On compactness estimates
for hyperbolic systems of conservation laws

Fabio Ancona [a,*], Olivier Glass [b], Khai T. Nguyen [c]

[a] *Dipartimento di Matematica, Università di Padova, Via Trieste 63, 35121 Padova, Italy*
[b] *Ceremade, Université Paris-Dauphine, CNRS UMR 7534, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France*
[c] *Department of Mathematics, Penn State University, 235, McAllister Building, PA 16802, USA*

Abstract

We study the compactness in L^1_{loc} of the semigroup mapping $(S_t)_{t>0}$ defining entropy weak solutions of general hyperbolic systems of conservation laws in one space dimension. We establish a lower estimate for the Kolmogorov ε-entropy of the image through the mapping S_t of bounded sets in $L^1 \cap L^\infty$, which is of the same order $1/\varepsilon$ as the ones established by the authors for scalar conservation laws. We also provide an upper estimate of order $1/\varepsilon$ for the Kolmogorov ε-entropy of such sets in the case of Temple systems with genuinely nonlinear characteristic families, that extends the same type of estimate derived by De Lellis and Golse for scalar conservation laws with convex flux. As suggested by Lax, these quantitative compactness estimates could provide a measure of the order of "resolution" of the numerical methods implemented for these equations.
© 2014 Elsevier Masson SAS. All rights reserved.

Résumé

Nous étudions la compacité dans L^1_{loc} du semi-groupe $(S_t)_{t>0}$ définissant les solutions faibles d'entropie de systèmes hyperboliques de lois de conservation généraux en dimension un d'espace. Nous établissons une estimée inférieure de l'ε-entropie de Kolmogorov de l'image par l'application S_t d'ensembles bornés dans $L^1 \cap L^\infty$, qui est du même ordre $1/\varepsilon$ que celles establies par les auteurs pour les lois de conservation scalaires. Nous obtenons aussi une estimée supérieure d'ordre $1/\varepsilon$ pour l'ε-entropie de Kolmogorov de tels ensembles dans le cas des systèmes de Temple avec des champs caractéristiques vraiment non linéaires, ce qui étend le même type d'estimées obtenues par De Lellis et Golse dans le cas des lois de conservation scalaires à flux convexe. Comme suggéré par Lax, ces estimées quantitatives pourraient donner une mesure de l'ordre de « résolution » de méthodes numériques mises en place pour ces équations.
© 2014 Elsevier Masson SAS. All rights reserved.

* Corresponding author.
E-mail addresses: ancona@math.unipd.it (F. Ancona), glass@ceremade.dauphine.fr (O. Glass), ktn2@math.psu.edu (K.T. Nguyen).

1. Introduction

Consider a general system of hyperbolic conservation laws in one space dimension

$$u_t + f(u)_x = 0, \quad t \geq 0, \, x \in \mathbb{R}, \tag{1}$$

where $u = u(t, x) \in \mathbb{R}^N$ represents the conserved quantities and the flux $f(u) = (f_1(u), \ldots, f_N(u))$ is a vector valued map of class C^2, defined on an open, connected domain $\Omega \subseteq \mathbb{R}^N$ containing the origin. Assume that the above system is strictly hyperbolic, i.e., that the Jacobian matrix $Df(u)$ has N real, distinct eigenvalues $\lambda_1(u) < \ldots < \lambda_N(u)$ for all $u \in \Omega$. Several laws of physics take the form of a conservation equation. A primary example of such systems is provided by the Euler equations of non-viscous gases (cf. [7]). The fundamental paper of Bianchini and Bressan [4] shows that (1) generates a unique (up to the domain) Lipschitz continuous semigroup $S : [0, \infty[\, \times \, \mathcal{D}_0 \to \mathcal{D}_0$ defined on a closed domain $\mathcal{D}_0 \subset L^1(\mathbb{R}, \mathbb{R}^N)$, with the properties:

(i)

$$\{v \in L^1(\mathbb{R}, \Omega) \mid \text{Tot.Var.}(v) \leq \delta_0\} \subset \mathcal{D}_0 \subset \{v \in L^1(\mathbb{R}, \Omega) \mid \text{Tot.Var.}(v) \leq 2\delta_0\}, \tag{2}$$

for suitable constant $\delta_0 > 0$.

(ii) For every $\bar{u} \in \mathcal{D}_0$, the semigroup trajectory $t \mapsto S_t \bar{u} \doteq u(t, \cdot)$ provides an entropy weak solution of the Cauchy problem for (1), with initial data

$$u(0, \cdot) = \bar{u}, \tag{3}$$

that satisfy the following admissibility criterion proposed by T.P. Liu in [15], which generalizes the classical stability conditions introduced by Lax [13].

Liu stability condition. A shock discontinuity of the i-th family (u^L, u^R), traveling with speed $\sigma_i[u^L, u^R]$, is *Liu admissible* if, for any state u lying on the i-th Hugoniot curve between u^L and u^R, the shock speed $\sigma_i[u^L, u]$ of the discontinuity (u^L, u) satisfies

$$\sigma_i[u^L, u] \geq \sigma_i[u^L, u^R]. \tag{4}$$

Thanks to the uniform BV-bound on the elements of $\mathcal{D}_0$, applying Helly's compactness theorem it follows that S_t is a compact mapping, for every $t > 0$. Aim of this paper is to provide a quantitative estimate of the compactness of such a mapping. Namely, following a suggestion of Lax [12], we wish to estimate the Kolmogorov ε-entropy in L^1 of the image of bounded sets in $\mathcal{D}_0$ through the map S_t. We recall that, given a metric space (X, d), and a totally bounded subset K of X, for every $\varepsilon > 0$ we define the Kolmogorov ε-entropy of K as follows. Let $N_\varepsilon(K \mid X)$ be the minimal number of sets in a cover of K by subsets of X having diameter no larger than 2ε. Then, the ε-entropy of K is defined as

$$H_\varepsilon(K \mid X) \doteq \log_2 N_\varepsilon(K \mid X).$$

Throughout the paper, we will call an ε-cover, a cover of K by subsets of X having diameter no larger than 2ε. Entropy numbers play a central roles in various areas of information theory and statistics as well as of learning theory. In the present setting, this concept could provide a measure of the order of "resolution" of a numerical method for (1), as suggested in [14].

In the case of scalar conservation laws ($N = 1$) with strictly convex (or concave) flux, De Lellis and Golse [8] obtained an upper bound of order $1/\varepsilon$ on the ε-entropy of $S_t(\mathcal{L})$, for sets $\mathcal{L} \subset L^1(\mathbb{R})$ of bounded, compactly supported functions, of the form

$$\mathcal{L}_{[I,m,M]} \doteq \{\bar{u} \in L^1(\mathbb{R}, \Omega) \mid \text{Supp}(\bar{u}) \subset I, \, \|\bar{u}\|_{L^1} \leq m, \|\bar{u}\|_{L^\infty} \leq M\}, \tag{5}$$

where I denotes a given interval of $\mathbb{R}$. This upper bound turns out to be optimal since we provided in [1] a lower bound of the same order for the ε-entropy of $S_t(\mathcal{L})$, for sets $\mathcal{L}$ as in (5), thus showing that such an ε-entropy is of size $\approx (1/\varepsilon)$ for scalar conservation laws with strictly convex (or concave) flux.

These estimates hold for classes of initial data with possibly unbounded total variation because the regularizing effect due to the convexity (or concavity) of the flux function f yields solutions $u(t, \cdot)$ of (1) that belong to $\text{BV}_{\text{loc}}(\mathbb{R})$ for any $t > 0$. This is no more true in the case of conservation laws with non-convex (or concave) flux and in the case

of systems of conservation laws with no monotonicity assumption on the eigenvalues of the Jacobian matrix $Df(u)$. On the other hand, the well-posedness theory for a general system of conservation laws has been established only for initial data with sufficiently small total variation. Therefore, aiming to establish estimates on the ε-entropy of solutions to general systems of conservation laws (1), it is natural to restrict our analysis to classes of initial data with uniformly bounded total variation. Namely, we shall provide estimates on the ε-entropy of $S_t(\mathcal{L} \cap \mathcal{D}_0)$, for sets $\mathcal{L}$ as in (5), with $\mathcal{D}_0$ as in (2). Specifically, we prove the following.

Theorem 1. *Let* $f : \Omega \to \mathbb{R}^N$ *be a* C^2 *map on an open, connected domain* $\Omega \subset \mathbb{R}^N$ *containing the origin, and assume that the system* (1) *is strictly hyperbolic. Let* $(S_t)_{\geq 0}$ *be the semigroup of entropy weak solutions generated by* (1) *defined on a domain* $\mathcal{D}_0$ *satisfying* (2). *Then, given any* $L, m, M, T > 0$, *for any interval* $I \subset \mathbb{R}$ *of length* $|I| = 2L$, *and for* $\varepsilon > 0$ *sufficiently small, the following estimates hold.*

(i)

$$H_\varepsilon\big(S_T(\mathcal{L}_{[I,m,M]} \cap \mathcal{D}_0) \,\big|\, L^1(\mathbb{R}, \Omega)\big) \geq \frac{N^2 L^2}{T} \cdot \frac{(\min\{c_1, c_2 \frac{T}{L}\})^2}{\max\{c_3, c_4 \frac{N^2 L}{T}, c_5 \frac{NL}{\delta_0 T}\}} \cdot \frac{1}{\varepsilon}, \tag{6}$$

where $c_3 \geq 0$, $c_l > 0$, $l = 1, 2, 4, 5$, *are constants given in* (195), (197), *which depend only on the eigenvalues* $\lambda_i(u)$ *of the Jacobian matrix* $Df(u)$, *on the corresponding right and left eigenvectors* $r_i(u), l_i(u)$, *and on their derivatives, in a neighborhood of the origin.*

(ii)

$$H_\varepsilon\big(S_T(\mathcal{L}_{[I,m,M]} \cap \mathcal{D}_0) \,\big|\, L^1(\mathbb{R}, \Omega)\big) \leq 48 N \delta_0 \cdot L_T \cdot \frac{1}{\varepsilon}, \tag{7}$$

where

$$L_T \doteq L + \frac{\Delta_\vee \lambda}{2} \cdot T, \qquad \Delta_\vee \lambda \doteq \sup\{\lambda_N(u) - \lambda_1(v); \ u, v \in \Omega\}. \tag{8}$$

Remark 1. If the bound δ_0 on the total variation of the initial data in the domain $\mathcal{D}_0$ satisfies the inequality $\delta_0 < \min\{\frac{c_5}{c_3} \cdot \frac{NL}{T}, \frac{c_5}{c_4} \cdot \frac{1}{L}\}$ (interpreting $1/c_3 \doteq \infty$ when $c_3 = 0$), then the lower estimate (6) takes the form

$$H_\varepsilon\big(S_T(\mathcal{L}_{[I,m,M]} \cap \mathcal{D}_0) \,\big|\, L^1(\mathbb{R}, \Omega)\big) \geq NL\delta_0 \cdot \frac{(\min\{c_1, c_2 \frac{T}{L}\})^2}{c_5} \cdot \frac{1}{\varepsilon}. \tag{9}$$

Therefore, in this case, upper and lower bounds (7), (9) of the ε-entropy turn out to have the same size $NL\delta_0 \cdot \frac{1}{\varepsilon}$. On the other hand, if $c_3 > 0$, in the case where $T \geq \max\{\frac{c_1}{c_2} \cdot L, \frac{c_4}{c_3} \cdot N^2 L, \frac{c_5}{c_3} \cdot \frac{NL}{\delta_0}\}$, we obtain by (6), (195), the estimate

$$H_\varepsilon\big(S_T(\mathcal{L}_{[I,m,M]} \cap \mathcal{D}_0) \,\big|\, L^1(\mathbb{R}, \Omega)\big) \geq \frac{N^2 L^2}{T} \cdot \frac{c_1^2}{c_3} \cdot \frac{1}{\varepsilon}, \tag{10}$$

with $c_3 \doteq 2 \sup\{|\nabla \lambda_i(u)|; \ |u| \leq \overline{d}, \ i = 1, \dots, N\}$ for some $\overline{d} > 0$. Hence, if $c_3 > 0$, for times T sufficiently large we obtain a lower bound on the ε-entropy of $S_T(\mathcal{L}_{[I,m,M]} \cap \mathcal{D}_0)$ which is of the same order $L^2/(|f''(0)| T) \cdot \frac{1}{\varepsilon}$ established in [1] for solutions to scalar conservation laws with strictly convex (or concave) flux f.

Remark 2. When $N = 1$, the semigroup map S_t is defined on the whole space $L^1(\mathbb{R})$ (cf. [5, Chapter 6], [11]). Thus, in this case we may analyze the ε-entropy of $S_t(\mathcal{L})$ for sets $\mathcal{L}$ of initial data with possibly unbounded total variation as in (5). In fact, for scalar conservation laws, with the same arguments used to establish Theorem 1(i), if $\overline{c} \doteq \sup\{|f''(u)|; \ |u| \leq \overline{d}\} > 0$ for some $\overline{d} > 0$, one can derive, for ε sufficiently small, the lower bound (cf. Remark 5 and Remark 6):

$$H_\varepsilon\big(S_T(\mathcal{L}_{[I,m,M]}) \,\big|\, L^1(\mathbb{R})\big) \geq \frac{L^2}{144 \cdot \ln(2) \cdot \overline{c} \cdot T} \cdot \frac{1}{\varepsilon}. \tag{11}$$

Thus, Theorem 1 provides in particular an extension of [1, Theorem 1.3] to the case of general scalar conservation laws with smooth, not necessarily convex (or concave) flux. Clearly, the lower bound (11) is significant only in the

case where $\inf\{|u|;\ |f''(u)| > 0\} = 0$, since otherwise one can easily see that the left-hand side of (11) equals $+\infty$ for small ε.

The upper bound (7) stated in Theorem 1 can be easily obtained relying on the upper estimates for the covering number of classes of functions with uniformly bounded total variation established in [2]. In fact, given any element $\varphi \in S_T(\mathcal{L}_{[I,m,M]} \cap \mathcal{D}_0)$, with $\mathcal{L}_{[I,m,M]}$ as in (5), $|I| = 2L$, by the finite speed of propagation along (generalized) characteristics (cf. [7, Chapter 10]) we have the bound $|\text{Supp}(\varphi)| \leq 2L_T$ on the support of φ, with L_T as in (8). Moreover, observe that, defining the total variation of a vector valued map $\varphi = (\varphi_1, \ldots, \varphi_p) : \mathbb{R} \to \mathbb{R}^p$ as $\text{Tot.Var.}(\varphi) \doteq \sum_i \text{Tot.Var.}(\varphi_i)$, and setting

$$\mathcal{M}_{[L,\delta_0,p]} \doteq \left\{ \varphi \in \text{BV}\left([0, 2L], \mathbb{R}^p\right) \mid \text{Tot.Var.}(\varphi) \leq 2\delta_0 \right\}, \tag{12}$$

one has

$$N_\varepsilon\left(\mathcal{M}_{[L,\delta_0,N]} \mid L^1\left([0, 2L], \mathbb{R}^N\right)\right) \leq N_\varepsilon\left(\mathcal{M}_{[NL,\delta_0,1]} \mid L^1\left([0, 2NL], \mathbb{R}\right)\right). \tag{13}$$

This is due to the fact that, if we let $\varphi\lceil_J$ denote the restriction of a map φ to a set J, for every ε-cover $\bigcup_\alpha E^\alpha$ of $\mathcal{M}_{[NL,\delta_0,1]}$ we can always consider the sets $E_1^\alpha \times \ldots \times E_N^\alpha$, with $E_i^\alpha \doteq \{\varphi(\cdot - (i - 1)L)\lceil_{[(i-1)L,iL]}; \ \varphi \in E^\alpha\}$, which provide an ε-cover $\bigcup_\alpha (E_1^\alpha \times \ldots \times \mathcal{E}_N^\alpha)$ of $\mathcal{M}_{[L,\delta_0,N]}$, with the same cardinality as $\bigcup_\alpha E^\alpha$. Thus, given any $L, m, M, T > 0$ and any interval $I \subset \mathbb{R}$ of length $|I| = 2L$, applying [2, Theorem 1], and relying on (2), (13), for $\varepsilon > 0$ sufficiently small we find the following upper bound on the minimal covering number

$$\begin{aligned} N_\varepsilon\left(S_T(\mathcal{L}_{[I,m,M]} \cap \mathcal{D}_0) \mid L^1(\mathbb{R})\right) &\leq N_\varepsilon\left(\mathcal{M}_{[L_T,\delta_0,N]} \mid L^1\left([0, 2L_T], \mathbb{R}^N\right)\right) \\ &\leq N_\varepsilon\left(\mathcal{M}_{[NL_T,\delta_0,N]} \mid L^1\left([0, 2NL_T], \mathbb{R}\right)\right) \\ &\leq 2^{\frac{48\delta_0 \cdot NL_T}{\varepsilon}}. \end{aligned} \tag{14}$$

One then clearly recovers (7) from (14).

Therefore, the main novelty of the estimates stated in Theorem 1 consists in the lower bound (6) that is independent on the total variation of the functions in $\mathcal{D}_0$, for times T sufficiently large (cf. Remark 1). Following the same strategy adopted in [1] we shall prove (6) in two steps:

1. For every i-th characteristic family, let $s \mapsto R_i(s)$ denote the integral curve of the i-th eigenvector r_i, starting at the origin. Consider a family of profiles of i-simple waves $\{\phi_i^\iota\}_\iota$ defined as parametrizations $s \mapsto \phi_i^\iota(s) \doteq R_i(\beta_\iota(s))$ of R_i through a suitable class of piecewise affine, compactly supported functions $\{\beta_\iota\}_\iota$. We will show that, at any given time T, any superposition $\phi^{\iota_1,\ldots,\iota_N}$ of simple waves $\phi_1^{\iota_1}, \ldots, \phi_N^{\iota_N}$, can be obtained as the value $u(T, \cdot) = S_T \bar{u}$ of an entropy admissible weak solution of (1), with initial data $\bar{u} \in \mathcal{L}_{[L,m,M]} \cap \mathcal{D}_0$.
2. We shall provide an optimal estimate of the maximum number of elements of the family $\{\phi^{\iota_1,\ldots,\iota_N}\}_{\iota_1,\ldots,\iota_N}$ contained in a subset of $S_T(\mathcal{L}_{[I,m,M]} \cap \mathcal{D}_0)$ of diameter 2ε. This estimate is established with a similar combinatorial argument as the one used in [2], and immediately yields a lower bound on the ε-entropy of the set $\{\phi^{\iota_1,\ldots,\iota_N}\}_{\iota_1,\ldots,\iota_N}$. In turn, from the lower bounds on $H_\varepsilon(\{\phi^{\iota_1,\ldots,\iota_N}\}_{\iota_1,\ldots,\iota_N} \mid L^1(\mathbb{R}, \Omega))$, we recover (6).

Next we focus our attention on a particular class of hyperbolic systems introduced by Temple [17,18], under the assumption that all characteristic families are genuinely nonlinear or linearly degenerate (see Definition 1 in Subsection 3.1). Systems of this type arise in traffic flow models, in multicomponent chromatography, as well as in problems of oil reservoir simulation. The special geometric features of such systems allow the existence of a continuous semigroup of solutions $S : [0, \infty[\times \mathcal{D} \to \mathcal{D}$ defined on domains $\mathcal{D}$ of L^∞-functions with possibly unbounded variation of the form

$$\mathcal{D} \doteq \left\{ v \in L^1(\mathbb{R}, \Omega) \mid W(v(x)) \in [a_1, b_1] \times \cdots \times [a_n, b_n] \text{ for all } x \in \mathbb{R} \right\}, \tag{15}$$

where $W(v) = (W_1(v), \ldots, W_N(v))$ denotes the Riemann coordinates of $v \in \Omega$ (see [6,3]).

Every trajectory of the semigroup $t \mapsto S_t \bar{u} \doteq u(t, \cdot)$ yields an entropy weak solution of (1), (3). When all characteristic families are genuinely nonlinear such a semigroup is Lipschitz continuous and the map $u(t, x) \doteq S_t \bar{u}(x)$ satisfies the following Oleĭnik-type inequalities [16] on the decay of positive waves (expressed in Riemann coordinates $w_i(t, \cdot) \doteq W_i(u(t, \cdot))$):

$$\frac{w_i(t, y) - w_i(t, x)}{y - x} \le \frac{1}{c\,t} \quad \forall x < y, \ t > 0, \ i = 1, \dots, N, \tag{16}$$

for some constant

$$0 < c \le \inf\big\{ \big|\langle \nabla\lambda_i(u), r_i(u)\rangle\big|; \ u \in W^{-1}(\Pi), \ i = 1, \dots, N\big\}, \tag{17}$$

where

$$\Pi \doteq [a_1, b_1] \times \cdots \times [a_n, b_n].$$

In this setting, it is natural to ask whether we can extend the estimates provided by Theorem 1 to classes of initial data with unbounded variation. The next result provides a positive answer to this question. Namely, relying on the analysis of the evolution of the Riemann coordinates along the characteristics and on the Oleǐnik-type inequalities, we will establish upper and lower estimates on the ε-entropy of solutions to genuinely nonlinear Temple systems which are the natural extension to this class of hyperbolic systems of the compactness estimates established in [1,8] for scalar conservation laws with strictly convex (or concave) flux. Specifically, letting $S_t^w \overline{w} \doteq W(u(t, \cdot))$ denote the Riemann coordinates expression of the solution of (1), (3), with $\overline{u} \doteq W^{-1} \circ \overline{w}$, determined by the semigroup map S, and adopting the norms $\|w\|_{L^1} \doteq \sum_i \|w_i\|_{L^1}$, $\|w\|_{L^\infty} \doteq \sup_i \|w_i\|_{L^\infty}$ on the space $L^1(\mathbb{R}, \Pi)$, we prove the following

Theorem 2. *In the same setting of Theorem 1, assume that (1) is a strictly hyperbolic system of Temple class, and that all characteristic families are genuinely nonlinear or linearly degenerate. Let $(S_t)_{\ge 0}$ be the semigroup of entropy weak solutions generated by (1) defined on a domain $\mathcal{D}$ as in (15). Then, given any $L, m, M, T > 0$, and any interval $I \subset \mathbb{R}$ of length $|I| = 2L$, setting*

$$\mathcal{L}_{[I,m,M]}^w \doteq \big\{ \overline{w} \in L^1(\mathbb{R}, \Pi) \,\big|\, \text{Supp}(\overline{w}) \subset I, \ \|\overline{w}\|_{L^1} \le m, \ \|\overline{w}\|_{L^\infty} \le M \big\}, \tag{18}$$

for $\varepsilon > 0$ sufficiently small, the following hold.

(i)

$$H_\varepsilon\big(S_T^w(\mathcal{L}_{[I,m,M]}^w) \,\big|\, L^1(\mathbb{R}, \Pi)\big) \ge \frac{N^2 L^2}{T} \cdot \frac{1}{\max\{c_6, \, c_7 \frac{NL}{T}\}} \cdot \frac{1}{\varepsilon}. \tag{19}$$

where c_6, c_7 are nonnegative constants given in (201), (202), which depend only on the gradient of the eigenvalues $\lambda_i(u)$ of the Jacobian matrix $Df(u)$ and on the corresponding right eigenvectors $r_i(u)$, in a neighborhood of the origin.

(ii) *If all characteristic families are genuinely nonlinear, one has*

$$H_\varepsilon\big(S_T^w(\mathcal{L}_{[I,m,M]}^w) \,\big|\, L^1(\mathbb{R}, \Pi)\big) \le \frac{32 N^2 L_T^2}{c\,T} \cdot \frac{1}{\varepsilon}, \tag{20}$$

where

$$L_T \doteq L + \sqrt{\frac{8NmT}{c}} \cdot \sup\big\{ \big|\langle \nabla\lambda_i(u), r_j(u)\rangle\big|; \ \big|W(u)\big| \le M, \ i, j = 1, \dots, N \big\}, \tag{21}$$

and c is the constant appearing in (16).

The paper is organized as follows. In Section 2 we first introduce a family of simple waves and then construct a class of classical solutions of (1) with initial data given by the profiles of N simple waves supported on disjoint sets. This analysis is in particular carried out with a finer accuracy for the special class of Temple systems. In Section 3 we establish a controllability result and a combinatorial computation both for general hyperbolic systems and for Temple systems, which yield the lower bound on the ε-entropy stated in Theorem 1 and Theorem 2. Finally, Section 4 contains the derivation of the upper bound on the ε-entropy for Temple systems stated in Theorem 2.

2. Simple waves and classical solutions

2.1. Simple waves

Let $f : \Omega \to \mathbb{R}^N$ be a C^2 map on an open, connected domain Ω, and assume that a neighborhood of the origin $B_{\bar{d}} \doteq \{u \in \mathbb{R}^n \mid |u| \leq \bar{d}\}$ is contained in Ω. We shall consider here a class of continuous, piecewise C^1 solutions of (1) that take values on the integral curves of the eigenvectors of the Jacobian matrix Df. Such solutions can be regarded as the nonlinear analogue of the elementary waves of each characteristic family in which it is decomposed a solution of a semilinear system (cf. [7, Section 7.6]). For every i-th characteristic family, let $s \mapsto R_i(s)$ denote the integral curve of the eigenvector r_i, passing through the origin. More precisely, we define $R_i(\cdot)$ as the unique solution of the Cauchy problem

$$\frac{du}{ds} = r_i\big(u(s)\big), \quad u(0) = 0, \tag{22}$$

that we may assume to be defined on the interval $]-\bar{d}, \bar{d}[$ of the same size of the neighborhood $B_{\bar{d}} \subset \Omega$. The curve R_i is called the *i-rarefaction curve* through 0. We may select the basis of right eigenvectors $r_i(u), i = 1, \ldots, N$, together with a basis of left eigenvectors $l_i, i = 1, \ldots, N$, so that

$$|r_i| \equiv 1, \qquad \langle l_i, r_j \rangle \equiv \delta_{i,j} \doteq \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{23}$$

where $u \cdot v$ denotes the inner product of the vectors $u, v \in \mathbb{R}^N$. It follows in particular that

$$\big|R_i(s)\big| \leq |s| \quad \forall s \in]-\bar{d}, \bar{d}[. \tag{24}$$

For every $b > 0$, $0 < d < \bar{d}$, we define the class of functions

$$\mathcal{PC}^1_{[d,b]} \doteq \big\{ \beta : \mathbb{R} \to [-d, d] \mid \beta \text{ is piecewise } C^1 \text{ and } \big|\dot{\beta}(x)\big| \leq b \big\}. \tag{25}$$

Here, we say that a map $\beta : \mathbb{R} \to [-d, d]$ is *piecewise C^1* if β is continuous on $\mathbb{R}$ and continuously differentiable on all but finitely many points of $\mathbb{R}$, while the bound on $\dot{\beta}$ in (25) is assumed to be satisfied at every point of differentiability of β. Given $\beta \in \mathcal{PC}^1_{[d,b]}$, consider the map

$$\phi_i^\beta(x) \doteq R_i\big(\beta(x)\big) \quad x \in \mathbb{R}, \tag{26}$$

and define the corresponding i-th characteristic starting at $y \in \mathbb{R}$ as:

$$x_i(t, y) \doteq y + \lambda_i\big(\phi_i^\beta(y)\big) \cdot t, \quad t \geq 0. \tag{27}$$

Observe that, by (22), one has

$$\frac{d}{dx}\phi_i^\beta(x) = \beta'(x) \cdot r_i\big(\phi_i^\beta(x)\big) \tag{28}$$

at every point x of differentiability of β. Hence, differentiating (27) w.r.t. y at a point where β is differentiable we find

$$\frac{\partial}{\partial y} x_i(t, y) = 1 + \big\langle \nabla \lambda_i\big(\phi_i^\beta(y)\big), r_i\big(\phi_i^\beta(y)\big) \big\rangle \cdot \dot{\beta}(y) \cdot t, \quad t \geq 0. \tag{29}$$

Set

$$\alpha_1 \doteq \sup\big\{ \big|\nabla \lambda_i(u)\big|; \ u \in B_{\bar{d}}, \ i = 1, \ldots, N \big\}. \tag{30}$$

Then, relying on (23), (24), (30), and because of the bound on $\dot{\beta}$ in (25), we derive from (29) the inequality

$$\frac{\partial}{\partial y} x_i(t, y) \geq 1 - \alpha_1 b \cdot t, \quad t \geq 0, \tag{31}$$

which, in turn, yields

$$\frac{\partial}{\partial y} x_i(t, y) \geq \frac{1}{2} \quad \forall t \in \left[0, 1/(2\alpha_1 \cdot b)\right]. \tag{32}$$

The inequality in (32), in particular, implies that the map $y \mapsto x_i(t, y)$ is increasing, hence injective. Moreover, since ϕ_i^β is continuous, from (32) we deduce also that the image of $y \mapsto x_i(t, y)$ is the whole line $\mathbb{R}$. Therefore, for every fixed $0 \leq t \leq 1/(2\alpha_1 \cdot b)$, we may define the inverse map of $x_i(t, \cdot)$ on $\mathbb{R}$. Then, set

$$z_i(t, \cdot) \doteq x_i^{-1}(t, \cdot), \tag{33}$$

and define the function

$$u(t, x) \doteq \phi_i^\beta\big(z_i(t, x)\big), \quad \forall (t, x) \in [0, T] \times \mathbb{R}, \tag{34}$$

with $T \leq 1/(2\alpha_1 \cdot b)$. The next lemma shows that $u(t, x)$ provides a classical solution of (1) on $[0, T] \times \mathbb{R}$, and we shall establish some a-priori estimates on $u(t, \cdot)$. We will say that the map $u(t, x)$ in (34) is an *i-th simple wave* with profile ϕ_i^β. We recall that a *classical solution* of a Cauchy problem (1), (3) is a locally Lipschitz continuous map $u : [0, T] \times \mathbb{R} \to \Omega$ that satisfies (1) almost everywhere and (3) for all $x \in \mathbb{R}$. A classical solution of (1), (3) is in particular an entropy weak solution of (1), (3) (see [7, Section 4.1]).

Lemma 1. *Given $T > 0$, $0 < d < \bar{d}$, $0 < b \leq 1/(2\alpha_1 \cdot T)$, with α_1 as in (30) (interpreting $1/\alpha_1 \doteq \infty$ when $\alpha_1 = 0$), for any fixed $i = 1, \ldots, N$, and for every $\beta \in \mathcal{PC}^1_{[d,b]}$, the map $u(t, x)$ defined in (34) provides a classical solution of the Cauchy problem*

$$u_t + f(u)_x = 0, \tag{35}$$
$$u(0, \cdot) = \phi_i^\beta, \tag{36}$$

on $[0, T] \times \mathbb{R}$. Moreover, for every $t \leq T$, there hold:

$$\left\| u(t, \cdot) \right\|_{L^\infty(\mathbb{R}, \Omega)} = \left\| \phi_i^\beta \right\|_{L^\infty(\mathbb{R}, \Omega)} \leq d, \qquad \left\| u_x(t, \cdot) \right\|_{L^\infty(\mathbb{R}, \Omega)} \leq 2 \cdot \left\| \frac{d}{dx} \phi_i^\beta \right\|_{L^\infty(\mathbb{R}, \Omega)} \leq 2b. \tag{37}$$

Proof. Observe first that, by the definitions (33), (34), and because of (28), (32), the map $u(t, x)$ is Lipschitz continuous, and it is differentiable at every point (t, x) lying outside the curves $t \mapsto (t, x_i(t, y_\ell))$, $\{y_\ell\}_\ell$ denoting the finite collection of points where β (and hence ϕ_i^β) is not differentiable. Moreover, one has

$$u\big(t, x_i(t, y)\big) = \phi_i^\beta(y), \quad \forall (t, y) \in [0, T] \times \big(\mathbb{R} \setminus \{y_\ell\}_\ell\big), \tag{38}$$

and, recalling (24), the first estimate in (37) holds. Taking the derivative with respect to t and y in both sides of (38), and recalling (27), (28), we obtain

$$u_t\big(t, x_i(t, y)\big) + \lambda_i\big(u\big(t, x_i(t, y)\big)\big) \cdot u_x\big(t, x_i(t, y)\big) = 0, \tag{39}$$

and

$$u_x\big(t, x_i(t, y)\big) \cdot \frac{\partial}{\partial y} x_i(t, y) = \beta'(y) \cdot r_i\big(u\big(t, x_i(t, y)\big)\big), \tag{40}$$

at every point $(t, y) \in [0, T] \times (\mathbb{R} \setminus \{y_\ell\}_\ell)$. We may divide both sides of (40) by $\frac{\partial}{\partial y} x_i(t, y)$ because of (32), and thus find

$$Df\big(u\big(t, x_i(t, y)\big)\big) \cdot u_x\big(t, x_i(t, y)\big) = \lambda_i\big(u\big(t, x_i(t, y)\big)\big) \cdot u_x\big(t, x_i(t, y)\big), \tag{41}$$

which, together with (39), yields

$$u_t(t, x) + Df\big(u(t, x)\big) \cdot u_x(t, x) = 0,$$

at every point $(t, x) \in ([0, T] \times \mathbb{R}) \setminus \bigcup_\ell \{(t, x_i(t, y_\ell)) \mid t \in [0, T]\}$. On the other hand, since by (27) $x(0, \cdot)$ is the identity map, it follows from (33), (34) that $u(0, x) = \phi_i^\beta(x)$ for all $x \in \mathbb{R}$. Therefore, $u(t, x)$ is a Lipschitz continuous map that satisfies Eq. (35) almost everywhere on $[0, T] \times \mathbb{R}$, together with the initial condition (36) at every $x \in \mathbb{R}$. Hence $u(t, x)$ provides a classical solution of (35)–(36). Moreover, relying on (23), (28), (32), (38), and because of the bound on $\dot{\beta}$ in (25), we recover from (40) the second estimates in (37), thus completing the proof of the lemma. $\square$

2.2. Superposition of simple waves

We wish to construct now a classical solution of (1), on a fixed time interval $[0, T]$, with initial data given by the profiles of N simple waves, one for each characteristic family, supported on disjoint sets. In order to analyze the behavior of the solution in the regions of interaction among simple waves we shall rely on uniform a-priori bounds on a classical solution $u(t, x)$ of (1) and on its spatial derivative, which can be derived by a standard technique (e.g. see [10, Section 4.2]) when the initial data has sufficiently small norms $\|u(0, \cdot)\|_{L^\infty(\mathbb{R},\Omega)}$, $\|u_x(0, \cdot)\|_{L^\infty(\mathbb{R},\Omega)}$. In order to state the next lemma that provides such a-priori estimates we need to introduce some further notation. Letting l^T denote the transpose (row) vector of a given (column) vector $l \in \mathbb{R}^N$, set

$$\Gamma_2(u) \doteq \sup_{i,j,k}\big\{\big|\lambda_k(u) - \lambda_i(u)\big|\big|l_i^T(u)Dr_j(u)\big|\big\},$$

$$\Gamma_3(u) \doteq \sup_{i,j,k}\big\{\big|\lambda_k(u) - \lambda_j(u)\big|\big|l_i^T(u)Dr_k(u)\big|\big\} + \sup_i\big|\nabla\lambda_i(u)\big|, \tag{42}$$

$$\Gamma_4(u) \doteq \sup_i\big|l_i(u)\big|,$$

$$\alpha_l \doteq \sup\big\{\Gamma_l(u); \, u \in B_{\bar{d}}\big\}, \quad l = 2, 3, 4. \tag{43}$$

Notice that (23) implies $\alpha_4 \geq 1$. Comparing (30), (42), (43), we deduce that

$$\alpha_1 \leq \alpha_3 \leq \alpha_3\alpha_4. \tag{44}$$

Lemma 2. *Given $T > 0$, $0 < d \leq (\bar{d}\, e^{-\alpha_2/\alpha_3})/(2\alpha_4 N)$, $0 < b \leq 1/(2\alpha_3\alpha_4 N^2 \cdot T)$, with α_l, $l = 1, 2, 3$, as in (30) and (43), consider a piecewise C^1 map $\phi : \mathbb{R} \to \Omega$ that satisfies*

$$\|\phi\|_{L^\infty(\mathbb{R},\Omega)} \leq d, \qquad \big\|\phi'\big\|_{L^\infty(\mathbb{R},\Omega)} \leq b. \tag{45}$$

Then, the Cauchy problem

$$u_t + f(u)_x = 0, \tag{46}$$

$$u(0, \cdot) = \phi, \tag{47}$$

admits a classical solution $u(t, x)$ on $[0, T] \times \mathbb{R}$ and, for every $t \leq T$, there hold

$$\big\|u(t, \cdot)\big\|_{L^\infty(\mathbb{R},\Omega)} \leq 2\alpha_4 N\, e^{\frac{\alpha_2}{\alpha_3}} \cdot d, \qquad \big\|u_x(t, \cdot)\big\|_{L^\infty(\mathbb{R},\Omega)} \leq 2\alpha_4 N \cdot b. \tag{48}$$

Proof. We provide here only a sketch of the proof. Further details can be found in [10, Section 4.2]. In order to prove the lemma it will be sufficient to show that, for every fixed time $T \leq 1/(2\alpha_3\alpha_4 N^2 \cdot b)$, and for every initial data ϕ satisfying (45), the estimates (48) hold on $[0, T]$ for a classical solution of (46)–(47). In fact, since by (45) we are assuming the initial bound

$$\|\phi\|_{L^\infty(\mathbb{R},\Omega)} \leq d \leq \frac{\bar{d}}{2\alpha_4 N}e^{-\frac{\alpha_2}{\alpha_3}}, \tag{49}$$

the first estimate in (48) guarantees in particular that $\|u(t, \cdot)\|_{L^\infty(\mathbb{R},\Omega)} \leq \bar{d}$ for all $t \in [0, T]$. As in the proof of [10, Theorem 4.2.5], relying on the a-priori bounds (48) one can then actually construct a classical solution of (46)–(47) on $[0, T]$, as limit of a Cauchy sequence of approximate solutions of the linearized problem.

Thus, assume that $u(t, x)$ is a classical solution of the Cauchy problem (46)–(47) on $[0, T] \times \mathbb{R}$, with a piecewise C^1 initial data ϕ satisfying (45). We may decompose u and u_x along the basis of right eigenvectors $r_1(u), \ldots, r_N(u)$, writing

$$u(t, x) = \sum_i p_i(t, x)r_i\big(u(t, x)\big), \qquad u_x(t, x) = \sum_i q_i(t, x)r_i\big(u(t, x)\big), \tag{50}$$

which, because of (23), is equivalent to set

$$p_i(t, x) \doteq \big\langle l_i\big(u(t, x)\big), u(t, x)\big\rangle, \qquad q_i(t, x) \doteq \big\langle l_i\big(u(t, x)\big), u_x(t, x)\big\rangle, \quad i = 1, \ldots, N. \tag{51}$$

Differentiating p_i, q_i along the i-th characteristic we find, for each i-th characteristic family, the equations

$$(p_i)_t + \lambda_i\big(u(t,x)\big)(p_i)_x = \sum_{j,k} \gamma^p_{i,j,k}\big(u(t,x)\big)\, p_j q_k,$$

$$(q_i)_t + \lambda_i\big(u(t,x)\big)(q_i)_x = \sum_{j,k} \gamma^q_{i,j,k}\big(u(t,x)\big)\, q_j q_k, \tag{52}$$

where

$$\gamma^p_{i,j,k}(u) \doteq \big(\lambda_k(u) - \lambda_i(u)\big)\big\langle l_i^T(u), Dr_j(u)r_k(u)\big\rangle,$$

$$\gamma^q_{i,j,k}(u) \doteq \frac{1}{2}\big(\lambda_k(u) - \lambda_j(u)\big)\big\langle l_i^T(u), \big[r_j(u), r_k(u)\big]\big\rangle - \delta_{i,k}\big\langle \nabla\lambda_i(u), r_j(u)\big\rangle \tag{53}$$

($\delta_{i,k}$ being the Kronecker symbol in (23) and $[r_j, r_k]$ denoting the Lie bracket of the vector fields r_j, r_k). Observe that, by definitions (42)–(43), one has

$$\max_{i,j,k}\big|\gamma^p_{i,j,k}(u)\big| \le \alpha_2, \qquad \max_{i,j,k}\big|\gamma^q_{i,j,k}(u)\big| \le \alpha_3 \quad \forall u \in B_{\bar d}. \tag{54}$$

Then, assuming that $\|u(t,\cdot)\|_{L^\infty(\mathbb{R},\Omega)} \le \bar d$ for all $t \in [0,T]$, it follows from the second equation in (52) integrated along the characteristics that, setting

$$Q(t) \doteq \sum_i \big\|q_i(t,\cdot)\big\|_{L^\infty(\mathbb{R},\Omega)}, \tag{55}$$

there holds

$$Q(t) \le Q(0) + \alpha_3 N \int_0^t \big(Q(s)\big)^2 ds \quad \forall t. \tag{56}$$

By a comparison argument one then derives from (56) that

$$Q(t) \le \frac{Q(0)}{1 - \alpha_3 Nt\, Q(0)} \quad \forall t \in \left[0, \frac{1}{\alpha_3 N\, Q(0)}\right[. \tag{57}$$

On the other hand, notice that by (23), (43), (50), (51), and recalling (23), one has

$$\big\|u_x(t,\cdot)\big\|_{L^\infty(\mathbb{R},\Omega)} \le Q(t) \le \alpha_4 N \big\|u_x(t,\cdot)\big\|_{L^\infty(\mathbb{R},\Omega)}. \tag{58}$$

Since we assume by (45) the initial bound

$$\big\|\phi'\big\|_{L^\infty(\mathbb{R},\Omega)} \le b \le \frac{1}{2\alpha_3\alpha_4 N^2 \cdot T}, \tag{59}$$

which, in turn, because of (58) implies

$$Q(0) \le \frac{1}{2\alpha_3 N \cdot T},$$

we obtain

$$Q(t) \le 2Q(0) \quad \forall t \in [0,T].$$

We deduce with (57), (58), that

$$\big\|u_x(t,\cdot)\big\|_{L^\infty(\mathbb{R},\Omega)} \le Q(t) \le 2Q(0) \le 2\alpha_4 N \cdot \big\|\phi'\big\|_{L^\infty(\mathbb{R},\Omega)} \quad \forall t \le T, \tag{60}$$

proving the second inequality in (48). Next, setting

$$P(t) \doteq \sum_i \big\|p_i(t,\cdot)\big\|_{L^\infty(\mathbb{R},\Omega)}, \tag{61}$$

and integrating the first equation in (52) along the characteristic, we derive

$$P(t) \leq P(0) + \alpha_2 N \int_0^t P(s) Q(s) ds \quad \forall t. \tag{62}$$

Then, applying Gronwall's lemma, we deduce from (62) that

$$P(t) \leq P(0) \exp\left(\alpha_2 N \int_0^t Q(s) ds \right) \quad \forall t. \tag{63}$$

On the other hand observe that since (59) implies $Q(0) \leq \frac{1}{2\alpha_3 N \cdot t}$ for all $t \leq T$, we deduce from (57) that

$$\int_0^t Q(s) ds \leq 2 Q(0) \, t \leq \frac{1}{\alpha_3 N} \quad \forall t \leq T. \tag{64}$$

Moreover, by (23), (43), (50), (51) there holds

$$\big\| u(t, \cdot) \big\|_{L^\infty(\mathbb{R}, \Omega)} \leq P(t) \leq \alpha_4 N \big\| u(t, \cdot) \big\|_{L^\infty(\mathbb{R}, \Omega)}. \tag{65}$$

Hence, (63)–(65) together yield

$$\big\| u(t, \cdot) \big\|_{L^\infty(\mathbb{R}, \Omega)} \leq P(t) \leq P(0) e^{\frac{\alpha_2}{\alpha_3}} \leq \alpha_4 N \, e^{\frac{\alpha_2}{\alpha_3}} \|\phi\|_{L^\infty(\mathbb{R}, \Omega)} \quad \forall t \leq T. \tag{66}$$

This completes the proof of the first inequality in (48), and hence of the lemma. □

Relying on Lemma 1 and Lemma 2 we shall construct now a classical solution $u(t, x)$ of (1) on a given time interval $[0, T]$, so that:

– the initial data $u(0, \cdot)$ is supported on N disjoint intervals I_i, $i = 1, \dots, N$, of the same length $|I_i| = L$, and on each interval I_i it coincides with the profile of a simple wave of the i-th characteristic family;
– the terminal value $u(T, \cdot)$ at time T is supported on an interval of length $\approx 2L$.

Namely, given $L, b > 0$, $0 < d < \bar{d}$ and

$$T \geq \frac{L}{\Delta_\wedge \lambda}, \qquad \Delta_\wedge \lambda \doteq \min_i \{ \lambda_{i+1}(0) - \lambda_i(0) \}, \tag{67}$$

set

$$\xi_i^- \doteq -L/2 - \lambda_i(0) \cdot T, \qquad \xi_i^+ \doteq \xi_i^- + L, \quad i = 1, \dots, N, \tag{68}$$

and consider the family of N-tuples of maps

$$\mathcal{PC}_{[L,d,b,T]}^{1,N} \doteq \big\{ \beta = (\beta_1, \dots, \beta_N) \in \big(\mathcal{PC}_{[d,b]}^1 \big)^N \mid \mathrm{Supp}(\beta_i) \subset \big[\xi_i^-, \xi_i^+ \big], \ i = 1, \dots, N \big\}, \tag{69}$$

where $\mathcal{PC}_{[d,b]}^1$ denotes the class of functions introduced in (25). Observe that, by (67), (68), one has

$$\xi_{i+1}^+ \leq \xi_i^- \quad \forall i = 1, \dots, N-1. \tag{70}$$

Then, let $\beta = (\beta_1, \dots, \beta_N) \in \mathcal{PC}_{[L,d,b,T]}^{1,N}$, and define the function $\phi^\beta : \mathbb{R} \to \Omega$, by setting

$$\phi^\beta(x) \doteq \sum_i \phi_i^{\beta_i}(x) = \begin{cases} \phi_i^{\beta_i}(x) & \text{if } x \in \mathrm{Supp}(\beta_i), \ i = 1, \dots, N, \\ 0 & \text{otherwise}, \end{cases} \tag{71}$$

where

$$\phi_i^{\beta_i}(x) \doteq R_i\big(\beta_i(x) \big) \tag{72}$$

denotes a map defined as in (26) in connection with $\beta_i \in \mathcal{PC}^1_{[d,b]}$. The next lemma shows that if we also assume

$$0 < d \leq \frac{1}{2\alpha_4 N e^{\alpha_2/\alpha_3}} \cdot \min\left\{\bar{d}, \frac{\Delta_\wedge \lambda}{2\alpha_1}\right\}, \qquad 0 < b \leq \min\left\{\frac{1}{2\alpha_1 \cdot T}, \frac{\Delta_\wedge \lambda}{4\alpha_3 \alpha_4 N^2 \cdot L}\right\} \tag{73}$$

(interpreting $1/\alpha_1 \doteq \infty$ when $\alpha_1 = 0$), for every given $\beta \in \mathcal{PC}^{1,N}_{[L,d,b,T]}$ we can apply Lemma 1 and Lemma 2 to derive the existence of a classical solution of (1) with initial data ϕ^β which possesses the desired properties.

Proposition 1. *Let $f : \Omega \to \mathbb{R}^N$ be a C^2 map defined on an open, connected domain $\Omega \subset \mathbb{R}^N$, $\Omega \supset B_{\bar{d}} \doteq \{u \in \mathbb{R}^n \mid |u| \leq \bar{d}\}$, and assume that the Jacobian matrix $Df(u)$ has N real, distinct eigenvalues $\lambda_1(u) < ... < \lambda_N(u)$. Given $L, T, d, b > 0$, satisfying (67), (73) (with α_1 as in (30), α_l, $l = 2, 3, 4$, as in (43), and $\Delta_\wedge \lambda$ as in (67)), let $\mathcal{PC}^{1,N}_{[L,d,b,T]}$ be the class of maps introduced in (68)–(69), and consider a map $\phi^\beta : \mathbb{R} \to \Omega$ as in (71), defined in connection with an N-tuple $\beta = (\beta_1, \ldots, \beta_N) \in \mathcal{PC}^{1,N}_{[L,d,b,T]}$. Then, there exists a classical solution $u(t, x)$ of the Cauchy problem*

$$u_t + f(u)_x = 0, \tag{74}$$
$$u(0, \cdot) = \phi^\beta, \tag{75}$$

on $[0, T] \times \mathbb{R}$. Moreover, setting

$$\alpha_5 \doteq \frac{\lambda_N(0) - \lambda_1(0)}{\Delta_\wedge \lambda}, \tag{76}$$

one has

$$\text{Supp}\big(u(T, \cdot)\big) \subseteq \big[-L \cdot (1 + \alpha_5),\ L \cdot (1 + \alpha_5)\big], \tag{77}$$

and, for every $t \leq T$, there hold:

$$\big\|u(t, \cdot)\big\|_{L^\infty(\mathbb{R}, \Omega)} \leq 2\alpha_4 N\, e^{\frac{\alpha_2}{\alpha_3}} \cdot d, \qquad \big\|u_x(t, \cdot)\big\|_{L^\infty(\mathbb{R}, \Omega)} \leq 4\alpha_4 N \cdot b. \tag{78}$$

Proof. We will prove the existence of a classical solution of the Cauchy problem (74)–(75) on $[0, T]$ satisfying (77)–(78), by first showing that such a solution is obtained on $[0, T - L/\Delta_\wedge \lambda]$ as a superposition of simple waves supported on disjoint set, and next deriving a-priori bounds on the solution and its support in the interval $[T - L/\Delta_\wedge \lambda,\ T]$.

1. Given $\beta = (\beta_1, \ldots, \beta_N) \in \mathcal{PC}^{1,N}_{[L,d,b,T]}$, define as in (27) the functions

$$x^\flat_i(t, y) \doteq y + \lambda_i\big(\phi^{\beta_i}_i(y)\big) \cdot t, \quad t \geq 0, \tag{79}$$

for each $i = 1, \ldots, N$. Since (73) implies $t \leq 1/(2\alpha_1 \cdot b)$ for all $t \in [0, T]$, by the inequality in (32) we deduce that the maps $y \mapsto x^\flat_i(t, y)$, $i = 1, \ldots, N$, are one-to-one in $\mathbb{R}$, for every fixed $t \in [0, T]$. Then, setting

$$z^\flat_i(t, \cdot) \doteq \big(x^\flat_i\big)^{-1}(t, \cdot), \quad i = 1, \ldots, N, \tag{80}$$

and letting $\phi^{\beta_i}_i$ be the map in (72), define the function

$$u^\flat(t, x) \doteq \begin{cases} \phi^{\beta_i}_i(z^\flat_i(t, x)) & \text{if } x \in [x^\flat_i(t, \xi^-_i), x^\flat_i(t, \xi^+_i)] \setminus \bigcup_{j \neq i}[x^\flat_j(t, \xi^-_j), x^\flat_j(t, \xi^+_j)], \quad i = 1, \ldots, N, \\ 0 & \text{otherwise}, \end{cases} \tag{81}$$

on $[0, T] \times \mathbb{R}$. Observe that, because of (69), (72), one has $\phi^{\beta_i}_i(\xi^\pm_i) = R_i(0) = 0$, for all $i = 1, \ldots, N$. Hence, recalling (67), (68), and by (79), there holds

$$x^\flat_{i+1}\big(t, \xi^\pm_{i+1}\big) \leq x^\flat_i\big(t, \xi^\pm_i\big) \quad \forall t \in \left[0, T - \frac{L}{\Delta_\wedge \lambda}\right], \ i = 1, \ldots, N - 1, \tag{82}$$

so that one has

$$u^\flat(t, x) = \begin{cases} \phi^{\beta_i}_i(z^\flat_i(t, x)) & \text{if } x \in [x^\flat_i(t, \xi^-_i), x^\flat_i(t, \xi^+_i)], \ i = 1, \ldots, N, \\ 0 & \text{otherwise}, \end{cases} \tag{83}$$

for all $(t, x) \in [0, T - L/\Delta_\wedge \lambda] \times \mathbb{R}$.

By (83) the restriction of $u^\flat(t, x)$ to the domain $[0, T - L/\Delta_\wedge \lambda] \times \mathbb{R}$ is a Lipschitz continuous map supported on the disjoint union of sets

$$D_i \doteq \left\{ (t, x) \mid t \in [0, T - L/\Delta_\wedge \lambda], \ x \in \left[x_i^\flat(t, \xi_i^-), x_i^\flat(t, \xi_i^+) \right] \right\}, \quad i = 1, \dots, N. \tag{84}$$

Since (73) implies $b < 1/(2\alpha_1 \cdot T)$, we know by Lemma 1 that $u^\flat(t, x)$ is a classical solution of (74) on each set D_i. Moreover, recalling that $z_i^\flat(0, \cdot)$ is the identity map, by (83) one has $u^\flat(0, x) = \phi_i^{\beta_i}(x)$, for all $x \in [\xi_i^-, \xi_i^+]$, $i = 1, \dots, N$. Therefore, looking at (69), (71), we deduce that (75) holds. Hence, it follows that $u^\flat(t, x)$ provides a classical solution of (74)–(75) on $[0, T - L/\Delta_\wedge \lambda] \times \mathbb{R}$.

Notice that, letting $u^\flat(t, \cdot)_{|D_i(t)}$ denote the restriction of $u^\flat(t, \cdot)$ to the set $D_i(t) \doteq [x_i^\flat(t, \xi_i^-), x_i^\flat(t, \xi_i^+)]$, we deduce from (71), (83), that for every $t \in [0, T - L/\Delta_\wedge \lambda]$ there holds

$$\left\| u^\flat(t, \cdot) \right\|_{L^\infty} = \max_i \left\| u^\flat(t, \cdot)_{|D_i(t)} \right\|_{L^\infty}, \qquad \left\| u_x^\flat(t, \cdot) \right\|_{L^\infty} = \max_i \left\| u^\flat(t, \cdot)_{|D_i(t)} \right\|_{L^\infty},$$

$$\left\| \phi^\beta \right\|_{L^\infty} = \max_i \left\| \phi_i^{\beta_i} \right\|_{L^\infty}, \qquad \left\| \frac{d}{dx} \phi^\beta \right\|_{L^\infty} = \max_i \left\| \frac{d}{dx} \phi_i^{\beta_i} \right\|_{L^\infty}. \tag{85}$$

Therefore, relying on the estimate (37) for each $u^\flat(t, \cdot)_{|D_i(t)}$, we derive from (85) the estimates

$$\left\| u^\flat(t, \cdot) \right\|_{L^\infty(\mathbb{R}, \Omega)} = \left\| \phi^\beta \right\|_{L^\infty(\mathbb{R}, \Omega)} \le d, \qquad \left\| u_x^\flat(t, \cdot) \right\|_{L^\infty(\mathbb{R}, \Omega)} \le 2 \left\| \frac{d}{dx} \phi^\beta \right\|_{L^\infty(\mathbb{R}, \Omega)} \le 2b, \tag{86}$$

for all $t \in [0, T - L/\Delta_\wedge \lambda]$.

2. Observe now that

$$\phi(x) \doteq u^\flat(T - L/\Delta_\wedge \lambda, x), \quad x \in \mathbb{R}, \tag{87}$$

is a piecewise C^1 map that satisfies the estimates (86), with d, b verifying the bounds (73). Thus, applying Lemma 2 we deduce the existence of a classical solution $u^\sharp(t, x)$ of (74) on the domain $[T - L/\Delta_\wedge \lambda, T] \times \mathbb{R}$, that assumes the initial data $u^\sharp(T - L/\Delta_\wedge \lambda, \cdot) = \phi$, at time $t = T - L/\Delta_\wedge \lambda$. Moreover, by (48), (86), there hold

$$\left\| u^\sharp(t, \cdot) \right\|_{L^\infty(\mathbb{R}, \Omega)} \le 2\alpha_4 N \, e^{\frac{\alpha_2}{\alpha_3}} \cdot d, \qquad \left\| u_x^\sharp(t, \cdot) \right\|_{L^\infty(\mathbb{R}, \Omega)} \le 4\alpha_4 N \cdot b, \tag{88}$$

for all $t \in [T - L/\Delta_\wedge \lambda, T]$. Therefore, the function defined by

$$u(t, x) \doteq \begin{cases} u^\flat(t, x) & \text{if } t \in [0, T - L/\Delta_\wedge \lambda], \\ u^\sharp(t, x) & \text{if } t \in]T - L/\Delta_\wedge \lambda, T], \end{cases} \tag{89}$$

provides a classical solution of (74), (75) that, because of (86), (88), satisfies the bounds (78) for all $t \in [0, T]$.

To conclude the proof of the proposition we shall derive now an estimate of the support of $u(T, x)$. Consider, for each i-th family, the i-th characteristic curve of u through a point $(\tau, y) \in [0, T] \times \mathbb{R}$, denoted by $t \mapsto x_i(t; \tau, y)$, $t \in [0, T]$, and defined as the (unique) solution of the Cauchy problem

$$\dot{x} = \lambda_i \big(u(t, x) \big), \qquad x(\tau) = y. \tag{90}$$

Set, for every $i = 1, \dots, N$,

$$\tau_i^- \doteq \inf \left\{ t \in [0, T]; \ x_i(t; 0, \xi_i^-) = x_j(t; 0, \xi_j^\pm) \text{ for some } j \ne i \right\}, \qquad y_i^- \doteq x_i(\tau_i^-; 0, \xi_i^-),$$

$$\tau_i^+ \doteq \inf \left\{ t \in [0, T]; \ x_i(t; 0, \xi_i^+) = x_j(t; 0, \xi_j^\pm) \text{ for some } j \ne i \right\}, \qquad y_i^+ \doteq x_i(\tau_i^+; 0, \xi_i^+), \tag{91}$$

where the equality $x_i(t; 0, \xi_i^-) = x_j(t; 0, \xi_j^\pm)$ is interpreted as $x_i(t; 0, \xi_i^-) = x_j(t; 0, \xi_j^-)$ or $x_i(t; 0, \xi_i^-) = x_j(t; 0, \xi_j^+)$, and analogously for $x_i(t; 0, \xi_i^+) = x_j(t; 0, \xi_j^\pm)$. Next, consider the union of the regions confined between the minimal and maximal characteristics emanating from the points $(\tau_i^\pm, y_i^\pm)$, $i = 1, \dots, N$ (see Fig. 1):

$$\Lambda \doteq \bigcup_i \left(\Lambda_i^- \cup \Lambda_i^+ \right), \qquad \begin{aligned} \Lambda_i^- &\doteq \left\{ (t, x) \in [\tau_i^-, T] \times \mathbb{R}; \ x_1(t; \tau_i^-, y_i^-) \le x \le x_N(t; \tau_i^-, y_i^-) \right\}, \\ \Lambda_i^+ &\doteq \left\{ (t, x) \in [\tau_i^+, T] \times \mathbb{R}; \ x_1(t; \tau_i^+, y_i^+) \le x \le x_N(t; \tau_i^+, y_i^+) \right\}. \end{aligned} \tag{92}$$
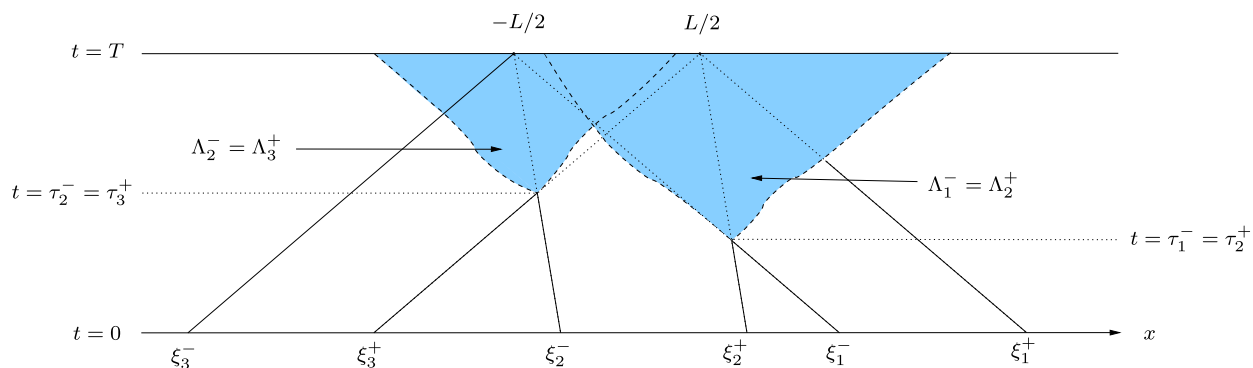
Fig. 1. The sets $\Lambda_i^\pm$.

Observe that $([0, T] \times \mathbb{R}) \setminus \Lambda$ is a domain of determinacy for the Cauchy problem (74)–(75), since, for every fixed $(\tau, y) \in ([0, T] \times \mathbb{R}) \setminus \Lambda$ and for any $i = 1, \ldots, N$, one has $\{(t, x_i(t; \tau, y)); \ 0 \leq t \leq \tau\} \subset ([0, T] \times \mathbb{R}) \setminus \Lambda$. Therefore, we deduce that the classical solution $u(t, x)$ of (74), (75) coincides with the function $u^\flat(t, x)$ defined in (81) on the whole region $([0, T] \times \mathbb{R}) \setminus \Lambda$, and that there hold

$$u(t, x) = \begin{cases} \phi_i^{\beta_i}(z_i^\flat(t, x)) & \text{if } x \in [x_i^\flat(t, \xi_i^-), x_i^\flat(t, \xi_i^+)], \ i = 1, \ldots, N, \\ 0 & \text{otherwise,} \end{cases} \tag{93}$$

for all $(t, x) \in ([0, T] \times \mathbb{R}) \setminus \Lambda$, and

$$x_i(t; 0, \xi_i^\pm) = x_i^\flat(t, \xi_i^\pm) \quad \forall t \in [0, \tau_i^\pm], \ i = 1, \ldots, N, \tag{94}$$

with obvious meaning of notations. Notice that, since by (69), (72) one has $\phi_i^{\beta_i}(\xi_i^\pm) = R_i(0) = 0$, it follows from (68), (79) that

$$x_i^\flat(T, \xi_i^\pm) = \pm L/2. \tag{95}$$

Thus, letting $u(T, \cdot)_{|D}$ denote the restriction of $u(T, \cdot)$ to a set D, we deduce from (93) that

$$\text{Supp}\big(u(T, \cdot)_{|\{x; \ (T,x)\notin \Lambda\}}\big) \subseteq [-L/2, L/2]. \tag{96}$$

On the other hand, observe that by (82), (91), (94), one has

$$\inf\{\tau_i^-, \tau_i^+; \ i = 1, \ldots, N\} \geq T - \frac{L}{\Delta_\wedge \lambda}. \tag{97}$$

Moreover, the first estimate in (78), together with the bound (73), implies in particular $\|u(t, \cdot)\|_{L^\infty} < \bar{d}$, for all $t \in [0, T]$, while (69), (72), (91), (93) yield

$$u(\tau_i^-, y_i^-) = \phi_i^{\beta_i}(\xi_i^-) = 0, \qquad u(\tau_i^+, y_i^+) = \phi_i^{\beta_i}(\xi_i^+) = 0. \tag{98}$$

Thus, relying on (30), (73), (78), (79), (90), (94), (95), (97), (98) we derive

$$\begin{aligned} x_N(T; \tau_i^\pm, y_i^\pm) &\leq y_i^\pm + \left(\lambda_N(0) + 2\alpha_1 \alpha_4 N e^{\frac{\alpha_2}{\alpha_3}} \cdot d\right) \cdot \left(T - \tau_i^\pm\right) \\ &= x_i^\flat(\tau_i^\pm, \xi_i^\pm) + \left(\lambda_N(0) + 2\alpha_1 \alpha_4 N e^{\frac{\alpha_2}{\alpha_3}} \cdot d\right) \cdot \left(T - \tau_i^\pm\right) \\ &\leq x_i^\flat(T, \xi_i^\pm) + \left(\left(\lambda_N(0) - \lambda_i(0)\right) + 2\alpha_1 \alpha_4 N e^{\frac{\alpha_2}{\alpha_3}} \cdot d\right) \cdot \frac{L}{\Delta_\wedge \lambda} \\ &\leq L \cdot \left(\frac{1}{2} + \frac{2\alpha_1 \alpha_4 N e^{\frac{\alpha_2}{\alpha_3}}}{\Delta_\wedge \lambda} \cdot d + \frac{\lambda_N(0) - \lambda_1(0)}{\Delta_\wedge \lambda}\right) \\ &\leq L \cdot \left(1 + \frac{\lambda_N(0) - \lambda_1(0)}{\Delta_\wedge \lambda}\right), \end{aligned} \tag{99}$$

and, analogously,

$$x_1\big(T; \tau_i^\pm, y_i^\pm\big) \geq -L \cdot \left(1 + \frac{\lambda_N(0) - \lambda_1(0)}{\Delta_\wedge \lambda}\right). \tag{100}$$

Then, recalling (76) and looking at the definition (92) of Λ, we deduce from (99)–(100) that there holds

$$\mathrm{Supp}\big(u(T, \cdot)_{|\{x;\,(T,x)\in\Lambda\}}\big) \subseteq \big[-L \cdot (1 + \alpha_5),\, L \cdot (1 + \alpha_5)\big]. \tag{101}$$

In turn, the inclusion (101) together with (96) yields (77), completing the proof of the proposition. □

Remark 3. Classical solutions of conservation laws coincide with the trajectory of the corresponding semigroup, whenever their initial data belongs to the domain of the semigroup. In fact, by the result in [7, Section 5.3], if (1)–(3) admit a classical solution, then such a solution coincides with any entropy weak solution of the same Cauchy problem. Therefore, if we consider a general system of conservation laws that generates a semigroup $(S_t)_{t\geq 0}$ of entropy weak solutions with a domain $\mathcal{D}_0$ as in (2), and we suppose that the map ϕ^β defined in (71) satisfies Tot.Var.$(\phi^\beta) \leq \delta_0$, it follows that the classical solution $u(t, \cdot)$ of the Cauchy problem (74)–(75) provided by Proposition 1 coincides with $S_t \phi^\beta$.

2.3. Simple waves for rich systems

Here we analyze the structure of simple waves for a class of systems, the so-called *rich systems*, that can be put in diagonal form with respect to Riemann coordinates. We recall that a system of conservation laws (1) is called a *rich system* (see [17]) if there exists a set of coordinates $w = (w_1, \ldots, w_N)$ consisting of Riemann invariants $w_i = W_i(u)$, $u \in \Omega$, associated to each characteristic field r_i. It is not restrictive to assume that the Riemann coordinates are chosen so that $W(0) = 0$. A necessary and sufficient condition for the existence of Riemann coordinates is the Frobenius involutive relation $[r_i, r_j] = \alpha_i^j r_i + \alpha_j^i r_j$, that must be satisfied, for some scalar functions α_i^j, α_j^i, for all $i, j = 1, \ldots, N$. When a system is endowed with a coordinate system of Riemann invariants it is convenient to normalize the eigenvectors $r_1, \ldots, r_N$ of Df so that there holds

$$\langle \nabla W_i, r_j \rangle \equiv \delta_{i,j} \tag{102}$$

instead of $|r_i| \equiv 1$ as in (23). In turn, (102) implies (cf. [7, Section 7.3]):

$$[r_i, r_j] \equiv 0 \quad \forall i, j = 1, \ldots, N. \tag{103}$$

Throughout the following, we will write $w_i(t, x) \doteq W_i(u(t, x))$ to denote the i-th Riemann coordinate of a solution $u = u(t, x)$ to (1), and we shall adopt the norms $\|w\|_{L^1} \doteq \sum_i \|w_i\|_{L^1}$, $\|w\|_{L^\infty} \doteq \max_i \|w_i\|_{L^\infty}$. Notice that, because of (102), multiplying (1) from the left by DW_i, $i = 1, \ldots, N$, we deduce that the system (1) is equivalent to the system in diagonal form

$$(w_i)_t + \lambda_i(w)(w_i)_x = 0, \quad i = 1, \ldots, N, \tag{104}$$

within the context of classical solutions. Thus, letting $t \mapsto x_i(t, y)$ denote the i-th characteristic of (104) starting at $y \in \mathbb{R}$, i.e. the solution of the Cauchy problem

$$\dot{x} = \lambda_i\big(w(t, x)\big), \qquad x(0) = y, \tag{105}$$

it follows that each i-th Riemann coordinate $w_i(t, x)$ of a classical solution to (1) remains constant along every i-th characteristic of (104). On the other hand, differentiating (104) w.r.t. x, and setting $q_i(t, x) \doteq (w_i(t, x))_x$, we find that

$$(q_i)_t + \lambda_i\big(w(t, x)\big)(q_i)_x = -\sum_j \frac{\partial}{\partial w_j}\lambda_i\big(w(t, x)\big)\, q_j q_i. \tag{106}$$

Observe that, by virtue of (102), the inverse map $u = W^{-1}(w)$ of $w = W(u) = (W_1(u), \ldots, W_N(u))$ satisfies $\partial u(w)/\partial w_i = r_i(u(w))$, for all $i = 1, \ldots, N$, and so the chain rule yields

$$\frac{\partial}{\partial w_j}\lambda_i(w)\bigg|_{w=W(u)} = \big\langle \nabla \lambda_i(u), r_j(u) \big\rangle \quad \forall i, j. \tag{107}$$

Next, set

$$\alpha_1' \doteq \sup\big\{\big|\langle \nabla\lambda_i(u), r_i(u)\rangle\big|;\ u \in B_{\bar{d}},\ i = 1, \dots, N\big\}, \tag{108}$$

$$\alpha_1'' \doteq \sup\big\{\big|\langle \nabla\lambda_i(u), r_j(u)\rangle\big|;\ u \in B_{\bar{d}},\ i, j = 1, \dots, N\big\}, \tag{109}$$

where $B_{\bar{d}}$ denotes as usual a ball centered in the origin and contained in the domain Ω of the flux function f. Since $W(0) = 0$, we may assume that

$$\big\{W^{-1}(w)\ \big|\ |w_i| \le \bar{d}'\big\} \subset B_{\bar{d}}, \tag{110}$$

for some $\bar{d}' > 0$. Thus, because of (107), (109), we have

$$\big|\nabla\lambda_i(w)\big| \le \sqrt{N}\,\bar{\alpha}_1'' \quad \forall w \in \big[-\bar{d}',\,\bar{d}'\,\big]^N,\ i = 1, \dots, N. \tag{111}$$

Then, with the same arguments of the proof of Lemma 2, we deduce the following sharper a-priori bounds on the Riemann coordinate expression of a classical solution of a rich system of conservation laws.

Lemma 3. *Assume that* (46) *is a strictly hyperbolic and rich system. Given* $T > 0$, $0 < d \le \bar{d}'$, $0 < b \le 1/(2\alpha_1'' N \cdot T)$, *with* α_1'' *as in* (109) *(interpreting* $1/\alpha_1'' \doteq \infty$ *when* $\alpha_1'' = 0$*), consider a piecewise* C^1 *map* $\phi : \mathbb{R} \to \Omega$ *that satisfies*

$$\big\|W \circ \phi\big\|_{L^\infty(\mathbb{R}, \Omega^w)} \le d, \qquad \bigg\|\frac{d}{dx}(W \circ \phi)\bigg\|_{L^\infty(\mathbb{R}, \Omega^w)} \le b, \tag{112}$$

where $\Omega^w \doteq \{w \in \mathbb{R}^N \mid w = W(u),\ u \in \Omega\}$. *Then, the Cauchy problem* (46)–(47), *admits a classical solution* $u(t, x)$ *on* $[0, T] \times \mathbb{R}$ *and, for every* $t \le T$, *letting* $w(t, x) \doteq W(u(t, x))$, *there hold*

$$\big\|w(t, \cdot)\big\|_{L^\infty(\mathbb{R}, \Omega^w)} \le d, \qquad \big\|w_x(t, \cdot)\big\|_{L^\infty(\mathbb{R}, \Omega^w)} \le 2 \cdot b. \tag{113}$$

Proof. Proceeding as in the proof of Lemma 2, it will be sufficient to show that, for any fixed time $T \le 1/(2\alpha_1'' N \cdot b)$, and for every initial data ϕ satisfying (112), the estimates (113) hold on $[0, T]$ for the Riemann coordinate expression $w(t, x)$ of a classical solution of (46)–(47). Observe that the first inequality in (113) is an immediate consequence of the invariance of each i-th Riemann coordinate $w_i(t, x)$ along the i-th characteristics of (104), and of the fact that $w(0, x) = W \circ \phi(x)$. Next, defining $Q(t) \doteq \sup_i \|q_i(t, \cdot)\|_{L^\infty}$, and relying on (106), (107), (109), (110), we derive as in (56)–(57) the bound

$$Q(t) \le \frac{Q(0)}{1 - \alpha_1'' N t\, Q(0)} \le 2 \cdot Q(0) \quad \forall 0 \le t < T, \tag{114}$$

provided that $Q(0) \le 1/(2\alpha_1'' N \cdot T)$. Thus, since $Q(t) = \|w_x(t, \cdot)\|_{L^\infty}$ by the definition of the L^∞-norm, and because $w_x(0, x) = \frac{d}{dx}(W \circ \phi)(x)$, if we assume $b \le 1/(2\alpha_1'' N \cdot T)$ we recover from (112), (114), the second inequality in (113). $\quad\square$

Observe now that as a consequence of (102) we deduce also that the rarefaction curve of the i-th family through 0 can be parametrized in Riemann coordinates as $s \mapsto R_i^{\mathcal{R}}(s) \doteq s\, e_i$, $s \in\ ]-\bar{d}', \bar{d}'[$, where e_i denotes the i-th element of the canonical basis of $\mathbb{R}^N$. Therefore, given $\beta \in \mathcal{PC}^1_{[d,b]}$, $d \le \bar{d}'$, the map ϕ_i^β in (26) takes the expression in Riemann coordinates:

$$W \circ \phi_i^\beta(x) \doteq \beta(x)\, e_i \quad x \in \mathbb{R}. \tag{115}$$

Similarly, the map ϕ^β in (71) defined in connection with an N-tuple $\beta = (\beta_1, \dots, \beta_N) \in \mathcal{PC}^{1,N}_{[L,d,b,T]}$, $d \le \bar{d}'$, is given in Riemann coordinates by

$$W \circ \phi^\beta(x) \doteq \sum_{i=1}^N W \circ \phi_i^{\beta_i}(x) = \sum_{i=1}^N \beta_i(x)\, e_i = \big(\beta_1(x), \dots, \beta_N(x)\big). \tag{116}$$

Notice that the supports of the simple waves $\phi_i^{\beta_i}$ may well overlap, because we are not assuming here that T satisfies the bound (67). However, by (116) the structure of the solution in Riemann coordinates can be viewed as a super-position of almost decoupled simple waves since each i-th simple wave has zero j-th Riemann component for every $j \neq i$. With similar arguments to the proof of Proposition 1 we then derive the sharper a-priori bound on the size of the support of $w(t, \cdot)$ provided by the following

Proposition 2. *In the same setting and with the notations of Proposition 1 and Lemma 3, assume that (74) is a strictly hyperbolic and rich system. Given $L, T > 0$, and $d, b > 0$ satisfying*

$$0 < d \leq \min\left\{ \bar{d}', \frac{\Delta_\wedge \lambda}{2\alpha_1'' \sqrt{N}} \right\}, \qquad 0 < b \leq \min\left\{ \frac{1}{2\alpha_1' \cdot T}, \frac{\Delta_\wedge \lambda}{2\alpha_1'' N \cdot L} \right\}, \tag{117}$$

with α_1', α_1'' as in (108), (109), and $\Delta_\wedge \lambda$ as in (67) (interpreting $1/\alpha_1' \doteq \infty$ when $\alpha_1' = 0$ and $1/\alpha_1'' \doteq \infty$ when $\alpha_1'' = 0$), let $\mathcal{PC}_{[L,d,b,T]}^{1,N}$ be the class of maps introduced in (68)–(69), and consider a map $\phi^\beta : \mathbb{R} \to \Omega$ as in (71), defined in connection with an N-tuple $\beta = (\beta_1, \dots, \beta_N) \in \mathcal{PC}_{[L,d,b,T]}^{1,N}$. Then, the Cauchy problem (74)–(75) admits a classical solution $u(t, x)$ on $[0, T] \times \mathbb{R}$. Moreover, letting $w(t, x) \doteq W(u(t, x))$, one has

$$\operatorname{Supp}\big(w(T, \cdot)\big) \subseteq [-L, L], \tag{118}$$

and, for every $t \leq T$, there hold:

$$\big\| w(t, \cdot) \big\|_{L^\infty(\mathbb{R}, \Omega^w)} \leq d, \qquad \big\| w_x(t, \cdot) \big\|_{L^\infty(\mathbb{R}, \Omega^w)} \leq 4b. \tag{119}$$

Proof. We shall first assume that $T \geq L/\Delta_\wedge \lambda$. In this case, as in the proof of Proposition 1, we will show that a classical solution of the Cauchy problem (74)–(75), satisfying (118), (119), is obtained on $[0, T - L/\Delta_\wedge \lambda]$ as a superposition of simple waves supported on disjoint set. Next, we will prove that such a solution can be extended to the interval $[T - L/\Delta_\wedge \lambda, T]$ relying on Lemma 3. Finally, we will discuss how to derive from Lemma 3 the existence of a classical solution of (74)–(75) verifying (118), (119) in the case where $T < L/\Delta_\wedge \lambda$.

1. Given $\beta = (\beta_1, \dots, \beta_N) \in \mathcal{PC}_{[L,d,b,T]}^{1,N}$, consider the functions $x_i^\flat(t, y)$, $i = 1, \dots, N$, defined in (79). Observe that, relying on (107)–(108), (117), by the same computations of Subsection 2.1 we derive the inequality

$$\frac{\partial}{\partial y} x_i^\flat(t, y) \geq 1 - \bar{\alpha}_1' b \cdot t \geq \frac{1}{2} \quad \forall t \in [0, T]. \tag{120}$$

It follows that the maps $y \mapsto x_i^\flat(t, y)$, $i = 1, \dots, N$, are one-to-one in $\mathbb{R}$, for every fixed $t \in [0, T]$. Thus, we may define the inverse map of $x_i^\flat(\cdot, y)$ on $\mathbb{R}$, and setting

$$z_i^\flat(t, \cdot) \doteq \big(x_i^\flat\big)^{-1}(t, \cdot), \quad i = 1, \dots, N, \tag{121}$$

we define the function

$$w_i^\flat(t, x) \doteq \begin{cases} \beta_i(z_i^\flat(t, x)) & \text{if } x \in [x_i^\flat(t, \xi_i^-), x_i^\flat(t, \xi_i^+)] \setminus \bigcup_{j \neq i}[x_j^\flat(t, \xi_j^-), x_j^\flat(t, \xi_j^+)], \\ 0 & \text{otherwise}, \end{cases} \quad i = 1, \dots, N, \tag{122}$$

on $[0, T] \times \mathbb{R}$. As in the proof of Proposition 1, notice that if we assume $T \geq L/\Delta_\wedge \lambda$ we derive

$$x_{i+1}^\flat\big(t, \xi_{i+1}^\pm\big) \leq x_i^\flat\big(t, \xi_i^\pm\big) \quad \forall t \in \left[0, T - \frac{L}{\Delta_\wedge \lambda}\right], i = 1, \dots, N - 1, \tag{123}$$

so that one has

$$w_i^\flat(t, x) = \begin{cases} \beta_i(z_i^\flat(t, x)) & \text{if } x \in [x_i^\flat(t, \xi_i^-), x_i^\flat(t, \xi_i^+)], \\ 0 & \text{otherwise}, \end{cases} \quad i = 1, \dots, N, \tag{124}$$

for all $(t, x) \in [0, T - L/\Delta_\wedge \lambda] \times \mathbb{R}$. Relying on (116), by the same arguments of the proof of Proposition 1 we then deduce that $w^\flat(t, x)$ is the Riemann coordinate expression of a classical solution of (74)–(75) on $[0, T - L/\Delta_\wedge \lambda] \times \mathbb{R}$.

Moreover, recalling that by definition (69) one has $\beta_i \in \mathcal{PC}^1_{[d,b]}$, for all $i = 1, \ldots, N$, and relying on (120), we deduce from (124) that

$$\left\| w^\flat(t, \cdot) \right\|_{L^\infty(\mathbb{R}, \Omega^w)} = \max_i \|\beta_i\|_{L^\infty} \leq d,$$

$$\left\| w^\flat_x(t, \cdot) \right\|_{L^\infty(\mathbb{R}, \Omega^w)} \leq \max_i \left\| \left(\frac{\partial}{\partial y} x^\flat_i(t, \cdot) \right)^{-1} \cdot \frac{d}{dx} \beta_i \right\|_{L^\infty} \leq 2b, \tag{125}$$

for all $t \in [0, T - L/\Delta_\wedge \lambda]$.

2. Relying on (117), (125), and applying Lemma 3, we deduce the existence of a classical solution of (74) on the domain $[T - L/\Delta_\wedge \lambda, T] \times \mathbb{R}$, that assumes the initial data

$$\phi(x) \doteq W^{-1}\big(w^\flat(T - L/\Delta_\wedge \lambda, x)\big), \quad x \in \mathbb{R}, \tag{126}$$

at time $t = T - L/\Delta_\wedge \lambda$. Moreover, the Riemann coordinate expression $w^\sharp(t, x)$ of such a solution satisfies the estimates

$$\left\| w^\sharp(t, \cdot) \right\|_{L^\infty(\mathbb{R}, \Omega^w)} \leq d, \qquad \left\| w^\sharp_x(t, \cdot) \right\|_{L^\infty(\mathbb{R}, \Omega^w)} \leq 4 \cdot b, \tag{127}$$

for all $t \in [T - L/\Delta_\wedge \lambda, T]$. Therefore, the function defined by

$$w(t, x) \doteq \begin{cases} w^\flat(t, x) & \text{if } t \in [0, T - L/\Delta_\wedge \lambda], \\ w^\sharp(t, x) & \text{if } t \in]T - L/\Delta_\wedge \lambda, T], \end{cases} \tag{128}$$

provides the Riemann coordinate expression of a classical solution of (74)–(75) on $[0, T] \times \mathbb{R}$ that, because of (125), (127) satisfies the bounds (119) for all $t \in [0, T]$.

In order to derive an estimate on the support of $w(T, \cdot)$, consider the i-th characteristic $t \mapsto x_i(t, y)$ starting at y at time $t = 0$, associated to $w(t, x)$, i.e. the solution of (105). Since $w(t, x)$ is the Riemann coordinate expression of a classical solution of (74) on $[0, T] \times \mathbb{R}$, it follows that the map $y \mapsto x_i(t, y)$ is a one-to-one correspondence on $\mathbb{R}$, for any $t \in [0, T]$. Hence, setting $z_i(t, \cdot) \doteq x_i^{-1}(t, \cdot)$, and recalling that each i-th Riemann coordinate $w_i(t, x)$ remains constant along the i-th characteristics, we may express $w_i(t, x)$ as

$$w_i(t, x) = \beta_i\big(z_i(t, x)\big) \quad \forall t \in [0, T], \ x \in \mathbb{R}. \tag{129}$$

Relying on (129), and because of (69), we deduce that in order to prove (118) it will be sufficient to show that the i-th characteristic map $x_i(T, \cdot)$ satisfies

$$\big[x_i\big(T, \xi_i^-\big), \, x_i\big(T, \xi^+\big)\big] \subseteq [-L, L], \tag{130}$$

for every $i = 1, \ldots, N$. To this end, let $t \mapsto x_i(t; \tau, y)$ denote the i-th characteristic starting at y at time $t = \tau$, i.e. the solution of

$$\dot{x} = \lambda_i\big(w(t, x)\big), \qquad x(\tau) = y, \tag{131}$$

and define the times $\tau_i^\pm$ and points $y_i^\pm$ as in (91). Then, recalling (79), thanks to (68), (95), (97), (98), (111), (119), (131), and because of (117), we find

$$\begin{aligned} x_i\big(T, \xi_i^\pm\big) &\leq y_i^\pm + \big(\lambda_i(0) + \sqrt{N}\,\bar{\alpha}''_1 \cdot d\big) \cdot \big(T - \tau_i^\pm\big) \\ &= x_i^\flat\big(T, \xi_i^\pm\big) + \sqrt{N}\,\bar{\alpha}''_1 \cdot d \cdot \big(T - \tau_i^\pm\big) \\ &\leq \frac{L}{2} + \sqrt{N}\,\bar{\alpha}''_1 \cdot d \cdot \frac{L}{\Delta_\wedge \lambda} < L. \end{aligned} \tag{132}$$

With similar arguments we derive $x_i(T, \xi_i^\pm) > -L$, which together with (132), yields (130). This completes the proof of the proposition in the case where $T \geq L/\Delta_\wedge \lambda$.

3. Assume $T < L/\Delta_\wedge \lambda$, and observe that by (117) one has $b \leq 1/(2\alpha''_1 N \cdot T)$. Then, applying Lemma 3, we deduce the existence of a classical solution of (74)–(75) on $[0, T] \times \mathbb{R}$ that satisfies the bounds (119) for all $t \in [0, T]$. Letting $w(t, x)$ denote the Riemann coordinate expression of such a solution, by the same arguments above we can show that (129), (130) hold, which, together with (69), yield (118), thus concluding the proof of the proposition. $\quad\square$

Remark 4. The same conclusion of Remark 3 holds if we consider a rich system that generates a semigroup of entropy weak solutions $(S_t)_{t \geq 0}$ with a domain $\mathcal{D}$ as in (15), and we assume that

$$]\bar{d}', \bar{d}'[\, \subset [a_i, b_i] \quad \forall i = 1, \ldots, N. \tag{133}$$

In fact, under this assumption it clearly follows that, for every given $\beta = (\beta_1, \ldots, \beta_N) \in \mathcal{PC}^{1,N}_{[L,d,b,T]}$, $d < \bar{d}'$, the map ϕ^β defined in (71) belongs to $\mathcal{D}$, and thus, relying on [7, Section 5.3], we deduce that the classical solution $u(t, \cdot)$ of the Cauchy problem (74)–(75) provided by Proposition 2 coincides with $S_t \phi^\beta$.

3. Lower compactness estimates for conservation laws

3.1. A controllability result

For arbitrary constants $L, b > 0$, $0 < M < \bar{d}$ ($\bar{d}$ being the radius of the ball contained in the domain of the flux function where condition (30) is verified), and $T > 0$ satisfying (67), recalling the definitions (68), (69), (71), let us consider the set

$$\mathcal{A}_{[L,M,b,T]} \doteq \left\{ \psi \in C(\mathbb{R}, \Omega) \mid \psi(x) = \phi^\beta(-x) \; \forall x \in \mathbb{R}, \text{ for some } \beta = (\beta_1, \ldots, \beta_N) \in \mathcal{PC}^{1,N}_{[L,M,b,T]} \right\}. \tag{134}$$

Notice that, because of (68), every map $\psi \in \mathcal{A}_{[L,M,b,T]}$ is supported on N disjoint intervals $[\xi_i^-, \xi_i^+]$, $i = 1, \ldots, N$, of length L. The next result shows that the elements of such a set can be obtained as the values $S_T \bar{u}$ at a fixed time T of the semigroup generated by (1), for initial data $\bar{u}$ varying in a set of the form (5).

Proposition 3. *Let $f : \Omega \to \mathbb{R}^N$ be a C^2 map on an open, connected domain $\Omega \subset \mathbb{R}^N$ containing the origin, and assume that the system (1) is strictly hyperbolic. Let $(S_t)_{\geq 0}$ be the semigroup of entropy weak solutions generated by (1) defined on a domain $\mathcal{D}_0$ satisfying (2). Then, given any $L, m, M, T > 0$, and setting*

$$\widetilde{L} \doteq \min \left\{ \frac{L}{(1 + \alpha_5)}, \; T \cdot \Delta_\wedge \lambda \right\} \tag{135}$$

($\Delta_\wedge \lambda$, α_5 being the constants in (67), (76)), for every

$$0 \leq b \leq \min \left\{ \frac{1}{2\alpha_1 \cdot T}, \; \frac{\Delta_\wedge \lambda}{4\alpha_3 \alpha_4 N^2 \cdot L}, \; \frac{\delta_0}{8\alpha_4 N L} \right\},$$
$$0 \leq h \leq \min \left\{ \frac{\bar{d}}{2\alpha_4 N \, e^{\alpha_2/\alpha_3}}, \; \frac{\Delta_\wedge \lambda}{4\alpha_1 \alpha_4 N \, e^{\alpha_2/\alpha_3}}, \; \frac{M}{2\alpha_4 N \, e^{\alpha_2/\alpha_3}}, \; \frac{m}{2L} \right\} \tag{136}$$

(α_l, $l = 1, 2, 3, 4$, being the constants in (30), (43), interpreting $1/\alpha_1 \doteq \infty$ when $\alpha_1 = 0$), there holds

$$\mathcal{A}_{[\widetilde{L},h,b,T]} \subset S_T(\mathcal{L}_{[I_L,m,M]} \cap \mathcal{D}_0), \quad I_L \doteq [-L, L], \tag{137}$$

where $\mathcal{A}_{[\widetilde{L},h,b,T]}$, $\mathcal{L}_{[I_L,m,M]}$ denote the sets defined as in (134), (5), respectively.

Proof. Following the same strategy adopted in [1], we will show that any element $\psi \in \mathcal{A}_{[\widetilde{L},h,b,T]}$ can be obtained as the value at time T of a classical solution to (1) by reversing the direction of time, and constructing a backward solution to (1) that starts at time T from ψ. Namely, given

$$\psi \in \mathcal{A}_{[\widetilde{L},h,b,T]}, \tag{138}$$

by definition (134) there will be an N-tuple of maps $\beta = (\beta_1, \ldots, \beta_N) \in \mathcal{PC}^{1,N}_{[\widetilde{L},h,b,T]}$, such that letting ϕ^β be the function defined in (71), one has $\psi(x) = \phi^\beta(-x)$, for all x. Notice that, by (135), one has

$$T \geq \frac{\widetilde{L}}{\Delta_\wedge \lambda}, \tag{139}$$

as in (67), while (136) imply that h, b satisfy the bounds (73) on d, b. Then, set

$$\omega_0(x) \doteq \psi(-x) = \phi^\beta(x) \quad \forall x \in \mathbb{R}, \tag{140}$$

and let $\omega : [0, T] \times \mathbb{R} \to \Omega$ denote the classical solution of the Cauchy problem (74)–(75), provided by Proposition 1. Next, consider the function

$$u(t, x) \doteq \omega(T - t, -x), \quad (t, x) \in [0, T] \times \mathbb{R}. \tag{141}$$

Observe that $u(t, x)$ is a classical solution of (1) with initial data $u(0, \cdot) = \omega(T, -\cdot)$ that, by (140), satisfies

$$u(T, \cdot) = \psi. \tag{142}$$

Moreover, by (77), (135) we have $|\mathrm{Supp}(\omega(T, -\cdot))| = |\mathrm{Supp}(\omega(T, \cdot))| \leq 2(1 + \alpha_5)\widetilde{L} \leq 2L$. Therefore, relying on the second estimate in (78) and on (136), we derive

$$
\begin{aligned}
\mathrm{Tot.Var.}\big(\omega(T, -\cdot)\big) &\leq \big\|\omega_x(T, \cdot)\big\|_{L^1(\mathbb{R}, \Omega)} \\
&\leq \big\|\omega_x(T, \cdot)\big\|_{L^\infty(\mathbb{R}, \Omega)} \cdot \big|\mathrm{Supp}\big(\omega(T, \cdot)\big)\big| \\
&\leq 4\alpha_4 N \cdot b \cdot 2L \leq \delta_0.
\end{aligned}
\tag{143}
$$

Thus, by (2) we deduce that $u(0, \cdot) = \omega(T, -\cdot) \in \mathcal{D}_0$, and hence, recalling Remark 3, we have $u(t, \cdot) = S_t \omega(T, -\cdot)$, for all $t \in [0, T]$. Because of (142), this implies in particular that $\psi = S_T \omega(T, -\cdot)$. To conclude the proof of

$$\psi \in S_T\big(\mathcal{L}_{[I_L, m, M]} \cap \mathcal{D}_0\big) \tag{144}$$

it thus remains to show that

$$\omega(T, -\cdot) \in \mathcal{L}_{[I_L, m, M]}. \tag{145}$$

Since ω is the classical solution of (74)–(75) provided by Proposition 1, recalling that $\psi(-\cdot) = \phi^\beta$, $\beta \in \mathcal{PC}^{1,N}_{[\widetilde{L}, h, b, T]}$, and relying on (77), (78), (135), (136), (138), we deduce that

$$
\begin{aligned}
&\mathrm{Supp}\big(\omega(T, -\cdot)\big) \subset [-L, L], \\
&\big\|\omega(T, -\cdot)\big\|_{L^\infty(\mathbb{R}, \Omega)} \leq 2\alpha_4 N \, e^{\frac{\alpha_2}{\alpha_3}} \cdot h \leq M, \\
&\big\|\omega(T, -\cdot)\big\|_{L^1(\mathbb{R}, \Omega)} \leq 2Lh \leq m.
\end{aligned}
\tag{146}
$$

Therefore, the inclusion (145) is verified because of (146), which completes the proof of the proposition. $\quad\square$

Remark 5. When $N = 1$, under the same assumptions as Proposition 3, assume also that $f'(0) = 0$ (possibly performing a space and flux transformation). Then, relying on Lemma 1 (where we may reach the same conclusion assuming that $b \leq 3/(4\bar{c} \cdot T)$, with $\bar{c} \doteq \sup\{|f''(u)|; \ |u| \leq \bar{d}\}$), we can show that the following holds. Given any $L, m, M, T > 0$, for every

$$0 \leq b \leq \frac{3}{4\bar{c} \cdot T}, \qquad 0 \leq h \leq \min\left\{\bar{d}', M, \frac{m}{2L}\right\},$$

one has

$$\mathcal{A}_{[L, h, b, T]} \subset S_T\big(\mathcal{L}_{[I_L, m, M]}\big), \quad I_L \doteq [-L/2, L/2], \tag{147}$$

where $\mathcal{A}_{[L, h, b, T]}$, $\mathcal{L}_{[I_L, m, M]}$ denote sets defined as in (134), (5), respectively.

We shall now extend the previous controllability results to class of functions with possibly unbounded total variation in the case of hyperbolic systems of conservation laws of Temple class. We recall that (see [7,17,18]):

Definition 1. A system of conservation laws (1) is called of *Temple class* if:

- it is a rich system, i.e. if it is endowed with a coordinates system $w = (w_1, \ldots, w_n)$ of Riemann invariants $w_i = W_i(u)$ associated to each characteristic field r_i;
- the level sets $\{u \in \Omega; \ W_i(u) = constant\}$ of every Riemann invariant are hyperplanes.

We shall assume that $W(0) = 0$ and that as w ranges within the product set $\Pi \doteq [a_1, b_1] \times \cdots \times [a_N, b_N]$, the corresponding state $u = W^{-1}(w)$ remains inside the domain Ω of the flux function f.

We also recall that a characteristic field r_i of a system (1) is said to be *genuinely nonlinear* (GNL) in the sense of Lax if $\langle \nabla \lambda_i(u), r_i(u) \rangle \neq 0$ for all $u \in \Omega$, while we say that r_i is *linearly degenerate* (LD) if $\langle \nabla \lambda_i(u), r_i(u) \rangle \equiv 0$ for all $u \in \Omega$.

As observed in the introduction, the results in [3,6] show that a Temple system with GNL or LD characteristic families admits a continuous semigroup of entropy weak solutions $S : [0, \infty[\times \mathcal{D} \to \mathcal{D}\}$ defined on domains $\mathcal{D}$ as in (15) of functions having possibly unbounded variation. We shall adopt the notation $S_t^w \overline{w} \doteq W(u(t, \cdot))$ for the Riemann coordinates expression of the solution of (1), (3), with $\overline{u} \doteq W^{-1} \circ \overline{w}$. Therefore, relying on the sharper a-priori bounds on the classical solutions of a rich system provided by Proposition 2, and setting

$$\mathcal{A}_{[L,M,b,T]}^w \doteq \left\{ \psi \in C(\mathbb{R}, \Pi) \mid \psi(x) = \beta(-x) \ \forall x \in \mathbb{R}, \text{ for some } \beta = (\beta_1, \ldots, \beta_N) \in \mathcal{PC}_{[L,M,b,T]}^{1,N} \right\}. \tag{148}$$

we establish the following

Proposition 4. *In the same setting of Proposition 3, assume that (1) is a strictly hyperbolic system of Temple class, and that all characteristic families are genuinely nonlinear or linearly degenerate. Let $(S_t)_{\geq 0}$ be the semigroup of entropy weak solutions generated by (1) defined on a domain $\mathcal{D}$ as in (15), and assume that (133) holds. Then, given any $L, m, M, T > 0$, for every b, h satisfying*

$$0 \leq b \leq \min\left\{ \frac{1}{2\alpha_1' \cdot T}, \frac{\Delta_\wedge \lambda}{2\alpha_1'' N \cdot L} \right\}, \qquad 0 \leq h \leq \min\left\{ \overline{d}', \frac{\Delta_\wedge \lambda}{2\alpha_1'' \sqrt{N}}, M, \frac{m}{2L} \right\} \tag{149}$$

(α_1', α_1'' being the constants in (108), (109) and $\Delta_\wedge \lambda$ as in (67), interpreting $1/\alpha_1' \doteq \infty$ when $\alpha_1' = 0$ and $1/\alpha_1'' \doteq \infty$ when $\alpha_1'' = 0$) there holds

$$\mathcal{A}_{[L,h,b,T]}^w \subset S_T^w\left(\mathcal{L}_{[I_L,m,M]}^w\right), \qquad I_L \doteq [-L, L], \tag{150}$$

where the sets $\mathcal{A}_{[L,h,b,T]}^w$, $\mathcal{L}_{[I_L,m,M]}^w$ are defined as in (148) and in (18), respectively.

Proof. The proof of Proposition 4 is entirely similar to that of Proposition 3, relying on Proposition 2 and Remark 4, and recalling (116), thus we omit it. $\square$

3.2. Lower compactness estimates on a family of simple waves

We shall provide now a lower estimate on the ε-entropy of the class $\mathcal{A}_{[L,M,b,T]}$ introduced in (134). To this end, set

$$\alpha_6 \doteq \sup\left\{ |Dr_i(u)|; \ u \in B_{\overline{d}}, \ i = 1, \ldots, N \right\}, \tag{151}$$

where $B_{\overline{d}}$ denotes as usual a ball centered in the origin and contained in the domain Ω of the flux function f. Following a similar strategy as the one pursued in [1] we then establish the following

Proposition 5. *In the same setting of Proposition 3, given $L, b > 0$, $0 < M < \overline{d}$, and $T > 0$ satisfying (67), for every $\varepsilon > 0$ satisfying*

$$\varepsilon \leq \min\left\{ \frac{LNM}{24}, \frac{LN}{48\alpha_6} \right\}, \tag{152}$$

(α_6 being the constant in (151)), letting $\mathcal{A}_{[L,M,b,T]}$ be the set defined in (134), one has

$$H_\varepsilon\left(\mathcal{A}_{[L,M,b,T]} \mid L^1(\mathbb{R}, \Omega)\right) \geq \frac{L^2 N^2 b}{216 \ln(2)} \cdot \frac{1}{\varepsilon}. \tag{153}$$

Proof. Towards a proof of (153), we shall first introduce a two-parameter family $\mathcal{F}_{n,h} \subset \mathcal{A}_{[L,M,b,T]}$, depending on $n \geq 2$ and $h > 0$, of superposition of simple waves ϕ^β, $\beta = (\beta_1, \ldots, \beta_N) \subset \mathcal{PC}_{[L,M,b,T]}^{1,N}$, defined as in (71) in
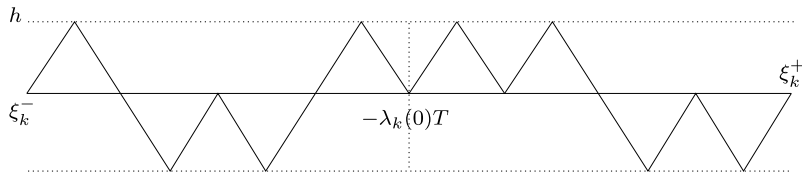
Fig. 2. The function β_k^ι for $n = 8$ and $\iota = (0, 1, 1, 0, 0, 0, 1, 1)$.

connection with piecewise affine, compactly supported maps $\beta_i \in \mathcal{PC}^1_{[M,b]}$. Next, we shall provide an optimal lower bound on the covering number $N_\varepsilon(\mathcal{F}_{n,h} \mid L^1(\mathbb{R}, \Omega))$, for a suitable choice of n, h, which, in turn, will yield the lower bound (153) on the ε-entropy of $\mathcal{A}_{[L,M,b,T]}$.

1. Given any integer $n \geq 2$ and any constant $h > 0$, for every k-th characteristic family and for any given n-tuple $\iota = (\iota_1, \ldots, \iota_n) \in \{0, 1\}^n$, we consider the function $\beta_k^\iota : \mathbb{R} \to [-h, h]$, with support contained in $[\xi_k^-, \xi_k^+]$, defined by setting (see Fig. 2)

$$\beta_k^\iota(x) \doteq (-1)^{\iota_\ell} \frac{2hn}{L} \cdot \left(\frac{L}{2n} - \left| x - \xi_k^- - (2\ell + 1) \cdot \frac{L}{2n} \right| \right) \quad \forall x \in \left[\xi_k^- + \frac{\ell L}{nN}, \ \xi_k^- + \frac{(\ell+1)L}{nN} \right], \tag{154}$$

for all $\ell \in \{0, \ldots, n - 1\}$. Recall that the quantities $\xi_k^\pm = \pm L/2 - \lambda_k(0) \cdot T$ were introduced in (68). Observe that, if we assume

$$0 < h \leq \min\left\{ M, \ \frac{Lb}{2n} \right\}, \tag{155}$$

by definition (25) it follows that $\beta_k^\iota \in \mathcal{PC}^1_{[M,b]}$, for every n-tuple $\iota = (\iota_1, \ldots, \iota_n) \in \{0, 1\}^n$. Therefore, for any given N-tuple of n-tuples $(\iota_1, \ldots, \iota_N) \in (\{0, 1\}^n)^N$, letting $\beta_k^{\iota_k}$, $k = 1, \ldots, N$, be the maps defined as in (154), and recalling definition (69), we deduce that $(\beta_1^{\iota_1}, \ldots, \beta_N^{\iota_N}) \in \mathcal{PC}^{1,N}_{[L,M,b,T]}$. Thus, for all $n \geq 2$ and h satisfying (155), setting

$$\mathcal{B}_{n,h} \doteq \left\{ (\beta_1^{\iota_1}, \ldots, \beta_N^{\iota_N}) \mid \beta_k^{\iota_k} : \mathbb{R} \to [-h, h] \text{ defined as in (154) with} \right.$$
$$\left. \mathrm{Supp}(\beta_k^{\iota_k}) \subset [\xi_k^-, \xi_k^+] \ \forall k, \ (\iota_1, \ldots, \iota_N) \in (\{0, 1\}^n)^N \right\}, \tag{156}$$

one has

$$\mathcal{B}_{n,h} \subset \mathcal{PC}^{1,N}_{[L,M,b,T]}. \tag{157}$$

Then, for any given N-tuple of n-tuples $(\iota_1, \ldots, \iota_N) \in (\{0, 1\}^n)^N$, let

$$\phi^{\iota_1, \ldots, \iota_N} \doteq \phi^{(\beta_1^{\iota_1}, \ldots, \beta_N^{\iota_N})}$$

denote the map defined as in (71) in connection with the N-tuple $(\beta_1^{\iota_1}, \ldots, \beta_N^{\iota_N}) \in \mathcal{B}_{n,h}$, and set

$$\mathcal{F}_{n,h} \doteq \left\{ \phi^{\iota_1, \ldots, \iota_N}(-\cdot) \mid (\iota_1, \ldots, \iota_N) \in (\{0, 1\}^n)^N \right\}. \tag{158}$$

Recalling definition (134), and because of (157), it follows that there holds

$$\mathcal{F}_{n,h} \subset \mathcal{A}_{[L,M,b,T]}, \tag{159}$$

for all $n \geq 2$ and $h > 0$ satisfying (155). Therefore, observing that (158) implies

$$H_\varepsilon\big(\mathcal{A}_{[L,M,b,T]} \mid L^1(\mathbb{R}, \Omega)\big) \geq H_\varepsilon\big(\mathcal{F}_{n,h} \mid L^1(\mathbb{R}, \Omega)\big), \tag{160}$$

we deduce that, in order to establish (153), it will be sufficient to show

$$H_\varepsilon\big(\mathcal{F}_{n,h} \mid L^1(\mathbb{R}, \Omega)\big) \geq \frac{L^2 N^2 b}{216 \ln(2)} \cdot \frac{1}{\varepsilon}, \tag{161}$$

for a suitable choice of $n \geq 2$ and $h > 0$ satisfying (155).

2. Towards an estimate of the covering number $N_\varepsilon(\mathcal{F}_{n,h} \mid L^1(\mathbb{R}, \Omega))$, let us denote with $\mathcal{C}_n^{\mathcal{F}}(\varepsilon)$ the maximum number of elements in $\mathcal{F}_{n,h}$ that have L^1-distance less than ε from a given element of $\mathcal{F}_{n,h}$. Namely, for any fixed $\phi^{\bar\iota_1,\dots,\bar\iota_N}(-\cdot) \doteq \phi^{(\beta_1^{\bar\iota_1},\dots,\beta_N^{\bar\iota_N})}(-\cdot) \in \mathcal{F}_{n,h}$, $\bar\iota \doteq (\bar\iota_1,\dots,\bar\iota_N) \in (\{0,1\}^n)^N$, define

$$\mathcal{C}_{n,\bar\iota}^{\mathcal{F}}(\varepsilon) \doteq \operatorname{Card}\{\phi^{\iota_1,\dots,\iota_N}(-\cdot) \in \mathcal{F}_{n,h} \mid \|\phi^{\iota_1,\dots,\iota_N} - \phi^{\bar\iota_1,\dots,\bar\iota_N}\|_{L^1(\mathbb{R},\Omega)} \leq \varepsilon\}, \tag{162}$$

and set

$$\mathcal{C}_n^{\mathcal{F}}(\varepsilon) \doteq \max\{\mathcal{C}_{n,\bar\iota}^{\mathcal{F}}(\varepsilon) \mid \bar\iota \doteq (\iota_1,\dots,\iota_N) \in (\{0,1\}^n)^N\}. \tag{163}$$

Observe that any element of an ε-cover of $\mathcal{F}_{n,h}$ contains at most $\mathcal{C}_n^{\mathcal{F}}(2\varepsilon)$ functions of $\mathcal{F}_{n,h}$. Thus, since the cardinality of $\mathcal{F}_{n,h}$ is the same of the set $\mathcal{B}_{n,h}$, which is 2^{nN}, it follows that the number of sets in an ε-cover of $\mathcal{F}_{n,h}$ is at least

$$N_\varepsilon(\mathcal{F}_{n,h} \mid L^1(\mathbb{R}, \Omega)) \geq \frac{2^{nN}}{\mathcal{C}_n^{\mathcal{F}}(2\varepsilon)}. \tag{164}$$

Therefore, we wish to provide now an upper bound on $\mathcal{C}_n^{\mathcal{F}}(2\varepsilon)$. To this end, consider any two N-tuples $\bar\iota \neq \iota \in (\{0,1\}^n)^N$, $\bar\iota \doteq (\bar\iota_1,\dots,\bar\iota_N)$, $\iota \doteq (\iota_1,\dots,\iota_N)$, and let $\phi^{\bar\iota_1,\dots,\bar\iota_N} \doteq \phi^{(\beta_1^{\bar\iota_1},\dots,\beta_N^{\bar\iota_N})}$, $\phi^{\iota_1,\dots,\iota_N} \doteq \phi^{(\beta_1^{\iota_1},\dots,\beta_N^{\iota_N})}$, denote the maps defined as in (71) in connection with the corresponding N-tuples $\beta^{\bar\iota_1,\dots,\bar\iota_N} \doteq (\beta_1^{\bar\iota_1},\dots,\beta_N^{\bar\iota_N})$ and $\beta^{\iota_1,\dots,\iota_N} \doteq (\beta_1^{\iota_1},\dots,\beta_N^{\iota_N})$ of $\mathcal{B}_{n,h}$. Recall that the eigenvectors are normalized so that $|r_k(u)| \equiv 1$, for all $k = 1,\dots,N$. Moreover, by definitions (71), (156), and because of (67), the maps $\phi^{\bar\iota_1,\dots,\bar\iota_N}$, $\phi^{\iota_1,\dots,\iota_N}$ and $\beta^{\bar\iota_1,\dots,\bar\iota_N}$, $\beta^{\iota_1,\dots,\iota_N}$ are supported on the disjoint union of sets $[\xi_k^-, \xi_k^+]$, $k = 1,\dots,N$. Thus, recalling (151), we find

$$\|\phi^{\bar\iota_1,\dots,\bar\iota_N} - \phi^{\iota_1,\dots,\iota_N}\|_{L^1(\mathbb{R},\Omega)} = \sum_{k=1}^N \int_{[\xi_k^-,\xi_k^+]} |R_k(\beta_k^{\bar\iota_k}(x)) - R_k(\beta_k^{\iota_k}(x))| dx$$

$$= \sum_{k=1}^N \int_{[\xi_k^-,\xi_k^+]} \left| \int_{\beta_k^{\iota_k}(x)}^{\beta_k^{\bar\iota_k}(x)} r_k(R_k(s)) ds \right| dx$$

$$\geq \sum_{k=1}^N \left[\int_{[\xi_k^-,\xi_k^+]} \left[|\beta_k^{\bar\iota_k}(x) - \beta_k^{\iota_k}(x)| - \left| \int_{\beta_k^{\iota_k}(x)}^{\beta_k^{\bar\iota_k}(x)} |r_k(R_k(s)) - r_k(0)| ds \right| \right] dx \right]$$

$$\geq \sum_{k=1}^N \int_{[\xi_k^-,\xi_k^+]} |\beta_k^{\bar\iota_k}(x) - \beta_k^{\iota_k}(x)| \cdot [1 - \|Dr_k\|_{L^\infty(B_h, M_n(\mathbb{R}))} \cdot h] dx$$

$$\geq \sum_{k=1}^N \|\beta_k^{\bar\iota_k} - \beta_k^{\iota_k}\|_{L^1(\mathbb{R},\mathbb{R})} \cdot [1 - \alpha_6 \cdot h], \tag{165}$$

where α_6 is the constant in (151). Hence, if we assume that

$$0 < h \leq \frac{1}{2\alpha_6}, \tag{166}$$

it follows from (165) that, adopting (with a slight abuse of notation) the L^1-distance

$$\|\beta^{\bar\iota_1,\dots,\bar\iota_N} - \beta^{\iota_1,\dots,\iota_N}\|_{L^1(\mathbb{R},\mathbb{R}^N)} \doteq \sum_{k=1}^N \|\beta_k^{\bar\iota_k} - \beta_k^{\iota_k}\|_{L^1(\mathbb{R},\mathbb{R})} \quad \forall(\bar\iota_1,\dots,\bar\iota_N),(\iota_1,\dots,\iota_N) \in (\{0,1\}^n)^N, \tag{167}$$

on the set $\mathcal{B}_{n,h}$ in (156), and the usual L^1-distance on the set $\mathcal{F}_{n,h}$ in (158), there holds

$$\left\| \beta^{\bar{\iota}_1,\ldots,\bar{\iota}_N} - \beta^{\iota_1,\ldots,\iota_N} \right\|_{L^1(\mathbb{R},\mathbb{R}^N)} \leq 2 \left\| \phi^{\bar{\iota}_1,\ldots,\bar{\iota}_N} - \phi^{\iota_1,\ldots,\iota_N} \right\|_{L^1(\mathbb{R},\Omega)}$$

$$\forall (\bar{\iota}_1,\ldots,\bar{\iota}_N),\, (\iota_1,\ldots,\iota_N) \in \left(\{0,1\}^n \right)^N. \tag{168}$$

Then, if we define $\mathcal{C}_n^{\mathcal{B}}(\varepsilon)$ as the maximum number of elements in $\mathcal{B}_{n,h}$ that have L^1-distance (defined as in (167)) less than ε from any given element of $\mathcal{B}_{n,h}$, we deduce from (168) that $\mathcal{C}_n^{\mathcal{F}}(2\varepsilon) \leq \mathcal{C}_n^{\mathcal{B}}(4\varepsilon)$. In turn, this inequality, together with (164), yields

$$N_\varepsilon\left(\mathcal{F}_{n,h} \mid L^1(\mathbb{R},\Omega) \right) \geq \frac{2^{nN}}{\mathcal{C}_n^{\mathcal{B}}(4\varepsilon)} \tag{169}$$

for all h satisfying (166).

In order to provide an upper estimate on $\mathcal{C}_n^{\mathcal{B}}(4\varepsilon)$, observe that, given any pair of nN-tuples $(\iota_1,\ldots,\iota_N)$, $(\bar{\iota}_1,\ldots,\bar{\iota}_N) \in \left(\{0,1\}^n \right)^N$, letting $(\beta_1^{\iota_1},\ldots,\beta_N^{\iota_N})$, $(\beta_1^{\bar{\iota}_1},\ldots,\beta_N^{\bar{\iota}_N})$ denote the corresponding N-tuples in $\mathcal{B}_{n,h}$, by definitions (154), (156), (167), and because every interval $[\xi_i^-, \xi_i^+]$ has length L, one has

$$\left\| \beta^{\bar{\iota}_1,\ldots,\bar{\iota}_N} - \beta^{\iota_1,\ldots,\iota_N} \right\|_{L^1(\mathbb{R},\mathbb{R}^N)} = \frac{Lh}{n} \cdot d\left((\iota_1,\ldots,\iota_N),(\bar{\iota}_1,\ldots,\bar{\iota}_N) \right), \tag{170}$$

where

$$d\left((\iota_1,\ldots,\iota_N),(\bar{\iota}_1,\ldots,\bar{\iota}_N) \right) \doteq \mathrm{Card}\left\{ (k,\ell) \in \{1,\ldots,N\} \times \{1,\ldots,n\} \mid (\iota_k)_\ell \neq (\bar{\iota}_k)_\ell \right\}.$$

Then, given any fixed nN-tuple $\bar{\iota} \doteq (\bar{\iota}_1,\ldots,\bar{\iota}_N) \in (\{0,1\}^n)^N$, define

$$\mathcal{C}_n^{\mathcal{I}}(\varepsilon) \doteq \mathrm{Card}\left\{ (\iota_1,\ldots,\iota_N) \in \left(\{0,1\}^n \right)^N \mid d\left((\iota_1,\ldots,\iota_N),(\bar{\iota}_1,\ldots,\bar{\iota}_N) \right) \leq \varepsilon \right\}. \tag{171}$$

Notice that the number $\mathcal{C}_n^{\mathcal{I}}(\varepsilon)$ is independent of the choice of $\bar{\iota} \doteq (\bar{\iota}_1,\ldots,\bar{\iota}_N) \in (\{0,1\}^n)^N$, and that, by (170), there holds

$$\mathcal{C}_n^{\mathcal{B}}(4\varepsilon) = \mathcal{C}_n^{\mathcal{I}}\left(\frac{4n\varepsilon}{Lh} \right). \tag{172}$$

We next derive an upper bound on $\mathcal{C}_n^{\mathcal{I}}(\varepsilon)$ following the same strategy as in the proof of Proposition 2.2 in [1]. Namely, by standard combinatorial properties, counting the nN-tuples that differ for a given number of entries, we compute

$$\mathcal{C}_n^{\mathcal{I}}\left(\frac{4n\varepsilon}{Lh} \right) = \sum_{\ell=0}^{\lfloor \frac{4n\varepsilon}{Lh} \rfloor} \binom{nN}{\ell}, \tag{173}$$

where $\lfloor \alpha \rfloor \doteq \max\{ z \in \mathbb{Z} \mid z \leq \alpha \}$ denotes the integer part of α. Next, observe that if $X_1,\ldots,X_{nN}$ are independent random variables with Bernoulli distribution $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \frac{1}{2}$, then for any integer $k \leq nN$ one has

$$\mathbb{P}(X_1 + \ldots + X_{nN} \leq k) = \frac{1}{2^{nN}} \sum_{\ell=0}^{k} \binom{nN}{\ell}. \tag{174}$$

Now, we recall Hoeffding's inequality [9, Theorem 2] which guarantees that, setting $S_{nN} \doteq X_1 + \ldots + X_{nN}$, for any fixed $\mu > 0$ there holds

$$\mathbb{P}\left(S_{nN} - \mathbb{E}(S_{nN}) \leq -\mu \right) \leq \exp\left(-\frac{2\mu^2}{nN} \right), \tag{175}$$

where $\mathbb{E}(S_{nN})$ denotes the expectation of S_{nN}. Notice that by the above assumptions on $X_1,\ldots,X_{nN}$, we have $\mathbb{E}(S_{nN}) = \frac{nN}{2}$. Hence, taking $k \doteq \lfloor \frac{4n\varepsilon}{Lh} \rfloor$, $\mu \doteq \frac{nN}{2} - \lfloor \frac{4n\varepsilon}{Lh} \rfloor$, and assuming

$$\varepsilon \leq \frac{NLh}{8}, \tag{176}$$

which implies $\mu > 0$, we deduce from (172)–(175) that there holds

$$\mathcal{C}_n^{\mathcal{B}}(4\varepsilon) \leq 2^{nN} \cdot \exp\left(-2\frac{(\frac{nN}{2} - \lfloor\frac{4n\varepsilon}{Lh}\rfloor)^2}{nN}\right)$$

$$\leq 2^{nN} \cdot \exp\left(-\frac{nN}{2}\left(1 - \frac{8\varepsilon}{LhN}\right)^2\right). \tag{177}$$

In turn, (177) together with (169) yields

$$N_\varepsilon\big(\mathcal{F}_{n,h} \,\big|\, L^1(\mathbb{R}, \Omega)\big) \geq \exp\left(\frac{nN}{2}\left(1 - \frac{8\varepsilon}{LhN}\right)^2\right), \tag{178}$$

for all $n \geq 2$ and h satisfying (166), (176). In order to derive the largest lower bound on the right-hand side of (178) we maximize the map

$$\Psi(h, n) \doteq \frac{nN}{2}\left(1 - \frac{8\varepsilon}{LhN}\right)^2,$$

with h, n, subject to (155), (166), (176). If we first fix $n \geq 2$, and then optimize the map $h \mapsto \Psi(h, n)$, when h satisfies the bound (155), we find that the maximum is attained for

$$h_n \doteq \frac{Lb}{2n}. \tag{179}$$

Next, optimizing the map $n \mapsto \Psi(h_n, n)$, with h_n satisfying (176), i.e. with $n \leq \frac{NL^2b}{16\varepsilon}$, we deduce that the maximum is attained for

$$\bar{n} \doteq \left\lfloor\frac{NL^2b}{48\varepsilon}\right\rfloor + 1. \tag{180}$$

One can check that

$$h_{\bar{n}} = \frac{Lb}{2\bar{n}} \leq \frac{24\varepsilon}{NL}, \qquad \frac{NLh_{\bar{n}}}{8} = \frac{NL^2b}{16\bar{n}} \geq \frac{3\varepsilon}{2},$$

so that, with $h_{\bar{n}}$, $\bar{n}$ defined by (179), (180), all conditions (155), (166), (176) are verified, provided that ε satisfies (152). Hence, we deduce from (178) that

$$N_\varepsilon\big(\mathcal{F}_{\bar{n},h_{\bar{n}}} \,\big|\, L^1(\mathbb{R}, \Omega)\big) \geq \exp\big(\Psi(h_{\bar{n}}, \bar{n})\big) = \exp\left(\frac{L^2N^2b}{216} \cdot \frac{1}{\varepsilon}\right), \tag{181}$$

which, in turn, yields

$$H_\varepsilon\big(\mathcal{F}_{\bar{n},h_{\bar{n}}} \,\big|\, L^1(\mathbb{R}, \Omega)\big) \geq \frac{L^2N^2b}{216\ln(2)} \cdot \frac{1}{\varepsilon} \tag{182}$$

for all ε satisfying (152). By the above observations at Point 1, recalling (160), this concludes the proof of the proposition. $\square$

Remark 6. When $N = 1$, under the same assumptions as Proposition 3 the following holds. Given $L, M, b, T > 0$, for every $0 < \varepsilon \leq LM/12$, letting $\mathcal{A}_{[L,M,b,T]}$ be the set defined in (134) one has

$$H_\varepsilon\big(\mathcal{A}_{[L,M,b,T]} \,\big|\, L^1(\mathbb{R})\big) \geq \frac{L^2b}{108\ln(2)} \cdot \frac{1}{\varepsilon}. \tag{183}$$

In order to analyze the ε-entropy of solutions to Temple systems of conservation laws, we shall now provide a lower bound on the ε-entropy of the class of maps $\mathcal{A}_{[L,M,b,T]}^w$ introduced in (148). Here, we are considering the ε-entropy of $\mathcal{A}_{[L,M,b,T]}^w$ related to the topology induced by the L^1-norm $\|w\|_{L^1} \doteq \sum_i \|w_i\|_{L^1}$.

Proposition 6. *Given* $L, M, b, T > 0$, *for every* $\varepsilon > 0$ *satisfying*

$$\varepsilon \leq \frac{LNM}{12}, \tag{184}$$

letting $\mathcal{A}^w_{[L,M,b,T]}$ *be the set defined in* (148), *one has*

$$H_\varepsilon\big(\mathcal{A}^w_{[L,M,b,T]} \mid L^1(\mathbb{R}, \Pi)\big) \geq \frac{L^2 N^2 b}{108 \ln(2)} \cdot \frac{1}{\varepsilon}. \tag{185}$$

Proof. The lower bound (185) is established with similar arguments to those of the proof of Proposition 5. Namely, given any integer $n \geq 2$ and any constant h satisfying (155), we consider the set $\mathcal{B}_{n,h}$ introduced in (156). Observe that by definitions (148), (156) one has

$$N_\varepsilon\big(\mathcal{A}_{[L,M,b,T]} \mid L^1(\mathbb{R}, \Pi)\big) \geq N_\varepsilon\big(\mathcal{B}_{n,h} \mid L^1(\mathbb{R}, \Pi)\big). \tag{186}$$

Next, let $\mathcal{C}^\mathcal{B}_n(\varepsilon)$ denote the maximum number of elements in $\mathcal{B}_{n,h}$ that have L^1-distance (defined as in (167)) less than ε from any given element of $\mathcal{B}_{n,h}$. With the same combinatory arguments of the proof of Proposition 5, for all

$$\varepsilon \leq \frac{NLh}{4}, \tag{187}$$

we derive

$$N_\varepsilon\big(\mathcal{B}_{n,h} \mid L^1(\mathbb{R}, \Pi)\big) \geq \frac{2^{nN}}{\mathcal{C}^\mathcal{B}_n(2\varepsilon)} \geq \exp\big(\Psi(h, n)\big), \tag{188}$$

with

$$\Psi(h, n) \doteq \frac{nN}{2}\left(1 - \frac{4\varepsilon}{LhN}\right)^2. \tag{189}$$

Maximizing the map (189) when h, n are subject to (155), (187), and combining (186), (188), we find

$$N_\varepsilon\big(\mathcal{A}^w_{[L,M,b,T]} \mid L^1(\mathbb{R}, \Pi)\big) \geq \exp\big(\Psi(h_{\bar{n}}, \bar{n})\big), \tag{190}$$

with

$$\bar{n} \doteq \left\lfloor \frac{NL^2 b}{24\varepsilon} \right\rfloor + 1, \qquad h_{\bar{n}} = \frac{Lb}{2\bar{n}} \leq \frac{12\varepsilon}{NL}. \tag{191}$$

Finally, observing that

$$\Psi(h_{\bar{n}}, \bar{n}) = \frac{L^2 N^2 b}{108} \cdot \frac{1}{\varepsilon}, \qquad \frac{NLh_{\bar{n}}}{4} \geq \frac{3\varepsilon}{2},$$

and taking the logarithm of both sides of (190), we recover the estimate (185) for all $\varepsilon > 0$ satisfying (184). ☐

3.3. Conclusion of the proofs of Theorem 1(i) and Theorem 2(i)

Proof of Theorem 1(i). We shall provide a proof of the lower bound (6) for sets of functions of the form (5) with support contained in the interval $I_L \doteq [-L, L]$. The case of sets of functions supported in any other given interval I of length $|I| = 2L$ can be recovered observing that every function in $S_T(\mathcal{L}_{[I,m,M]})$ is obtained by shifting horizontally a corresponding-function in $S_T(\mathcal{L}_{[I_L,m,M]})$ by a fixed constant. Thus, the ε-entropy of the two sets turns out to be the same.

Combining Proposition 3 and Proposition 5 we find that, for every

$$0 < \varepsilon \leq \min\left\{\frac{L\bar{d}}{48\alpha_4\, e^{\alpha_2/\alpha_3}}, \frac{L\,\Delta_\wedge\lambda}{96\alpha_1\,\alpha_4\, e^{\alpha_2/\alpha_3}}, \frac{LM}{48\alpha_4\, e^{\alpha_2/\alpha_3}}, \frac{Nm}{48}, \frac{LN}{48\alpha_6}\right\} \cdot \min\left\{\frac{1}{1+\alpha_5}, \frac{T\cdot\Delta_\wedge\lambda}{L}\right\} \tag{192}$$

($\alpha_l, l = 2, \ldots, 6$ and $\Delta_\wedge\lambda$ being the constants defined in (43), (76), (151) and (67), respectively) there holds

$$H_\varepsilon\big(S_T(\mathcal{L}_{[I_L,m,M]} \cap \mathcal{D}_0) \mid L^1(\mathbb{R}, \Omega)\big) \geq \frac{\widetilde{L}^2 N^2 b}{216 \ln(2)} \cdot \frac{1}{\varepsilon}, \tag{193}$$

with

$$\widetilde{L} = L \cdot \min\left\{\bar{c}_1, \bar{c}_2 \frac{T}{L}\right\}, \qquad b = \frac{1}{T} \cdot \frac{1}{\max\{\bar{c}_3, \bar{c}_4 \frac{N^2 L}{T}, \bar{c}_5 \frac{NL}{\delta_0 T}\}}, \tag{194}$$

where

$$\bar{c}_1 \doteq \frac{\bar{c}_2}{\bar{c}_2 + \lambda_N(0) - \lambda_1(0)} \qquad \bar{c}_2 \doteq \min_i\{\lambda_{i+1}(0) - \lambda_i(0)\},$$

$$\bar{c}_3 \doteq 2\sup\{|\nabla\lambda_i(u)|;\ u \in B_{\bar{d}},\ i = 1, \ldots, N\}, \qquad \bar{c}_5 \doteq 8\sup\{|l_i(0)|;\ u \in B_{\bar{d}},\ i = 1, \ldots, N\} \tag{195}$$

$$\bar{c}_4 \doteq \frac{\bar{c}_5}{2\bar{c}_2} \cdot \big(\sup\{|\lambda_k(u) - \lambda_j(u)| |l_i^T(u) Dr_k(u)|;\ u \in B_{\bar{d}},\ i, j, k \in \{1, \ldots, N\}\}$$
$$+ \sup\{|\nabla\lambda_i(u)|;\ u \in B_{\bar{d}},\ i = 1, \ldots, N\}\big). \tag{196}$$

Thus, (193)–(195)–(196) together yield (6), taking

$$c_l = \bar{c}_l, \quad \text{for } l = 1, 2, \qquad c_l = 216 \ln(2) \cdot \bar{c}_l, \quad \text{for } l = 3, 4, 5. \qquad \square \tag{197}$$

Proof of Theorem 2(i). As for the proof of Theorem 1(ii), it will be sufficient to establish the lower bound (19) for sets of functions of the form (18) with support contained in the interval $I_L \doteq [-L, L]$.

Combining Proposition 4 and Proposition 6 we find that, for every

$$0 < \varepsilon \leq \min\left\{\frac{LN\bar{d}'}{12}, \frac{L\sqrt{N}\Delta_\wedge\lambda}{24\alpha_1''}, \frac{LNM}{12}, \frac{Nm}{24}\right\} \tag{198}$$

(α_1'', $\Delta_\wedge\lambda$ being the constants defined in (109), (67), respectively) there holds

$$H_\varepsilon\big(S_T^w\big(\mathcal{L}_{[I_L,m,M]}^w\big) \mid L^1(\mathbb{R}, \Pi)\big) \geq \frac{L^2 N^2 b}{108 \ln(2)} \cdot \frac{1}{\varepsilon}, \tag{199}$$

with

$$b = \frac{1}{T} \cdot \frac{1}{\max\{\bar{c}_6, \bar{c}_7 \frac{NL}{T}\}}, \tag{200}$$

where

$$\bar{c}_6 \doteq 2\sup\{|\langle\nabla\lambda_i(u), r_i(u)\rangle|;\ u \in B_{\bar{d}},\ i = 1, \ldots, N\},$$

$$\bar{c}_7 \doteq \frac{2}{\bar{c}_2} \cdot \sup\{|\langle\nabla\lambda_i(u), r_j(u)\rangle|;\ u \in B_{\bar{d}},\ i, j = 1, \ldots, N\}. \tag{201}$$

Thus, (199)–(201) together yield (19), taking

$$c_l = 108 \ln(2) \cdot \bar{c}_l, \quad \text{for } l = 6, 7. \qquad \square \tag{202}$$

4. Upper compactness estimates for genuinely nonlinear Temple systems

Assume that (1) is a strictly hyperbolic system of Temple class, and that all characteristic families are genuinely nonlinear (cf. Subsection 3.1). Let $(S_t^w)_{t\geq 0}$ be the Riemann coordinate expression of the semigroup of entropy weak solutions generated by (1), defined on a domain $L^1(\mathbb{R}, \Pi)$ with $\Pi \doteq [a_1, b_1] \times \cdots \times [a_N, b_N]$. In connection with a class of initial data $\mathcal{L}_{[I,m,M]}^w \subset L^1(\mathbb{R}, \Pi)$ as in (18), consider the sets of i-th components of elements of $S_T^w(\mathcal{L}_{[I,m,M]}^w)$, at a fixed time $T > 0$:

$$S_{T,i}^w\big(\mathcal{L}_{[I,m,M]}^w\big) \doteq \big\{\varphi_i \mid (\varphi_1, \ldots, \varphi_N) \in S_T^w\big(\mathcal{L}_{[I,m,M]}^w\big)\big\}, \quad i = 1, \ldots, N. \tag{203}$$

Thanks to the Oleĭnik-type inequalities (16), we may establish an upper estimate on the ε-entropy for $S_{T,i}^w(\mathcal{L}_{[I,m,M]}^w)$ following the same strategy adopted in [8] for scalar conservation laws with convex flux, relying on the upper bound on the ε-entropy for classes of nondecreasing functions provided by:

Lemma 4. *(See [8, Lemma 3.1].) Given any,* $L, M > 0$*, setting*

$$\mathcal{I}_{[L,M]} \doteq \left\{ v : [0, L] \to [0, M] \mid v \text{ is nondecreasing} \right\}, \tag{204}$$

for $0 < \varepsilon < \frac{LM}{6}$*, there holds*

$$H_\varepsilon \left(\mathcal{I}_{[L,M]} \mid L^1([0, L]) \right) \leq \frac{4LM}{\varepsilon}.$$

In order to obtain an a-priori bound on size of the support of solutions to (1), expressed in terms of the L^1-norm of their initial data, we will use the next technical lemma derived in [1].

Lemma 5. *(See [1, Lemma 4.2].) Given* $v \in \mathrm{BV}(\mathbb{R})$*, compactly supported and satisfying*

$$Dv \leq B \quad \text{in the sense of measures}, \tag{205}$$

for some constant $B > 0$*, there holds*

$$\|v\|_{L^\infty} \leq \sqrt{2B\|v\|_{L^1}}. \tag{206}$$

Proof of Theorem 2(ii). As for the proof of Theorem 2(i), it will be sufficient to establish the upper bound (20) for sets of functions of the form (18) with support contained in the interval $I_L \doteq [-L, L]$. As stated in the introduction, we adopt the norms $\|w\|_{L^1} \doteq \sum_i \|w_i\|_{L^1}$, $\|w\|_{L^\infty} \doteq \sup_i \|w_i\|_{L^\infty}$ on the space $L^1(\mathbb{R}, \Pi)$.

1. Given any initial data $\overline{w} \in \mathcal{L}^w_{[I_L, m, M]}$, let $w(t, x) \doteq S^w_t \overline{w}(x)$ be the corresponding entropy weak solution of (1) satisfying the Oleĭnik-type inequalities (16). Observe that, by the properties of solutions of Temple systems (cf. [6]), and because $\overline{w} \in \mathcal{L}^w_{[I_L, m, M]}$, for all $t \geq 0$, $i = 1, \dots, N$, one has

$$\left\| w_i(t, \cdot) \right\|_{L^\infty(\mathbb{R}, \Pi)} \leq \|\overline{w}_i\|_{L^\infty(\mathbb{R}, \Pi)} \leq M, \qquad \left\| w_i(t, \cdot) \right\|_{L^1(\mathbb{R}, \Pi)} \leq \|\overline{w}_i\|_{L^1(\mathbb{R}, \Pi)}. \tag{207}$$

On the other hand, notice that $w_i(t, \cdot)$ is compactly supported, and that by virtue of (16), (207), one has $w_i(t, \cdot) \in \mathrm{BV}(\mathbb{R})$ for all $t > 0$ and

$$Dw_i(t, \cdot) \leq \frac{1}{ct} \quad \forall t > 0, \ i = 1, \dots, N. \tag{208}$$

Thus, invoking Lemma 5 and relying on (207), (208), we derive

$$\sum_i \left\| w_i(t, \cdot) \right\|_{L^\infty} \leq \sum_i \sqrt{\frac{2\|\overline{w}_i\|_{L^1}}{ct}} \leq \sqrt{\frac{2N}{ct}} \sqrt{\sum_i \|\overline{w}_i\|_{L^1}} \leq \sqrt{\frac{2Nm}{ct}}, \quad \forall t > 0. \tag{209}$$

Moreover, applying the theory of generalized characteristics (see [7, Section 10.2]), letting $\xi^-_{(t,z)}(\cdot), \xi^+_{(t,z)}(\cdot)$ denote the minimal and maximal backward characteristics emanating from (t, z), and setting

$$l^-(t) \doteq \inf\{ z \mid \xi^+_{(t,z)}(0) \geq -L \}, \qquad l^+(t) \doteq \sup\{ z \mid \xi^-_{(t,z)}(0) \leq L \}, \tag{210}$$

we find

$$\mathrm{Supp}\big(w_i(t, \cdot)\big) \subset \left[l^-_i(t), l^+_i(t) \right], \tag{211}$$

for all $t \geq 0$, $i = 1, \dots, N$. Then, recalling that the minimal backward characteristic $\xi^-_{(t,z)}(\cdot)$ is a solution of

$$\dot{\xi}(s) = \lambda_i\big(w(s, \xi(s)-)\big) \quad \text{a.e. } s \in [0, t], \tag{212}$$

setting

$$\alpha_{i,j} \doteq \sup\left\{ \left| \langle \nabla\lambda_i(u), r_j(u) \rangle \right|; \ |W(u)| \leq M \right\}, \tag{213}$$

and relying on (209), we derive

$$l_i^+(t) \leq L + \lambda_i(0) \cdot t + \sum_j \alpha_{i,j} \int_0^t \|w_j(s,\cdot)\|_{L^\infty} \, ds$$

$$\leq L + \lambda_i(0) \cdot t + \sup_j \alpha_{i,j} \int_0^t \sum_j \|w_j(s,\cdot)\|_{L^\infty} \, ds$$

$$\leq L + \lambda_i(0) \cdot t + \sup_j \alpha_{i,j} \sqrt{\frac{2Nm}{c}} \int_0^t \frac{1}{\sqrt{s}} \, ds$$

$$\leq L + \lambda_i(0) \cdot t + \sup_j \alpha_{i,j} \sqrt{\frac{8Nmt}{c}}, \tag{214}$$

for all $t \geq 0$, $i = 1, \ldots, N$. Analogously, observing that the maximal backward characteristic $\xi_{(t,z)}^+(\cdot)$ is a solution of

$$\dot{\xi}(s) = \lambda_i\big(w(s, \xi(s)+)\big) \quad \text{a.e. } s \in [0, t], \tag{215}$$

with the same arguments above we derive

$$l_i^-(t) \geq -L + \lambda_i(0) \cdot t - \sup_j \alpha_{i,j} \sqrt{\frac{8Nmt}{c}} \quad \forall t \geq 0, \ i = 1, \ldots, N, \tag{216}$$

which, together with (211), (214), yields

$$\text{Supp}\big(w_i(t,\cdot)\big) \subset \big[-L_t + \lambda_i(0) \cdot t, \ L_t + \lambda_i(0) \cdot t\big], \qquad L_t \doteq \left(L + \sup_{i,j} \alpha_{i,j} \sqrt{\frac{8Nmt}{c}}\right), \tag{217}$$

for all $t \geq 0$, $i = 1, \ldots, N$. Finally, observing that by (17), (213) we have $c \leq \sup_{i,j} \alpha_{i,j}$, and combining (209) with (217), we find

$$\|w_i(t,\cdot)\|_{L^\infty} \leq \frac{\sqrt{2Nm\,ct}}{ct}$$

$$\leq \frac{\sqrt{2Nm\,t}}{ct} \cdot \frac{\sup_{i,j} \alpha_{i,j}}{\sqrt{c}}$$

$$\leq \frac{L_t}{ct} \quad \forall t > 0, \ i = 1, \ldots, N. \tag{218}$$

2. In connection with any given $\psi \in S_T^w(\mathcal{L}_{[I_L,m,M]}^w)$, consider the function $\varphi_i^\natural : [0, 2L_T] \to \mathbb{R}$ defined by setting

$$\varphi_i^\natural(x) \doteq \frac{x}{cT} - \psi\big(x + \lambda_i(0) \cdot T - L_T\big) + \frac{L_T}{cT}, \tag{219}$$

with L_T as in (216). Notice that, by virtue of (208), $\varphi_i^\natural$ is nondecreasing and, thanks to (218), one has

$$0 \leq \varphi_i^\natural(x) \leq \frac{4L_T}{cT} \quad \forall x \in [0, 2L_T]. \tag{220}$$

Hence, recalling the definition (204), we have

$$\varphi_i^\natural \in \mathcal{I}_{[2L_T, \frac{4L_T}{cT}]}.$$

Finally, observe that since $\varphi_i^\natural$ is obtained from φ_i by a change of sign, a translation by a fixed function, and a shift of a fixed constant, it follows that, setting

$$\mathcal{U}_i^\natural \doteq \big\{\varphi_i^\natural \mid \varphi \in S_T^w\big(\mathcal{L}_{[I_L,m,M]}^w\big)\big\},$$

recalling (203), there holds

$$N_\varepsilon\big(S_{T,i}^w\big(\mathcal{L}_{[I,m,M]}^w\big) \,\big|\, L^1\big(\big[-L_T + \lambda_i(0) \cdot T, \ L_T + \lambda_i(0) \cdot T\big]\big)\big) = N_\varepsilon\big(\mathcal{U}_i^\natural \,\big|\, L^1\big([0, 2L_T]\big)\big)$$
$$\leq N_\varepsilon\big(\mathcal{I}_{[2L_T, \frac{4L_T}{cT}]} \,\big|\, L^1\big([0, 2L_T]\big)\big). \tag{221}$$

On the other hand, by virtue of (216), one has

$$N_\varepsilon\big(S_{T,i}^w\big(\mathcal{L}_{[I,m,M]}^w\big) \,\big|\, L^1(\mathbb{R})\big) = N_\varepsilon\big(S_{T,i}^w\big(\mathcal{L}_{[I,m,M]}^w\big) \,\big|\, L^1\big(\big[-L_T + \lambda_i(0) \cdot T, \ L_T + \lambda_i(0) \cdot T\big]\big)\big). \tag{222}$$

Thus, applying Lemma 4, and relying on (221), (222), we find

$$N_\varepsilon\big(S_{T,i}^w\big(\mathcal{L}_{[I,m,M]}^w\big) \,\big|\, L^1(\mathbb{R})\big) \leq 2^{\frac{32L_T^2}{cT} \cdot \frac{1}{\varepsilon}} \quad \forall i = 1, \ldots, N, \tag{223}$$

which, in turn, yields

$$N_\varepsilon\big(S_T^w\big(\mathcal{L}_{[I,m,M]}^w\big) \,\big|\, L^1(\mathbb{R}, \Pi)\big) \leq \prod_{i=1}^N N_{\frac{\varepsilon}{N}} N_\varepsilon\big(S_{T,i}^w\big(\mathcal{L}_{[I,m,M]}^w\big) \,\big|\, L^1(\mathbb{R})\big) \leq 2^{\frac{32N^2 L_T^2}{cT} \cdot \frac{1}{\varepsilon}}, \tag{224}$$

proving the upper bound (20). □

Conflict of interest statement

The authors declare that there is no conflict of interest.

Acknowledgements

References

[1] F. Ancona, O. Glass, K.T. Nguyen, Lower compactness estimates for scalar balance laws, Commun. Pure Appl. Math. 65 (9) (2012) 1303–1329.
[2] P.L. Bartlett, S.R. Kulkarni, S.E. Posner, Covering numbers for real-valued function classes, IEEE Trans. Inf. Theory 43 (5) (1997) 1721–1724.
[3] S. Bianchini, Stability of L^∞ solutions for hyperbolic systems with coinciding shocks and rarefactions, SIAM J. Math. Anal. 33 (4) (2001) 959–981.
[4] S. Bianchini, A. Bressan, Vanishing viscosity solutions to nonlinear hyperbolic systems, Ann. Math. 161 (2005) 223–342.
[5] A. Bressan, Hyperbolic Systems of Conservation Laws, Oxford Lecture Series in Mathematics and Its Applications, vol. 20, Oxford University Press, Oxford, 2000.
[6] A. Bressan, P. Goatin, Stability of L^∞ solutions of Temple class systems, Differ. Integral Equ. 13 (10–12) (2000) 1503–1528.
[7] C.M. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, second edition, Grundlehren Math. Wiss., vol. 325, Springer Verlag, 2005.
[8] C. De Lellis, F. Golse, A quantitative compactness estimate for scalar conservation laws, Commun. Pure Appl. Math. 58 (7) (2005) 989–998.
[9] W. Hoeffding, Probability inequalities for sums of bounded random variables, J. Am. Stat. Assoc. 58 (1963) 13–30.
[10] L. Hörmander, Lectures on Nonlinear Hyperbolic Differential Equations, Math. Appl., vol. 26, Springer Verlag, Berlin, 1997.
[11] S.N. Kružkov, First order quasilinear equations with several independent variables, Mat. Sb. 81 (123) (1970) 228–255 (in Russian). English translation in: Mat. Sb. 10 (2) (1970) 217–243.
[12] P.D. Lax, Weak solutions of nonlinear hyperbolic equations and their numerical computation, Commun. Pure Appl. Math. 7 (1954) 159–193.
[13] P.D. Lax, Hyperbolic systems of conservation laws II, Commun. Pure Appl. Math. 10 (1957) 537–566.
[14] P.D. Lax, Accuracy and resolution in the computation of solutions of linear and nonlinear equations, in: Recent Advances in Numerical Analysis, Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1978, in: Publ. Math. Res. Cent. Univ. Wis.-Madison, Academic Press, New York, 1978, pp. 107–117.
[15] T.P. Liu, The Riemann problem for general systems of conservation laws, J. Differ. Equ. 18 (1975) 218–234.
[16] O.A. Oleinik, Discontinuous solutions of non-linear differential equations, Usp. Mat. Nauk 12 (3(75)) (1957) 3–73 (in Russian). English translation in: Transl. Am. Math. Soc. Ser. 2 26 (1957) 95–172.
[17] D. Serre, Systèmes de Lois de Conservation. II, Diderot Editeur, 1996.
[18] B. Temple, Systems of conservation laws with invariant submanifolds, Trans. Am. Math. Soc. 280 (1983) 781–795.