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Asymptotic bifurcation and second order elliptic equations on $\mathbb{R}^N$

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Abstract

This paper deals with asymptotic bifurcation, first in the abstract setting of an equation $G(u) = \lambda u$, where G acts between real Hilbert spaces and $\lambda \in \mathbb{R}$, and then for square-integrable solutions of a second order non-linear elliptic equation on $\mathbb{R}^N$. The novel feature of this work is that G is not required to be asymptotically linear in the usual sense since this condition is not appropriate for the application to the elliptic problem. Instead, G is only required to be Hadamard asymptotically linear and we give conditions ensuring that there is asymptotic bifurcation at eigenvalues of odd multiplicity of the H-asymptotic derivative which are sufficiently far from the essential spectrum. The latter restriction is justified since we also show that for some elliptic equations there is no asymptotic bifurcation at a simple eigenvalue of the H-asymptotic derivative if it is too close to the essential spectrum.

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1. Introduction

Let X and Y be real Banach spaces with $X \subset Y$ and consider a function $G : X \to Y$. There is asymptotic bifurcation at $\mu \in \mathbb{R}$ for the equation

$$G(u) = \lambda u \tag{1.1}$$

if there is a sequence of solutions $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times X$ such that $\lambda_n \to \mu$ and $\|u_n\|_X \to \infty$ as $n \to \infty$. The purpose of this paper is twofold.

(1) To provide general criteria for determining whether there is or is not asymptotic bifurcation at a point μ.
(2) To treat the particular case of the nonlinear elliptic eigenvalue problem

$$-\Delta u + V u + g(u) = \lambda u \quad \text{for } u \in H^2(\mathbb{R}^N), \tag{1.2}$$

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where

$$V \in L^{\infty}(\mathbb{R}^N) \quad \text{and} \quad g \in C^1(\mathbb{R}^N) \quad \text{with } g(0)=0 \text{ and } \lim_{|s|\to\infty} g'(s) = \ell \in \mathbb{R}. \tag{1.3}$$

The study of asymptotic bifurcation (or bifurcation from infinity, as it is sometimes called) has a long history and it appears in the classical texts by M.A. Krasnoselskii [13,14] in the context of equations of the form (1.1) where $G : X \to Y$ is asymptotically linear. In 1973 Rabinowitz [15] and Toland [23] introduced independently the use of the inversion $u \mapsto v = u/\|u\|_X^2$ to deal with asymptotic bifurcation for asymptotically linear problems and since then it has become the standard tool for dealing with such problems, see [1,3,27] for example. Setting $G^*(v) = \|v\|_X^2 G(v/\|v\|_X^2)$ for $v \neq 0$ and $G^*(0) = 0$, there is asymptotic bifurcation at μ for the equation $G(u) = \lambda u$ if and only if μ is a bifurcation point for the equation $G^*(v) = \lambda v$ in the usual sense. Furthermore, $G^* : X \to Y$ is Fréchet differentiable at 0 if and only if G is asymptotically linear. In this way, under appropriate compactness conditions allowing them to use the Leray–Schauder degree, Rabinowitz and Toland obtained results about asymptotic bifurcation for abstract equations of the form $G(u) = \lambda u$ and applied them to elliptic boundary-value problems on bounded domains. It turns out that the situation concerning unbounded domains is quite different and this is a main theme of the present paper.

Under our hypotheses (1.3), $g : \mathbb{R} \to \mathbb{R}$ is asymptotically linear with $\lim_{|s|\to\infty} \frac{g(s)}{s} = \ell$ and $G \in C(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ for all $p \in [1, \infty)$ where $G(u) = -\Delta u + Vu + g(u)$. However, as was shown in [19], $G : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is asymptotically linear if and only if $g(s) \equiv \ell s$ for all $s \in \mathbb{R}$, and its inversion is Fréchet differentiable at 0 only in the case where (1.2) is a linear equation. It was also shown in [19] that $G^* : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is differentiable at 0 for all $p \in [1, \infty)$ in the weaker sense called Hadamard differentiability [9] under our hypotheses. Therefore one might hope that the results in [7,8,17] concerning bifurcation for problems that are only Hadamard differentiable can be applied to the inverted version of (1.2). However, all those results involve hypotheses about concavity and compactness which are not always satisfied by (1.2). Furthermore, after some work one finds that the lack of asymptotic linearity means that the variational approach to asymptotic bifurcation via inversion, as developed in [26,4], also breaks down. This situation was a major motivation for the development in [20] of a new approach to bifurcation for Hadamard differentiable problems, avoiding all assumptions about concavity and compactness. In the present paper we exploit these new results in the context of asymptotic bifurcation, first for the abstract equation (1.1) and then for (1.2).

The rest of this paper is organised as follows. In Section 2, various definitions concerning differentiability and asymptotic linearity that are relevant for our results are recalled. Section 3 contains some preliminary results about quantities required to formulate the hypotheses of the main theorems. First of all we present the notion of essential conditioning number for a Fredholm operator of index zero. Then we consider the Lipschitz modulus and its estimation under inversion. In Section 4 we turn to the main issue of asymptotic bifurcation for (1.1). Using inversion this is reduced to the study of bifurcation from the line of trivial solutions for a problem which is Hadamard differentiable, but not necessarily Fréchet differentiable, at 0. The section begins with the statement of a special case of our recent results in [20] on this topic, which is then applied to the inversion of (1.1) to obtain our conclusions about asymptotic bifurcation. It should be noted that [20] deals with nonlinear eigenvalue problems in the general form $F(\lambda, u) = 0$ where $F : \mathbb{R} \times X \to Y$ and not just the special case $F(\lambda, u) = G(u) - \lambda u$. Although we do not undertake it here, since the hypotheses then take a somewhat less intuitive form, it would be easy to derive results about asymptotic bifurcation for this general form, starting from [20] and following the same procedure as in Section 4.

Asymptotic bifurcation for the elliptic equation (1.2) is the subject of Section 5. The nonlinearity $g \in C(\mathbb{R})$ is required to be asymptotically linear but the context is somewhat broader than (1.3). In Section 5.1 we use the results in Section 4 to show that there is continuous asymptotic bifurcation at an isolated eigenvalue of odd multiplicity of the H-asymptotic derivative and that there is no asymptotic bifurcation at regular points of this operator. However, as in the abstract situation treated in Section 4.2, both these conclusions are proved under the assumption that the point under consideration is sufficiently far from the essential spectrum of the H-asymptotic derivative. This restriction does not appear in the analogous results for asymptotically linear problems. The remainder of the paper is devoted to showing that it is not merely a technical hypothesis and that the conclusions obtained in Sections 4.2 and 5.1 concerning asymptotic bifurcation can fail when it is not satisfied. In Section 5.2 we show that there are special cases of (1.2) for which there is no asymptotic bifurcation at a simple eigenvalue of the H-asymptotic derivative which is too close to its essential spectrum, even if it is the infimum of the spectrum. Then, in Section 5.3, we show that there are also cases where asymptotic bifurcation can occur at points which are not in the spectrum of the H-asymptotic derivative, in fact, even at a point lying below the whole spectrum. The situations presented in Sections 5.2 and 5.3 are not just relevant to asymptotic bifurcation since, through inversion, they provide examples of problems where there is not bifurcation at a simple eigenvalue and there is bifurcation at a regular point of the Hadamard derivative at 0 which is sufficiently close to the essential spectrum. Thus they provide additional evidence, to what is already available in [20,21], that a restriction like (4.1) below, or (5.1) in [20], is necessary in order to draw conclusions about bifurcation for problems that are only Hadamard differentiable at 0.

Finally, we should comment on earlier work concerning problems similar to (1.2). Soon after his early work using inversion, Toland [24,25] treated one-dimensional problems like (1.2) on unbounded intervals. However, the nonlinearity $g(u)$ is replaced by a term $g(x,u)$ with a non-trivial x-dependence which renders the associated operator asymptotically linear and hence ensures Fréchet differentiability of the inversion. More recent work has exploited this approach as a preliminary step to dealing with an autonomous nonlinearity $g(u)$. In [22] it is shown in Appendix B that, if $g(u)$ is replaced by $\ell u+\psi(x)\{g(u)-\ell u\}$ where $\psi\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, then the associated operator is asymptotically linear from $W^{2,p}(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$ for $p>\min\{1,N/2\}$ with asymptotic derivative ℓu. This is used to establish asymptotic bifurcation for a variant of (1.2) with a nonlinearity $\ell u+\chi_n(x)\{g(u)-\ell u\}$ where χ_n is the characteristic function of the ball $B(0,n)\subset\mathbb{R}^N$. For the autonomous nonlinearity $g(u)$, asymptotic bifurcation from the lowest eigenvalue of the asymptotic derivative is deduced from this by letting $n\to\infty$ under favourable assumptions about the behaviour of the nonlinearity g. By a similar approach based on approximation and passage to a limit, Genoud [10,11] deals with the general form $\Delta u+f(x,u)u=\lambda u$ and proves asymptotic bifurcation of positive solutions under certain hypotheses about the behaviour of f. When his results are applied to (1.2) they imply asymptotic bifurcation from the lowest eigenvalue Λ of the H-asymptotic derivative without any restriction on the distance of Λ from the essential spectrum. This is because, to satisfy his assumptions (for example (f4) in [11]) about the term $f(x,u)$ requires that $g(s)/s\geq\ell$ for all $s\neq 0$, whereas no condition of this kind is required for the results obtained here in Section 5.1. Indeed, although Λ is always an asymptotic bifurcation point for the truncated problems, the results in Section 5.2 show that it may not be an asymptotic bifurcation point for the original problem and the passage to the limit can fail when $g(s)/s$ is increasing on $(0,\infty)$. Furthermore, our discussion of asymptotic bifurcation is not restricted to the lowest eigenvalue Λ of the H-asymptotic derivative, but covers any isolated eigenvalue of odd multiplicity, possibly lying in a gap in the essential spectrum, so long as the distance from the essential spectrum is large enough. Nonetheless, even for the simplest equations of the type (1.2), such as the example discussed in Section 5.4, there are cases not covered by any of the results we have mentioned and there remains plenty of scope for further research on this problem.

2. Differentiability and asymptotic linearity

As mentioned in the Introduction, differentiability at the origin and asymptotic linearity are related through inversion. In this section we give a more precise account of these notions, starting with inversion.

Let X and Y be real Banach spaces. For a function $M:X\to Y$, the map $M^*:X\to Y$ defined by

$$M^*(u)=\|u\|_X^2 M\left(\frac{u}{\|u\|_X^2}\right)\quad\text{for } u\neq 0,\qquad M^*(0)=0$$

will be called the inversion of M.

Note that

$$M^{**}(u)=\|u\|_X^2 M^*\left(\frac{u}{\|u\|_X^2}\right)=M(u)\quad\text{for all } u\neq 0$$

and so $M^{**}=M\Leftrightarrow M(0)=0$.

Clearly, u is solution with large norm of the equation $M(u)=0$ if and only if $v=\frac{u}{\|u\|_X^2}$ is a small solution of $M^*(v)=0$ with $v\neq 0$.

We now turn to differentiability and asymptotic linearity.

Definition. A map $M : X \to Y$ is Fréchet differentiable at $u \in X$ if there exists $T \in B(X, Y)$ such that

$$\left\| M(u+v) - M(u) - Tv \right\|_Y / \|v\|_X \to 0 \quad \text{as } \|v\|_X \to 0.$$

Then T is unique and $T = M'(u)$ is the Fréchet derivative of M at u.

Definition. A map $M : X \to Y$ is asymptotically linear if there exists $L \in B(X, Y)$ such that

$$\left\| M(u) - L(u) \right\|_Y / \|u\|_X \to 0 \quad \text{as } \|u\|_X \to \infty.$$

In this case, L is unique and is denoted $M'(\infty)$, the asymptotic derivative of M. The following result is well-known and it appears already in [13].

Proposition 2.1. *A map $M : X \to Y$ is asymptotically linear $\Leftrightarrow$ $M^* : X \to Y$ is Fréchet differentiable at 0. In this case, $M'(\infty) = (M^*)'(0)$.*

For a function g satisfying

(F) $g \in C(\mathbb{R}, \mathbb{R})$, $\lim_{|s|\to\infty} \frac{g(s)}{s} = \ell$ for some $\ell \in \mathbb{R}$ and $\limsup_{s\to 0} |\frac{g(s)}{s}| < \infty$,

there is a constant C such that $|g(s)| \le C|s|$ for all $s \in \mathbb{R}$ and so an operator $G : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ can be defined for any $p \in [1, \infty)$ by setting

$$G(u) = -\Delta u + Vu + g(u),$$

provided that $V \in L^\infty(\mathbb{R}^N)$ and g satisfies (F). Since $g : \mathbb{R} \to \mathbb{R}$ is asymptotically linear, one might expect G to inherit this property. However, as is shown in Theorem 5.2 of [19] this is not so, except in the trivial case where g is linear.

Proposition 2.2. *Let $V \in L^\infty(\mathbb{R}^N)$ and g satisfy the condition (F). For $1 \le p < \infty$,*
(a) $G : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is continuous and bounded, but
(b) $G : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is asymptotically linear if and only if it is linear (i.e. $g(s) = \ell s$ for all $s \in \mathbb{R}$).

Confronted by this situation, we introduced in [19] a weaker notion of asymptotic linearity related to Hadamard differentiability of the inversion at the origin.

Definition. A function $M : X \to Y$ is Hadamard differentiable at $u \in X$ if there exists $T \in B(X, Y)$ such that

$$\left\| \frac{M(u + t_n v_n) - M(u)}{t_n} - Tv \right\|_Y \to 0$$
$$\left(\Leftrightarrow \left\| M(u + t_n v_n) - M(u) - T(t_n v_n) \right\|_Y / |t_n| \to 0\right)$$

for all sequences $\{t_n\} \subset \mathbb{R}\backslash\{0\}$ and $\{v_n\} \subset X$ such that $t_n \to 0$ and $\|v_n - v\|_X \to 0$ for some $v \in X$.

In this case, T is unique and is denoted $M'(u)$, the Hadamard derivative of M at u. When $\dim X < \infty$, Hadamard differentiability and Fréchet differentiability are equivalent. Hadamard differentiability is discussed at length in Chapter 4 of [9].

The following weaker version of asymptotic linearity, which will be referred to as Hadamard asymptotic linearity, was proposed and investigated in [19], together with several variants.

Definition. A map $M : X \to Y$ is H-asymptotically linear if there exits $L \in B(X, Y)$ such that $\| \frac{M(t_n u_n)}{t_n} - Lu\|_Y \to 0$ for all sequences $\{t_n\} \subset \mathbb{R}$ and $\{u_n\} \subset X$ such that $\|t_n u_n\| \to \infty$ and $\|u_n - u\|_X \to 0$ for some $u \in X$.

Then L is unique and is denoted $M'(\infty)$, the H-asymptotic derivative of M. Theorem A.1 in [19] shows that it is equivalent to requiring that

(i) $\limsup_{\|u\|\to\infty} \|M(u)\|/\|u\| < \infty$, and

(ii) there exists $L \in B(X,Y)$ such that $\|\frac{M(t_n u_n)}{t_n} - Lu\|_Y \to 0$ for all sequences $\{t_n\} \subset \mathbb{R}$ and $\{u_n\} \subset X$ such that $|t_n| \to \infty$ and $\|u_n - u\|_X \to 0$ for some $u \in X\backslash\{0\}$.

Asymptotic linearity implies H-asymptotic linearity with the same asymptotic derivative. Furthermore, if $\dim X < \infty$, then (ii) implies (i) in the definition of H-asymptotic linearity and H-asymptotic linearity is equivalent to asymptotic linearity.

Proposition 2.3. *A map $M : X \to Y$ is H-asymptotically linear $\Leftrightarrow$ $M^* : X \to Y$ is Hadamard differentiable at* 0. *In this case, $M'(\infty) = (M^*)'(0)$.*

This is Theorem 3.1 in [19] and from Theorem 5.2 of that paper we obtain the following result concerning the left hand side of (1.2).

Proposition 2.4. *Let $V \in L^\infty(\mathbb{R}^N)$ and g satisfy the condition (F). For $1 \le p < \infty$, $G : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is H-asymptotically linear with $G'(\infty) = -\Delta + V + \ell$.*

Combining these results we see that $G^* : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is Hadamard differentiable at 0.

3. Two important quantities

This section introduces two quantities which are used to formulate a crucial hypothesis for our abstract results on bifurcation and asymptotic bifurcation. The first one concerns Fredholm operators of index zero and the second one concerns mappings which are Lipschitz continuous in some neighbourhood of the origin.

3.1. The essential conditioning number

Let X and Y be real Hilbert spaces. For linear operators mapping X into Y we use the notation

- $B(X,Y)$ for the bounded operators
- $Iso(X,Y)$ for the isomorphisms
- $\Phi_0(X,Y)$ for the bounded Fredholm operators of index zero
- $F(X,Y)$ for the bounded operators of finite rank.

The following quantities are discussed in Section 5 of [20]. For $L \in B(X,Y)$, $[L] = \{T : T - L \in F(X,Y)\}$ and $[L]_r = [L] \cap Iso(X,Y)$.

For $L \in \Phi_0(X,Y)$, $[L]_r \neq \emptyset$ and, when X is a Hilbert space, the quantity $\gamma(L) \equiv \inf\{\|T^{-1}\| : T \in [L]_r\}$ is the **essential conditioning number** of L.

Accurate estimation of $\gamma(L)$ is important in order to make the best use of the results in [20] concerning bifurcation for Hadamard differentiable problems. We can do this when L is a self-adjoint operator.

3.1.1. Self-adjoint operators and the graph space

Let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a real Hilbert space. For a self-adjoint operator $S : D(S) \subset H \to H$ acting in H, the graph norm of S on $D(S)$ is defined by

$$\|u\|_S = \left\{\|u\|^2 + \|Su\|^2\right\}^{1/2} \quad \text{for } u \in D(S).$$

Recall that since S is closed, the graph space $(D(S), \|\cdot\|_S)$ is a Hilbert space. The following result is an easy consequence of the closed graph theorem, see Section 5 of [20].

Proposition 3.1. *Let $S : D(S) \subset H \to H$ and $T : D(T) \subset H \to H$ be two self-adjoint operators having the same domain $X = D(S) = D(T)$. Then $\|\cdot\|_S$ and $\|\cdot\|_T$ are equivalent norms on the subspace X and $S, T \in B(X,H)$ for any of these norms.*

For a self-adjoint operator $S : D(S) \subset H \to H$ and any $\varepsilon > 0$, $\|\cdot\|_S$ and $\|\cdot\|_{\varepsilon S}$ are equivalent norms on $D(S)$ and $\|\cdot\| \le \|\cdot\|_{\varepsilon S}$. The spectrum and essential spectrum of S are denoted by $\sigma(S)$ and $\sigma_e(S)$, respectively.

- $\sigma(S) = \{\lambda \in \mathbb{R} : S - \lambda I \notin Iso(D(S), H)\}$
- $\sigma_e(S) = \{\lambda \in \sigma(S) : S - \lambda I \notin \Phi_0(D(S), H)\}$
- $S - \lambda I \in \Phi_0(D(S), H) \Leftrightarrow \lambda \notin \sigma_e(S)$.

Proposition 3.2. *Let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a real Hilbert space, $S : D(S) \subset H \to H$ a self-adjoint operator and $\lambda_0 \notin \sigma_e(S)$. Then there exists a constant $K > 0$ such that*

$$\gamma^\varepsilon(S - \lambda_0 I) \le \frac{1}{d(\lambda_0, \sigma_e(S))} + \varepsilon K \quad \text{for all } \varepsilon > 0,$$

where $\gamma^\varepsilon(S - \lambda_0 I)$ denotes the essential conditioning number of $S - \lambda_0 I \in \Phi_0(X, H)$ and X is $D(S)$ with the graph norm, $\|\cdot\|_{\varepsilon S}$.

This is Corollary 5.5 in [20].

3.2. The Lipschitz modulus and inversion

Consider a function $G : X \to Y$ where X and Y are real Banach spaces. The quantity $L(G)$ defined by

$$L(G) = \lim_{\delta \to 0} \sup_{\substack{u,v \in B(0,\delta) \\ u \ne v}} \frac{\|G(u) - G(v)\|_Y}{\|u - v\|_X} = \limsup_{\substack{u,v \to 0 \\ u \ne v}} \frac{\|G(u) - G(v)\|_Y}{\|u - v\|_X}$$

is called the **Lipschitz modulus** of G at 0. Here $B(0, \delta)$ denotes the open ball in X with centre 0 and radius δ.

Recall that G is strictly Fréchet differentiable at 0 if there exists $T \in B(X, Y)$ such that

$$\lim_{\substack{u,v \to 0 \\ u \ne v}} \frac{\|G(u) - G(v) - T(u - v)\|_Y}{\|u - v\|_X} = 0,$$

and that this implies that G is Fréchet differentiable at 0 with $G'(0) = T$.

From the discussion of the Lipschitz modulus in [5,20] we recall the following points.

(1) $L(G) < \infty \Leftrightarrow G$ is Lipschitz continuous on some neighbourhood of 0. Hence, $L(G) < \infty$ does not imply even Gâteaux differentiability of G at 0.

(2) However, $L(G) = 0 \Leftrightarrow$

$$\lim_{\substack{u,v \to 0 \\ u \ne v}} \frac{\|G(u) - G(v)\|_Y}{\|u - v\|_X} = 0$$

$\Leftrightarrow G$ is strictly Fréchet differentiable at 0 with $G'(0) = 0$.

Thus G is strictly Fréchet differentiable at $0 \Leftrightarrow$ there exists $T \in B(X, Y)$ such that $L(G - T) = 0$.

(3) If $F \in C^1(U, Y)$ for some open neighbourhood U of 0 in X, then $L(R) = 0$ where $R : X \to Y$ is the remainder, $R(u) = F(u) - \{F(0) + F'(0)u\}$. Also F is strictly Fréchet differentiable at 0 and $L(F) = \|F'(0)\|$.

(4) G strictly Fréchet differentiable at 0 implies that $L(G) = \|G'(0)\|$. But G being Fréchet differentiable at 0 with $L(G) < \infty$ does not imply that G is strictly Fréchet differentiable at 0. Also G being Fréchet differentiable at 0 with $G'(0) = 0$ does not imply that $L(G) < \infty$.

For the present work, it is important to be able to estimate $L(G^*)$ using properties of G. To avoid cumbersome notation we use $\|\cdot\|$ to denote the norms in both X and Y.

Lemma 3.3. *Consider $M : X \to Y$ and its inversion M^* where X is a Hilbert space. Then, for any $T \in B(X, Y)$,*

$$L(M^*) \le \|T\| + 2\limsup_{\|u\| \to \infty} \frac{\|N(u)\|}{\|u\|} + \lim_{R \to \infty} \sup_{\substack{\|u\|, \|v\| \ge R \\ u \ne v}} \frac{\|N(u) - N(v)\|}{\|u - v\|}$$

where $N = M - T$.

(i) Choosing $T=0$, we get the estimate,

$$L\left(M^*\right) \le 2\limsup_{\|u\|\to\infty} \frac{\|M(u)\|}{\|u\|} + \lim_{R\to\infty} \sup_{\substack{\|u\|,\|v\|\ge R\\ u\ne v}} \frac{\|M(u)-M(v)\|}{\|u-v\|} \tag{3.1}$$

from which it follows that

$$L\left(M^*\right) \le 3 \lim_{R\to\infty} \sup_{\substack{\|u\|,\|v\|\ge R\\ u\ne v}} \frac{\|M(u)-M(v)\|}{\|u-v\|} \le 3 \sup_{\substack{u,v\in X\\ u\ne v}} \frac{\|M(u)-M(v)\|}{\|u-v\|}. \tag{3.2}$$

(ii) If $M: X\to Y$ is asymptotically linear, then

$$L\left(M^*\right) \le \left\|M'(\infty)\right\| + \lim_{R\to\infty} \sup_{\substack{\|u\|,\|v\|\ge R\\ u\ne v}} \frac{\|N(u)-N(v)\|}{\|u-v\|}, \tag{3.3}$$

where $N = M - M'(\infty)$. In particular, if $M'(\infty)=0$ we have

$$L\left(M^*\right) \le \sup_{\substack{u,v\in X\\ u\ne v}} \frac{\|M(u)-M(v)\|}{\|u-v\|}. \tag{3.4}$$

Remark 1. In Example 2 after the proof, we show that (3.3) and (3.4) can fail if M is only H-asymptotically linear.

Remark 2. The inversion M^* and the denominator in definition of the Lipschitz modulus are defined using the same norm on X.

Remark 3. Hence it follows from part (i) of Lemma 3.3 that M^* is Lipschitz continuous on a neighbourhood of 0 whenever M is uniformly Lipschitz continuous on the complement of some bounded set.

Proof of Lemma 3.3. We may as well assume that there exists $R>0$ such that

$$\sup_{\|u\|\ge R} \frac{\|N(u)\|}{\|u\|} < \infty \quad \text{and} \quad \sup_{\substack{\|u\|,\|v\|\ge R\\ u\ne v}} \frac{\|N(u)-N(v)\|}{\|u-v\|} < \infty.$$

Note that, for any $T\in B(X,Y)$, $M^* = N^* + T$ and so

$$\frac{\|M^*(u)-M^*(v)\|}{\|u-v\|} \le \frac{\|N^*(u)-N^*(v)\|}{\|u-v\|} + \|T\|.$$

For $u, v \ne 0$,

$$\begin{aligned}\left\|\frac{u}{\|u\|^2} - \frac{v}{\|v\|^2}\right\|^2 &= \frac{1}{\|u\|^2} + \frac{1}{\|v\|^2} - \frac{2\langle u, v\rangle}{\|u\|^2\|v\|^2}\\ &= \frac{1}{\|u\|^2} + \frac{1}{\|v\|^2} + \frac{\|u-v\|^2 - \|u\|^2 - \|v\|^2}{\|u\|^2\|v\|^2} = \frac{\|u-v\|^2}{\|u\|^2\|v\|^2}\end{aligned}$$

and so

$$\left\|\frac{u}{\|u\|^2} - \frac{v}{\|v\|^2}\right\| \le \frac{\|u-v\|}{\|u\|\|v\|}. \tag{3.5}$$

Consider $u, v\in B(0,1/R)\backslash\{0\}$ with $u\ne v$ and $\|u\|\ge\|v\|>0$. Then, setting $z = u/\|u\|^2$ and $w = v/\|v\|^2$,

$$\frac{N^*(u)-N^*(v)}{\|u-v\|} = \frac{\|u\|^2 N(z) - \|v\|^2 N(w)}{\|u-v\|} = \frac{(\|u\|^2 - \|v\|^2)N(z) + \|v\|^2(N(z)-N(w))}{\|u-v\|}$$

and hence

$$\frac{\|N^*(u)-N^*(v)\|}{\|u-v\|} \le \|u+v\|\|z\|\frac{\|N(z)\|}{\|z\|} + \frac{\|v\|^2\|z-w\|}{\|u-v\|}\frac{\|N(z)-N(w)\|}{\|z-w\|}$$
$$\le 2\sup_{\|x\|\ge R}\frac{\|N(x)\|}{\|x\|} + \sup_{\substack{\|x\|,\|y\|\ge R\\ x\ne y}}\frac{\|N(x)-N(y)\|}{\|x-y\|}$$

since $\|u\|^2-\|v\|^2 = \langle u+v, u-v\rangle \le \|u+v\|\|u-v\| \le 2\|u\|\|u-v\| = 2\|u-v\|/\|z\|$ and

$$\frac{\|v\|^2\|z-w\|}{\|u-v\|} \le \frac{\|v\|^2}{\|u\|\|v\|} \le 1,$$

by (3.5). Furthermore, for $u \in B(0,1/R)\backslash\{0\}$, we have that

$$\frac{\|N^*(u)-N^*(0)\|}{\|u\|} = \frac{\|N(z)\|}{\|z\|} \le \sup_{\|x\|\ge R}\frac{\|N(x)\|}{\|x\|}.$$

The first estimate of $L(M^*)$ follows by letting $R\to\infty$.

If the right hand side of (3.1) is finite there exist constants $K>0$ and $R>0$ such that

$$\|M(u)-M(v)\| \le K\|u-v\| \quad \text{for all } u,v\in X \text{ with } \|u\|\ge R \text{ and } \|v\|\ge R.$$

Fixing v and letting $\|u\|\to\infty$, this implies that $\limsup_{\|u\|\to\infty}\|M(u)\|/\|u\| \le K$. Thus,

$$\lim\sup_{\|u\|\to\infty}\|M(u)\|/\|u\| \le \lim_{R\to\infty}\sup_{\substack{\|u\|,\|v\|\ge R\\ u\ne v}}\frac{\|M(u)-M(v)\|}{\|u-v\|}$$

and we obtain (3.1) and (3.2).

For part (ii), it suffices to choose $T = M'(\infty)$, since we then have that $\limsup_{\|u\|\to\infty}\frac{\|N(u)\|}{\|u\|} = 0$ when M is asymptotically linear. □

The following examples shed some light on the estimates obtained in Lemma 3.3.

Example 1. Consider the function $m:\mathbb{R}\to\mathbb{R}$ defined by

$$m(t) = t \quad \text{for } |t|\le 1 \quad \text{and} \quad m(t) = \frac{1}{t} \quad \text{for } |t|>1.$$

For its inversion we find

$$m^*(t) = t^3 \quad \text{for } |t|\le 1 \quad \text{and} \quad m^*(t) = t \quad \text{for } |t|>1.$$

It is easy to see that for the Lipschitz moduli of these functions we have $L(m)=1$ and $L(m^*)=0$. Also

$$\lim_{|t|\to\infty}\frac{|m(t)|}{|t|} = \lim_{R\to\infty}\sup_{\substack{|t|,|s|\ge R\\ t\ne s}}\frac{|m(t)-m(s)|}{|t-s|} = 0,$$

confirming the estimate (3.1) of the lemma. Note also that

$$|m(t)-m(s)| \le |t-s| \quad \text{and} \quad |m^*(t)-m^*(s)| \le 3|t-s| \quad \text{for all } t,s\in\mathbb{R}.$$

Since $m(0)=0$, $m^{**}=m$. However,

$$\lim_{|t|\to\infty}\frac{|m^*(t)|}{|t|} = \lim_{R\to\infty}\sup_{\substack{|t|,|s|\ge R\\ t\ne s}}\frac{|m^*(t)-m^*(s)|}{|t-s|} = 1,$$

and so (3.1) yields $L(m^{**})\le 3$ whereas we know that $L(m^{**}) = L(m) = 1$. Clearly, m^* is asymptotically linear with $(m^*)'(\infty) = 1$ and $m^*(t) - (m^*)'(\infty)t = 0$ for $|t|\ge 1$. Hence applying the estimate (3.4) to m^* gives $L(m^{**}) \le |(m^*)'(\infty)| = 1$ which is the correct value.

The next example shows that when M is not asymptotically linear, but still H-asymptotically linear, the estimates (3.3) and (3.4) can fail and that (3.2), and hence also (3.1), can be sharp. It suffices to consider the Nemytskii operator associated with the function m used in Example 1.

Example 2. Consider the function $M : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $M(u)(x) = m(u(x))$ for $u \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$, where m is the function discussed in Example 1. Since $|m(t)| \le |t|$ and $|m(t) - m(s)| \le |t - s|$ for all $t, s \in \mathbb{R}$, it follows that $\|M(u) - M(v)\| \le \|u - v\|$ for all $u, v \in L^2(\mathbb{R})$, where $\|\cdot\|$ is the usual norm on $L^2(\mathbb{R})$, so (3.2) yields $L(M^*) \le 3$. We now show that in fact $L(M^*) = 3$.

For $k > 0$, let $\chi_{(0,k)}$ denote the characteristic function of the interval $(0, k)$. Set $u^k = \frac{1}{k}\chi_{(0,k)}$ and $u_n^k = \frac{1-\frac{1}{n}}{k}\chi_{(0,k)}$ for $n \in \mathbb{N}$. Then $\|u^k\| = \frac{1}{\sqrt{k}}$, $\|u_n^k\| = \frac{1-\frac{1}{n}}{\sqrt{k}}$ and $\|u^k - u_n^k\| = \frac{1}{n\sqrt{k}}$. Also $M^*(u^k) = \frac{1}{k}M(\frac{u^k}{\|u^k\|^2}) = \frac{1}{k}m(\chi_{(0,k)}) = \frac{1}{k}m^*(1)\chi_{(0,k)}$ and $M^*(u_n^k) = \frac{(1-\frac{1}{n})^2}{k}m(\frac{1}{(1-\frac{1}{n})}\chi_{(0,k)}) = \frac{1}{k}m^*((1-\frac{1}{n}))\chi_{(0,k)}$, so that

$$\frac{\|M^*(u^k) - M^*(u_n^k)\|}{\|u^k - u_n^k\|} = \frac{|m^*(1) - m^*((1-\frac{1}{n}))|}{1/n} \to \big|(m^*)'(1-)\big| = 3$$

as $n \to \infty$ for all $k > 0$. Given $\delta > 0$, choose k so that $\frac{1}{\sqrt{k}} < \delta$. Then $u^k, u_n^k \in B(0, \delta)$ for all $n \ge 2$ and so

$$\sup_{\substack{u,v\in B(0,\delta)\\ u\neq v}} \frac{\|M^*(u) - M^*(v)\|}{\|u - v\|} \ge 3 \quad \text{for all } \delta > 0.$$

Hence $L(M^*) \ge 3$ and we have shown that $L(M^*) = 3$.

Note that $M : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is H-asymptotically linear with $M'(\infty) = 0$ by Theorem 5.2 of [19], but M is not asymptotically linear. Furthermore, $\|M(u) - M(v)\| \le \|u - v\|$ for all $u, v \in L^2(\mathbb{R})$ so the right hand side of (3.4) is 1 whereas $L(M^*) = 3$, showing that (3.3) and (3.4) need not hold when M is only H-asymptotically linear.

To get an estimate of $L(M^*)$ using Lemma 3.3, M has to be uniformly Lipschitz continuous on the complement of some bounded subset of X. For mappings of this kind, H-asymptotic linearity follows from a simpler property. Note in passing that for any H-asymptotically linear map $M : X \to Y$,

$$\lim_{|t|\to\infty} \frac{M(tu)}{t} = M'(\infty)u \quad \text{for all } u \in X, \text{ where } M'(\infty) \in B(X, Y).$$

Lemma 3.4. *Let X and Y be real Banach spaces and consider a mapping $M : X \to Y$ for which there exist constants $K > 0$ and $R > 0$ such that*

$$\big\|M(u) - M(v)\big\| \le K\|u - v\| \quad \textit{for all } u, v \in X \textit{ with } \|u\| \ge R \textit{ and } \|v\| \ge R.$$

If there exists a linear operator $L : X \to Y$ such that

$$\lim_{|t|\to\infty} \frac{M(tu)}{t} = Lu \quad \textit{for all } u \in X,$$

then $L \in B(X, Y)$ and $M : X \to Y$ is H-asymptotically linear with $M'(\infty) = L$.

Proof. First we prove that $L \in B(X, Y)$ with $\|L\| \le K$. For $u \in X$ with $\|u\| = 1$ and for $t > R$,

$$Lu = Lu - \frac{M(tu)}{t} + \frac{M(tu) - M(Ru)}{t} + \frac{M(Ru)}{t}$$

and so

$$\|Lu\| \le \left\|Lu - \frac{M(tu)}{t}\right\| + \frac{K|t - R|}{|t|} + \left\|\frac{M(Ru)}{t}\right\|.$$

Letting $t \to \infty$ yields $\|Lu\| \le K$, as required.

Now consider sequences $\{t_n\}$ and $\{u_n\}$ such that $\|t_n u_n\| \to \infty$ and $\|u_n - u\| \to 0$ as $n \to \infty$, for some $u \in X$. If $u \neq 0$, then

$$\frac{M(t_n u_n)}{t_n} - Lu = \frac{M(t_n u)}{t_n} - Lu + \frac{M(t_n u_n) - M(t_n u)}{t_n}$$

and there exists n_0 such that $\|t_n u_n\| \geq R$ and $\|t_n u\| \geq R$ for all $n \geq n_0$. Hence, for $n \geq n_0$,

$$\left\| \frac{M(t_n u_n)}{t_n} - Lu \right\| \leq \left\| \frac{M(t_n u)}{t_n} - Lu \right\| + K\|u_n - u\|,$$

showing that $\| \frac{M(t_n u_n)}{t_n} - Lu\| \to 0$ as $n \to \infty$.

If $u = 0$, then $\|u_n\| \to 0$. As in the proof of (3.2), we have that $\limsup_{n\to\infty} \|M(t_n u_n)\|/\|t_n u_n\| \leq K$ and hence $\frac{M(t_n u_n)}{t_n} \to 0 = Lu$. □

4. Bifurcation and asymptotic bifurcation

Starting from a recent result concerning bifurcation for problems that are only Hadamard differentiable at the origin, inversion is used to obtain conclusions about asymptotic bifurcation under the assumption of H-asymptotic linearity.

4.1. Bifurcation

Let X and Y be real Banach spaces and consider the equation $F(\lambda, u) = 0$ where $F : \mathbb{R} \times X \to Y$ with $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Setting $\mathcal{S} = \{(\lambda, u) \in \mathbb{R} \times X : F(\lambda, u) = 0 \text{ and } u \neq 0\}$, μ is called a **bifurcation point** for the equation $F(\lambda, u) = 0$ if there exists a sequence $\{(\lambda_n, u_n)\} \subset \mathcal{S}$ such that $\lambda_n \to \mu$ and $\|u_n\| \to 0$ as $n \to \infty$. There is **continuous bifurcation** at μ if there exists a bounded connected subset $\mathcal{C}$ of $\mathcal{S}$ such that $\overline{\mathcal{C}} \cap [\mathbb{R} \times \{0\}] = \{(\mu, 0)\}$.

Proposition 4.1. *Let $(Y, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a real Hilbert space and let X be the graph space of some self-adjoint operator acting in Y. Consider the equation $M(u) = \lambda u$ where the function $M : X \to Y$ has the following properties.*

(H1) $M(0) = 0$.
(H2) *M is Hadamard differentiable at 0 and $S = M'(0) \in B(X, Y)$ is a self-adjoint operator acting in Y with domain X.*
(H3) *$\mu \notin \sigma_e(S)$ and there exists $\varepsilon > 0$ such that*

$$\gamma^{\varepsilon}(S - \mu I)L^{\varepsilon}(R) < 1, \tag{4.1}$$

where $R : M - M'(0) : X \to Y$ and both the essential conditioning number $\gamma^{\varepsilon}(S - \mu I)$ and the Lipschitz modulus $L^{\varepsilon}(R)$ are calculated using the norm $\|\cdot\|_{\varepsilon S}$ on X.

Then we have the following conclusions.

(i) *If* $\ker\{M'(0) - \mu I\} = \{0\}$, *$\mu$ is not a bifurcation point.*
(ii) *If* $\dim \ker\{M'(0) - \mu I\}$ *is odd, there is continuous bifurcation at μ.*
(iii) *If* $\ker\{M'(0) - \mu I\} = span\{\phi\}$ *where $\|\phi\| = 1$, there is continuous bifurcation at μ and, for any sequence of solutions $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times X$ such that $\lambda_n \to \mu$ and $\|u_n\|_S \to 0$, we have that $u_n = \langle u_n, \phi\rangle\{\phi + w_n\}$ where $\langle w_n, \phi\rangle = 0$ and $\|w_n\|_S \to 0$.*

The statements about bifurcation in parts (i) to (iii) refer to the graph norm, or any equivalent norm, on X.

Proof. Setting $F(\lambda, u) = M(u) - \lambda u$, this follows from Theorems 6.3 and 6.4 of [20]. □

4.2. Asymptotic bifurcation

Let X and Y be real Banach spaces and consider the equation $F(\lambda,u)=0$ where $F:\mathbb{R}\times X\to Y$. Setting $\mathcal{S}=\{(\lambda,u)\in\mathbb{R}\times X: F(\lambda,u)=0 \text{ and } u\neq 0\}$, μ is called an **asymptotic bifurcation point** for the equation $F(\lambda,u)=0$ if there exists a sequence $\{(\lambda_n,u_n)\}\subset\mathcal{S}$ such that $\lambda_n\to\mu$ and $\|u_n\|_X\to\infty$ as $n\to\infty$. There is **continuous asymptotic bifurcation** at μ if there exists an unbounded connected subset $\mathcal{C}$ of $\mathcal{S}$ such that whenever $\{(\lambda_n,u_n)\}$ is a sequence in $\mathcal{C}$ and $|\lambda_n|+\|u_n\|_X\to\infty$, then $\lambda_n\to\mu$ and $\|u_n\|_X\to\infty$.

Theorem 4.2. *Let $(Y,\langle\cdot,\cdot\rangle,\|\cdot\|)$ be a real Hilbert space and let X be the graph space of some self-adjoint operator acting in Y. Consider the equation $G(u)=\lambda u$ where $G:X\to Y$ is H-asymptotically linear and $G'(\infty)\in B(X,Y)$ is also a self-adjoint operator S with domain X acting in Y.*

Set $\rho=G-G'(\infty)$ and then, for $\varepsilon>0$, let $\rho^{*,\varepsilon}:X\to Y$ denote the inversion of ρ defined using the graph norm $\|\cdot\|_{\varepsilon S}$ on X.

Suppose that $\mu\notin\sigma_e(S)$ and that there exists $\varepsilon>0$ such that

$$\gamma^{\varepsilon}(S-\mu I)L^{\varepsilon}\left(\rho^{*,\varepsilon}\right)<1, \tag{4.2}$$

where both the essential conditioning number and the Lipschitz modulus are calculated using the norm $\|\cdot\|_{\varepsilon S}$ on X.

(i) If $\ker(S-\mu I)=\{0\}$, then μ is not an asymptotic bifurcation point.
(ii) If $\dim\ker(S-\mu I)$ is odd, there is continuous asymptotic bifurcation at μ.
(iii) If $\ker(S-\mu I)=span\{\phi\}$ where $\|\phi\|=1$, there is continuous asymptotic bifurcation at μ and, any sequence of solutions $\{(\lambda_n,u_n)\}$ such that $\lambda_n\to\mu$ and $\|u_n\|_S\to\infty$ has the property that

$$\left|\langle u_n,\phi\rangle\right|\to\infty \quad \text{and} \quad u_n=\langle u_n,\phi\rangle\{\phi+w_n\}$$

where $w_n\in X$ with $\langle w_n,\phi\rangle=0$ and $\|w_n\|_S\to 0$.

The statements about asymptotic bifurcation in parts (i) to (iii) refer to the graph norm, or any equivalent norm, on X.

Remark 1. By Lemma 3.3(i) and since $\|u\|\le\|u\|_{\varepsilon S}$ for all $u\in X$,

$$\begin{aligned}
L^{\varepsilon}\left(\rho^{*,\varepsilon}\right) &\le 2\limsup_{\|u\|_{\varepsilon S}\to\infty}\frac{\|\rho(u)-\rho(0)\|}{\|u\|_{\varepsilon S}}+\lim_{r\to\infty}\sup_{\substack{\|u\|_{\varepsilon S},\|v\|_{\varepsilon S}\ge r\\ u\neq v}}\frac{\|\rho(u)-\rho(v)\|}{\|u-v\|_{\varepsilon S}}\\
&\le 2\sup_{u\in X\setminus\{0\}}\frac{\|\rho(u)-\rho(0)\|}{\|u\|}+\sup_{\substack{u,v\in X\\ u\neq v}}\frac{\|\rho(u)-\rho(v)\|}{\|u-v\|}\\
&\le 3\sup_{\substack{u,v\in X\\ u\neq v}}\frac{\|\rho(u)-\rho(v)\|}{\|u-v\|}
\end{aligned}$$

for all $\varepsilon>0$. Using Proposition 3.2, condition (4.2) is satisfied provided that

$$d\left(\mu,\sigma_e(S)\right)>2A+\Gamma, \tag{4.3}$$

where

$$A=\sup_{u\in X\setminus\{0\}}\frac{\|\rho(u)-\rho(0)\|}{\|u\|} \quad\text{and}\quad \Gamma=\sup_{\substack{u,v\in X\\ u\neq v}}\frac{\|\rho(u)-\rho(v)\|}{\|u-v\|}.$$

Remark 2. Part (iii) concerns asymptotic bifurcation at a simple eigenvalue of the H-asymptotic derivative, giving some information about the form of large solutions. For asymptotically linear problems, Dancer [2] formulated some additional hypotheses under which he was able to show that, near a simple eigenvalue of the asymptotic derivative, the large solutions not only admit such a development, but form a continuous curve.

Proof of Theorem 4.2. Choose $\varepsilon > 0$ such that (4.2) is satisfied and let $N = G^{*,\varepsilon} : X \to Y$ be the inversion of G defined using the norm $\|\cdot\|_{\varepsilon S}$ on X. Then $N(0) = 0$ and N is Hadamard differentiable at 0 with $N'(0) = G'(\infty) = S$. Furthermore, for $R = N - N'(0)$ we have that

$$R = G^{*,\varepsilon} - G'(\infty) = \left[G - G'(\infty)\right]^{*,\varepsilon} = \rho^{*,\varepsilon},$$

and so (4.2) implies that (4.1) is satisfied. Hence the hypotheses of Proposition 4.1 are satisfied for the equation

$$N(v) = \lambda v \quad \Leftrightarrow \quad G^{*,\varepsilon}(v) = \lambda v.$$

Part (i) follows immediately from Proposition 4.1 since $\|u_n\|_{\varepsilon S} \to \infty$ implies that $\|v_n\|_{\varepsilon S} \to 0$ for $v_n = u_n/\|u_n\|^2_{\varepsilon S}$.

(ii) By part (ii) of Proposition 4.1 there is a bounded connected subset $\mathcal{C}$ of $\{(\lambda, v) \in \mathbb{R} \times X : N(v) = \lambda v$ and $v \neq 0\}$ such that $\bar{\mathcal{C}} \cap [\mathbb{R} \times \{0\}] = \{(\mu, 0)\}$. Setting $\mathcal{D} = \{(\lambda, v/\|v\|^2_{\varepsilon S}) : (\lambda, v) \in \mathcal{C}\}$, we have that $G(u) = \lambda u$ and $u \neq 0$ for all $(\lambda, u) \in \mathcal{D}$. Furthermore, $\mathcal{D}$ is connected since $(\lambda, v) \mapsto (\lambda, v/\|v\|^2_{\varepsilon S})$ is a continuous map of $\mathbb{R} \times [X\backslash\{0\}]$ onto itself. Since there is a sequence $\{(\lambda_n, v_n)\} \subset \mathcal{C}$ such that $(\lambda_n, v_n) \to (\mu, 0)$ in $\mathbb{R} \times X$, $\mathcal{D}$ is unbounded. If $\{(\lambda_n, u_n)\}$ is a sequence in $\mathcal{D}$ such that $|\lambda_n| + \|u_n\|_{\varepsilon S} \to \infty$, $\{(\lambda_n, u_n/\|u_n\|^2_{\varepsilon S})\} \subset \mathcal{C}$ so $\{\lambda_n\}$ is bounded and hence $\|u_n\|_{\varepsilon S} \to \infty$. It now follows from the properties of $\mathcal{C}$ that $\lambda_n \to \mu$.

(iii) Consider any sequence of solutions $\{(\lambda_n, u_n)\}$ of $G(u) = \lambda u$ such that $\lambda_n \to \mu$ and $\|u_n\|_{\varepsilon S} \to \infty$ and set $v_n = u_n/\|u_n\|^2_{\varepsilon S}$. By part (iii) of Proposition 4.1, $v_n = \langle v_n, \phi\rangle\{\phi + w_n\}$ where $\langle w_n, \phi\rangle = 0$ and $\|w_n\|_{\varepsilon S} \to 0$. Hence $u_n = \langle u_n, \phi\rangle\{\phi + w_n\}$ and $|\langle u_n, \phi\rangle| \to \infty$ since $\|u_n\|_{\varepsilon S} \to \infty$ and $\|\phi + w_n\|_{\varepsilon S} \to \|\phi\|_{\varepsilon S} < \infty$. □

Using the condition (4.3) we can formulate a special case of Theorem 4.2 and provide some additional information about asymptotic bifurcation.

Corollary 4.3. *Let $(Y, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a real Hilbert space and $S : D(S) \subset Y \to Y$ a self-adjoint operator. Suppose that*

(1) *$\rho : Y \to Y$ is H-asymptotically linear with $\rho'(\infty) = 0$,*
(2) *$\rho : Y \to Y$ is Lipschitz continuous with $\|\rho(u) - \rho(v)\| \le \Gamma\|u - v\|$ for all $u, v \in Y$,*
(3) *$d(\mu, \sigma_e(S)) > 2A + \Gamma$ where $A = \sup_{u \in X\backslash\{0\}} \frac{\|\rho(u)-\rho(0)\|}{\|u\|}$.*

Setting $G = S + \rho$ on the graph space X of S, the hypotheses of Theorem 4.2 are satisfied and, in addition to the conclusions (i) to (iii) of that result we also have asymptotic bifurcation with respect to the weaker norm, $\|\cdot\|$.

(iv) For any sequence $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times X$ such that $G(u_n) = \lambda_n u_n$ with $\{\lambda_n\}$ bounded and $\|u_n\|_S \to \infty$, we have that $\|u_n\| \to \infty$.

Remark. It follows from Lemma 3.4 that when (2) is satisfied, then (1) holds provided that $\lim_{|t|\to\infty} \|\rho(tu)\|/t = 0$ for all $u \in Y$.

Proof of Corollary 4.3. Since $\lambda_n u_n = G(u_n) = Su_n + \rho(u_n)$, we have that

$$\|Su_n\| = \left\|\lambda_n u_n - \rho(u_n)\right\| \le \left(|\lambda_n| + \Gamma\right)\|u_n\| + \left\|\rho(0)\right\|,$$

showing that $\{u_n\}$ would be bounded in the graph norm if $\{\|u_n\|\}$ were bounded. □

5. Asymptotic bifurcation for an elliptic equation on $\mathbb{R}^N$

Consider the nonlinear elliptic equation,

$$-\Delta u + Vu + g(u) + h = \lambda u \tag{5.1}$$

where $V \in L^\infty(\mathbb{R}^N)$, $h \in L^2(\mathbb{R}^N)$ and

(G) $g \in C(\mathbb{R})$ with $g(0) = \lim_{|s|\to\infty} \frac{g(s)}{s} = 0 \in \mathbb{R}$ and $|g(s) - g(t)| \le \Gamma|s - t|$ for all $s, t \in \mathbb{R}$.

The case where $\lim_{|s|\to\infty}\frac{g(s)}{s}=\ell\neq 0$ can be reduced to (G) by replacing V by $V+\ell$ and $g(u)$ by $g(u)-\ell u$. Hence (5.1) constitutes a generalisation of (1.2) under weaker hypotheses than (1.3).

For $V\in L^\infty(\mathbb{R}^N)$, we have that

$$S=-\Delta+V:D(S)\subset L^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)$$

is a self-adjoint operator with domain $D(S)=H^2(\mathbb{R}^N)$. Its spectrum, essential spectrum and discrete spectrum are denoted by $\sigma(S)$, $\sigma_e(S)$ and $\sigma_d(S)$, respectively. We recall that

$$\inf\sigma_e(S)\geq\liminf_{|x|\to\infty}V(x)=\lim_{R\to\infty}\operatorname*{ess\,inf}_{|x|\geq R}V(x)$$

and that $\sigma_d(S)$ is the set of all isolated eigenvalues of finite multiplicity. See [16], for example. The graph norm on $D(S)=H^2(\mathbb{R}^N)$ is equivalent to the usual Sobolev norm on this space and $S\in B(H^2(\mathbb{R}^N),L^2(\mathbb{R}^N))$.

Let $h\in L^2(\mathbb{R}^N)$ and g satisfy (G). Setting

$$M(u)=g(u)+h,$$

it follows that $M(0)=h\in L^2(\mathbb{R}^N)$ and that

$$\|M(u)-M(v)\|_{L^2}\leq\Gamma\|u-v\|_{L^2}\quad\text{for all } u,v\in L^2(\mathbb{R}^N).$$

Thus $M:L^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)$ is bounded and uniformly Lipschitz continuous. Using Lemma 3.4 and dominated convergence it is easy to see that $M:L^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)$ is H-asymptotically linear with H-asymptotic derivative $M'(\infty)=0$. Hence $M^*:L^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)$ is Hadamard differentiable at 0 with $(M^*)'(0)=0$ by Proposition 2.3. However, it follows from Proposition 2.2 that M is asymptotically linear only when $g\equiv 0$.

Most of our discussion of (5.1) concerns solutions $u\in H^2(\mathbb{R}^N)$. Note however that if $u\in L^2(\mathbb{R}^N)$, then our hypotheses imply that $Vu+g(u)+h\in L^2(\mathbb{R}^N)$ since $|g(s)|\leq\Gamma|s|$ for all $s\in\mathbb{R}$. Hence, if $u\in L^2(\mathbb{R}^N)$ satisfies (5.1) in the sense of distributions, then $u\in H^2(\mathbb{R}^N)$. In fact, for distributional solutions $u\in L^2(\mathbb{R}^N)$, Eq. (5.1) is equivalent to

$$G(u)=\lambda u\quad\text{for } u\in D(S)=H^2(\mathbb{R}^N),$$

where the mapping $G:H^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)$, defined by

$$G(u)=-\Delta u+Vu+g(u)+h=Su+M(u),$$

is H-asymptotically linear with $G'(\infty)=S=-\Delta+V$.

In the setting of Section 4 with $X=H^2(\mathbb{R}^N)$ and $Y=L^2(\mathbb{R}^N)$, there is asymptotic bifurcation at μ for Eq. (5.1) when there exists a sequence $\{(\lambda_n,u_n)\}\subset\mathbb{R}\times H^2(\mathbb{R}^N)$ of solutions such that $\lambda_n\to\mu$ and $\|u_n\|_{H^2}\to\infty$. As in Corollary 4.3, the properties of G imply that $\|u_n\|_{H^2}\to\infty$ can be replaced by the seemingly stronger statement that $\|u_n\|_{L^2}\to\infty$ in this context. Indeed, using elliptic regularity theory and a boot-strap argument as in the proof of Lemma 6.2 of [7] (see also [18]), we obtain the following additional information about the regularity of solutions and the meaning of asymptotic bifurcation for (5.1).

Proposition 5.1. *Suppose that $V\in L^\infty(\mathbb{R}^N)$, $h\in L^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and that g satisfies (G). If $(\lambda,u)\in\mathbb{R}\times H^2(\mathbb{R}^N)$ is a solution of (5.1), then $u\in W^{2,p}(\mathbb{R}^N)$ for all $p\in[2,\infty)$. In particular, $u\in C^1(\mathbb{R}^N)$ and $\lim_{|x|\to\infty}\{|u(x)|+|\nabla u(x)|\}=0$. Furthermore, for all $D>0$ and $p\in[2,\infty)$, there exist constants $C(N,D,p)$ and $c(N,D)$ such that*

$$\|u\|_{W^{2,p}}\leq C(N,D,p)\|u\|_{L^2}\quad\text{and}\quad\|u\|_{L^\infty}\leq c(N,D)\|u\|_{L^2}$$

for all solutions (λ,u) of (5.1) with $|\lambda|\leq D$.

It follows from this result that if μ is not an asymptotic bifurcation point for (5.1) with respect to the L^2-norm, then there is not asymptotic bifurcation at μ with respect to the L^∞- and $W^{2,p}$-norms for $p\in[2,\infty)$, either. On the other hand, the results below show that in some cases asymptotic bifurcation at μ in the L^2 sense implies L^∞-asymptotic bifurcation (see Theorem 5.3), whereas in other cases it does not (see Theorem 5.7).

5.1. Asymptotic bifurcation at eigenvalues of odd multiplicity of $G'(\infty)$

We begin the discussion of asymptotic bifurcation for (5.1) by exploiting the results formulated in Section 4.2, dealing first with an arbitrary eigenvalue of odd multiplicity of $G'(\infty)$ and then obtaining some extra information about what happens at $\Lambda = \inf \sigma(G'(\infty))$ when $\Lambda < \inf \sigma_e(G'(\infty))$.

For a function g satisfying (G), setting $A = \sup_{s \neq 0} |\frac{g(s)}{s}|$, we have that $A \leq \Gamma$ and $\|M(u) - M(0)\|_{L^2} \leq A\|u\|_{L^2}$ for all $u \in L^2(\mathbb{R}^N)$.

The following result is now an immediate consequence of Corollary 4.3 with $Y = L^2(\mathbb{R}^N)$.

Theorem 5.2. *Suppose that $V \in L^\infty(\mathbb{R}^N)$, $h \in L^2(\mathbb{R}^N)$ and that g satisfies (G). Consider Eq. (5.1) and μ such that $d(\mu, \sigma_e(S)) > 2A + \Gamma$ where $S = G'(\infty) = -\Delta + V : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$.*

- (i) *If $\ker(S - \mu I) = \{0\}$, μ is not an asymptotic bifurcation point.*
- (ii) *If $\dim \ker(S - \mu I)$ is odd, there is continuous asymptotic bifurcation at μ with respect to the $H^2(\mathbb{R}^N)$ norm. Furthermore, $\|u_n\|_{L^2} \to \infty$ as $n \to \infty$ whenever $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times H^2(\mathbb{R}^N)$ is a sequence of solutions of (5.1) such that $\lambda_n \to \mu$ and $\|u_n\|_{H^2} \to \infty$ as $n \to \infty$.*
- (iii) *If $\ker(S - \mu I) = \mathrm{span}\{\phi\}$ where $\|\phi\|_{L^2} = 1$, for any sequence $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times H^2(\mathbb{R}^N)$ of solutions of (5.1) such that $\lambda_n \to \mu$ and $\|u_n\|_{H^2} \to \infty$ as $n \to \infty$, we have that*

$$\left|\langle u_n, \phi\rangle\right| \to \infty \quad \text{and} \quad u_n = \langle u_n, \phi\rangle\{\phi + w_n\}$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product on $L^2(\mathbb{R}^N)$, $w_n \in H^2(\mathbb{R}^N)$ with $\langle w_n, \phi\rangle = 0$ and $\|w_n\|_{H^2} \to 0$. In fact, if $h \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we have that ϕ, u_n and $w_n \in W^{2,p}(\mathbb{R}^N)$ with $\|w_n\|_{W^{2,p}} \to 0$ for all $p \in [2, \infty)$.

Proof. The only part that is not a direct consequence of Corollary 4.3 is the statement concerning $W^{2,p}$ for $p > 2$. Suppose therefore that $h \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. By Proposition 5.1, $u_n \in W^{2,p}(\mathbb{R}^N)$ for all $p \in [2, \infty)$. Similarly $\phi \in W^{2,p}(\mathbb{R}^N)$ for all $p \in [2, \infty)$ and since $t_n = \langle u_n, \phi\rangle \neq 0$, we have the same thing for w_n. Substituting $u_n = t_n\{\phi + w_n\}$ into (5.1) and dividing by t_n yields

$$\Delta w_n = (V - \lambda_n)w_n + (\mu - \lambda_n)\phi + \frac{M(t_n[\phi + w_n])}{t_n}$$

where $M(u) = g(u) + h$ and hence

$$\|\Delta w_n\|_{L^p} \leq \big(\|V\|_{L^\infty} + |\lambda_n|\big)\|w_n\|_{L^p} + |\mu - \lambda_n|\|\phi\|_{L^p} + \left\|\frac{M(t_n[\phi + w_n])}{t_n}\right\|_{L^p}.$$

But (G) and Theorem 5.2 of [19] imply that $M : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is H-asymptotically linear with $M'(\infty) = 0$ for all $p \in [1, \infty)$. Thus, it follows that, if $\|w_n\|_{L^p} \to 0$ for some $p \in [2, \infty)$, then $\|\frac{M(t_n[\phi+w_n])}{t_n}\|_{L^p} \to 0$ since $|t_n| \to \infty$ and consequently $\|\Delta w_n\|_{L^p} \to 0$ since $\lambda_n \to \mu$. This means that, if $\|w_n\|_{L^p} \to 0$ for some $p \in [2, \infty)$, then $\|w_n\|_{W^{2,p}} \to 0$ for that value of p. We already know from Corollary 4.3(iii) that $\|w_n\|_{H^2} \to 0$ which implies that $\|w_n\|_{L^p} \to 0$ for an interval of values of p greater than 2. Boot-strapping then shows that all values in $[2, \infty)$ can be reached. □

With reference to the remarks at the end of the introduction concerning asymptotic bifurcation from the lowest eigenvalue of $G'(\infty) = S$, observe that Theorem 5.2 deals with any isolated eigenvalue μ of odd multiplicity, even ones lying in gaps in the essential spectrum, provided that $d(\mu, \sigma_e(-\Delta + V)) > 2A + \Gamma$. In the case where $\Lambda = \inf \sigma(S) < \inf \sigma_e(S)$ and $h \equiv 0$, we can sharpen the conclusion of Theorem 5.2 concerning asymptotic bifurcation at the simple eigenvalue Λ by exploiting elliptic regularity theory and the maximum principle. If $\Lambda < \inf \sigma_e(S)$, Λ is a simple eigenvalue of S with an eigenfunction $\phi \in H^2(\mathbb{R}) \cap C^1(\mathbb{R}^N)$ such that $\phi > 0$ on $\mathbb{R}^N$ and $\|\phi\|_{L^2} = 1$. In general, $\inf V \leq \Lambda$ and $\liminf_{|x|\to\infty} V(x) \leq \inf \sigma_e(S)$, but if $\lim_{|x|\to\infty} V(x) = V(\infty)$ exists, then $\sigma_e(S) = [V(\infty), \infty)$. See [16], for example. Hence, for such potentials, $d(\Lambda, \sigma_e(S)) = V(\infty) - \Lambda$.

Theorem 5.3. *Suppose that $V \in L^\infty(\mathbb{R}^N)$ and that g satisfies (G). Suppose also that $2A + \Gamma < \inf \sigma_e(S) - \Lambda$ and that $A < \liminf_{|x|\to\infty} V(x) - \Lambda$.*

(a) There exist $\varepsilon > 0$ and $K > 0$ such that, for all solutions $(\lambda, u) \in \mathbb{R} \times H^2(\mathbb{R}^N)$ of (1.2) with $\lambda < \Lambda + \varepsilon$ and $\|u\|_{H^2} > K$, either $u > 0$ on $\mathbb{R}^N$ or $u < 0$ on $\mathbb{R}^N$.
(b) For any sequence $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times H^2(\mathbb{R}^N)$ of solutions of (1.2) such that $\lambda_n \to \Lambda$ and $\|u_n\|_{H^2} \to \infty$, there exists n_0 such that for all $n \geq n_0$, either $u_n > 0$ on $\mathbb{R}^N$ or $u_n < 0$ on $\mathbb{R}^N$. Furthermore, $|u_n| \to \infty$, uniformly on compact subsets of $\mathbb{R}^N$ as $n \to \infty$.
(c) There exist two sequences $\{(\lambda_n^{\pm}, u_n^{\pm})\} \subset \mathbb{R} \times [H^2(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)]$ of solutions of (1.2) such that $\lambda_n^{\pm} \to \Lambda$, $u_n^+ > 0$, $u_n^- < 0$ on $\mathbb{R}^N$ and $u_n^{\pm} \to \pm\infty$, uniformly on compact subsets of $\mathbb{R}^N$.

Proof. (a) Suppose that for all $n \in \mathbb{N}$, there exists a solution (λ_n, u_n) with $\lambda_n < \Lambda + \frac{1}{n}$, $\|u_n\|_{H^2} > n$ and $u_n(x_n) = 0$ for some point $x_n \in \mathbb{R}^N$.

By Theorem 5.2(iii) with $\mu = \Lambda$ we have that

$$u_n = t_n\{\phi + w_n\} \quad \text{and} \quad t_n = \langle u_n, \phi\rangle, \quad \text{where } \ker(S - \Lambda I) = span\{\phi\}$$
$$\text{for } \phi \in H^2(\mathbb{R}^N) \cap C^1(\mathbb{R}^N) \text{ with } \|\phi\|_{L^2} = 1 \text{ and } \phi > 0 \text{ on } \mathbb{R}^N,$$
$$w_n \in W^{2,p}(\mathbb{R}^N) \quad \text{and} \quad \|w_n\|_{W^{2,p}} \to 0 \quad \text{for all } p \in [2, \infty), \langle w_n, \phi\rangle = 0.$$

Setting $\gamma = \liminf_{|x|\to\infty} V(x) - \Lambda - A$, we have $\gamma > 0$ by hypothesis and so there exists $R > 0$ such that $V(x) - \Lambda - A \geq \gamma/2$ whenever $|x| \geq R$. Thus there exists n_0 such that, for all $n \geq n_0$ and $|x| \geq R$, $V(x) - \lambda_n - A \geq V(x) - \Lambda - 1/n - A \geq \gamma/4$.

Furthermore, $m(R) \equiv \min\{\phi(x) : |x| \leq R\} > 0$ and we can choose n_0 such that $\phi(x) + w_n(x) \geq m(R)/2$ for $|x| \leq R$ since $\|w_n\|_{L^\infty} \to 0$. Consider first $n \geq n_0$ for which $t_n > 0$. In this case we have that $u_n \geq t_n m(R)/2$ for $|x| \leq R$.

Let $\Omega(n, R) = \{x \in \mathbb{R}^N : |x| > R \text{ and } u_n(x) < 0\}$. Since $u_n \in C(\mathbb{R}^N)$, Ω is an open set and $u_n(x) = 0$ for $x \in \partial\Omega(n, R)$ because $u_n(x) > 0$ for $|x| = R$. We also have that $u_n(x) \to 0$ as $|x| \to \infty$. For $x \in \Omega(n, R)$,

$$V(x) - \lambda_n + \frac{g(u_n(x))}{u_n(x)} \geq V(x) - \lambda_n - A \geq \gamma/4$$

and

$$\Delta u_n(x) = \left\{V(x) - \lambda_n + \frac{g(u_n(x))}{u_n(x)}\right\} u_n(x) \leq \frac{\gamma}{4} u_n(x) < 0.$$

The weak maximum principle now implies that $\Omega(n, R) = \emptyset$ and hence $u_n \geq 0$ on $\mathbb{R}^N$. Setting $c_n(x) = V(x) - \lambda_n + \frac{g(u_n(x))}{u_n(x)}$, we then have that

$$-\Delta u_n + c_n^+ u_n = c_n^- u_n \geq 0 \quad \text{on } \mathbb{R}^N \quad \text{and} \quad u_n(x) \to 0 \quad \text{as } |x| \to \infty$$

where $c_n^+ = \max\{0, c_n\}$ and $c_n = c_n^+ - c_n^-$. By the strong maximum principle, $u_n > 0$ on $\mathbb{R}^N$ for all $n \geq n_0$ for which $t_n > 0$.

A similar argument shows that $u_n < 0$ on $\mathbb{R}^N$ for all $n \geq n_0$ for which $t_n < 0$. But $\|u_n\|_{L^2} > 0$ for all n and hence our initial assumption leads to a contradiction. Therefore there exists $m \in \mathbb{N}$ such that if (λ, u) is a solution with $\lambda < \Lambda + \frac{1}{m}$ and $\|u\|_{H^2} > m$ then u has no zeros in $\mathbb{R}^N$.

(b) The first statement is an immediate consequence of part (a). Furthermore, $t_n = \langle u_n, \phi\rangle \neq 0$ for $n \geq n_0$. By Theorem 5.2(iii), $|t_n| \to \infty$ and $u_n = t_n\{\phi + w_n\}$ where $\|w_n\|_{W^{2,p}} \to 0$ as $n \to \infty$ for all $p \in [2, \infty)$. Hence $\|w_n\|_{L^\infty} \to 0$ and, since ϕ is strictly positive on compact subsets on $\mathbb{R}^N$, it follows that $|u_n| \to \infty$ on uniformly compact subsets as $n \to \infty$.

(c) To get a suitable sequence $\{(\lambda_n^+, u_n^+)\}$ we begin by considering an equation like (1.2) which has the same positive solutions. Define $g_+ : \mathbb{R} \to \mathbb{R}$ by

$$g_+(s) = g(s) \quad \text{for } s \geq 0 \quad \text{and} \quad g_+(s) = -g(-s) \quad \text{if } s < 0.$$

Then g_+ also satisfies (G) with the same constant Γ. For $st \geq 0$ this is obvious, whereas for $st < 0$ it suffices to note that, since $g(0) = 0$,

$$\left|g_+(s) - g_+(t)\right| \leq \left|g_+(s)\right| + \left|g_+(t)\right| \leq \Gamma\left(|s| + |t|\right) = \Gamma|s - t|.$$

Applying Theorem 5.2 to the equation

$$-\Delta u + Vu + g_+(u) = \lambda u \tag{5.2}$$

we get a sequence $\{(\lambda_n, v_n)\}$ with the properties $\lambda_n \to \Lambda$, $|\langle v_n, \phi\rangle| \to \infty$ and $v_n = \langle v_n, \phi\rangle\{\phi + w_n\}$ where $w_n \in W^{2,p}(\mathbb{R}^N)$ with $\langle w_n, \phi\rangle = 0$ and $\|w_n\|_{W^{2,p}} \to 0$ for all $p \in [2,\infty)$. But g_+ is an odd function and so $(\lambda_n, -v_n)$ also satisfies (5.2). Replacing the subsequence $\{(\lambda_n^-, v_n^-)\}$ given by part (b) by $\{(\lambda_n^-, -v_n^-)\}$ we obtain a sequence $\{(\lambda_n, u_n)\} = \{(\lambda_n^+, v_n^+)\} \cup \{(\lambda_n^-, -v_n^-)\}$ of solutions of (5.2) such that $u_n > 0$ for all $n \geq n_0$. Hence $\{(\lambda_n, u_n) : n \geq n_0\}$ is a sequence of solutions of (1.2) which we can relabel as $\{(\lambda_n^+, u_n^+)\}$, having the required properties.

The existence of the sequence $\{(\lambda_n^-, u_n^-)\}$ can easily be deduced by considering Eq. (1.2) with g replaced by g_- where $g_-(s) = -g(-s)$ for all $s \in \mathbb{R}$. This function g_- satisfies (G) with the same constants Γ and A as g. By what has just been proved there exists a sequence $\{(\mu_n^+, z_n^+)\}$ of positive solutions for this modified problem with $\mu_n^+ \to \Lambda$ and $z_n^+ \to \infty$ uniformly on compact subsets of $\mathbb{R}^N$. Setting $\lambda_n^- = \mu_n^+$ and $u_n^- = -z_n^+$ yields a sequence of negative solutions of (1.2) having the required properties. □

5.2. *No asymptotic bifurcation at a simple eigenvalue of $G'(\infty)$*

The purpose of this section is to show that there are potentials $V \in L^\infty(\mathbb{R}^N)$ and nonlinearities g satisfying the condition (G) for which the H-asymptotic derivative $S = G'(\infty)$ of (1.2) can have isolated eigenvalues of odd multiplicity, even simple eigenvalues, that are not asymptotic bifurcation points for Eq. (1.2). At these eigenvalues the assumption $d(\mu, \sigma_e(S)) > 2A + \Gamma$ in Theorem 5.2 is not satisfied and the conclusions (ii) and (iii) fail.

We deal first with the case $N = 1$ since we can give an elementary proof with only a minor assumption about the behaviour of V as $x \to \infty$. This approach also shows that for $N \geq 2$, there may be no asymptotic bifurcation of positive or negative solutions of the kind given by Theorem 5.3. In order to treat the case $N \geq 2$ without any such restriction on the nodal structure of possible solutions, we use work of Koch and Tataru [12] concerning the absence of embedded eigenvalues for the linear Schrödinger operator. This requires some additional restrictions on V and g.

We begin with the case $N = 1$.

Theorem 5.4. *Suppose that $V \in L^\infty(\mathbb{R})$ and that g satisfies (G). Suppose also that there exists $R > 0$ such that $V \in C^1((R,\infty))$ with $V'(x) \leq 0$ for $x > R$. Let*

$$\xi = \lim_{x\to\infty} V(x) + \limsup_{s\to 0} \frac{g(s)}{s}.$$

If $u \in H^2(\mathbb{R})$ satisfies (1.2), then $u \equiv 0$ when $\lambda > \xi$ and consequently there are no asymptotic bifurcation points for (1.2) in the interval (ξ, ∞).

If, in addition,

$$\lim_{x\to\infty} V(x) = \Lambda \quad \text{and} \quad g'(0) \text{ exists} \quad \text{with } \frac{g(s)}{s} > g'(0) \text{ for all } s \neq 0,$$

then $u \equiv 0$ for all $\lambda \in \mathbb{R}$.

Remarks. When g is differentiable at 0, $\xi = \lim_{x\to\infty} V(x) + g'(0)$ and $\xi \geq \inf \sigma_e(S) + g'(0)$. However, in cases where $g'(0) < 0$, we can have $\xi < \Lambda = \inf \sigma(S)$ and then $\sigma(S) \subset (\xi, \infty)$ so no eigenvalue of the H-asymptotic derivative $S = G'(\infty)$ is an asymptotic bifurcation point. More generally, (ξ, ∞) may contain some eigenvalues of S but not others. Examples are given after the proof.

Proof of Theorem 5.4. Suppose that $u \in H^2(\mathbb{R})$ satisfies (1.2) and that $u \not\equiv 0$. The first step is to show that u has no zeros in the interval (R, ∞). Under the hypotheses, $u \in C^2((R,\infty))$ and $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} u'(x) = 0$.

Let $\gamma(t) = \int_0^t g(s)ds$ and then set

$$J(u)(x) = \frac{1}{2}\{u'(x)^2 + \lambda u(x)^2 - V(x)u(x)^2\} - \gamma(u(x)).$$

Then $J(u) \in C^1((R,\infty))$ and

$$\frac{d}{dx}J(u) = \{u'' + \lambda u - Vu - g(u)\}u' - \frac{1}{2}V'u^2 = -\frac{1}{2}V'u^2 \geq 0 \quad \text{on } (R, \infty),$$

whereas $\lim_{x\to\infty} J(u)(x) = 0$. Hence $J(u)(x) \leq 0$ on (R, ∞). Thus if $u(y) = 0$ for some $y > R$ we have $0 \geq J(u)(y) = \frac{1}{2}u'(y)^2$ and so $u(y) = u'(y) = 0$. But then (1.2) and the Lipschitz continuity of g imply that $u \equiv 0$ on $\mathbb{R}$, a contradiction. Hence $u(x) \neq 0$ when $x \in (R, \infty)$, as claimed.

Consider now $\lambda > \xi$ and set $\varepsilon = \lambda - \xi > 0$. By the definition of ξ, there exist $Z > R$ and $T > 0$ such that

$$V(x) + \frac{g(s)}{s} \leq \xi + \frac{\varepsilon}{2} = \lambda - \frac{\varepsilon}{2} \quad \text{for } |x| \geq Z \text{ and } 0 < |s| \leq T.$$

But $u(x) \to 0$ as $x \to \infty$, so by increasing Z, we can assume that $|u(x)| \leq T$ on (Z, ∞).

Let $k = \sqrt{\frac{\varepsilon}{2}}$ and then set $a \equiv \frac{2n\pi}{k}$ and $b \equiv \frac{(2n+1)\pi}{k}$ with $n \in \mathbb{N}$ such that $a > Z$. We have

$$\int_a^b u''(x)\sin kx\, dx = k\{u(b) + u(a)\} - k^2 \int_a^b u(x)\sin kx\, dx$$

where $u'' = \{V + \frac{g(u)}{u} - \lambda\}u$ so

$$k\{u(b) + u(a)\} \int_a^b \left\{V + \frac{g(u)}{u} - \lambda + k^2\right\} u \sin kx\, dx = k^2\{u(b) + u(a)\}^2 > 0,$$

since u has no zeros in (Z, ∞). But on (a, b), $\{V + \frac{g(u)}{u} - \lambda + k^2\}\sin kx \leq 0$ and hence $k\{u(b) + u(a)\}\int_a^b \{V + \frac{g(u)}{u} - \lambda + k^2\}u \sin kx\, dx \leq 0$. This contradiction implies that $u \equiv 0$ is the only solution of (1.2) in $H^2(\mathbb{R}^N)$ for $\lambda > \xi$.

Suppose next that $u \not\equiv 0$ satisfies (1.2) for some $\lambda \in \mathbb{R}$. Using the additional assumption on g, we have that

$$\lambda \int_{-\infty}^{\infty} u^2 dx = \int_{-\infty}^{\infty} (u')^2 + Vu^2 + g(u)u\, dx > \int_{-\infty}^{\infty} (u')^2 + Vu^2 + g'(0)u^2\, dx.$$

Hence

$$\lambda - g'(0) > \frac{\int_{-\infty}^{\infty}(u')^2 + Vu^2 dx}{\int_{-\infty}^{\infty} u^2 dx} \geq \inf\left\{\frac{\int_{-\infty}^{\infty}(v')^2 + Vv^2 dx}{\int_{-\infty}^{\infty} v^2 dx} : v \in H^1(\mathbb{R}^N)\backslash\{0\}\right\} = \Lambda,$$

showing that $\lambda > \Lambda + g'(0) = \xi$ since $V(\infty) = \Lambda$. By the first part of the theorem, this proves that $u \equiv 0$. □

Example 1. For $N = 1$, $V = -\frac{1}{2}\chi_{(-\pi/2,\pi/2)}$ and $g(s) = -As/(1+s^2)$ with $A > 0$, we have that $\sigma(S) = \{-\frac{1}{4}\} \cup [0, \infty)$ and g satisfies (G) with $A = \Gamma$. Then $\Lambda = -1/4$, $V(\infty) = 0$ and $\xi = -A$. Eq. (1.2) has no non-trivial solutions $u \in L^2(\mathbb{R})$ for $\lambda > -A$. If $A > 1/4$, this implies that there is no asymptotic bifurcation at Λ. On the other hand, for $0 < A < \frac{1}{12}$, Λ is an asymptotic bifurcation point and both Theorems 5.2 and 5.3 apply.

As the potential well deepens, the number of negative eigenvalues increases and we encounter situations where there is asymptotic bifurcation at some eigenvalues but not at others.

Example 2. For $n \in \mathbb{N}$, let $B_n = (2n + 1)^2$ and $S_n u = -u'' + V_n u$ where $V_n = -B_n\chi_{(-\pi/2,\pi/2)}$. Now $\sigma(S_n) = \{\lambda_1, ..., \lambda_{2n+1}\} \cup [0, \infty)$ where $\lambda_1 < \lambda_2 < .... < \lambda_{2n+1} < 0$ are simple eigenvalues. For $1 \leq k \leq 2n + 1$, $\lambda_k = -B_n + (\frac{2\theta_k}{\pi})^2$ where θ_{2j+1} is the unique solution of the equation $\tan\theta = \sqrt{\frac{B_n\pi^2}{4\theta^2} - 1}$ in the interval $(j\pi, (j + \frac{1}{2})\pi)$ for $0 \leq j \leq n$ and θ_{2j} is the unique solution of the equation $\tan\theta = -\sqrt{\frac{B_n\pi^2}{4\theta^2} - 1}$ in the interval $((j - \frac{1}{2})\pi, j\pi)$ for $1 \leq j \leq n$. It follows that

$$-(2n + 1)^2 + (k - 1)^2 < \lambda_k < -(2n + 1)^2 + k^2 \quad \text{for } 1 \leq k \leq 2n + 1.$$

Consider a nonlinearity satisfying the condition (G). By Theorem 5.2 there is continuous asymptotic bifurcation at λ_k if $k^2 < (2n + 1)^2 - (2A + \Gamma)$ whereas, by Theorem 5.4 there is no asymptotic bifurcation at λ_k if $g'(0)$ exists and

$(k-1)^2 > (2n+1)^2 + g'(0)$. In particular, there is asymptotic bifurcation at $\Lambda = \lambda_1$ provided that $4n(n+1) > 2A + \Gamma$ and there is no asymptotic bifurcation at λ_{2n+1} if $g'(0) < -(4n+1)$.

For the case where $g(s) = -As/(1+s^2)$ with $A > 0$, these estimates imply that there is asymptotic bifurcation at λ_k for $1 \le k \le 2n+1$ such that $3A + k^2 < (2n+1)^2$ and there is no asymptotic bifurcation at λ_k for $1 \le k \le 2n+1$ such that $(2n+1)^2 < A + (k-1)^2$.

The first step in the proof of Theorem 5.4 shows that a non-trivial solution of (1.2) cannot have a zero in some neighbourhood of infinity. For $N \ge 2$ this fails, but we can still obtain a result about solutions of (1.2) which have the appropriate nodal structure.

Theorem 5.5. *Suppose that $V \in L^\infty(\mathbb{R}^N)$ and that g satisfies (G). Let*

$$\xi = \lim_{R\to\infty} \operatorname*{ess\,sup}_{|x|\ge R} V(x) + \limsup_{s\to 0} \frac{g(s)}{s}.$$

(i) If $(\lambda, u) \in (\xi, \infty) \times H^2(\mathbb{R}^N)$ satisfies (1.2) and there exists $R > 0$ such that $u(x)u(y) \ge 0$ for all $|x|, |y| \ge R$, then $u \equiv 0$.

(ii) Suppose also that $V(\infty) = \lim_{|x|\to\infty} V(x)$ exists and that g is differentiable at 0. Then $\xi = V(\infty) + g'(0)$ and, when $\Lambda > \xi$, there are no sequences $\{(\lambda_n^\pm, u_n^\pm)\}$ of the type given by part (c) of Theorem 5.3, even when $\Lambda < V(\infty) = \inf \sigma_e(S)$ and Λ so is a simple eigenvalue of S.

Proof. (i) Since $\varepsilon \equiv \lambda - \xi > 0$, there exist $R_1 > R$ and $T > 0$ such that

$$V(x) + \frac{g(s)}{s} \le \xi + \frac{\varepsilon}{2} = \lambda - \frac{\varepsilon}{2} \quad \text{for } |x| \ge R_1 \text{ and } 0 < |s| \le T.$$

It follows from Proposition 5.1 that $u \in C^1(\mathbb{R}^N)$ and that $u(x) \to 0$ as $|x| \to \infty$, so by increasing the value of R_1, we can assume that $|u(x)| < T$ for $|x| \ge R_1$.

Let

$$\Lambda_r = \inf\left\{ \int_{|x|<r} |\nabla v|^2 dx : v \in H_0^1\big(B(0,r)\big) \text{ with } \int_{|x|<r} v^2 dx = 1 \right\}.$$

Then $\Lambda_r > 0$ is the lowest eigenvalue of the Dirichlet Laplacian in $B(0,r)$ and there exists an eigenfunction $\phi_r \in C^2(\overline{B(0,r)})$ such that

$$-\Delta\phi_r = \Lambda_r\phi_r \quad \text{and} \quad \phi_r > 0 \quad \text{on } B(0,r)$$
$$\text{whereas} \quad \phi_r = 0 \quad \text{and} \quad \frac{\partial\phi_r}{\partial n} < 0 \quad \text{on } \partial B(0,r),$$

where $\frac{\partial}{\partial n}$ denotes the derivative in the direction of the outward unit normal. A simple scaling argument shows that $\Lambda_r \to 0$ as $r \to \infty$. Choose r so that $\Lambda_r < \frac{\varepsilon}{4}$, and then set

$$B = B(x_0, r) \quad \text{and} \quad \psi(x) = \phi_r(x - x_0) \quad \text{for } x \in \overline{B},$$

where $x_0 = (R_1 + r, 0, \dots, 0)$. Then $-\Delta\psi = \Lambda_r\psi$ on B and we find that

$$\int_{\partial B} u\frac{\partial\psi}{\partial n} ds = \int_B \{u\Delta\psi - \psi\Delta u\} dx = \int_B u\psi\left\{-\Lambda_r - V - \frac{g(u)}{u} + \lambda\right\} dx$$

since $\psi = 0$ on ∂B. Noting that $|x| > R_1$ for all $x \in B$, we have that $-\Lambda_r - V - \frac{g(u)}{u} + \lambda \ge \frac{\varepsilon}{4} > 0$ on B.

Our hypothesis about the nodal structure of u allows two cases: either $u(x) \ge 0$ for all $|x| > R$ or $u(x) \le 0$ for all $|x| > R$. Let $c(x) = V(x) + \frac{g(u(x))}{u(x)} - \lambda$ for x such that $u(x) \ne 0$ so that (1.2) can be written as $-\Delta u + cu = 0$ and hence $-\Delta u + c^+u = c^-u$ where $c^+ = \max\{0, c\}$ and $c^- = \max\{0, -c\}$. Suppose that $u \not\equiv 0$. Using the strong maximum principle, we conclude that either $u(x) > 0$ for all $|x| > R$ or $u(x) < 0$ for all $|x| > R$. Recall that $R_1 > R$.

Thus, in the first case, $u\frac{\partial\psi}{\partial n}<0$ on ∂B whereas $u\psi>0$ in B, leading to a contradiction. The other case also leads to a contradiction. Hence $u\equiv 0$.

(ii) This follows immediately from part (i). □

To deal with the case $N\geq 2$ without making any assumption about the nodal structure of solutions like that used in part (i) of Theorem 5.5 a much deeper analysis appears to be necessary. One possibility is to exploit what is known about the absence of embedded eigenvalues for the linear Schrödinger operator. The following result is an easy consequence of the penetrating work by Koch and Tataru [12] on this topic and I am very grateful to Professors M. Lewin and D. Smets for bringing [12] to my attention.

Theorem 5.6. *For $N\geq 2$, consider Eq. (1.2) where $V\in L^\infty(\mathbb{R}^N)$ and g satisfies the condition (G). Suppose also that*

(i) there exists a constant $V(\infty)$ such that $V-V(\infty)\in L^{\frac{N+1}{2}}(\mathbb{R}^N)$ and
(ii) g is differentiable at 0 and there exist positive constants K and s_0 such that $|g(s)-g'(0)s|\leq K|s|^{\frac{N+5}{N+1}}$ for $|s|\leq s_0$.

If $(\lambda,u)\in\mathbb{R}\times H^2(\mathbb{R}^N)$ satisfies (1.2), then $u\equiv 0$ when $\lambda>\xi=V(\infty)+g'(0)$ and consequently there are no asymptotic bifurcation points for (1.2) in the interval (ξ,∞).

If, in addition,

(iii) $\Lambda=V(\infty)$ and $\frac{g(s)}{s}>g'(0)$ for all $s\neq 0$,

then $u\equiv 0$ for all $\lambda\in\mathbb{R}$.

Remarks. If g is C^2 in a neighbourhood of 0 and $N\geq 3$, then the condition (ii) is satisfied. Note also that the function $g(s)=\frac{s}{1+|s|^\gamma}$ satisfies (G) and (ii) provided that $\gamma\geq\frac{4}{N+1}$. The interval (ξ,∞) contains the whole spectrum of the H-asymptotic derivative $G'(\infty)$ if $\Lambda>\xi$. In particular, this occurs when $\inf_{\mathbb{R}^N}V-V(\infty)>g'(0)$, which is one of the hypotheses of Theorem 5.7 below.

Proof of Theorem 5.6. Suppose that $(\lambda,u)\in\mathbb{R}\times H^2(\mathbb{R}^N)$ satisfies (1.2) and that $u\not\equiv 0$. Setting

$$W_u(x)=V(x)-V(\infty)+\frac{g(u(x))}{u(x)}-g'(0)\quad\text{when } u(x)\neq 0$$

and $W_u(x)=V(x)-V(\infty)$ when $u(x)=0$, we have that u satisfies

$$-\Delta u+W_u u=\big\{\lambda-V(\infty)-g'(0)\big\}u.$$

By Proposition 5.1, $u\in L^\infty(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$ and $u(x)\to 0$ as $|x|\to\infty$. Using property (ii), this implies that there exists $R>0$ such that $|u(x)|\leq s_0$ for $|x|\geq R$ and hence that

$$\left|\frac{g(u(x))}{u(x)}-g'(0)\right|\leq K\big|u(x)\big|^{\frac{4}{N+1}}\quad\text{when } u(x)\neq 0 \text{ and } |x|\geq R.$$

Since $u\in L^\infty(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$, this estimate and the hypothesis (i) ensure that $W_u\in L^{\frac{N+1}{2}}(\mathbb{R}^N)$. It follows from Theorem 3 in [12] that $u\equiv 0$ if $\lambda-V(\infty)-g'(0)>0$.

Suppose now that (iii) also holds and that $(\lambda,u)\in\mathbb{R}\times H^2(\mathbb{R}^N)$ satisfies (1.2) with $u\not\equiv 0$. Using (iii) we have that

$$\begin{aligned}\lambda-V(\infty)-g'(0)&=\frac{\int_{\mathbb{R}^N}|\nabla u|^2+W_u u^2dx}{\int_{\mathbb{R}^N}u^2dx}>\frac{\int_{\mathbb{R}^N}|\nabla u|^2+[V-V(\infty)]u^2dx}{\int_{\mathbb{R}^N}u^2dx}\\&\geq\inf\left\{\frac{\int_{\mathbb{R}^N}|\nabla v|^2+[V-V(\infty)]v^2dx}{\int_{\mathbb{R}^N}v^2dx}:v\in H^1\big(\mathbb{R}^N\big)\backslash\{0\}\right\}=\Lambda-V(\infty)=0.\end{aligned}$$

Hence $\lambda>\xi$ and, by the first part of the proof, this implies that $u\equiv 0$, a contradiction. □

5.3. Asymptotic bifurcation at a point not in $\sigma(G'(\infty))$

In this section we show that continuous asymptotic bifurcation can occur at a point which lies strictly below the whole spectrum of the H-asymptotic derivative. In fact, this situation is covered by work with A. Edelson [6] concerning the global behaviour of a branch of positive solutions of (1.2), but the connection with asymptotic bifurcation is not mentioned in [6]. In order to satisfy the smoothness assumptions made in [6], which deals with classical solutions of equations like (1.2), some additional regularity of V and g are required in this section. However, these restrictions could easily be relaxed by following, in the context of strong solutions, the same arguments as used in [6].

In [6], asymptotic bifurcation is proved by comparing the given equation with a radially symmetric majorant, which has a branch of positive, radially symmetric solutions lying below the branch of positive solutions for the given problem. For the radial solutions we could obtain precise decay rates as $|x| \to \infty$ for solutions in $L^p(\mathbb{R}^N)$, from which asymptotic bifurcation can be deduced for both problems. The branches of positive solutions are constructed using the method of sub- and super-solutions.

For $V \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, define $V^R(x) : \mathbb{R}^N \to \mathbb{R}$ by

$$V^R(x) = \max_{|y|=|x|} V(y)$$

and then define S^R and Λ^R by

$$S^R = -\Delta + V^R \quad \text{and} \quad \Lambda^R = \inf \sigma\big(S^R\big).$$

Then $S^R : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is self-adjoint and $S^R \geq S = -\Delta + V$. Hence $\Lambda = \inf \sigma(S) \leq \Lambda^R$.

Theorem 5.7. *Consider $V \in L^\infty(\mathbb{R}^N)$ and g satisfying (G). Suppose, in addition, that*

(i) $V \in C^1(\mathbb{R}^N)$,
(ii) $V(\infty) = \lim_{|x|\to\infty} V(x)$ *exists and* $\lim_{|x|\to\infty} |x|^{2+\varepsilon}\{V(x) - V(\infty)\} = 0$ *for some* $\varepsilon > 0$,
(iii) $\Lambda^R < V(\infty)$,
(iv) $g \in C^1(\mathbb{R})$ *with* $g(s)/s$ *strictly increasing on* $(0, \infty)$ *and* $g'(0) + V(\infty) < \inf_{x\in\mathbb{R}^N} V(x)$,
(v) *there exist* $\sigma, M \in (0, \infty)$ *such that*

$$\lim_{s\to 0+} \frac{g(s) - g'(0)s}{s^{\sigma+1}} = M.$$

Then for every λ in the interval $I = (\Lambda + g'(0), V(\infty) + g'(0))$ there is a unique positive solution $u_\lambda \in H^2(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ of (1.2). Furthermore, $u_\lambda \in W^{2,p}(\mathbb{R}^N)$ for all $p \in (1, \infty)$ and the map $\lambda \mapsto u_\lambda$ is continuous from I into E with $\|u_\lambda\|_E \to 0$ as $\lambda \to \Lambda + g'(0)$ where E is any of the spaces $C^1(\mathbb{R}^N)$ or $W^{2,p}(\mathbb{R}^N)$ for $p \in (1, \infty)$ with the usual norms. For $\lambda, \mu \in I$ with $\lambda < \mu$, $0 < u_\lambda(x) < u_\mu(x)$ for all $x \in \mathbb{R}^N$. Finally, $\|u_\lambda\|_{L^2} \to \infty$ as $\lambda \to V(\infty) + g'(0)$ provided that $N \leq 4$ and $\sigma > 4/N$ in condition (v).

Remark 1. Since $\inf V \leq \Lambda \leq \Lambda^R$, the conditions (iii) and (iv) imply that $g'(0) < \inf_{x\in\mathbb{R}^N} V(x) - V(\infty) < 0$. By (G) we also have that $\lim_{s\to\infty} \frac{g(s)}{s} = 0$. Hence (iv) also implies that $g(s) < 0$ for all $s > 0$.

Remark 2. Recall from Proposition 2.4 that $G : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, where $G(u) = -\Delta u + Vu + g(u)$, is H-asymptotically linear with $G'(\infty) = S = -\Delta + V$. For $N \leq 4$ and $\sigma > 4/N$, Theorem 5.7 shows that there is continuous asymptotic bifurcation with respect to the L^2-norm at $\mu \equiv V(\infty) + g'(0)$ where $\mu < \inf V < \Lambda = \inf \sigma(S)$ and so $G'(\infty) - \mu I : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is an isomorphism.

Note that by (iv), $d(\Lambda, \sigma_e(S)) = V(\infty) - \Lambda \leq V(\infty) - \inf V < -g'(0) = |g'(0)| \leq \sup\{|\frac{g(s)}{s}| : s \neq 0\} = A$, and so Theorem 5.2 cannot be applied to establish asymptotic bifurcation at Λ. Indeed, Theorem 5.6 shows that, for $N \geq 2$ and $\sigma \geq \frac{4}{N+1}$ in (v), $u \equiv 0$ is the only solution in $H^2(\mathbb{R}^N)$ for $\lambda > V(\infty) + g'(0)$ and so there is no asymptotic bifurcation at Λ nor at any other point in $\sigma(G'(\infty))$.

Remark 3. The asymptotic bifurcation at $\mu = V(\infty) + g'(0)$ is of a different type from what occurs at Λ in Theorem 5.3, where the solutions tend to infinity uniformly on compact subsets of $\mathbb{R}^N$. The solutions u_λ given by Theorem 5.7 remain uniformly bounded as $\lambda \to \mu$. Indeed by Theorem 14 of [6], there exists a function $U \in W^{2,p}(\mathbb{R}^N)$ for all p satisfying $N/p < \max\{N-2, 2/\sigma\}$ such that $0 < u_\lambda(x) < U(x)$ for all $x \in \mathbb{R}^N$ and $\lambda \in I$ and $\|u_\lambda - U\|_{W^{2,p}} \to 0$ as $\lambda \to \mu$. Thus u_λ converges uniformly to U, but $U \notin L^2(\mathbb{R}^N)$.

Proof of Theorem 5.7. The paper [6] deals with elliptic equations of the form

$$-\Delta u + fu + h(u)u = \mu u$$

where $f(x) \to 0$ as $|x| \to \infty$ and $h(0) = 0$. We express (1.2) in this form by setting

$$f(x) = V(x) - V(\infty), \qquad h(s) = \frac{g(s)}{s} - g'(0) \quad \text{for } s \neq 0, \quad h(0) = 0,$$

and

$$\mu = \lambda - V(\infty) - g'(0).$$

Note the function $g(x)$ in Eq. (1) of [6] is identically 1 in our context.

Since $\lim_{s\to\infty} h(s) = -g'(0)$, the assumptions (A1), (A3), (A5) and (A5)* of [6] are clearly satisfied. Furthermore, $h \in C^1((0,\infty))$ and

$$h'(s) = \frac{g'(s)s - g(s)}{s^2} \quad \text{for } s > 0.$$

Since $g(0) = 0$, it follows easily that $\lim_{s\to 0+} h(s) = \lim_{s\to 0+} sh'(s) = 0$, showing that (A2) of [6] is also satisfied. The condition (A4)(i) is satisfied with $\beta = 0$ and (A4)(ii) is precisely our hypothesis (v).

Noting that $\inf \sigma(-\Delta + f) = \Lambda - V(\infty)$, Theorem 10 of [6] now yields the following information. For every $\mu \in (\Lambda - V(\infty), 0) = J$, there is a unique solution, w_μ, of the problem,

$$w \in C^2(\mathbb{R}^N) \quad \text{and} \quad -\Delta w + fw + h(w)w = \mu w \quad \text{on } \mathbb{R}^N$$
$$\text{with} \lim_{|x|\to\infty} w(x) = 0 \quad \text{and} \quad w(x) > 0 \quad \text{for all } x \in \mathbb{R}^N.$$

Furthermore, $w_s(x) < w_t(x)$ for all $x \in \mathbb{R}^N$ if $s, t \in J$ with $s < t$. Also $w \in W^{2,p}(\mathbb{R}^N)$ for all $p \in (1,\infty)$ and the map $\mu \mapsto w_\mu$ is continuous from J into E with $\|w_\mu\|_E \to 0$ as $\mu \to \Lambda - V(\infty)$ where E is any of the spaces $C^1(\mathbb{R}^N)$ or $W^{2,p}(\mathbb{R}^N)$ for $p \in (1,\infty)$ with the usual norms.

Our condition (ii) ensures that the extra assumption required for Theorem 21 in [6] is satisfied, so we have that $\|w_\mu\|_{L^2} \to \infty$ as $\mu \to 0$ provided that $N \le 4$ and $\sigma > 4/N$.

Setting $u_\lambda = w_{\lambda - V(\infty) - g'(0)}$, we obtain a curve of solutions of (1.2) having the required properties. □

5.4. A special case of (1.2)

We consider a typical example of potential and a nonlinearity satisfying the condition (G) for which it is easy to compare the situations discussed in Sections 5.1 to 5.3.

(A1) $V(x) = W(|x|)$ where $W \in C^1(\mathbb{R})$ is an even function with compact support such that

$$\Lambda \equiv \inf\left\{ \int_{\mathbb{R}^N} |\nabla u|^2 + Vu^2 dx : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} u^2 dx = 1 \right\} < 0.$$

(A2) $g(s) = Cs/(1+|s|^\sigma)$ where $C \in \mathbb{R}$ and $\sigma > 0$.

Eq. (1.2) is now

$$-\Delta u + Vu + \frac{Cu}{1+|u|^\sigma} = \lambda u. \tag{5.3}$$

Clearly $g \in C^1(\mathbb{R})$ and g is an odd function with $g'(0) = C$ and $\lim_{s\to\infty} g'(s) = 0$. Hence g satisfies the condition (G) with $\Gamma = \max_{s\geq 0} |g'(s)| \geq |C|$. By assumption (A1), $S = -\Delta + V : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is a self-adjoint operator with $\Lambda = \inf \sigma(S) < 0 = \inf \sigma_e(S)$.

Setting $G(u) = -\Delta u + Vu + g(u)$, it follows from Propositions 2.2 and 2.4 that $G : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is continuous and H-asymptotically linear with $G'(\infty) = -\Delta + V = S$, but for $C \neq 0$, G is not asymptotically linear.

First of all we point out that Genoud's work in [11] establishes asymptotic bifurcation at Λ when $C > 0$ and provides a good deal of extra information.

Case 1. Suppose that $C > 0$. Set $k = C + \sup_{x\in\mathbb{R}^N} V(x)$ and then write (5.3) in the form

$$\Delta u + f(x, u)u = \tilde{\lambda} u, \tag{5.4}$$

where $\tilde{\lambda} = k - \lambda$ and $f(x, s) = k - V(x) - \frac{C}{1+|s|^\sigma} = \sup V(x) - V(x) + \frac{C|s|^\sigma}{1+|s|^\sigma}$. For $C > 0$, the hypotheses (f1) to (f6) of [11] are satisfied and so Theorem 1 of [11] shows that for $N \leq 3$, there is continuous asymptotic bifurcation at $\tilde{\lambda}_\infty \equiv k - \Lambda$ for Eq. (5.4) and hence at Λ for (5.3).

This conclusion cannot be obtained using our Theorem 5.2 which requires Λ to be sufficiently far from $\sigma_e(S)$. On the other hand, if $C < 0$ then the condition (f4) in [11] cannot be satisfied and Λ may not be an asymptotic bifurcation point for (5.3).

Case 2. For $N = 1$, Theorem 5.4 shows that there are no non-trivial solutions of (5.3) with $\lambda > g'(0) = C$ and hence asymptotic bifurcation cannot occur at points in (C, ∞). By Theorem 5.6 this is also true for $N \geq 2$, provided that $\sigma \geq \frac{4}{N+1}$. Hence, if $C < \Lambda$, there is no asymptotic bifurcation at the simple eigenvalue Λ of the H-asymptotic derivative $G'(\infty)$, nor at any point in $\sigma(G'(\infty))$.

The condition $C < \Lambda$ implies that $C < 0$ and that $|\Lambda| = d(\Lambda, \sigma_e(S)) < |C|$ so Theorem 5.2 cannot be applied in this case. However, if $|\Lambda|$ is large enough relative to $|C|$, the hypotheses of Theorems 5.2 and 5.3 are satisfied and, even for $C > 0$, we obtain information about asymptotic bifurcation not contained in [11], which deals only with Λ.

Case 3. Suppose that $\sigma \leq 3 + 2\sqrt{2}$ and that $|\Lambda| > 3|C|$. The restriction on σ ensures that $|C| = \max_{s\in\mathbb{R}} |g'(s)|$ and so, in the notation of Theorem 5.2, $A = \Gamma = |C|$. Since $d(\Lambda, \sigma_e(S)) = |\Lambda|$, it follows from Theorems 5.2 and 5.3 that there is continuous asymptotic bifurcation at Λ and we have some additional information about the solutions. In fact, $d(\mu, \sigma_e(S)) > 2A + \Gamma$ provided that $\mu < -3|C|$ and so Theorem 5.2 shows that all points in the interval $(-\infty, -3|C|)$ at which asymptotic bifurcation occurs must be eigenvalues of $S = G'(\infty)$ and there is asymptotic bifurcation at all eigenvalues of odd multiplicity in this interval.

When $\sigma > 3 + 2\sqrt{2}$, we still have $A = |C|$ but now $\max_{s\in\mathbb{R}} |g'(s)| = |C| \frac{(\sigma-1)^2}{4\sigma} > |C|$ so the condition $d(\mu, \sigma(G'(\infty))) > 2A + \Gamma$ in Theorem 5.2 is satisfied for $\mu < -|C|\{2 + \frac{(\sigma-1)^2}{4\sigma}\}$.

Cases 2 and 3 show that for $C < 0$, Λ may or may not be an asymptotic bifurcation point. When there is no asymptotic bifurcation at Λ it may occur at a point lying below the spectrum of the H-asymptotic derivative.

Case 4. Suppose that $N \leq 4$, that $C < \inf_{x\in\mathbb{R}^N} V(x)$ and that $\sigma > 4/N$. Then $C < \Lambda < 0$ and the hypotheses of Theorem 5.7 are satisfied with $\Lambda^R = \Lambda$ in condition (iii) and $M = -C$ in condition (v). Hence there is continuous asymptotic bifurcation at $g'(0) = C$. Since $\Lambda \geq \inf_{x\in\mathbb{R}^N} V(x) > C$, $C \notin \sigma(G'(\infty))$. By Corollary 11 of [6], the solutions u_λ are radially symmetric and they remain uniformly bounded as $\lambda \to C$.

Conflict of interest statement

I, Charles A. Stuart, hereby certify that there are no conflicts of interest.

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