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A proof of Alexandrov's uniqueness theorem for convex surfaces in $\mathbb{R}^3$

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Abstract

We give a new proof of a classical uniqueness theorem of Alexandrov [4] using the weak uniqueness continuation theorem of Bers–Nirenberg [8]. We prove a version of this theorem with the minimal regularity assumption: the spherical Hessians of the corresponding convex bodies as Radon measures are nonsingular.

MSC: 53A05; 53C24

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We give a new proof of the following uniqueness theorem of Alexandrov, using the weak unique continuation theorem of Bers–Nirenberg [8].

Theorem 1. *(See Theorem 9 in [4].) Suppose M_1 and M_2 are two closed strictly convex C^2 surfaces in $\mathbb{R}^3$, suppose $f(y_1, y_2) \in C^1$ is a function such that $\frac{\partial f}{\partial y_1}\frac{\partial f}{\partial y_2} > 0$. Denote by $\kappa_1 \geq \kappa_2$ the principal curvatures of surfaces, and denote by ν_{M_1} and ν_{M_2} the Gauss maps of M_1 and M_2 respectively. If*

$$f\left(\kappa_1\left(\nu_{M_1}^{-1}(x), \kappa_2\left(\nu_{M_1}^{-1}(x)\right)\right)\right) = f\left(\kappa_1\left(\nu_{M_2}^{-1}(x), \kappa_2\left(\nu_{M_2}^{-1}(x)\right)\right)\right), \quad \forall x \in \mathbb{S}^2 \tag{1}$$

then M_1 is equal to M_2 up to a translation.

This classical result was first proved for analytical surfaces by Alexandrov in [3], for C^4 surfaces by Pogorelov in [20], and Hartman and Wintner [14] reduced regularity to C^3, see also [21]. Pogorelov [22,23] published certain uniqueness results for C^2 surfaces, these general results would imply Theorem 1 in C^2 case. It was pointed out

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in [19] that the proof of Pogorelov is erroneous, it contains an uncorrectable mistake (see pp. 301–302 in [19]). There is a counter-example of Martinez-Maure [15] (see also [19]) to the main claims in [22,23]. The results by Han–Nadirashvili–Yuan [13] imply two proofs of Theorem 1, one for C^2 surfaces and another for $C^{2,\alpha}$ surfaces. The problem is often reduced to a uniqueness problem for linear elliptic equations in appropriate settings, either on $\mathbb{S}^2$ or in $\mathbb{R}^3$, we refer to [4,21]. Here we will concentrate on the corresponding equation on $\mathbb{S}^2$, as in [11]. The advantage in this setting is that it is globally defined.

If M is a strictly convex surface with support function u, then the principal curvatures at $\nu^{-1}(x)$ are the reciprocals of the principal radii λ_1, λ_2 of M, which are the eigenvalues of spherical Hessian $W_u(x) = (u_{ij}(x) + u(x)\delta_{ij})$ where u_{ij} are the covariant derivatives with respect to any given local orthonormal frame on $\mathbb{S}^2$. Set

$$\tilde{F}(W_u) =: f\left(\frac{1}{\lambda_1(W_u)}, \frac{1}{\lambda_2(W_u)}\right) = f(\kappa_1, \kappa_2). \tag{2}$$

In view of Lemma 1 in [5], if f satisfies the conditions in Theorem 1, then $\tilde{F}^{ij} = \frac{\partial \tilde{F}}{\partial w_{ij}} \in L^\infty$ is uniformly elliptic. In the case $n = 2$, it can be read off from the explicit formulas

$$\lambda_1 = \frac{\sigma_1(W_u) - \sqrt{\sigma_1(W_u)^2 - 4\sigma_2(W_u)}}{2}, \qquad \lambda_2 = \frac{\sigma_1(W_u) + \sqrt{\sigma_1(W_u)^2 - 4\sigma_2(W_u)}}{2}.$$

As noted by Alexandrov in [5], $\tilde{F}^{ij}$ in general is not continuous if $f(y_1, y_2)$ is not symmetric (even f is analytic).

We want to address when Theorem 1 remains true for convex bodies in $\mathbb{R}^3$ with weakened regularity assumption. In the Brunn–Minkowski theory, the uniqueness of Alexandrov–Fenchel–Jessen [1,2,10] states that, if two bounded convex bodies in $\mathbb{R}^{n+1}$ have the same kth area measures on $\mathbb{S}^n$, then these two bodies are the same up to a rigidity motion in $\mathbb{R}^{n+1}$. Though for a general convex body, the principal curvatures of its boundary may not be defined. But one can always define the support function u, which is a function on $\mathbb{S}^2$. By the convexity, then $W_u = (u_{ij} + u\delta_{ij})$ is a Radon measure on $\mathbb{S}^2$. Also, by Alexandrov's theorem for the differentiability of convex functions, W_u is defined for almost every point $x \in \mathbb{S}^2$. Denote $\mathcal{N}$ to be the space of all positive definite 2×2 matrices, and let G be a function defined on $\mathcal{N}$. For a support function u of a bounded convex body Ω_u, $G(W_u)$ is defined for *a.e.* $x \in \mathbb{S}^2$. For fixed support functions u^l of Ω_{u^l}, $l = 1, 2$, there is $\Omega \subset \mathbb{S}^2$ with $|\mathbb{S}^2 \setminus \Omega| = 0$ such that W_{u^1}, W_{u^2} are pointwise finite in Ω. Set $P_{u^1,u^2} = \{W \in \mathcal{N} \mid \exists x \in \Omega,\ W = W_{u^1}(x),\ \text{or } W = W_{u^2}(x)\}$, let $\mathcal{P}_{u^1,u^2}$ be the convex hull of P_{u^1,u^2} in $\mathcal{N}$.

We establish the following slightly more general version of Theorem 1.

Theorem 2. *Suppose Ω_1 and Ω_2 are two bounded convex bodies in $\mathbb{R}^3$. Let u^l, $l = 1, 2$ be the corresponding supporting functions respectively. Suppose the spherical Hessians $W_{u^l} = (u^l_{ij} + \delta_{ij}u^l)$ (in the weak sense) are two non-singular Radon measures. Let $G : \mathcal{N} \to \mathbb{R}$ be a $C^{0,1}$ function such that*

$$\Lambda I \geq (G^{ij})(W) := \left(\frac{\partial G}{\partial W_{ij}}\right)(W) \geq \lambda I > 0, \quad \forall W \in \mathcal{P}_{u^1,u^2},$$

for some positive constants Λ, λ. If

$$G(W_{u^1}) = G(W_{u^2}), \tag{3}$$

at almost every parallel normal $x \in \mathbb{S}^2$, then Ω_1 is equal to Ω_2 up to a translation.

Suppose u^1, u^2 are the support functions of two convex bodies Ω_1, Ω_2 respectively, and suppose W_{u^l}, $l = 1, 2$ are defined and they satisfy Eq. (3) at some point $x \in \mathbb{S}^2$. Then, for $u = u^1 - u^2$, $W_u(x)$ satisfies equation

$$F^{ij}(x)\big(W_u(x)\big) = 0, \tag{4}$$

with $F^{ij}(x) = \int_0^1 \frac{\partial \tilde{F}}{\partial W_{ij}}(tW_{u^1}(x) + (1-t)W_{u^2}(x))dt$. By the convexity, W_{u^l}, $l = 1, 2$ exist almost everywhere on $\mathbb{S}^2$. If they satisfy Eq. (3) almost everywhere, Eq. (4) is verified almost everywhere. Note that u may *not be a solution (even in a weak sense) of partial differential equation (4)*. The classical elliptic theory (e.g., [16,18,8]) requires $u \in W^{2,2}$ in order to make sense of u as a weak solution of (4). A main step in the proof of Theorem 2 is to show that with the assumptions in the theorem, $u = u^1 - u^2$ is indeed in $W^{2,2}(\mathbb{S}^2)$. The proof will appear in the last part of the paper.

Let's now focus on $W^{2,2}$ solutions of differential equation (4), with general uniformly elliptic condition on tensor F^{ij} on $\mathbb{S}^2$:

$$\lambda|\xi|^2 \le F^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \quad \forall x \in \mathbb{S}^2,\ \xi \in \mathbb{R}^2, \tag{5}$$

for some positive numbers λ, Λ. The aforementioned proofs of Theorem 1 [20,14,21,13] all reduce to the statement that any solution of (5) is a linear function, under various regularity assumptions on F^{ij} and u. Eq. (4) is also related to minimal cone equation in $\mathbb{R}^3$ [13]. The following result was proved in [13].

Theorem 3. *(See Theorem 1.1 in [13].) Suppose $F^{ij}(x) \in L^\infty(\mathbb{S}^2)$ satisfies (5), suppose $u \in W^{2,2}(\mathbb{S}^2)$ is a solution of (4). Then, $u(x) = a_1x_1 + a_2x_2 + a_3x_3$ for some $a_i \in \mathbb{R}$.*

There the original statement in [13] is for 1-homogeneous $W^{2,2}_{loc}(\mathbb{R}^3)$ solution v of equation

$$\sum_{i,j=1}^{3} a^{ij}(X)v_{ij}(X) = 0. \tag{6}$$

These two statements are equivalent. To see this, set $u(x) = \frac{v(X)}{|X|}$ with $x = \frac{X}{|X|}$. By the homogeneity assumption, the radial direction corresponds to null eigenvalue of $\nabla^2 v$, the other two eigenvalues coincide the eigenvalues of the spherical Hessian of $W = (u_{ij} + u\delta_{ij})$. $v(X) \in W^{2,2}_{loc}(\mathbb{R}^3)$ is a solution to (6) if and only if $u \in W^{2,2}(\mathbb{S}^2)$ is a solution to (4) with $F^{ij}(x) = \langle e_i, Ae_j\rangle$, where $A = (a^{ij}(\frac{X}{|X|}))$ and (e_1, e_2) is any orthonormal frame on $\mathbb{S}^2$.

The proof in [13] uses gradient maps and support planes introduced by Alexandrov, as in [3,20,21]. We give a different proof of Theorem 3 using the maximum principle for smooth solutions and the unique continuation theorem of Bers–Nirenberg [8], working purely on solutions of Eq. (4) on $\mathbb{S}^2$.

Note that F in Theorem 2 (and Theorem 1) is not assumed to be symmetric. The weak assumption $F^{ij} \in L^\infty$ is needed to deal with this case. This assumption also fits well with the weak unique continuation theorem of Bers–Nirenberg. This beautiful result of Bers–Nirenberg will be used in a crucial way in our proof. If $u \in W^{2,2}(\mathbb{S}^2)$, $u \in C^\alpha(\mathbb{S}^2)$ for some $0 < \alpha < 1$ by the Sobolev embedding theorem. Eq. (4) and $C^{1,\alpha}$ estimates for 2-d linear elliptic PDE (e.g., [16,18,8]) imply that u is in $C^{1,\alpha}(\mathbb{S}^2)$ for some $\alpha > 0$ depending only on $\|u\|_{C^0}$ and the ellipticity constants of F^{ij}. This fact will be assumed in the rest of the paper.

The following lemma is elementary.

Lemma 4. *Suppose $F^{ij} \in L^\infty(\mathbb{S}^2)$ satisfies (5), suppose at some point $x \in \mathbb{S}^2$, $W_u(x) = (u_{ij}(x)+u(x)\delta_{ij})$ satisfies (4). Then,*

$$|W_u|^2(x) \le -\frac{2\Lambda}{\lambda}\det W_u(x).$$

Proof. At x, by Eq. (4),

$$\det W_u = -\frac{1}{F^{22}}\left(F^{11}W_{11}^2 + 2F^{12}W_{11}W_{12} + F^{22}W_{12}^2\right) \le -\frac{\lambda}{\Lambda}\left(W_{11}^2 + W_{12}^2\right), \tag{7}$$

and similarly, $\det W_u \le -\frac{\lambda}{\Lambda}(W_{22}^2 + W_{21}^2)$. Thus,

$$\left(W_{11}^2 + W_{12}^2 + W_{21}^2 + W_{22}^2\right) \le -\frac{2\Lambda}{\lambda}\det W_u. \quad \square \tag{8}$$

For each $u \in C^1(\mathbb{S}^2)$, set $X_u = \sum_i u_ie_i + ue_{n+1}$. For any unit vector E in $\mathbb{R}^3$, define

$$\phi_E(x) = \langle E, X_u(x)\rangle, \quad \text{and} \quad \rho_u(x) = |X_u(x)|^2, \tag{9}$$

where $\langle\,,\rangle$ is the standard inner product in $\mathbb{R}^3$. The function ρ was introduced by Weyl in his study of Weyl's problem [25]. It played important role in Nirenberg's solution of Weyl's problem in [17]. Our basic observation is that there is a maximum principle for ρ_u and ϕ_E.

Lemma 5. *Suppose $U \subset \mathbb{S}^2$ is an open set, $F^{ij} \in C^1(U)$ is a tensor in U and $u \in C^3(U)$ satisfies Eq. (4), then there are two constants C_1, C_2 depending only on the C^1-norm of F^{ij} such that*

$$F^{ij}(\rho_u)_{ij} \geq -C_1|\nabla \rho_u|, \qquad F^{ij}(\phi_E)_{ij} \geq -C_2|\nabla \phi_E| \quad \text{in } U. \tag{10}$$

Proof. Picking any orthonormal frame e_1, e_2, we have

$$(X_u)_i = W_{ij}e_j, \qquad (X_u)_{ij} = W_{ijk}e_k - W_{ij}\vec{x}. \tag{11}$$

By Codazzi property of W and (4),

$$\frac{1}{2}F^{ij}(\rho_u)_{ij} = \langle X_u, F^{ij}W_{ijk}e_k\rangle + F^{ij}W_{ik}W_{kj} = -u_k F^{ij}_{,k}W_{ij} + F^{ij}W_{ik}W_{kj}.$$

On the other hand, $\nabla\rho_u = 2W \cdot (\nabla u)$. At the non-degenerate points (i.e., $\det W \neq 0$), $\nabla u = \frac{1}{2}W^{-1} \cdot \nabla \rho_u$, where W^{-1} denotes the inverse matrix of W. Now,

$$2u_k F^{ij}_{,k}W_{ij} = W^{kl}(\rho_u)_l F^{ij}_{,k}W_{ij} = (\rho_u)_l F^{ij}_{,k}\frac{A^{kl}W_{ij}}{\det W}, \tag{12}$$

where A^{kl} denotes the co-factor of W_{kl}.

The first inequality in (10) follows from (8) and (12).

The proof for ϕ_E follows the same argument and the following facts:

$$F^{ij}(\phi_E)_{ij} = -\langle E, e_k\rangle F^{ij}_{,k}W_{ij}, \qquad \nabla\phi_E = W \cdot \langle E, e_k\rangle. \qquad \square$$

Lemma 5 yields immediately Theorem 1 in C^3 case, which corresponds to the Hartman–Wintner theorem [14].

Corollary 6. *Suppose $f \in C^2$ and is symmetric, M_1, M_2 are two closed convex C^3 surfaces satisfy conditions in Theorem 1, then the surfaces are the same up to a translation.*

Proof. Since $f \in C^2$ is symmetric, F^{ij} in (4) is in $C^1(\mathbb{S}^2)$ and $u \in C^3(\mathbb{S}^2)$. By Lemma 5 and the strong maximum principle, X_u is a constant vector. $\square$

To precede further, set

$$\mathcal{M} = \Big\{p \in \mathbb{S}^2 : \rho_u(p) = \max_{q\in\mathbb{S}^2}\rho_u(q)\Big\},$$

for each unit vector $E \in \mathbb{R}^3$,

$$\mathcal{M}_E = \Big\{p \in \mathbb{S}^2 : \phi_E(p) = \max_{q\in\mathbb{S}^2}\phi_E(q)\Big\}.$$

Lemma 7. *$\mathcal{M}$ and $\mathcal{M}_E$ have no isolated points.*

Proof. We prove the lemma for $\mathcal{M}$, the proof for $\mathcal{M}_E$ is the same. If point $p_0 \in \mathcal{M}$ is an isolated point, we may assume $p_0 = (0, 0, 1)$. Pick $\bar{U}$ a small open geodesic ball centered at p_0 such that $\bar{U}$ is properly contained in $\mathbb{S}^2_+$, and pick a sequence of smooth 2-tensor $(F^{ij}_\epsilon) > 0$ which is convergent to (F^{ij}) in L^∞-norm in $\bar{U}$. Consider

$$\begin{cases} F^{ij}_\epsilon\big(u^\epsilon_{ij} + u^\epsilon\delta_{ij}\big) = 0 & \text{in } \bar{U} \\ u^\epsilon = u & \text{on } \partial\bar{U}. \end{cases} \tag{13}$$

Since $x_3 > 0$ in $\mathbb{S}^2_+$, one may write $u^\epsilon = x_3 v^\epsilon$ in $\bar{U}$. As $(x_3)_{ij} = -x_3\delta_{ij}$, it easy to check that v^ϵ satisfies

$$F^{ij}_\epsilon v^\epsilon_{ij} + b_k v^\epsilon_k = 0 \quad \text{in } \bar{U}.$$

Therefore, (13) is uniquely solvable.

Since $p_0 \in \mathcal{M}$ is an isolated point, there are open geodesic balls $\bar{U}' \subset \bar{U}$ centered at p_0 and a small $\delta > 0$ such that

$$\rho_u(p_0) - \rho_u(p) \geq \delta \quad \text{for } \forall p \in \partial \bar{U}'. \tag{14}$$

By the $C^{1,\alpha}$ estimates for linear elliptic equation in dimension two and the uniqueness of the Dirichlet problem [16,8,18], $\exists \epsilon_k$ such that

$$\|u - u^{\epsilon_k}\|_{C^{1,\alpha}(\bar{U}')} \to 0, \qquad \|\rho_u - \rho_{u^{\epsilon_k}}\|_{C^{\alpha}(\bar{U}')} \to 0.$$

Together with (14), if ϵ_k is small enough, there is a local maximal point of $\rho_{u^{\epsilon_k}}$ in $\bar{U}' \subset \bar{U}$. Since $u^{\epsilon_k}, F_\epsilon^{ij} \in C^\infty(\bar{U}')$ satisfy (13), it follows from Lemma 5 and the strong maximum principle that $\rho_{u^{\epsilon_k}}$ must be constant in $\bar{U}'$, when ϵ_k is small enough. This implies that ρ is constant in $\bar{U}'$. A contradiction. □

We now prove Theorem 3.

Proof of Theorem 3. For any $p_0 \in \mathcal{M}$, if $\rho_u(p_0) = 0$, then $u \equiv 0$. We may assume $\rho_u(p_0) > 0$. Set $E := \frac{X_u(p_0)}{|X_u(p_0)|}$. Choose another two unit constant vectors β_1, β_2 with $\langle \beta_i, \beta_j \rangle = \delta_{ij}, \beta_i \perp E$ for $i, j = 1, 2$. Under these orthogonal coordinates in $\mathbb{R}^3$,

$$X_u(p) = a(p)E + b_1(p)\beta_1 + b_2(p)\beta_2, \quad \forall p \in \mathcal{M}_E. \tag{15}$$

On the other hand, $\phi_E(p) = \rho_u^{1/2}(p_0)$, $\forall p \in \mathcal{M}_E$. Thus,

$$a(p) = \rho_u^{1/2}(p_0), \qquad b_1(p) = b_2(p) = 0, \quad \forall p \in \mathcal{M}_E. \tag{16}$$

Consider the function $\tilde{u}(x) = u(x) - \rho_u^{1/2}(p_0)E \cdot x$. (15) and (16) yield, $\forall p \in \mathcal{M}_E$,

$$\nabla_{e_i}\tilde{u}(p) = \nabla_{e_i}u(p) - \rho_u^{1/2}(p_0)\langle E, e_i\rangle = \langle X_u(p), e_i\rangle - \rho_u^{1/2}(p_0)\langle E, e_i\rangle = 0. \tag{17}$$

Moreover, $\tilde{u}(x)$ also satisfies Eq. (4). As pointed out in [8], if $\tilde{u}$ satisfies an elliptic equation, $\nabla\tilde{u}$ satisfies an elliptic system of equations. Lemma 7, (17) and the unique continuation theorem of Bers–Nirenberg (p. 113 in [7]) imply $\nabla\tilde{u} \equiv 0$. Thus, $\tilde{u}(x) \equiv \tilde{u}(p_0) = 0$ and $u(x)$ is a linear function on $\mathbb{S}^2$. □

Theorem 1 is a direct consequence of Theorem 3. We now prove Theorem 2.

Proof of Theorem 2. The main step is to show $u = u^1 - u^2 \in W^{2,2}(\mathbb{S}^2)$, using the assumption that $W_{u^l}, l = 1, 2$ are non-singular Radon measures. It follows from the convexity, the spherical Hessians $W_{u^l}, l = 1, 2$ and W_u are defined almost everywhere on $\mathbb{S}^2$ (Alexandrov's theorem). So, we can define $G(W_{u^l}), l = 1, 2$ almost everywhere in $\mathbb{S}^2$. As $W_u^l, l = 1, 2$ are nonsingular Radon measures, $W_{u^l} \in L^1(\mathbb{S}^2)$ (see [9]), we also have $W_u \in L^1(\mathbb{S}^2)$. Since u^1, u^2 satisfy $G(W_{u^1}) = G(W_{u^2})$ for almost every parallel normal $x \in \mathbb{S}^2$, there is $\Omega \subset \mathbb{S}^2$ with $|\mathbb{S}^2 \setminus \Omega| = 0$, such that W_u satisfies the following equation *pointwise* in Ω,

$$G^{ij}(x)\big(u_{ij}(x) + u(x)\delta_{ij}\big) = 0, \quad x \in \Omega,$$

where $G^{ij} = \int_0^1 \frac{\partial G}{\partial w_{ij}}(tW_u^1 + (1-t)W_u^2)dt$. By Lemma 4, we can obtain that

$$|W_u|^2 = W_{11}^2 + W_{12}^2 + W_{21}^2 + W_{22}^2 \leq -\frac{2\Lambda}{\lambda}\det W_u, \quad x \in \Omega.$$

On the other hand,

$$\det W_u \leq \det W_{\tilde{u}},$$

where $\tilde{u} = u^1 + u^2$. Thus, to prove $u \in W^{2,2}(\mathbb{S}^2)$, it suffices to get an upper bound for $\int_{\mathbb{S}^2} \det W_{\tilde{u}}$.

Recall that $W_{u^l} \in L^1(\mathbb{S}^2)$, so $u^l \in W^{2,1}(\mathbb{S}^2)$, $l = 1, 2$ and the same for $\tilde{u}$. This allows us to choose two sequences of smooth convex bodies Ω_ϵ^l with supporting functions u_ϵ^l such that $\|\tilde{u}_\epsilon - \tilde{u}\|_{W^{2,1}(\mathbb{S}^2)} \to 0$ as $\epsilon \to 0$. By Fatou's Lemma and continuity of the area measures,

$$\int_{\mathbb{S}^2} \det W_{\tilde{u}} = \int_{\Omega} \det W_{\tilde{u}} \le \liminf_{\epsilon\to 0} \int_{\mathbb{S}^2} \det W_{\tilde{u}_\epsilon} \le V(\Omega^1) + V(\Omega^2) + 2V(\Omega^1, \Omega^2),$$

where $V(\Omega^1)$, $V(\Omega^2)$ denote the volumes of the convex bodies Ω^1 and Ω^2 respectively and $V(\Omega^1, \Omega^2)$ is the mixed volume.

It follows that $W_u \in L^2(\mathbb{S}^2)$ and thus, $u \in W^{2,2}(\mathbb{S}^2)$. This implies that u is a $W^{2,2}$ weak solution of the differential equation

$$G^{ij}(x)\big(u_{ij}(x) + u(x)\delta_{ij}\big) = 0, \quad \forall x \in \mathbb{S}^2.$$

Finally, the theorem follows directly from Theorem 3. □

Remark 8. Alexandrov proved in [3] that, if u is a homogeneous degree 1 analytic function in $\mathbb{R}^3$ with $\nabla^2 u$ definite nowhere, then u is a linear function. As a consequence, Alexandrov proved in [6] that if an analytic closed convex surface in $\mathbb{R}^3$ satisfies the condition $(\kappa_1 - c)(\kappa_2 - c) \le 0$ at every point for some constant c, then it is a sphere. Martinez-Maure gave a C^2 counter-example in [15] to this statement, see also [19]. The counter-examples in [15,19] indicate that Theorem 3 is not true if F^{ij} is merely assumed to be degenerate elliptic. It is an interesting question that under what degeneracy condition on F^{ij} so that Theorem 3 is still true, even in smooth case. This question is related to similar questions in this nature posted by Alexandrov [4] and Pogorelov [21].

We shall wrap up this paper by mention a stability type result related with uniqueness. Indeed, by using the uniqueness property proved in Theorem 3, we can prove the following stability theorem via compactness argument.

Proposition 9. *Suppose $F^{ij}(x) \in L^\infty(\mathbb{S}^2)$ satisfies (5), and $u(x) \in W^{2,2}(\mathbb{S}^2)$ is a solution of the following equation*

$$F^{ij}(x)(W_u)_{ij} = f(x), \quad \forall x \in \mathbb{S}^2. \tag{18}$$

Assume that $f(x) \in L^\infty(\mathbb{S}^2)$ and there exists a point $x_0 \in \mathbb{S}^2$ such that $\rho_u(x_0) = 0$ (see (9) for the definition of ρ_u). Then,

$$\|u\|_{L^\infty(\mathbb{S}^2)} \le C_3 \|f\|_{L^\infty(\mathbb{S}^2)} \tag{19}$$

holds for some positive constant C_3 only depending on the ellipticity constants λ, Λ.

Proof. As mentioned above, we will prove this proposition by a compactness argument. Suppose the desired estimate (19) does not hold, then there exists a sequence of functions $\{f_n(x)\}_{n=1}^\infty$ on $\mathbb{S}^2$ with $\|f\|_{L^\infty(\mathbb{S}^2)} \le C_4$ and a sequence of points $\{x_n\}_{n=1}^\infty \subset \mathbb{S}^2$ such that $\rho_{u_n}(x_n) = 0$ and $K_n := \frac{\|u\|_{L^\infty(\mathbb{S}^2)}}{\|f\|_{L^\infty(\mathbb{S}^2)}} \to +\infty$, where $u_n(x)$ is the solution of Eq. (18) with right hand side replaced by $f_n(x)$.

Let $v_n(x) = \frac{u_n(x)}{K_n \|f\|_{L^\infty(\mathbb{S}^2)}}$, then $\|v_n\|_{L^\infty(\mathbb{S}^2)} = 1$ and $v_n(x)$ satisfies

$$F^{ij}(x)(W_{v_n})_{ij} = \tilde{f}_n := \frac{f_n(x)}{K_n \|f_n\|_{L^\infty(\mathbb{S}^2)}}. \tag{20}$$

By the interior $C^{1,\alpha}$ estimates for linear elliptic equation in dimension two [16,8,18], we have

$$\|v_n\|_{C^{1,\alpha}(\mathbb{S}^2)} \le C_5\big(\|v_n\|_{L^\infty(\mathbb{S}^2)} + \|\tilde{f}_n\|_{L^\infty(\mathbb{S}^2)}\big) \le 2C_5$$

for some positive constant $C_5 = C_5(\lambda, \Lambda)$. In particular, this gives that $\|\nabla v_n\|_{L^\infty(\mathbb{S}^2)} \le C_6$. Now, apply the *a priori* $W^{2,2}$ estimate for linear elliptic equation in dimension two [16,8,18,12], we see that $\|v_n\|_{W^{2,2}(\mathbb{S}^2)} \le C_7$ for some constant $C_7 = C_7(\lambda, \Lambda, C_6)$. It follows from this uniform estimate that, up to a subsequence, $\{v_n(x)\}_{n=1}^\infty$ converges to some function $v(x) \in W^{2,2}(\mathbb{S}^2)$ and $v(x)$ satisfies

$$F^{ij}(x)(W_v)_{ij} = 0, \quad a.e.\ x \in \mathbb{S}^2.$$

Then, the previous uniqueness result Theorem 3 tells that $v(x)$ must be a linear function, i.e., there exists a constant vector $\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ such that $v(x) = a_1x_1 + a_2x_2 + a_3x_3$.

On the other hand, recall that, by the assumption at the beginning, there exists $x_n \in \mathbb{S}^2$ such that $\rho_{v_n}(x_n) = 0$. Then, up to a subsequence, $x_n \to x_\infty \in \mathbb{S}^2$ and $\rho_v(x_\infty) = 0$. This together with the linear property of $v(x)$ imply that $v(x) \equiv 0$. However, this contradicts with the fact that $\|v\|_{L^\infty(\mathbb{S}^2)} = 1$ as $\|v_n\|_{L^\infty(\mathbb{S}^2)} = 1$. □

As a direct corollary, we have the following stability property for convex surfaces.

Theorem 10. *Suppose M_1, M_2 and f satisfy the same assumptions as in Theorem 3. Define $\mu_1(x) := f(\kappa_1(\nu_{M_1}^{-1}(x), \kappa_2(\nu_{M_1}^{-1}(x)))$ and $\mu_2(x) := f(\kappa_1(\nu_{M_2}^{-1}(x), \kappa_2(\nu_{M_2}^{-1}(x)))$ for $\forall x \in \mathbb{S}^2$. If $\|\mu_1 - \mu_2\|_{L^\infty(\mathbb{S}^2)} < \epsilon$, then, module a linear translation, M_1 is very close to M_2. More precisely, suppose u_1, u_2 are the supporting functions of M_1 and M_2 after module the linear translation, then there exists a constant C such that*

$$\|u_1 - u_2\|_{L^\infty(\mathbb{S}^2)} \le C\|\mu_1 - \mu_2\|_{L^\infty(\mathbb{S}^2)}. \tag{21}$$

Finally, it is worth to remark that there are many stability type results for convex surfaces proved in the literature (see [24]). However, almost all the proofs need to use the assumption that $f(\kappa_1, \kappa_2, \cdots, \kappa_n)$ satisfies divergence property. Here, we do not make such kind assumption in this dimension two case. There is one drawback in the above stability result: one could not get the sharp constant via the compactness argument. It would be an interesting question to derive a sharp estimate for (21).

Conflict of interest statement

There is no conflict of interest.

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