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Traveling wave solutions of Allen–Cahn equation with a fractional Laplacian

Changfeng Gui *, Mingfeng Zhao

Department of Mathematics, University of Connecticut, 196 Auditorium Road, Unit 3009, Storrs, CT 06269-3009, United States

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Abstract

In this paper, we show the existence and qualitative properties of traveling wave solutions to the Allen–Cahn equation with fractional Laplacians. A key ingredient is the estimation of the traveling speed of traveling wave solutions.

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1. Introduction

Front propagation is a natural phenomenon which has appeared in phase transition, chemical reaction, combustion, biological spreading, etc. The mechanism of front propagation is often the competing effects of diffusion and reaction. Traveling wave solutions are typical profiles of physical states near the propagating fronts, and therefore are of great importance in the study of reaction diffusion processes. There has been a tremendous amount of literature on traveling wave solutions in mathematics as well as in various branches of applied sciences (see [41,66,2,4,39,9,10,8,48,14] and references therein). Traveling wave solutions are essential building blocks in various phase field models, and play an important role in pattern formation and phase separation (see, e.g., [5,28,38], etc., for the classical model and [62,80,81] for nonlocal models with fractional Laplacians). Other nonlocal phase transition models and related traveling wave solutions have been studied in [33,29,6,7,88], and others, where the kernels of convolution in the nonlocal operators are bounded, and in [43,44,31] where the kernels are periodic.

In the study of front propagation, traditionally the diffusion process is quite standard and normal, in the sense that the concerned particles or objects are engaged in a Brownian motion with a uniformly changed random variable. The resulting diffusion effect on the physical state, when represented by a function mathematically, is the operation of Laplacian on this function. Therefore, the difference of various reaction diffusion systems relies on the nonlinear

* Corresponding author.
E-mail addresses: gui@math.uconn.edu (C. Gui), mingfeng.zhao@uconn.edu (M. Zhao).

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reaction effect which varies in combustion, chemical reaction, phase transition, biological pattern formation, etc. In general, a typical reaction diffusion system is in the form of

$$u_t - \Delta u = f(u) \tag{1.1}$$

where $f(u)$ is a nonlinear function.

Recently, however, there has been a fast increasing number of studies on front propagation of reaction diffusion systems with an anomalous diffusion such as super diffusion, which plays important roles in various physical, chemical, biological and geological processes. (See, e.g., [75] for a brief summary and references therein.) Mathematically, such a super diffusion is related to Lévy process and may be modeled by a fractional Laplace operator $(-\Delta)^s u$ with $0 < s < 1$, whose Fourier transformation is $(2\pi|\xi|)^{2s}\widehat{u}$. (See [65] and [72], etc.) Below an exact definition of fractional Laplacians will be given.

In this paper, we study the traveling wave solutions of Allen–Cahn equation with a fractional Laplacian, where the nonlinear reaction is a bistable potential. If the front of a solution in large time propagates at constant speed, the solution is typically close to a profile depending on the distance away from the traveling fronts. Therefore we shall study only traveling wave solutions of one spatial variable, although more complicated traveling waves solutions do exist (see, e.g., [9,10,30,40,51,55–59,68,76,77,84,85,87,89,90,51] and references therein). More precisely, we are going to study the traveling wave solutions of the following reaction diffusion equation:

$$u_t(t,y) + (-\Delta)^s u(t,y) = f\big(u(t,y)\big), \quad \forall t>0,\ y\in\mathbf{R}, \tag{1.2}$$

where $0 < s < 1$, and $f \in C^2(\mathbf{R})$ is a bistable potential satisfying

$$f(-1) = f(1) = 0, \qquad f'(-1) < 0, \qquad f'(1) < 0. \tag{1.3}$$

Let $G(u) := -\int_{-1}^{u} f(t)dt$ and t_0 be the zero in $(-1,1)$ of $f = -G'$ closest to 1, from (1.3), it is easy to see that $G(t_0) > G(1)$. We shall focus on the unbalanced case where $G(1) > G(-1) = 0$ and consider the following condition

$$G(u) > G(-1) = 0, \quad \forall u \in (-1,1) \quad \text{and} \quad f(u) < 0, \quad \forall u \in \Sigma(G) := \big\{u \in (-1,1) : G(u) \le G(1)\big\}. \tag{1.4}$$

This condition means that G at all critical points in $(-1,1)$ of f has value greater than $G(1)$.

The fractional Laplacian is often defined by Fourier transformation, for any $0 < s < 1$ and $u \in \mathcal{S}(\mathbf{R}^n)$, the Schwartz space of rapidly decaying smooth functions, the fractional Laplacian $(-\Delta)^s u$ is defined in [67] by

$$\widehat{(-\Delta)^s u}(y) = \big(2\pi|y|\big)^{2s}\hat{u}(y), \quad \forall y \in \mathbf{R}^n.$$

It is well known that equivalently we have

$$(-\Delta)^s u(y) = C_{n,s}\,\text{P.V.}\int_{\mathbf{R}^n} \frac{u(y)-u(z)}{|y-z|^{n+2s}}\,dz, \quad \forall y \in \mathbf{R}^n, \tag{1.5}$$

where $C_{n,s} = \frac{s2^{2s}\Gamma(\frac{n+2s}{2})}{\pi^{\frac{n}{2}}\Gamma(1-s)}$ is a normalized constant. The above integral definition of fractional Laplacian can be used for more general functions, in particular, for $u \in C^2(\mathbf{R}^n)$.

Fractional Laplacian can also be defined as a Dirichlet to Neumann map. Define the n-dimensional *fractional Poisson kernel* $P^{n,s}$ as

$$P^{n,s}(x,y) = \frac{\Gamma(\frac{n+2s}{2})}{\pi^{\frac{n}{2}}\Gamma(s)} \frac{x^{2s}}{[x^2+|y|^2]^{\frac{n+2s}{2}}}, \quad \forall (x,y) \in \mathbf{R}^+ \times \mathbf{R}^n = \mathbf{R}^{n+1}_+.$$

The *s-harmonic extension* $\bar{u}$ of $u \in C^2(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ in $\mathbf{R}^{n+1}_+$ is given by

$$\bar{u}(x,y) = P^s(x,\cdot) * u(y), \quad \forall (x,y) \in \mathbf{R}^{n+1}_+.$$

By L'Hospital's rule and the dominated convergence theorem, we can get

$$\lim_{x\searrow 0} -x^{1-2s}\bar{u}_x(x,y) = d_n(s)(-\Delta)^s u(y), \quad \forall y \in \mathbf{R}^n, \tag{1.6}$$

where $d_n(s) = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$. This fact is well-known for $s = \frac{1}{2}$ and recently proved for all $s \in (0, 1)$ by Caffarelli and Silvestre in [26], where a key discovery is that $\bar{u}(x, y)$ satisfies

$$\begin{cases} \operatorname{div}\left[x^{1-2s}\nabla\bar{u}(x, y)\right] = 0, & \forall (x, y) \in \mathbf{R}^{n+1}_+, \\ \lim\limits_{x\searrow 0} \bar{u}(x, y) = u(y), & \forall y \in \mathbf{R}^n. \end{cases} \tag{1.7}$$

In this paper, both the s-harmonic extension form and the integral form of fractional Laplacians shall be used for different purposes.

We say that $u \in C^2(\mathbf{R}^2)$, is a *traveling wave solution* of the PDE (1.2) if u has the special form

$$u(t, y) = g(y - \mu t), \quad \forall (t, y) \in \mathbf{R}^2,$$

where g is a function of one variable. The constant μ is called the *speed* of the traveling wave, and the function g is called the profile of the traveling wave. It is easy to see that g satisfies

$$(-\Delta)^s g(y) - \mu g'(y) = f\big(g(y)\big), \quad \forall y \in \mathbf{R}. \tag{1.8}$$

Since the Allen–Cahn equation models a phenomenon of one stable state invading the other stable state, one may consider the following limiting condition for the wave profile

$$\lim_{y\to -\infty} g(y) = -1, \qquad \lim_{y\to\infty} g(y) = 1. \tag{1.9}$$

Traveling wave solutions for reaction diffusion equations with fractional Laplacians have been considered in many articles in last few years (see, e.g., [24,71,18,74,75]). When the nonlinear reaction f is combustive, i.e., there exists some $0 < \theta < 1$ such that

$$f(u) = 0, \quad \forall u \in [0, \theta] \cup \{1\}, \qquad f(u) > 0, \quad \forall u \in (\theta, 1), \qquad f'(1) < 0. \tag{1.10}$$

It is shown in [71] that when $s \in (1/2, 1)$, there exists a unique traveling wave solution (μ, g) to (1.8) with

$$\lim_{y\to -\infty} g(y) = 0, \qquad \lim_{y\to\infty} g(y) = 1. \tag{1.11}$$

On the other hand, recently it is shown in [53] that there does not exist a traveling wave solution for the combustion model if $s \in (0, 1/2]$. We note that the above limit is usually used for the combustion model or Fisher–KPP model below, since the concerned states are represented by 0 and 1 with only 1 being stable (monostable model); while the concerned states in Allen–Cahn model are often represented by -1 and 1 with both states being stable (bistable model). In the Allen–Cahn model, there is also another nodal point t_0 of f in $(-1, 1)$. However, this nodal point may represent an unstable state which is not the concerned state, since otherwise the equation may be regarded as Fisher–KPP equation by restricting u on $(t_0, 1)$, which will be discussed below.

Several traveling waves problems similar to the fractional diffusion reaction models are investigated recently. For example, in [24] and [18], traveling waves solutions to a nonlinear boundary reaction problem

$$\begin{cases} \bar{u}_t - \Delta\bar{u} = 0, & y \in \mathbf{R},\ x \geq 0, \\ \dfrac{\partial\bar{u}}{\partial\nu} = -f(\bar{u}), & y \in \mathbf{R},\ x = 0, \\ \lim\limits_{y\to -\infty} \bar{u}(0, y) = 0, & \lim\limits_{y\to\infty} \bar{u}(0, y) = 1, \end{cases} \tag{1.12}$$

are considered. Let $\bar{u}(x, y, t) = \bar{u}(x, y - \mu t)$, we know $\bar{u}$ satisfies

$$\begin{cases} \Delta\bar{u} + \mu\bar{u}_y = 0, & y \in \mathbf{R},\ x \geq 0, \\ \dfrac{\partial\bar{u}}{\partial\nu} = -f(\bar{u}), & y \in \mathbf{R},\ x = 0, \\ \lim\limits_{y\to -\infty} \bar{u}(0, y) = 0, & \lim\limits_{y\to\infty} \bar{u}(0, y) = 1. \end{cases} \tag{1.13}$$

It is shown in [71] that when f is a combustive nonlinearity, there exists a unique pair of solution $(\mu, \bar{u})$ to (1.13), which gives a traveling waves solution $\bar{u}$ to (1.12). Similar result for (1.12) with a bistable nonlinearity f is shown in [18]. Note that (1.13) and (1.7) with $s=\frac{1}{2}$ are very similar in nature. We recall that the classical combustion equation or Allen–Cahn equation both have a unique traveling wave solution u with a unique speed μ.

Another important nonlinear reaction is Fisher–KPP type, i.e.,

$$f(0)=f(1)=0, \qquad f(u)>0, \quad \forall u\in(0,1), \qquad f'(0)>0, \qquad f'(1)<0. \tag{1.14}$$

For the classical Fisher–KPP equation with the usual Laplacian, it is well-known that there exists a traveling wave solution u for any speed μ larger than or equal to some minimum speed $\mu_0>0$. It is shown that the front propagation speed could be very fast depending on initial values (see [12,60], etc.). However, Fisher–KPP equation with a fractional Laplacian displays a very different behavior, due to the super diffusion process involved. It was discovered numerically in [34,35,69,70] that the front propagation can accelerate exponentially in time. This phenomenon is rigorously studied and proved in [19] and [20]. Since a traveling wave front propagates linearly in t, it is an immediate consequence that there is no traveling wave solution for Fisher–KPP equation with a fractional Laplacian.

When the bistable nonlinearity is balanced, i.e., the associated double well potential G has two wells with equal depths $G(1)=G(-1)=0$, a traveling wave solution with one spatial variable for the classical Allen–Cahn equation is indeed a standing wave, i.e., the speed μ must be zero. Such a solution sometimes is called a layer solution as it describes a transition layer structure near the interface between two physical states (phases). If the regular Laplacian is replaced by fractional Laplacians $(-\Delta)^s$, it is shown in [23] for $s=1/2$ and [21,22] for $s\in(0,1)$ that a standing wave solution exists in high dimensions by the s-harmonic extension (see also [80,81] and [78]). In particular, it is shown that a multidimensional standing wave solution must be one dimensional (without counting the dimension of extension) for certain low dimensions and s (see also [16,17]). These results are analogues of a well studied phenomenon for the classical Allen–Cahn equation and usually referred to as the De Giorgi conjecture (see [32,45,3,46,79,36]). Numerical computations and asymptotical analysis performed in [49] also show that a layer solution exists for $s>\frac{1}{2}$.

A natural question is whether or not there is a traveling wave solution to the unbalanced fractional Allen–Cahn equation where $G(-1)\neq G(1)$.

In [75], when f is a piecewise linear bistable nonlinearity, exact traveling wave solutions for fractional Allen–Cahn equations are computed for $s\in[\frac{1}{2},1)$. On the other hand, using the extension formulation of the fractional Laplacian $(-\partial_{yy})^{\frac{1}{2}}u(y)=-\bar{u}_x(0,y)$ and an explicit harmonic function

$$\bar{u}(x,y)=\frac{2}{\pi}\arctan\frac{y}{x+1},$$

straightforward computations show that $u(y)=\bar{u}(0,y)=\frac{2}{\pi}\arctan y$ is a solution to

$$(-\partial_{yy})^{\frac{1}{2}}u-\mu u_y-f(\mu,u)=0,$$

where

$$f(\mu,u)=\frac{1}{\pi}\big\{\sin(\pi u)-\mu\big[\cos(\pi u)+1\big]\big\}$$

is a bistable nonlinearity for any constant μ. However, it is not clear how the existence of a traveling wave solution can be achieved for a general fractional Allen–Cahn equation. To see the difficulties, let us briefly recall the methods used for the construction of various traveling wave solutions. The classical method adapted in many text books (see, e.g., [37,86]) for the existence of traveling wave solutions of ordinary differential equation is phase plane analysis. This method actually works for very general nonlinear reaction including bistable, Fisher–KPP and combustive models. The local feature of this method obviously does not allow the generalization of the method to problems with nonlocal fractional Laplacians. The super sub solution method sometimes can be used to construct traveling wave solutions in both classical cases or nonlocal cases (see [11,71], etc.), however, it does not seem to work easily for the fractional Allen–Cahn model. Variational methods can sometimes be exploited for the construction of traveling wave solutions, in particular as minimizers of certain functionals. For example, in the classical Allen–Cahn equation, one may consider a functional

$$E(u):=\int_{\mathbf{R}}e^{\mu y}\left[\frac{1}{2}|u'|^2+G(u)\right]dy$$

for a proper μ, where u is among functions of a suitable set of function spaces such as $\phi_0 + H^1(\mathbf{R})$ with a smooth function ϕ_0 and

$$\phi_0(y) = -1, \quad \forall y \le -1; \qquad \phi_0(y) = 1, \quad \forall y \ge 1.$$

(See [1] for some recent work and reference therein.)

In [23,21,22], existence of standing wave solutions is shown by using a functional in the extension form of fractional Laplacians

$$\mathcal{E}_s(u) := \int_{\mathbf{R}} \frac{1}{2}\left|(-\partial_{yy}u)^{s/2}u(y)\right|^2 + G\big(u(y)\big)dy = \int_{\mathbf{R}^2_+} \frac{1}{2}x^{1-2s}\left|\nabla\bar{u}(x,y)\right|^2 dxdy + \int_{\mathbf{R}} G\big(u(y)\big)dy.$$

For the unbalanced fractional Allen–Cahn equation, one may try to add an exponential weight to the above functional as follows

$$\mathcal{E}_{\mu,s}(u) := \int_{\mathbf{R}^2_+} \frac{1}{2}e^{\mu y}x^{1-2s}\left|\nabla\bar{u}(x,y)\right|^2 dxdy + \int_{\mathbf{R}} e^{\mu y}G\big(u(y)\big)dy.$$

This, however, leads only to a different type of traveling wave solutions, namely, a solution to (1.13) for a heat equation with a nonlinear boundary reaction as in (1.12). This type of traveling wave solutions is indeed considered in [24] and [18].

In this paper, we shall use a continuation argument to show the existence of traveling wave solutions to (1.8). To be more precise, let G be a fixed unbalanced doubwell potential with $f = -G'$ such that $G(0) > G(u)$ for all $u \in (-1,0) \cup (0,1)$, and $G_0(u) = \frac{1}{4}(1-u^2)^2$, we shall consider a family of double well potentials $G_\theta, \theta \in [0,1]$ with $G_\theta(u) = (1-\theta)G_0(u) + \theta G(u)$. It is easy to see that for any $\theta \in [0,1]$, G_θ is a double well potential and $G_\theta(0) > G(u)$ for all $u \in (-1,0) \cup (0,1)$. Let $f_\theta(u) = -G'_\theta(u) = (1-\theta)(u-u^3) + \theta f(u)$ be the corresponding bistable nonlinear reaction. We shall consider a family of fractional Allen–Cahn equations

$$\begin{cases} (-\partial_{yy})^s u - \mu u_y - f_\theta(u) = 0, & y \in \mathbf{R}, \\ \lim\limits_{y\to\pm\infty} u(y) = \pm 1. \end{cases} \tag{1.15}$$

When $\theta = 0$, (1.15) has a standing wave solution, which is shown in [23,22] in general. Indeed, one can construct explicitly a traveling wave solution for any $s \in (0,1)$ with a corresponding nonlinear G_0 as shown in the example before, where for a special case $s = 1/2$ such a solution is computed explicitly. Based on the invertibility of the linearized equation of (1.15), using the implicit function theorem we can find a local branch of traveling wave solutions (μ_θ, u_θ) in a suitable function space for θ sufficiently small. Once we can show that the branch of solutions can be extended to $\theta = 1$, we obtain a traveling wave solution for (1.8) as desired. The key to this continuation argument is to have an apriori uniform bound of μ_θ for all possible traveling wave solutions (μ_θ, u_θ). To demonstrate the idea, let us look at the classical case of ordinary differential equation

$$\begin{cases} -u_{yy} - \mu u_y - f_\theta(u) = 0, & y \in \mathbf{R}, \\ \lim\limits_{y\to\pm\infty} u(y) = \pm 1. \end{cases} \tag{1.16}$$

By multiplying u_y on the both sides of the first equation in (1.16) and integrate, it is easy to see that

$$u_y^2(y) + 2\mu \int_{-\infty}^{y} u_y^2(s)ds = 2G_\theta\big(u(y)\big), \quad \forall y \in \mathbf{R}. \tag{1.17}$$

Letting $y \to \infty$, we have

$$\mu \int_{-\infty}^{\infty} u_y^2(s)ds = G_\theta(1) = \theta G(1) \ge 0. \tag{1.18}$$

Without loss of generality, assuming $u(0)=0$ and taking $y=0$ in (1.17), we have

$$u_y^2(0)\ge 2\big[G_\theta(0)-G_\theta(1)\big]=\frac{1-\theta}{2}+2\theta\big[G(0)-G(1)\big]>0.$$

Now let's assume that $u_y(y^0)=\sup_{y\in\mathbf{R}}u_y(y)\ge u_y(0)$, we get $u_{yy}(y_0)=0$. From (1.16), we see

$$\mu\le\frac{\sup_{u\in[-1,1]}[-f_\theta(u)]}{\sqrt{\frac{1-\theta}{2}+2\theta[G(0)-G(1)]}}\le\frac{2+\|f\|_{C([-1,1])}}{\sqrt{\min\{\frac{1}{2},2[G(0)-G(1)]\}}}=:\mu^*<\infty,\quad \forall\theta\in[0,1],\tag{1.19}$$

where μ^* only depends on G.

We note that continuation arguments have also been used in the study of nonlocal problems in [7,42], where a family of operators is used to connect nonlocal operators to the classical elliptic operators. Other related issues may be found in [15,25,27,30,40,50,52,54,61,68,73,86,87].

The main theorem can be stated as follows.

Theorem 1.1. *For any $0<s<1$, let $f=-G'\in C^2(\mathbf{R})$ be a bistable nonlinearity, i.e., condition (1.3) holds. Then there is a unique monotonically increasing traveling wave solution to (1.8) and (1.9) if (1.4) holds.*

The organization of this paper is as follows. In Section 2, we shall collect some useful facts on fractional Laplacian operators with an advection term. In Section 3, we shall show some basic properties of traveling wave solutions such as uniqueness, monotonicity, asymptotic behavior and nondegeneracy, etc. Some interesting identities related to traveling wave solutions shall be discussed in Section 4. Finally in Section 5, we shall show an uniform bound of the speeds of traveling wave solutions, and as a consequence prove the main theorem on the existence of a traveling wave solution.

2. Preliminaries

In this section, we shall collect some basic properties regarding the extension form of fractional Laplacians for all $0<s<1$.

In the following, for any $R>0$ and $y^0\in\mathbf{R}$, we use notations

$$B_R^+(0,y^0)=B_R(0,y^0)\cap\mathbf{R}_+^2\quad\text{and}\quad\Gamma_R^0(0,y^0)=\{(x,y)\in B_R(0,y^0):x=0\}.$$

The following form of Hopf lemma for degenerate elliptic equations is quoted from [21].

Proposition 2.1 *(Hopf lemma). (See Proposition 4.11 in [21].) Assume v satisfies*

$$\begin{cases}\operatorname{div}\big[x^{1-2s}\nabla v(x,y)\big]\le 0, & \forall(x,y)\in B_R^+(0,y^0),\\ v(x,y)>v(0,y^0), & \forall(x,y)\in B_R^+(0,y^0).\end{cases}$$

Then $\limsup_{x\searrow 0}-x^{1-2s}D_xv(x,y^0)<0$.

In order to study traveling wave solutions, we now state a slight variation of a maximum principle in [21] to include an advection term.

Proposition 2.2. *Let $c\in C(\mathbf{R})$, $d\in L^\infty(\mathbf{R})$ and $v\in C^2(\mathbf{R})$ satisfy*

$$\begin{cases}(-\Delta)^s v(y)-c(y)v_y(y)+d(y)v(y)\ge 0, & \forall y\in\mathbf{R},\\ \lim\limits_{|y|\to\infty}v(y)=0.\end{cases}$$

Assume there exists some subset $H\subset\mathbf{R}$ (maybe empty set) such that

$$v(y)\ge 0,\quad\forall y\in H\quad\text{and}\quad d(y)\ge 0,\quad\forall y\notin H.$$

Then either $v(y)>0$ in $\mathbf{R}$ or $v(y)\equiv 0$ in $\mathbf{R}$.

Proof. Case I: $v(y) \equiv 0$ in $\mathbf{R}$, we are done.

Case II: $v(y) \geq 0$ but $v(y) \not\equiv 0$ in $\mathbf{R}$. Since $\lim_{|y|\to\infty} v(y) = 0$, there exists some $y^0 \in \mathbf{R}$ such that $v(y^0) = \inf_{y\in\mathbf{R}} v(y)$. If $v(y^0) > 0$, we get $v(y) \geq v(y^0) > 0$ for all $y \in \mathbf{R}$, we are done. If $v(y^0) = 0$, since $v(y) \not\equiv 0$ in $\mathbf{R}$, we get $v_y(y^0) = 0$ and $(-\Delta)^s v(y^0) < 0$, which implies that

$$(-\Delta)^s v\big(y^0\big) + c\big(y^0\big)v_y\big(y^0\big) + d\big(y^0\big)v\big(y^0\big) < 0.$$

We get a contradiction.

Case III: $v(y^1) < 0$ for some $y^1 \in \mathbf{R}$. Since $\lim_{|y|\to\infty} v(y) = 0$, we know that u cannot be a constant function in $\mathbf{R}$ and there exists some $y^2 \in \mathbf{R}$ such that $v(y^2) = \inf_{y\in\mathbf{R}} v(y) < 0$. Hence we have

$$v_y\big(y^2\big) = 0 \quad \text{and} \quad (-\Delta)^s v\big(y^2\big) < 0.$$

Since $v(y^2) < 0$, we know $y^2 \notin H$, which implies that $d(y^2) \geq 0$. So we get

$$(-\Delta)^s v\big(y^2\big) + c\big(y^2\big)v_y\big(y^2\big) + d\big(y^2\big)v\big(y^2\big) < 0,$$

which contradicts with the assumption.

In summary, we know that either $u(y) > 0$ in $\mathbf{R}$ or $u(y) \equiv 0$ in $\mathbf{R}$. □

3. Uniqueness, asymptotic behavior and nondegeneracy of traveling wave solutions

In this section, we shall assume the existence of a traveling wave solution and show some basic properties of traveling wave solution such as uniqueness and asymptotic behavior. These properties will be used later to study the existence of solutions by a continuation method. Many of these properties are similar to those in [21], where standing wave solutions with $\mu = 0$ are considered.

Let $0 < s < 1$, $f \in C^2(\mathbf{R})$ be a bistable nonlinearity. We shall consider the following problem

$$\begin{cases} (-\Delta)^s u(y) - \mu u'(y) = f\big(u(y)\big), & \forall y \in \mathbf{R}, \\ u'(y) > 0, & \forall y \in \mathbf{R}, \\ \lim\limits_{y\to\pm\infty} u(y) = \pm 1. \end{cases} \tag{3.1}$$

3.1. Uniqueness of speed and solution

Theorem 3.1. *For $i = 1, 2$, let $u_i \in C^2(\mathbf{R})$ be solutions of (3.1) with speeds μ_i, respectively. Assume that $u_1(0) = u_2(0)$. Then $\mu_1 = \mu_2$ and $u_1(y) \equiv u_2(y)$ in $\mathbf{R}$.*

Proof. Without loss of generality, we can assume $\mu_2 \geq \mu_1$ and $u_1(0) = u_2(0) = 0$. For any $t \geq 0$, let $u_2^t(y) = u_2(y+t)$, $w(y) = u_2(y) - u_1(y)$ and $w^t(y) = u_2^t(y) - u_1(y)$ in $\mathbf{R}$, hence we know that $\lim_{|y|\to\infty} w^t(y) = 0$ and

$$(-\Delta)^s w^t(y) - \mu_2\big(w^t\big)'(y) - \big[f\big(u_2^t(y)\big) - f\big(u_1(y)\big)\big] = (\mu_2 - \mu_1)u_1'(y) \geq 0, \quad \forall y \in \mathbf{R}.$$

By the mean value theorem, we know that for any $y \in \mathbf{R}$, there exists some $0 < \theta^t(y) < 1$ such that

$$f\big(u_2^t(y)\big) - f\big(u_1(y)\big) = f'\big(u_2^t(y) + \theta^t(y)\big(u_2^t(y) - u_1(y)\big)\big)\big[u_2^t(y) - u_1(y)\big].$$

Define $d^t(y) := -f'(u_2^t(y) + \theta^t(y)(u_2^t(y) - u_1(y)))$ in $\mathbf{R}$, we have $\|d^t\|_{L^\infty(\mathbf{R})} \leq \|f'\|_{C([-1,1])}$ and

$$(-\Delta)^s w^t(y) - \mu_2\big(w^t\big)'(y) + d^t(y)w^t(y) \geq 0, \quad \forall y \in \mathbf{R}. \tag{3.2}$$

Claim I. *There exists some large $T_0 > 0$ such that for all $t \geq T_0$, we have $w^t(y) > 0$ in $\mathbf{R}$.*

Since $f'(\pm 1) < 0$, we can find some $0 < \tau < 1$ such that

$$f'(t) < 0, \quad \text{for all } \tau \leq |t| \leq 1.$$

For $i = 1, 2$, since $\lim_{y\to\pm\infty} u_i(y) = \pm 1$, there exists some large $Y_0 > 0$ such that $\tau < u_i(y) < 1$ for all $y \ge Y_0$, $-1 < u_i(y) < -\tau$ for all $y \le -Y_0$ and

$$A_0 := \max\left\{ \sup_{y\in[-Y_0,Y_0]} u_1(y), \sup_{y\in[-Y_0,Y_0]} u_2(y) \right\} < 1.$$

Since $\lim_{y\to\infty} u_2(y) = 1$, we can find some large $Y_1 > 0$ such that $u_2(y) > A_0$ for all $y \ge Y_1$. Let $T_0 = Y_1 + Y_0 > 0$, for all $t \ge T_0$ and all $y \in [-Y_0, Y_0]$, we have $y + t \ge Y_1$, which implies $u_2^t(y) = u_2(t + y) > A_0 \ge u_1(y)$ in $[-Y_0, Y_0]$, that is, $w^t(y) > 0$ in $[-Y_0, Y_0]$.

For any fixed $t \ge T_0$, consider the set $H^t := \{y \in \mathbf{R}\colon\ w^t(y) > 0\}$, we have $[-Y_0, Y_0] \subset H^t$, in particular, $H^t \ne \emptyset$, and for any fixed $y \notin H^t$, we have $|y| > Y_0$. If $y > Y_0$, we have $\tau < u_1(y), u_2^t(y) < 1$, hence $d^t(y) \ge 0$. If $y < Y_0$, we get $-1 < u_2^t(y), u_1(y) < -\tau$, which implies that $d^t(y) \ge 0$. In summary, we know that

$$w^t(y) > 0, \quad \forall y \in H^t \quad \text{and} \quad d^t(y) \ge 0, \quad \forall y \notin H^t.$$

By Proposition 2.2, we derive that $w^t(y) > 0$ in $\mathbf{R}$.

Claim II. *For any fixed $t > 0$ such that $w^t(y) > 0$ in $\mathbf{R}$, we can find some ϵ_t such that $t > \epsilon_t > 0$ and $w^{t+h}(y) > 0$ in $\mathbf{R}$ for all $|h| < \epsilon_t$.*

Since $u_2^t(y), u_1(y) \to \pm 1$, as $y \to \pm\infty$, and $0 < \tau < 1$, there exists some large $Y_2 > 0$ such that $Y_2 > Y_1$, $1 > u_2^t(y), u_1(y) > \frac{1+\tau}{2}$ for all $y \ge Y_2$ and $-1 < u_2^t(y), u_1(y) < -\frac{1+\tau}{2}$ for all $y \le -Y_2$. Consider the set

$$K^t := \left\{ y \in \mathbf{R}\colon\ \left|u_2^t(y)\right| \le \frac{1+\tau}{2} \text{ or } \left|u_1(y)\right| \le \frac{1+\tau}{2} \right\} \subset [-Y_2, Y_2].$$

Since $u_1(0) = 0$, then $0 \in K^t$, in particular, $K^t \ne \emptyset$. Since $w^t(y) > 0$ in $\mathbf{R}$, we have

$$A_2 := \inf_{y\in K^t} w^t(y) > 0.$$

Since $u_2 \in C^2(\mathbf{R})$, then u_2^t is uniformly continuous on $\mathbf{R}$, in particular, there exists some small $\epsilon_t > 0$ such that for all $t > \epsilon_t > 0$ and all $y, z \in \mathbf{R}$ with $|y - z| < \epsilon_t$, we have

$$\left|u_2^t(y) - u_2^t(z)\right| < \min\left\{ \frac{A_2}{2}, \frac{1-\tau}{2} \right\}.$$

Hence for all $h \in \mathbf{R}$ such that $|h| < \epsilon_t$, we have

$$\left|u_2^{t+h}(y) - u_2^t(y)\right| = \left|u_2^t(y+h) - u_2^t(y)\right| < \min\left\{ \frac{A_2}{2}, \frac{1-\tau}{2} \right\}, \quad \forall y \in \mathbf{R}.$$

For all $y \in K^t$, we have $w^t(y) \ge A_2$ and

$$w^{t+h}(y) = u_2^{t+h}(y) - u_2^t(y) + w^t(y) > -\frac{A_2}{2} + A_2 = \frac{A_2}{2} > 0.$$

For any fixed h such that $|h| < \epsilon_t$, define the set $H^{t+h} := \{y \in \mathbf{R}\colon\ w^{t+h}(y) > 0\}$, so we get $\emptyset \ne K^t \subset H^{t+h}$. For all fixed $y \notin H^{t+h}$, we get $h \notin K^t$, which implies that $|u_1(y)| \ge \frac{1+\tau}{2}$. If $u_1(y) \ge \frac{1+\tau}{2}$, since $w^t(y) > 0$, we have $u_2^t(y) > \frac{1+\tau}{2}$ and

$$u_2^{t+h}(y) = u_2^{t+h}(y) - u_2^t(y) + u_2^t(y) > -\frac{1-\tau}{2} + \frac{1+\tau}{2} = \tau.$$

Hence $d^{t+h}(y) > 0$. If $u_1(y) \le -\frac{1+\tau}{2}$, since $y \notin H^{t+h}$, we have $u_2^{t+h}(y) \le u_1(y) \le -\frac{1+\tau}{2} < -\tau$, which implies that $d^{t+h}(y) > 0$.

In summary, we have $w^{t+h}(y) > 0$ for all $y \in H^{t+h}$ and $d^{t+h}(y) > 0$ for all $y \notin H^{t+h}$. Since $\lim_{|y|\to\infty} w^{t+h}(y) = 0$ and $|h| < t$, by Proposition 2.2, we know that $w^{t+h}(y) > 0$ in $\mathbf{R}$.

Claim III. *For any $t > 0$ such that $w^t(y) \ge 0$ in $\mathbf{R}$, we have $w^t(y) > 0$ in $\mathbf{R}$.*

Case I: If $w^t(y) \equiv 0$ in $\mathbf{R}$ means $u_2(t+y) \equiv u_1(y)$ in $\mathbf{R}$. Since $u_1(0) = u_2(0) = 0$, we have $u_2(t) = 0$ with $t > 0$, which contradicts with $u_2'(y) > 0$ in $\mathbf{R}$.

Case II: If $w^t(y) \not\equiv 0$ in $\mathbf{R}$ and $w^t(y^1) = 0$ for some $y^1 \in \mathbf{R}$, that is, y^1 is a global minimum point of w^t in $\mathbf{R}$, which implies that $(w^t)'(y^1) = 0$ and $(-\Delta)^s w^t(y^1) < 0$. By (3.2), we get $(-\Delta)^s w^t(y^1) \geq 0$. we get a contraction.

In summary, we must have $w^t(y) > 0$ in $\mathbf{R}$.

Claim IV. *For all $t > 0$, we have $w^t(y) > 0$ in $\mathbf{R}$.*

Consider the set

$$S = \{t > 0: \ w^t(y) > 0 \text{ in } \mathbf{R}\}.$$

Claim I says that $[T_0, \infty) \subset S$, in particular, $S \neq \emptyset$. Claim II implies that S is open. Here we claim the closeness of S. Suppose that there exists some sequence $\{t_n\}_{n=1}^{\infty} \subset S$ such that $t_n \to t > 0$, as $n \to \infty$. For all $n \geq 1$, $u_2^{t_n}(y) = u_2(y + t_n) > u_1(y)$ in $\mathbf{R}$, we get $u_2^t(y) = u_2(y+t) \geq u_1(y)$ in $\mathbf{R}$. From Claim III, we know that $u_2^t(y) > u_1(y)$ in $\mathbf{R}$, that is, $t \in S$. Hence S is closed with respect to $(0, \infty)$. In summary, S is a nonempty, open and closed subset of $(0, \infty)$, which implies that $S = (0, \infty)$, that is, for all $t > 0$, we have $w^t(y) > 0$ in $\mathbf{R}$.

Claim IV implies that $w(y) = u_2(y) - u_1(y) \geq 0$ in $\mathbf{R}$. In summary, we have known that w satisfies

$$\begin{cases} (-\Delta)^s w(y) - \mu_2 w'(y) + d(y) w(y) \geq 0, & \forall y \in \mathbf{R}, \\ w(y) \geq 0, \quad \forall y \in \mathbf{R}, \\ \lim\limits_{|y| \to \infty} w(y) = 0, \qquad w(0) = 0. \end{cases}$$

By Proposition 2.2, we have $w(y) \equiv 0$ in $\mathbf{R}$, that is, $u_2(y) \equiv u_1(y)$ in $\mathbf{R}$. In particular, $\mu_1 = \mu_2$. This completes the proof of Theorem 3.1. $\square$

Remark 3.1. By similar arguments as in the proof of Theorem 3.1, we can show that if u satisfies

$$\begin{cases} (-\Delta)^s u(y) - \mu u'(y) = f\big(u(y)\big), & \forall y \in \mathbf{R}, \\ \big|u(y)\big| \leq 1, \quad \forall y \in \mathbf{R}, \\ \lim\limits_{y \to \pm\infty} u(y) = \pm 1. \end{cases}$$

Then $u'(y) > 0$ in $\mathbf{R}$, in particular, u is monotonically increasing in $\mathbf{R}$.

3.2. Asymptotic behavior of traveling wave solutions at infinity

Proposition 3.2. *Let $u \in C^2(\mathbf{R})$ be a solution of (3.1). Then there exists some constants $B > A > 0$ such that*

$$\frac{A}{|y|^{1+2s}} \leq u'(y) \leq \frac{B}{|y|^{1+2s}}, \quad \textit{for all } |y| \geq 1.$$

As a consequence, we have $u' \in L^p(\mathbf{R})$ for any $1 \leq p \leq \infty$, and

$$\frac{A}{y^{2s}} \leq 1 - u(y) \leq \frac{B}{y^{2s}}, \quad \forall y > 1 \quad \textit{and} \quad \frac{A}{|y|^{2s}} \leq 1 + u(y) \leq \frac{B}{|y|^{2s}}, \quad \forall y < -1.$$

Proof. For any $t > 0$, we define the functions

$$p_s^t(y) = \frac{1}{\pi} \int_0^{\infty} \cos(yr) e^{-tr^{2s}} \, dr \quad \text{and} \quad v_s^t(y) = -1 + 2 \int_{-\infty}^{y} p_s^t(r) \, dr \quad \text{in } \mathbf{R}.$$

By Theorem 3.1 in [22], we know that $p_s^t, v_s^t \in C^{\infty}(\mathbf{R})$ and there exists some $f_s^t \in C^2([-1, 1])$ which is an odd function in $[-1, 1]$ and satisfies

$$\begin{cases} f_s^t(0) = f_s^t(1) = 0, \qquad \big(f_s^t\big)'(\pm 1) = -\dfrac{1}{t} < 0, \\ f_s^t(u) > 0, \quad \forall u \in (0,1), \\ \big(f_s^t\big)''(1) = -\dfrac{\Gamma(4s)}{[\Gamma(2s)]^2}\dfrac{\pi}{t}\dfrac{\cos(\pi s)}{\sin(\pi s)}, \end{cases}$$

such that v_s^t is a layer solution in $\mathbf{R}$ with nonlinearity f_s^t, that is,

$$\begin{cases} (-\Delta)^s v_s^t(y) = f_s^t\big(v_s^t(y)\big), \quad y \in \mathbf{R}, \\ \big(v_s^t\big)'(y) > 0, \quad y \in \mathbf{R}, \\ \lim_{y\to\pm\infty} v_s^t(y) = \pm 1. \end{cases}$$

Moreover, we have

$$\lim_{|y|\to\infty} |y|^{1+2s}\big(v_s^t\big)'(y) = \frac{4ts\Gamma(2s)}{\pi}\sin(\pi s) > 0,$$
$$\lim_{y\to\pm\infty} |y|^{2s}\big|v_s^t(y) \mp 1\big| = \frac{2t\Gamma(2s)}{\pi}\sin(\pi s) > 0.$$

Let $\varphi_s^t(y) = (v_s^t)'(y)$ in $\mathbf{R}$, by Corollary 3.3 in [22], we know that there exists some $R_s^t > 0$ such that for all $y \in \mathbf{R}$ with $|y| \ge R_s^t > 0$, we have

$$(-\Delta)^s\varphi_s^t(y) + \frac{1}{2t}\varphi_s^t(y) \le 0 \le (-\Delta)^s\varphi_s^t(y) + \frac{2}{t}\varphi_s^t(y).$$

Since $f'(\pm 1) < 0$, then there exists some large $T_0 > 0$ such that for all $t \ge T_0$, we have

$$\frac{3}{t} < \min\big\{-f'(-1), -f'(1)\big\}.$$

For any $\delta > 0$, we define

$$w_{\delta,s}^t(y) = \delta\varphi_s^t(y) - u'(y), \quad \forall y \in \mathbf{R}.$$

Then

$$\begin{aligned}(-\Delta)^s w_{\delta,s}^t(y) &- \mu\big(w_{\delta,s}^t\big)'(y) + \frac{3}{t}w_{\delta,s}^t(y) \\ &= \delta\varphi_s^t(y)\left[\frac{2}{t} + \big(f_s^t\big)'\big(v_s^t(y)\big)\right] - u'(y)\left[\frac{3}{t} + f'\big(u(y)\big)\right] + \delta\left[\frac{1}{t}\varphi_s^t(y) - \mu\big(\varphi_s^t\big)'(y)\right].\end{aligned}$$

Since $\lim_{y\to\pm\infty} v_s^t(y) = \pm 1$, and $(f_s^t)'(\pm 1) = -\frac{1}{t} < 0$, then

$$\lim_{y\to\pm\infty}\left[\frac{2}{t} + \big(f_s^t\big)'\big(v_s^t(y)\big)\right] = \frac{1}{t} > 0.$$

Since $\lim_{y\to\pm\infty} u(y) = \pm 1$, and $\frac{3}{T_0} + f'(\pm 1) < 0$, then

$$\lim_{y\to\pm\infty}\left[\frac{3}{T_0} + f'\big(u(y)\big)\right] = \frac{3}{T_0} + f'(\pm 1) < 0.$$

Hence, there exists some $R_1 > 0$ such that for all $|y| \ge R_1$, we have

$$\frac{2}{T_0} + \big(f_s^{T_0}\big)'\big(v_s^{T_0}(y)\big) > 0 \quad \text{and} \quad \frac{3}{T_0} + f'\big(u(y)\big) < 0.$$

Since $\varphi_s^{T_0}(y) = (v_s^{T_0})'(y) > 0$ and $u'(y) > 0$ in $\mathbf{R}$, then for all $y \in \mathbf{R}$ such that $|y| \ge R_1$, we have

$$\delta\varphi_s^{T_0}(y)\left[\frac{2}{T_0} + \big(f_s^{T_0}\big)'\big(v_s^{T_0}(y)\big)\right] - u'(y)\left[\frac{3}{T_0} + f'\big(u(y)\big)\right] > 0.$$

By Corollary 3.3 in [22], then $v_s^{T_0}(y) = v_s^1({T_0}^{-\frac{1}{2s}} y)$ in $\mathbf{R}$ and

$$\varphi_s^{T_0}(y) = {T_0}^{-\frac{1}{2s}} \varphi_s^1\big({T_0}^{-\frac{1}{2s}} y\big) \quad \text{and} \quad \big(\varphi_s^{T_0}\big)'(y) = {T_0}^{-\frac{1}{s}} \big(\varphi_s^1\big)'\big({T_0}^{-\frac{1}{2s}} y\big) \quad \text{in } \mathbf{R}.$$

Notice that for all $y \in \mathbf{R}\backslash\{0\}$, we have

$$\big(\varphi_s^1\big)'(y) = -\frac{2}{\pi} \int_0^\infty r \sin(yr) e^{-r^{2s}}\, dr = -\frac{2 \sin y}{\pi} \int_0^\infty r \sin\big(|y| r\big) e^{-r^{2s}}\, dr = -\frac{2}{\pi} \frac{\operatorname{sign} y}{y^2} \int_0^\infty z \sin z e^{-(\frac{z}{|y|})^{2s}}\, dz.$$

So $(\varphi_s^1)'$ is an odd function in $\mathbf{R}\backslash\{0\}$. Now for any $y > 0$, using integration by parts, we have

$$\begin{aligned}
-\int_0^\infty z \sin z e^{-(\frac{z}{y})^{2s}}\, dz &= z e^{-(\frac{z}{y})^{2s}} \cos z\big|_0^\infty - \int_0^\infty \cos z \left[e^{-(\frac{z}{y})^{2s}} - z e^{-(\frac{z}{y})^{2s}} 2s \left(\frac{z}{y}\right)^{2s-1} \frac{1}{y}\right] dz \\
&= -\int_0^\infty \cos z e^{-(\frac{z}{y})^{2s}}\, dz + \frac{2s}{y^{2s}} \int_0^\infty z^{2s} \cos z e^{-(\frac{z}{y})^{2s}}\, dz \\
&= -e^{-(\frac{z}{y})^{2s}} \sin z\big|_0^\infty - \int_0^\infty \sin z e^{-(\frac{z}{y})^{2s}} 2s \left(\frac{z}{y}\right)^{2s-1} \frac{1}{y}\, dz \\
&\quad + \frac{2s}{y^{2s}} \left\{ z^{2s} e^{-(\frac{z}{y})^{2s}} \sin z\big|_0^\infty - \int_0^\infty \sin z \left[2s z^{2s-1} e^{-(\frac{z}{y})^{2s}} - z^{2s} e^{-(\frac{z}{y})^{2s}} 2s \left(\frac{z}{y}\right)^{2s-1} \frac{1}{y}\right] dz \right\} \\
&= -\frac{2s + 4s^2}{y^{2s}} \int_0^\infty z^{2s-1} \sin z e^{-(\frac{z}{y})^{2s}}\, dz + \frac{4s^2}{y^{4s}} \int_0^\infty z^{4s-1} \sin z e^{-(\frac{z}{y})^{2s}}\, dz.
\end{aligned}$$

Then we get

$$\big(\varphi_s^1\big)'(y) = -\frac{4s(1+2s)}{\pi} \frac{1}{y^{2+2s}} \int_0^\infty z^{2s-1} \sin z e^{-(\frac{z}{y})^{2s}}\, dz + \frac{8s^2}{\pi} \frac{1}{y^{2+4s}} \int_0^\infty z^{4s-1} \sin z e^{-(\frac{z}{y})^{2s}}\, dz, \quad \forall y > 0.$$

By taking $\kappa = 2, 4$ in Lemma 3.4 in [22], respectively, we can get

$$\lim_{y\to\infty} \int_0^\infty z^{2s-1} \sin z e^{-(\frac{z}{y})^{2s}}\, dz = \Gamma(s) \sin(s\pi),$$

$$\lim_{y\to\infty} \int_0^\infty z^{4s-1} \sin z e^{-(\frac{z}{y})^{2s}}\, dz = \Gamma(2s) \sin(2s\pi).$$

Hence,

$$\lim_{y\to\infty} y^{2+2s} \big(\varphi_s^1\big)'(y) = -\frac{4s(1+2s)}{\pi} \Gamma(s) \sin(s\pi).$$

Since $(\varphi_s^1)'$ is an odd function in $\mathbf{R}\backslash\{0\}$, then

$$\lim_{y\to-\infty} |y|^{2+2s} \big(\varphi_s^1\big)'(y) = -\frac{4s(1+2s)}{\pi} \Gamma(s) \sin(s\pi).$$

Hence, there exists some $C_1 > 0$ such that $\big||y|^{2+2s} (\varphi_s^1)'(y)\big| \le C_1$ in $\mathbf{R}$, that is,

$$\big|\big(\varphi_s^1\big)'(y)\big| \le \frac{C_1}{|y|^{2+2s}}, \quad \forall y \neq 0.$$

Since $(\varphi_s^{T_0})'(y) = T_0{}^{-\frac{1}{s}}(\varphi_s^1)'(T_0{}^{-\frac{1}{2s}}y)$ in $\mathbf{R}$, then we have

$$\left|\left(\varphi_s^{T_0}\right)'(y)\right| \le T_0{}^{-\frac{1}{s}} \frac{C_1}{(T_0{}^{-\frac{1}{2s}}y)^{2+2s}} = \frac{C_1 T_0}{|y|^{2+2s}}, \quad \forall y \neq 0.$$

Since $\lim_{|y|\to\infty} |y|^{1+2s}\varphi_s^{T_0}(y) = \frac{4T_0 s\Gamma(2s)}{\pi}\sin(s\pi) > 0$, then there exists some $R_2 > 0$ such that $R_2 > R_1 > 0$ and for all $y \in \mathbf{R}$ with $|y| \ge R_2$, we have

$$|y|^{1+2s}\varphi_s^{T_0}(y) \ge \frac{\frac{4T_0 s\Gamma(2s)}{\pi}\sin(s\pi)}{2} = \frac{2T_0 s\Gamma(2s)}{\pi}\sin(s\pi) > 0.$$

Then there exists some constant $C_2 > 0$ such that for all $y \in \mathbf{R}$ with $|y| \ge R_2$, we have

$$\frac{1}{T_0}\varphi_s^{T_0}(y) \ge \frac{C_2}{|y|^{1+2s}}.$$

Now look at $\frac{1}{T_0}\varphi_s^{T_0}(y) - \mu(\varphi_s^{T_0})'(y)$, for all $y \in \mathbf{R}$ with $|y| \ge R_2$,

$$\frac{1}{t}\varphi_s^{T_0}(y) - \mu\left(\varphi_s^{T_0}\right)'(y) \ge \frac{C_2}{|y|^{1+2s}} - \frac{\mu C_1 T_0}{|y|^{2+2s}} \ge \frac{C_2}{|y|^{2+2s}}\left[|y| - \frac{|\mu| C_1 T_0}{C_2}\right].$$

Taking $R_3 = 2\max\{R_2, \frac{|\mu|C_1T_0}{C_2}\}$, for all $y \in \mathbf{R}$ with $|y| \ge R_3$, we have

$$\frac{1}{T_0}\varphi_s^{T_0}(y) - \mu\left(\varphi_s^{T_0}\right)'(y) > 0.$$

In summary, we know that for all $\delta > 0$ and all $y \in \mathbf{R}$ with $|y| \ge R_3$, we have

$$(-\Delta)^s w_{\delta,s}^{T_0}(y) - \mu\left(w_{\delta,s}^{T_0}\right)'(y) + \frac{3}{T_0}w_{\delta,s}^{T_0}(y) > 0.$$

Since $\varphi_s^{T_0}(y) > 0$ in $\mathbf{R}$, then there exists some large $\delta_0 > 0$ such that for all $\delta \ge \delta_0$ and all $y \in \mathbf{R}$ with $|y| \le R_3 + 1$, we have $w_{\delta,s}^{T_0}(y) = \delta\varphi_s^{T_0}(y) - u'(y) \ge 1$. Hence $w_{\delta,s}^{T_0}$ satisfies

$$\begin{cases} (-\Delta)^s w_{\delta,s}^{T_0}(y) - \mu\left(w_{\delta,s}^{T_0}\right)'(y) + \dfrac{3}{T_0}w_{\delta,s}^{T_0}(y) > 0, & \text{if } |y| > R_3, \\ w_{\delta,s}^{T_0}(y) \ge 1 > 0, & \text{if } |y| \le R_3 + 1, \\ \lim\limits_{|y|\to\infty} w_{\delta,s}^{T_0}(y) = 0. \end{cases}$$

For any $y \in \mathbf{R}$ with $|y| \le R_3$, then $w_{\delta,s}^{T_0}(y) \ge 1$. Hence we can define

$$d_{\delta,s}^{T_0}(y) = \begin{cases} -\frac{(-\Delta)^s w_{\delta,s}^{T_0}(y) - c(w_{\delta,s}^{T_0})'(y)}{w_{\delta,s}^{T_0}(y)}, & \text{if } |y| \le R_3, \\ \frac{3}{T_0} > 0, & \text{if } |y| > R_3. \end{cases}$$

Hence, we get $d \in L^\infty(\mathbf{R})$, and for all $y \in \mathbf{R}$ with $|y| \le R_3$, we have

$$(-\Delta)^s w_{\delta,s}^{T_0}(y) - \mu\left(w_{\delta,s}^{T_0}\right)'(y) + d_{\delta,s}^{T_0}(y)w_{\delta,s}^{T_0}(y) = 0.$$

So $w_{\delta,s}^{T_0}$ satisfies

$$\begin{cases} (-\Delta)^s w_{\delta,s}^{T_0}(y) - \mu\left(w_{\delta,s}^{T_0}\right)'(y) + d_{\delta,s}^{T_0}(y)w_{\delta,s}^{T_0}(y) \ge 0, & y \in \mathbf{R}, \\ w_{\delta,s}^{T_0}(y) \ge 1 > 0, & \text{if } |y| < R_3, \\ d_{\delta,s}^{T_0}(y) > 0, & \text{if } |y| \ge R_3, \\ \lim\limits_{|y|\to\infty} w_{\delta,s}^{T_0}(y) = 0. \end{cases}$$

By Proposition 2.2, we know that $w_{\delta,s}^{T_0}(y) > 0$ in $\mathbf{R}$, that is, for all $\delta \geq \delta_0$, we have $u'(y) < \delta\varphi_s^{T_0}(y)$ in $\mathbf{R}$. In particular, there exists some $B > 0$ such that for all $y \in \mathbf{R}$ with $|y| \geq 1$, we have

$$u'(y) \leq \frac{B}{|y|^{1+2s}}.$$

On the other hand, for any $\delta > 0$, we may define

$$\tilde{w}_{\delta,s}^t(y) = -w_{\delta,s}^t(y), \quad y \in \mathbf{R}.$$

Then

$$(-\Delta)^s \tilde{w}_{\delta,s}^t(y) - \mu\big(\tilde{w}_{\delta,s}^t\big)'(y) + \frac{1}{4t}\tilde{w}_{\delta,s}^t(y)$$
$$= -\delta\varphi_s^t(y)\left[\frac{1}{2t} + \big(f_s^t\big)'\big(v_s^t(y)\big)\right] + u'(y)\left[\frac{1}{4t} + f'\big(u(y)\big)\right] + \delta\left[\frac{1}{4t}\varphi_s^t(y) + \mu\big(\varphi_s^t\big)'(y)\right], \quad y \in \mathbf{R}.$$

Since $v_s^t(y) \to \pm 1$, as $y \to \pm\infty$, and $(f_s^t)'(\pm 1) = -\frac{1}{t} < 0$, then

$$\lim_{y \to \pm\infty}\left[\frac{1}{2t} + \big(f_s^t\big)'\big(v_s^t(y)\big)\right] = -\frac{1}{2t} < 0.$$

Since $f'(\pm 1) < 0$, then there exists some small $T_1 > 0$ such that for all $t > 0$ with $t \leq T_1$, we have

$$\frac{1}{4t} > \max\big\{-f'(-1), -f'(1)\big\}.$$

Since $\lim_{y\to\pm\infty} u(y) = \pm 1$, then

$$\lim_{y \to \pm\infty}\left[\frac{1}{4T_1} + f'\big(u(y)\big)\right] = \frac{1}{4T_1} + f'(\pm 1) > 0.$$

Hence, there exists some large $R_3 > 0$ such that $R_4 > R_3$ and for all $y \in \mathbf{R}$ with $|y| \geq R_4$, we have

$$\frac{1}{2T_1} + \big(f_s^{T_1}\big)'\big(v_s^{T_1}(y)\big) < 0 \quad \text{and} \quad \frac{1}{4T_1} + f'\big(u(y)\big) > 0.$$

Since $u'(y) > 0$ and $\varphi_s^{T_1}(y) = (v_s^{T_1})'(y) > 0$ in $\mathbf{R}$, then for all $\delta > 0$ and all $y \in \mathbf{R}$ with $|y| \geq R_4$, we have

$$-\delta\varphi_s^{T_1}(y)\left[\frac{1}{2T_1} + \big(f_s^{T_1}\big)'\big(v_s^{T_1}(y)\big)\right] + u'(y)\left[\frac{1}{4T_1} + f'\big(u(y)\big)\right] > 0.$$

Now look at $\frac{1}{4T_1}\varphi_s^{T_1}(y) + \mu(\varphi_s^{T_1})'(y)$, for all $y \in \mathbf{R}$ with $|y| \geq R_4$, we have

$$\frac{1}{4T_1}\varphi_s^{T_1}(y) + \mu\big(\varphi_s^{T_1}\big)'(y) \geq \frac{C_4}{4|y|^{1+2s}} - \frac{|\mu|C_3T_1}{|y|^{2+2s}} \geq \frac{C_4}{4|y|^{2+2s}}\left[|y| - \frac{4|\mu|C_3T_1}{C_4}\right].$$

Taking $R_5 = 2\max\{R_4, \frac{4|\mu|C_3T_1}{C_4}\}$, then for all $y \in \mathbf{R}$ with $|y| \geq R_3$, we have

$$\frac{1}{4T_1}\varphi_s^{T_1}(y) + \mu\big(\varphi_s^{T_1}\big)'(y) > 0.$$

In summary, we know that for all $\delta > 0$ and all $y \in \mathbf{R}$ with $|y| \geq R_5$, we have

$$(-\Delta)^s \tilde{w}_{\delta,s}^{T_1}(y) - \mu\big(\tilde{w}_{\delta,s}^{T_1}\big)'(y) + \frac{1}{4T_1}\tilde{w}_{\delta,s}^{T_1}(y) > 0.$$

Since $u'(y) > 0$ for all $y \in \mathbf{R}$, then there exists some $\delta_1 > 0$ such that for all $\delta \leq \delta_1$ and all $y \in \mathbf{R}$ with $|y| \leq R_5 + 1$, we have

$$w_{\delta,s}^{T_1}(y) = \delta\varphi_s^{T_1}(y) - u'(y) < 0.$$

Let $\epsilon_{\delta_1,s}^{T_1} = \sup_{|y|\le R_5+1} w_{\delta_1,s}^{T_1}(y) < 0$, then for all $\delta \le \delta_1$ and all $y \in \mathbf{R}$ with $|y| \le R_5 + 1$, we have

$$\tilde{w}_{\delta,s}^{T_1}(y) = -w_{\delta,s}^{T_1}(y) \ge -\epsilon_{\delta_1,s}^{T_1} > 0.$$

Then $\tilde{w}_{\delta,s}^{T_1}$ satisfies

$$\begin{cases} (-\Delta)^s \tilde{w}_{\delta,s}^{T_1}(y) - \mu\big(\tilde{w}_{\delta,s}^{T_1}\big)'(y) + \dfrac{1}{4T_1}\tilde{w}_{\delta,s}^{T_1}(y) > 0, & \text{for all } |y| > R_5, \\ \tilde{w}_{\delta,s}^{T_1}(y) \ge -\epsilon_{\delta_1,s}^{T_1} > 0, & \text{for all } |y| < R_5 + 1, \\ \lim\limits_{|y|\to\infty} \tilde{w}_{\delta,s}^{T_1}(y) = 0. \end{cases}$$

For all $y \in \mathbf{R}$ with $|y| \le R_5$, since $\tilde{w}_{\delta,s}^{T_1}(y) \ge -\epsilon_{\delta_1,s}^{T_1}$, then we can define

$$d_{\delta,s}^{T_1}(y) = \begin{cases} -\dfrac{(-\Delta)^s \tilde{w}_{\delta,s}^{T_1}(y) - \mu(\tilde{w}_{\delta,s}^{T_1})'(y)}{\tilde{w}_{\delta,s}^{T_1}(y)}, & \text{if } |y| \le R_5, \\ \frac{1}{4T_1} > 0, & \text{if } |y| > R_5. \end{cases}$$

Hence, we get $d \in L^\infty(\mathbf{R})$ and $\tilde{w}_{\delta,s}^{T_1}$ satisfies the PDE:

$$\begin{cases} (-\Delta)^s \tilde{w}_{\delta,s}^{T_1}(y) - \mu\big(\tilde{w}_{\delta,s}^{T_1}\big)'(y) + d_{\delta,s}^{T_1}(y)\tilde{w}_{\delta,s}^{T_1}(y) \ge 0, & \text{for all } y \in \mathbf{R}, \\ \tilde{w}_{\delta,s}^{T_1}(y) \ge -\epsilon_{\delta_1,s}^{T_1} > 0, & \text{for all } |y| < R_5, \\ d_{\delta,s}^{T_1}(y) > 0, & \text{for all } |y| \ge R_5, \\ \lim\limits_{|y|\to\infty} \tilde{w}_{\delta,s}^{T_1}(y) = 0. \end{cases}$$

By Proposition 2.2, then $\tilde{w}_{\delta,s}^{T_1}(y) > 0$ in $\mathbf{R}$, that is, for all $0 < \delta \le \delta_1$, we have $u'(y) > \delta\varphi_s^{T_1}(y)$ in $\mathbf{R}$. In particular, there exists some $A > 0$ such that for all $y \in \mathbf{R}$ with $|y| \ge 1$, we have

$$u'(y) \ge \frac{A}{|y|^{1+2s}}.$$

This completes the proof. □

Remark 3.2. Let $1 \le p \le \infty$, by Proposition 3.2, we know that $u' \in L^p(\mathbf{R})$ for all $0 < s < 1$, but $f(u(\cdot)) \in L^p(\mathbf{R})$ if and only if $\frac{1}{2p} < s < 1$.

Proposition 3.3. *Let $s = \frac{1}{2}$, $u \in C^2(\mathbf{R})$ be a solution of (3.1), and $\bar{u}$ be the harmonic extension in $\mathbf{R}_+^2$ of u. Then we have*

$$\int_{\mathbf{R}} \big[\bar{u}_x(x,y)\big]^2\,dy = \int_{\mathbf{R}} \big[\bar{u}_y(x,y)\big]^2\,dy, \quad \forall x \ge 0.$$

Proof. Since $-\bar{u}_x(0,y) = (-\Delta)^{\frac{1}{2}}u(y) = \mu u'(y) + f(u(y))$ in $\mathbf{R}$, $u' \in L^2(\mathbf{R})$ and Remark 3.2, we know that $\bar{u}_x(0,\cdot) \in L^2(\mathbf{R})$. Since $\bar{u}$ is harmonic in $\mathbf{R}_+^2$, then $\bar{u}_x(x,\cdot) \in L^2(\mathbf{R})$ for all $x \ge 0$. In particular, we can consider the function

$$\psi(x) = \int_{\mathbf{R}} \big(\big[\bar{u}_x(x,y)\big]^2 - \big[\bar{u}_y(x,y)\big]^2\big)dy, \quad \forall x \ge 0.$$

Notice that $\bar{u} \in C^2(\overline{\mathbf{R}_+^2})$ and $\bar{u}_y(x,y) \to 0$, as $|y| \to \infty$. Differentiating ψ and using integration by parts, we get

$$\begin{aligned}\psi'(x) &= 2\int_{\mathbf{R}} \bar{u}_x(x,y)\bar{u}_{xx}(x,y)\,dy - 2\int_{\mathbf{R}} \bar{u}_y(x,y)\bar{u}_{xy}(x,y)\,dy \\ &= 2\int_{\mathbf{R}} \bar{u}_x(x,y)\bar{u}_{xx}(x,y)\,dy + 2\int_{\mathbf{R}} \bar{u}_x(x,y)\bar{u}_{yy}(x,y)\,dy \\ &= 2\int_{\mathbf{R}} \bar{u}_x(x,y)\Delta\bar{u}(x,y)\,dy = 0, \quad \forall x \ge 0.\end{aligned}$$

Claim I. $\lim_{x\to\infty}\int_{\mathbf{R}} |\nabla\bar{u}(x,y)|^2\,dy = 0$.

Let $P(x,y) = P^{\frac{1}{2}}(x,y) = \frac{1}{\pi}\cdot\frac{x}{x^2+y^2}$ in $\mathbf{R}^2_+$, we get

$$\|P(x,\cdot)\|^2_{L^2(\mathbf{R})} = \frac{1}{\pi^2}\int_{\mathbf{R}} \frac{x^2}{[x^2+y^2]^2}\,dy = \frac{1}{\pi^2 x}\int_{\mathbf{R}} \frac{1}{[1+y^2]^2}\,dy = \frac{1}{2\pi x} \to 0, \quad \text{as } x\to\infty.$$

By Young's inequality, we have

$$\|\bar{u}_y(x,\cdot)\|_{L^2(\mathbf{R})} = \|P(x,\cdot)*u'\|_{L^2(\mathbf{R})} \le \|P(x,\cdot)\|_{L^2(\mathbf{R})}\cdot\|u'\|_{L^1(\mathbf{R})} \to 0, \quad \text{as } x\to\infty.$$

Look at $\bar{u}_x(x,y)$, by Proposition 3.2, we know that there exists some constant $C_1 > 0$ such that

$$|\bar{u}_x(0,y)| = |(-\Delta)^{\frac{1}{2}}u(y)| = |\mu u'(y) + f(u(y))| \le \frac{C_1}{1+|y|}, \quad \forall y\in\mathbf{R}.$$

Since $\bar{u}_x$ is harmonic in $\mathbf{R}^2_+$, so there exists some constant $C > C_1 > 0$ such that

$$|\bar{u}_x(x,y)| \le \frac{C}{1+x}, \quad \forall (x,y)\in\mathbf{R}^2_+.$$

Since $\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$, we know that $\frac{x}{x^2+y^2}$ and $\frac{y}{x^2+y^2}$ are harmonic in $\mathbf{R}^2_+$. Let $v(x,y) = \frac{C(x+1+y)}{(x+1)^2+y^2}$, we know that v is harmonic in $\mathbf{R}^2$ and $\pm\bar{u}_x(0,y) \le \frac{C_1}{1+y} \le \frac{C(1+y)}{1+y^2} = v(0,y)$ for all $y \ge 0$ and $\pm\bar{u}_x(x,0) \le \frac{C}{1+x} = v(x,0)$ for all $x \ge 0$. By the weak maximum principle, we know that

$$\pm\bar{u}_x(x,y) \le v(x,y) = \frac{C(x+1+y)}{(x+1)^2+y^2}, \quad \forall x\ge 0,\ \forall y \ge 0.$$

Let $w(x,y) = \frac{C(x+1-y)}{(x+1)^2+y^2}$, we get w is harmonic in $\mathbf{R}^2$ and $\pm\bar{u}_x(0,y) \le \frac{C_1}{1-y} \le \frac{C(1-y)}{1+y^2} = w(0,y)$ for all $y \le 0$ and $\pm\bar{u}_x(x,0) \le \frac{C}{1+x} = w(x,0)$ for all $x \ge 0$. By the weak maximum principle, we know that

$$\pm\bar{u}_x(x,y) \le w(x,y) = \frac{C(x+1-y)}{(x+1)^2+y^2}, \quad \forall x\ge 0,\ \forall y \le 0.$$

In summary, we have

$$|\bar{u}_x(x,y)| \le \frac{C(x+1+|y|)}{(x+1)^2+y^2} \le \frac{C}{|(x+1,y)|}, \quad \forall (x,y)\in\mathbf{R}^2_+.$$

Notice that

$$\int_{\mathbf{R}} \frac{1}{|(x+1,y)|^2}\,dy \le \int_{\mathbf{R}} \frac{1}{x^2+y^2}\,dy = \frac{1}{x}\int_{\mathbf{R}} \frac{1}{1+y^2}\,dy = \frac{\pi}{x} \to 0, \quad \text{as } x\to\infty.$$

So we know that $\|\bar{u}_x(x,\cdot)\|_{L^2(\mathbf{R})} \to 0$, as $x\to\infty$. In summary, we have $\lim_{x\to\infty}\int_{\mathbf{R}} |\nabla\bar{u}(x,y)|^2\,dy = 0$.
By Claim I, we get $\psi(x) \equiv 0$ in $\mathbf{R}^+$, which completes the proof. □

3.3. Nondegeneracy

Proposition 3.4. *Let $u \in C^2(\mathbf{R})$ be a solution of (3.1), $h \in \mathbf{R}$ and $\phi \in C^2(\mathbf{R})$ such that $\phi(0) = 0$, $\lim_{|y|\to\infty} \phi(y) = 0$ and*

$$(-\Delta)^s \phi(y) - \mu\phi'(y) - f'\big(u(y)\big)\phi(y) + hu'(y) = 0, \quad \forall y \in \mathbf{R}.$$

Then $h = 0$ and $\phi(y) \equiv 0$ in $\mathbf{R}$.

Proof. Case I: If $h \geq 0$. Let $v(y) = u'(y) > 0$ in $\mathbf{R}$, we have

$$(-\Delta)^s \phi(y) - \mu\phi'(y) - f'\big(u(y)\big)\phi(y) = -hv(y) \leq 0 = (-\Delta)^s v(y) - \mu v'(y) - f'\big(u(y)\big)v(y), \quad \forall y \in \mathbf{R}.$$

For any $\delta > 0$, we consider the function $w_\delta(y) := v(y) - \delta\phi(y)$ in $\mathbf{R}$, we have $\lim_{|y|\to\infty} w_\delta(y) = 0$. Since $f'(\pm1) < 0$ and $\lim_{|y|\to\infty} u(y) = \pm1$, we can find some fixed large $R_0 > 0$ such that for all y with $|y| \geq R_0$, we have $-f'(u(y)) > 0$. Since $v(y) > 0$ in $\mathbf{R}$, there exists some fixed small $\epsilon_0 > 0$ such that for all δ with $0 < |\delta| \leq \epsilon_0$ and all y with $|y| \leq R_0$, we have $w_\delta(y) > 0$. By Proposition 2.2, we know that

$$w_\delta(y) > 0, \quad \forall y \in \mathbf{R}.$$

Consider

$$\Lambda = \sup\big\{\epsilon > 0 : w_\delta(y) > 0 \text{ in } \mathbf{R},\ \forall \delta \in (0, \epsilon)\big\} \geq \epsilon_0.$$

Subcase I: If $\Lambda = \infty$, in this subcase, we know that for all $\delta > 0$ and all $y \in \mathbf{R}$, we have $w_\delta(y) > 0$, which implies that $\phi(y) \leq 0$ in $\mathbf{R}$. Since $\phi(0) = 0$, by Proposition 2.2, we know that $\phi(y) \equiv 0$ in $\mathbf{R}$. Hence $h \equiv 0$.

Subcase II: If $\Lambda < \infty$, then $w_\Lambda(y) \geq 0$ in $\mathbf{R}$. Since $\phi(0) = 0$, $v(y) > 0$ in $\mathbf{R}$, Proposition 2.2, we know $w_\Lambda(y) > 0$ in $\mathbf{R}$. Using the previous argument, we can find a larger $\Lambda' > \Lambda$ such that for all $\delta \in (0, \Lambda')$, we have $w_\delta(y) > 0$ in $\mathbf{R}$, which contradicts with the definition of Λ.

Hence, in this case, we know that $h = 0$ and $\phi(y) \equiv 0$ in $\mathbf{R}$.

Case II: If $h \leq 0$. In this case, let $k = -h \geq 0$ and $\psi(y) = -\phi(y)$ in $\mathbf{R}$, then k and ψ satisfies $\lim_{|y|\to\infty} \psi(y) = 0$ and

$$(-\Delta)^s \psi(y) - \mu\psi'(y) - f'\big(u(y)\big)\psi(y) + ku'(y) = 0, \quad \forall y \in \mathbf{R}.$$

Applying the result of Case I, we know that $k = 0$ and $\psi(y) \equiv 0$ in $\mathbf{R}$, which implies that, $h = 0$ and $\phi(y) \equiv 0$ in $\mathbf{R}$.

In summary, we can conclude that $h = 0$ and $\phi(y) \equiv 0$ in $\mathbf{R}$. $\square$

4. Hamiltonian identity and Modica-type estimate

In this section, we will assume that $0 < s < 1$, $\mu \in \mathbf{R}$, $f \in C^2(\mathbf{R})$ is a bistable potential, $G \in C^3(\mathbf{R})$ with $G' = -f$, $u \in C^2(\mathbf{R})$ and $\bar{u}$ is the s-harmonic extension of u, and u satisfies

$$\begin{cases} (-\Delta)^s u(y) - \mu u'(y) = f\big(u(y)\big), & \forall y \in \mathbf{R}, \\ u'(y) > 0, & \forall y \in \mathbf{R}, \\ \lim_{y\to\pm\infty} u(y) = \pm1. \end{cases} \tag{4.1}$$

Following [21], we show similar Hamiltonian identity and Modica-type estimate for traveling wave solution.

Proposition 4.1 *(Hamiltonian identity). For all $y \in \mathbf{R}$, we have*

$$\frac{1}{2}\int_0^\infty \big[\bar{u}_y^2(t, y) - \bar{u}_x^2(t, y)\big]t^{1-2s}\,dt = d_1(s)\left[-\mu\int_{-\infty}^y \big[\bar{u}_y(0, r)\big]^2\,dr + G\big(\bar{u}(0, y)\big) - G(-1)\right],$$

and

$$\mu \int_{-\infty}^{\infty} \left[\bar{u}_y(0,r)\right]^2 dr = \mu \int_{\mathbf{R}} |u'(y)|^2 dy = G(1) - G(-1).$$

Proof. Since $\bar{u}$ is the s-harmonic extension in $\mathbf{R}^2_+$ of u, then $\bar{u} \in C^2(\mathbf{R}^2_+) \cap C(\overline{\mathbf{R}^2_+})$ and satisfies

$$\begin{cases} \operatorname{div}\left[x^{1-2s}\nabla \bar{u}(x,y)\right] = 0, & \forall (x,y) \in \mathbf{R}^2_+, \\ \lim_{x \searrow 0} -x^{1-2s}\bar{u}_x(x,y) = d_1(s)\left[\mu \bar{u}_y(0,y) + f\left(\bar{u}(0,y)\right)\right], & \forall y \in \mathbf{R}, \\ \bar{u}(0,y) = u(y), & y \in \mathbf{R}. \end{cases}$$

In the view of Lemma 5.1 in [21], we can define

$$v(y) = \frac{1}{2}\int_0^{\infty} \left[\bar{u}_y^2(t,y) - \bar{u}_x^2(t,y)\right] t^{1-2s}\, dt, \quad \forall y \in \mathbf{R}.$$

By Lemma 5.1 in [21] again, we know that $\lim_{|y|\to\infty} \int_0^\infty |\nabla \bar{u}(t,y)|^2 t^{1-2s}\, dt = 0$, which implies that

$$\lim_{|y|\to\infty} v(y) = 0 \quad \text{and} \quad \lim_{t\to\infty} t^{1-2s}\bar{u}_x(t,y)\bar{u}_y(t,y) = 0.$$

In particular, $v \in L^\infty(\mathbf{R})$. Differentiating v with respect to y, we have

$$v'(y) = \int_0^{\infty} \left[\bar{u}_y(t,y)\bar{u}_{yy}(t,y) - \bar{u}_x(t,y)\bar{u}_{xy}(t,y)\right] t^{1-2s}\, dt, \quad \forall y \in \mathbf{R}.$$

Since $\operatorname{div}[x^{1-2s}\nabla \bar{u}(x,y)] = 0$ in $\mathbf{R}^2_+$, then

$$\bar{u}_{yy}(x,y) = -x^{2s-1} D_x\left[x^{1-2s}\bar{u}_x(x,y)\right], \quad \text{in } \mathbf{R}^2_+.$$

Hence, using integration by parts, we can get

$$\begin{aligned} v'(y) &= \int_0^{\infty} \left[-\bar{u}_y(t,y)t^{2s-1} D_x\left[t^{1-2s}\bar{u}_x(t,y)\right] - \bar{u}_x(t,y)\bar{u}_{xy}(t,y)\right] t^{1-2s}\, dt \\ &= -\int_0^{\infty} \bar{u}_y(t,y) D_x\left[t^{1-2s}\bar{u}_x(t,y)\right] dt - \int_0^{\infty} \bar{u}_x(t,y)\bar{u}_{xy}(t,y) t^{1-2s}\, dt \\ &= \lim_{t\searrow 0} t^{1-2s}\bar{u}_x(t,y)\bar{u}_y(t,y) \\ &= -d_1(s)\bar{u}_y(0,y)\left[\mu \bar{u}_y(0,y) + f\left(\bar{u}(0,y)\right)\right] \\ &= d_1(s)\left[-\mu \bar{u}_y^2(0,y) + \frac{d}{dy} G\left(\bar{u}(0,y)\right)\right], \quad \forall y \in \mathbf{R}. \end{aligned}$$

Then there exists some constant $C_0 > 0$ such that

$$v(y) = d_1(s)\left[-\mu \int_{-\infty}^{y} \bar{u}_y^2(0,r)\, dr + G\left(\bar{u}(0,y)\right) + C_0\right], \quad \forall y \in \mathbf{R}.$$

By taking $y \to -\infty$, we know $C_0 = -G(-1)$. □

Proposition 4.2 *(Modica type estimate). For all $(x,y) \in \overline{\mathbf{R}^2_+}$, we have*

$$\frac{1}{2}\int_0^x \left[\bar{u}_y^2(t,y)-\bar{u}_x^2(t,y)\right]t^{1-2s}\,dt < d_1(s)\left[-\mu\int_{-\infty}^{y}\bar{u}_y^2(0,r)\,dr + G\big(\bar{u}(0,y)\big)-G(-1)\right].$$

As a consequence, we have

$$G\big(u(y)\big)-G(-1) > \mu\int_{-\infty}^{y}\left|u'(r)\right|^2\,dr,\quad \forall y\in\mathbf{R}. \tag{4.2}$$

Proof. Consider the function

$$v(x,y)=\frac{1}{2}\int_0^x \left[\bar{u}_y^2(t,y)-\bar{u}_x^2(t,y)\right]t^{1-2s}\,dt,\quad \forall (x,y)\in\overline{\mathbf{R}_+^2}.$$

By Lemma 5.1 in [21], we know that $\lim_{|y|\to\infty}\int_0^\infty |\nabla\bar{u}(t,y)|^2 t^{1-2s}\,dt=0$, which implies that

$$\lim_{|y|\to\infty} v(x,y)=0,\quad \text{uniformly in } x\ge 0.$$

In particular, we get $v\in L^\infty(\mathbf{R}_+^2)$. Differentiate v with respect to x and y, respectively, we get

$$v_x(x,y)=\frac{1}{2}\left[\bar{u}_y^2(x,y)-\bar{u}_x^2(x,y)\right]x^{1-2s},$$

$$v_y(x,y)=\int_0^x \left[\bar{u}_y(t,y)\bar{u}_{yy}(t,y)-\bar{u}_x(t,y)\bar{u}_{xy}(t,y)\right]t^{1-2s}\,dt.$$

Since $\operatorname{div}[x^{1-2s}\nabla\bar{u}(x,y)]=0$, using integration by parts, we obtain

$$\begin{aligned}
v_y(x,y)&=\int_0^x \left[-\bar{u}_y(t,y)t^{2s-1}D_x\left[t^{1-2s}\bar{u}_x(x,y)\right]-\bar{u}_x(t,y)\bar{u}_{xy}(t,y)\right]t^{1-2s}\,dt\\
&=-\int_0^x \bar{u}_y(t,y)D_x\left[t^{1-2s}\bar{u}_x(t,y)\right]dt-\int_0^x \bar{u}_x(t,y)\bar{u}_{xy}(t,y)t^{1-2s}\,dt\\
&=\bar{u}_y(0,y)\lim_{t\searrow 0} t^{1-2s}\bar{u}_x(t,y)-x^{1-2s}\bar{u}_x(x,y)\bar{u}_y(x,y)\\
&=-\bar{u}_y(0,y)d_1(s)\left[\mu\bar{u}_y(0,y)+f\left(\bar{u}(0,y)\right)\right]-x^{1-2s}\bar{u}_x(x,y)\bar{u}_y(x,y)\\
&=d_1(s)\left[-\mu\left[\bar{u}_y(0,y)\right]^2+\frac{d}{dy}G\big(\bar{u}(0,y)\big)\right]-x^{1-2s}\bar{u}_x(x,y)\bar{u}_y(x,y).
\end{aligned}$$

Consider the function

$$w(x,y)=d_1(s)\left[-\mu\int_{-\infty}^{y}\left[\bar{u}_y(0,r)\right]^2\,dr+G\big(\bar{u}(0,y)\big)-G(-1)\right]-v(x,y),\quad \text{for all } (x,y)\in\overline{\mathbf{R}_+^2}.$$

Then we have

$$w_x(x,y)=\frac{1}{2}\left[\bar{u}_x^2(x,y)-\bar{u}_y^2(x,y)\right]x^{1-2s}\quad\text{and}\quad w_y(x,y)=x^{1-2s}\bar{u}_x(x,y)\bar{u}_y(x,y).$$

By Proposition 4.1, we have $\lim_{x\to\infty}w(x,y)=0$ in $\mathbf{R}$, and

$$\lim_{y\to\infty}\left[-\mu\int_{-\infty}^{y}\left[\bar{u}_y(0,r)\right]^2\,dr+G\big(\bar{u}(0,y)\big)-G(-1)\right]=0.$$

Since $v(x, y) \to 0$ uniformly in $x \geq 0$, as $|y| \to \infty$, then $w(x, y) \to 0$ uniformly in $x \geq 0$, as $|y| \to \infty$. Since $v \in L^\infty(\mathbf{R}^2_+)$, $|\bar{u}(0, y)| \leq 1$ in $\mathbf{R}$, and $\int_{\mathbf{R}} |D_y \bar{u}(0, r)|^2\, dr < \infty$, then we get $w \in L^\infty(\mathbf{R}^2_+)$.

A direct computation show that for all $(x, y) \in \mathbf{R}^2_+$, we have

$$\operatorname{div}\left[x^{1-2s} \nabla w(x, y)\right] = (2s - 1)x^{1-4s} \bar{u}_y^2(x, y), \tag{4.3}$$

$$\operatorname{div}\left[x^{2s-1} \nabla w(x, y)\right] = (2s - 1)x^{-1} \bar{u}_x^2(x, y). \tag{4.4}$$

Claim I. *w is not a constant function in $\mathbf{R}^2_+$.*

Assume $w(x, y) \equiv w(0, 0)$ in $\overline{\mathbf{R}^2_+}$. Since $v(0, y) \equiv 0$ in $\mathbf{R}$, we have

$$w(0, 0) = w(0, y) = d_1(s)\left[-\mu \int_{-\infty}^{y} \bar{u}_y^2(0, r)\, dr + G\big(\bar{u}(0, y)\big) - G(-1)\right], \quad \text{in } \mathbf{R}.$$

This implies that

$$\mu\left|D_y \bar{u}(0, y)\right|^2 = \frac{d}{dy} G\big(\bar{u}(0, y)\big) = -f\big(\bar{u}(0, y)\big)\bar{u}_y(0, y), \quad \text{in } \mathbf{R}.$$

Since $\bar{u}_y(0, y) = u'(y) > 0$ and $\bar{u}(0, y) = u(y)$ in $\mathbf{R}$, then we get

$$cu'(y) = -f\big(u(y)\big), \quad \text{in } \mathbf{R}.$$

But we know that $(-\Delta)^s u(y) - \mu u'(y) = f(u(y))$ in $\mathbf{R}$, then $(-\Delta)^s u(y) = 0$ for all $y \in \mathbf{R}$. Since $u \in L^\infty(\mathbf{R})$, by Liouville's theorem for the fractional harmonic functions, then u is a constant function in $\mathbf{R}$, which contradicts with $u(y) \to \pm 1$, as $y \to \pm\infty$. Hence w is a nonconstant function in $\mathbf{R}^2_+$.

Since w is a bounded function in $\mathbf{R}^2_+$, then we can define

$$A = \inf_{(x,y)\in \mathbf{R}^2_+} w(x, y).$$

Claim II. *$w(x, y) > A$ for all $(x, y) \in \overline{\mathbf{R}^2_+}$.*

If Claim II is not true, then there exists some $(x^0, y^0) \in \overline{\mathbf{R}^2_+}$ such that

$$w\big(x^0, y^0\big) = A = \inf_{(x,y)\in \mathbf{R}^2_+} w(x, y).$$

Case I: Assume that $(x^0, y^0) \in \mathbf{R}^2_+$, that is, (x^0, y^0) is a global interior infimum of w in $\mathbf{R}^2_+$. Since $u'(y) > 0$ in $\mathbf{R}$, then $\bar{u}_y(x, y) > 0$ for all $(x, y) \in \overline{\mathbf{R}^2_+}$. Recall $w_y(x, y) = x^{1-2s} \bar{u}_x(x, y)\bar{u}_y(x, y)$ in $\mathbf{R}^2_+$. So we have $\bar{u}_x(x, y) = x^{2s-1} \frac{w_y(x,y)}{\bar{u}_y(x,y)}$, which implies that

$$\operatorname{div}\left[x^{2s-1} \nabla w(x, y)\right] = (2s - 1)x^{2s-2} \frac{\bar{u}_x(x, y)}{\bar{u}_y(x, y)} w_y(x, y), \quad \text{in } \mathbf{R}^2_+.$$

Hence w satisfies an locally uniformly elliptic equation in $\mathbf{R}^2_+$. But (x^0, y^0) is an interior minimum, by the strong maximum principle, w must be a constant function in $\mathbf{R}^2_+$, which is a contradiction.

Case II: Assume that $(x^0, y^0) \in \partial\mathbf{R}^2_+$, that is, $(0, y^0)$ is a boundary minimum point of w in $\overline{\mathbf{R}^2_+}$. For $w(0, \cdot)$, y^0 is an interior minimum of $w(0, \cdot)$ in $\mathbf{R}$, we get $w_y(0, y^0) = 0$. Since

$$w(0, y) = d_1(s)\left[-\mu \int_{-\infty}^{y} \bar{u}_y^2(0, r)\, dr + G\big(\bar{u}(0, y)\big) - G(-1)\right], \quad \text{in } \mathbf{R}.$$

Then we have

$$0 = -f\big(\bar{u}(0, y^0)\big)\bar{u}_y\big(0, y^0\big) - \mu \bar{u}_y^2\big(0, y^0\big).$$

Since $D_y \bar{u}(x, y) > 0$ in $\overline{\mathbf{R}^2_+}$, then $-\mu D_y \bar{u}(0, y^0) = f(\bar{u}(0, y^0))$, which implies that

$$\lim_{x \searrow 0} -x^{1-2s} \bar{u}_x\big(x, y^0\big) = 0. \tag{4.5}$$

We then consider several sub cases.

Subcase I: If $0 < s \le \frac{1}{2}$, by (4.3), then $\operatorname{div}[x^{1-2s} \nabla w(x, y)] \le 0$ in $\mathbf{R}^2_+$. Since $w(x, y) > w(0, y^0)$ in $\mathbf{R}^2_+$, by Hopf lemma, Proposition 2.1, we know $\limsup_{x \searrow 0} -x^{1-2s} \bar{u}_x(x, y^0) < 0$, which contradicts with (4.5).

Subcase II: If $s > \frac{1}{2}$, then $\lim_{x \searrow 0} \bar{u}_x(x, y) = 0$ in $\mathbf{R}$. Since $w(x, y) > w(0, y^0)$ in $\mathbf{R}^2_+$, then

$$0 \ge \liminf_{x \searrow 0} -x^{2s-1} w_x\big(x, y^0\big).$$

Recall $w_x(x, y) = \frac{1}{2}[\bar{u}_x^2(x, y) - \bar{u}_y^2(x, y)]x^{1-2s}$ in $\mathbf{R}^2_+$. We have

$$0 \ge \liminf_{x \searrow 0} \big[\bar{u}_y^2\big(x, y^0\big) - \bar{u}_x^2\big(x, y^0\big)\big] = \bar{u}_y^2\big(0, y^0\big) - \limsup_{x \searrow 0} \bar{u}_x^2\big(x, y^0\big) = \bar{u}_y^2\big(0, y^0\big) > 0.$$

Therefore, we get a contradiction.

In summary, we know that for all $0 < s < 1$, $w(x, y) > A$ in $\overline{\mathbf{R}^2_+}$.

Claim III. $A \ge 0$.

Assume $A < 0$. Since $w(x, y) \to 0$ uniformly in $x \ge 0$, as $|y| \to \infty$, and $w(x, y) \to 0$, as $x \to \infty$, then there exists some $(x^1, y^1) \in \overline{\mathbf{R}^2_+}$ such that $w(x^1, y^1) = \inf_{(x,y) \in \mathbf{R}^2_+} w(x, y) < 0$, which contradicts with Claim II.

Therefore, we can conclude that $w(x, y) > 0$ in $\overline{\mathbf{R}^2_+}$, that is, for all $(x, y) \in \overline{\mathbf{R}^2_+}$, we have

$$\frac{1}{2}\int_0^x \big[\big(\bar{u}_y^2(t, y) - \bar{u}_x^2(t, y)\big)t^{1-2s}\big]\,dt < d_1(s)\left[-\mu \int_{-\infty}^{y} \bar{u}_y^2(0, r)\,dr + G\big(\bar{u}(0, y)\big) - G(-1)\right]. \qquad \Box$$

Remark 4.1. From inequality (4.2), we know that if $\mu \ge 0$, we have

$$G(t) > G(-1), \quad \forall t \in (-1, 1).$$

Furthermore, we have

$$\int_{-\infty}^{y} (-\Delta)^s u(r) u'(r)\,dr = G(-1) - G\big(u(y)\big) + \mu \int_{-\infty}^{y} \big|u'(r)\big|^2\,dr < 0, \quad \forall y \in \mathbf{R}. \tag{4.6}$$

5. Proof of the main theorem

In this section, we will assume that $0 < s < 1$, $G_1 \in C^3(\mathbf{R})$ is a unbalanced bistable potential, i.e., $f_1 = -G_1'$ satisfies (1.3). Moreover, G_1 satisfies (1.4). Let t_0 be the zero in $(-1, 1)$ of f_1 closest to 1.

Now take any fixed $G_0 \in C^3(\mathbf{R})$ which is a balanced double well potential such that $G_0(t) \equiv G(t)$ for all $t \in [-1, t_0]$ and t_0 is the zero in $(-1, 1)$ of $f_0 = -G_0'$ closet to 1. For any $0 \le \theta \le 1$, we let

$$G_\theta(t) = (1 - \theta)G_0(t) + \theta G_1(t) = G_0(t) + \theta\big[G_1(t) - G_0(t)\big], \quad t \in \mathbf{R}.$$

It is easy to see that for all $\theta \in (0, 1)$, G_θ is a unbalanced bistable potential, i.e., $f_\theta = -G_\theta'$ satisfies (1.3). Moreover, G_θ satisfies (1.4). Since G_0 is a balanced bistable potential, by Theorem 2.4 in [22], we know that there exists a unique solution $g = u_0 \in C^2(\mathbf{R})$ of the following problem:

$$\begin{cases} (-\Delta)^s g(y) = f_0\big(g(y)\big), & \forall y \in \mathbf{R}, \\ g'(y) > 0, \quad \forall y \in \mathbf{R}, \\ \lim_{y\to\pm\infty} g(y) = \pm 1, & g(t_0) = 0. \end{cases}$$

Let $\mu_\theta \in \mathbf{R}$, $u_\theta \in C^2(\mathbf{R})$ and $\overline{u_\theta}$ be the s-harmonic extension of u_θ, we will consider the following problem:

$$\begin{cases} (-\Delta)^s u_\theta(y) - \mu_\theta u_\theta'(y) = f_\theta\big(u_\theta(y)\big), & \forall y \in \mathbf{R}, \\ u_\theta'(y) > 0, \quad \forall y \in \mathbf{R}, \\ \lim_{y\to\pm\infty} u_\theta(y) = \pm 1, & u_\theta(0) = t_0. \end{cases} \tag{5.1}$$

5.1. Upper bound of speeds

In order to get the uniform upper bounds of C^2 norm for the solutions to (5.1), it suffices to get the uniform upper bound of the speeds.

Theorem 5.1. *Let $0 < s < 1$ and $u_\theta \in C^2(\mathbf{R})$ be a solution of (5.1), then there exists some constant $C > 0$ which only depends on s, G_0 and G_1 such that*

$$0 \le \mu_\theta \le C, \quad \forall \theta \in [0, 1].$$

Proof. Since $\theta G_1(1) = G_\theta(1) \ge G_\theta(-1) = 0$, by Proposition 4.1, we know that $\mu_0 = 0$ and $0 < \mu_\theta$ for all $\theta \in (0, 1]$. In the following, we only work on the upper bound for $\theta \in (0, 1]$. Both sides of (5.1) are multiplied by u_θ' and integrate over $(-\infty, 0)$, by Proposition 4.1,we can get

$$\begin{aligned} -\int_{-\infty}^{0} (-\Delta)^s u_\theta(y) u_\theta'(y)\, dy &= -\int_{-\infty}^{0} f_\theta\big(u_\theta(y)\big) u_\theta'(y)\, dy - \mu_\theta \int_{-\infty}^{0} \big|u_\theta'(y)\big|^2\, dy \\ &= G_\theta(t_0) - \mu_\theta \int_{-\infty}^{0} \big|u_\theta'(y)\big|^2\, dy \\ &\ge G_\theta(t_0) - \mu_\theta \int_{\mathbf{R}} \big|u_\theta'(y)\big|^2\, dy \\ &= G_1(t_0) - G_\theta(1) \\ &\ge G_1(t_0) - G_1(1). \end{aligned} \tag{5.2}$$

Case I: Critical Case, i.e., $s = \frac{1}{2}$. Let $\overline{u_\theta}$ be the s-harmonic extension in $\mathbf{R}^2_+$ of u_θ, we know that

$$(-\Delta)^{\frac{1}{2}} u_\theta(y) = -\overline{u_\theta}_x(0, y), \quad \forall y \in \mathbf{R}.$$

By the Cauchy–Schwarz inequality and Proposition 3.3, we have

$$\begin{aligned} -\int_{-\infty}^{0} (-\Delta)^{\frac{1}{2}} u_\theta(y) u_\theta'(y)\, dy &= \int_{-\infty}^{0} \overline{u_\theta}_x(0, y) u_\theta'(y)\, dy \\ &\le \Big[\int_{\mathbf{R}} \big|\overline{u_\theta}_x(0, y)\big|^2\, dy\Big]^{\frac{1}{2}} \Big[\int_{\mathbf{R}} \big|u_\theta'(y)\big|^2\, dy\Big]^{\frac{1}{2}} \\ &= \int_{\mathbf{R}} \big|u_\theta'(y)\big|^2\, dy \\ &= \frac{G_\theta(1)}{\mu_\theta} \\ &\le \frac{G_1(1)}{\mu_\theta}. \end{aligned} \tag{5.3}$$

By (5.2) and (5.3), we have

$$\mu_\theta \le \frac{G_1(1)}{G_1(t_0) - G_1(1)}, \quad \forall \theta \in [0,1].$$

Case II: Subcritical case, i.e., $\frac{1}{2} < s < 1$. By the definition of G_θ, we can find some constant $C_0 \ge 1$ which only depends on G_0 and G_1 such that

$$\|G_\theta\|_{C^2([-1,1])} \le C_0, \quad \forall \theta \in [0,1].$$

By Remark 3.2, we know $(-\Delta)^s u \in L^2(\mathbf{R})$. Both sides of (5.1) are multiplied by $(-\Delta)^s u_\theta$ and integrate over $(-\infty, 0)$

$$\int_{-\infty}^{0} \big[(-\Delta)^s u_\theta(y)\big]^2\, dy = \mu_\theta \int_{-\infty}^{0} (-\Delta)^s u_\theta(y) u_\theta'(y)\, dy + \int_{-\infty}^{0} f_\theta\big(u_\theta(y)\big)(-\Delta)^s u_\theta(y)\, dy \ge 0.$$

By (5.2), we have

$$0 < \mu_\theta\big[G_1(t_0) - G_1(1)\big] \le -\mu_\theta \int_{-\infty}^{0} (-\Delta)^s u_\theta(y) u_\theta'(y)\, dy \le \int_{-\infty}^{0} f_\theta\big(u_\theta(y)\big)(-\Delta)^s u_\theta(y)\, dy.$$

In particular, we know that

$$\int_{-\infty}^{0} \big[(-\Delta)^s u_\theta(y)\big]^2\, dy < \int_{-\infty}^{0} f_\theta\big(u_\theta(y)\big)(-\Delta)^s u_\theta(y)\, dy.$$

Now let's estimate $\int_{-\infty}^{0} f_\theta(u_\theta(y))(-\Delta)^s u_\theta(y)\, dy$, for any $R > 0$, we have

$$\begin{aligned}
&\int_{-\infty}^{0} f_\theta\big(u_\theta(y)\big)(-\Delta)^s u_\theta(y)\, dy \\
&= C_{1,s}\left[\int_{-\infty}^{0} f_\theta\big(u_\theta(y)\big) \int_{|y-z|>R} \frac{u_\theta(y) - u_\theta(z)}{|y-z|^{1+2s}}\, dz dy + \int_{-\infty}^{0} f_\theta\big(u_\theta(y)\big)\left(\mathrm{P.V.} \int_{|y-z|\le R} \frac{u_\theta(y) - u_\theta(z)}{|y-z|^{1+2s}}\, dz\right) dy\right].
\end{aligned}$$

For the first term on the right hand, by Fubini–Tonelli theorem, we have

$$\begin{aligned}
&\int_{-\infty}^{0} f_\theta\big(u_\theta(y)\big) \int_{|y-z|>R} \frac{u_\theta(y) - u_\theta(z)}{|y-z|^{1+2s}}\, dz dy \\
&= \int_{|w|>R} \int_{-\infty}^{0} \frac{f_\theta(u_\theta(y))[u_\theta(y) - u_\theta(y+w)]}{|w|^{1+2s}}\, dy dw \\
&= \int_{|w|>R} \int_{-\infty}^{0} \int_{0}^{1} \frac{G_\theta'(u_\theta(y)) u_\theta'(y+tw) w}{|w|^{1+2s}}\, dt dy dw \\
&\le \big\|G_\theta'\big\|_{C([-1,1])} \int_{|w|>R} |w|^{-2s} \int_{\mathbf{R}} u_\theta'(y+tw)\, dy\, dw \\
&= \frac{4}{2s-1}\big\|G_\theta'\big\|_{C([-1,1])} \cdot R^{1-2s} \\
&\le \frac{4C_0}{2s-1} \cdot R^{1-2s}.
\end{aligned}$$

For the second term, by Fubini–Tonelli theorem, we have

$$
\begin{aligned}
&\int_{-\infty}^{0} f_\theta\big(u_\theta(y)\big)\bigg(\text{P.V.}\int_{|y-z|\le R} \frac{u_\theta(y)-u_\theta(z)}{|y-z|^{1+2s}}\,dz\bigg)\,dy \\
&\quad = -\int_{-\infty}^{0} f_\theta\big(u_\theta(y)\big)\int_{|w|<R} \frac{u_\theta(y+w)-u_\theta(y)-u_\theta'(y)w}{|w|^{1+2s}}\,dwdy \\
&\quad = -\int_{-\infty}^{0} f_\theta\big(u_\theta(y)\big)\int_{|w|<R}\int_0^1\int_0^1 \frac{u_\theta''(y+rtw)\cdot tw^2}{|w|^{1+2s}}\,drdtdwdy \\
&\quad = -\int_{|w|<R}\int_0^1\int_0^1 t|w|^{1-2s}\int_{-\infty}^{0} f_\theta\big(u_\theta(y)\big)u_\theta''(y+rtw)dydrdtdw.
\end{aligned}
$$

We note that $f_\theta(u_\theta(0)) = f_\theta(t_0) = 0$, which implies

$$
\begin{aligned}
-\int_{-\infty}^{0} f_\theta\big(u_\theta(y)\big)u_\theta''(y+rtw)dy &= -f_\theta\big(u_\theta(0)\big)u_\theta'(rtw) + \int_{-\infty}^{0} f_\theta'\big(u_\theta(y)\big)u_\theta'(y)u_\theta'(y+rtw)\,dy \\
&= \int_{-\infty}^{0} f_\theta'\big(u_\theta(y)\big)u_\theta'(y)u_\theta'(y+rtw)\,dy.
\end{aligned}
$$

By the Cauchy–Schwarz inequality and Proposition 3.3, we get

$$
\begin{aligned}
&\int_{-\infty}^{0} f_\theta\big(u_\theta(y)\big)\bigg(\text{P.V.}\int_{|y-z|\le R} \frac{u_\theta(y)-u_\theta(z)}{|y-z|^{1+2s}}\,dz\bigg)\,dy \\
&\quad = \int_{|w|<R}\int_0^1\int_0^1 t|w|^{1-2s}\int_{-\infty}^{0} f_\theta'\big(u_\theta(y)\big)u_\theta'(y)u_\theta'(y+r)\,dydrdtdw \\
&\quad \le \big\|f_\theta'\big\|_{C([-1,1])}\int_{|w|<R}\int_0^1\int_0^1 |w|^{1-2s}\int_{\mathbf{R}} u_\theta'(y)u_\theta'(y+rtw)dydrdtdw \\
&\quad \le C_0\int_{|w|<R} |w|^{1-2s}\,dw\cdot\big\|u_\theta'\big\|_{L^2(\mathbf{R})}^2 \\
&\quad = \frac{C_0}{2-2s}R^{2-2s}\cdot\frac{G_\theta(1)}{\mu_\theta} \\
&\quad \le \frac{C_0}{2-2s}R^{2-2s}\cdot\frac{G_1(1)}{\mu_\theta}.
\end{aligned}
$$

In summary, we get

$$
\mu_\theta\big[G_1(t_0)-G_1(1)\big] \le \frac{4C_0}{2s-1}R^{1-2s} + \frac{C_0}{2-2s}R^{2-2s}\cdot\frac{G_1(1)}{\mu_\theta}.
$$

Taking $R = 1$, we know that there exists a constant $C > 0$ which is independent of θ, such that

$$
\mu_\theta \le C, \quad \forall\theta \in [0,1].
$$

Case III: Supercritical Case, i.e., $0 < s < \frac{1}{2}$. For any $R > 0$, we know that

$$\begin{aligned}-(-\Delta)^s u_\theta(y) &= C_{n,s}\int_{\mathbf{R}} \frac{u_\theta(y+z)-u_\theta(y)}{|z|^{1+2s}}\,dz \\ &= C_{1,s}\int_{|z|\ge R} \frac{u_\theta(y+z)-u_\theta(y)}{|z|^{1+2s}}\,dy + C_{1,s}\int_{|z|<R}\int_0^1 \frac{z}{|z|^{1+2s}}u'_\theta(y+tz)\,dt\,dz \\ &\le \frac{2C_{1,s}}{s}R^{-2s} + C_{1,s}\int_{|z|<R}\int_0^1 \frac{1}{|z|^{2s}}u'_\theta(y+tz)\,dt\,dz.\end{aligned}$$

By Fubini–Tonelli theorem, the Cauchy–Schwarz inequality and Proposition 4.1, we have

$$\begin{aligned}&-\int_{-\infty}^0 (-\Delta)^s u_\theta(y)u'_\theta(y)\,dy \\ &\le \frac{2C_{1,s}}{s}R^{-2s}\int_{-\infty}^0 u'_\theta(y)\,dy + C_{1,s}\int_0^1\int_{|z|<R}\frac{1}{|z|^{2s}}\int_{-\infty}^0 u'_\theta(y+tz)u'_\theta(y)\,dy\,dz\,dt \\ &\le \frac{4C_{1,s}}{s}R^{-2s} + \frac{2C_{1,s}}{1-2s}R^{1-2s}\|u'_\theta\|^2_{L^2(\mathbf{R})} \\ &= \frac{4C_{1,s}}{s}R^{-2s} + \frac{2C_{1,s}}{1-2s}R^{1-2s}\frac{G_\theta(1)}{\mu_\theta} \\ &\le \frac{4C_{1,s}}{s}R^{-2s} + \frac{2C_{1,s}}{1-2s}R^{1-2s}\frac{G_1(1)}{\mu_\theta}.\end{aligned}$$

By (5.2), we have

$$G_1(t_0) - G_1(1) \le \frac{4C_{1,s}}{s}R^{-2s} + \frac{2C_{1,s}}{1-2s}R^{1-2s}\frac{G_1(1)}{\mu_\theta}, \quad \forall R > 0.$$

Find some R such that $\frac{4C_{1,s}}{s}R^{-2s} = \frac{G_1(t_0)-G_1(1)}{2}$, we can get

$$\mu_\theta \frac{G_1(t_0)-G_1(1)}{2} \le \frac{2C_{1,s}}{1-2s}R^{1-2s}G_1(1).$$

Let $C = \frac{\frac{4C_{1,s}}{1-2s}R^{1-2s}G_1(1)}{G_1(t_0)-G_1(1)} > 0$, we have

$$\mu_\theta \le C, \quad \forall \theta \in [0,1].$$

In summary, for all $0 < s < 1$, there exists some $C > 0$ which only depends on s, G_0 and G_1 such that

$$0 \le \mu_\theta \le C, \quad \forall \theta \in [0,1]. \qquad \square$$

5.2. *Continuity method*

Let $v_\theta(y) = u_\theta(y) - g(y)$ in $\mathbf{R}$ where g is the solution associated to $\theta = 0$. Then (5.1) is equivalent to the following problem:

$$\begin{cases} (-\Delta)^s v_\theta(y) - \mu_\theta v'_\theta(y) - \mu_\theta g'(y) + (-\Delta)^s g(y) - f_\theta\big(v_\theta(y)+g(y)\big) = 0, & y \in \mathbf{R}, \\ v_\theta(0) = 0, \qquad \lim_{|y|\to\infty} v'_\theta(y) = 0. \end{cases}$$

For any $0 < \alpha < 1$, we consider the function space:

$$C_g^{1,\alpha}(\mathbf{R}) = \Big\{ v \in C^{1,\alpha}(\mathbf{R})\colon\ v(0) = 0,\ \lim_{|y|\to\infty} v(y) = 0,\ -1 \le v(y) + g(y) \le 1,\ \text{in } \mathbf{R} \Big\}.$$

We define the following operator $S : [0,1] \times \mathbf{R} \times C_g^{1,\alpha}(\mathbf{R}) \to C^\alpha(\mathbf{R})$ given by: for any $0 \le \theta \le 1$, $\mu \in \mathbf{R}$ and $v \in C_g^{1,\alpha}(\mathbf{R})$, we have

$$S(\theta, \mu, v) = v - \mu(-\Delta)^{-s} v' - \mu(-\Delta)^{-s} g' + g - (-\Delta)^{-s}\big[f_\theta(v+g)\big].$$

It is easy to see that S is C^1, and for all $h \in \mathbf{R}$ and $\phi \in C_g^{1,\alpha}(\mathbf{R})$, we have

$$D_{\mu,v} S(\theta, \mu, v)[h, \phi] = \phi - h(-\Delta)^{-s} v' - \mu(-\Delta)^{-s}\phi' - h(-\Delta)^{-s} g' - (-\Delta)^{-s}\big[f'_\theta(v+g)\phi\big].$$

Define the solution set Σ as:

$$\Sigma = \big\{\theta \in [0,1]\colon \text{There exists } \mu_\theta \in \mathbf{R} \text{ and } v_\theta \in C_g^{1,\alpha}(\mathbf{R}) \text{ such that } S(\theta, \mu_\theta, v_\theta) = 0\big\}.$$

Theorem 5.2. $\Sigma = [0,1]$.

Proof. By taking $\theta = \mu = 0$ and $v(y) \equiv 0$ in $\mathbf{R}$, by the assumption of g, we see that $S(0,0,0) = 0$, that is, $(0,0,0) \in \Sigma$, in particular, we know that Σ is a nonempty subset of $[0,1]$.

Claim I. *Σ is open in* $[0,1]$.

If $\theta \in \Sigma$, let $u_\theta(y) = v_\theta(y) + g(y)$ in $\mathbf{R}$, that is, u_θ is a solution of (5.1). Assume that there exist $h \in \mathbf{R}$ and $\phi \in C_g^{1,\alpha}(\mathbf{R})$ such that $D_{\mu,v_\theta} S(\theta, \mu_\theta, v_\theta)[h, \phi] = 0$, that is,

$$(-\Delta)^s \phi(y) - \mu_\theta \phi'(y) - h u'_\theta(y) - f'_\theta\big(u_\theta(y)\big)\phi(y) = 0, \quad y \in \mathbf{R}.$$

By Proposition 3.4, we have $h = 0$ and $\phi(y) \equiv 0$ in $\mathbf{R}$, which implies that $D_{\mu,v} S(\theta, \mu_\theta, u_\theta)$ is injective. By the implicit function theorem, Σ is open in $[0,1]$.

Claim II. *Σ is closed in* $[0,1]$.

Assume that there exists a sequence $\{\theta_k\}_{k=1}^\infty \subset \Sigma$ such that $\theta_k \to \theta$, as $k \to \infty$. Let $u_k(y) = v_{\theta_k}(y) + g(y)$ in $\mathbf{R}$. By Theorem 3.1, we know that for each θ_k, the speed μ_{θ_k} and the solution v_{θ_k} are unique. By Proposition 4.1, we know that $\mu_k \ge 0$ for all $k \ge 1$. By Theorem 5.1, we know that there exists some constant $C_1 > 0$ which just depends on s, G_0 and G_1 such that

$$0 < \mu_{\theta_k} \le C_1 < \infty, \quad \text{for all } k \ge 1.$$

By the regularity theory for fractional Laplacians (see [13,47,63,64,82] and [83]), for any $0 < \alpha < 1$, there exists some constant $C > 0$ which just depends on s, G_0 and G_1 such that

$$\|u_k\|_{C^{2,\alpha}(\mathbf{R}^n)} \le C, \quad \forall k \ge 1.$$

Taking any fixed $t_1 \in (t_0, 1)$, since $\lim_{y\to\pm\infty} u_k(y) = \pm 1$, there exists some $y_k > 0$ such that $u_k(y_k) = t_1$ for all $k \ge 1$. Let $w_k(y) = u_k(y + y_k)$ in $\mathbf{R}$, w_k solves the following problem:

$$\begin{cases} (-\Delta)^s w_k(y) - \mu_{\theta_k} w'_k(y) = f_{\theta_k}\big(w_k(y)\big), & y \in \mathbf{R}, \\ w'_k(y) > 0, \quad y \in \mathbf{R}, \\ \lim\limits_{y\to\pm\infty} w_k(y) = \pm 1, \qquad w_k(0) = t_1. \end{cases}$$

Moreover, we know that $\|v_k\|_{C^{2,\alpha}(\mathbf{R}^n)} \le C$ for all $k \ge 1$. By Ascoli–Arzela theorem, there exists a subsequence of $\mu_\theta \ge 0$ and $\{w_k\}_{k=1}^\infty$, which is still denoted the same, such that $\mu_{\theta_k} \to \mu_\theta$ and $w_k \to w$ in $C^2_{loc}(\mathbf{R})$, as $k \to \infty$. In particular, w solves the problem:

$$\begin{cases} (-\Delta)^s w(y) - \mu_\theta w'(y) = f_\theta\big(w(y)\big), & y \in \mathbf{R}, \\ \big|w(y)\big| \le 1, & y \in \mathbf{R}, \\ w'(y) \ge 0, \quad y \in \mathbf{R}, & w(0) = t_1. \end{cases}$$

Since $w'(y) \ge 0$ and $|w(y)| \le 1$ in $\mathbf{R}$, there exist some constants $L^\pm$ such that $w(y) \to L^\pm$, as $y \to \pm\infty$, and $-1 \le L^- \le t_1 \le L^+ \le 1$. By a compactness argument, we also see that $f(L^\pm) = 0$. Hence $L^+ = 1$. Since $\mu_\theta \ge 0$, by Proposition 4.1 we get

$$\mu_\theta \int_{\mathbf{R}} \big|w'(y)\big|^2 \, dy = G_\theta\big(L^+\big) - G_\theta\big(L^-\big).$$

By (1.4), we conclude $L^- = -1$. Therefore, $\theta \in \Sigma$, that is, Σ is closed in [0, 1].

By Claim I and Claim II, we know that $\Sigma = [0, 1]$. The theorem is proven. □

6. Note added in proof

The results of the paper were also reported in the PDE Seminar at University of Minnesota in April, 2012, the International Conference on Variational Methods at Chern Institute in May, 2012, the Workshop on Recent Trends in Geometric and Nonlinear Analysis at BIRS, Banff, Canada in August 2012, with a publicly available online video of the lecture, and on numerous other occasions.

Conflict of interest statement

This article has no conflict with other publishers or institutes.

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