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Asymptotic analysis of solutions to a gauged O(3) sigma model

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Abstract

We analyze an elliptic equation arising in the study of the gauged O(3) sigma model with the Chern–Simons term. In this paper, we study the asymptotic behavior of solutions and apply it to prove the uniqueness of stable solutions. However, one of the features of this nonlinear equation is the existence of stable nontopological solutions in $\mathbb{R}^2$, which implies the possibility that a stable solution which blows up at a vortex point exists. To exclude this kind of blow up behavior is one of the main difficulties which we have to overcome.

Keywords: Gauged O(3) sigma models; Blow up analysis; Pohozaev type identity; Stable solutions

1. Introduction

The classical O(3) sigma model in $2+1$ dimension originated to describe physical phenomena such as planar ferromagnet [3]. However, the solitons in this model are not suitable for particle models due to their scale invariance which makes particles have arbitrary size. This problem was overcome by Schroers in [23], where a U(1) gauge field was added and the dynamics was governed by the Maxwell term. After his work, there have been many studies on U(1) gauged O(3) sigma model, where the gauge field dynamics was governed by the Chern–Simons term [1,14,18,19] or both of Maxwell and Chern–Simons terms [18].

In this paper, we consider another Chern–Simons gauged O(3) sigma model whose Lagrangian is defined by

$$\mathcal{L} = \frac{\kappa}{4}\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho + \frac{1}{2}D_\mu\phi \cdot D^\mu\phi - \frac{1}{2\kappa^2}(\gamma + \mathbf{n}\cdot\phi)^2|\mathbf{n}\times\phi|^2.$$

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The unknowns are the spin vector $\phi = (\phi_1, \phi_2, \phi_3) : \mathbb{R}^{1,2} \to S^2 \subset \mathbb{R}^3$ and the gauge field $A_\rho : \mathbb{R}^{1,2} \to \mathbb{R}$ with $\rho = 0, 1, 2$. The gauge covariant derivative is defined by

$$D_j\phi = \partial_j\phi + A_j(\mathbf{n} \times \phi),$$

and the curvature $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Moreover, $\mathbf{n} = (0, 0, 1)$ is the north pole of S^2 and $\varepsilon^{\alpha\beta\gamma}$ is the totally antisymmetric tensor with $\varepsilon^{012} = 1$. The constant $\kappa > 0$ represents the strength of the Chern–Simons action, and the constant $\gamma \in [-1, 1]$ is a free parameter which determines the vacuum manifold of the potential. Since the Euler–Lagrangian equation is very complicated to study even for stationary solution, we restrict to consider energy minimizers only. The static energy from the Lagrangian is

$$\mathcal{E}(\phi, A) = \int_{\mathbb{R}^2} e(\phi, A)\, dx,$$

where the energy density $e(\phi, A)$ is given by

$$e(\phi, A) = \frac{1}{2}\left(\frac{\kappa^2 F_{12}^2}{|\mathbf{n} \times \phi|^2} + |D_1\phi|^2 + |D_2\phi|^2 + \frac{1}{\kappa^2}(\gamma + \mathbf{n}\cdot\phi)^2|\mathbf{n} \times \phi|^2\right).$$

We see that

$$\begin{aligned}\mathcal{E}(\phi, A) = \frac{1}{2}\int_{\mathbb{R}^2}\left\{\left(\frac{\kappa F_{12}}{|\mathbf{n}\times\phi|} \pm \frac{1}{\kappa}(\gamma + \mathbf{n}\cdot\phi)|\mathbf{n}\times\phi|\right)^2 + |D_1\phi \pm \phi \times D_2\phi|^2\right\} dx \\ \pm \int_{\mathbb{R}^2} \phi\cdot(D_1\phi \times D_2\phi) - F_{12}(\gamma + \mathbf{n}\times\phi)\, dx.\end{aligned}$$

Then the self-dual equations for solutions minimizing the static energy are given by

$$\begin{aligned} &D_1\phi + \phi \times D_2\phi = 0, \\ &F_{12} + \frac{1}{\kappa^2}(\gamma + \mathbf{n}\cdot\phi)|\mathbf{n}\times\phi|^2 = 0. \end{aligned} \tag{1.1}$$

If we set $u = \ln[(1 + \phi_3)/(1 - \phi_3)]$ and prescribe

$$\phi^{-1}(\mathbf{s}) = \{p_{1,1}, \dots, p_{d_1,1}\}, \qquad \phi^{-1}(\mathbf{n}) = \{p_{1,2}, \dots, p_{d_2,2}\},$$

where $\mathbf{s} = (0, 0, -1)$ is the south pole of S^2, then we can reduce the system (1.1) to the following equation:

$$\Delta u + \frac{1}{\varepsilon^2}\frac{e^u(1 - e^u)}{(\tau + e^u)^3} = 4\pi\sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} - 4\pi\sum_{j=1}^{d_2} m_{j,2}\delta_{p_{j,2}} \quad \text{on } \mathbb{R}^2,$$

where $\tau = \frac{1+\gamma}{1-\gamma} \in (0, \infty)$, $m_{j,i} \in \mathbb{N} \cup \{0\}$, and δ_p stands for the Dirac measure concentrated at p. For the details of derivation of the above equation from (1.1), we refer the readers to [8].

In this paper, we want to consider the above equation in a flat 2-dimensional torus Ω:

$$\Delta u + \frac{1}{\varepsilon^2}\frac{e^u(1 - e^u)}{(\tau + e^u)^3} = 4\pi\sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} - 4\pi\sum_{j=1}^{d_2} m_{j,2}\delta_{p_{j,2}} \quad \text{on } \Omega. \tag{1.2}$$

This consideration is physically meaningful, due to the theory suggested by 't Hooft in [25]. We also refer to [4,6,7,11] for more developments of Eq. (1.2).

Before we go further, we shall make some remarks about our nonlinear term $f_\tau(u) \equiv \frac{e^u(1-e^u)}{(\tau+e^u)^3}$. As $\varepsilon \to 0$, heuristically, solutions u_ε of (1.2) might tend to $\pm\infty$. If $u_\varepsilon \to -\infty$, then (1.2) tends to:

$$\Delta u + e^u = \text{a sum of Dirac measure.}$$

On the other hand, if $u \to +\infty$, then Eq. (1.2) tends to

$$\Delta u - e^{-u} = \text{a sum of Dirac measure},$$

in other words, $(-u)$ satisfies the Liouville equation again. Thus one of the limiting equation is the Liouville equation, which shares the same property of the well-known Chern–Simons–Higgs (CSH) equation:

$$\Delta u + \frac{1}{\varepsilon^2} e^u\left(1 - e^u\right) = 4\pi \sum_{j=1}^{d} m_j \delta_{p_j} \quad \text{on } \Omega. \tag{1.3}$$

The CSH model has been proposed more than twenty years ago in [16] and independently in [17] to describe vortices in high temperature superconductivity. Actually, (1.3) was derived from the Euler–Lagrange equations of the CSH model via a vortex ansatz, see [16,17,27,28]. We also refer to [9,10,20–22] for more developments.

In a recent paper [26], Tarantello proved the following theorem:

Theorem A. *For given $\{p_j\}$ and $m_j \in \mathbb{N}$, there exists $\varepsilon_0 \equiv \varepsilon_0(p_j, m_j) > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, then there exists a unique topological solution u_ε for (1.3), i.e. a unique solution which satisfies $u_\varepsilon \to 0$ a.e. in Ω as $\varepsilon \to 0$.*

It is natural to ask whether Theorem A also holds for Eq. (1.2). In [26], Tarantello proved that if u_ε is a topological solution of (1.3), then u_ε is strictly stable solution. As a consequence of this fact, the uniqueness of the topological solutions was established. In this paper, we study the uniqueness of *stable solutions* instead of topological solutions, because the definition of a topological solution depends on a sequence of solutions, not only the solution itself. Here u is called a stable solution of (1.2) if the linearized equation of (1.2) at u has nonnegative eigenvalues.

Our main purpose is to prove the equivalence of stable solutions and topological solutions under certain assumptions. To state our result, we need the following conditions:

(H1): $N_1 \neq N_2$ where $N_i \equiv \sum_{j=1}^{d_i} m_{j,i}$;
(H2): either $\tau = 1$ or, if $N_i > N_k$, then $m_{j,i} \in [0,1]$ for all $1 \leqslant j \leqslant d_i$.

Then we have the following theorem.

Theorem 1.1. *Let u_ε be a sequence of solutions of (1.2) with $\varepsilon > 0$.*

(i) *if $u_\varepsilon \to 0$ a.e. in $\Omega \setminus \bigcup_{j,i}\{p_{j,i}\}$ as $\varepsilon \to 0$, then u_ε is a strictly stable solution for sufficiently small $\varepsilon > 0$.*
(ii) *if (H1)–(H2) hold and u_ε is a sequence of stable solutions, then $u_\varepsilon \to 0$ a.e. in $\Omega \setminus \{p_{j,i}\}$ as $\varepsilon \to 0$.*

Remark 1.2. A nontopological entire solution of the CSH equation (1.3) is always unstable (see Appendix A). Hence for a sequence of stable solutions u_ε of the CSH equation (1.3), we can prove that u_ε is a topological solution for small $\varepsilon > 0$. The proof is simpler than (ii) of Theorem 1.1.

As a consequence of Theorem 1.1, we also have the following result about the uniqueness of stable solutions of (1.2).

Theorem 1.3. *Let u_ε be a sequence of solutions of (1.2) with $\varepsilon > 0$. If (H1)–(H2) hold, then there exists $\varepsilon_0 := \varepsilon_0(p_{j,i}, m_{j,i}) > 0$ such that there exists a unique stable solution of (1.2) for each $\varepsilon \in (0, \varepsilon_0)$.*

We remark that the uniqueness of topological solutions of (1.2) always holds even without the assumptions (H1)–(H2). Indeed, this result and (i) of Theorem 1.1 can be proved by a suitable adaptation of the argument in [26]. Roughly speaking, this is due to the fact that the behavior of a topological solution is the same no matter whether it is a solution of (1.3) or of (1.2). See either Proposition 4.8 in [26] or Lemma 5.1 below.

However, there are dramatic differences between these two equations when stable solutions are considered. First of all, the asymptotic analysis is relatively easier for the CSH equation (1.3). By the maximum principle, any solution u of the CSH equation (1.3) is always negative, thus $e^u(1 - e^u)$ is always positive. On the contrary, a solution $u(x)$ of

Eq. (1.2) could tend to either $+\infty$ or $-\infty$ as x converges to a vortex point in case $N_1 \neq 0$ and $N_2 \neq 0$. This fact readily implies that the nonlinear term $f_\tau(u)$ must change sign in Ω and this is of course the cause of a lot of difficulties in the study of the asymptotic behavior of u_ε as $\varepsilon \to 0$.

Secondly, any nontopological entire solution of the CSH equation (1.3) is always unstable. This might not be true for Eq. (1.2). Indeed, it has been proved that any nontopological radially symmetric entire solution of (1.2) is unstable provided that either $\tau = 1$ or $m_{j,i} \in [0, 1]$ for all i, j. Hence if $\tau \neq 1$ and $m_{j,i} > 1$ for some i, j, then there might exist nontopological stable entire solutions for (1.2). Of course, this fact might complicate our analysis, because stable solutions might be bubbling even at a vortex point $p_{j,i}$, where $\tau \neq 1$ and $m_{j,i} > 1$. Our condition (H2) partly reflects this fact. However, (H2) still allows the possibility that $m_{j,k} > 1$ as far as the global condition $N_i > N_k$ is satisfied, since in this case one can prove that stable solutions cannot blow up at $p_{j,k}$. But it is still an interesting open problem to see whether those conditions are necessary or not and we will discuss it in another paper.

Remark 1.4. If any one of the N_i's is zero, then Theorems 1.1 and 1.3 hold even without the assumptions (H1)–(H2).

To understand the asymptotic behavior of solutions of (1.2) as $\varepsilon \to 0$, we also ask whether or not there might exist a sequence of solutions u_ε for (1.2) such that

$$\lim_{\varepsilon\to 0}\Big(\sup_K u_\varepsilon\Big) = \infty \quad \text{and} \quad \lim_{\varepsilon\to 0}\Big(\inf_K u_\varepsilon\Big) = -\infty, \tag{1.4}$$

where $K = \Omega \setminus \bigcup_{i,j} B_r(p_{j,i})$ for any fixed $r > 0$. The following theorem tells us that the kind of blow-up behavior as introduced in (1.4) cannot occur.

Theorem 1.5. *Let $Z \equiv \bigcup_{j,i}\{p_{j,i}\}$ and $Z_i \equiv \bigcup_j \{p_{j,i}\}$ for $i = 1, 2$. We assume that $\{u_\varepsilon\}$ is a sequence of solutions of (1.2). Then, up to subsequences, one of the following holds true:*

(a) *$u_\varepsilon \to 0$ uniformly on any compact subset of $\Omega \setminus Z$;*
(b) *for any compact subset $K \subset \Omega \setminus Z_2$, there exists $\nu_K > 0$ such that*

$$\lim_{\varepsilon\to 0}\Big(\sup_K u_\varepsilon\Big) \leqslant -\nu_K;$$

(c) *for any compact subset $K \subset \Omega \setminus Z_1$, there exists $\nu_K > 0$ such that*

$$\lim_{\varepsilon\to 0}\Big(\inf_K u_\varepsilon\Big) \geqslant \nu_K.$$

Besides the application to our analysis, we believe that the above alternative could be useful in further studies of (1.2).

We also remark that it is important to use a suitable Pohozaev type identity for handling solutions with different asymptotic behavior. The following antiderivatives of $f_\tau(u)$ are used to this purpose depending on the situations at hand:

$$F_{1,\tau}(u) \equiv \frac{-(1-e^u)^2}{2(\tau+1)(\tau+e^u)^2},$$

and

$$F_{2,\tau}(u) \equiv \frac{e^u((1-\tau)e^u + 2\tau)}{2\tau^2(\tau+e^u)^2}.$$

Moreover, we denote by G the Green's function on Ω which satisfies

$$-\Delta_x G(x,y) = \delta_y - \frac{1}{|\Omega|}, \quad x, y \in \Omega \quad \text{and} \quad \int_\Omega G(x,y)\,dx = 0, \tag{1.5}$$

and by $\gamma(x,y) = G(x,y) + \frac{1}{2\pi}\ln|x-y|$ its regular part. We also define

$$u_0^+(x) \equiv -4\pi \sum_{j=1}^{d_1} m_{j,1} G(x, p_{j,1}), \qquad u_0^-(x) \equiv -4\pi \sum_{j=1}^{d_2} m_{j,2} G(x, p_{j,2}), \qquad u_0 \equiv u_0^+ - u_0^-,$$

and therefore we see that it holds

$$\Delta u_0 = -\frac{4\pi(N_1 - N_2)}{|\Omega|} + 4\pi \sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} - 4\pi \sum_{j=1}^{d_2} m_{j,2}\delta_{p_{j,2}} \quad \text{on } \Omega. \tag{1.6}$$

In this paper, we consider only a domain which is a subset of $\mathbb{R}^2$ because of not only physical background but also mathematical tools (Pohozaev Identity, Green's function, etc.). We note that our results cannot be generalized to higher dimensional case.

The rest of this paper is devoted to the proof of the above theorems. In Section 2, we discuss some preliminary results. In Section 3, we investigate the asymptotic behavior of solutions of (1.2) as $\varepsilon \to 0$. In Sections 4–6, we study the asymptotic behavior of stable solutions. The main purpose is to prove some identities involving data coming from different regions, one being a neighborhood of the vortex point and the other one its complement. The more subtle part is the asymptotic analysis of the bubbling behavior of stable solutions at vortex points. Finally, we prove Theorems 1.1 and 1.3.

2. Preliminaries

We consider the following limiting problem for (1.2) when Z is empty,

$$\Delta u + \frac{e^u(1-e^u)}{(\tau + e^u)^3} = 0 \quad \text{in } \mathbb{R}^2, \tag{2.1}$$

and we also define (recall $f_\tau(u) = \frac{e^u(1-e^u)}{(\tau+e^u)^3}$)

$$\beta \equiv \frac{1}{2\pi} \int_{\mathbb{R}^2} f_\tau(u)\, dx. \tag{2.2}$$

By applying the method of moving planes as introduced in [12] and improved in [5] and [24], we obtain the following lemma.

Lemma 2.1. *Let u be a solution of (2.1). Assume that there exists a constant $c \in \mathbb{R}$ such that*

$$\textit{either} \quad u \leqslant c \quad \textit{or} \quad u \geqslant c \quad \textit{or} \quad \limsup_{|x|\to\infty} \frac{|u|}{|x|^2} \leqslant c.$$

If $f_\tau(u) \in L^1(\mathbb{R}^2)$, then u is radially symmetric about some point $x_0 \in \mathbb{R}^2$.

Proof. The proof of Lemma 2.1 is standard and we just provide a sketch for reader's convenience. First of all, we observe that $f_\tau(u) \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Next we define

$$v(x) \equiv \frac{1}{2\pi} \int_{\mathbb{R}^2} \big(\ln|x-y| - \ln\big(|y|+1\big)\big) f_\tau\big(u(y)\big)\, dy, \tag{2.3}$$

so that $\Delta v = f_\tau(u)$ and by known elliptic estimates

$$\lim_{|x|\to\infty} \frac{v(x)}{\ln|x|} = \beta. \tag{2.4}$$

At this point we may define $h = u + v$ and then observe that $\Delta h = 0$.

Step 1. Now we claim that h is constant in $\mathbb{R}^2$. If $u \leqslant c$ or $u \geqslant c$ in $\mathbb{R}^2$ for some constant $c \in \mathbb{R}$, then (2.4) implies that either $h \leqslant c_1(\ln(|x|+1)+1)$ or $h \geqslant c_1(\ln(|x|+1)+1)$ for some constant $c_1 \in \mathbb{R}$. Then, by Liouville's theorem, $h(x) = u(x) + v(x) \equiv$ constant. Now we consider the case $\limsup_{|x|\to\infty} \frac{|u|}{|x|^2} \leqslant c$. Then, we also see that $\limsup_{|x|\to\infty} \frac{|h|}{|x|^2}$ is bounded. By the mean value theorem, there exist constants $c_1, c_2 \in \mathbb{R}$ such that

$$\sup_{B_{\frac{R}{2}}(y)} \left|D^\alpha h\right| \leqslant \frac{c_1}{R^2} \sup_{B_R(y)} |h| \leqslant c_2,$$

for any $y \in \mathbb{R}^2$, $R = \frac{|y|}{2}$ and $|\alpha| = 2$ (see Theorem 2.10 in [13]). Then $D^\alpha h$ is a constant for $|\alpha| = 2$ since $D^\alpha h$ is bounded and harmonic in $\mathbb{R}^2$. After a coordinates transformation, we can assume that either $h(x) = a(x_1^2 - x_2^2) + b$ or $h(x) = cx_1 + dx_2 + e$ for some constants $a, b, c, d, e \in \mathbb{R}$ where $x = (x_1, x_2)$. Hence (2.4) implies that either

$$u(x) = a\left(x_1^2 - x_2^2\right) - \left(\beta + o(1)\right)\ln|x| + b = \left(a + o(1)\right)\left(x_1^2 - x_2^2\right) + b \quad \text{as } |x| \to \infty, \tag{2.5}$$

or

$$u(x) = cx_1 + dx_2 - \left(\beta + o(1)\right)\ln|x| + e = \left(c + o(1)\right)x_1 + \left(d + o(1)\right)x_2 + e \quad \text{as } |x| \to \infty. \tag{2.6}$$

For a fixed $\delta \in (0, 1)$ we can find a constant $C_\delta > 0$ such that

$$\infty > \int_{\mathbb{R}^2} \left|f_\tau(u)\right| dx \geqslant \int_{\delta\leqslant u\leqslant 2\delta} \left|f_\tau(u)\right| dx \geqslant \int_{\delta\leqslant u\leqslant 2\delta} C_\delta\, dx = C_\delta\left|\left\{x \in \mathbb{R}^2 \;\middle|\; \delta \leqslant u(x) \leqslant 2\delta\right\}\right|. \tag{2.7}$$

Therefore, by using (2.5) and (2.6), we see that $|\{x \in \mathbb{R}^2 \mid \delta \leqslant u(x) \leqslant 2\delta\}| = \infty$ unless h is constant which proves the claim. Then, as a consequence of (2.4), we see that

$$\lim_{|x|\to\infty} \frac{u(x)}{\ln|x|} = -\beta. \tag{2.8}$$

Step 2. We claim that if $\beta = 0$ then $u \equiv 0$. Suppose that there exists $x_0 \in \mathbb{R}^2$ such that $u(x_0) < 0$. Then there exists $r > 0$ such that

$$u|_{B_r(x_0)} < 0. \tag{2.9}$$

Let us set $v_\delta(x) = \delta \ln(\frac{|x-x_0|}{r})$ on $\mathbb{R}^2 \setminus B_r(x_0)$. Then we see that $v_\delta \geqslant u$ on $\partial B_r(x_0)$. Since $u = o(\ln|x|)$ as $|x| \to \infty$ (which is of course a consequence of (2.8) and $\beta = 0$), then there exists $R_\delta > 0$ such that $v_\delta > u$ on $\mathbb{R}^2 \setminus B_{R_\delta}(0)$. We claim that $v_\delta \geqslant u$ on $B_{R_\delta}(0) \setminus B_r(x_0)$. If not, there exists $x_1 \in B_{R_\delta}(0) \setminus B_r(x_0)$ such that $u(x_1) - v_\delta(x_1) = \max_{B_{R_\delta}(0)\setminus B_r(x_0)}(u - v_\delta) > 0$. Then by the maximum principle, we see that

$$0 \geqslant \Delta(u - v_\delta)(x_1) = -f_\tau\left(u(x_1)\right) > 0 \quad \text{since } u(x_1) > v_\delta(x_1) \geqslant 0.$$

Thus, $v_\delta = \delta \ln(\frac{|x-x_0|}{r}) \geqslant u$ on $\mathbb{R}^2 \setminus B_r(x_0)$. Since $\delta > 0$ is arbitrary, we conclude that

$$u(x) \leqslant 0 \quad \text{on } \mathbb{R}^2 \setminus B_r(x_0). \tag{2.10}$$

Now we see that (2.9) and (2.10) contradict (2.2) with $\beta = 0$. Therefore we have $u \geqslant 0$ on $\mathbb{R}^2$, and then, by using (2.2) with $\beta = 0$, we conclude that $u \equiv 0$ on $\mathbb{R}^2$.

Step 3. From now on, we consider the case $\beta \neq 0$. By using the strong maximum principle and (2.8), we conclude that

$$\begin{cases} u > 0, & f_\tau(u) < 0 \quad \text{if } \beta < 0, \\ u < 0, & f_\tau(u) > 0 \quad \text{if } \beta > 0. \end{cases} \tag{2.11}$$

In view of (2.11), we can use the maximum principle to show that

$$\begin{cases} u \leqslant -\beta \ln|x| + C & \text{if } \beta < 0, \\ u \geqslant -\beta \ln|x| + C & \text{if } \beta > 0, \end{cases} \tag{2.12}$$

for large $|x|$ and a suitable constant $C \in \mathbb{R}$. By using (2.12), then $f_\tau(u) \in L^1(\mathbb{R}^2)$ implies that $|\beta| > 2$ and then we deduce the sharper estimate

$$u(x) = -\beta \ln|x| + O(1) \quad \text{as } |x| \to +\infty. \tag{2.13}$$

At this point, the method of moving planes to be used with (2.13) shows that u is radially symmetric. Since the proof is standard we skip it here and refer to [5,12] for further details. Therefore, the proof of Lemma 2.1 is completed. □

Let $u(r;s)$ be the solution of the following initial value problem

$$\begin{cases} u'' + \frac{1}{r}u' + \frac{e^u(1-e^u)}{(\tau+e^u)^3} = 0 \quad \text{for } r>0, \\ u(0;s) = s, \quad u'(0;s) = 0, \end{cases} \tag{2.14}$$

where u' denotes $\frac{du}{dr}(r;s)$ and let us set

$$\beta(s) \equiv \frac{1}{2\pi}\int_{\mathbb{R}^2} f_\tau\big(u(r;s)\big) = \int_0^\infty f_\tau\big(u(r;s)\big) r\,dr. \tag{2.15}$$

It turns out that the solutions of (2.14) admit only three kinds of limiting conditions as $r \to \infty$:

$$\begin{cases} \text{topological boundary condition: } u \to 0, \\ \text{nontopological boundary condition of type I: } u \to -\infty, \\ \text{nontopological boundary condition of type II: } u \to \infty. \end{cases} \tag{2.16}$$

We will use the following lemma recently obtained in [8].

Lemma 2.2. *Let $u(r;s)$ be a solution of (2.14). Then, we have*

(i) $\beta(0) = 0$. *In this case, $u(r;0) \equiv 0$ is the unique topological solution of (2.14);*
(ii) $\beta : (-\infty, 0) \to (4, \infty)$ *is strictly increasing and bijective and*

$$\lim_{s\to 0_-} \beta(s) = \infty \quad \textit{and} \quad \lim_{s\to -\infty} \beta(s) = 4.$$

In this case, $u(r;s)$ is a nontopological solution of type I;

(iii) $\beta : (0, \infty) \to (-\infty, -4)$ *is strictly increasing and bijective and*

$$\lim_{s\to 0_+} \beta(s) = -\infty \quad \textit{and} \quad \lim_{s\to \infty} \beta(s) = -4.$$

In this case, $u(r;s)$ is a nontopological solution of type II.

3. Proof of Theorem 1.5: the asymptotic behavior of solutions

One of the main steps in the proof of Theorem 1.5 is to obtain a uniform bound for

$$\int_\Omega \left| \frac{1}{\varepsilon^2} \frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3} \right| dx.$$

Toward this goal we have the following lemma.

Lemma 3.1. *Let u_ε be a sequence of solutions of (1.2). Then, there exists a constant $M_0 \in (0,\infty)$ such that*

$$\int_\Omega \left| \frac{1}{\varepsilon^2} \frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3} \right| dx \leqslant M_0.$$

Proof. We observe that, for any $a \in (0,\infty)$, it holds

$$\Delta u_\varepsilon \left(\frac{1-e^{u_\varepsilon}}{a+e^{u_\varepsilon}} \right) = \operatorname{div}\left[\nabla u_\varepsilon \left(\frac{1-e^{u_\varepsilon}}{a+e^{u_\varepsilon}} \right) \right] + \frac{(a+1)|\nabla u_\varepsilon|^2 e^{u_\varepsilon}}{(a+e^{u_\varepsilon})^2}. \tag{3.1}$$

Then, multiplying both sides of Eq. (1.2) by $\frac{1-e^{u_\varepsilon}}{a+e^{u_\varepsilon}}$ and integrating over Ω, we conclude that

$$\int_\Omega \frac{(a+1)|\nabla u_\varepsilon|^2 e^{u_\varepsilon}}{(a+e^{u_\varepsilon})^2} + \frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})^2}{(\tau+e^{u_\varepsilon})^3(a+e^{u_\varepsilon})}\,dx = 4\pi\left(\frac{N_1}{a}+N_2\right). \tag{3.2}$$

Let us fix $a=1$. Then there exist some constants $M_1, M_2 \geqslant 0$ such that

$$\int_\Omega \frac{|\nabla u_\varepsilon|^2 e^{u_\varepsilon}}{(1+e^{u_\varepsilon})^2}\,dx \leqslant M_1, \tag{3.3}$$

and

$$\int_\Omega \frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})^2}{(\tau+e^{u_\varepsilon})^3(1+e^{u_\varepsilon})}\,dx \leqslant M_2. \tag{3.4}$$

We also see that there exists $\delta_{\varepsilon,1} \in (1,2)$ such that

$$\int_{\{u_\varepsilon=-\delta_{\varepsilon,1}\}} |\nabla u_\varepsilon|\,dS = \int_{-2}^{-1}\left(\int_{\{u_\varepsilon=r\}} |\nabla u_\varepsilon|\,dS\right)dr, \tag{3.5}$$

and there exists a constant $c_0>0$ such that

$$\int_{\{-2\leqslant u_\varepsilon\leqslant 0\}} |\nabla u_\varepsilon|^2\,dx \leqslant c_0 \int_{\{-2\leqslant u_\varepsilon\leqslant 0\}} \frac{|\nabla u_\varepsilon|^2 e^{u_\varepsilon}}{(1+e^{u_\varepsilon})^2}\,dx \leqslant c_0 M_1. \tag{3.6}$$

Hence we also have

$$\int_{\{u_\varepsilon=-\delta_{\varepsilon,1}\}} |\nabla u_\varepsilon|\,dS = \int_{-2}^{-1}\left(\int_{\{u_\varepsilon=r\}} |\nabla u_\varepsilon|\,dS\right)dr = \int_{\{-2\leqslant u_\varepsilon\leqslant -1\}} |\nabla u_\varepsilon|^2\,dx \leqslant c_0 M_1. \tag{3.7}$$

Let ν be an exterior unit normal vector to $\partial\{x\in\Omega \mid -\delta_{\varepsilon,1}\leqslant u_\varepsilon\leqslant 0\}$. By using $\frac{\partial u_\varepsilon}{\partial\nu}|_{u_\varepsilon=0}\geqslant 0$ and (3.7), we see that

$$\begin{aligned} 0 &\leqslant \int_{\{-\delta_{\varepsilon,1}\leqslant u_\varepsilon\leqslant 0\}} \frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3}\,dx = -\int_{\{-\delta_{\varepsilon,1}\leqslant u_\varepsilon\leqslant 0\}} \Delta u_\varepsilon\,dx \\ &= -\int_{\{u_\varepsilon=-\delta_{\varepsilon,1}\}} \frac{\partial u_\varepsilon}{\partial\nu}\,dS - \int_{\{u_\varepsilon=0\}} \frac{\partial u_\varepsilon}{\partial\nu}\,dS \leqslant \int_{\{u_\varepsilon=-\delta_{\varepsilon,1}\}} |\nabla u_\varepsilon|\,dS \leqslant c_0 M_1. \end{aligned} \tag{3.8}$$

The same argument with minor changes shows that we can find constants $\delta_{\varepsilon,2}\in(1,2)$ and $c_1>0$ such that

$$\int_{\{0\leqslant u_\varepsilon\leqslant \delta_{\varepsilon,2}\}} \left|\frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3}\right|dx \leqslant c_1 M_1. \tag{3.9}$$

Moreover, there exist constants $c_2, c_3>0$ such that

$$\int_{\{u_\varepsilon\leqslant -\delta_{\varepsilon,1}\}} \left|\frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3}\right|dx \leqslant c_2 \int_{\{u_\varepsilon\leqslant -\delta_{\varepsilon,1}\}} \frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})^2}{(\tau+e^{u_\varepsilon})^3(1+e^{u_\varepsilon})}\,dx \leqslant c_2 M_2, \tag{3.10}$$

and

$$\int\limits_{\{u_\varepsilon \geqslant \delta_{\varepsilon,2}\}} \left| \frac{1}{\varepsilon^2} \frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3} \right| dx \leqslant c_3 \int\limits_{\{u_\varepsilon \geqslant \delta_{\varepsilon,2}\}} \frac{1}{\varepsilon^2} \frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})^2}{(\tau+e^{u_\varepsilon})^3(1+e^{u_\varepsilon})} dx \leqslant c_3 M_2. \tag{3.11}$$

The desired conclusion follows by using (3.8), (3.9), (3.10) and (3.11). □

Let us recall the following form of the Harnack inequality which will be widely used in the sequel (see [2] and [13]).

Lemma 3.2. *Let $D \subseteq \mathbb{R}^2$ be a smooth bounded domain and v satisfy:*

$$-\Delta v = f \quad \text{in } D,$$

with $f \in L^p(D)$, $p > 1$. For any subdomain $D' \subset\subset D$, there exist two positive constants $\sigma \in (0,1)$ and $\gamma > 0$, depending on D' only such that:

(a) *if $\sup_{\partial D} v \leqslant C$, then $\sup_{D'} v \leqslant \sigma \inf_{D'} v + (1+\sigma)\gamma \|f\|_{L^p} + (1-\sigma)C$,*
(b) *if $\inf_{\partial D} v \geqslant -C$, then $\sigma \sup_{D'} v \leqslant \inf_{D'} v + (1+\sigma)\gamma \|f\|_{L^p} + (1-\sigma)C$.*

Moreover, we have the following lemmas.

Lemma 3.3. *Let u_ε be a sequence of solutions of (1.2). Let K be a compact subset such that $K \subset \Omega \setminus Z$. Then there exist constants $a, b > 0$ such that $|u_\varepsilon(x_\varepsilon) - u_\varepsilon(z_\varepsilon)| \leqslant ar^2 + b$ for any $r > 0$ and $z_\varepsilon \in B_{\varepsilon r}(x_\varepsilon) \subseteq K$.*

Proof. By using the Green's representation formula for a solution u_ε of (1.2), we see that for $x \in K \subset\subset \Omega \setminus Z$,

$$\begin{aligned} u_\varepsilon(x) &= \frac{1}{|\Omega|}\int\limits_\Omega u_\varepsilon(y)\,dy + \int\limits_\Omega G(x,y)\big(-\Delta u_\varepsilon(y)\big)\,dy \\ &= \frac{1}{|\Omega|}\int\limits_\Omega u_\varepsilon(y)\,dy + \int\limits_\Omega G(x,y)\left(\frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3} - 4\pi\sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} + 4\pi\sum_{j=1}^{d_2} m_{j,2}\delta_{p_{j,2}}\right) dy \\ &= \frac{1}{|\Omega|}\int\limits_\Omega u_\varepsilon(y)\,dy + \int\limits_\Omega G(x,y)\frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3}\,dy + O(1). \end{aligned} \tag{3.12}$$

Then,

$$u_\varepsilon(x) - u_\varepsilon(z) = \int\limits_\Omega \big(G(x,y) - G(z,y)\big)\frac{1}{\varepsilon^2}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3}\,dy + O(1) \quad \text{for } x, z \in K. \tag{3.13}$$

In view of Lemma 3.1, we see that

$$u_\varepsilon(x) - u_\varepsilon(z) = \frac{1}{2\pi\varepsilon^2}\int\limits_\Omega \ln\left(\frac{|z-y|}{|x-y|}\right)\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3}\,dy + O(1) \quad \text{for } x, z \in K. \tag{3.14}$$

For fixed $r > 0$, we assume that $z_\varepsilon \in B_{\varepsilon r}(x_\varepsilon) \subseteq K$. By the mean value theorem, there exists $\theta = \theta(\varepsilon, y) \in (0,1)$ such that

$$\big|\ln|z_\varepsilon - y| - \ln|x_\varepsilon - y|\big| = \frac{||z_\varepsilon - y| - |x_\varepsilon - y||}{\theta|z_\varepsilon - y| + (1-\theta)|x_\varepsilon - y|} \leqslant \frac{|x_\varepsilon - z_\varepsilon|}{\theta|z_\varepsilon - y| + (1-\theta)|x_\varepsilon - y|}. \tag{3.15}$$

For any $y \in \Omega \setminus B_{2\varepsilon r}(x_\varepsilon)$, we have $|z_\varepsilon - y| \geqslant \varepsilon r$ and $|x_\varepsilon - y| \geqslant 2\varepsilon r$. Thus, we see that

$$\big|\ln|z_\varepsilon - y| - \ln|x_\varepsilon - y|\big| \leqslant \frac{\varepsilon r}{\theta\varepsilon r + (1-\theta)2\varepsilon r} = \frac{1}{2-\theta} \leqslant 1 \quad \text{on } \Omega \setminus B_{2\varepsilon r}(x_\varepsilon). \tag{3.16}$$

At this point, Lemma 3.1 implies that

$$\frac{1}{2\pi\varepsilon^2}\int_{\Omega\setminus B_{2\varepsilon r}(x_\varepsilon)}\left|\ln\left(\frac{|z_\varepsilon-y|}{|x_\varepsilon-y|}\right)\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3}\right|dy=O(1). \tag{3.17}$$

We also see that

$$\begin{aligned}\int_{B_{2\varepsilon r}(x_\varepsilon)}\left|\ln\left(\frac{|z_\varepsilon-y|}{|x_\varepsilon-y|}\right)\right|dy &\leqslant \int_{B_{2\varepsilon r}(x_\varepsilon)}\frac{|x_\varepsilon-z_\varepsilon|}{\theta|z_\varepsilon-y|+(1-\theta)|x_\varepsilon-y|}\,dy\\ &\leqslant \int_{B_{2\varepsilon r}(x_\varepsilon)}\frac{|x_\varepsilon-z_\varepsilon|}{\min\{|z_\varepsilon-y|,|x_\varepsilon-y|\}}\,dy\\ &\leqslant \int_{B_{2\varepsilon r}(x_\varepsilon)}\frac{|x_\varepsilon-z_\varepsilon|}{|z_\varepsilon-y|}+\frac{|x_\varepsilon-z_\varepsilon|}{|x_\varepsilon-y|}\,dy\\ &\leqslant \int_{B_{4\varepsilon r}(z_\varepsilon)}\frac{|x_\varepsilon-z_\varepsilon|}{|z_\varepsilon-y|}\,dy+\int_{B_{2\varepsilon r}(x_\varepsilon)}\frac{|x_\varepsilon-z_\varepsilon|}{|x_\varepsilon-y|}\,dy\\ &\leqslant 2\int_{B_{4\varepsilon r}(0)}\frac{|x_\varepsilon-z_\varepsilon|}{|y|}\,dy\leqslant 16r^2\varepsilon^2\pi.\end{aligned} \tag{3.18}$$

Therefore we conclude that

$$\frac{1}{2\pi\varepsilon^2}\int_{B_{2\varepsilon r}(x_\varepsilon)}\left|\ln\left(\frac{|z_\varepsilon-y|}{|x_\varepsilon-y|}\right)\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})}{(\tau+e^{u_\varepsilon})^3}\right|dy\leqslant 8r^2\sup_{t\in\mathbb{R}}\left|\frac{e^t(1-e^t)}{(\tau+e^t)^3}\right|, \tag{3.19}$$

and we readily obtain constants $a, b > 0$ such that for any $r > 0$, it holds

$$\left|u_\varepsilon(x_\varepsilon)-u_\varepsilon(z_\varepsilon)\right|\leqslant ar^2+b \quad \text{for } z_\varepsilon\in B_{\varepsilon r}(x_\varepsilon)\subseteq K. \qquad \square \tag{3.20}$$

Lemma 3.4. *Let K be a connected compact set such that $K\subset\Omega\setminus Z$. Suppose that there exists a sequence of solutions $\{u_\varepsilon\}$ of (1.2) such that*

$$\lim_{\varepsilon\to 0}\left(\inf_K|u_\varepsilon|\right)=0.$$

Then, we have $\|u_\varepsilon\|_{L^\infty(K)}\to 0$ as $\varepsilon\to 0$.

Proof. Choose a sequence of points $\{x_\varepsilon\}\subseteq K$ such that $|u_\varepsilon(x_\varepsilon)|=\inf_K|u_\varepsilon|$. Passing to a subsequence (still denoted by u_ε), we may assume that $\lim_{\varepsilon\to 0}x_\varepsilon=x_0\in K$. We argue by contradiction. Suppose that there exists a positive constant $c_K>0$ and a sequence $\{z_\varepsilon\}\subseteq K$ such that $\sup_K|u_\varepsilon|=|u_\varepsilon(z_\varepsilon)|\geqslant c_K$ for small $\varepsilon>0$. We will use the constant $M_0\geqslant 0$ obtained in Lemma 3.1. If $u_\varepsilon(z_\varepsilon)\leqslant -c_K$ then, by using Lemma 2.2, we can choose $s_1<0$ such that

$$\beta(s_1)>\frac{M_0}{\pi}\quad\text{and}\quad -c_K<s_1<0.$$

If $u_\varepsilon(z_\varepsilon)\geqslant c_K$ then, by using Lemma 2.2, we can choose $s_1>0$ such that

$$\beta(s_1)<-\frac{M_0}{\pi}\quad\text{and}\quad 0<s_1<c_K.$$

We can also choose $y_\varepsilon\in K$ such that $u_\varepsilon(y_\varepsilon)=s_1$ by the intermediate value theorem. Let $\bar{u}_\varepsilon(x)=u_\varepsilon(\varepsilon x+y_\varepsilon)$ for $x\in\Omega_{\varepsilon,y_\varepsilon}\equiv\{x\in\mathbb{R}^2 \mid \varepsilon x+y_\varepsilon\in K_1\}$ where K_1 is a compact subset such that $K\subset\operatorname{int}(K_1)\subset\Omega\setminus Z$. Then $\bar{u}_\varepsilon$ satisfies

$$\begin{cases}\Delta\bar{u}_\varepsilon+\dfrac{e^{\bar{u}_\varepsilon}(1-e^{\bar{u}_\varepsilon})}{(\tau+e^{\bar{u}_\varepsilon})^3}=0 \quad \text{on } \Omega_{\varepsilon,y_\varepsilon},\\ \bar{u}_\varepsilon(0)=s_1,\\ \displaystyle\int_{\Omega_{\varepsilon,y_\varepsilon}}\left|\frac{e^{\bar{u}_\varepsilon}(1-e^{\bar{u}_\varepsilon})}{(\tau+e^{\bar{u}_\varepsilon})^3}\right|dx\leqslant M_0.\end{cases} \tag{3.21}$$

By using Lemma 3.3, we see that $\bar{u}_\varepsilon$ is bounded in $C^0_{\mathrm{loc}}(\Omega_{\varepsilon,y_\varepsilon})$. Passing to a subsequence, we may assume that $\bar{u}_\varepsilon$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to a function u_* which is a solution of

$$\begin{cases} \Delta u_* + \dfrac{e^{u_*}(1-e^{u_*})}{(\tau+e^{u_*})^3} = 0 \quad \text{on } \mathbb{R}^2, \\ u_*(0) = s_1, \\ \displaystyle\int_{\mathbb{R}^2} \left|\frac{e^{u_*}(1-e^{u_*})}{(\tau+e^{u_*})^3}\right| dx \leqslant M_0. \end{cases} \tag{3.22}$$

By using Lemma 3.3 and Lemma 2.1, we conclude that u_* is radially symmetric with respect to some point $\bar{p}$ in $\mathbb{R}^2$ and u_* does not change sign. Hence Lemma 2.2 shows that

$$M_0 \geqslant \left|\int_{\mathbb{R}^2} \frac{e^{u_*}(1-e^{u_*})}{(\tau+e^{u_*})^3}\, dx\right| = 2\pi\left|\beta\big(u_*(\bar{p})\big)\right| \geqslant 2\pi\left|\beta(s_1)\right| > 2M_0, \tag{3.23}$$

which is the desired contradiction. Therefore, $\lim_{\varepsilon\to 0}\|u_\varepsilon\|_{L^\infty(K)} = 0$. □

As a corollary of Lemma 3.4, we obtain the following proposition.

Proposition 3.5. *Let u_ε be a sequence of solutions of (1.2). Then, up to subsequences, one of the following holds true:*

(a) *$u_\varepsilon \to 0$ uniformly on any compact subset of $\Omega \setminus Z$;*
(b) *for any compact subset $K \subset \Omega \setminus Z$, there exists $\nu_K > 0$ such that*

$$\lim_{\varepsilon\to 0}\Big(\sup_K u_\varepsilon\Big) \leqslant -\nu_K;$$

(c) *for any compact subset $K \subset \Omega \setminus Z$, there exists $\nu_K > 0$ such that*

$$\lim_{\varepsilon\to 0}\Big(\inf_K u_\varepsilon\Big) \geqslant \nu_K.$$

Proof. In view of Lemma 3.4, it suffices to show that (a) holds whenever both (b) and (c) fail to hold. Suppose that (b) and (c) do not hold. Then, we can take compact sets $K_1, K_2 \subset \Omega \setminus Z$ and sequences $\{x_{1,\varepsilon}\} \subset K_1$, $\{x_{2,\varepsilon}\} \subset K_2$ such that

$$\lim_{\varepsilon\to 0} u_\varepsilon(x_{1,\varepsilon}) \geqslant 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} u_\varepsilon(x_{2,\varepsilon}) \leqslant 0.$$

For any compact set $K \subset \Omega \setminus Z$, taking a connected compact set $\tilde{K} \subset \Omega \setminus Z$ such that

$$\tilde{K} \supseteq K \cup K_1 \cup K_2,$$

and using the intermediate value theorem, we can obtain a sequence $\{x_\varepsilon\} \subseteq \tilde{K}$ satisfying

$$\lim_{\varepsilon\to 0}\left|u_\varepsilon(x_\varepsilon)\right| = 0.$$

Hence, Lemma 3.4 yields that $\lim_{\varepsilon\to 0}\|u_\varepsilon\|_{L^\infty(\tilde{K})} = 0$, which completes the proof. □

Proof of Theorem 1.5 completed. First of all, we assume that (b) in Proposition 3.5 holds. In this case, we also suppose that there exists $r \in (0, \frac{1}{3}\operatorname{dist}(Z_1, Z_2))$ such that $B_{2r}(p_{i,1}) \cap B_{2r}(p_{j,1}) = \emptyset$ when $i \neq j$ and $\lim_{\varepsilon\to 0}(\sup_{\bigcup_{j=1}^{d_1} B_r(p_{j,1})} u_\varepsilon) \geqslant 0$. By using $\lim_{x\to p_{j,1}} u_\varepsilon(x) = -\infty$ and the intermediate value theorem, we see that there exists $x_\varepsilon \in \overline{\bigcup_{j=1}^{d_1} B_r(p_{j,1})}$ such that $|u_\varepsilon(x_\varepsilon)| = \inf_{\bigcup_{j=1}^{d_1} B_r(p_{j,1})} |u_\varepsilon| \to 0$ as $\varepsilon \to 0$. Let $x_0 \in \overline{\bigcup_{j=1}^{d_1} B_r(p_{j,1})}$ be the limit point of x_ε. Passing to a subsequence, only one of the following two possibilities can be satisfied: either $x_0 \notin Z_1$ or $x_0 \in Z_1$.

Case 1: $x_0 \notin Z_1$.

Let us fix a constant $d \in (0, \frac{1}{3}\operatorname{dist}(x_0, Z))$. Since $\overline{B_d(x_0)} \subset \Omega \setminus Z$ and in particular $\lim_{\varepsilon\to 0}(\inf_{\overline{B_d(x_0)}} |u_\varepsilon|) = 0$, then, in view of Lemma 3.4, we see that $\lim_{\varepsilon\to 0}(\sup_{\overline{B_d(x_0)}} |u_\varepsilon|) = 0$. This is a contradiction since we are assuming that Proposition 3.5(b) holds.

Case 2: $x_0 \in Z_1$.

For the sake of simplicity, we assume that $x_0 = 0 \in Z_1$. Since we are assuming that Proposition 3.5(b) holds, then there exists $\gamma > 0$ such that $\lim_{\varepsilon\to 0}(\sup_{|x|=r} u_\varepsilon) < -\gamma$. By the maximum principle, we see that $\sup_{|x|\leqslant r} u_\varepsilon \leqslant 0$. We claim that

$$\lim_{\varepsilon\to 0} \frac{|x_\varepsilon|}{\varepsilon} = \infty. \tag{3.24}$$

We argue by contradiction and suppose that $\liminf_{\varepsilon\to 0} \frac{|x_\varepsilon|}{\varepsilon} < \infty$. Hence, passing to a subsequence, we could assume that $\frac{|x_\varepsilon|}{\varepsilon} \leqslant c$ for some constant $c > 0$ and small $\varepsilon > 0$. Note that $u_\varepsilon(x) = 2m_{j,1}\ln|x| + v_\varepsilon(x)$ near $x = 0$ for some smooth function v_ε and $1 \leqslant j \leqslant d_1$. Let $\hat{v}_\varepsilon(x) = v_\varepsilon(|x_\varepsilon|x) + 2m_{j,1}\ln|x_\varepsilon|$ for $|x| < \frac{r}{|x_\varepsilon|}$. Then $\hat{v}_\varepsilon$ satisfies

$$\Delta\hat{v}_\varepsilon + \frac{|x_\varepsilon|^2}{\varepsilon^2}\frac{|x|^{2m_{j,1}}e^{\hat{v}_\varepsilon}(1 - |x|^{2m_{j,1}}e^{\hat{v}_\varepsilon})}{(\tau + |x|^{2m_{j,1}}e^{\hat{v}_\varepsilon})^3} = 0 \quad \text{on } B_{\frac{r}{|x_\varepsilon|}}(0). \tag{3.25}$$

We also observe that

$$\hat{v}_\varepsilon(x) = u_\varepsilon\big(|x_\varepsilon|x\big) - 2m_{j,1}\ln|x| \leqslant -2m_{j,1}\ln|x| \quad \text{for } |x| \leqslant \frac{r}{|x_\varepsilon|}, \tag{3.26}$$

and

$$\lim_{\varepsilon\to 0}\hat{v}_\varepsilon\left(\frac{x_\varepsilon}{|x_\varepsilon|}\right) = \lim_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) = 0. \tag{3.27}$$

Since $\frac{|x_\varepsilon|}{\varepsilon} \leqslant c$ and $\sup_{t\geqslant 0}|\frac{t(1-t)}{(\tau+t)^3}| < \infty$, then for any $p > 1$ and $R > 0$, there exists a constant $C_{p,R} > 0$ such that $\lim_{\varepsilon\to 0}\|\Delta\hat{v}_\varepsilon\|_{L^p(B_R(0))} \leqslant C_{p,R}$. By using (3.26), (3.27), and Lemma 3.2, we see that for large $R > 0$, there exist $\sigma \in (0, 1)$ and $\gamma > 0$, independent of $\varepsilon > 0$, such that

$$o(1) = \hat{v}_\varepsilon\left(\frac{x_\varepsilon}{|x_\varepsilon|}\right) \leqslant \sup_{B_{R/2}(0)}\hat{v}_\varepsilon \leqslant \sigma \inf_{B_{R/2}(0)}\hat{v}_\varepsilon + (1+\sigma)\gamma\|\Delta\hat{v}_\varepsilon\|_{L^p(B_R(0))} - (1-\sigma)2m_{j,1}\ln R.$$

Hence $\hat{v}_\varepsilon$ is bounded in $C^0_{\mathrm{loc}}(B_{\frac{r}{|x_\varepsilon|}}(0))$. Passing to a subsequence, we may assume that $\lim_{\varepsilon\to 0}\frac{x_\varepsilon}{|x_\varepsilon|} = y_0 \in S^1$, $\lim_{\varepsilon\to 0}\frac{|x_\varepsilon|}{\varepsilon} = c_0 \geqslant 0$, and $\hat{v}_\varepsilon$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to a function $\hat{v}$ satisfying

$$\Delta\hat{v} + \frac{c_0^2|x|^{2m_{j,1}}e^{\hat{v}}(1 - |x|^{2m_{j,1}}e^{\hat{v}})}{(\tau + |x|^{2m_{j,1}}e^{\hat{v}})^3} = 0 \quad \text{in } \mathbb{R}^2. \tag{3.28}$$

Then the function $\hat{u} = \hat{v} + 2m_{j,1}\ln|x| \leqslant 0$ satisfies

$$\Delta\hat{u} + \frac{c_0^2 e^{\hat{u}}(1 - e^{\hat{u}})}{(\tau + e^{\hat{u}})^3} = 4\pi m_{j,1}\delta_0 \quad \text{in } \mathbb{R}^2. \tag{3.29}$$

Since $\hat{u} \leqslant 0$, we have $c_0 > 0$ and since $\hat{u}(y_0) = \lim_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) = 0$, we have $\hat{u} \equiv 0$ by the strong maximum principle. This is of course a contradiction and (3.24) is proved.

At this point, let us fix a constant $s_2 < 0$ such that $\beta(s_2) \geqslant \frac{M_0}{\pi}$ (see (2.15) and Lemma 3.1) and $-\gamma < s_2 < 0$. We can choose y_ε on a line segment joining x_ε to $\frac{r x_\varepsilon}{|x_\varepsilon|}$ such that $u_\varepsilon(y_\varepsilon) = s_2$ and $|y_\varepsilon| \geqslant |x_\varepsilon|$ by the intermediate value theorem. Let $\hat{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x + y_\varepsilon)$ on $B_{\frac{|x_\varepsilon|}{2\varepsilon}}(0)$. We note that $0 \notin B_{\frac{|x_\varepsilon|}{2}}(y_\varepsilon)$. Then $\hat{u}_\varepsilon$ satisfies

$$\begin{cases} \Delta \hat{u}_\varepsilon + \dfrac{e^{\hat{u}_\varepsilon}(1-e^{\hat{u}_\varepsilon})}{(\tau + e^{\hat{u}_\varepsilon})^3} = 0 \quad \text{in } B_{\frac{|x_\varepsilon|}{2\varepsilon}}(0), \\ \hat{u}_\varepsilon(0) = s_2, \\ \displaystyle\int\limits_{B_{\frac{|x_\varepsilon|}{2\varepsilon}}(0)} \left| \dfrac{e^{\hat{u}_\varepsilon}(1-e^{\hat{u}_\varepsilon})}{(\tau + e^{\hat{u}_\varepsilon})^3} \right| dx \leqslant M_0. \end{cases} \tag{3.30}$$

By using the fact that $\hat{u}_\varepsilon \leqslant 0$ and $\hat{u}_\varepsilon(0) = s_2$ with Lemma 3.2, then we see that for large $R > 0$ there exist $\sigma \in (0,1)$ and $\gamma > 0$, independent of $\varepsilon > 0$, such that

$$s_2 = \hat{u}_\varepsilon(0) \leqslant \sup_{B_{R/2}(0)} \hat{u}_\varepsilon \leqslant \sigma \inf_{B_{R/2}(0)} \hat{u}_\varepsilon + (1+\sigma)\gamma \|\Delta \hat{u}_\varepsilon\|_{L^p(B_R(0))},$$

and $\hat{u}_\varepsilon$ is bounded in $C^0_{\mathrm{loc}}(B_{\frac{|x_\varepsilon|}{2\varepsilon}}(0))$. Then $\hat{u}_\varepsilon$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to a function u_* satisfying

$$\begin{cases} \Delta u_* + \dfrac{e^{u_*}(1-e^{u_*})}{(\tau + e^{u_*})^3} = 0 \quad \text{in } \mathbb{R}^2, \\ u_*(0) = s_2, \quad u_* \leqslant 0, \\ \displaystyle\int\limits_{\mathbb{R}^2} \left| \dfrac{e^{u_*}(1-e^{u_*})}{(\tau + e^{u_*})^3} \right| dx \leqslant M_0. \end{cases} \tag{3.31}$$

By using Lemma 2.1, we see that u_* is radially symmetric about some point. Then, we see that $|\int_{\mathbb{R}^2} \frac{e^{u_*}(1-e^{u_*})}{(\tau+e^{u_*})^3}|\,dx \geqslant 2\pi\beta(s_2) \geqslant 2M_0$ from Lemma 2.2 which is once more a contradiction.

At this point, by using the above results, we see that $\lim_{\varepsilon\to 0}(\sup_{\bigcup_{j=1}^{d_1} B_r(p_{j,1})} u_\varepsilon) < -c$ for some constant $c > 0$, which shows that (b) in Theorem 1.5 holds whenever (b) in Proposition 3.5 holds.

The proof of (c) in Theorem 1.5 follows essentially by the same argument and we skip it here to avoid repetitions. □

4. Proof of Theorem 1.1: stable solution ⇒ topological solution

In this section, we will prove one of the implications in the statement of Theorem 1.1, that is, stable solution ⇒ topological solution whenever (H1)–(H2) hold. Let u_ε be a sequence of stable solutions of (1.2). To prove Theorem 1.1, we argue by contradiction and suppose that u_ε does not converge to 0 almost everywhere. Then either (b) or (c) of Theorem 1.5 would occur. Since $f_\tau(u) = -\frac{f_{\tau^{-1}}(-u)}{\tau^3}$, without loss of generality we can assume that u_ε has the profile (b) of Theorem 1.5.

If $u_\varepsilon - 2\ln\varepsilon$ has a bubble at some point in $\Omega \setminus Z_2$, then there are two possibilities. One is that the limiting equation is the mean field equation and it is easy to see that the solution is not stable. Another one is that the limiting equation is (1.2), but defined in the whole $\mathbb{R}^2$, and after a suitable scaling, u_ε tends to a nontopological solution u such that $\lim_{|x|\to\infty} u(x) = -\infty$. Again, this is also unstable. The proof is not difficult. But for the sake of completeness, we put the proof in Appendix A. To the best of our knowledge, even for CSH (1.3), this result has not been written in the literature.

Therefore, from now on, we may assume that for any small $r > 0$, there exists $c_r > 0$ such that

$$w_\varepsilon \equiv u_\varepsilon - 2\ln\varepsilon < c_r \quad \text{on } \Omega \setminus \bigcup_j \big(B_r(p_{j,2})\big). \tag{4.1}$$

Now we consider

$$\begin{aligned} \mu_\varepsilon &\equiv \inf_{\phi \in W^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla\phi|^2 - \frac{1}{\varepsilon^2} f'_\tau(u_\varepsilon)\phi^2\,dx}{\|\phi\|^2_{L^2(\Omega)}} \\ &\leqslant \frac{1}{|\Omega|^2} \int\limits_\Omega -\frac{1}{\varepsilon^2} f'_\tau(u_\varepsilon)\,dx = \frac{1}{|\Omega|^2} \int\limits_\Omega \frac{e^{u_\varepsilon}(-\tau + 2(\tau+1)e^{u_\varepsilon} - e^{2u_\varepsilon})}{\varepsilon^2(\tau + e^{u_\varepsilon})^4}\,dx. \end{aligned} \tag{4.2}$$

To derive a contradiction, we want to prove that for small $\varepsilon > 0$,

$$\int_\Omega \frac{e^{u_\varepsilon}(-\tau + 2(\tau+1)e^{u_\varepsilon} - e^{2u_\varepsilon})}{\varepsilon^2(\tau + e^{u_\varepsilon})^4}\,dx < 0. \tag{4.3}$$

To prove (4.3), we need to compute the integral over a small neighborhood of each $p_{j,2} \in Z_2$. Let us first show a simple fact about w_ε.

Lemma 4.1. *w_ε satisfies*

$$\textit{either}\quad \lim_{\varepsilon\to 0}\|w_\varepsilon - u_0\|_{L^\infty(\Omega\setminus\bigcup_j(B_r(p_{j,2})))} < \infty \quad \textit{or} \quad \lim_{\varepsilon\to 0}\Big(\sup_{\Omega\setminus\bigcup_j(B_r(p_{j,2}))} w_\varepsilon\Big) = -\infty. \tag{4.4}$$

Moreover, for any small $r > 0$, there exists $C_r > 0$ such that

$$\sup_{\Omega\setminus\bigcup_j(B_r(p_{j,2}))} \big|\nabla(w_\varepsilon - u_0)\big| \leqslant C_r. \tag{4.5}$$

Proof. We note that w_ε satisfies the following equation

$$\Delta w_\varepsilon + \frac{e^{w_\varepsilon}(1-\varepsilon^2 e^{w_\varepsilon})}{(\tau + \varepsilon^2 e^{w_\varepsilon})^3} = 4\pi\sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} - 4\pi\sum_{j=1}^{d_2} m_{j,2}\delta_{p_{j,2}} \quad \text{on } \Omega. \tag{4.6}$$

We also see that

$$\Delta(w_\varepsilon - u_0) + \frac{e^{w_\varepsilon}(1-\varepsilon^2 e^{w_\varepsilon})}{(\tau + \varepsilon^2 e^{w_\varepsilon})^3} = \frac{4\pi(N_1 - N_2)}{|\Omega|} \quad \text{on } \Omega.$$

By using (4.1) and Lemma 3.2, we readily obtain (4.4).

Next, by using the Green's representation formula for a solution w_ε of (4.6), we see that for $x \in \Omega$,

$$w_\varepsilon(x) - u_0(x) = \frac{1}{|\Omega|}\int_\Omega w_\varepsilon(y)\,dy + \int_\Omega G(x,y)\frac{e^{w_\varepsilon}(1-\varepsilon^2 e^{w_\varepsilon})}{(\tau + e^{w_\varepsilon})^3}\,dy. \tag{4.7}$$

By using Lemma 3.1, we conclude that there exists a constant $C > 0$, independent of $\varepsilon > 0$ and $r > 0$, such that for $x \in \Omega \setminus \bigcup_j(B_r(p_{j,2}))$, it holds

$$\begin{aligned}
\big|\nabla\big(w_\varepsilon(x) - u_0(x)\big)\big| &\leqslant \frac{1}{2\pi}\int_\Omega \frac{1}{|x-y|}\left|\frac{e^{w_\varepsilon}(1-\varepsilon^2 e^{w_\varepsilon})}{(\tau + \varepsilon^2 e^{w_\varepsilon})^3}\right| dy + C \\
&\leqslant \frac{1}{2\pi}\left\{\sup_{\Omega\setminus\bigcup_j(B_{\frac{r}{2}}(p_{j,2}))}\left|\frac{e^{w_\varepsilon}(1-\varepsilon^2 e^{w_\varepsilon})}{(\tau + \varepsilon^2 e^{w_\varepsilon})^3}\right| \int_{B_{\frac{r}{2}}(x)} \frac{1}{|x-y|}\,dy \right. \\
&\qquad \left. + \frac{2}{r}\int_{\Omega\setminus B_{\frac{r}{2}}(x)} \left|\frac{e^{w_\varepsilon}(1-\varepsilon^2 e^{w_\varepsilon})}{(\tau + \varepsilon^2 e^{w_\varepsilon})^3}\right| dy\right\} + C.
\end{aligned} \tag{4.8}$$

By using (4.1) and Lemma 3.1, we obtain (4.5) which concludes the proof of our lemma. □

If $\lim_{\varepsilon\to 0}\|w_\varepsilon - u_0\|_{L^\infty(\Omega\setminus\bigcup_j(B_r(p_{j,2})))} < \infty$ for any small $r > 0$, then there exists a function w satisfying

$$w_\varepsilon \to w \quad \text{in } C^2_{\mathrm{loc}}(\Omega \setminus Z_2).$$

By using Lemma 3.1, we also see that w satisfies

$$\Delta w + \frac{e^w}{\tau^3} = 4\pi\sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} + 4\pi\sum_{j=1}^{d_2} \beta_{j,2}\delta_{p_{j,2}} \quad \text{on } \Omega \text{ where } \beta_{j,2} > -1. \tag{4.9}$$

If $\lim_{\varepsilon\to 0}(\sup_{\Omega\setminus\bigcup_j(B_r(p_{j,2}))} w_\varepsilon)=-\infty$, then for fixed $x_0\in\Omega\setminus Z$, and by using (4.5), we see that there exists a function g satisfying

$$g_\varepsilon\equiv w_\varepsilon-w_\varepsilon(x_0)\to g\quad\text{in } C^2_{\mathrm{loc}}(\Omega\setminus Z_2),$$

and

$$\Delta g=4\pi\sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}}+4\pi\sum_{j=1}^{d_2}\beta_{j,2}\delta_{p_{j,2}}\quad\text{on }\Omega\text{ where }\beta_{j,2}\in\mathbb{R}. \tag{4.10}$$

Clearly (4.10) implies $N_1+\sum_{j=1}^{d_2}\beta_{j,2}=0$.

Next we have the following property.

Lemma 4.2. *For any* $1\leqslant j\leqslant d_2$,

$$\lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{B_r(p_{j,2})}\frac{f_\tau(u_\varepsilon)}{\varepsilon^2}\,dx=-4\pi(m_{j,2}+\beta_{j,2}), \tag{4.11}$$

and

$$\lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{B_r(p_{j,2})}\frac{F_{2,\tau}(u_\varepsilon)}{\varepsilon^2}\,dx=2\pi\left(\beta_{j,2}^2-m_{j,2}^2\right), \tag{4.12}$$

where $F_{2,\tau}(u)\equiv\frac{e^u((1-\tau)e^u+2\tau)}{2\tau^2(\tau+e^u)^2}$.

Proof. For the sake of simplicity, we assume that $p_{j,2}=0$. We consider the following two cases.

Case 1. $w_\varepsilon\to w$ in $C^2_{\mathrm{loc}}(\Omega\setminus Z_2)$.

We integrate (4.6) on $B_r(0)$ and take the limit as $\varepsilon\to 0$ to conclude that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{B_r(0)}\frac{f_\tau(u_\varepsilon)}{\varepsilon^2}\,dx&=-\lim_{\varepsilon\to 0}\left(4\pi m_{j,2}+\int_{\partial B_r(0)}\frac{\partial w_\varepsilon}{\partial\nu}\,d\sigma\right)=-\left(4\pi m_{j,2}+\int_{\partial B_r(0)}\frac{\partial w}{\partial\nu}\,d\sigma\right)\\&=-4\pi(m_{j,2}+\beta_{j,2})+\int_{B_r(0)}\frac{e^w}{\tau^3}\,dx.\end{aligned}$$

Clearly Lemma 3.1 implies that

$$\lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{B_r(0)}\frac{f_\tau(u_\varepsilon)}{\varepsilon^2}\,dx=-4\pi(m_{j,2}+\beta_{j,2}).$$

At this point we consider the function $v\equiv w-2\beta_{j,2}\ln|x|$ which satisfies

$$\Delta v+\frac{e^w}{\tau^3}=0\quad\text{on } B_r(0). \tag{4.13}$$

Multiplying (4.13) by $\nabla w\cdot x$ and integrating over $B_r(0)$, we conclude that

$$\int_{\partial B_r(0)}\left[\left(\nabla v\cdot\frac{x}{|x|}\right)(\nabla v\cdot x)-\frac{|\nabla v|^2|x|}{2}+\frac{e^w|x|}{\tau^3}\right]d\sigma=\int_{B_r(0)}\frac{(2+2\beta_{j,2})e^w}{\tau^3}\,dx. \tag{4.14}$$

Let us also consider the function $v_\varepsilon(x)\equiv u_\varepsilon(x)+2m_{j,2}\ln|x|$ which satisfies

$$\Delta v_\varepsilon+\frac{f_\tau(u_\varepsilon)}{\varepsilon^2}=0\quad\text{on } B_r(0). \tag{4.15}$$

Multiplying (4.15) by $\nabla u_\varepsilon\cdot x$ and integrating over $B_r(0)$, we have

$$\int_{B_r(0)} \frac{2F_{2,\tau}(u_\varepsilon)}{\varepsilon^2}\,dx = \int_{\partial B_r(0)} \left[\left(\nabla v_\varepsilon \cdot \frac{x}{|x|}\right)(\nabla v_\varepsilon \cdot x) - \frac{|\nabla v_\varepsilon|^2|x|}{2} + \frac{F_{2,\tau}(u_\varepsilon)|x|}{\varepsilon^2} - 2m_{j,2}\frac{\nabla v_\varepsilon \cdot x}{|x|}\right]d\sigma. \tag{4.16}$$

Hence, as $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0}\int_{B_r(0)} \frac{2F_{2,\tau}(u_\varepsilon)}{\varepsilon^2}\,dx = \int_{\partial B_r(0)} \left[\frac{(\nabla v \cdot x + 2(m_{j,2}+\beta_{j,2}))^2}{|x|} - \left|\nabla v + \frac{2(m_{j,2}+\beta_{j,2})x}{|x|^2}\right|^2 \frac{|x|}{2} + \frac{e^w|x|}{\tau^3} - \frac{2m_{j,2}\{\nabla v \cdot x + 2(m_{j,2}+\beta_{j,2})\}}{|x|}\right]d\sigma.$$

By using (4.14), we also see that

$$\lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{B_r(0)} \frac{F_{2,\tau}(u_\varepsilon)}{\varepsilon^2}\,dx = 2\pi\left(\beta_{j,2}^2 - m_{j,2}^2\right),$$

which is (4.12).

Case 2. $g_\varepsilon = w_\varepsilon - w_\varepsilon(x_0) \to g$ in $C^2_{\mathrm{loc}}(\Omega \setminus Z_2)$.

We integrate (4.6) on $B_r(0)$ and take the limit as $\varepsilon \to 0$ to conclude that

$$\lim_{\varepsilon\to 0}\int_{B_r(0)} \frac{f_\tau(u_\varepsilon)}{\varepsilon^2}\,dx = -\lim_{\varepsilon\to 0}\left(4\pi m_{j,2} + \int_{\partial B_r(0)} \frac{\partial g_\varepsilon}{\partial \nu}\,d\sigma\right) = -\left(4\pi m_{j,2} + \int_{\partial B_r(0)} \frac{\partial g}{\partial \nu}\,d\sigma\right) = -4\pi(m_{j,2}+\beta_{j,2}).$$

Let us consider the function $h \equiv g - 2\beta_{j,2}\ln|x|$. Then h satisfies

$$\Delta h = 0 \quad \text{on } B_r(0). \tag{4.17}$$

Next we also define $h_\varepsilon(x) \equiv g_\varepsilon(x) + 2m_{j,2}\ln|x|$ which satisfies

$$\Delta h_\varepsilon + \frac{f_\tau(u_\varepsilon)}{\varepsilon^2} = 0 \quad \text{on } B_r(0). \tag{4.18}$$

Multiplying (4.18) by $\nabla u_\varepsilon \cdot x$ and integrating over $B_r(0)$, we see that

$$\lim_{\varepsilon\to 0}\int_{B_r(0)} \frac{2F_{2,\tau}(u_\varepsilon)}{\varepsilon^2}\,dx = \int_{\partial B_r(0)} \left[\frac{(\nabla h \cdot x + 2(m_{j,2}+\beta_{j,2}))^2}{|x|} - \left|\nabla h + \frac{2(m_{j,2}+\beta_{j,2})x}{|x|^2}\right|^2 \frac{|x|}{2} - \frac{2m_{j,2}\{\nabla h \cdot x + 2(m_{j,2}+\beta_{j,2})\}}{|x|}\right]d\sigma.$$

By using (4.17), we also conclude that

$$\lim_{\varepsilon\to 0}\int_{B_r(0)} \frac{F_{2,\tau}(u_\varepsilon)}{\varepsilon^2}\,dx = 2\pi\left(\beta_{j,2}^2 - m_{j,2}^2\right),$$

which is (4.11). □

Let $\hat{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x)$ which satisfies

$$\Delta \hat{u}_\varepsilon + f_\tau(\hat{u}_\varepsilon) = -4\pi m_{j,2}\delta_{p_{j,2}} \quad \text{on } B_{\frac{r}{\varepsilon}}(p_{j,2}).$$

Moreover we have:

Lemma 4.3. *There exists a constant $c > 0$, independent of $r > 0$ and $\varepsilon > 0$, such that*

$$\left|\nabla \hat{u}_\varepsilon(x) + \frac{2m_{j,2}(x - p_{j,2})}{|x - p_{j,2}|^2}\right| \leqslant c \quad \text{on } B_{\frac{r}{\varepsilon}}(p_{j,2}). \tag{4.19}$$

Proof. For the sake of simplicity, we assume that $p_{j,2}=0$. By using the Green's representation formula for a solution u_ε of (1.2) (see (3.12)) and Lemma 3.1, we see that for $x \in B_r(0)$,

$$\begin{aligned}\left|\nabla u_\varepsilon(x)+\frac{2m_{j,2}x}{|x|^2}\right| &\leqslant C+\frac{1}{2\pi\varepsilon^2}\int_\Omega \frac{|f_\tau(u_\varepsilon)|}{|x-y|}\,dy\\ &= C+\frac{1}{2\pi\varepsilon^2}\left(\int_{B_\varepsilon(x)}\frac{|f_\tau(u_\varepsilon)|}{|x-y|}\,dy+\int_{\Omega\setminus B_\varepsilon(x)}\frac{|f_\tau(u_\varepsilon)|}{|x-y|}\,dy\right)\\ &\leqslant C+\frac{C'}{\varepsilon},\end{aligned}$$

for some constants $C, C'>0$, independent of $r>0$ and $\varepsilon>0$. The desired conclusion follows by the substitution $\hat{u}_\varepsilon(x)=u_\varepsilon(\varepsilon x)$. □

As mentioned above, we have to study the behavior of $\hat{u}_\varepsilon$ as $\varepsilon\to 0$. This is the most delicate part of our proof. Here, the Pohozaev identity (4.12) is used.

Lemma 4.4. *If $\tau=1$ or $m_{j,2}\in[0,1]$, then for any $\eta>0$,*

$$\lim_{\varepsilon\to 0}\Big(\sup_{|x-p_{j,2}|=\eta}\hat{u}_\varepsilon(x)\Big)=\lim_{\varepsilon\to 0}\Big(\sup_{|x-p_{j,2}|=\eta}u_\varepsilon(\varepsilon x)\Big)=-\infty. \tag{4.20}$$

Moreover, if $w_\varepsilon\to w$ in $C^2_{\mathrm{loc}}(\Omega\setminus Z_2)$, then (4.20) always holds without any further assumptions for τ and $m_{j,2}$.

Proof. For the sake of simplicity, we assume that $p_{j,2}=0$. We divide the proof of our lemma into two steps.

Step 1. We claim that for any $\eta>0$, there exists $c_\eta>0$ such that for small $\varepsilon>0$,

$$\sup_{|x|=\eta}\hat{u}_\varepsilon(x)=\sup_{|x|=\eta}u_\varepsilon(\varepsilon x)<c_\eta. \tag{4.21}$$

We argue by contradiction and suppose that there exists $\eta_0>0$ such that

$$\lim_{\varepsilon\to 0}\Big(\sup_{|x|=\eta_0}\hat{u}_\varepsilon(x)\Big)=\lim_{\varepsilon\to 0}\Big(\sup_{|x|=\eta_0}u_\varepsilon(\varepsilon x)\Big)=+\infty.$$

Since $|\nabla\hat{u}_\varepsilon|$ is locally bounded in $B_{\frac{r}{\varepsilon}}(0)\setminus\{0\}$, we also see that

$$\lim_{\varepsilon\to 0}\Big(\inf_{|x|=\eta_0}\hat{u}_\varepsilon(x)\Big)=\lim_{\varepsilon\to 0}\Big(\inf_{|x|=\eta_0}u_\varepsilon(\varepsilon x)\Big)=+\infty. \tag{4.22}$$

Fix $c\in(-\infty,0)$ and $n\in\mathbb{N}$. Since $\lim_{\varepsilon\to 0}\sup_{\partial B(0,r/\varepsilon)}\hat{u}_\varepsilon=-\infty$, then (4.22) implies that there exists $y^i_\varepsilon=(r^i_\varepsilon\cos\theta_i, r^i_\varepsilon\sin\theta_i)$ such that $\hat{u}_\varepsilon(y^i_\varepsilon)=c$ where $\theta_i=\frac{2\pi i}{n}$ and

$$\lim_{\varepsilon\to 0}r^i_\varepsilon=+\infty,\qquad \lim_{\varepsilon\to 0}\big(\varepsilon r^i_\varepsilon\big)=0\quad\text{for all } 1\leqslant i\leqslant n.$$

In view of (4.19), we see that the function $\bar{u}^i_\varepsilon(x)=\hat{u}_\varepsilon(x+y^i_\varepsilon)$ satisfies

$$\Delta\bar{u}^i_\varepsilon+f_\tau\big(\bar{u}^i_\varepsilon\big)=0,\qquad \big|\nabla\bar{u}^i_\varepsilon\big|\leqslant C_1\quad\text{on } B_{\frac{r^i_\varepsilon}{2}}(0),\qquad \bar{u}^i_\varepsilon(0)=c<0,$$

for some constant $C_1>0$. Then $\{\bar{u}^i_\varepsilon\}$ is uniformly bounded in $L^\infty_{\mathrm{loc}}(B_{\frac{r^i_\varepsilon}{2}}(0))$ and there exists a function $\bar{u}^i$ such that $\bar{u}^i_\varepsilon\to\bar{u}^i$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ and

$$\Delta\bar{u}^i+f_\tau\big(\bar{u}^i\big)=0,\qquad \big|\nabla\bar{u}^i\big|\leqslant C_1\quad\text{on } \mathbb{R}^2,\qquad \bar{u}^i(0)=c<0.$$

By using Lemma 2.1, we see that $\bar{u}^i$ is radially symmetric with respect to some point $\bar{p}^i$ in $\mathbb{R}^2$ and $\bar{u}^i$ does not change sign. Hence Lemma 3.1 and Lemma 2.2 imply that there exists a large $R>0$ such that $\int_{B_R(0)}|f_\tau(\bar{u}^i_\varepsilon)|\,dx\geqslant 4\pi$. Then,

$$M_0 \geqslant \int_{B_{\frac{r}{\varepsilon}}(0)} \left|f_\tau(\hat{u}_\varepsilon)\right| dx \geqslant \sum_{i=1}^{n} \int_{B_R(y_\varepsilon^i)} \left|f_\tau(\hat{u}_\varepsilon)\right| dx = \sum_{i=1}^{n} \int_{B_R(0)} \left|f_\tau\big(\bar{u}_\varepsilon^i\big)\right| dx \geqslant 4\pi n \quad \text{for any } n \in \mathbb{N},$$

which is a contradiction. Therefore (4.21) holds as claimed.

Moreover, by using (4.1), (4.21) and the maximum principle, we obtain

$$\sup_{\eta \leqslant |x| \leqslant \frac{r}{\varepsilon}} \hat{u}_\varepsilon(x) = \sup_{\eta \leqslant |x| \leqslant \frac{r}{\varepsilon}} u_\varepsilon(\varepsilon x) < c_\eta. \tag{4.23}$$

Step 2. To prove our lemma, we argue by contradiction and suppose that $\{\hat{u}_\varepsilon\}$ is uniformly bounded in $L^\infty_{\mathrm{loc}}(B_{\frac{r}{\varepsilon}}(0) \setminus \{0\})$. Then, since $\sup_{t\in\mathbb{R}} |f_\tau(t)| < \infty$ and by using (4.19) and (4.23), we see that there exists a function $\hat{u}$ such that $\hat{u}_\varepsilon \to \hat{u}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$ and

$$\begin{cases} \Delta \hat{u} + f_\tau(\hat{u}) = -4\pi m_{j,2}\delta_0 \quad \text{on } \mathbb{R}^2, \\ f_\tau(\hat{u}) \in L^1(\mathbb{R}^2), \quad \sup_{|x|\geqslant 1} \hat{u}(x) \leqslant C, \quad \sup_{|x|\geqslant 1} \left|\nabla \hat{u}(x)\right| \leqslant C, \end{cases} \tag{4.24}$$

for some constant $C > 0$. Let $\hat{\beta} = \frac{1}{2\pi} \int_{\mathbb{R}^2} f_\tau(\hat{u})\, dx$. Then we obtain

$$\lim_{|x|\to\infty} \frac{\hat{u}(x)}{\ln|x|} = -2m_{j,2} - \hat{\beta}.$$

In view of (4.24), we also see that we cannot have $\lim_{|x|\to\infty} \hat{u}(x) > 0$. Moreover, since $f_\tau(\hat{u}) \in L^1(\mathbb{R}^2)$ and $\sup_{|x|\geqslant 1} |\nabla \hat{u}(x)| \leqslant C$, then we see that

$$\text{either} \quad \lim_{|x|\to\infty} \hat{u}(x) = 0 \quad \text{or} \quad \lim_{|x|\to\infty} \hat{u}(x) = -\infty. \tag{4.25}$$

Indeed, if there exists a sequence $x_n \in \mathbb{R}^2$ such that

$$\lim_{n\to\infty} |x_n| \to +\infty, \qquad \lim_{n\to\infty} \hat{u}(x_n) = c \notin \{0, -\infty\},$$

then since $\sup_{|x|\geqslant 1} |\nabla \hat{u}(x)| \leqslant C$, there exist small $r_0 > 0$ and $c_0 > 0$, independent of n, such that

$$\left|f_\tau(\hat{u})\right| \geqslant c_0 > 0 \quad \text{on } B_{r_0}(x_n).$$

Then $\int_{\mathbb{R}^2} |f_\tau(\hat{u})|\, dx \geqslant \sum_{n=1}^\infty \int_{B_{r_0}(x_n)} |f_\tau(\hat{u})|\, dx = +\infty$ which proves (4.25).

If $\lim_{|x|\to\infty} \hat{u}(x) = 0$, then (4.1) and the maximum principle imply that there exist $c_\tau > 0$ and $R_0 > 0$ such that

$$\hat{u}_\varepsilon < c_\tau \quad \text{on } B_{\frac{r}{\varepsilon}}(0) \setminus B_{R_0}(0), \tag{4.26}$$

which implies that

$$F_{2,\tau}(\hat{u}_\varepsilon) > 0 \quad \text{on } B_{\frac{r}{\varepsilon}}(0) \setminus B_{R_0}(0).$$

In view of $\sup_{t\in\mathbb{R}} |F_{2,\tau}(t)| < \infty$, (4.12) and (4.26), we also see that, for any $R \in (R_0, \infty)$, we have

$$\begin{aligned} 2\pi\left(\beta_{j,2}^2 - m_{j,2}^2\right) &= \lim_{r,\eta\to 0} \lim_{\varepsilon\to 0} \int_{B_{\frac{r}{\varepsilon}}(0)\setminus B_\eta(0)} F_{2,\tau}(\hat{u}_\varepsilon)\, dx \\ &\geqslant \lim_{\eta\to 0} \lim_{\varepsilon\to 0} \int_{B_R(0)\setminus B_\eta(0)} F_{2,\tau}(\hat{u}_\varepsilon)\, dx = \int_{B_R(0)} F_{2,\tau}(\hat{u})\, dx. \end{aligned} \tag{4.27}$$

Since $\lim_{|x|\to\infty} \hat{u}(x) = 0$, we see that

$$\lim_{|x|\to\infty} F_{2,\tau}(\hat{u}) = \lim_{|x|\to\infty} \frac{e^{\hat{u}}((1-\tau)e^{\hat{u}} + 2\tau)}{2\tau^2(\tau + e^{\hat{u}})^2} = \frac{1}{2(\tau+1)\tau^2} \neq 0,$$

which shows that the right hand side of (4.27) could be arbitrarily large, which is impossible. Hence the first case in (4.25) cannot occur.

If $\lim_{|x|\to\infty}\hat{u}(x)=-\infty$, then in view of (4.1) and the maximum principle, there exists $R_0>0$ such that

$$\hat{u}_\varepsilon<0 \quad \text{on } B_{\frac{r}{\varepsilon}}(0)\setminus B_{R_0}(0). \tag{4.28}$$

By using Lemma 4.2 and (4.28), we see that

$$2\pi\hat{\beta}=\lim_{R\to\infty}\int_{|x|\leqslant R} f_\tau(\hat{u})\,dx=\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{|x|\leqslant R} f_\tau(\hat{u}_\varepsilon)\,dx\leqslant\lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{|x|\leqslant\frac{r}{\varepsilon}} f_\tau(\hat{u}_\varepsilon)\,dx=-4\pi(m_{j,2}+\beta_{j,2}).$$

Hence we conclude that

$$\hat{\beta}\leqslant -2(m_{j,2}+\beta_{j,2}),$$

and in particular that

$$\lim_{|x|\to\infty}\frac{\hat{u}(x)}{\ln|x|}=-2m_{j,2}-\hat{\beta}\geqslant 2\beta_{j,2}. \tag{4.29}$$

By using (4.24) and $\lim_{|x|\to\infty}\hat{u}(x)=-\infty$, we see that $e^{\hat{u}}\in L^1(\mathbb{R}^2\setminus B_1(0))$ and then

$$\lim_{|x|\to\infty}\frac{\hat{u}(x)}{\ln|x|}<-2. \tag{4.30}$$

At this point, the method of moving planes can be used with (4.30) to prove that $\hat{u}$ is radially symmetric (see [5,12]). Moreover, (4.29) and (4.30) imply that

$$\beta_{j,2}<-1. \tag{4.31}$$

If $w_\varepsilon\to w$ in $C^2_{\mathrm{loc}}(\Omega\setminus Z_2)$, then (4.31) contradicts (4.9). Moreover, if $\tau=1$ or $m_{j,2}\in[0,1]$, then Theorem 3.4 in [8] imply that $\hat{u}$ cannot be stable solution, which yields a contradiction and completes the proof of our lemma. □

Lemma 4.5. *If*

$$\lim_{\varepsilon\to 0}\Big(\sup_{|x-p_{j,2}|=\eta}\hat{u}_\varepsilon(x)\Big)=\lim_{\varepsilon\to 0}\Big(\sup_{|x-p_{j,2}|=\eta}u_\varepsilon(\varepsilon x)\Big)=-\infty, \tag{4.32}$$

then

$$\lim_{\varepsilon\to 0}\Big(\sup_{\eta\leqslant|x-p_{j,2}|\leqslant\frac{r}{\varepsilon}}\hat{u}_\varepsilon(x)\Big)=\lim_{\varepsilon\to 0}\Big(\sup_{\eta\leqslant|x-p_{j,2}|\leqslant\frac{r}{\varepsilon}}u_\varepsilon(\varepsilon x)\Big)=-\infty. \tag{4.33}$$

Moreover, $m_{j,2}+\beta_{j,2}=0$.

Proof. For the sake of simplicity, we assume that $p_{j,2}=0$. We divide the proof of our lemma into the following steps.

Step 1. To prove (4.33), we argue by contradiction and suppose that for some constant $c\in(-\infty,0)$, there exists $y_\varepsilon\in B_{\frac{r}{\varepsilon}}(0)\setminus B_\eta(0)$ such that $\hat{u}_\varepsilon(y_\varepsilon)=c$. In view of (4.1), (4.19) and (4.32), we see that

$$\lim_{\varepsilon\to 0}|y_\varepsilon|=\infty \quad \text{and} \quad \lim_{\varepsilon\to 0}\big(\varepsilon|y_\varepsilon|\big)=0.$$

Moreover, by using (4.19), we see that the function $\bar{u}_\varepsilon(x)=\hat{u}_\varepsilon(x+y_\varepsilon)$ satisfies

$$\begin{cases}\Delta\bar{u}_\varepsilon+f_\tau(\bar{u}_\varepsilon)=0, \quad |\nabla\bar{u}_\varepsilon|\leqslant C_1 \quad \text{on } B_{\frac{|y_\varepsilon|}{2}}(0),\\ \bar{u}_\varepsilon(0)=c<0, \quad f_\tau(\bar{u}_\varepsilon)\in L^1\big(B_{\frac{|y_\varepsilon|}{2}}(0)\big),\end{cases}$$

for some constant $C_1>0$. Then $\{\bar{u}_\varepsilon\}$ is uniformly bounded in $L^\infty_{\mathrm{loc}}(B_{\frac{|y_\varepsilon|}{2}}(0))$ and there exists a function $\bar{u}$ such that $\bar{u}_\varepsilon\to\bar{u}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ and

$$\begin{cases} \Delta\bar u + f_\tau(\bar u) = 0, & |\nabla \bar u| \leqslant C_1 \quad \text{on } \mathbb{R}^2, \\ \bar u(0) = c < 0, & f_\tau(\bar u) \in L^1(\mathbb{R}^2). \end{cases}$$

By using Lemma 2.1, we conclude that $\bar u$ is a nontopological radially symmetric solution. Then Theorem 3.4 in [8] shows that $\bar u$ cannot be a stable solution which proves (4.33).

Step 2. By using Lemma 4.2 and (4.33) we see that

$$\begin{aligned} -4\pi(m_{j,2}+\beta_{j,2}) &= \lim_{r,\eta\to 0}\lim_{\varepsilon\to 0}\int\limits_{B_{\frac{r}{\varepsilon}}(0)\setminus B_\eta(0)} f_\tau(\hat u_\varepsilon)\,dx = \lim_{r,\eta\to 0}\lim_{\varepsilon\to 0}\int\limits_{B_{\frac{r}{\varepsilon}}(0)\setminus B_\eta(0)} \frac{e^{\hat u_\varepsilon}}{\tau^3}\,dx \\ &= \lim_{r,\eta\to 0}\lim_{\varepsilon\to 0}\int\limits_{B_{\frac{r}{\varepsilon}}(0)\setminus B_\eta(0)} F_{2,\tau}(\hat u_\varepsilon)\,dx = 2\pi\left(\beta_{j,2}^2 - m_{j,2}^2\right), \end{aligned} \tag{4.34}$$

which implies

$$\begin{cases} m_{j,2}+\beta_{j,2} \leqslant 0, \quad \text{and} \\ \text{either} \quad m_{j,2}+\beta_{j,2} = 0 \quad \text{or} \quad m_{j,2}-\beta_{j,2} = 2. \end{cases}$$

To prove our lemma, we argue by contradiction and suppose that

$$m_{j,2}+\beta_{j,2} < 0, \tag{4.35}$$

which implies

$$m_{j,2}-\beta_{j,2} = 2, \qquad m_{j,2} < 1 < -\beta_{j,2}. \tag{4.36}$$

If $w_\varepsilon \to w$ in $C^2_{\mathrm{loc}}(\Omega \setminus Z_2)$, then (4.36) contradicts $\beta_{j,2} > -1$ in (4.9), and we obtain that $m_{j,2}+\beta_{j,2} = 0$ in this case.

Therefore, from now on, we assume that

$$\lim_{\varepsilon\to 0}\Big(\sup_{\Omega\setminus\bigcup_j (B_r(p_{j,2}))} w_\varepsilon\Big) = -\infty \quad \text{for any small } r > 0. \tag{4.37}$$

Then (4.32) and (4.37) imply that for any $r, \eta > 0$,

$$\lim_{\varepsilon\to 0}\Big(\sup_{x\in\partial B_{\frac{r}{\varepsilon}}(0)\cup\partial B_\eta(0)} e^{\hat u_\varepsilon(x)}|x|^2\Big) = 0. \tag{4.38}$$

Step 3. We claim that for any $r, \eta > 0$,

$$\lim_{\varepsilon\to 0}\Big(\sup_{x\in B_{\frac{r}{\varepsilon}}(0)\setminus B_\eta(0)} e^{\hat u_\varepsilon(x)}|x|^2\Big) = 0. \tag{4.39}$$

Let us choose $y_\varepsilon \in B_{\frac{r}{\varepsilon}}(0) \setminus B_\eta(0)$ such that

$$e^{\hat u_\varepsilon(y_\varepsilon)}|y_\varepsilon|^2 = \Big(\sup_{x\in B_{\frac{r}{\varepsilon}}(0)\setminus B_\eta(0)} e^{\hat u_\varepsilon(x)}|x|^2\Big).$$

We consider the function $\tilde u_\varepsilon(x) \equiv \hat u_\varepsilon(|y_\varepsilon|x) + 2\ln|y_\varepsilon|$. Then $\tilde u_\varepsilon$ satisfies

$$\Delta\tilde u_\varepsilon + \frac{e^{\tilde u_\varepsilon}(1-e^{\tilde u_\varepsilon}/|y_\varepsilon|^2)}{(\tau + e^{\tilde u_\varepsilon}/|y_\varepsilon|^2)^3} = -4\pi m_{j,2}\delta_0 \quad \text{on } B_{\frac{r}{\varepsilon|y_\varepsilon|}}(0).$$

Moreover,

$$\tilde u_\varepsilon(x) = \hat u_\varepsilon\big(|y_\varepsilon|x\big) + 2\ln\big(|y_\varepsilon||x|\big) - 2\ln|x| \leqslant \tilde u_\varepsilon\left(\frac{y_\varepsilon}{|y_\varepsilon|}\right) - 2\ln|x| \quad \text{on } B_{\frac{r}{\varepsilon|y_\varepsilon|}}(0) \setminus B_{\frac{\eta}{|y_\varepsilon|}}(0). \tag{4.40}$$

To prove the claim (4.39), we argue by contradiction and consider the following two cases.

Case 1: Suppose that

$$\lim_{\varepsilon\to 0}\left(e^{\hat{u}_\varepsilon(y_\varepsilon)}|y_\varepsilon|^2\right)=+\infty.$$

Then we see that, in view of (4.38), we have $\lim_{\varepsilon\to 0}|y_\varepsilon|=+\infty$ and $\lim_{\varepsilon\to 0}(\varepsilon|y_\varepsilon|)=0$.

Moreover, we see that

$$s_\varepsilon\equiv\exp\left(-\frac{1}{2}\tilde{u}_\varepsilon\left(\frac{y_\varepsilon}{|y_\varepsilon|}\right)\right)\to 0\quad\text{as }\varepsilon\to 0.$$

In view of (4.40), we see that for any $x\in B_{\frac{1}{\delta}}(0)\setminus B_\delta(0)$,

$$\tilde{u}_\varepsilon(x)\leqslant\tilde{u}_\varepsilon\left(\frac{y_\varepsilon}{|y_\varepsilon|}\right)-2\ln\delta. \tag{4.41}$$

By using (4.33) and $\lim_{\varepsilon\to 0}|y_\varepsilon|=+\infty$, we also see that

$$\lim_{\varepsilon\to 0}\left(\frac{e^{\tilde{u}_\varepsilon(x)}}{|y_\varepsilon|^2}\right)=\lim_{\varepsilon\to 0}e^{\hat{u}_\varepsilon(|y_\varepsilon|x)}=0\quad\text{on }B_{\frac{1}{\delta}}(0)\setminus B_\delta(0). \tag{4.42}$$

Let $\bar{w}_\varepsilon(x)\equiv\tilde{u}_\varepsilon(s_\varepsilon x+\frac{y_\varepsilon}{|y_\varepsilon|})+2\ln s_\varepsilon$ for $|x|<\frac{\delta}{2s_\varepsilon}$. For small $\varepsilon,\delta>0$, $\bar{w}_\varepsilon$ satisfies

$$\Delta\bar{w}_\varepsilon+\frac{e^{\bar{w}_\varepsilon}(1-\frac{e^{\bar{w}_\varepsilon}}{s_\varepsilon^2|y_\varepsilon|^2})}{(\tau+\frac{e^{\bar{w}_\varepsilon}}{s_\varepsilon^2|y_\varepsilon|^2})^3}=0\quad\text{on }B_{\frac{\delta}{2s_\varepsilon}}(0).$$

By using (4.41), we see that

$$\begin{cases}\bar{w}_\varepsilon(x)\leqslant\tilde{u}_\varepsilon\big(y_\varepsilon/|y_\varepsilon|\big)-2\ln\delta+2\ln s_\varepsilon=-2\ln\delta\quad\text{for }|x|<\dfrac{\delta}{2s_\varepsilon},\\ \bar{w}_\varepsilon(0)=\tilde{u}_\varepsilon\big(y_\varepsilon/|y_\varepsilon|\big)+2\ln s_\varepsilon=0.\end{cases} \tag{4.43}$$

In view of (4.42), we also conclude that $\lim_{\varepsilon\to 0}(\frac{1}{s_\varepsilon^2|y_\varepsilon|^2})=0$ and for small $\varepsilon>0$,

$$0\leqslant-\Delta\bar{w}_\varepsilon\leqslant\frac{1}{\delta^2\tau^3}\quad\text{on }B_{\frac{\delta}{2s_\varepsilon}}(0). \tag{4.44}$$

By using (4.43), (4.44), and Lemma 3.2, we see that for any $p>1$ and $R>0$, there exist constants $\sigma\in(0,1)$ and $\gamma>0$, depending on $R>0$ only such that

$$0=\bar{w}_\varepsilon(0)\leqslant\sup_{B_R(0)}\bar{w}_\varepsilon\leqslant\sigma\inf_{B_R(0)}\bar{w}_\varepsilon+(1+\sigma)\gamma\|\Delta\bar{w}_\varepsilon\|_{L^p(B_{2R}(0))}-2(1-\sigma)\ln\delta,$$

which implies that $\bar{w}_\varepsilon$ is bounded in $C^0_{\mathrm{loc}}(B_{\frac{\delta}{2s_\varepsilon}}(0))$. Then there exists a function w_* such that $\bar{w}_\varepsilon\to w_*$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$. By Lemma 3.1, w_* satisfies

$$\begin{cases}\Delta w_*+\dfrac{e^{w_*}}{\tau^3}=0\quad\text{in }\mathbb{R}^2,\\ w_*(0)=0,\quad e^{w_*}\in L^1(\mathbb{R}^2).\end{cases}$$

However we see that w_* cannot be a stable solution, which yields the desired contradiction and rules out Case 1.

Case 2: Suppose that there exists $c>0$ such that

$$e^{-c}<\lim_{\varepsilon\to 0}e^{\hat{u}_\varepsilon(y_\varepsilon)}|y_\varepsilon|^2<e^c. \tag{4.45}$$

Then, in view of (4.38), we see that $\lim_{\varepsilon\to 0}|y_\varepsilon|=+\infty$ and $\lim_{\varepsilon\to 0}(\varepsilon|y_\varepsilon|)=0$. By using (4.40) and (4.45), we also conclude that

$$\begin{cases}\tilde{u}_\varepsilon(x)\leqslant\tilde{u}_\varepsilon\left(\dfrac{y_\varepsilon}{|y_\varepsilon|}\right)-2\ln|x|\leqslant c-2\ln|x|\quad\text{for }x\in B_{\frac{1}{\delta}}(0)\setminus B_\delta(0),\\ -c\leqslant\tilde{u}_\varepsilon\big(y_\varepsilon/|y_\varepsilon|\big).\end{cases} \tag{4.46}$$

By using (4.33) and $\lim_{\varepsilon\to 0}|y_\varepsilon| = +\infty$, we also have

$$\lim_{\varepsilon\to 0}\left(\frac{e^{\tilde{u}_\varepsilon(x)}}{|y_\varepsilon|^2}\right) = \lim_{\varepsilon\to 0} e^{\hat{u}_\varepsilon(|y_\varepsilon|x)} = 0 \quad \text{on } B_{\frac{1}{\delta}}(0)\setminus B_\delta(0). \tag{4.47}$$

Then (4.47) implies that for small $\varepsilon > 0$,

$$0 \leqslant -\Delta\tilde{u}_\varepsilon = \frac{e^{\tilde{u}_\varepsilon}(1-e^{\tilde{u}_\varepsilon}/|y_\varepsilon|^2)}{(\tau+e^{\tilde{u}_\varepsilon}/|y_\varepsilon|^2)^3} \leqslant \frac{e^c}{\delta^2\tau^3} \quad \text{on } B_{\frac{1}{\delta}}(0)\setminus B_\delta(0). \tag{4.48}$$

By using (4.46), (4.48), and Lemma 3.2, we see that for any $p > 1$ and $\delta > 0$, there exist constants $\sigma \in (0,1)$ and $\gamma > 0$, depending only on $\delta > 0$ such that

$$\begin{aligned} -c \leqslant \tilde{u}_\varepsilon\big(y_\varepsilon/|y_\varepsilon|\big) &\leqslant \sup_{B_{\frac{1}{\delta}}(0)\setminus B_\delta(0)} \tilde{u}_\varepsilon \\ &\leqslant \sigma \inf_{B_{\frac{1}{\delta}}(0)\setminus B_\delta(0)} \tilde{u}_\varepsilon + (1+\sigma)\gamma\|\Delta\tilde{u}_\varepsilon\|_{L^p(B_{\frac{2}{\delta}}(0)\setminus B_{\frac{\delta}{2}}(0))} + (1-\sigma)\left(c - 2\ln\left(\frac{\delta}{2}\right)\right), \end{aligned}$$

which implies that $\tilde{u}_\varepsilon$ is bounded in $C^0_{\mathrm{loc}}(B_{\frac{r}{\varepsilon|y_\varepsilon|}}(0)\setminus\{0\})$. Let

$$\alpha = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{B_\delta(0)} \frac{e^{\tilde{u}_\varepsilon}(1-e^{\tilde{u}_\varepsilon}/|y_\varepsilon|^2)}{(\tau+e^{\tilde{u}_\varepsilon}/|y_\varepsilon|^2)^3}\,dx = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{B_{\delta|y_\varepsilon|}(0)} \frac{e^{\hat{u}_\varepsilon}(1-e^{\hat{u}_\varepsilon})}{(\tau+e^{\hat{u}_\varepsilon})^3}\,dx.$$

In view of Lemma 3.1, we see that there exists a function w_* such that $\tilde{u}_\varepsilon \to w_*$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2\setminus\{0\})$ and

$$\begin{cases} \Delta w_* + \dfrac{e^{w_*}}{\tau^3} = (-\alpha - 4\pi m_{j,2})\delta_0 \quad \text{in } \mathbb{R}^2, \\ (-\alpha - 4\pi m_{j,2}) > -4\pi, \\ w_* \leqslant c - 2\ln|x|, \quad e^{w_*} \in L^1\big(\mathbb{R}^2\big). \end{cases}$$

However, w_* cannot be a stable solution, which yields once more a contradiction and concludes the proof of (4.39) as claimed.

Step 4. For any $d \in (0, -(m_{j,2}+\beta_{j,2}))$, there exists $r_\varepsilon := r_\varepsilon(d) \in (0, \frac{r}{\varepsilon})$ such that

$$\lim_{\varepsilon\to 0}\int_{B_{r_\varepsilon}(0)} f_\tau(\hat{u}_\varepsilon)\,dx = 4\pi d. \tag{4.49}$$

Now we claim that

$$\lim_{\varepsilon\to 0}\int_{B_{r_\varepsilon}(0)} F_{2,\tau}(\hat{u}_\varepsilon)\,dx = 2\pi d(d+2m_{j,2}). \tag{4.50}$$

By using (4.33) and (4.39), we see that $\lim_{\varepsilon\to 0} r_\varepsilon = +\infty$ and $\lim_{\varepsilon\to 0}(\varepsilon r_\varepsilon) = 0$. Let us consider the function $\hat{\hat{u}}_\varepsilon(x) \equiv \hat{u}_\varepsilon(r_\varepsilon x) + 2\ln r_\varepsilon$ which satisfies

$$\Delta\hat{\hat{u}}_\varepsilon + \frac{e^{\hat{\hat{u}}_\varepsilon}(1-e^{\hat{\hat{u}}_\varepsilon}/r_\varepsilon^2)}{(\tau+e^{\hat{\hat{u}}_\varepsilon}/r_\varepsilon^2)^3} = -4\pi m_{j,2}\delta_0 \quad \text{on } B_{\frac{r}{\varepsilon r_\varepsilon}}(0).$$

We claim that for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$\big|\hat{\hat{u}}_\varepsilon(x_1) - \hat{\hat{u}}_\varepsilon(x_2)\big| \leqslant C_\delta \quad \text{for any } x_1, x_2 \in B_{\frac{1}{\delta}}(0)\setminus B_\delta(0). \tag{4.51}$$

By using the Green's representation formula for a solution u_ε of (1.2) and Lemma 3.1, we see that for any $x_1, x_2 \in B_{\frac{1}{\delta}}(0) \setminus B_\delta(0)$,

$$\begin{aligned}
\hat{\hat{u}}_\varepsilon(x_1) - \hat{\hat{u}}_\varepsilon(x_2) &= u_\varepsilon(\varepsilon r_\varepsilon x_1) - u_\varepsilon(\varepsilon r_\varepsilon x_2) \\
&= \int_\Omega \big(G(\varepsilon r_\varepsilon x_1, y) - G(\varepsilon r_\varepsilon x_2, y)\big)\big(-\Delta u_\varepsilon(y)\big)\,dy \\
&= O(1) + \int_\Omega \big(G(\varepsilon r_\varepsilon x_1, y) - G(\varepsilon r_\varepsilon x_2, y)\big)\frac{f_\tau(u_\varepsilon(y))}{\varepsilon^2}\,dy \\
&= O(1) + \frac{1}{2\pi}\int_\Omega \ln\left(\frac{|\varepsilon r_\varepsilon x_2 - y|}{|\varepsilon r_\varepsilon x_1 - y|}\right)\frac{f_\tau(u_\varepsilon(y))}{\varepsilon^2}\,dy.
\end{aligned}$$

By the mean value theorem, there exists $\theta = \theta(\varepsilon, y) \in (0, 1)$ such that

$$\begin{aligned}
\big|\ln|\varepsilon r_\varepsilon x_2 - y| - \ln|\varepsilon r_\varepsilon x_1 - y|\big| &= \frac{||\varepsilon r_\varepsilon x_2 - y| - |\varepsilon r_\varepsilon x_1 - y||}{\theta|\varepsilon r_\varepsilon x_2 - y| + (1-\theta)|\varepsilon r_\varepsilon x_1 - y|} \\
&\leqslant \frac{|\varepsilon r_\varepsilon(x_1 - x_2)|}{\theta|\varepsilon r_\varepsilon x_2 - y| + (1-\theta)|\varepsilon r_\varepsilon x_1 - y|}.
\end{aligned} \tag{4.52}$$

For any $y \in \Omega \setminus B_{\frac{2\varepsilon r_\varepsilon}{\delta}}(0)$, we have $|\varepsilon r_\varepsilon x_i - y| \geqslant \frac{\varepsilon r_\varepsilon}{\delta}$ for $i = 1, 2$. Then by using Lemma 3.1 and (4.52), we see that

$$\int_{\Omega \setminus B_{\frac{2\varepsilon r_\varepsilon}{\delta}}(0)} \left|\ln\left(\frac{|\varepsilon r_\varepsilon x_2 - y|}{|\varepsilon r_\varepsilon x_1 - y|}\right)\frac{f_\tau(u_\varepsilon)}{\varepsilon^2}\right| dy = O(1). \tag{4.53}$$

By using the fact that $|\varepsilon r_\varepsilon x_i - y| \geqslant \frac{\varepsilon r_\varepsilon \delta}{2}$ for $i = 1, 2$ for any $y \in B_{\frac{\varepsilon r_\varepsilon \delta}{2}}(0)$ with Lemma 3.1 and (4.52), we also have

$$\int_{B_{\frac{\varepsilon r_\varepsilon \delta}{2}}(0)} \left|\ln\left(\frac{|\varepsilon r_\varepsilon x_2 - y|}{|\varepsilon r_\varepsilon x_1 - y|}\right)\frac{f_\tau(u_\varepsilon)}{\varepsilon^2}\right| dy = O(1). \tag{4.54}$$

Moreover, by using (4.33) and (4.39), we see that

$$\begin{aligned}
\int_{B_{\frac{2\varepsilon r_\varepsilon}{\delta}}(0) \setminus B_{\frac{\varepsilon r_\varepsilon \delta}{2}}(0)} \ln\left(\frac{|\varepsilon r_\varepsilon x_2 - y|}{|\varepsilon r_\varepsilon x_1 - y|}\right)\frac{f_\tau(u_\varepsilon)}{\varepsilon^2}\,dy &= \int_{B_{\frac{2r_\varepsilon}{\delta}}(0) \setminus B_{\frac{r_\varepsilon \delta}{2}}(0)} \ln\left(\frac{|r_\varepsilon x_2 - y|}{|r_\varepsilon x_1 - y|}\right) f_\tau\big(\hat{u}_\varepsilon(y)\big)\,dy \\
&\leqslant 2r_\varepsilon|x_1 - x_2| \sup_{B_{\frac{2r_\varepsilon}{\delta}}(0) \setminus B_{\frac{r_\varepsilon \delta}{2}}(0)} \big(\big|f_\tau(\hat{u}_\varepsilon)\big|\big) \int_{B_{\frac{4r_\varepsilon}{\delta}}(0)} \frac{1}{|y|}\,dy \\
&= \frac{16\pi|x_1 - x_2|}{\delta} \sup_{B_{\frac{2r_\varepsilon}{\delta}}(0) \setminus B_{\frac{r_\varepsilon \delta}{2}}(0)} \big(r_\varepsilon^2\big|f_\tau(\hat{u}_\varepsilon)\big|\big) \\
&\leqslant \frac{32\pi|x_1 - x_2|}{\delta\tau^3} \sup_{B_{\frac{2r_\varepsilon}{\delta}}(0) \setminus B_{\frac{r_\varepsilon \delta}{2}}(0)} \big(r_\varepsilon^2 e^{\hat{u}_\varepsilon}\big) = O(1).
\end{aligned} \tag{4.55}$$

At this point, (4.51) follows by using (4.53), (4.54), and (4.55).

Now we fix $y_0 \in \mathbb{R}^2 \setminus \{0\}$. Then, in view of (4.39), (4.49) and (4.51), we see that there exists a function h such that $h_\varepsilon \equiv \hat{\hat{u}}_\varepsilon - \hat{\hat{u}}_\varepsilon(y_0) \to h$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$ and

$$\Delta h = -4\pi(m_{j,2} + d)\delta_0 \quad \text{in } \mathbb{R}^2.$$

We also conclude that the function $v(x) = h(x) + 2(m_{j,2} + d)\ln|x|$ satisfies

$$\Delta v = 0 \quad \text{in } \mathbb{R}^2. \tag{4.56}$$

We consider the function $v_\varepsilon(x) \equiv h_\varepsilon(x) + 2m_{j,2} \ln|x|$ which satisfies

$$\Delta v_\varepsilon(x) + \frac{r_\varepsilon^2 e^{\hat{u}_\varepsilon(r_\varepsilon x)}(1 - e^{\hat{u}_\varepsilon(r_\varepsilon x)})}{(\tau + e^{\hat{u}_\varepsilon(r_\varepsilon x)})^3} = 0 \quad \text{on } B_{\frac{r}{\varepsilon r_\varepsilon}}(0).$$

Let $v_\varepsilon(x) = \hat{v}_\varepsilon(r_\varepsilon x)$. Then, we see that

$$\Delta \hat{v}_\varepsilon + \frac{e^{\hat{u}_\varepsilon}(1 - e^{\hat{u}_\varepsilon})}{(\tau + e^{\hat{u}_\varepsilon})^3} = 0 \quad \text{on } B_{\frac{r}{\varepsilon}}(0). \tag{4.57}$$

Multiplying (4.57) by $\nabla \hat{u}_\varepsilon \cdot x$ and integrating over $B_{r_\varepsilon}(0)$, we conclude that

$$\int\limits_{B_{r_\varepsilon}(0)} 2F_{2,\tau}(\hat{u}_\varepsilon)\,dx = \int\limits_{\partial B_{r_\varepsilon}(0)} \left[\left(\nabla \hat{v}_\varepsilon \cdot \frac{x}{|x|}\right)(\nabla \hat{v}_\varepsilon \cdot x) - \frac{|\nabla \hat{v}_\varepsilon|^2|x|}{2} + F_{2,\tau}(\hat{u}_\varepsilon)|x| - 2m_{j,2}\frac{\nabla \hat{v}_\varepsilon \cdot x}{|x|}\right] d\sigma. \tag{4.58}$$

Hence (4.39) and (4.56) imply

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int\limits_{B_{r_\varepsilon}(0)} 2F_{2,\tau}(\hat{u}_\varepsilon)\,dx \\
&\quad = \lim_{\varepsilon\to 0} \int\limits_{\partial B_1(0)} \left[\left(\nabla v_\varepsilon \cdot \frac{x}{|x|}\right)(\nabla v_\varepsilon \cdot x) - \frac{|\nabla v_\varepsilon|^2|x|}{2} + r_\varepsilon^2 F_{2,\tau}\big(\hat{u}_\varepsilon(r_\varepsilon x)\big) - 2m_{j,2}\frac{\nabla v_\varepsilon \cdot x}{|x|}\right] d\sigma \\
&\quad = \int\limits_{\partial B_1(0)} \left\{\frac{(\nabla v \cdot x - 2d)^2}{|x|} - \left(\nabla v - \frac{2d x}{|x|^2}\right)^2 \frac{|x|}{2} - \frac{2m_{j,2}(\nabla v \cdot x - 2d)}{|x|}\right\} d\sigma \\
&\quad = 4\pi d(d + 2m_{j,2}),
\end{aligned} \tag{4.59}$$

and we complete the proof of our claim (4.50). At this point, in view of (4.33), (4.49) and (4.59), we see that

$$\begin{aligned}
4\pi d &= \lim_{\eta\to 0}\lim_{\varepsilon\to 0} \int\limits_{B_{r_\varepsilon}(0)\setminus B_\eta(0)} f_\tau(\hat{u}_\varepsilon)\,dx = \lim_{\eta\to 0}\lim_{\varepsilon\to 0} \int\limits_{B_{r_\varepsilon}(0)\setminus B_\eta(0)} \frac{e^{\hat{u}_\varepsilon}}{\tau^3}\,dx \\
&= \lim_{\eta\to 0}\lim_{\varepsilon\to 0} \int\limits_{B_{r_\varepsilon}(0)\setminus B_\eta(0)} F_{2,\tau}(\hat{u}_\varepsilon)\,dx = 2\pi d(d + 2m_{j,2}),
\end{aligned}$$

which implies $d + 2m_{j,2} = 2$. Since $d > 0$ can be chosen arbitrarily, we obtain a contradiction which concludes the proof of $m_{j,2} + \beta_{j,2} = 0$ under the assumption (4.32). □

Remark 4.6. It turns out that Lemma 4.4 and Lemma 4.5 yield the following result. Suppose that $u_\varepsilon - 2\ln\varepsilon$ is uniformly bounded in any compact subset of $\Omega \setminus Z_2$ and $u_\varepsilon - 2\ln\varepsilon$ converges to w in $C^2_{\mathrm{loc}}(\Omega \setminus Z_2)$ as $\varepsilon \to 0$, then $0 \leqslant m_{j,2} < 1$ for $1 \leqslant j \leqslant d_2$, $N_1 > N_2$, and w satisfies

$$\Delta w + \frac{e^w}{\tau^3} = 4\pi \sum_{j=1}^{d_1} m_{j,1}\delta_{p_{j,1}} - 4\pi \sum_{j=1}^{d_2} m_{j,2}\delta_{p_{j,2}} \quad \text{on } \Omega.$$

At this point, we are ready to prove one part of Theorem 1.1.

Proof of Theorem 1.1: stable solution ⇒ topological solution. To prove our theorem, we consider the following cases.

Case 1. If $\tau=1$, then Lemma 4.4 and Lemma 4.5 imply that for any $r,\eta>0$,

$$\begin{cases} \lim_{\varepsilon\to 0}\Big(\sup_{\eta\leqslant|x-p_{j,2}|\leqslant\frac{r}{\varepsilon}}\hat{u}_\varepsilon(x)\Big)=\lim_{\varepsilon\to 0}\Big(\sup_{\eta\leqslant|x-p_{j,2}|\leqslant\frac{r}{\varepsilon}}u_\varepsilon(\varepsilon x)\Big)=-\infty, \\ m_{j,2}+\beta_{j,2}=0 \quad \text{for all } 1\leqslant j\leqslant d_2. \end{cases} \tag{4.60}$$

By using (4.60) and Lemma 4.2, we see that

$$\begin{aligned} \lim_{r,\eta\to 0}\Big(\lim_{\varepsilon\to 0}\int_{B_{\frac{r}{\varepsilon}}(p_{j,2})\setminus B_\eta(p_{j,2})} f_\tau'(\hat{u}_\varepsilon)\,dx\Big) &= \lim_{r,\eta\to 0}\Big(\lim_{\varepsilon\to 0}\int_{B_{\frac{r}{\varepsilon}}(p_{j,2})\setminus B_\eta(p_{j,2})} \frac{e^{\hat{u}_\varepsilon}}{\tau^3}\,dx\Big) \\ &= \lim_{r,\eta\to 0}\Big(\lim_{\varepsilon\to 0}\int_{B_{\frac{r}{\varepsilon}}(p_{j,2})\setminus B_\eta(p_{j,2})} f_\tau(\hat{u}_\varepsilon)\,dx\Big) \\ &= -4\pi(m_{j,2}+\beta_{j,2})=0. \end{aligned} \tag{4.61}$$

If $\lim_{\varepsilon\to 0}(\sup_{\Omega\setminus\bigcup_j(B_r(p_{j,2}))}w_\varepsilon)=-\infty$ for any small $r>0$, then in view of (4.10) and (4.60), we see that

$$\Delta g = 4\pi\sum_{j=1}^{d_1}m_{j,1}\delta_{p_{j,1}}-4\pi\sum_{j=1}^{d_2}m_{j,2}\delta_{p_{j,2}} \quad \text{on } \Omega,$$

thus $N_1=N_2$ which contradicts (H1).

On the other hand, if $w_\varepsilon\to w$ in $C^2_{\mathrm{loc}}(\Omega\setminus Z_2)$, then in view of (4.2) and (4.61), we see that

$$\begin{aligned} \lim_{\varepsilon\to 0}\mu_\varepsilon &\leqslant \frac{1}{|\Omega|^2}\lim_{\varepsilon\to 0}\int_\Omega -\frac{f_\tau'(u_\varepsilon)}{\varepsilon^2}\,dx \\ &= \frac{1}{|\Omega|^2}\lim_{r\to 0}\lim_{\varepsilon\to 0}\Bigg[\int_{\Omega\setminus\bigcup_j(B_r(p_{j,2}))}\frac{e^{w_\varepsilon}(-\tau+2(\tau+1)\varepsilon^2e^{w_\varepsilon}-\varepsilon^4e^{2w_\varepsilon})}{(\tau+\varepsilon^2e^{w_\varepsilon})^4}\,dx-\sum_{j=1}^{d_2}\int_{B_{\frac{r}{\varepsilon}}(p_{j,2})}f_\tau'(\hat{u}_\varepsilon)\,dx\Bigg] \\ &= -\frac{1}{|\Omega|^2}\int_\Omega\frac{e^w}{\tau^3}\,dx<0, \end{aligned}$$

which implies $\mu_\varepsilon<0$ for small $\varepsilon>0$. Then u_ε cannot be a stable solution of (1.2) which is once more a contradiction.

Case 2. If $N_2>N_1$ then, in view of (H2), we have $m_{j,2}\in[0,1]$ for all $1\leqslant j\leqslant d_2$. Then by using Lemma 4.4 and Lemma 4.5 we obtain (4.60). By using the same arguments as in Case 1, we can prove that u_ε cannot be stable solution of (1.2). We skip the details of this part to avoid repetitions.

Case 3. If $N_2<N_1$, then we define the following set

$$J_0\equiv\Big\{l\;\Big|\;1\leqslant l\leqslant d_2,\qquad \lim_{\varepsilon\to 0}\Big(\sup_{|x-p_{l,2}|=\eta}\hat{u}_\varepsilon(x)\Big)=-\infty\Big\}.$$

If $J_0=\{1,\dots,d_2\}$, then the desired conclusion will follow by the same argument adopted in Case 1.

Therefore we suppose that $J_0\neq\{1,\dots,d_2\}$ and define $J_1\equiv\{1,\dots,d_2\}\setminus J_0\neq\emptyset$. By using Lemma 4.4, we see that $\lim_{\varepsilon\to 0}(\sup_{\Omega\setminus\bigcup_j(B_r(p_{j,2}))}w_\varepsilon)=-\infty$ for any small $r>0$. Then by (4.10), we have

$$N_2=\sum_{j\in J_0}m_{j,2}+\sum_{j\in J_1}m_{j,2}<N_1=\sum_{j\in J_0}(-\beta_{j,2})+\sum_{j\in J_1}(-\beta_{j,2}).$$

By using Lemma 4.5, we see that there exists $j_0\in J_1$ such that

$$m_{j_0,2}<-\beta_{j_0,2}.$$

For the sake of simplicity, we assume that $p_{j_0,2}=0$. In view of Lemma 4.2, we see that

$$\lim_{r\to 0}\lim_{\varepsilon\to 0}\int\limits_{B_{\frac{r}{\varepsilon}}(0)} f_\tau(\hat{u}_\varepsilon)\,dx = -4\pi(m_{j_0,2}+\beta_{j_0,2}) > 0.$$

Since $j_0 \in J_1$, the same argument adopted in the proof of Lemma 4.4 shows that there exists a function $\hat{u}$ such that $\hat{u}_\varepsilon \to \hat{u}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$ and

$$\begin{cases} \Delta\hat{u} + f_\tau(\hat{u}) = -4\pi m_{j_0,2}\delta_0 \quad \text{on } \mathbb{R}^2, \\ \lim\limits_{|x|\to\infty} \hat{u}(x) = -\infty, \\ f_\tau(\hat{u}) \in L^1(\mathbb{R}^2), \quad e^{\hat{u}} \in L^1(\mathbb{R}^2 \setminus B_1(0)). \end{cases}$$

Let

$$\hat{\beta} = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} f_\tau(\hat{u})\,dx. \tag{4.62}$$

Then we conclude that

$$\lim_{|x|\to\infty}\frac{\hat{u}(x)}{\ln|x|} = -2m_{j_0,2} - \hat{\beta} < -2. \tag{4.63}$$

Moreover, by similar arguments adopted in Step 1 in the proof of Lemma 4.5, (4.1), and (4.63), we can show that there exist ν and $R_0 > 0$ such that

$$\hat{u}_\varepsilon < -\nu \quad \text{on } B_{\frac{r}{\varepsilon}}(0) \setminus B_{R_0}(0). \tag{4.64}$$

Let $\bar{\hat{u}}_\varepsilon(\rho) \equiv \frac{1}{2\pi\rho}\int_{\partial B_\rho(0)} \hat{u}_\varepsilon d\sigma$. Then $\bar{\hat{u}}_\varepsilon$ satisfies

$$\rho\frac{d\bar{\hat{u}}_\varepsilon}{d\rho} + \frac{1}{2\pi}\int\limits_{B_\rho(0)} f_\tau(\hat{u}_\varepsilon)\,dx = -2m_{j_0,2}. \tag{4.65}$$

Then (4.62), (4.63), (4.64) and (4.65) imply that there exists $\sigma > 0$ such that for large $\rho > 0$,

$$\rho\frac{d\bar{\hat{u}}_\varepsilon}{d\rho} \leqslant -(2+\sigma). \tag{4.66}$$

We claim that there exists a constant $C > 0$ such that

$$\left|\hat{u}_\varepsilon(x) - \bar{\hat{u}}_\varepsilon(|x|)\right| \leqslant C \quad \text{for } x \in B_{\frac{r}{\varepsilon}}(0) \setminus B_{R_0}(0). \tag{4.67}$$

Indeed, since $j_0 \in J_1$ and in view of (4.21), we see that $\{\hat{u}_\varepsilon\}$ is uniformly bounded in $L^\infty_{\mathrm{loc}}(B_{\frac{r}{\varepsilon}}(0) \setminus \{0\})$. Then we have

$$\lim_{\varepsilon\to 0}\Big(\sup_{x\in\partial B_{\frac{r}{\varepsilon}}(0)\cup\partial B_{R_0}(0)} e^{\hat{u}_\varepsilon(x)}|x|^2\Big) < +\infty.$$

Then, by using (4.64) and the similar argument adopted in Step 3 in the proof of Lemma 4.5, we conclude that

$$\lim_{\varepsilon\to 0}\Big(\sup_{x\in B_{\frac{r}{\varepsilon}}(0)\setminus B_{R_0}(0)} e^{\hat{u}_\varepsilon(x)}|x|^2\Big) < +\infty. \tag{4.68}$$

Moreover, by using the Green's representation formula for a solution u_ε of (1.2) and by arguing as in the proof of (4.51), we obtain (4.67). In view of (4.66) and (4.67) we can find a constant $c > 0$ such that

$$\lim_{\varepsilon\to 0}\int\limits_{B_{\frac{r}{\varepsilon}}(0)\setminus B_R(0)} f_\tau(\hat{u}_\varepsilon)\,dx \leqslant cR^{-\sigma}.$$

Now we see that

$$2\pi\hat{\beta} = \lim_{R\to\infty}\int\limits_{|x|\leqslant R} f_\tau(\hat{u})\,dx = \lim_{R\to\infty}\lim_{\varepsilon\to 0}\int\limits_{|x|\leqslant R} f_\tau(\hat{u}_\varepsilon)\,dx$$

$$= \lim_{R\to\infty}\lim_{\varepsilon\to 0}\Bigg(\int\limits_{|x|\leqslant\frac{r}{\varepsilon}} f_\tau(\hat{u}_\varepsilon)\,dx - \int\limits_{B_{\frac{r}{\varepsilon}}(0)\setminus B_R(0)} f_\tau(\hat{u}_\varepsilon)\,dx\Bigg) = -4\pi(m_{j_0,2}+\beta_{j_0,2}) > 0.$$

Moreover, the method of moving planes to be used with (4.63) shows that $\hat{u}$ is radially symmetric (see [5,12]). Now by using Theorem 3.4 in [8] and $\hat{\beta} > 0$, we see that $\hat{u}$ cannot be stable solution.

At this point, we complete the proof of one part of Theorem 1.1: stable solution ⇒ topological solution under the assumptions (H1)–(H2). □

5. Proof of Theorem 1.1: topological solution ⇒ strictly stable solution

In this section, we prove the other implication in the statement of Theorem 1.1, that is, topological solution ⇒ strictly stable solution. We assume that u_ε is a sequence of topological solutions of (1.2) with a sequence $\varepsilon > 0$. Although we use arguments similar to those in [26], we still need to carry out a subtle analysis to control the solution's sign changes.

Lemma 5.1. *Let u_ε be a sequence of topological solutions of (1.2) with $\varepsilon > 0$. Then, as $\varepsilon \to 0$, we have*

(i) *$u_\varepsilon \to 0$ in $C^m_{\mathrm{loc}}(\Omega\setminus Z)$ for any $m\in\mathbb{Z}^+$ and faster than any power of ε;*
(ii) *$\frac{(1-e^{u_\varepsilon})^2}{\varepsilon^2(\tau+e^{u_\varepsilon})^2} \to 4(\tau+1)\pi\sum_{i=1,2}\sum_{j=1}^{d_i} m_{j,i}^2\delta_{p_{j,i}}$, weakly in the sense of measures in Ω.*

Proof. Let $\Omega_\delta \equiv \{x\in\Omega \mid \operatorname{dist}(x,Z)\geqslant\delta\}$. In view of Theorem 1.5 we have $u_\varepsilon \to 0$ uniformly on any compact subset of $\Omega\setminus Z$ as $\varepsilon\to 0$. Then we see that for any small $\delta > 0$,

$$\Delta\big(|u_\varepsilon|^2\big) = 2|\nabla u_\varepsilon|^2 + 2u_\varepsilon\Delta u_\varepsilon = 2|\nabla u_\varepsilon|^2 + \frac{2u_\varepsilon e^{u_\varepsilon}(e^{u_\varepsilon}-1)}{\varepsilon^2(\tau+e^{u_\varepsilon})^3} \geqslant 0 \quad \text{on } \Omega_\delta, \tag{5.1}$$

since $t(e^t-1)\geqslant 0$ for any $t\in\mathbb{R}$. Moreover, we see that

$$\Delta\big(|\nabla u_\varepsilon|^2\big) = \sum_{i,j=1}^{2} 2\left|\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\right|^2 + \frac{2|\nabla u_\varepsilon|^2 e^{u_\varepsilon}(-e^{2u_\varepsilon}+2(\tau+1)e^{u_\varepsilon}-\tau)}{\varepsilon^2(\tau+e^{u_\varepsilon})^4}$$

$$\geqslant \frac{2|\nabla u_\varepsilon|^2 e^{u_\varepsilon}(\tau+1+o(1))}{\varepsilon^2(\tau+e^{u_\varepsilon})^4} \geqslant 0 \quad \text{on } \Omega_\delta \text{ as } \varepsilon\to 0. \tag{5.2}$$

We have the following inequality,

$$\frac{|t|}{1+|t|} \leqslant \big|1-e^t\big| \quad \text{for any } t\in\mathbb{R}. \tag{5.3}$$

By using (3.2), (5.1), (5.3), and the mean value theorem, we see that there exists a constant $c > 0$ such that

$$\sup_{\Omega_{2\delta}}\big(|u_\varepsilon|^2\big) \leqslant \frac{1}{|\Omega_\delta|}\int\limits_{\Omega_\delta}|u_\varepsilon|^2\,dx$$

$$\leqslant \frac{(1+\|u_\varepsilon\|_{L^\infty(\Omega_\delta)})^2}{|\Omega_\delta|}\int\limits_{\Omega_\delta}\frac{|u_\varepsilon|^2}{(1+|u_\varepsilon|)^2}\,dx$$

$$\leqslant \frac{(1+\|u_\varepsilon\|_{L^\infty(\Omega_\delta)})^2}{|\Omega_\delta|}\Big\|\frac{(\tau+e^{u_\varepsilon})^4}{e^{u_\varepsilon}}\Big\|_{L^\infty(\Omega_\delta)}\int\limits_{\Omega_\delta}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})^2}{(\tau+e^{u_\varepsilon})^4}\,dx$$

$$\leqslant \frac{c\varepsilon^2(1+\|u_\varepsilon\|_{L^\infty(\Omega_\delta)})^2}{|\Omega_\delta|}\left\|\frac{(\tau+e^{u_\varepsilon})^4}{e^{u_\varepsilon}}\right\|_{L^\infty(\Omega_\delta)}, \tag{5.4}$$

for small $\varepsilon > 0$. In view of (3.2), (5.2), and the mean value theorem, we can find a constant $C > 0$ such that

$$\begin{aligned}\sup_{\Omega_{2\delta}}\big(|\nabla u_\varepsilon|^2\big) &\leqslant \frac{1}{|\Omega_\delta|}\int_{\Omega_\delta}|\nabla u_\varepsilon|^2\,dx\\ &\leqslant \frac{1}{|\Omega_\delta|}\left\|\frac{(\tau+e^{u_\varepsilon})^2}{e^{u_\varepsilon}}\right\|_{L^\infty(\Omega_\delta)}\int_\Omega \frac{|\nabla u_\varepsilon|^2 e^{u_\varepsilon}}{(\tau+e^{u_\varepsilon})^2}\,dx\\ &\leqslant \frac{C}{|\Omega_\delta|}\left\|\frac{(\tau+e^{u_\varepsilon})^2}{e^{u_\varepsilon}}\right\|_{L^\infty(\Omega_\delta)},\end{aligned} \tag{5.5}$$

for small $\varepsilon > 0$. Let $\phi \in C^\infty(\overline{\Omega})$ be such that $\phi = 0$ in $\{x \in \Omega \mid \operatorname{dist}(x, Z) \leqslant \delta\}$, $\phi = 1$ in $\Omega_{2\delta}$ and $0 \leqslant \phi \leqslant 1$. Since $u_\varepsilon \to 0$ uniformly on any compact subset of $\Omega \setminus Z$ as $\varepsilon \to 0$, we note that there exists some constant $C_\delta > 0$, independent of $\varepsilon > 0$, such that

$$\left|\frac{1-e^{u_\varepsilon}}{\tau+e^{u_\varepsilon}}\right| \leqslant C_\delta|u_\varepsilon| \quad \text{on } \Omega_\delta. \tag{5.6}$$

Next, by using (5.4), (5.5) and (5.6), we conclude that

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\Omega_{2\delta}}\frac{e^{u_\varepsilon}(1-e^{u_\varepsilon})^2}{(\tau+e^{u_\varepsilon})^4}\,dx &\leqslant \frac{1}{\varepsilon^2}\int_\Omega \frac{e^{u_\varepsilon}(e^{u_\varepsilon}-1)}{(\tau+e^{u_\varepsilon})^3}\left[\frac{(e^{u_\varepsilon}-1)\phi}{(\tau+e^{u_\varepsilon})}\right]dx\\ &= \int_\Omega \Delta u_\varepsilon\left[\frac{(e^{u_\varepsilon}-1)\phi}{(\tau+e^{u_\varepsilon})}\right]dx = \int_\Omega u_\varepsilon\Delta\left[\frac{(e^{u_\varepsilon}-1)\phi}{(\tau+e^{u_\varepsilon})}\right]dx\\ &= \int_\Omega u_\varepsilon\left[\Delta\left(\frac{e^{u_\varepsilon}-1}{\tau+e^{u_\varepsilon}}\right)\phi + \frac{2(\tau+1)e^{u_\varepsilon}\nabla u_\varepsilon\cdot\nabla\phi}{(\tau+e^{u_\varepsilon})^2} + \frac{(e^{u_\varepsilon}-1)\Delta\phi}{(\tau+e^{u_\varepsilon})}\right]dx\\ &= \int_\Omega\left[\frac{-(\tau+1)e^{u_\varepsilon}|\nabla u_\varepsilon|^2\phi}{(\tau+e^{u_\varepsilon})^2} + \frac{(\tau+1)e^{u_\varepsilon}u_\varepsilon\nabla u_\varepsilon\cdot\nabla\phi}{(\tau+e^{u_\varepsilon})^2} + \frac{(e^{u_\varepsilon}-1)u_\varepsilon\Delta\phi}{(\tau+e^{u_\varepsilon})}\right]dx\\ &\leqslant \int_\Omega\left[\frac{(\tau+1)e^{u_\varepsilon}u_\varepsilon\nabla u_\varepsilon\cdot\nabla\phi}{(\tau+e^{u_\varepsilon})^2} + \frac{(e^{u_\varepsilon}-1)u_\varepsilon\Delta\phi}{(\tau+e^{u_\varepsilon})}\right]dx\\ &= \int_\Omega\left[\frac{-(\tau+1)e^{u_\varepsilon}u_\varepsilon^2\Delta\phi}{2(\tau+e^{u_\varepsilon})^2} + \frac{(\tau+1)e^{u_\varepsilon}(e^{u_\varepsilon}-\tau)u_\varepsilon^2\nabla u_\varepsilon\cdot\nabla\phi}{2(\tau+e^{u_\varepsilon})^3} + \frac{(e^{u_\varepsilon}-1)u_\varepsilon\Delta\phi}{(\tau+e^{u_\varepsilon})}\right]dx\\ &\leqslant c_\delta\|u_\varepsilon\|^2_{L^2(\Omega_\delta)} \leqslant C_\delta\varepsilon^2,\end{aligned} \tag{5.7}$$

for some constants $c_\delta, C_\delta > 0$. By a suitable iteration of (5.4), (5.7), and the elliptic estimates, we deduce that (i) holds. In other words, for any small $\delta > 0$ and any $m, n \in \mathbb{Z}^+$, there exists a constant $c_{\delta,m,n} > 0$ such that

$$\sup_{\Omega_{2\delta}}\left(\sum_{|\alpha|=0}^m\big|D^\alpha u_\varepsilon\big|\right) \leqslant c_{\delta,m,n}\varepsilon^n. \tag{5.8}$$

Moreover, we see that $v_\varepsilon(x) = u_\varepsilon(x) + (-1)^i 2m_{j,i}\ln|x - p_{j,i}|$ satisfies

$$\Delta v_\varepsilon + \frac{f_\tau(u_\varepsilon)}{\varepsilon^2} = 0 \quad \text{on } B_r(p_{j,i}). \tag{5.9}$$

For the sake of simplicity, we assume that $p_{j,i} = 0$. Multiplying (5.9) by $\nabla u_\varepsilon \cdot x$ and integrating over $B_r(0)$, we obtain the Pohozaev type identity

$$\int_{\partial B_r(0)} \left[\left(\nabla v_\varepsilon \cdot \frac{x}{|x|}\right)(\nabla v_\varepsilon \cdot x) - \frac{|\nabla v_\varepsilon|^2}{2}|x| + \frac{1}{\varepsilon^2} F_{1,\tau}(u_\varepsilon)|x|\right] d\sigma = \int_{B_r(0)} \frac{2F_{1,\tau}(u_\varepsilon)}{\varepsilon^2} + \frac{(-1)^{i-1} 2m_{j,i} f_\tau(u_\varepsilon)}{\varepsilon^2}\, dx,$$

where $F_{1,\tau}(u) = \frac{-(1-e^u)^2}{2(\tau+1)(\tau+e^u)^2}$. By using (5.8), we have

$$\lim_{\varepsilon\to 0} \int_{\partial B_r(0)} \frac{1}{\varepsilon^2} F_{1,\tau}(u_\varepsilon)|x|\, d\sigma = 0,$$

thus

$$\lim_{\varepsilon\to 0} \int_{B_r(0)} \frac{2F_{1,\tau}(u_\varepsilon)}{\varepsilon^2}\, dx = -4\pi m_{j,i}^2,$$

$$\lim_{\varepsilon\to 0} \int_{B_r(0)} \frac{(1-e^{u_\varepsilon})^2}{\varepsilon^2(\tau+e^{u_\varepsilon})^2}\, dx = 4(\tau+1)\pi m_{j,i}^2,$$

for any small $r > 0$ which concludes the proof of our lemma. □

For a solution u_ε of (1.2), let

$$\mu_\varepsilon \equiv \inf_{\phi \in W^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla\phi|^2 - \frac{1}{\varepsilon^2} f_\tau'(u_\varepsilon)\phi^2\, dx}{\|\phi\|^2_{L^2(\Omega)}}, \tag{5.10}$$

and ϕ_ε be the corresponding first eigenfunction with $\phi_\varepsilon > 0$ in Ω and $\|\phi_\varepsilon\|_{L^2(\Omega)} = 1$,

$$\mu_\varepsilon = \int_\Omega |\nabla\phi_\varepsilon|^2 - \frac{1}{\varepsilon^2} f_\tau'(u_\varepsilon)\phi_\varepsilon^2\, dx, \tag{5.11}$$

and

$$-\Delta\phi_\varepsilon - \frac{1}{\varepsilon^2} f_\tau'(u_\varepsilon)\phi_\varepsilon = \mu_\varepsilon\phi_\varepsilon. \tag{5.12}$$

We note that $\varepsilon^2\mu_\varepsilon$ is bounded from below:

$$\varepsilon^2\mu_\varepsilon \geqslant -\int_\Omega f_\tau'(u_\varepsilon)\phi_\varepsilon^2\, dx \geqslant -\sup_{t\in\mathbb{R}} \left|f_\tau'(t)\right|.$$

To prove Theorem 1.1, we argue by contradiction and suppose that, along a subsequence (still denoted in the same way), we have a sequence of topological solutions u_ε of (1.2) with a sequence $\varepsilon > 0$ such that

$$\lim_{\varepsilon\to 0} \varepsilon^2\mu_\varepsilon = \mu_0 \leqslant 0. \tag{5.13}$$

In view of (i) of Lemma 5.1 and (5.13), we have the following lemma.

Lemma 5.2. *There exist $p_{j_0,i_0} \in Z$ and $r_0 > 0$ such that for any $r \in (0, r_0)$, there exists a constant $a_r > 0$ such that*

$$\lim_{\varepsilon\to 0} \int_{B_r(p_{j_0,i_0})} \phi_\varepsilon^2\, dx \geqslant a_r.$$

Proof. Suppose that there exists a small $r > 0$ such that

$$\lim_{\varepsilon\to 0} \int_{\bigcup_{j,i} B_r(p_{j,i})} \phi_\varepsilon^2\, dx = 0. \tag{5.14}$$

Then

$$\lim_{\varepsilon\to 0}\left|\int_{\bigcup_{j,i} B_r(p_{j,i})} f_\tau'(u_\varepsilon)\phi_\varepsilon^2\,dx\right| \leqslant \sup_{t\in\mathbb{R}}\left|f_\tau'(t)\right| \lim_{\varepsilon\to 0}\int_{\bigcup_{j,i} B_r(p_{j,i})} \phi_\varepsilon^2\,dx = 0.$$

By using (i) of Lemma 5.1, we see that

$$\int_{\Omega\setminus\bigcup_{j,i} B_r(p_{j,i})} f_\tau'(u_\varepsilon)\phi_\varepsilon^2\,dx = \int_{\Omega\setminus\bigcup_{j,i} B_r(p_{j,i})} \left(-\frac{1}{(\tau+1)^3}+o(1)\right)\phi_\varepsilon^2\,dx \quad \text{as } \varepsilon\to 0. \tag{5.15}$$

Next, by using (5.13), (5.14) and (5.15), we see that

$$0\geqslant \lim_{\varepsilon\to 0}\varepsilon^2\mu_\varepsilon = \lim_{\varepsilon\to 0}\int_\Omega \varepsilon^2|\nabla\phi_\varepsilon|^2 - f_\tau'(u_\varepsilon)\phi_\varepsilon^2\,dx \geqslant \lim_{\varepsilon\to 0}\int_\Omega -f_\tau'(u_\varepsilon)\phi_\varepsilon^2\,dx = \frac{1}{(\tau+1)^3}.$$

This is the desired contradiction which concludes the proof of our lemma. □

Since $f_\tau(u) = -f_{\tau^{-1}}(-u)/\tau^3$, we can assume without loss of generality that $i_0 = 2$, $p_{j_0,i_0} = 0$, and $\nu \equiv m_{j_0,i_0}$ in Lemma 5.2. We consider the scaled function

$$\hat{u}_\varepsilon(y) = u_\varepsilon(\varepsilon y) \quad \text{in } B_{\frac{r_0}{\varepsilon}}(0). \tag{5.16}$$

Then $\hat{u}_\varepsilon$ satisfies

$$\Delta\hat{u}_\varepsilon + \frac{e^{\hat{u}_\varepsilon}(1-e^{\hat{u}_\varepsilon})}{(\tau+e^{\hat{u}_\varepsilon})^3} = -4\pi\nu\delta_0 \quad \text{in } B_{\frac{r_0}{\varepsilon}}(0).$$

Now we have the following lemma.

Lemma 5.3. $\lim_{\varepsilon\to 0}(\sup_{B_{\frac{r_0}{\varepsilon}}(0)}|\hat{u}_\varepsilon - u|) = 0$, *where u is a topological solution of*

$$\begin{cases} \Delta u + \dfrac{e^u(1-e^u)}{(\tau+e^u)^3} = -4\pi\nu\delta_0 \quad \text{in } \mathbb{R}^2, \\ \sup\limits_{\mathbb{R}^2\setminus B_1(0)} |\nabla u| < +\infty, \\ \dfrac{e^u(1-e^u)}{(\tau+e^u)^3}, \dfrac{(1-e^u)^2}{(\tau+e^u)^2} \in L^1(\mathbb{R}^2). \end{cases} \tag{5.17}$$

Proof. We decompose

$$\hat{u}_\varepsilon(y) = -2\nu\ln|y| + \hat{v}_\varepsilon(y). \tag{5.18}$$

Then $\hat{v}_\varepsilon$ satisfies

$$\Delta\hat{v}_\varepsilon + \frac{|y|^{-2\nu}e^{\hat{v}_\varepsilon}(1-|y|^{-2\nu}e^{\hat{v}_\varepsilon})}{(\tau+|y|^{-2\nu}e^{\hat{v}_\varepsilon})^3} = 0 \quad \text{in } B_{\frac{r_0}{\varepsilon}}(0). \tag{5.19}$$

By using Lemma 5.1, $\lim_{x\to p_{j,2}} u_\varepsilon(x) = +\infty$ and the maximum principle, we conclude that there exists $c>0$ such that for small $\varepsilon>0$,

$$\inf_{B_r(p_{j,2})} u_\varepsilon \geqslant -c. \tag{5.20}$$

In view of (5.18) and (5.20), we have

$$\hat{v}_\varepsilon|_{\partial B_R(0)} \geqslant -c + 2\nu\ln R \quad \text{for any } R>0.$$

By using the Green's representation formula for a solution u_ε of (1.2) (see (3.12) and (4.19)), we see that there exists $c_0 > 0$ such that

$$\left|\nabla \hat{v}_\varepsilon(x)\right| \leqslant c_0 \quad \text{on } B_{\frac{r_0}{\varepsilon}}(0). \tag{5.21}$$

We claim that $\hat{v}_\varepsilon$ is uniformly bounded in the $C^{2,\alpha}$ topology. To prove our claim, we argue by contradiction and suppose that there exists $R_0 > 0$ such that $\lim_{\varepsilon\to 0}(\sup_{B_{R_0}(0)} \hat{v}_\varepsilon) = +\infty$. Then (5.21) implies that $\lim_{\varepsilon\to 0}(\inf_{B_R(0)} \hat{v}_\varepsilon) = +\infty$ for any $R \geqslant R_0$. Clearly Lemma 5.1 shows that, for any $R \geqslant R_0$,

$$4(\tau+1)\pi\nu^2 \geqslant \lim_{\varepsilon\to 0} \int\limits_{B_R(0)} \frac{(1-|x|^{-2\nu}e^{\hat{v}_\varepsilon})^2}{(\tau+|x|^{-2\nu}e^{\hat{v}_\varepsilon})^2}\,dx = \pi R^2. \tag{5.22}$$

Since the right hand side of (5.22) could be arbitrarily large, we obtain a contradiction which proves our claim.

Then we obtain a subsequence $\hat{v}_\varepsilon$ (still denoted in the same way) such that

$$\hat{v}_\varepsilon \to v \quad \text{uniformly in } C^2_{\mathrm{loc}}(\mathbb{R}^2). \tag{5.23}$$

Let us define $u(y) \equiv -2\nu \ln|y| + v(y)$. In view of (5.21), Lemma 3.1 and Lemma 5.1, we see that u satisfies (5.17). Since $\sup_{\mathbb{R}^2\setminus B_1(0)} |\nabla u| < +\infty$ and $\frac{(1-e^u)^2}{(\tau+e^u)^2} \in L^1(\mathbb{R}^2)$, we see that u is a topological solution in $\mathbb{R}^2$. Moreover, by using a Pohozaev type identity (see Lemma 5.1), we have

$$\int\limits_{\mathbb{R}^2} \frac{(1-e^u)^2}{(\tau+e^u)^2}\,dx = 4(\tau+1)\pi\nu^2. \tag{5.24}$$

Now we claim that a stronger convergence property holds, namely

$$\lim_{\varepsilon\to 0}\Big(\sup_{B_{\frac{r_0}{\varepsilon}}(0)} |\hat{u}_\varepsilon - u|\Big) = 0.$$

In view of (5.23), we have

$$\lim_{\varepsilon\to 0}\Big(\sup_{B_1(0)} |\hat{u}_\varepsilon - u|\Big) = 0. \tag{5.25}$$

We also see that

$$\int\limits_{B_{\frac{r_0}{\varepsilon}}(0)} \frac{(e^{\hat{u}_\varepsilon}-e^u)^2}{(\tau+e^{\hat{u}_\varepsilon})^2}\,dx = \int\limits_{B_{\frac{r_0}{\varepsilon}}(0)} \frac{(e^{\hat{u}_\varepsilon}-1)^2}{(\tau+e^{\hat{u}_\varepsilon})^2} + \frac{(e^u-1)^2}{(\tau+e^{\hat{u}_\varepsilon})^2} - \frac{2(1-e^{\hat{u}_\varepsilon})(1-e^u)}{(\tau+e^{\hat{u}_\varepsilon})^2}\,dx.$$

At this point Lemma 5.1, (5.24), and the dominated convergence theorem imply that

$$\lim_{\varepsilon\to 0} \int\limits_{B_{\frac{r_0}{\varepsilon}}(0)} \frac{(e^{\hat{u}_\varepsilon}-e^u)^2}{(\tau+e^{\hat{u}_\varepsilon})^2}\,dx = 0. \tag{5.26}$$

By using (5.17), (5.20), (5.21), (5.25), and (5.26), we obtain the desired conclusion. □

At this point, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1: topological solution ⇒ strictly stable solution. In view of the strong convergence property as stated in Lemma 5.3, we can deduce information about the limiting problem of the linearized equation of (1.2) at u_ε. With this purpose, we define

$$\hat{\psi}_\varepsilon(y) \equiv \varepsilon\phi_\varepsilon(\varepsilon y) \quad \text{on } B_{\frac{r_0}{\varepsilon}}(0). \tag{5.27}$$

Then we have

$$\begin{cases} -\Delta\hat{\psi}_\varepsilon - f_\tau'(\hat{u}_\varepsilon)\hat{\psi}_\varepsilon = \varepsilon^2\mu_\varepsilon\hat{\psi}_\varepsilon & \text{on } B_{\frac{r_0}{\varepsilon}}(0), \\ \hat{\psi}_\varepsilon > 0 & \text{in } B_{\frac{r_0}{\varepsilon}}(0), \end{cases} \tag{5.28}$$

and $\|\nabla\hat{\psi}_\varepsilon\|_{L^2(B_{\frac{r_0}{\varepsilon}}(0))} + \|\hat{\psi}_\varepsilon\|_{L^2(B_{\frac{r_0}{\varepsilon}}(0))} \leqslant C$ for some constant $C > 0$. By using standard elliptic estimates, we see that $\hat{\psi}_\varepsilon$ is uniformly bounded in the $C^{2,\alpha}_{\mathrm{loc}}$ topology. Hence, by passing to a subsequence (still denoted in the same way), we see that there exists $\hat{\psi} \geqslant 0$ such that

$$\hat{\psi}_\varepsilon \to \hat{\psi} \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^2),$$

and

$$\begin{cases} -\Delta\hat{\psi} - f_\tau'(u)\hat{\psi} = \mu_0\hat{\psi} & \text{in } \mathbb{R}^2, \\ \hat{\psi} \in W^{1,2}(\mathbb{R}^2), \quad \hat{\psi} \geqslant 0. \end{cases} \tag{5.29}$$

Since u decays exponentially at infinity, then $f_\tau'(u) + \frac{1}{(\tau+1)^3}$ has exponentially decay at infinity. Hence by Lemma 5.3, we see that

$$\lim_{\varepsilon\to 0} \int\limits_{B_{\frac{r_0}{2\varepsilon}}(0)} \left(f_\tau'(\hat{u}_\varepsilon) + \frac{1}{(\tau+1)^3}\right)\hat{\psi}_\varepsilon^2\,dx = \int\limits_{B_R(0)} \left(f_\tau'(u) + \frac{1}{(\tau+1)^3}\right)\hat{\psi}^2\,dx + O\left(e^{-\delta_0 R}\right)$$

for some $\delta_0 > 0$. Hence by using (5.27), (5.28), and Lemma 5.2, we can prove that for large $R > 0$,

$$\begin{aligned} \int\limits_{B_R(0)} \left(f_\tau'(u) + \frac{1}{(\tau+1)^3}\right)\hat{\psi}^2\,dx &\geqslant \lim_{\varepsilon\to 0}\left(-\varepsilon^2\mu_\varepsilon + \frac{1}{(\tau+1)^3}\right) \int\limits_{B_{\frac{r_0}{2\varepsilon}}(0)} \hat{\psi}_\varepsilon^2\,dx + O\left(e^{-\delta_0 R}\right) \\ &\geqslant \left(|\mu_0| + \frac{1}{(\tau+1)^3}\right)a_{\frac{r_0}{2}} + O\left(e^{-\delta_0 R}\right) > 0, \end{aligned}$$

which implies $\hat{\psi} \neq 0 \in W^{1,2}(\mathbb{R}^2)$ (see Lemma 4.15 in [26] for further details). On the other side, by arguing as in Proposition 4.16 in [26], we see that the problem (5.29) admits only the trivial solution and we obtain a contradiction. This observation concludes the proof of Theorem 1.1: topological solution $\Rightarrow$ strictly stable solution. □

6. Uniqueness of stable solution

In this section, we deduce Theorem 1.3 from Theorem 1.1.

Proof of Theorem 1.3. The existence of stable solution can be proved by well known monotone iteration schemes and therefore we will skip it here. Hence, to prove Theorem 1.3, it suffices to prove the uniqueness property. We argue by contradiction and suppose that there exist two sequences of distinct stable solutions $u_{\varepsilon,1}$ and $u_{\varepsilon,2}$ of (1.2). From Theorem 1.1, up to the extraction of subsequences, we have $u_{\varepsilon,i} \to 0$ uniformly in any compact subset of $\Omega \setminus Z$ as $\varepsilon \to 0$ for $i = 1, 2$. Since $u_{\varepsilon,1} - u_{\varepsilon,2}$ is not identically zero, we can define $\phi_\varepsilon \equiv \frac{u_{\varepsilon,1}-u_{\varepsilon,2}}{\|u_{\varepsilon,1}-u_{\varepsilon,2}\|_{L^2(\Omega)}}$ which satisfies

$$\Delta\phi_\varepsilon + \frac{1}{\varepsilon^2}f_\tau'(\eta_\varepsilon)\phi_\varepsilon = 0 \quad \text{on } \Omega,$$

where η_ε is some real number between $u_{\varepsilon,1}$ and $u_{\varepsilon,2}$. By using the proof of Lemma 5.2, we see that there exist $p_{j_0,i_0} \in Z$ and $r_0 > 0$ such that for any $r \in (0, r_0)$, there exists a constant $a_r > 0$ such that

$$\lim_{\varepsilon\to 0} \int\limits_{B_r(p_{j_0,i_0})} \phi_\varepsilon^2\,dx \geqslant a_r.$$

Since $f_\tau(u) = -f_{\tau^{-1}}(-u)/\tau^3$, we can assume without loss of generality that $i_0 = 2$, $p_{j_0,i_0} = 0$, and $\nu \equiv m_{j_0,i_0}$. We consider the scaled function

$$\hat{u}_{\varepsilon,i}(y) = u_{\varepsilon,i}(\varepsilon y) \quad \text{in } B_{\frac{r_0}{\varepsilon}}(0) \equiv \left\{ y \in \mathbb{R}^2 \;\middle|\; |y| < \frac{r_0}{\varepsilon} \right\}.$$

In view of Lemma 5.3, we obtain

$$\hat{u}_{\varepsilon,i} \to u_i \quad \text{uniformly in } C^2_{\text{loc}}(\mathbb{R}^2) \text{ for } i = 1, 2,$$

where u_i is a topological solution of

$$\Delta u_i + \frac{e^{u_i}(1 - e^{u_i})}{(\tau + e^{u_i})^3} = -4\pi\nu\delta_0 \quad \text{in } \mathbb{R}^2.$$

Moreover, we can apply the method of moving planes (see [12,15]) to conclude that u_i is radially symmetric about the origin. Since radially symmetric and topological solutions are unique (see [8]), we conclude that $u_1 = u_2$ in $\mathbb{R}^2$. Let us set $u \equiv u_1$. We can find $\hat{\psi}$ such that

$$\varepsilon\phi_\varepsilon(\varepsilon y) \to \hat{\psi}(y) \quad \text{in } C^2_{\text{loc}},$$

and

$$\begin{cases} -\Delta\hat{\psi} - f'_\tau(u)\hat{\psi} = 0 & \text{in } \mathbb{R}^2, \\ \hat{\psi} \in W^{1,2}(\mathbb{R}^2). \end{cases}$$

By arguing as in the proof of Theorem 1.1 (see Section 5), we see that $\hat{\psi} \neq 0$. Then,

$$\mu^* \equiv \inf_{\psi \in W^{1,2}(\mathbb{R}^2)\setminus\{0\}} \frac{\int_{\mathbb{R}^2} |\nabla\psi|^2 - f'_\tau(u)\psi^2\,dx}{\int_{\mathbb{R}^2}(1 - e^u)\psi^2\,dx} \leqslant 0. \tag{6.1}$$

Then Lemma 5.3 shows that the infimum of (6.1) is attained at some $\psi_0 \in W^{1,2}(\mathbb{R}^2) \setminus \{0\}$ satisfying

$$-\Delta\psi_0 - f'_\tau(u)\psi_0 = \mu^*(1 - e^u)\psi_0, \qquad \psi_0 > 0 \quad \text{in } \mathbb{R}^2.$$

At this point Theorem 3.4 in [8] shows that $\mu^* < 0$. However, by arguing as in Proposition 4.16 in [26], we can show that $\psi_0 \equiv 0$ which is the desired contradiction. Therefore there exists a unique stable solution of (1.2) for sufficiently small $\varepsilon > 0$. □

Conflict of interest statement

We confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. We further confirm that the order of authors listed in the manuscript has been approved by all of us.

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Appendix A

In this appendix, we discuss nontopological solutions of the following equation:

$$\begin{cases} \Delta u + f_\tau(u) = 4\pi\nu\delta_0 & \text{in } \mathbb{R}^2, \\ f_\tau(u) \in L^1(\mathbb{R}^2), \\ \lim_{|x|\to\infty} u(x) = -\infty. \end{cases} \tag{A.1}$$

As we mentioned in Section 4, we need to analyze a solution u_ε of (1.2), such that $u_\varepsilon - 2\ln\varepsilon$ has a bubble at some point in $\Omega \setminus Z_2$ and u_ε (after a suitable scaling) tends to a nontopological solution u of (A.1). It is not difficult to check that it is enough to our purposes to consider the case $\nu \geqslant 0$. Concerning this problem, we have the following proposition.

Proposition A.1. *Let u be a solution of (A.1) and $\nu \geqslant 0$. Then u is unstable.*

Proof. By using the maximum principle, we always have $u < 0$. Moreover, if u is radially symmetric, then Theorem 3.4 in [8] shows that u is unstable. In particular, if $\nu = 0$, then Lemma 2.1 shows that u is a radially symmetric function. Thus, we only need to prove the instability of u in the case where $\nu > 0$ and u is not radially symmetric. Let us set

$$\frac{\partial}{\partial\theta} = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}.$$

Then we see that

$$\Delta(\partial_\theta u) + f_\tau'(u)(\partial_\theta u) = 0 \quad \text{in } \mathbb{R}^2.$$

Let $\beta = \frac{1}{2\pi}\int_{\mathbb{R}^2} f_\tau(u)\,dx$. Since $u < 0$, we see that $e^u \in L^1(\mathbb{R}^2)$ and $\lim_{|x|\to\infty}\frac{u(x)}{\ln|x|} = -\beta + 2\nu < -2$. Moreover, by using the results in [5], we obtain the sharper estimate $u(x) = (-\beta + 2\nu)\ln|x| + C + O(|x|^{-\gamma})$, $u_\theta(x) = O(|x|^{-1})$ as $|x| \to +\infty$ where C is a constant and γ is a positive constant. We also note that there exist a local maximum point and a local minimum point of u on each sphere of radius r since u is not radially symmetric. Thus $\partial_\theta u$ changes signs, which implies at least that the first eigenvalue of the linearized equation of (1.2) at u is negative. Therefore we see that u is unstable which was the desired conclusion. □

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