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Large solutions to elliptic equations involving fractional Laplacian

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Abstract

The purpose of this paper is to study boundary blow up solutions for semi-linear fractional elliptic equations of the form

$$\begin{cases} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \bar{\Omega}^c, \\ \lim\limits_{x\in\Omega, x\to\partial\Omega} u(x) = +\infty, \end{cases} \tag{0.1}$$

where $p > 1$, Ω is an open bounded C^2 domain of $\mathbb{R}^N$, $N \geq 2$, the operator $(-\Delta)^\alpha$ with $\alpha \in (0, 1)$ is the fractional Laplacian and $f : \Omega \to \mathbb{R}$ is a continuous function which satisfies some appropriate conditions. We obtain that problem (0.1) admits a solution with boundary behavior like $d(x)^{-\frac{2\alpha}{p-1}}$, when $1 + 2\alpha < p < 1 - \frac{2\alpha}{\tau_0(\alpha)}$, for some $\tau_0(\alpha) \in (-1, 0)$, and has infinitely many solutions with boundary behavior like $d(x)^{\tau_0(\alpha)}$, when $\max\{1 - \frac{2\alpha}{\tau_0} + \frac{\tau_0(\alpha)+1}{\tau_0}, 1\} < p < 1 - \frac{2\alpha}{\tau_0}$. Moreover, we also obtained some uniqueness and non-existence results for problem (0.1).

Keywords: Large solutions; Fractional Laplacian; Existence; Uniqueness; Non-existence infinite existence

1. Introduction

In their pioneering work, Keller [22] and Osserman [27] studied the existence of solutions to the nonlinear reaction diffusion equation

$$\begin{cases} -\Delta u + h(u) = 0, & \text{in } \Omega, \\ u = +\infty, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain in $\mathbb{R}^N$, $N \ge 2$, and h is a nondecreasing positive function. They independently proved that this equation admits a solution if and only if h satisfies

$$\int_1^{+\infty} \frac{ds}{\sqrt{H(s)}} < +\infty, \tag{1.2}$$

where $H(s) = \int_0^s h(t)dt$, that in the case of $h(u) = u^p$ means $p > 1$. This integral condition on the non-linearity is known as the Keller–Osserman criteria. The solution of (1.1) found in [22] and [27] exists as a consequence of the interaction between the reaction and the diffusion term, without the influence of an external source that blows up at the boundary. Solutions exploding at the boundary are usually called boundary blow up solutions or large solutions. From then on, more general boundary blow-up problem:

$$\begin{cases} -\Delta u(x) + h(x,u) = f(x), & x \in \Omega, \\ \lim\limits_{x\in\Omega,\ x\to\partial\Omega} u(x) = +\infty \end{cases} \tag{1.3}$$

has been extensively studied, see [2–4,11–14,20,24–26,29]. It has being extended in various ways, weakened the assumptions on the domain and the nonlinear terms, extended to more general class of equations and obtained more information on the uniqueness and the asymptotic behavior of solution at the boundary.

During the last years there has been a renewed and increasing interest in the study of linear and nonlinear integral operators, especially, the fractional Laplacian, motivated by great applications and by important advances on the theory of nonlinear partial differential equations, see [5–7,10,15,17–19,28,32] for details.

In a recent work, Felmer and Quaas [15] considered an analog of (1.1) where the Laplacian is replaced by the fractional Laplacian

$$\begin{cases} (-\Delta)^\alpha u + |u|^{p-1}u = f, & \text{in } \Omega, \\ u = g, & \text{in } \ \bar{\Omega}^c, \\ \lim\limits_{x\in\Omega,\ x\to\partial\Omega} u(x) = +\infty, \end{cases} \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \ge 2$, with boundary $\partial\Omega$ of class C^2, $p > 1$ and the fractional Laplacian operator is defined as

$$(-\Delta)^\alpha u(x) = -\frac{1}{2}\int_{\mathbb{R}^N} \frac{\delta(u,x,y)}{|y|^{N+2\alpha}}dy, \quad x \in \Omega,$$

with $\alpha \in (0,1)$ and $\delta(u,x,y) = u(x+y) + u(x-y) - 2u(x)$. The authors proved the existence of a solution to (1.4) provided that g explodes at the boundary and satisfies other technical conditions. In case the function g blows up with an explosion rate as $d(x)^\beta$, with $\beta \in (-\frac{2\alpha}{p-1}, 0)$ and $d(x) = dist(x, \partial\Omega)$, the solution satisfies

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\beta} \le \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} < +\infty.$$

In [15] the explosion is driven by the function g. The external source f has a secondary role, not intervening in the explosive character of the solution. f may be bounded or unbounded, in latter case the explosion rate has to be controlled by $d(x)^{-2\alpha p/(p-1)}$.

One interesting question not answered in [15] is the existence of a boundary blow up solution without external source, that is assuming $g = 0$ in $\bar{\Omega}^c$ and $f = 0$ in Ω, thus extending the original result by Keller and Osserman, where solutions exists due to the pure interaction between the reaction and the diffusion terms. It is the purpose of this article to answer positively this question and to better understand how the non-local character influences the large solutions of (1.4) and what is the structure of the large solutions of (1.4) with or without sources. Comparing with the Laplacian case, where well possedness holds for (1.4), a much richer structure for the solution set appears for the non-local case, depending on the parameters and the data f and g. In particular, Theorem 1.1 shows that existence, uniqueness, non-existence and infinite existence may occur at different values of p and α.

Our first result is on the existence of blowing up solutions driven by the sole interaction between the diffusion and reaction term, assuming the external value g vanishes. Thus we will be considering the equation

$$
\begin{aligned}
&(-\Delta)^\alpha u + |u|^{p-1}u = f \quad \text{in } \Omega,\\
&u = 0 \quad \text{in } \Omega^c,\\
&\lim_{x\in\Omega, x\to\partial\Omega} u(x) = +\infty.
\end{aligned} \tag{1.5}
$$

On the external source f we will assume the following hypotheses

(H1) The external source $f : \Omega \to \mathbb{R}$ is a $C^\beta_{loc}(\Omega)$, for some $\beta > 0$.
(H2) Defining $f_-(x) = \max\{-f(x), 0\}$ and $f_+(x) = \max\{f(x), 0\}$ we have

$$
\limsup_{x\in\Omega, x\to\partial\Omega} f_+(x)d(x)^{\frac{2\alpha p}{p-1}} < +\infty \quad \text{and} \quad \lim_{x\in\Omega, x\to\partial\Omega} f_-(x)d(x)^{\frac{2\alpha p}{p-1}} = 0.
$$

A related condition that we need for non-existence results

(H2*) The function f satisfies

$$
\limsup_{x\in\Omega, x\to\partial\Omega} \left|f(x)\right|d(x)^{2\alpha} < +\infty.
$$

Now we are in a position to state our first theorem.

Theorem 1.1. *Assume that Ω is an open, bounded and connected domain of class C^2 and $\alpha \in (0,1)$. Then we have:*
Existence: *Assume that f satisfies (H1) and (H2), then there exists $\tau_0(\alpha) \in (-1,0)$ such that for every p satisfying*

$$
1 + 2\alpha < p < 1 - \frac{2\alpha}{\tau_0(\alpha)}, \tag{1.6}
$$

Eq. (1.5) possesses at least one solution u satisfying

$$
0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} \le \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} < +\infty. \tag{1.7}
$$

Uniqueness: *If f further satisfies $f \ge 0$ in Ω, then $u > 0$ in Ω and u is the unique solution of (1.5) satisfying (1.7).*
Nonexistence: *If f satisfies (H1), (H2*) and $f \ge 0$, then in the following three cases:*

i) *For any $\tau \in (-1,0) \setminus \{-\frac{2\alpha}{p-1}, \tau_0(\alpha)\}$ and p satisfying (1.6) or*
ii) *For any $\tau \in (-1,0)$ and*

$$
p \ge 1 - \frac{2\alpha}{\tau_0(\alpha)} \quad \textit{or} \tag{1.8}
$$

iii) *For any $\tau \in (-1,0) \setminus \{\tau_0(\alpha)\}$ and*

$$
1 < p \le 1 + 2\alpha, \tag{1.9}
$$

Eq. (1.5) does not have a solution u satisfying

$$
0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\tau} \le \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\tau} < +\infty. \tag{1.10}
$$

Special existence for $\tau = \tau_0(\alpha)$. *Assume $f(x) \equiv 0$, $x \in \Omega$ and that*

$$
\max\left\{1 - \frac{2\alpha}{\tau_0(\alpha)} + \frac{\tau_0(\alpha)+1}{\tau_0(\alpha)}, 1\right\} < p < 1 - \frac{2\alpha}{\tau_0(\alpha)}. \tag{1.11}
$$

Then, there exist constants $C_1 \ge 0$ and $C_2 > 0$, such that for any $t > 0$ there is a positive solution u of Eq. (1.5) satisfying

$$
C_1 d(x)^{\min\{\tau_0(\alpha)p+2\alpha,0\}} \le t d(x)^{\tau_0(\alpha)} - u(x) \le C_2 d(x)^{\min\{\tau_0(\alpha)p+2\alpha,0\}}. \tag{1.12}
$$

Remark 1.1. We remark that hypothesis (H2) and (H2*) are satisfied when $f \equiv 0$, so this theorem answer the question on existence rised in [15]. We also observe that a function f satisfying (H2) may also satisfy

$$\lim_{x\in\Omega,\ x\to\partial\Omega} f(x) = -\infty,$$

what matters is that the rate of explosion is smaller than $\frac{2\alpha p}{p-1}$.

For proving the existence part of this theorem we will construct appropriate super and sub-solutions. This construction involves the one dimensional truncated Laplacian of power functions given by

$$C(\tau) = \int_0^{+\infty} \frac{\chi_{(0,1)}(t)|1-t|^\tau + (1+t)^\tau - 2}{t^{1+2\alpha}} dt, \tag{1.13}$$

for $\tau \in (-1, 0)$ and where $\chi_{(0,1)}$ is the characteristic function of the interval $(0, 1)$. The number $\tau_0(\alpha)$ appearing in the statement of our theorems is precisely the unique $\tau \in (-1, 0)$ satisfying $C(\tau) = 0$. See Proposition 3.1 for details.

Remark 1.2. For the uniqueness, we would like to mention that, by using iteration technique, Kim in [23] has proved the uniqueness of solution to the problem

$$\begin{cases} -\Delta u + u_+^p = 0, & \text{in } \Omega, \\ u = +\infty, & \text{in } \partial\Omega, \end{cases} \tag{1.14}$$

where $u_+ = \max\{u, 0\}$, under the hypotheses that $p > 1$ and Ω is bounded and satisfying $\partial\Omega = \partial\bar{\Omega}$. García-Melián in [20,21] introduced some improved iteration technique to obtain the uniqueness for problem (1.14) with replacing nonlinear term by $a(x)u^p$. However, there is a big difficulty for us to extend the iteration technique to our problem (1.4) involving fractional Laplacian, which is caused by the nonlocal character.

In the second part, we are also interested in considering the existence of blowing up solutions driven by external source f on which we assume the following hypothesis

(H3) There exists $\gamma \in (-1 - 2\alpha, 0)$ such that

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} f(x)d(x)^{-\gamma} \le \limsup_{x\in\Omega, x\to\partial\Omega} f(x)d(x)^{-\gamma} < +\infty.$$

Depending on the size of γ we will say that the external source is weak or strong. In order to gain in clarity, in this case we will state separately the existence, uniqueness and non-existence theorem in this source-driven case.

Theorem 1.2 *(Existence). Assume that Ω is an open, bounded and connected domain of class C^2. Assume that f satisfies (H1) and let $\alpha \in (0, 1)$ then we have:*

(i) (weak source) If f satisfies (H3) with

$$-2\alpha - \frac{2\alpha}{p-1} \le \gamma < -2\alpha, \tag{1.15}$$

then, for every p such that (1.8) holds, Eq. (1.5) possesses at least one solution u, with asymptotic behavior near the boundary given by

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\gamma-2\alpha} \le \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\gamma-2\alpha} < +\infty. \tag{1.16}$$

(ii) (strong source) If f satisfies (H3) with

$$-1 - 2\alpha < \gamma < -2\alpha - \frac{2\alpha}{p-1} \tag{1.17}$$

then, for every p such that

$$p > 1 + 2\alpha, \tag{1.18}$$

Eq. (1.5) possesses at least one solution u, with asymptotic behavior near the boundary given by

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\frac{\gamma}{p}} \le \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\frac{\gamma}{p}} < +\infty. \tag{1.19}$$

As we already mentioned, in Theorem 1.1 the existence of blowing up solutions results from the interaction between the reaction u^p and the diffusion term $(-\Delta)^\alpha$, while the role of the external source f is secondary. In contrast, in Theorem 1.2 the existence of blowing up solutions results on the interaction between the external source, and the diffusion term in case of weak source and the interaction between the external source and the reaction term in case of strong source.

Regarding uniqueness result for solutions of (1.5), as in Theorem 1.1 we will assume that f is non-negative, hypothesis that we need for technical reasons. We have

Theorem 1.3 *(Uniqueness). Assume that Ω is an open, bounded and connected domain of class C^2, $\alpha \in (0,1)$ and f satisfies (H1) and $f \ge 0$. Then we have*

i) *(weak source) the solution of (1.5) satisfying (1.16) is positive and unique, and*
ii) *(strong source) the solution of (1.5) satisfying (1.19) is positive and unique.*

We complete our theorems with a non-existence result for solution with a previously defined asymptotic behavior, as we saw in Theorem 1.1. We have

Theorem 1.4 *(Non-existence). Assume that Ω is an open, bounded and connected domain of class C^2, $\alpha \in (0,1)$ and f satisfies (H1), (H3) and $f \ge 0$. Then we have*

i) *(weak source) Suppose that p satisfies (1.8), γ satisfies (1.15) and $\tau \in (-1,0) \setminus \{\gamma + 2\alpha\}$. Then Eq. (1.5) does not have a solution u satisfying (1.10).*
ii) *(strong source) Suppose that p satisfies (1.18), γ satisfies (1.17) and $\tau \in (-1,0) \setminus \{\frac{\gamma}{p}\}$. Then, Eq. (1.5) does not have a solution u satisfying (1.10).*

All theorems stated so far deal with Eq. (1.4) in the case $g \equiv 0$, but they may also be applied when $g \not\equiv 0$ and, in particular, these result improve those given in [15]. In what follows we describe how to obtain this. We start with some notation, we consider $L^1_\omega(\bar\Omega^c)$ the weighted L^1 space in $\bar\Omega^c$ with weight

$$\omega(y) = \frac{1}{1+|y|^{N+2\alpha}}, \quad \text{for all } y \in \mathbb{R}^N.$$

Our hypothesis on the external values g is the following

(H4) The function $g : \bar\Omega^c \to \mathbb{R}$ is measurable and $g \in L^1_\omega(\bar\Omega^c)$.

Given g satisfying (H4), we define

$$G(x) = \frac{1}{2}\int_{\mathbb{R}^N} \frac{\tilde g(x+y)}{|y|^{N+2\alpha}}dy, \quad x \in \Omega, \tag{1.20}$$

where

$$\tilde g(x) = \begin{cases} 0, & x \in \bar\Omega, \\ g(x), & x \in \bar\Omega^c. \end{cases} \tag{1.21}$$

We observe that

$$G(x) = -(-\Delta)^\alpha \tilde g(x), \quad x \in \Omega.$$

Hypothesis (H4) implies that G is continuous in Ω as seen in Lemma 2.1 and has an explosion of order $d(x)^{\beta-2\alpha}$ towards the boundary $\partial\Omega$, if g has an explosion of order $d(x)^{\beta}$ for some $\beta \in (-1, 0)$, as we shall see in Proposition 3.3. We observe that under the hypothesis (H4), if u is a solution of Eq. (1.4), then $u - \tilde{g}$ is the solution of

$$\begin{cases} (-\Delta)^{\alpha} u(x) + |u|^{p-1} u(x) = f(x) + G(x), & x \in \Omega, \\ u(x) = 0, & x \in \bar{\Omega}^c, \\ \lim\limits_{x \in \Omega,\ x \to \partial\Omega} u(x) = +\infty \end{cases} \tag{1.22}$$

and vice versa, if v is a solution of (1.22), then $v + \tilde{g}$ is a solution of (1.4).

Thus, using Theorems 1.1–1.4, we can state the corresponding results of existence, uniqueness and non-existence for (1.4), combining f with g to define a new external source

$$F(x) = G(x) + f(x), \quad x \in \Omega. \tag{1.23}$$

With this we can state appropriate hypothesis for g and thus we can write theorems, one corresponding to each of Theorem 1.1, 1.2, 1.3 and 1.4. Even though, at first sight we need that $G(x)$ is $C^{\beta}_{loc}(\Omega)$, actually continuity of g is sufficient, as we discuss Remark 4.1.

Moreover, in Remark 4.2 we explain how our results in this paper allows to give a different proof of those obtained by Felmer and Quaas in [15], generalizing them.

After this paper was completed, we have learned of a preprint of Abatangelo [1] were different, but related results are obtained.

This article is organized as follows. In Section 2 we present some preliminaries to introduce the notion of viscosity solutions, comparison and stability theorems in case of explosion at the boundary. Then we prove an existence theorem for the nonlinear problem with blow up at the boundary, assuming the existence of ordered. Section 3 is devoted to obtain crucial estimates used to construct super and sub-solutions. In Section 4 we prove the existence of solution to (1.5) in Theorem 1.1 and Theorem 1.2. In Section 5, we give the proof of the uniqueness of solution to (1.5) in Theorem 1.1 and Theorem 1.3. Finally, the nonexistence related to Theorem 1.1 and Theorem 1.4 are shown in Section 6.

2. Preliminaries and existence theorem

The purpose of this section is to introduce some preliminaries and prove an existence theorem for blow-up solutions assuming the existence of ordered super-solution and sub-solution which blow up at the boundary. In order to prove this theorem we adapt the theory of viscosity to allow for boundary blow up.

We start this section by defining the notion of viscosity solution for non-local equation, allowing blow up at the boundary, see for example [7]. We consider the equation of the form:

$$(-\Delta)^{\alpha} u = h(x, u) \quad \text{in } \Omega, \qquad u = g \quad \text{in } \Omega^c. \tag{2.1}$$

Definition 2.1. We say that a function $u : (\partial\Omega)^c \to \mathbb{R}$, continuous in Ω and in $L^1_{\omega}(\mathbb{R}^N)$ is a viscosity super-solution (sub-solution) of (2.1) if

$$u \geq g \quad (\text{resp. } u \leq g) \text{ in } \bar{\Omega}^c$$

and for every point $x_0 \in \Omega$ and some neighborhood V of x_0 with $\bar{V} \subset \Omega$ and for any $\phi \in C^2(\bar{V})$ such that $u(x_0) = \phi(x_0)$ and

$$u(x) \geq \phi(x) \quad \big(\text{resp. } u(x) \leq \phi(x)\big) \text{ for all } x \in V,$$

defining

$$\tilde{u} = \begin{cases} \phi & \text{in } V, \\ u & \text{in } V^c, \end{cases}$$

we have

$$(-\Delta)^\alpha \tilde{u}(x_0) \ge h\big(x_0, u(x_0)\big) \quad \big(\text{resp. } (-\Delta)^\alpha \tilde{u}(x_0) \le h\big(x_0, u(x_0)\big)\big).$$

We say that u is a viscosity solution of (2.1) if it is a viscosity super-solution and also a viscosity sub-solution of (2.1).

Remark 2.1. This definition is equivalent in the case of super-solution to take ϕ such that $u-\phi$ has a zero at x_0 that is a global minimum $(-\Delta)^\alpha \tilde{u}(x_0) \ge h(x_0, u(x_0))$ for other definitions of super-solution and their equivalence can be found in [9].

It will be convenient for us to have also a notion of classical solution.

Definition 2.2. We say that a function $u : (\partial\Omega)^c \to \mathbb{R}$, continuous in Ω and in $L^1_\omega(\mathbb{R}^N)$ is a classical solution of (2.1) if $(-\Delta)^\alpha u(x)$ is well defined for all $x \in \Omega$,

$$(-\Delta)^\alpha u(x) = h\big(x, u(x)\big), \quad \text{for all } x \in \Omega$$

and $u(x) = g(x)$ a.e. in $\overline{\Omega}^c$. Classical super and sub-solutions are defined similarly.

Next we have our first regularity theorem.

Theorem 2.1. *Let $g \in L^1_\omega(\mathbb{R}^N)$ and $f \in C^\beta_{loc}(\Omega)$, with $\beta \in (0,1)$, and u be a viscosity solution of*

$$(-\Delta)^\alpha u = f \quad \text{in } \Omega, \qquad u = g \quad \text{in } \Omega^c,$$

then there exists $\gamma > 0$ such that $u \in C^{2\alpha+\gamma}_{loc}(\Omega)$

Proof. Here we use ideas of [32]. Suppose without loss of generality that $B_1 \subset \Omega$ and $f \in C^\beta(B_1)$. Let η be a non-negative, smooth function with support in B_1, such that $\eta = 1$ in $B_{1/2}$. Now we look at the equation

$$-\Delta w = \eta f \quad \text{in } \mathbb{R}^N.$$

By Hölder regularity theory for the Laplacian we find $w \in C^{2,\beta}$, so that $(-\Delta)^{1-\alpha} w \in C^{2\alpha+\beta}$, see [33] or Theorem 3.1 in [16]. Then, since

$$(-\Delta)^\alpha \big(u - (-\Delta)^{1-\alpha} w\big) = 0 \quad \text{in } B_{1/2},$$

we can use Theorem 1.1 and Remark 9.4 of [8] (see also Theorem 4.1 there), to obtain that there exist $\tilde{\beta}$ such that $u - (-\Delta)^{1-\alpha} w \in C^{2\alpha+\tilde{\beta}}(B_{1/2})$, from where we conclude. □

The Maximum and the Comparison Principles are key tools in the analysis, we present them here for completion.

Theorem 2.2. *(Maximum Principle) Let $\mathcal{O}$ be an open and bounded domain of $\mathbb{R}^N$ and u be a classical solution of*

$$(-\Delta)^\alpha u \le 0 \quad \text{in } \mathcal{O}, \tag{2.2}$$

continuous in $\bar{\mathcal{O}}$ and bounded from above in $\mathbb{R}^N$. Then $u(x) \le M$, for all $x \in \mathcal{O}$, where $M = \sup_{x \in \mathcal{O}^c} u(x) < +\infty$.

Proof. If the conclusion is false, then there exists $x' \in \mathcal{O}$ such that $u(x') > M$. By continuity of u, there exists $x_0 \in \mathcal{O}$ such that

$$u(x_0) = \max_{x \in \mathcal{O}} u(x) = \max_{x \in \mathbb{R}^N} u(x)$$

and then $(-\Delta)^\alpha u(x_0) > 0$, which contradicts (2.2). □

Remark 2.2. Maximum Principle for less regular solution can be found for example in [31].

Theorem 2.3. *(Comparison Principle) Let u and v be classical super-solution and sub-solution of*

$$(-\Delta)^\alpha u + h(u) = f \quad \text{in } \mathcal{O},$$

respectively, where $\mathcal{O}$ is an open, bounded domain, the functions $f : \mathcal{O} \to \mathbb{R}$ is continuous and $h : \mathbb{R} \to \mathbb{R}$ is increasing. Suppose further that u and v are continuous in $\bar{\mathcal{O}}$ and $v(x) \le u(x)$ for all $x \in \mathcal{O}^c$. Then

$$u(x) \ge v(x), \quad x \in \mathcal{O}.$$

Proof. Suppose by contradiction that $w = u - v$ has a negative minimum in $x_0 \in \mathcal{O}$, then $(-\Delta)^\alpha w(x_0) < 0$ and so, by assumptions on u and v, $h(u(x_0)) > h(v(x_0))$, which contradicts the monotonicity of h. □

We devote the rest of the section to the proof of the existence theorem through super and sub-solutions. We prove the theorem by an approximation procedure for which we need some preliminary steps. We need to deal with a Dirichlet problem involving fractional Laplacian operator and with exterior data which blows up away from the boundary. Precisely, on the exterior data g, we assume the following hypothesis, given an open, bounded set $\mathcal{O}$ in $\mathbb{R}^N$ with C^2 boundary:

(G) $g : \mathcal{O}^c \to \mathbb{R}$ is in $L^1_\omega(\mathcal{O}^c)$ and it is of class C^2 in $\{z \in \mathcal{O}^c, dist(z, \partial\mathcal{O}) \le \delta\}$, where $\delta > 0$.

In studying the nonlocal problem (1.4) with explosive exterior source, we have to adapt the stability theorem and the existence theorem for the linear Dirichlet problem. The following lemma is important in this direction.

Lemma 2.1. *Assume that $\mathcal{O}$ is an open, bounded domain in $\mathbb{R}^N$ with C^2 boundary. Let $w : \mathbb{R}^N \to \mathbb{R}$:*

(i) *If $w \in L^1_\omega(\mathbb{R}^N)$ and w is of class C^2 in $\{z \in \mathbb{R}^N, d(z, \mathcal{O}) \le \delta\}$ for some $\delta > 0$, then $(-\Delta)^\alpha w$ is continuous in $\bar{\mathcal{O}}$.*
(ii) *If $w \in L^1_\omega(\mathbb{R}^N)$ and w is of class C^2 in $\mathcal{O}$, then $(-\Delta)^\alpha w$ is continuous in $\mathcal{O}$.*
(iii) *If $w \in L^1_\omega(\mathbb{R}^N)$ and $w \equiv 0$ in $\mathcal{O}$, then $(-\Delta)^\alpha w$ is continuous in $\mathcal{O}$.*

Proof. We first prove (ii). Let $x \in \Omega$ and $\eta > 0$ such that $B(x, 2\eta) \subset \Omega$. Then we consider

$$(-\Delta)^\alpha u(x) = L_1(x) + L_2(x),$$

where

$$L_1(x) = \int\limits_{B(0,\eta)} \frac{\delta(u, x, y)}{|y|^{N+2\alpha}} dy \quad \text{and} \quad L_2(x) = \int\limits_{B(0,\eta)^c} \frac{\delta(u, x, y)}{|y|^{N+2\alpha}} dy.$$

Since w is of class C^2 in $\mathcal{O}$, we may write L_1 as

$$L_1(x) = \int_0^\eta \left\{ \int\limits_{S^{N-1}} \int_{-1}^{1} \int_{1}^{1} t\omega^t D^2 w(x + str\omega)\omega dt ds d\omega \right\} r^{1-\alpha} dr,$$

where the term inside the brackets is uniformly continuous in (x, r), so the resulting function L_1 is continuous. On the other hand we may write L_2 as

$$L_2(x) = -2w(x) \int\limits_{B(0,\eta)^c} \frac{dy}{|y|^{N+2\alpha}} - 2 \int\limits_{B(x,\eta)^c} \frac{w(z)dz}{|z - x|^{N+2\alpha}},$$

from where L_2 is also continuous. The proof of (i) and (iii) are similar. □

The next theorem gives the stability property for viscosity solutions in our setting.

Theorem 2.4. *Suppose that $\mathcal{O}$ is an open, bounded and C^2 domain and $h : \mathbb{R} \to \mathbb{R}$ is continuous. Assume that (u_n), $n \in \mathbb{N}$ is a sequence of functions, bounded in $L^1_\omega(\mathcal{O}^c)$ and f_n and f are continuous in $\mathcal{O}$ such that:*

$(-\Delta)^\alpha u_n + h(u_n) \ge f_n$ (resp. $(-\Delta)^\alpha u_n + h(u_n) \le f_n$) in $\mathcal{O}$ in viscosity sense,
$u_n \to u$ locally uniformly in $\mathcal{O}$,
$u_n \to u$ in $L^1_\omega(\mathbb{R}^N)$, and
$f_n \to f$ locally uniformly in $\mathcal{O}$.

Then, $(-\Delta)^\alpha u + h(u) \ge f$ (resp. $(-\Delta)^\alpha u + h(u) \le f$) in $\mathcal{O}$ in viscosity sense.

Proof. If $|u_n| \le C$ in $\mathcal{O}$ then we use Lemma 4.3 of [7]. If u_n is unbounded in $\mathcal{O}$, then u_n is bounded in $\mathcal{O}_k = \{x \in \mathcal{O}, dist(x, \partial\mathcal{O}) \ge \frac{1}{k}\}$, since u_n is continuous in $\mathcal{O}$, and then by Lemma 4.3 of [7], u is a viscosity solution of $(-\Delta)^\alpha u + h(u) \ge f$ in $\mathcal{O}_k$ for any k. Thus u is a viscosity solution of $(-\Delta)^\alpha u + h(u) \ge f$ in $\mathcal{O}$ and the proof is completed. □

An existence result for the Dirichlet problem is given as follows:

Theorem 2.5. *Suppose that $\mathcal{O}$ is an open, bounded and C^2 domain, $g : \mathcal{O}^c \to \mathbb{R}$ satisfies (G), $f : \bar{\mathcal{O}} \to \mathbb{R}$ is continuous, $f \in C^\beta_{loc}(\mathcal{O})$, with $\beta \in (0, 1)$, and $p > 1$. Then there exists a classical solution u of*

$$\begin{cases} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = f(x), & x \in \mathcal{O}, \\ u(x) = g(x), & x \in \mathcal{O}^c, \end{cases} \tag{2.3}$$

which is continuous in $\bar{\mathcal{O}}$.

In proving Theorem 2.5, we will use the following lemma:

Lemma 2.2. *Suppose that $\mathcal{O}$ is an open, bounded and C^2 domain, $f : \bar{\mathcal{O}} \to \mathbb{R}$ is continuous and $C > 0$. Then there exist a classical solution of*

$$\begin{cases} (-\Delta)^\alpha u(x) + Cu(x) = f(x), & x \in \mathcal{O}, \\ u(x) = 0, & x \in \mathcal{O}^c, \end{cases} \tag{2.4}$$

which is continuous in $\bar{\mathcal{O}}$.

Proof. For the existence of a viscosity solution u of (2.4), that is continuous in $\bar{\mathcal{O}}$, we refers to Theorem 3.1 in [15]. Now we apply Theorem 2.6 of [7] to conclude that u is $C^\theta_{loc}(\mathcal{O})$, with $\theta > 0$, and then we use Theorem 2.1 to conclude that the solution is classical (see also Proposition 1.1 and 1.4 in [30]). □

Using Lemma 2.2, we find $\bar{V}$, a classical solution of

$$\begin{cases} (-\Delta)^\alpha \bar{V}(x) = -1, & x \in \mathcal{O}, \\ \bar{V}(x) = 0, & x \in \mathcal{O}^c, \end{cases} \tag{2.5}$$

which is continuous in $\bar{\mathcal{O}}$ and negative in $\mathcal{O}$. It is classical since we apply Theorem 2.6 of [7] to conclude that u is $C^\theta_{loc}(\mathcal{O})$, with $\theta > 0$, and then we use Theorem 2.1 to conclude that the solution is classical (see also Propositions 1.1 and 1.4 in [30]).

Now we prove Theorem 2.5.

Proof of Theorem 2.5. Under assumption (G) and in view of the hypothesis on $\mathcal{O}$, we may extend g to $\bar{g}$ in $\mathbb{R}^N$ as a C^2 function in $\{z \in \mathbb{R}^N, d(z, \mathcal{O}) \le \delta\}$. We certainly have $\bar{g} \in L^1_\omega(\mathbb{R}^N)$ and, by Lemma 2.1 $(-\Delta)^\alpha \bar{g}$ is continuous in $\bar{\mathcal{O}}$. Next we use Lemma 2.2 to find a solution v of Eq. (2.4) with $f(x)$ replaced by $f(x) - (-\Delta)^\alpha \bar{g}(x) - C\bar{g}(x)$, where $C > 0$. Then we define $u = v + \bar{g}$ and we see that u is continuous in $\bar{\mathcal{O}}$ and it satisfies in the viscosity sense

$$\begin{cases} (-\Delta)^\alpha u(x) + Cu(x) = f(x), & x \in \mathcal{O}, \\ u(x) = g(x), & x \in \mathcal{O}^c. \end{cases}$$

Now we use Theorem 2.6 in [7] and then Theorem 2.1 to conclude that u is a classical solution. Continuing the proof, we find super and sub-solutions for (2.3). We define

$$u_\lambda(x) = \lambda \bar{V}(x) + \bar{g}(x), \quad x \in \mathbb{R}^N,$$

where $\lambda \in \mathbb{R}$ and $\bar{V}$ is given in (2.5). We see that $u_\lambda(x) = g(x)$ in $\mathcal{O}^c$ for any λ and for λ large (negative), u_λ satisfies

$$(-\Delta)^\alpha u_\lambda(x) + \left|u_\lambda(x)\right|^{p-1} u_\lambda(x) - f(x) \geq (-\Delta)^\alpha \bar{g}(x) - \lambda - f(x) - \left|\bar{g}(x)\right|^p,$$

for $x \in \mathcal{O}$. Since $(-\Delta)^\alpha \bar{g}$, $\bar{g}$ and f are bounded in $\bar{\mathcal{O}}$, choosing $\lambda_1 < 0$ large enough we find that $u_{\lambda_1} \geq 0$ is a super-solution of (2.3) with $u_{\lambda_1} = g$ in $\mathcal{O}^c$.

On the other hand, for $\lambda > 0$ we have

$$(-\Delta)^\alpha u_\lambda(x) + |u_\lambda|^{p-1} u_\lambda(x) - f(x) \leq (-\Delta)^\alpha \bar{g}(x) - \lambda + |\bar{g}|^{p-1} \bar{g}(x) - f(x).$$

As before, there is $\lambda_2 > 0$ large enough, so that u_{λ_2} is a sub-solution of (2.3) with $u_{\lambda_2} = g$ in $\mathcal{O}^c$. Moreover, we have that $u_{\lambda_2} < u_{\lambda_1}$ in $\mathcal{O}$ and $u_{\lambda_2} = u_{\lambda_1} = g$ in $\mathcal{O}^c$.

Let $u_0 = u_{\lambda_2}$ and define iteratively, using the above argument, the sequence of functions u_n $(n \geq 1)$ as the classical solutions of

$$\begin{aligned} &(-\Delta)^\alpha u_n(x) + C u_n(x) = f(x) + C u_{n-1}(x) - |u_{n-1}|^{p-1} u_{n-1}(x), \quad x \in \mathcal{O}, \\ &u_n(x) = g(x), \quad x \in \mathcal{O}^c, \end{aligned}$$

where $C > 0$ is so that the function $r(t) = Ct - |t|^{p-1} t$ is increasing in the interval $[\min_{x \in \bar{\mathcal{O}}} u_{\lambda_2}(x), \max_{x \in \bar{\mathcal{O}}} u_{\lambda_1}(x)]$. Next, using Theorem 2.3 we get

$$u_{\lambda_2} \leq u_n \leq u_{n+1} \leq u_{\lambda_1} \quad \text{in } \mathcal{O}, \text{ for all } n \in \mathbb{N}.$$

Then we define $u(x) = \lim_{n \to +\infty} u_n(x)$, for $x \in \mathcal{O}$ and $u(x) = g(x)$, for $x \in \mathcal{O}^c$ and we have

$$u_{\lambda_2} \leq u \leq u_{\lambda_1} \quad \text{in } \mathcal{O}. \tag{2.6}$$

Moreover, $u_{\lambda_1}, u_{\lambda_2} \in L^1_\omega(\mathbb{R}^N)$ so that $u_n \to u$ in $L^1_\omega(\mathbb{R}^N)$, as $n \to \infty$.

By interior estimates as given in [6], for any compact set K of $\mathcal{O}$, we have that u_n has uniformly bounded $C^\theta(K)$ norm. So, by Ascoli–Arzela Theorem we have that u is continuous in K and $u_n \to u$ uniformly in K. Taking a sequence of compact sets $K_n = \{z \in \mathcal{O}, d(z, \partial\mathcal{O}) \geq \frac{1}{n}\}$, and $\mathcal{O} = \bigcup_{n=1}^{+\infty} K_n$, we find that u is continuous in $\mathcal{O}$ and, by Theorem 2.4, u is a viscosity solution of (2.3). Now we apply Theorem 2.6 of [7] to find that u is $C^\theta_{loc}(\mathcal{O})$, and then we use Theorem 2.1 con conclude that u is a classical solution. In addition, u is continuous up to the boundary by (2.6). □

Remark 2.3. In the above limiting proceeding half relaxed limits can be use instead of C^θ regularity.

Now we are in a position to prove the main theorem of this section. We prove the existence of a blow-up solution of (1.5) assuming the existence of suitable super and sub-solutions.

Theorem 2.6. *Assume that Ω is an open, bounded domain of class C^2, $p > 1$ and f satisfy (H1). Suppose there exists a super-solution $\bar{U}$ and a sub-solution $\underline{U}$ of (1.5) such that $\bar{U}$ and $\underline{U}$ are of class C^2 in Ω, $\underline{U}$, $\bar{U} \in L^1_\omega(\mathbb{R}^N)$,*

$$\bar{U} \geq \underline{U} \quad \text{in } \Omega, \qquad \liminf_{x \in \Omega, x \to \partial\Omega} \underline{U}(x) = +\infty \quad \text{and} \quad \bar{U} = \underline{U} = 0 \quad \text{in } \bar{\Omega}^c.$$

Then there exists at least one solution u of (1.5) in the viscosity sense and

$$\underline{U} \leq u \leq \bar{U} \quad \text{in } \Omega.$$

Additionally, if $f \geq 0$ in Ω, then $u > 0$ in Ω.

Proof. Let us consider $\Omega_n = \{x \in \Omega : d(x) > 1/n\}$ and use Theorem 2.5 to find a solution u_n of

$$\begin{cases} (-\Delta)^\alpha u(x) + |u|^{p-1} u(x) = f(x), & x \in \Omega_n, \\ u(x) = \underline{U}(x), & x \in \Omega_n^c, \end{cases} \tag{2.7}$$

We just replace $\mathcal{O}$ by Ω_n and define $\delta = \frac{1}{4n}$, so that $\underline{U}(x)$ satisfies assumption (G). We notice that Ω_n is of class C^2 for $n \geq N_0$, for certain N_0 large. Next we show that u_n is a sub-solution of (2.7) in Ω_{n+1}. In fact, since u_n is the solution of (2.7) in Ω_n and $\underline{U}$ is a sub-solution of (2.7) in Ω_n, by Theorem 2.3,

$$u_n \geq \underline{U} \quad \text{in } \Omega_n.$$

Additionally, $u_n = \underline{U}$ in Ω_n^c. Then, for $x \in \Omega_{n+1} \setminus \Omega_n$, we have

$$(-\Delta)^\alpha u_n(x) = -\frac{1}{2} \int_{\mathbb{R}^N} \frac{\delta(u_n, x, y)}{|y|^{N+2\alpha}} dy \leq (-\Delta)^\alpha \underline{U}(x),$$

so that u_n is a sub-solution of (2.7) in Ω_{n+1}. From here and since u_{n+1} is the solution of (2.7) in Ω_{n+1} and $\bar{U}$ is a super-solution of (2.7) in Ω_{n+1}, by Theorem 2.3, we have $u_n \leq u_{n+1} \leq \bar{U}$ in Ω_{n+1}. Therefore, for any $n \geq N_0$,

$$\underline{U} \leq u_n \leq u_{n+1} \leq \bar{U} \quad \text{in} \quad \Omega.$$

Then we can define the function u as

$$u(x) = \lim_{n \to +\infty} u_n(x), \quad x \in \Omega \quad \text{and} \quad u(x) = 0, \quad x \in \bar{\Omega}^c$$

and we have

$$\underline{U}(x) \leq u(x) \leq \bar{U}(x), \quad x \in \Omega.$$

Since $\underline{U}$ and $\bar{U}$ belong to $L^1_\omega(\mathbb{R}^N)$, we see that $u_n \to u$ in $L^1_\omega(\mathbb{R}^N)$, as $n \to \infty$. Now we repeat the arguments of the proof of Theorem 2.5 to find that u is a classical solution of (1.5). Finally, if f is positive we easily find that u is positive, again by a contradiction argument. □

3. Some estimates

In order to prove our existence theorems we will use Theorem 2.6, so that it is crucial to have available super and sub-solutions to (1.4). In this section we provide the basic estimates that will allow to obtain in the next section the necessary super and sub-solutions.

To this end, we use appropriate powers of the distance function d and the main result in this section are the estimates given in Proposition 3.2, that provides the asymptotic behavior of the fractional operator applied to d.

But before going to this estimates, we describe the behavior of the function C defined in (1.13), which is a C^2 defined in $(-1, 2\alpha)$. We have:

Proposition 3.1. *For every $\alpha \in (0,1)$ there exists a unique $\tau_0(\alpha) \in (-1,0)$ such that $C(\tau_0(\alpha)) = 0$ and*

$$C(\tau)\big(\tau - \tau_0(\alpha)\big) < 0, \quad \text{for all } \tau \in (-1,0) \setminus \big\{\tau_0(\alpha)\big\}. \tag{3.1}$$

Moreover, the function τ_0 satisfies

$$\lim_{\alpha \to 1^-} \tau_0(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to 0^+} \tau_0(\alpha) = -1. \tag{3.2}$$

Proof. We first observe that $C(0) < 0$ since the integrand in (1.13) is zero in $(0,1)$ and negative in $(1, +\infty)$. Next easily see that

$$\lim_{\tau \to -1^+} C(\tau) = +\infty, \tag{3.3}$$

since, as τ approaches -1, the integrand loses integrability at 0. Next we see that $C(\cdot)$ is strictly convex in $(-1,0)$, since

$$C'(\tau) = \int_0^{+\infty} \frac{|1-t|^\tau \chi_{(0,1)}(t) \log|1-t| + (1+t)^\tau \log(1+t)}{t^{1+2\alpha}} dt$$

and

$$C''(\tau) = \int_0^{+\infty} \frac{|1-t|^\tau[\chi_{(0,1)}(t)\log|1-t|]^2 + (1+t)^\tau[\log(1+t)]^2}{t^{1+2\alpha}} dt > 0.$$

The convexity $C(\cdot)$, $C(0) < 0$ and (3.3) allow to conclude the existence and uniqueness of $\tau_0(\alpha) \in (-1, 0)$ such that (3.1) holds. To prove the first limit in (3.2), we proceed by contradiction, assuming that for $\{\alpha_n\}$ converging to 1 and $\tau_0 \in (-1, 0)$ such that

$$\tau_0(\alpha_n) \le \tau_0 < 0.$$

Then, for a constant $c_1 > 0$ we have

$$\lim_{\alpha_n \to 1^-} \int_0^{\frac{1}{2}} \frac{(1-t)^{\tau_0(\alpha_n)} + (1+t)^{\tau_0(\alpha_n)} - 2}{t^{1+2\alpha_n}} dt \ge c_1 \lim_{\alpha_n \to 1^-} \int_0^{\frac{1}{2}} t^{1-2\alpha_n} dt = +\infty$$

and, for a constant c_2 independent of n, we have

$$\int_{\frac{1}{2}}^{+\infty} \left| \frac{\chi_{(0,1)}(t)(1-t)^{\tau_0(\alpha_n)} + (1+t)^{\tau_0(\alpha_n)} - 2}{t^{1+2\alpha_n}} \right| dt \le c_2,$$

contradicting the fact that $C(\tau_0(\alpha_n)) = 0$. For the second limit in (3.2), we proceed similarly, assuming that for $\{\alpha_n\}$ converging to 0 and $\bar{\tau}_0 \in (-1, 0)$ such that

$$\tau_0(\alpha_n) \ge \bar{\tau}_0 > -1.$$

There are positive constants c_1 and c_2 we have such that

$$\int_0^2 \left| \frac{\chi_{0,1}(t)(1-t)^{\tau_0(\alpha_n)} + (1+t)^{\tau_0(\alpha_n)} - 2}{t^{1+2\alpha_n}} \right| dt \le c_1$$

and

$$\lim_{n\to\infty} \int_2^{+\infty} \frac{(1+t)^{\tau_0(\alpha_n)} - 2}{t^{1+2\alpha_n}} dt \le -c_2 \lim_{n\to\infty} \int_2^{+\infty} \frac{1}{t^{1+2\alpha_n}} dt = -\infty,$$

contradicting again that $C(\tau_0(\alpha_n)) = 0$. $\square$

Next we prove our main result in this section. We assume that $\delta > 0$ is such that the distance function $d(\cdot)$ is of class C^2 in $A_\delta = \{x \in \Omega, d(x) < \delta\}$ and we define

$$V_\tau(x) = \begin{cases} l(x), & x \in \Omega \setminus A_\delta, \\ d(x)^\tau, & x \in A_\delta, \\ 0, & x \in \Omega^c, \end{cases} \tag{3.4}$$

where τ is a parameter in $(-1, 0)$ and the function l is positive such that V_τ is C^2 in Ω. We have the following

Proposition 3.2. *Assume Ω is a bounded, open subset of $\mathbb{R}^N$ with a C^2 boundary and let $\alpha \in (0, 1)$. Then there exists $\delta_1 \in (0, \delta)$ and a constant $C > 1$ such that:*

(i) *If $\tau \in (-1, \tau_0(\alpha))$, then*

$$\frac{1}{C} d(x)^{\tau-2\alpha} \le -(-\Delta)^\alpha V_\tau(x) \le C d(x)^{\tau-2\alpha}, \quad \textit{for all } x \in A_{\delta_1}.$$

(ii) *If* $\tau \in (\tau_0(\alpha), 0)$*, then*

$$\frac{1}{C} d(x)^{\tau-2\alpha} \le (-\Delta)^\alpha V_\tau(x) \le C d(x)^{\tau-2\alpha}, \quad \textit{for all } x \in A_{\delta_1}.$$

(iii) *If* $\tau = \tau_0(\alpha)$*, then*

$$\left|(-\Delta)^\alpha V_\tau(x)\right| \le C d(x)^{\min\{\tau_0(\alpha), 2\tau_0(\alpha)-2\alpha+1\}}, \quad \textit{for all } x \in A_{\delta_1}.$$

Proof. By compactness we prove that the corresponding inequality holds in a neighborhood of any point $\bar{x} \in \partial\Omega$ and without loss of generality we may assume that $\bar{x} = 0$. For a given $0 < \eta \le \delta$, we define

$$Q_\eta = \left\{z = \left(z_1, z'\right) \in \mathbb{R} \times \mathbb{R}^{N-1}, |z_1| < \eta, \left|z'\right| < \eta\right\}$$

and $Q_\eta^+ = \{z \in Q_\eta, z_1 > 0\}$. Let $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ be a C^2 function such that $(z_1, z') \in \Omega \cap Q_\eta$ if and only if $z_1 \in (\varphi(z'), \eta)$ and moreover, $(\varphi(z'), z') \in \partial\Omega$ for all $|z'| < \eta$. We further assume that $(-1, 0, \cdots, 0)$ is the outer normal vector of Ω at $\bar{x}$.

In the proof of our inequalities, we let $x = (x_1, 0)$, with $x_1 \in (0, \eta/4)$, be then a generic point in $A_{\eta/4}$. We observe that $|x - \bar{x}| = d(x) = x_1$. By definition we have

$$-(-\Delta)^\alpha V_\tau(x) = \frac{1}{2} \int_{Q_\eta} \frac{\delta(V_\tau, x, y)}{|y|^{N+2\alpha}} dy + \frac{1}{2} \int_{\mathbb{R}^N \setminus Q_\eta} \frac{\delta(V_\tau, x, y)}{|y|^{N+2\alpha}} dy \tag{3.5}$$

and we see that

$$\left| \int_{\mathbb{R}^N \setminus Q_\eta} \frac{\delta(V_\tau, x, y)}{|y|^{N+2\alpha}} dy \right| \le c|x|^\tau, \tag{3.6}$$

where the constant c is independent of x. Thus we only need to study the asymptotic behavior of the first integral, that from now on we denote by $E(x_1)/2$.

Our first goal is to get a lower bound for $E(x_1)$. For that purpose we first notice that, since $\tau \in (-1, 0)$, we have that

$$d(z)^\tau \ge \left|z_1 - \varphi\left(z'\right)\right|^\tau, \quad \text{for all } z \in Q_\delta \cap \Omega. \tag{3.7}$$

Now we assume that $0 < \eta \le \delta/2$, then for all $y \in Q_\eta$ we have $x \pm y \in Q_\delta$. Thus $x \pm y \in \Omega \cap Q_\delta$ if and only if $\varphi(\pm y') < x_1 \pm y_1 < \delta$ and $|y'| < \delta$. Then, by (3.7), we have that

$$V_\tau(x + y) = d(x + y)^\tau \ge \left[x_1 + y_1 - \varphi\left(y'\right)\right]^\tau, \quad x + y \in Q_\delta \cap \Omega \tag{3.8}$$

and

$$V_\tau(x - y) = d(x - y)^\tau \ge \left[x_1 - y_1 - \varphi\left(-y'\right)\right]^\tau, \quad x - y \in Q_\delta \cap \Omega. \tag{3.9}$$

On the other side, for $y \in Q_\eta$ we have that if $x \pm y \in Q_\delta \cap \Omega^c$ then, by definition of V_τ, we have $V_\tau(x \pm y) = 0$. Now, for $y \in Q_\eta$ we define the intervals

$$I_+ = \left(\varphi\left(y'\right) - x_1, \eta - x_1\right) \quad \text{and} \quad I_- = \left(x_1 - \eta, x_1 - \varphi\left(-y'\right)\right) \tag{3.10}$$

and the functions

$$\begin{aligned}
I(y) &= \chi_{I_+}(y_1)\left|x_1 + y_1 - \varphi\left(y'\right)\right|^\tau + \chi_{I_-}(y_1)\left|x_1 - y_1 - \varphi\left(-y'\right)\right|^\tau - 2x_1^\tau, \\
J(y_1) &= \chi_{(x_1-\eta, x_1)}(y_1)|x_1 - y_1|^\tau + \chi_{(-x_1, \eta-x_1)}(y_1)|x_1 + y_1|^\tau - 2x_1^\tau, \\
I_1(y) &= \left\{\chi_{I_+}(y_1) - \chi_{(-x_1, \eta-x_1)}(y_1)\right\}|x_1 + y_1|^\tau, \\
I_2(y) &= \chi_{I_+}(y_1)\left(\left|x_1 + y_1 - \varphi\left(y'\right)\right|^\tau - |x_1 + y_1|^\tau\right),
\end{aligned}$$

where χ_A denotes the characteristic function of the set A. Then, using these definitions and inequalities (3.8) and (3.9), we have that

$$E(x_1) \geq \int\limits_{Q_\eta} \frac{I(y)}{|y|^{N+2\alpha}} dy = \int\limits_{Q_\eta} \frac{J(y_1)}{|y|^{N+2\alpha}} dy + E_1(x_1) + E_2(x_1), \tag{3.11}$$

where

$$E_i(x_1) = \int\limits_{Q_\eta} \frac{I_i(y) + I_{-i}(y)}{|y|^{N+2\alpha}} dy, \quad i = 1, 2. \tag{3.12}$$

Here we have considered that

$$I_{-1}(y) = \left\{\chi_{I_-}(y_1) - \chi_{(x_1-\eta,x_1)}(y_1)\right\}|x_1 - y_1|^\tau$$

and

$$I_{-2}(y) = \chi_{I_-}(y_1)\left(\left|x_1 - y_1 - \varphi\left(-y'\right)\right|^\tau - |x_1 - y_1|^\tau\right),$$

for $y = (y_1, y') \in \mathbb{R}^N$. We start studying the first integral in the right hand side in (3.11). Changing variables we see that

$$\int\limits_{Q_\eta} \frac{J(y_1)}{|y|^{N+2\alpha}} dy = x_1^{\tau-2\alpha} \int\limits_{Q_{\frac{\eta}{x_1}}} \frac{J(x_1 z_1) x_1^{-\tau}}{|z|^{N+2\alpha}} dz = 2x_1^{\tau-2\alpha}(R_1 - R_2),$$

where

$$R_1 = \int\limits_{Q^+_{\frac{\eta}{x_1}}} \frac{\chi_{(0,1)}(z_1)|1 - z_1|^\tau + (1 + z_1)^\tau - 2}{|z|^{N+2\alpha}} dz$$

and

$$R_2 = \int\limits_{Q^+_{\frac{\eta}{x_1}}} \frac{\chi_{(\frac{\eta}{x_1}-1,\frac{\eta}{x_1})}(z_1)(1 + z_1)^\tau}{|z|^{N+2\alpha}} dz.$$

Next we estimate these last two integrals. For R_1 we see that, for appropriate positive constants c_1 and c_2

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N_+} \frac{\chi_{(0,1)}(z_1)|1 - z_1|^\tau + (1 + z_1)^\tau - 2}{|z|^{N+2\alpha}} dz \\
&= \int\limits_0^{+\infty} \frac{\chi_{(0,1)}(z_1)|1 - z_1|^\tau + (1 + z_1)^\tau - 2}{z_1^{1+2\alpha}} dz_1 \int\limits_{\mathbb{R}^{N-1}} \frac{1}{(|z'|^2 + 1)^{\frac{N+2\alpha}{2}}} dz' \\
&= c_1\, C(\tau)
\end{aligned}$$

and

$$\int\limits_{(Q^+_{\frac{\eta}{x_1}})^c} \frac{\chi_{(0,1)}(z_1)|1 - z_1|^\tau + (1 + z_1)^\tau - 2}{|z|^{N+2\alpha}} dz = -c_2\, x_1^{2\alpha}\left(1 + o(1)\right).$$

Consequently we have, for some constant c that

$$R_1 = c_1\left(C(\tau) + c x_1^{2\alpha} + o\left(x_1^{2\alpha}\right)\right). \tag{3.13}$$

For R_2 we have that

$$R_2 = \int_{\frac{\eta}{x_1}-1}^{\frac{\eta}{x_1}} \frac{(1+z_1)^\tau}{z_1^{1+2\alpha}} \int_{B_{\frac{\eta}{x_1}}} \frac{1}{(1+|z'|^2)^{\frac{N+2\alpha}{2}}} dz' dz_1 \le c_3 x_1^{2\alpha-\tau+1}, \tag{3.14}$$

where $c_3 > 0$. Here and in what follows we denote by B_σ the ball of radius σ in $\mathbb{R}^{N-1}$. From (3.13) and (3.14) we then conclude that

$$\int_{Q_\eta} \frac{J(y_1)}{|y|^{N+2\alpha}} dy = c_1 x_1^{\tau-2\alpha}\big(C(\tau) + c x_1^{2\alpha} + o\big(x_1^{2\alpha}\big)\big). \tag{3.15}$$

Continuing with our analysis we estimate $E_1(x_1)$. We only consider the term $I_1(y)$, since the estimate for $I_{-1}(y)$ is similar. We have

$$\int_{Q_\eta} \frac{I_1(y)}{|y|^{N+2\alpha}} dy = -\int_{B_\eta} \int_{-x_1}^{\varphi(y')-x_1} \frac{|x_1+y_1|^\tau}{|y|^{N+2\alpha}} dy_1 dy' = -x_1^{\tau-2\alpha} F_1(x_1), \tag{3.16}$$

where

$$F_1(x_1) = \int_{B_{\frac{\eta}{x_1}}} \int_0^{\frac{\varphi(x_1 z')}{x_1}} \frac{|z_1|^\tau}{((z_1-1)^2+|z'|^2)^{(N+2\alpha)/2}} dz_1 dz'. \tag{3.17}$$

In what follows we write $\varphi_-(y') = \min\{\varphi(y'), 0\}$ and $\varphi_+(y') = \varphi(y') - \varphi_-(y')$. Next we see that assuming that $0 \le \varphi_+(y') \le C|y'|^2$ for $|y'| \le \eta$, for given (z_1, z') satisfying $0 \le z_1 \le \frac{\varphi_+(x_1 z')}{x_1}$ and $|z'| \le \frac{\eta}{x_1}$ then

$$(1-z_1)^2 + |z'|^2 \ge \frac{1}{4}\big(1+|z'|^2\big), \tag{3.18}$$

if we assume η small enough. Thus

$$\begin{aligned} F_1(x_1) &\le C \int_{B_{\frac{\eta}{x_1}}} \int_0^{\frac{\varphi_+(x_1 z')}{x_1}} \frac{|z_1|^\tau}{(1+|z'|^2)^{(N+2\alpha)/2}} dz_1 dz' \\ &\le C x_1^{\tau+1} \int_{B_{\frac{\eta}{x_1}}} \frac{|z'|^{2(\tau+1)}}{(1+|z'|^2)^{(N+2\alpha)/2}} dz' \\ &\le C x_1^{\tau+1}\big(x_1^{-2\tau+2\alpha-1}+1\big) \le C x_1^{\min\{\tau+1, 2\alpha-\tau\}}. \end{aligned}$$

Thus we have obtained

$$E_1(x_1) \ge -C x_1^{\tau-2\alpha} x_1^{\min\{\tau+1, 2\alpha-\tau\}}. \tag{3.19}$$

We continue with the estimate of $E_2(x_1)$. As before we only consider the term $I_2(y)$,

$$\begin{aligned} \int_{Q_\eta} \frac{I_2(y)}{|y|^{N+2\alpha}} dy &= \int_{B_\eta} \int_{\varphi(y')-x_1}^{\eta-x_1} \frac{|x_1+y_1-\varphi(y')|^\tau - |x_1+y_1|^\tau}{(y_1^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy_1 dy' \\ &\ge \int_{B_\eta} \int_{\varphi_-(y')-x_1}^{\eta-x_1} \frac{|x_1+y_1-\varphi_-(y')|^\tau - |x_1+y_1|^\tau}{(y_1^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy_1 dy' \end{aligned}$$

$$
\begin{aligned}
&= \int_{B_\eta} \int_{\varphi_-(y')}^{\eta} \frac{|z_1 - \varphi_-(y')|^\tau - |z_1|^\tau}{((z_1 - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{2}}} dz_1 dy' \\
&\geq \int_{B_\eta} \int_0^{\eta} \frac{|z_1 - \varphi_-(y')|^\tau - |z_1|^\tau}{((z_1 - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{2}}} dz_1 dy' + \int_{B_\eta} \int_{\varphi_-(y')}^{0} \frac{-|z_1|^\tau}{((z_1 - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{2}}} dz_1 dy' \\
&= E_{21}(x_1) + E_{22}(x_1). \qquad (3.20)
\end{aligned}
$$

We observe that $E_{22}(x_1)$ is similar to $F_1(x_1)$. In order to estimate $E_{21}(x_1)$ we use integration by parts

$$
\begin{aligned}
E_{21}(x_1) &= \frac{1}{\tau+1} \int_{B_\eta} \left\{ \frac{(\eta - \varphi_-(y'))^{\tau+1} - \eta^{\tau+1}}{((\eta - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{2}}} - \frac{(-\varphi_-(y'))^{\tau+1}}{(x_1^2 + |y'|^2)^{\frac{N+2\alpha}{2}}} \right\} dy' \\
&\quad + \frac{N+2\alpha}{\tau+1} \int_{B_\eta} \int_0^{\eta} \frac{(z_1 - \varphi_-(y'))^{\tau+1} - (z_1)^{\tau+1}}{((z_1 - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{2}+1}} (z_1 - x_1) dz_1 dy' \\
&= A_1 + A_2.
\end{aligned}
$$

For the first integral we have

$$
\begin{aligned}
A_1 &\geq \frac{1}{\tau+1} \int_{B_\eta} \left\{ \frac{-\eta^{\tau+1}}{((\eta - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{2}}} - \frac{(-\varphi_-(y'))^{\tau+1}}{(x_1^2 + |y'|^2)^{\frac{N+2\alpha}{2}}} \right\} dy' \\
&\geq -C(\eta) - C \int_{B_\eta} \frac{|y'|^{2\tau+2}}{(x_1^2 + |y'|^2)^{\frac{N+2\alpha}{2}}} dy' \\
&\geq -C x_1^{\tau - 2\alpha + \tau + 1} - C.
\end{aligned}
$$

For the second integral, since $\tau \in (-1, 0)$ and $(z_1 - \varphi_-(y'))^{\tau+1} - |z_1|^{\tau+1} > 0$, we have that

$$
\begin{aligned}
A_2 &\geq \frac{N+2\alpha}{\tau+1} \int_{B_\eta} \int_0^{x_1} \frac{(z_1 - \varphi_-(y'))^{\tau+1} - |z_1|^{\tau+1}}{((z_1 - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{2}+1}} (z_1 - x_1) dz_1 dy' \\
&\geq \frac{N+2\alpha}{(\tau+1)^2} \int_{B_\eta} \int_0^{x_1} \frac{-\varphi_-(y') z_1^\tau}{((z_1 - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{2}+1}} (z_1 - x_1) dz_1 dy' \\
&\geq C_3 x_1^{2\tau - 2\alpha + 1} \int_{B_{\eta/x_1}} \int_0^{1} \frac{|z'|^2 z_1^\tau}{((z_1 - 1)^2 + |z'|^2)^{\frac{N+2\alpha}{2}+1}} (z_1 - 1) dz_1 dz' \\
&\geq -C_4 x_1^{2\tau - 2\alpha + 1}, \qquad (3.21)
\end{aligned}
$$

where $C_3, C_4 > 0$ independent of x_1 and the second inequality used $a = z_1$ and $b = -\varphi_-(y')$ in the fact that $(a+b)^{\tau+1} - a^{\tau+1} \leq \frac{a^\tau b}{\tau+1}$ for $a > 0, b \geq 0$.

Thus, we have obtained

$$
E_2(x_1) \geq -C x_1^{\tau - 2\alpha} x_1^{\min\{\tau+1, 2\alpha - \tau\}}. \qquad (3.22)
$$

The next step is to obtain the other inequality for $E(x_1)$. By choosing δ smaller if necessary, we can prove that

Lemma 3.1. *Under the regularity conditions on the boundary and with the arrangements given at the beginning of the proof of Proposition 3.2, that is, that the boundary is locally described by φ after a rotation. We have that there is $\eta > 0$ and $C > 0$ such that*

$$
d(z) \geq (z_1 - \varphi(z'))(1 - C|z'|^2) \quad \textit{for all } (z_1, z') \in \Omega \cap Q_\eta.
$$

Proof. Since φ is C^2 and $\nabla\varphi(0)=0$, there exist $\eta_1\in(0,1/8)$ small and $C_1>0$ such that $C_1\eta_1<1/4$ and

$$\left|\varphi(y')\right|<C_1\left|y'\right|^2,\qquad \left|\nabla\varphi(y')\right|\le C_1\left|y'\right|,\quad \forall\, y'\in B_{\eta_1}. \tag{3.23}$$

Choosing $\eta_2\in(0,\eta_1)$ such that for any $z=(z_1,z')\in Q_{\eta_2}\cap\Omega$, there exists y' satisfying $(\varphi(y'),y')\in\partial\Omega\cap Q_{\eta_1}$ and $d(z)=|z-(\varphi(y'),y')|$.

We observe that y' mentioned above, is the minimizer of

$$H(z')=\left(z_1-\varphi(z')\right)^2+\left|z'-y'\right|^2,\quad \left|z'\right|<\eta_1,$$

then

$$-\left(z_1-\varphi(y')\right)\nabla\varphi(y')+\left(z'-y'\right)=0,$$

which, together with (3.23) implies that

$$\begin{aligned}\left|y'\right|-\left|z'\right|&\le\left|z'-y'\right|=\left|\left(z_1-\varphi(y')\right)\nabla\varphi(y')\right|\le\left(|z_1|+C_1\left|y'\right|^2\right)\left|\nabla\varphi(y')\right|\\&\le C_1\left(\eta_2+C_1\eta_1^2\right)\left|y'\right|\le 2C_1\eta_1\left|y'\right|<\frac12\left|y'\right|.\end{aligned}$$

Then

$$\left|y'\right|\le 2\left|z'\right|. \tag{3.24}$$

Denote the points z, $(\varphi(y'),y')$, $(\varphi(z'),z')$ by A,B,C, respectively, and let θ be the angle between the segment BC and the hyper plane with normal vector $e_1=(1,0,\ldots,0)$ and containing C. Then the angle $\angle C=\frac{\pi}{2}-\theta$. Denotes the arc from B to C in the plane ABC by $\mathrm{arc}(BC)$. By the geometry, there exists some point $x=(\varphi(x'),x')\in\mathrm{arc}(BC)$ such that line BC parallels the tangent line of $\mathrm{arc}(BC)$ at point x. Then, from (3.24) we have $|x'|\le\max\{|z'|,|y'|\}\le 2|z'|$ and so, from (3.23) we obtain

$$\tan(\theta)=\left|\frac{y'-z'}{|y'-z'|}\cdot\nabla\varphi(x')\right|\le\left|\nabla\varphi(x')\right|\le C_1\left|x'\right|\le 2C_1\left|z'\right|,$$

which implies that for some $C>0$,

$$\cos(\theta)\ge 1-C\left|z'\right|^2. \tag{3.25}$$

Then we complete the proof using Sine Theorem and (3.25)

$$\begin{aligned}d(z)&=\frac{\sin(\angle C)}{\sin(\angle B)}\left(z_1-\varphi(z')\right)\ge\left(z_1-\varphi(z')\right)\sin\left(\frac{\pi}{2}-\theta\right)\\&=\left(z_1-\varphi(z')\right)\cos(\theta)\ge\left(z_1-\varphi(z')\right)\left(1-C\left|z'\right|^2\right).\qquad\square\end{aligned}$$

From this lemma, by making C and η smaller if necessary we obtain that

$$d^\tau(z)\le\left(z_1-\varphi(z')\right)^\tau\left(1+C\left|z'\right|^2\right)\quad\text{for all } z\in\Omega\cap Q_\eta. \tag{3.26}$$

With $x=(x_1,0)$ satisfying $x_1\in(0,\eta/4)$ as at the beginning of the proof, we have that $d(x)=x_1$ and for any $y\in Q_\eta$ we see that $x\pm y\in Q_\delta$. We also see that $x\pm y\in\Omega\cap Q_\delta$ if and only if $\varphi(\pm y')<x_1\pm y_1<\delta$ and $|y'|<\delta$. Then, for $x\pm y\in\Omega\cap Q_\delta$, by (3.26) we have,

$$V_\tau(x\pm y)=d(x\pm y)^\tau\le\left(x_1\pm y_1-\varphi(\pm y')\right)^\tau\left(1+C\left|y'\right|^2\right). \tag{3.27}$$

For $y\in Q_\eta$, we define

$$I_3(y)=C\left|y'\right|^2\chi_{I_+}(y_1)\left|x_1+y_1-\varphi(y')\right|^\tau$$

and

$$I_3(-y) = C\left|y'\right|^2 \chi_{I_-}(y_1)\left|x_1 - y_1 - \varphi\left(-y'\right)\right|^\tau,$$

where I_+ and I_- were defined in (3.10). Using (3.27) as in (3.11) we find

$$\begin{aligned} E(x_1) &= \int_{Q_\eta} \frac{\delta(V_\tau, x, y)}{|y|^{N+2\alpha}} dy \le \int_{Q_\eta} \frac{I(y)}{|y|^{N+2\alpha}} dy + E_3(x_1) \\ &= \int_{Q_\eta} \frac{J(y)}{|y|^{N+2\alpha}} dy + E_1(x_1) + E_2(x_1) + E_3(x_1), \end{aligned} \tag{3.28}$$

where E_1 and E_2 were defined in (3.12) and

$$E_3(x_1) = \int_{Q_\eta} \frac{I_3(y) + I_3(-y)}{|y|^{N+2\alpha}} dy. \tag{3.29}$$

We estimate $E_3(x_1)$ and for that we observe that it is enough to estimate the integral with one of the terms in (3.29) (the other is similar), say

$$\begin{aligned} \int_{Q_\eta} \frac{I_3(y)}{|y|^{N+2\alpha}} dy &= \int_{B_\eta} \int_{\varphi(y')-x_1}^{\eta - x_1} \frac{C|y'|^2|x_1 + y_1 - \varphi(y')|^\tau}{|y|^{N+2\alpha}} dy_1 dy' \\ &= C x_1^{\tau - 2\alpha + 2} \int_{B_{\frac{\eta}{x_1}}} \int_{\frac{\varphi(x_1 z')}{x_1}}^{\frac{\eta}{x_1}} \frac{|z'|^2 |z_1 - \frac{\varphi(x_1 z')}{x_1}|^\tau}{((z_1 - 1)^2 + |z'|^2)^{(N+2\alpha)/2}} dz_1 dz' \\ &= C x_1^{\tau - 2\alpha + 2}(A_1 + A_2), \end{aligned} \tag{3.30}$$

where A_1 and A_2 are integrals over properly chosen subdomains, estimated separately.

$$\begin{aligned} A_1 &= \int_{B_{\frac{\eta}{x_1}}} \int_{\frac{\varphi(x_1 z')}{x_1}}^{\frac{\varphi(x_1 z')}{x_1} + \frac{1}{2}} \frac{|z'|^2 |z_1 - \frac{\varphi(x_1 z')}{x_1}|^\tau}{((z_1 - 1)^2 + |z'|^2)^{(N+2\alpha)/2}} dz_1 dz' \\ &\le \frac{c}{(\tau + 1)2^{\tau + 1}} \int_{B_{\frac{\eta}{x_1}}} \frac{|z'|^2}{(1 + |z'|^2)^{(N+2\alpha)/2}} dz' \qquad (3.31) \\ &\le c' \left(\frac{\eta}{x_1}\right)^{-2\alpha + 1}. \qquad (3.32) \end{aligned}$$

The inequality in (3.31) is obtained noticing that the ball $B((1, 0), 1/2)$ in R^N does not touch the band

$$\left\{(z_1, z') \,/\, \left|z'\right| \le \eta, \frac{\varphi(x_1 z')}{x_1} \le z_1 \le \frac{\varphi(x_1 z')}{x_1} + 1/2\right\}$$

if x_1 is small enough, and so $(z_1 - 1)^2 + |z'|^2 \ge \frac{1}{8} + \frac{1}{2}|z'|^2$. Then simple integration gives the next term. Next we estimate A_2

$$A_2 = \int_{B_{\frac{\eta}{x_1}}} \int_{\frac{\varphi(x_1 z')}{x_1} + \frac{1}{2}}^{\frac{\eta}{x_1}} \frac{|z'|^2 |z_1 - \frac{\varphi(x_1 z')}{x_1}|^\tau}{((z_1 - 1)^2 + |z'|^2)^{(N+2\alpha)/2}} dz_1 dz'$$

$$\leq \frac{1}{2^\tau} \int\limits_{B_{\frac{\eta}{x_1}}} \int\limits_{\frac{\varphi(x_1 z')}{x_1}+\frac{1}{2}}^{\frac{\eta}{x_1}} \frac{|z'|^2}{((z_1-1)^2+|z'|^2)^{(N+2\alpha)/2}} dz_1 dz'$$

$$\leq c' \left(\frac{\eta}{x_1}\right)^{-2\alpha+2}. \tag{3.33}$$

Putting together (3.30), (3.32), (3.33) and (3.29) we obtain

$$E_3(x_1) = \int\limits_{Q_\eta} \frac{(I_3(y)+I_3(-y))}{|y|^{N+2\alpha}} dy \leq c x_1^\tau. \tag{3.34}$$

From (3.16) but using the other inequality for F_1, that is,

$$F_1(x_1) \geq C \int\limits_{B_{\frac{\eta}{x_1}}} \int\limits_0^{\frac{\varphi_-(x_1 z')}{x_1}} \frac{|z_1|^\tau}{(1+|z'|^2)^{(N+2\alpha)/2}} dz_1 dz'$$

and arguing similarly we obtain as in (3.19)

$$E_1(x_1) \leq C x_1^{\tau-2\alpha} x_1^{\min\{\tau+1,2\alpha\}}. \tag{3.35}$$

Then we look at $E_2(x_1)$ and, as in (3.20), we only consider the term $I_2(y)$:

$$\int\limits_{Q_\eta} \frac{I_2(y)}{|y|^{N+2\alpha}} dy \leq \int\limits_{B_\eta} \int\limits_{\varphi_+(y')}^{\eta} \frac{|z_1-\varphi_+(y')|^\tau - |z_1|^\tau}{((z_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dz_1 dy' = \tilde{E}_{21}(x_1).$$

In order to estimate $\tilde{E}_{21}(x_1)$ we use integration by parts

$$\tilde{E}_{21}(x_1) = \frac{1}{\tau+1} \int\limits_{B_\eta} \left\{ \frac{(\eta-\varphi_+(y'))^{\tau+1} - \eta^{\tau+1}}{((\eta-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} - \frac{(\varphi_+(y'))^{\tau+1}}{((\varphi_+(y')-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} \right\} dy'$$

$$+ \frac{N+2\alpha}{\tau+1} \int\limits_{B_\eta} \int\limits_{\varphi_+(y')}^{\eta} \frac{(z_1-\varphi_+(y'))^{\tau+1} - z_1^{\tau+1}}{((z_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}+1}} (z_1-x_1) dz_1 dy'$$

$$\leq \frac{N+2\alpha}{\tau+1} \int\limits_{B_\eta} \int\limits_{\min\{\varphi_+(y'),x_1\}}^{x_1} \frac{(z_1-\varphi_+(y'))^{\tau+1} - z_1^{\tau+1}}{((z_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}+1}} (z_1-x_1) dz_1 dy'.$$

This integral can be estimated in a similar way as E_{21}, see (3.21) and the estimates given before. We then obtain

$$E_2(x_1) \leq C x_1^{2\tau-2\alpha+1}. \tag{3.36}$$

Then we conclude from (3.5), (3.11), (3.15), (3.19), (3.22), (3.28), (3.34), (3.35) and (3.36) that

$$-(-\Delta)^\alpha V_\tau(x) = C x_1^{\tau-2\alpha} \left(C(\tau) + O\left(x_1^{\min\{\tau+1,2\alpha\}}\right)\right), \tag{3.37}$$

where there exists a constant $c > 0$ so that

$$\left|O\left(x_1^{\min\{\tau+1,2\alpha\}}\right)\right| \leq c x_1^{\min\{\tau+1,2\alpha\}}, \quad \text{for all small } x_1 > 0.$$

From here, depending on the value of $\tau \in (-1,0)$, conditions (i), (ii) and (iii) follows and the proof of the proposition is complete. □

We end this section with an estimate we need when dealing with Eq. (1.4) when the external value g is not zero. We have the following proposition

Proposition 3.3. *Assume that Ω is a bounded, open and C^2 domain in $\mathbb{R}^N$. Assume that $g \in L^1_\omega(\Omega^c)$. Assume further that there are numbers $\beta \in (-1, 0)$, $\eta > 0$ and $c > 1$ such that*

$$\frac{1}{c} \le g(x)d(x)^{-\beta} \le c, \quad x \in \bar{\Omega}^c \text{ and } d(x) \le \eta.$$

Then there exist $\eta_1 > 0$ and $C > 1$ such that G, defined in (1.20), satisfies

$$\frac{1}{C} d(x)^{\beta-2\alpha} \le G(x) \le C d(x)^{\beta-2\alpha}, \quad x \in A_{\eta_1}. \tag{3.38}$$

Proof. The proof of this proposition requires estimates similar to those in the proof of Proposition 3.2 so we omit it. However, the function C used there and defined in (1.13), needs to be replaced here by $\tilde{C} : (-1, 0) \to \mathbb{R}$ given by

$$\tilde{C}(\beta) = \int_1^\infty \frac{|t-1|^\beta}{t^{1+2\alpha}} dt.$$

We observe that this function is always positive. □

4. Proof of existence results

In this section, we will give the proof of existence of large solution to (1.5). By Theorem 2.6 we only need to find ordered super and sub-solution, denoted by U and W, for (1.5) under our various assumptions. We begin with a simple lemma that reduce the problem to find them only in A_δ.

Lemma 4.1. *Let U and W be classical ordered super and sub-solution of (1.5) in the sub-domain A_δ. Then there exists λ large such that $U_\lambda = U - \lambda \bar{V}$ and $W_\lambda = W + \lambda \bar{V}$, where $\bar{V}$ is the solution of (2.5), with $\mathcal{O} = \Omega$, are ordered super and sub-solution of (1.5).*

Proof. Notice that by negativity $\bar{V}$ in Ω, we have that $U_\lambda \ge U$ and $W_\lambda \le W$, so they are still ordered in A_δ. In addition U_λ satisfies

$$(-\Delta)^\alpha U_\lambda + |U_\lambda|^{p-1} U_\lambda - f(x) \ge (-\Delta)^\alpha U + |U|^{p-1} U - f(x) + \lambda > 0, \quad \text{in } \Omega.$$

This inequality holds because of our assumption in A_δ, the fact that $(-\Delta)^\alpha U + |U|^{p-1} U - f(x)$ is continuous in $\Omega \setminus A_\delta$ and by taking λ large enough.

By the same type of arguments we find the W_λ is a sub-solution of the first equation in (1.5) and we complete the proof. □

Now we are in position to prove our existence results that we already reduced to find ordered super and sub-solution of (1.5) with the first equation in A_δ with the desired asymptotic behavior.

Proof of Theorem 1.1 (Existence). Define

$$U_\mu(x) = \mu V_\tau(x) \quad \text{and} \quad W_\mu(x) = \mu V_\tau(x), \tag{4.1}$$

with $\tau = -\frac{2\alpha}{p-1}$. We observe that $\tau = -\frac{2\alpha}{p-1} \in (-1, \tau_0(\alpha))$ and $\tau p = \tau - 2\alpha$. Then by Proposition 3.2 and (H2) we find that for $x \in A_\delta$ and $\delta > 0$ small

$$(-\Delta)^\alpha U_\mu(x) + U_\mu^p(x) - f(x) \ge -C\mu d(x)^{\tau-2\alpha} + \mu^p d(x)^{\tau p} - C d(x)^{\tau p},$$

for some $C > 0$. Then there exists a large $\mu > 0$ such that U_μ is a super-solution of (1.5) with the first equation in A_δ with the desired asymptotic behavior. Now by Proposition 3.2 we have that for $x \in A_\delta$ and $\delta > 0$ small

$$(-\Delta)^\alpha W_\mu(x) + W_\mu^p(x) - f(x) \le -\frac{\mu}{C} d(x)^{\tau-2\alpha} + \mu^p d(x)^{\tau p} - f(x) \le 0,$$

in the last inequality we have used (H2) and $\mu > 0$ small. Then, by Theorem 2.6 there exists a solution, with the desired asymptotic behavior. □

Proof of Theorem 1.1 (Special case $\tau = \tau_0(\alpha)$). We define for $t > 0$,

$$U_\mu(x) = tV_{\tau_0(\alpha)}(x) - \mu V_{\tau_1}(x) \quad \text{and} \quad W_\mu(x) = tV_{\tau_0(\alpha)}(x) - \mu V_{\tau_1}(x), \tag{4.2}$$

where $\tau_1 = \min\{\tau_0(\alpha)p + 2\alpha, 0\}$. If $\tau_1 = 0$, we write $V_0 = \chi_\Omega$ and we have

$$(-\Delta)^\alpha V_0(x) = \int_{\mathbb{R}^N\setminus\Omega} \frac{1}{|z-x|^{N+2\alpha}} dz, \quad x \in \Omega.$$

By direct computation, there exists $C > 1$ such that

$$\frac{1}{C} d(x)^{-2\alpha} \le (-\Delta)^\alpha V_0(x) \le C d(x)^{-2\alpha}, \quad x \in \Omega. \tag{4.3}$$

We see that $\tau_1 \in (\tau_0(\alpha), 0]$ and, if $\tau_1 < 0$, we have $\tau_1 - 2\alpha = \tau_0(\alpha)p$ and

$$\tau_1 - 2\alpha < \min\{\tau_0(\alpha), \tau_0(\alpha) - 2\alpha + \tau_0(\alpha) + 1\}.$$

Then, by Proposition 3.2 and (4.3), for $x \in A_\delta$, it follows that

$$(-\Delta)^\alpha U_\mu(x) + |U_\mu(x)|^{p-1} U_\mu(x) \ge -Ctd(x)^{\min\{\tau_0(\alpha), \tau_0(\alpha)-2\alpha+\tau_0(\alpha)+1\}} - C\mu d(x)^{\tau_1 - 2\alpha} + t^p d(x)^{\tau_0(\alpha)p}.$$

Thus, letting $\mu = t^p/(2C)$ if $\tau_1 < 0$ and $\mu = 0$ if $\tau_1 = 0$, for a possible smaller $\delta > 0$, we obtain

$$(-\Delta)^\alpha U_\mu(x) + |U_\mu(x)|^{p-1} U_\mu(x) \ge 0, \quad x \in A_\delta.$$

For the sub-solution, by Proposition 3.2 and (4.3), for $x \in A_\delta$, we have

$$(-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1} W_\mu(x) \le Ctd(x)^{\min\{\tau_0(\alpha), \tau_0(\alpha)-2\alpha+\tau_0(\alpha)+1\}} - \frac{\mu}{C} d(x)^{\tau_1 - 2\alpha} + t^p d(x)^{\tau_0(\alpha)p},$$

where $C > 1$. Then, for $\mu \ge 2Ct^p$ and a possibly smaller $\delta > 0$

$$(-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1} W_\mu(x) \le 0, \quad x \in A_\delta,$$

completing the proof. □

Proof of Theorem 1.2. We define U_μ and W_μ as in (4.1). In the case of a weak source, we take $\tau = \gamma + 2\alpha$ and we observe that $\gamma + 2\alpha \ge -\frac{2\alpha}{p-1} \ge \tau_0(\alpha)$ and $p(\gamma + 2\alpha) \ge \gamma$. Using Proposition 3.2 and (H3) we find that U_μ is a super-solution for $\mu > 0$ large (resp. W_μ is a sub-solution for $\mu > 0$ small) of (1.5) with the first equation in A_δ for $\delta > 0$ small. In the case of a strong source, we take $\tau = \frac{\gamma}{p}$ and observe that $\gamma < \frac{\gamma}{p} - 2\alpha$. Using Proposition 3.2 we find

$$|(-\Delta)^\alpha U_\mu|, |(-\Delta)^\alpha U_\mu| \le Cd(x)^{\frac{\gamma}{p} - 2\alpha}.$$

By (H3) we find that U_μ is a super-solution for μ large (resp. W_μ is a sub-solution for μ small) of (1.5) with the first equation in A_δ for δ small. □

Remark 4.1. In order to obtain the above existence results for classical solution to (1.4), that is, when g is not necessarily zero, we only need use the above results with F as a right hand side as given in (1.23). Here we only need to assume that g satisfies (H4). In fact, as above we find super and sub-solutions for (1.5), with f replaced by F. Then, as in the proof of Theorem 2.6, we find a viscosity solution of (1.5) and then $v = u + \tilde{g}$ is a viscosity solution of (1.4). Next we use Theorem 2.6 in [7] and then we use Theorem 2.1 to obtain that v is a classical solution of (1.4).

Remark 4.2. Now we compare Theorem 1.1 with the result in [15]. Let us assume that f and g satisfies hypothesis (F0)–(F2) and (G0)–(G3), respectively, given in [15]. We first observe that the function F, as defined above, satisfies (H1) thanks to (G0), (G3) and (F0). Next we see that F satisfies (H2), since (G2), (F1) and (F2) holds. Here we have to use Proposition 3.3. In the range of p given by (1.6), we then may apply Theorem 1.1 to obtain existence of a blow-up solution as given in Theorem 1.1 in [15]. We see that the existence is proved here, without assuming hypothesis (G1), thus we generalized this earlier result. Moreover, here we obtain a uniqueness and nonexistence of blow-up solution, if we further assume hypotheses on f and g, guaranteeing hypothesis (H2*) in Theorem 1.1. The complementary range of p is obtained using Theorem 1.2 for the existence of solutions as given in Theorem 1.1 in [15] and uniqueness and non-existence as in Theorem 1.3 and 1.4 are truly new results. The hypotheses needed on g to obtain (H3) for the function F are a bit stronger, since we are requiring in (H3) that the explosion rate is the same from above and from below, while in (G2) and (G4) they may be different.

5. Proof of uniqueness results

In this section we prove our uniqueness results, which are given in Theorem 1.1 and Theorem 1.3. These results are for positive solutions, so we assume that the external source f is non-negative. We assume that there are two positive solutions u and v of (1.5) and then define the set

$$\mathcal{A}=\{x\in\Omega,\ u(x)>v(x)\}. \tag{5.1}$$

This set is open, $\mathcal{A}\subset\Omega$ and we only need to prove that $\mathcal{A}=\emptyset$, to obtain that $u=v$, by interchanging the roles of u and v.

We will distinguish three cases, depending on the conditions satisfying u and v: Case a) u and v satisfy (1.6) and (1.7) (uniqueness part of Theorem 1.1), Case b) u and v (1.15) and (1.16) (weak source in Theorem 1.3) and Case c) u and v with (1.17)–(1.19) (strong source in Theorem 1.3).

We start our proof considering an auxiliary function

$$V(x)=\begin{cases} c(1-|x|^2)^3, & x\in B_1(0),\\ 0, & x\in B_1^c(0),\end{cases} \tag{5.2}$$

where the constant c may be chosen so that V satisfies

$$(-\Delta)^\alpha V(x)\le 1 \quad\text{and}\quad 0<V(0)=\max_{x\in\mathbb{R}^N}V(x). \tag{5.3}$$

In order to prove the uniqueness result in the three cases, we need first some preliminary lemmas.

Lemma 5.1. *Given $k>1$, if $\mathcal{A}_k=\{x\in\Omega, u(x)-kv(x)>0\}\neq\emptyset$, then*

$$\partial\mathcal{A}_k\cap\partial\Omega\neq\emptyset. \tag{5.4}$$

Proof. If (5.4) is not true, there exists $\bar{x}\in\Omega$ such that

$$u(\bar{x})-kv(\bar{x})=\max_{x\in\mathbb{R}^N}(u-kv)(x)>0,$$

Then, we have

$$(-\Delta)^\alpha(u-kv)(\bar{x})\ge 0,$$

which contradicts

$$\begin{aligned}(-\Delta)^\alpha(u-kv)(\bar{x})&=-u^p(\bar{x})+kv^p(\bar{x})-(k-1)f(\bar{x})\\ &\le -\big(k^p-k\big)v^p(\bar{x})<0. \qquad \square\end{aligned}$$

Lemma 5.2. *If $\mathcal{A}_k\neq\emptyset$, for $k>1$, then*

$$\sup_{x\in\Omega}(u-kv)(x)=+\infty. \tag{5.5}$$

Proof. Assume that $\bar{M} = \sup_{x\in\Omega}(u - kv)(x) < +\infty$. We see that $\bar{M} > 0$ and there is no point $\bar{x} \in \Omega$ achieving the supreme of $u - kv$, by the same argument given above. Let us consider $x_0 \in \mathcal{A}_k$, $r = d(x_0)/2$ and define

$$w_k = u - kv \quad \text{in } \mathbb{R}^N. \tag{5.6}$$

Under the conditions of Case a) and b) (resp. Case c)), for all $x \in B_r(x_0) \cap \mathcal{A}_k$ we have

$$(-\Delta)^\alpha w_k(x) = -u^p(x) + kv^p(x) + (1-k)f(x) \le -K_1 r^{\tau-2\alpha}, \tag{5.7}$$

(resp. $\le -K_1 r^\gamma$). Here we have used that $\tau = -2\alpha/(p-1)$ and, in Case a) (1.7) for v, in Case b) (H3) and (1.15) and in Case c) (H3). Moreover, in Case a) we have considered $K_1 = C(k^p - k)$ and in Cases b) and c) $K_1 = C(k-1)$ for some constant C. Now we define

$$w(x) = \frac{2\bar{M}}{V(0)} V\left(\frac{x - x_0}{r}\right)$$

for $x \in \mathbb{R}^N$, where V is given in (5.2), and we see that

$$w(x_0) = 2\bar{M} \tag{5.8}$$

and

$$(-\Delta)^\alpha w \le \frac{2\bar{M}}{V(0)} r^{-2\alpha}, \quad \text{in } B_r(x_0). \tag{5.9}$$

Since $\tau < 0$ ($\gamma < -2\alpha$ in the Case c)), by Lemma 5.1 we can take $x_0 \in \mathcal{A}_k$ close to $\partial\Omega$, so that

$$\frac{2\bar{M}}{V(0)} \le K_1 r^\tau \quad \left(\frac{2\bar{M}}{V(0)} \le K_1 r^{\gamma+2\alpha}, \text{ in Case c)}\right).$$

From here, combining (5.7) with (5.9), we have that

$$(-\Delta)^\alpha (w_k + w)(x) \le 0, \qquad x \in B_r(x_0) \cap \mathcal{A}_k.$$

Then, by the Maximum Principle, we obtain

$$w_k(x_0) + w(x_0) \le \max\left\{\bar{M}, \sup_{x\in B_r(x_0)\cap\mathcal{A}_k^c} (w_k + w)\right\}. \tag{5.10}$$

In case we have

$$\bar{M} < \sup_{x\in B_r(x_0)\cap\mathcal{A}_k^c} (w_k + w), \tag{5.11}$$

then

$$\begin{aligned} w(x_0) < (w_k + w)(x_0) &\le \sup_{x\in B_r(x_0)\cap\mathcal{A}_k^c} (w_k + w)(x) \\ &\le \sup_{x\in B_r(x_0)\cap\mathcal{A}_k^c} w(x) \le w(x_0), \end{aligned} \tag{5.12}$$

which is impossible. So that (5.11) is false and then, from (5.10) we get

$$w(x_0) < w_k(x_0) + w(x_0) \le \bar{M},$$

which is impossible in view of (5.8), completing the proof. □

Lemma 5.3. *There exists a sequence $\{C_n\}$, with $C_n > 0$, satisfying*

$$\lim_{n\to+\infty} C_n = 0 \tag{5.13}$$

and such that for all $x_0 \in \mathcal{A}_k$ and $k > 1$ we have

$$0 < \int_{Q_n} \frac{w_k(z) - M_n}{|z - x|^{N+2\alpha}} dz \le C_n r^{\tau - 2\alpha}, \quad \forall x \in B_r(x_0),$$

where we consider $r = d(x_0)/2$, $Q_n = \{z \in A_{r/n} \,/\, w_k(z) > M_n\}$ *and* $M_n = \max_{x \in \Omega \setminus A_{r/n}} w_k(x)$.

Proof. In Case a): we see that $Q_n \subset A_{r/n}$ and $\lim_{n\to+\infty} |Q_n| = 0$, so that using (1.10) we directly obtain

$$\int_{Q_n} \frac{w_k(z) - M_n}{|z - x|^{N+2\alpha}} dz \le C_0 r^{-N-2\alpha} \int_{A_{r/n}} d(z)^\tau dz \le C r^{-N-2\alpha} \int_0^{r/n} t^\tau t^{N-1} dt \le \frac{C}{n^{N+\tau}} r^{\tau-2\alpha},$$

where C depends on C_0 and $\partial\Omega$. We complete the proof defining $C_n = \frac{C}{n^{N+\tau}}$.

In Case b) we argue similarly using (1.16) and define C_n as before, while in Case c) we argue similarly using (1.19), but defining $C_n = \frac{C}{n^{N+\gamma/p}}$. $\square$

Now we are in a position to prove our non-existence results.

Proof of uniqueness results in Cases a), b) and c). We assume that $\mathcal{A} \neq \emptyset$, then there exists $k > 1$ such that $\mathcal{A}_k \neq \emptyset$. By Lemma 5.2 there exists $x_0 \in \mathcal{A}_k$ such that

$$w_k(x_0) = \max\{w_k(x) \,/\, x \in \Omega \setminus A_{d(x_0)}\}.$$

Proceeding as in Lemma 5.2 with the function

$$w(x) = \frac{K_1}{2} r^\tau V\left(\frac{x - x_0}{r}\right) \quad \text{and} \quad w(x) = \frac{K_1}{2} r^{\gamma+2\alpha} V\left(\frac{x - x_0}{r}\right), \text{ in Case c)},$$

we see that

$$(-\Delta)^\alpha (w_k + w)(x) \le -\frac{K_1}{2} r^{\tau-2\alpha}, \quad x \in B_r(x_0) \cap \mathcal{A}_k. \tag{5.14}$$

$$\text{and} \quad (-\Delta)^\alpha (w_k + w)(x) \le -\frac{K_1}{2} r^\gamma, \quad \text{in Case c)}. \tag{5.15}$$

With M_n, as given in Lemma 5.3, we define

$$\bar{w}_n(x) = \begin{cases} (w_k + w)(x), & \text{if } w_k(x) \le M_n, \\ M_n, & \text{if } w_k(x) > M_n, \end{cases} \tag{5.16}$$

for $n > 1$. By Lemma 5.3 we find n_0 such that

$$\begin{aligned}(-\Delta)^\alpha \bar{w}_{n_0}(x) &= (-\Delta)^\alpha (w_k + w)(x) + 2 \int_{Q_{n_0}} \frac{w_k(z) - M_{n_0}}{|z - x|^{N+2\alpha}} dz \\ &\le 0, \quad \text{in } B_r(x_0) \cap \mathcal{A}_k.\end{aligned}$$

In Case b) we have use (1.15) and in Case c) we have use (1.17), to get similar conclusion. Then, by the Maximum Principle, we get

$$\bar{w}_{n_0}(x_0) \le \max\Big\{M_{n_0}, \sup_{x \in B_r(x_0) \cap \mathcal{A}_k^c} (w_{k_0} + w)\Big\}.$$

Using the same argument as in (5.12), we conclude that

$$\sup_{x \in B_r(x_0) \cap \mathcal{A}_k^c} (w_{k_0} + w) > M_{n_0}$$

does not hold and therefore

$$\bar{w}_{n_0}(x_0) = w_k(x_0) + w(x_0) \le M_{n_0}. \tag{5.17}$$

Next, by the definition of M_n, we choose $x_1 \in \Omega \setminus A_{r/n_0}$ such that $w_k(x_1) = M_{n_0}$. But then we have

$$w_k(x_0) + w(x_0) \geq w(x_0) = \frac{K_1}{2} V(0) r^{\tau} \quad \text{in Case a) and b)}$$

$$\text{and} \quad w_k(x_0) + w(x_0) \geq w(x_0) = \frac{K_1}{2} V(0) r^{\gamma+2\alpha} \quad \text{in Case c).}$$

Thus, by the asymptotic behavior of v, (1.6) in Case a), (1.15) in Case b) and (1.17) in Case c), we have

$$r^{\tau} \geq n_0^{\tau} C v(x_1) \quad \text{and} \quad r^{\gamma+2\alpha} \geq r^{\gamma/p} \geq n_0^{\gamma/p} C v(x_1) \quad \text{in Case c).}$$

We recall that in Case a) $K_1 = C(k^p - k)$, so from (5.17)

$$u(x_1) > (1 + c_0) k v(x_1), \tag{5.18}$$

where $c_0 > 0$ is a constant, not depending on x_0 and increasing in k. Now we repeat this process above initiating by x_1 and $k_1 = k(1 + c_0)$. Proceeding inductively, we can find a sequence $\{x_m\} \subset \mathcal{A}$ such that

$$u(x_m) > (1 + c_0)^m k v(x_m),$$

which contradicts the common asymptotic behavior of u and v.

In the Case b) and c) recall that $K_1 = C(k - 1)$ and, as before, we can proceed inductively to find a sequence $\{x_m\} \subset \mathcal{A}$ such that

$$u(x_m) > (k + m c_0) v(x_m),$$

which again contradicts the common asymptotic behavior of u and v. □

6. Proof of our non-existence results

In this section we prove our non-existence results. Our arguments are based on the construction of some special super and sub-solutions and some ideas used in Section 5. The main portion of our proof is based on the following proposition that we state and prove next.

Proposition 6.1. *Assume that Ω is an open, bounded and connected domain of class C^2, $\alpha \in (0, 1)$, $p > 1$ and f is nonnegative. Suppose that U is a sub or super-solution of (1.5) satisfying $U = 0$ in Ω^c and (1.10) for some $\tau \in (-1, 0)$. Moreover, if $\tau > -\frac{2\alpha}{p-1}$, assume there are numbers $\epsilon > 0$ and $\delta > 0$ such that, in case U is a sub-solution of (1.5),*

$$(-\Delta)^{\alpha} U(x) \leq -\epsilon d(x)^{\tau-2\alpha} \quad \text{or} \quad f(x) \geq \epsilon d(x)^{\tau-2\alpha}, \quad \text{for } x \in A_{\delta}, \tag{6.1}$$

and in case U is a super-solution of (1.5),

$$(-\Delta)^{\alpha} U(x) \geq \epsilon d(x)^{\tau-2\alpha} \quad \text{and} \quad f(x) \leq \frac{\epsilon}{2} d(x)^{\tau-2\alpha}, \quad \text{for } x \in A_{\delta}. \tag{6.2}$$

Then there is no solution u of (1.5) such that, in case U is a sub-solution,

$$\begin{aligned} 0 < \liminf_{x \in \Omega,\ x \to \partial\Omega} u(x) d(x)^{-\tau} &\leq \limsup_{x \in \Omega,\ x \to \partial\Omega} u(x) d(x)^{-\tau} \\ &< \liminf_{x \in \Omega,\ x \to \partial\Omega} U(x) d(x)^{-\tau} \end{aligned} \tag{6.3}$$

or in case U is a super-solution,

$$\begin{aligned} 0 < \limsup_{x \in \Omega,\ x \to \partial\Omega} U(x) d(x)^{-\tau} &< \liminf_{x \in \Omega,\ x \to \partial\Omega} u(x) d(x)^{-\tau} \\ &\leq \limsup_{x \in \Omega,\ x \to \partial\Omega} u(x) d(x)^{-\tau} < \infty. \end{aligned} \tag{6.4}$$

We prove this proposition by a contradiction argument, so we assume that u is a solution of (1.5) satisfying (6.3) or (6.4), depending on the fact that U is a sub-solution or a super-solution. Since f is non-negative we have that $u > 0$ in Ω and by our assumptions on U, there is a constant $C_0 \geq 1$ so that, in case U is a sub-solution

$$C_0^{-1} \leq u(x)d(x)^{-\tau} < U(x)d(x)^{-\tau} \leq C_0, \quad x \in A_\delta \tag{6.5}$$

and, in case U is a super-solution

$$C_0^{-1} \leq U(x)d(x)^{-\tau} < u(x)d(x)^{-\tau} \leq C_0, \quad x \in A_\delta. \tag{6.6}$$

Here δ is decreased if necessary so that (6.1), (6.2), (6.5) and (6.6) hold. We define

$$\pi_k(x) = \begin{cases} U(x) - ku(x), & \text{in case } U \text{ is a sub-solution,} \\ u(x) - kU(x), & \text{in case } U \text{ is a super-solution,} \end{cases} \tag{6.7}$$

where $k \geq 0$. In order to prove Proposition 6.1, we need the following two preliminary lemmas.

Lemma 6.1. *Under the hypotheses of Proposition 6.1. If $\mathcal{A}_k = \{x \in \Omega \,/\, \pi_k(x) > 0\} \neq \emptyset$, for $k > 1$. Then,*

$$\partial\mathcal{A}_k \cap \partial\Omega \neq \emptyset. \tag{6.8}$$

The proof of this lemma follows the same arguments as the proof of Lemma 5.1 so we omit it.

Lemma 6.2. *Under the hypotheses of Proposition 6.1. If $\mathcal{A}_k \neq \emptyset$, for $k > 1$, then*

$$\sup_{x\in\Omega} \pi_k(x) = +\infty. \tag{6.9}$$

Proof. If (6.9) fails, then we have $M = \sup_{x\in\Omega} \pi_k(x) < +\infty$. We see that $M > 0$ and, as in Lemma 5.2, there is no point $\bar{x} \in \Omega$ achieving M. By Lemma 6.1 we may choose $x_0 \in \mathcal{A}_k$ and $r = d(x_0)/4$ such that $B_r(x_0) \subset A_\delta$, where r could be chosen as small as we want. Here δ is as in (6.1) and (6.2).

In what follows we consider $x \in B_r(x_0) \cap \mathcal{A}_k$ and we notice that $3r < d(x) < 5r$. We first analyze the case U is a sub-solution and $\tau \leq -\frac{2\alpha}{p-1}$. We have

$$\begin{aligned}(-\Delta)^\alpha \pi_k(x) &\leq -U^p(x) + ku^p(x) - (k-1)f(x) \\ &\leq -\big(k^{p-1} - 1\big)ku^p(x) \\ &\leq -C_0^{-p}\big(k^{p-1} - 1\big)kd(x)^{\tau p} \leq -K_1 r^{\tau - 2\alpha},\end{aligned}$$

where we have used $f \geq 0$, $k > 1$, (6.5), $K_1 = 5^{\tau-2\alpha}C_0^{-p}(k^{p-1} - 1)k > 0$ and C_0 is taken from (6.5). Next we consider the case U is a sub-solution and $\tau > -\frac{2\alpha}{p-1}$. By the first inequality in (6.1), we have

$$\begin{aligned}(-\Delta)^\alpha \pi_k(x) &\leq -\epsilon d(x)^{\tau-2\alpha} + ku^p(x) - kf(x) \\ &\leq -\big(\epsilon - kC_0^p r^{2\alpha-\tau+\tau p}\big)d(x)^{\tau-2\alpha} \leq -K_1 r^{\tau-2\alpha},\end{aligned}$$

where the last inequality is achieved by choosing r small enough so that $(\epsilon - kC_0^p r^{2\alpha-\tau+\tau p}) \geq \frac{\epsilon}{2}$ and $K_1 = 5^{\tau-2\alpha}\frac{\epsilon}{2}$. On the other hand, if the second inequality in (6.1) holds, we have

$$\begin{aligned}(-\Delta)^\alpha \pi_k(x) &\leq ku^p(x) - (k-1)\epsilon d(x)^{\tau-2\alpha} \\ &\leq -\big((k-1)\epsilon - kC_0^p r^{2\alpha-\tau+\tau p}\big)d(x)^{\tau-2\alpha} \leq -K_1 r^{\tau-2\alpha},\end{aligned}$$

where r satisfies $(k-1)\epsilon - kC_0^p r^{2\alpha-\tau+\tau p} \geq \frac{k-1}{2}\epsilon$ and $K_1 = 5^{\tau-2\alpha}\frac{k-1}{2}\epsilon$.

In case U is a super-solution and $\tau \leq -\frac{2\alpha}{p-1}$, we argue similarly to obtain

$$(-\Delta)^\alpha \pi_k(x) \leq -u^p(x) + kU^p(x) - (k-1)f(x) \leq -K_1 r^{\tau-2\alpha},$$

where $K_1 = 5^{\tau-2\alpha}C_0^{-p}(k^{p-1} - 1)k > 0$. Finally, in case U is a super-solution and $\tau > -\frac{2\alpha}{p-1}$, using (6.2) we find

$$(-\Delta)^\alpha \pi_k(x) \le -u^p(x) - k\epsilon d(x)^{\tau-2\alpha} + f(x) \le -K_1 r^{\tau-2\alpha},$$

with $K_1 = 5^{\tau-2\alpha}\frac{k}{2}\epsilon > 0$. Thus, in all cases we have obtained

$$(-\Delta)^\alpha \pi_k(x) \le -K_1 r^{\tau-2\alpha}, \quad x \in B_r(x_0) \cap \mathcal{A}_k, \tag{6.10}$$

for some $K_1 = K_1(k) > 0$ non-decreasing with k. From here we can argue as in Lemma 5.2 to get a contradiction □

Now proof of Proposition 6.1 is easy.

Proof of Proposition 6.1. From (6.10), recalling that K_1 non-decreasing with k, we can argue as in the proof of uniqueness result in Case b) to get a sequence (x_m) in A_δ such that, for some $k_0 > 1$ and $\bar{k} > 0$, in case U is a sub-solution we have

$$U(x_m) > (k_0 + m\bar{k})u(x_m)$$

and, in case U is a super-solution we have

$$u(x_m) > (k_0 + m\bar{k})U(x_m).$$

From here we obtain a contradiction with (6.5) or (6.6), for m large. □

Proof of non-existence part of Theorem 1.1. For any $t > 0$ we construct a sub-solution or super-solution U of (1.5) such that

$$\lim_{x\in\Omega, x\to\partial\Omega} U(x)d(x)^{-\tau} = t, \tag{6.11}$$

and U satisfies the assumption of Proposition 6.1, for different combinations of the parameters p and τ. For $t > 0$ and $\mu \in \mathbb{R}$ we define

$$U_{\mu,t} = tV_\tau + \mu V_0 \quad \text{in } \mathbb{R}^N, \tag{6.12}$$

where $V_0 = \chi_\Omega$ is the characteristic function of Ω and V_τ is defined in (3.4). It is obvious that (6.11) holds for $U_{\mu,t}$ for any $\mu \in \mathbb{R}$. To complete proof we show that for any $t > 0$, there is $\mu(t)$ such that $U_{\mu(t),t}$ is a sub-solution or super-solution of (1.5), depending on the zone to which (p, τ) belongs.

Zone 1: We consider $p > 1$ and $\tau \in (\tau_0(\alpha), 0)$. By Proposition 3.2 (ii), there exist $\delta_1 > 0$ and $C_1 > 0$ such that

$$(-\Delta)^\alpha V_\tau(x) > C_1 d(x)^{\tau-2\alpha}, \quad x \in A_{\delta_1}. \tag{6.13}$$

Combining with (H2*), for any $\mu > 0$, there exists $\delta_1 > 0$ depending on t such that

$$(-\Delta)^\alpha U_{\mu,t}(x) + U_{\mu,t}^p(x) - f(x) > C_1 t d(x)^{\tau-2\alpha} - Cd(x)^{-2\alpha} \ge 0, \quad x \in A_{\delta_1}.$$

On the other hand, since V_τ is of class C^2, f is continuous in Ω and $\Omega \setminus A_{\delta_1}$ is compact, there exists $C_2 > 0$ such that

$$|f|, \left|(-\Delta)^\alpha V_\tau(x)\right| \le C_2, \quad x \in \Omega \setminus A_{\delta_1}. \tag{6.14}$$

Then, using (4.3), there exists $\mu > 0$ such that

$$(-\Delta)^\alpha U_{\mu,t}(x) + U_{\mu,t}^p(x) - f(x) > -2C_2 + C_0\mu \ge 0, \quad x \in \Omega \setminus A_{\delta_1}. \tag{6.15}$$

We conclude that for any $t > 0$, there exists $\mu(t) > 0$ such that $U_{\mu(t),t}$ is a super-solution of (1.5) and, by (H2*) and (6.13), it satisfies (6.2).

Zone 2: We consider $p > 1 + 2\alpha$ and $\tau \in (-1, -\frac{2\alpha}{p-1})$. By Proposition 3.2 (i) and (ii), there exists $\delta_1 > 0$ depending on t such that

$$(-\Delta)^\alpha U_{\mu,t}(x) + U_{\mu,t}^p(x) - f(x) \ge -C_1 t d(x)^{\tau-2\alpha} + t^p d(x)^{\tau p} - Cd(x)^{-2\alpha} \ge 0, \tag{6.16}$$

for $x \in A_{\delta_1}$ and for any $\mu > 0$, where we used that $0 > \tau - 2\alpha > \tau p$. On the other hand, for $x \in \Omega \setminus A_{\delta_1}$, (6.15) holds for some $\mu > 0$ and so we have constructed a super-solution of (1.5).

Zone 3: We consider $1+2\alpha < p \le 1 - \frac{2\alpha}{\tau_0(\alpha)}$ and $\tau \in (-\frac{2\alpha}{p-1}, \tau_0(\alpha))$, which implies that $\tau p > \tau - 2\alpha$. By Proposition 3.2 (i) and $f \ge 0$ in Ω, there exists $\delta_1 > 0$ so that for all $\mu \le 0$

$$(-\Delta)^\alpha U_{\mu,t}(x) + U^p_{\mu,t}(x) - f(x) \le -C_1 t d(x)^{\tau-2\alpha} + t^p d(x)^{\tau p} \le 0, \tag{6.17}$$

for $x \in A_{\delta_1}$. Then, using (4.3) and (6.14), there exists $\mu = \mu(t) < 0$ such that

$$(-\Delta)^\alpha U_{\mu,t}(x) + U^p_{\mu,t}(x) - f(x) < 2C_2 + C_0\mu \le 0, \quad x \in \Omega \setminus A_{\delta_1}. \tag{6.18}$$

We conclude that for any $t > 0$, there exists $\mu(t) < 0$ such that $U_{\mu(t),t}$ is a sub-solution of (1.5) and it satisfies (6.1).

We see that Zone 1, 2 and 3 cover the range of parameters in part (i) of Theorem 1.1, completing the proof in the case.

Zone 4: To cover part (ii) of Theorem 1.1 we only need to consider $p = 1 - \frac{2\alpha}{\tau_0(\alpha)}$ with $\tau = \tau_0(\alpha) = -\frac{2\alpha}{p-1}$, which implies that $\tau p = \tau - 2\alpha < \min\{\tau - 2\alpha + \tau + 1, \tau\}$. By Proposition 3.2 (iii), there exists $\delta_1 > 0$ depending on t such that

$$\begin{aligned}(-\Delta)^\alpha U_{\mu,t}(x) + U^p_{\mu,t}(x) - f(x) \ge{}& -C_1 t d(x)^{\min\{\tau-2\alpha+\tau+1,\tau\}} + t^p d(x)^{\tau p} \\ &-Cd(x)^{-2\alpha} \ge 0, \quad x \in A_{\delta_1}\end{aligned}$$

for any $\mu > 0$. For $x \in \Omega \setminus A_{\delta_1}$, (6.15) holds for some $\mu > 0$, so we have constructed a super-solution of (1.5).

We see that Zones 1, 2 and 4 cover the parameters in part (ii) of Theorem 1.1, so the proof is complete in this case too.

Zone 5: We consider $1 < p \le 1 + 2\alpha$ and $\tau \in (-1, \tau_0(\alpha))$, which implies that $\tau p > \tau - 2\alpha$. By Proposition 3.2 (i) and $f \ge 0$ in Ω, there exists $\delta_1 > 0$ such that for all $\mu \le 0$ and $x \in A_{\delta_1}$, inequality (6.17) holds. Then, using (4.3) and (6.14), there exists $\mu = \mu(t) < 0$ such that (6.18) holds and we conclude that for any $t > 0$, there exists $\mu(t) < 0$ such that $U_{\mu(t),t}$ satisfies the first inequality of (6.1) and it is a sub-solution of (1.5).

We see that Zones 1 and 5 cover the parameters in part (iii) of Theorem 1.1. This completes the proof. □

Proof of Theorem 1.4. Here again we construct sub or super-solutions satisfying Proposition 6.1 to prove the theorem. In the case of a weak source, that is, part (i) of Theorem 1.4, we have $p \ge 1 - \frac{2\alpha}{\tau_0(\alpha)}$ and $-2\alpha - \frac{2\alpha}{p-1} \le \gamma < -2\alpha$, which implies that $-1 < \tau_0(\alpha) \le -\frac{2\alpha}{p-1} \le \gamma + 2\alpha < 0$. We consider two zones depending on τ.

Zone 1: we consider $\tau \in (\gamma + 2\alpha, 0)$, so we have $\gamma < \tau p$ and $\gamma < \tau - 2\alpha$. By Proposition 3.2 (ii) and (H3), we have that, for any $t > 0$ there exist $\delta_1 > 0$, $C_1 > 0$ and $C_2 > 0$ such that

$$(-\Delta)^\alpha U_{\mu,t}(x) + U^p_{\mu,t}(x) - f(x) \le C_1 t d(x)^{\tau-2\alpha} + t^p d(x)^{\tau p} - C_2 d(x)^\gamma \le 0, \tag{6.19}$$

for $x \in A_{\delta_1}$ and any $\mu \le 0$. On the other hand, using (4.3) and (6.14) we find $\mu = \mu(t) < 0$ such that (6.18) holds for $x \in \Omega \setminus A_{\delta_1}$. We conclude that for any $t > 0$, there exists $\mu(t) < 0$ such that $U_{\mu(t),t}$ is a sub-solution of (1.5) and by (H3), it satisfies (6.1).

Zone 2: We consider $\tau \in (-1, \gamma + 2\alpha)$. For $\tau \in (\tau_0(\alpha), \gamma + 2\alpha)$ in case $\tau_0(\alpha) < \gamma + 2\alpha$, by Proposition 3.2 (i) there exists $\delta_1 > 0$, depending on t, such that

$$(-\Delta)^\alpha U_{\mu,t}(x) + U^p_{\mu,t}(x) - f(x) \ge C_1 t d(x)^{\tau-2\alpha} - C_2 d(x)^\gamma \ge 0, \tag{6.20}$$

for $x \in A_{\delta_1}$ and any $\mu \ge 0$. For $\tau \in (-1, \tau_0(\alpha)] \cap (-1, \gamma + 2\alpha)$, we have $\tau p < \gamma$ and $\tau p < \tau - 2\alpha$, so by Proposition 3.2 (i) and (iii), there exists $\delta_1 > 0$ dependent of t such that (6.16) holds for any $\mu \ge 0$, while for $x \in \Omega \setminus A_{\delta_1}$, (6.15) holds for some $\mu > 0$. We conclude that for any $t > 0$, there exists $\mu(t) > 0$ such that $U_{\mu(t),t}$ is a super-solution of (1.5) and by (H3) it satisfies (6.2), completing the proof in the weak source case.

Next we consider the case of strong source, that is part (ii) of Theorem 1.4. Here we have that

$$-1 < \frac{\gamma}{p} < -\frac{2\alpha}{p-1} < 0.$$

Here again we have two zones, depending on the parameter τ.

Zone 1: We consider $\tau \in (\frac{\gamma}{p}, 0)$, in which case we have $\tau - 2\alpha > \gamma$ and $\tau p > \gamma$. Then there exist $\delta_1 > 0$, $C_1 > 0$ and $C_2 > 0$ such that (6.19) holds for any $\mu \le 0$ and using (4.3) and (6.14), there exists $\mu = \mu(t) < 0$ such that (6.18) holds for $x \in \Omega \setminus A_{\delta_1}$. Thus, for any $t > 0$ there exists $\mu(t) < 0$ such that $U_{\mu(t),t}$ is a sub-solution of (1.5) and (H3) implies the first inequality of (6.1).

Zone 2: we consider $\tau \in (-1, \frac{\gamma}{p})$, in which case we have $\tau p < \tau - 2\alpha$ and $\tau p < \gamma$. Then there exist $\delta_1 > 0$, $C_1 > 0$ and $C_2 > 0$ such that (6.20) holds for $x \in A_{\delta_1}$ and $\mu \geq 0$. We see also that for $x \in \Omega \setminus A_{\delta_1}$, inequality (6.15) holds for some $\mu > 0$ and so for any $t > 0$, there exists $\mu(t) > 0$ such that $U_{\mu(t),t}$ is a super-solution of (1.5).

This completes the proof of the theorem. $\square$

Conflict of interest statement

We certify that there is no conflict of interest with any financial organization regarding the material discussed in the manuscript.

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