathcal{N}_0(u_1,u_2,u_3,u_4) \|_{Y_T^{\frac{1}{2}+\delta,-1-\delta}} \\
&\lesssim \prod_{j=1}^{4} \| u_j \|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\frac{3\delta}{2}}} + \| \mathcal{N}_0(u_1,u_2,u_3,u_4) \|_{Y_T^{\frac{1}{2}+\delta,-1}} \\
&\lesssim T^{2\delta-} \prod_{j=1}^{4} \| u_j \|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}.
\end{aligned}
$$

We proceed to justify (6.122), which is equivalent to

$$\left\|\mathcal{N}_0\left(u_1,u_2,u_3,u_4\right)\right\|_{X^{\frac{1}{2}+\delta,-\frac{1}{2}-\delta}} \lesssim T^{\theta}\prod_{j=1}^{4}\|u_j\|_{X^{\frac{1}{2}-\delta,\frac{1}{2}-\frac{3\delta}{2}}}, \tag{6.123}$$

where each u_j satisfies $u_j=\chi_{[0,T]}(t)u_j$.

Using the condition $|\sigma|\gtrsim|n_{\max}|^2$, we have

$$\frac{|n|^{\frac{1}{2}+\delta}|n_1|}{\langle\sigma\rangle^{\frac{1}{2}+\delta}|n_1|^{\frac{1}{2}-\delta}}\lesssim 1. \tag{6.124}$$

With (6.124) it suffices to show that

$$\|w_1u_2u_3u_4\|_{L^2_{x,t}}\lesssim\|w_1\|_{0,\frac{1}{2}-\delta}\prod_{j=2}^{4}\|u_j\|_{X^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}. \tag{6.125}$$

The inequality (6.125) is obtained with Hölder's inequality, (2.8), (2.10) and Lemma 2.4,

$$\begin{aligned}\|w_1u_2u_3u_4\|_{L^2_{x,t}}&\le\|w_1\|_{L^4_{x,t}}\prod_{j=2}^{4}\|u_j\|_{L^{12}_{x,t}}\\&\lesssim\|w_1\|_{X^{0,\frac{1}{3}}}\prod_{j=2}^{4}\|u_j\|_{X^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}},\end{aligned}$$

for $\delta>0$ sufficiently small. The proof of (6.122) (and thus of (3.8)) is complete.

It remains to justify (3.10) and (3.11). We consider the case $k=1$, the adaptation to other cases is straightforward using the techniques of previous cases. With the condition $|\sigma_1|\gtrsim|n_{\max}|^2$, we have

$$\frac{|n|^{\frac{1}{2}+\delta}|n_1|}{|n_1|^{\frac{1}{2}-2\delta}|\sigma_1|^{\frac{1}{2}+2\delta}}\lesssim\frac{1}{|n|^{\delta}}. \tag{6.126}$$

With (6.126) and duality, (3.10) follows from

$$\left|\int v\cdot f_1u_2u_3u_4\,dxdt\right|\lesssim\|v\|_{\delta,\frac{1}{2}-\delta,T}\|f_1\|_{L^2_{x,t\in[0,T]}}\prod_{j=2}^{4}\|u_j\|_{X_T^{\frac{1}{2}-\delta,\frac{1}{2}-\delta}}. \tag{6.127}$$

Then (6.127) is established using Hölder, (2.8), (2.10) and (2.12) (as in various proofs above). The proof of (3.11) is omitted, as it follows the same strategy. The proof of Proposition 3.3 is complete. □

Conflict of interest statement

The author has no conflict of interest to declare.

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Appendix A

A.1. Proofs of lemmata

Here we present the proofs of Lemmas 6.10, 6.11, 6.12 and 6.13.

Proof of Lemma 6.10. We will prove that the estimate $|S(n_4,\mu)| < (N^0)^3$ is satisfied by enumerating the set $S(n_4,\mu)$ with a specific algorithm. This algorithm will count the number of elements in $S(n_4,\mu)$ by selectively fixing the integers $\{n, n_1, n_2, n_3, n', n'_1, n'_2, m_1, m_2, m_3, m'_1, m'_2\}$ one at a time, ensuring that the conditions required to remain in $S(n_4,\mu)$ are satisfied at each stage. This process will terminate once we have fixed an element of $S(n_4,\mu)$; from here we work backwards, counting how many choices were made to determine the size of $|S(n_4,\mu)|$. This algorithm is designed to keep the number of elements in our count below $(N^0)^3$, and this is clearly satisfied as long as no more than three of the integers in the set $\{n, n_1, n_2, n_3, n', n'_1, n'_2, m_1, m_2, m_3, m'_1, m'_2\}$ (all with magnitude $\le N^0$) need to be selected in order for the algorithm to terminate. Therefore, in the analysis that follows, as we enumerate $S(n_4,\mu)$, we simply need to ensure that we are required to select at most *three* integers.

Fix n and n_3 (from here we will only select one more integer in the set $\{n_1, n_2, n', n'_1, n'_2, m_1, m_2, m_3, m'_1, m'_2\}$). Then n_2 is determined by n_1 (since $n = n_1 + \cdots + n_4$), and n_1 satisfies a non-degenerate (since $n_1 \neq -n_2$) quadratic equation with at most two roots (see (6.35)). We recall the following identity, which can be proven using the moment generating function of the complex Gaussian:

$$\mathbb{E}\big(g_n(\omega)^k \overline{g_n(\omega)}^l\big) = \delta_{k,l} k!, \tag{A.1}$$

where $\delta_{k,l}$ is the Dirac-delta function. Using (A.1), in order for the non-zero expectation condition

$$\mathbb{E}\big(g_{n_1}(\omega) g_{n_2}(\omega) g_{n'_1}(\omega) g_{n'_2}(\omega) \overline{g_{m_1}}(\omega) \overline{g_{m_2}}(\omega) \overline{g_{m'_1}}(\omega) \overline{g_{m'_2}}(\omega)\big) \neq 0$$

to be satisfied, it must be possible to arrange the indices $\{n_1, n_2, n'_1, n'_2, -m_1, -m_2, -m'_1, -m'_2\}$ into pairs, such that the sum of the indices is zero in each pair. Since $n_1 \neq -n_2$, these integers cannot be paired with each other. We have therefore determined (at least) two of the integers in the set $\{n'_1, n'_2, -m_1, -m_2, -m'_1, -m'_2\}$, and we proceed with various subcases, listed as follows:

- **CASE 1:** n'_1 and n'_2 are determined.

Notice that n' has already been fixed since $n' = n'_1 + n'_2 + n_3 + n_4$. Fix m_3 (we cannot fix any more integers); then m_1 is determined by m_2, while m_2 satisfies a non-degenerate (since $m_1 \neq -m_2$) quadratic equation with at most two roots. The same is true for m'_1 and m'_2.

- **CASE 2:** m_1 and m_2 are determined.

Notice that m_3 is determined in this case by $m_3 = n - m_1 - m_2 - n_4$. Fix n' (we cannot fix any more integers). Having specified n', n_3 and n_4: n'_2 is determined by n'_1, while n'_1 satisfies a non-degenerate (since $n'_1 \neq -n'_2$) quadratic equation with at most two roots. With m_3 determined, the same is true for m'_1 and m'_2.

- **CASE 3:** m'_1 and m'_2 are determined.

Having specified m'_1, m'_2 and n_4, n' is determined by m_3, and m_3 satisfies a non-degenerate (since $n' \neq m_3$) quadratic equation with at most two roots. From here all integers are determined, and we only count $(N^0)^2$ elements in this case. Indeed, with n, m_3 and n_4 specified, m_1 is determined by m_2, and m_2 satisfies a non-degenerate ($m_1 \neq -m_2$) quadratic equation with at most two roots. The same reasoning applies to the pair n'_1 and n'_2.

- **CASE 4:** n'_1 and m_1 are determined.

Fix m_3 (we cannot select any more integers). Having specified n, m_1, m_3 and n_4, we observe that m_2 is determined by $m_2 = n - m_1 - m_3 - n_4$. Then having specified n'_1, n_3 and n_4, the integer n' is determined by n'_2, which satisfies a non-degenerate ($n' \neq n'_2$) quadratic equation with at most two roots. Lastly, with n', m_3, and n_4 specified, m'_1 and m'_2 are determined by the same argument (since $m'_1 \neq -m'_2$).

Remaining cases will be similar to case 4 (with two integers in distinct quintuples determined). We can always proceed as in case 4: by fixing m_3, and determining all remaining integers as the solutions to non-degenerate quadratic equations. The details are omitted, and the proof of Lemma 6.10 is complete. □

Proof of Lemma 6.11. This proof will rely on the property (6.20), which is satisfied here by the definition of $*(n_4, \mu)$ (see (6.29)–(6.30)). The main advantage of this property is that the integers n_1, n_2 and n_3 (and m_1, m_2 and m_3) now play symmetric roles in Lemma 6.11, and thus in the remainder of this proof. Indeed, it now suffices to show that $\#\{S(n, mu)\} < N_1 N_2$.

As in the proof of Lemma 6.10, we will prove that $\#\{S(n,\mu)\} < N_1 N_2$ by enumerating all elements of the set $S(n,\mu)$ with a specific algorithm. This algorithm will require that we select $|n_1| \sim N_1$ and $|n_2| \sim N_2$, and the remaining integers $\{n_3, n_4, m_1, m_2, m_3\}$ will be determined by the conditions required to remain in $S(n,\mu)$ (the estimate $\#\{S(n,\mu)\} < N_1 N_2$ follows).

Select $|n_1| \sim N_1$ and $|n_2| \sim N_2$ (we will not select any more integers). With n, n_1 and n_2 determined, n_4 is determined by n_3, which satisfies a non-degenerate (since $n_3 \neq -n_4$ by (6.20)) quadratic equation with at most 2 roots. Then, due to (A.1), the non-zero expectation condition

$$\mathbb{E}\big(g_{n_1}(\omega) g_{n_2}(\omega) g_{n_3}(\omega) \overline{g_{m_1}}(\omega) \overline{g_{m_2}}(\omega) \overline{g_{m_3}}(\omega)\big) \neq 0$$

requires that the indices in the set $\{n_1, n_2, n_3, -m_1, -m_2, -m_3\}$ can be organized into pairs such that the sum of the integers is zero in each pair. By the condition (6.20), with n_1, n_2 and n_3 determined, it follows that m_1, m_2 and m_3 are determined. All integers have been determined, and we have enumerated $S(n,\mu)$ with only $N_1 N_2$ selections. The proof of Lemma 6.11 is complete. □

Next we prove Lemma 6.13, then we will use a similar method to prove Lemma 6.12.

Proof of Lemma 6.13. If n and n_4 are determined, there are two cases to consider.

- **CASE 1:** $n = n^0$ or $n = n^1$.

Without loss of generality, suppose $n = n^0$. Since $|n_4| \le |n_3| \le |n_2|$, we cannot have $n_4 = n^0$ or $n_4 = n^1$. Thus, we must have $n_k = n^1$ for some $k \in \{1, 2, 3\}$. Suppose $n_1 = n^1$. By the condition $|n + n_1| = |n^0 + n^1| \ll (N^0)^{\frac{16\delta}{1-4\delta}}$, with n fixed, there at most $(N^0)^{\frac{16\delta}{1-4\delta}}$ possible values for n_1; we make this choice and fix n_1. Having determined n, n_1 and n_4, n_2 is determined by n_3, which satisfies a non-degenerate (since $n_2 \neq -n_3$, recall (6.20)) quadratic equation with at most two roots. All integers have now been determined, and we proceed to the next case.

- **CASE 2:** $n = n^2$, $n = n^3$ or $n = n^4$.

Since $|n_4| \le |n_3| \le |n_2|$, we must have $n_4 = n^k$ for some $k \in \{2, 3, 4\}$. In particular, with n and n_4 fixed in this case, two of the integers in the triple (n^2, n^3, n^4) have been determined. Suppose n_2 is the remaining element in this triple that has not been determined. Then by the condition $|n + n_2 + n_4| = |n^2 + n^3 + n^4| \ll (N^0)^{\frac{16\delta}{1-4\delta}}$, with n and n_4 determined, there are at most $(N^0)^{\frac{16\delta}{1-4\delta}}$ possible values n_2; we make this choice, and fix n_2. Then n_1 is determined by n_3 which satisfies a non-degenerate (by (6.20)) quadratic equation with at most two roots. Having determined all integers with at most $(N^0)^{\frac{16\delta}{1-4\delta}}$ selections, the proof of Lemma 6.13 is complete. □

Remark A.1. We remark that, in the proof of Lemma 6.12 below, we will intentionally avoid using the restriction $N_2 \ge N_3 \ge N_4$. More than that, we will not impose any conditions on n_4 to distinguish it from n_1, n_2, n_3, and thus one can restate (and prove) lemmata analogous to Lemma 6.12 with the role of n_4 swapped with n_1, n_2, or n_3.

Proof of Lemma 6.12. The crucial ingredient to this proof is the probabilistic restriction

$$\mathbb{E}\big(g_{n_1}(\omega) g_{n_2}(\omega) g_{n_3}(\omega) \overline{g_{n_1'}}(\omega) \overline{g_{n_2'}}(\omega) \overline{g_{n_3'}}(\omega) \overline{g_{m_1}}(\omega) \overline{g_{m_2}}(\omega) \overline{g_{m_3}}(\omega) g_{m_1'}(\omega) g_{m_2'}(\omega) g_{m_3'}(\omega)\big) \neq 0.$$

According to (6.17), this restriction allows us to pair the indices in the set $\{n_1, n_2, n_3, -n_1', -n_2', -n_3' - m_1, -m_2, -m_3, m_1', m_2', m_3'\}$ such that the sum of the indices is zero in each pair. Once again, due to (6.20), two elements of the same quintuple (e.g. $\{-n, n_1, n_2, n_3, n_4\}$) cannot belong to the same pair.

Fix n and n_4. As long as we do not have $\{-n, n_4\} = \{n^0, n^1\}$, the method used in the proof of Lemma 6.13 applies, and all integers in the set $\{-n, n_1, n_2, n_3, n_4\}$ are determined by at most $(N^0)^{\frac{16\delta}{1-4\delta}}$ selections. If $\{-n, n_4\} = \{n^0, n^1\}$, we fix one of the smaller variables $\{n_1, n_2, n_3\} = \{n^2, n^3, n^4\}$ and use $n = n_1 + n_2 + n_3 + n_4$, and $|n^2 + n^3 + n^4| \ll (N^0)^{\frac{16\delta}{1-4\delta}}$. Only six undetermined elements remain in the set $\{n_1', n_2', n_3' m_1, m_2, m_3, m_1', m_2', m_3'\}$. It is easily verified that, in all cases, we can fix one more variable such that only $(N^0)^{\frac{16\delta}{1-4\delta}}$ choices remain afterward. The size estimate produced by this algorithm is $\#\{S(\mu)\} \lesssim (N^0)^{3+\frac{32\delta}{1-4\delta}}$, and the proof of Lemma 6.12 is complete. □

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