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Extremal domains of big volume for the first eigenvalue of the Laplace–Beltrami operator in a compact manifold

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Abstract

We prove the existence of new extremal domains for the first eigenvalue of the Laplace–Beltrami operator in some compact Riemannian manifolds of dimension $n \geqslant 2$. The volume of such domains is close to the volume of the manifold. If the first eigenfunction ϕ_0 of the Laplace–Beltrami operator over the manifold is a nonconstant function, these domains are close to the complement of geodesic balls centered at a nondegenerate critical point of ϕ_0. If ϕ_0 is a constant function and $n \geqslant 4$, these domains are close to the complement of geodesic balls centered at a nondegenerate critical point of the scalar curvature.
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1. Introduction and statement of the main result

Let (M, g) be an n-dimensional Riemannian manifold, Ω a (connected and open) domain in M with smooth boundary, and λ_Ω the first eigenvalue of $-\Delta_g$ (the Laplace–Beltrami operator) in Ω with 0 Dirichlet boundary condition. The domain $\Omega_0 \subset M$ is said to be extremal if $\Omega \mapsto \lambda_\Omega$ is critical at Ω_0 with respect to variations of the domain Ω_0 which preserve its volume.

P.R. Garabedian and M. Schiffer proved in [9] that a domain Ω_0 is extremal in the Euclidean space $\mathbb{R}^n$ if and only if its first eigenfunction of the Laplacian with 0 Dirichlet boundary condition has a constant Neumann data at the boundary. In the Euclidean space, extremal domains are then characterized as the domains for which the *over-determined* system

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = \text{constant} & \text{on } \partial\Omega \end{cases} \tag{1}$$

has a positive solution (here ν is the outward unit normal vector field along $\partial\Omega$). By a classical result due to J. Serrin the only domains for which the system (1) has a positive solution are round balls, see [20]. In the Euclidean space, round balls are in fact not only extremal domains, but also minimizers for the first eigenvalue of the Laplacian with 0 Dirichlet boundary condition. This follows from the Faber–Krahn inequality,

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$$\lambda_\Omega \geqslant \lambda_{B^n(\Omega)} \tag{2}$$

where $B^n(\Omega)$ is a round ball of $\mathbb{R}^n$ with the same volume as Ω, because equality holds in (2) if and only if $\Omega = B^n(\Omega)$, see [8] and [10]. The result of J. Serrin, based on the moving plane argument introduced by A.D. Alexandrov in [1], use strongly the symmetry of the Euclidean space, and naturally it fails in other geometries. The classification of extremal domains is then achieved in the Euclidean space $\mathbb{R}^n$, but it is completely open in a general Riemannian manifold.

Some new examples of extremal domains for the first eigenvalue of the Laplace–Beltrami operator in some Riemannian manifolds have been obtained in [16] by F. Pacard and P. Sicbaldi. Such new domains have small volume and are close to geodesic balls centered at a nondegenerate critical point of the scalar curvature of the manifold (the existence of at least a nondegenerate critical point of the scalar curvature is required in order to build such domains). Such result has been generalized to a general compact Riemannian manifold by E. Delay and P. Sicbaldi [4], by eliminating the assumption of the existence of a nondegenerate critical point of the scalar curvature of the manifold. In fact, it was quite natural to expect that a small domain close to a geodesic ball could be an extremal domain in a Riemannian manifold, because a Riemannian metric is locally close to the Euclidean one. The real difficulty was to find the point of the manifold where such small topological ball had to be centered in order to be an extremal domain, and this point is a nondegenerate critical point of the scalar curvature if it exists (see [16]) or the critical point of an other special function depending on curvatures (see [4]).

The previous results have been inspired by some parallel results on the isoperimetric problem. The solutions of the isoperimetric problem

$$I_\kappa := \min_{\Omega \subset M:\ \mathrm{Vol}\,\Omega = \kappa} \mathrm{Vol}\,\partial\Omega$$

are (where they are smooth enough) constant mean curvature hypersurfaces. O. Druet proved in [5] that for small volumes (i.e. $\kappa > 0$ small), the solutions of the isoperimetric problem are close to geodesic spheres of small radius centered at a point where the scalar curvature is maximal. Independently, R. Ye built in [24] constant mean curvature topological spheres which are close to geodesic spheres of small radius centered at a nondegenerate critical point of the scalar curvature, and F. Pacard and X. Xu generalized such a construction in compact manifolds that do not have any nondegenerate critical point of the scalar curvature, see [17]. Now, it is well known (see [8,10,11]) that the determination of the isoperimetric profile I_κ is related to Faber–Krahn minimizers, where one looks for the least value of the first eigenvalue of the Laplace–Beltrami operator amongst domains with prescribed volume

$$FK_\kappa := \min_{\Omega \subset M:\ \mathrm{Vol}\,\Omega = \kappa} \lambda_\Omega.$$

Observe that a solution to this minimizing problem (when it is smooth) is an extremal domain. The result of F. Pacard and P. Sicbaldi can be considered the parallel of the result of R. Ye in the context of extremal domains, as the result of E. Delay and P. Sicbaldi is in some sense the parallel of the result of F. Pacard and X. Xu. Moreover, paralleling his result about the isoperimetric problem, O. Druet obtained in [6] that for small volumes (i.e. $\kappa > 0$ small), the Faber–Krahn minimizers are close to geodesic balls of small radius centered at a point where the scalar curvature is maximal.

For arbitrary volume, the situation is much more complex and very few results are known (see for example the proof of the existence of new nontrivial extremal domains in flat tori in [22], the study of the shape of such domains in [21], and the concavity condition for extremal domains in flat tori obtained in [18]). In this paper we give an existence result for extremal domains of big volume in a compact Riemannian manifold. We build new examples of extremal domains, that cannot be topological balls because of the condition on the volume. In fact, the examples of extremal domains we build are the complement of small topological balls. In particular, the novelty is that the geometry and the topology of such domains can be arbitrary.

We will present now the main result of this paper. The manifold M is supposed to be compact and can be a manifold with or without boundary. If $\partial M \neq \emptyset$, then ∂M is supposed to be an $(n-1)$-dimensional Riemannian manifold with the induced metric. Let Ω_0 be a domain in the interior $\mathring{M}$ of M and let us consider the domain $M \backslash \overline{\Omega}_0$, where $\overline{\Omega}_0$ denotes the closure of Ω_0.

Definition 1.1. We say that $\{M\backslash\overline{\Omega}_t\}_{t\in(-t_0,t_0)}$, $\overline{\Omega}_t \subseteq \mathring{M}$, is a deformation of $M\backslash\overline{\Omega}_0$ if there exists a vector field Ξ (such that $\Xi(\partial M) \subseteq T(\partial M)$, where $T(\partial M)$ is the tangent bundle of ∂M) for which $M\backslash\overline{\Omega}_t = \xi(t, M\backslash\overline{\Omega}_0)$ where $\xi(t,\cdot)$ is the flow associated to Ξ, namely

$$\frac{d\xi}{dt}(t,p) = \Xi\big(\xi(t,p)\big) \quad \text{and} \quad \xi(0,p) = p.$$

The deformation is said to be volume preserving if the volume of $M\backslash\overline{\Omega}_t$ does not depend on t.

Let us denote by λ_t the first eigenvalue of $-\Delta_g$ on $M\backslash\overline{\Omega}_t$ with 0 Dirichlet boundary condition on $\partial\Omega_t$. If $\partial M \neq \emptyset$, we ask also one of the following boundary conditions:

(1) 0 Dirichlet boundary condition on ∂M, or
(2) 0 Neumann boundary condition on ∂M.

We will suppose the regularity of ∂M. Observe that both $t \mapsto \lambda_t$ and the associated eigenfunction $t \mapsto u_t$ (normalized to have $L^2(M\backslash\overline{\Omega}_t)$-norm equal to 1) are continuously differentiable, and we can give the following:

Definition 1.2. The domain $M\backslash\overline{\Omega}_0$ is an *extremal domain* for the first eigenvalue of $-\Delta_g$ if for any volume preserving deformation $\{M\backslash\overline{\Omega}_t\}_{t\in(-t_0,t_0)}$ of $M\backslash\overline{\Omega}_0$, we have

$$\frac{d\lambda_t}{dt}\bigg|_{t=0} = 0.$$

Let ϕ_0 be the first eigenfunction of the Laplace–Beltrami operator over the manifold M, i.e. the positive solution in M of

$$\Delta_g\phi_0 + \lambda_0\phi_0 = 0$$

for a nonnegative constant λ_0, normalized to have L^2-norm equal to 1. If $\partial M \neq \emptyset$, then we take the same boundary condition on ∂M considered in the definition of extremal domains. Here λ_0 is the first eigenvalue of $-\Delta_g$ on M under the boundary condition that has been chosen. If the volume of Ω is very small, it is natural to expect that the first eigenfunction of the Laplace–Beltrami operator over $M \setminus \overline{\Omega}$ is close to ϕ_0. We remark that we have to distinguish two cases of behavior of ϕ_0 (and then also of the first eigenfunction over $M \setminus \overline{\Omega}$), according with the condition at the boundary:

- CASE 1. If $\partial M \neq \emptyset$ and ϕ_0 satisfies the 0 Dirichlet condition on ∂M then ϕ_0 is a positive nonconstant function. Moreover $\lambda_0 > 0$.
- CASE 2. If $\partial M = \emptyset$, or if $\partial M \neq \emptyset$ and ϕ_0 satisfies the 0 Neumann condition on ∂M, then ϕ_0 is a constant function
$$\phi_0 = \frac{1}{\sqrt{\mathrm{Vol}_g(M)}}$$
and $\lambda_0 = 0$.

As we said previously, for the first eigenfunction of the Laplace–Beltrami operator over $M \setminus \overline{\Omega}$, where $\Omega \subset \mathring{M}$, we take the same boundary condition of ϕ_0 at ∂M, and we will distinguish the two cases above, CASE 1 and CASE 2.

For all $\epsilon > 0$ small enough, we denote by $B_\epsilon(p) \subset M$ the geodesic ball of center $p \in M$ and radius ϵ. We denote by $\mathring{B}_\epsilon \subset \mathbb{R}^n$ the Euclidean ball of radius ϵ centered at the origin.

We can state the main result of our paper:

Theorem 1.3. *In CASE* 1 *assume that p_0 is a nondegenerate critical point of the first eigenfunction ϕ_0 of the Laplace–Beltrami operator over M, and in CASE* 2 *assume that p_0 is a nondegenerate critical point of* Scal*, the scalar curvature function of (M,g). In CASE* 2 *we assume also $n \geqslant 4$. Then, for all $\epsilon > 0$ small enough, say $\epsilon \in (0,\epsilon_0)$, there exists a smooth domain $\Omega_\epsilon \subset M$ such that*:

(i) *The volume of* Ω_ϵ *is equal to the Euclidean volume of* $\mathring{B}_\epsilon$.
(ii) *The domain* $M\backslash\overline{\Omega}_\epsilon$ *is extremal in the sense of Definition* 1.2.

Moreover there exists a constant $c > 0$ *and for all* $\epsilon \in (0, \epsilon_0)$ *there exists a point* $p_\epsilon \in M$ *such that the boundary of* Ω_ϵ *is a normal graph over* $\partial B_\epsilon(p_\epsilon)$ *for some function* w_ϵ, *with*

$$\operatorname{dist}(p_\epsilon, p_0) \leqslant c\epsilon$$

and

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}(\partial B_\epsilon(p_\epsilon))} \leqslant c\epsilon^2 \quad \text{in CASE 1 and } n \geqslant 3,$$
$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}(\partial B_\epsilon(p_\epsilon))} \leqslant c\epsilon^2 \log\epsilon \quad \text{in CASE 1 and } n = 2,$$
$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}(\partial B_\epsilon(p_\epsilon))} \leqslant c\epsilon^3 \quad \text{in CASE 2 and } n \geqslant 5,$$
$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}(\partial B_\epsilon(p_\epsilon))} \leqslant c\epsilon^3 \log\epsilon \quad \text{in CASE 2 and } n = 4.$$

Let us digress slightly. Firstly, with respect to the result of F. Pacard and P. Sicbaldi in [16], a new phenomena appears: there are two types of extremal domains, those that are the complement of a small perturbed geodesic ball centered at a nondegenerate critical point of the function ϕ_0 and those that are the complement of a small perturbed geodesic ball centered at a nondegenerate critical point of the scalar curvature. It is important to remark that the construction of the first type of domains depends on a global condition (the existence of a nondegenerate critical point of ϕ_0) while the construction of the second type of domains depends on a local condition (the existence of a nondegenerate critical point of the scalar curvature of the manifold). Although the statement of the result in CASE 2 appears very similar to the result of F. Pacard and P. Sicbaldi in [16], it is quite surprising the fact that the global geometry of the manifold does not have a rôle in the construction of such last domains. Moreover, for CASE 2 the construction of extremal domains is different with respect to that of [16] and technically much more difficult. The technique used in [16] is based on the fact that the first eigenfunction of the Laplace–Beltrami operator on the perturbation of a small geodesic ball is a perturbation of the first eigenfunction of the Euclidean Laplacian on a small ball, and such a function is very well known. But these facts fail when the domain is the complement of a small ball in a Riemannian manifold, and an other approach is needed. We remark also that the construction of the second type of domains requires the existence of the nondegenerate critical point of the scalar curvature function. For example, our result in CASE 2 cannot be applied when the manifold M is a bounded region of $\mathbb{R}^n$. For this last case, the global geometry of the domain appears.

To complete this section, we present two open problems, linked to the previous result.

Open problem 1. Theorem 1.3 does not give any information in CASE 2 for the dimensions 2 and 3. In fact, in order to prove the main theorem for CASE 2, we need some local estimations of a Green function on the manifold M. When the dimension of M is at least 4, we are able to compute the first coefficients of the local expansion of such Green function, but for the dimensions 2 and 3, other terms (depending on the global geometry of the manifold) appear (see Section 6). It will be interesting to adapt the proof of Theorem 1.3 to CASE 2 for the dimensions 2 and 3, and we suspect that the global geometry of the manifold plays an important rôle in such cases.

Open problem 2. It will be interesting to know if the obtained extremal domains are or not Faber–Krahn minimizers, in the class of domains with the same volume. We recall that the existence of minimizers for the first eigenvalue of the Laplace–Beltrami operator was proved by G. Buttazzo and G. Dal Maso in [3] when the manifold is a bounded domain of the Euclidean space $\mathbb{R}^n$, and the proof of this result should be working also for a compact Riemannian manifold.

2. Characterization of the problem

In order to prove our theorem we need the following result that characterizes extremal domains of the form $M\backslash\overline{\Omega}$, where Ω is a bounded domain in a Riemannian manifold M. The following result gives a formula for the first variation of the first eigenvalue for some mixed problems under variations of the domain. A similar result is obtained in [7]. Our proof is based on some arguments of D.Z. Zanger contained in [25].

Keeping in mind the notation of the previous section, we have

Proposition 2.1. *The derivative of $t \mapsto \lambda_t$ at $t = 0$ is given by*

$$\left.\frac{d\lambda_t}{dt}\right|_{t=0} = -\int_{\partial\Omega_0} \big(g(\nabla u_0, \nu_0)\big)^2 g(\Xi, \nu_0)\,\mathrm{dvol}_g,$$

where dvol_g *is the volume element on* $\partial\Omega_0$ *for the metric induced by* g *and* ν_0 *is the normal vector field about* $\partial\Omega_0$.

Proof. We denote by ξ the flow associated to Ξ. By definition, we have

$$u_t\big(\xi(t,p)\big) = 0 \tag{3}$$

for all $p \in \partial\Omega_0$. Moreover, if we take a 0 Dirichlet boundary condition on ∂M, then Eq. (3) holds also on ∂M. On the other hand, if we take a 0 Neumann condition on ∂M, then we have

$$g\big(\nabla u_t\big(\xi(t,p)\big), \nu_t\big) = 0 \tag{4}$$

for all $p \in \partial M$, where ν_t is the unit normal vector about ∂M.

Differentiating (3) with respect to t and evaluating the result at $t = 0$ we obtain

$$\partial_t u_0 = -g(\nabla u_0, \Xi)$$

on $\partial\Omega_0$. Now, $u_0 \equiv 0$ on $\partial\Omega_0$, and hence only the normal component of Ξ plays a rôle in this formula. Therefore, we have

$$\partial_t u_0 = -g(\nabla u_0, \nu_0) g(\Xi, \nu_0) \tag{5}$$

on $\partial\Omega_0$. The same reasoning holds on ∂M if we take a 0 Dirichlet boundary condition on ∂M. In this case, by the fact that $\Xi(\partial M) \subseteq T(\partial M)$, we have

$$\partial_t u_0 = 0 \tag{6}$$

on ∂M. On the other hand, if we take a 0 Neumann condition on ∂M, then it is possible to choose a system of coordinates $x = (x^1, \dots, xn)$ such that $\nu_t = -\partial_{x^1}$ on ∂M and differentiating (4) with respect to t and evaluating the result at $t = 0$ we obtain

$$0 = -\partial_{x^1}\partial_t u_0 - g(\nabla\partial_{x^1} u_0, \Xi) = -\partial_{x^1}\partial_t u_0 = g(\nabla\partial_t u_0, \nu_0) \tag{7}$$

on ∂M, where we used the fact that ν_t does not depend on t on ∂M together with the facts that $\partial_{x^1} u_0 = 0$ on ∂M and that $g(\Xi, \nu_0) = 0$ on ∂M because $\Xi(\partial M) \subseteq T(\partial M)$.

Now, we differentiate with respect to t the identity

$$\Delta_g u_t + \lambda_t u_t = 0 \tag{8}$$

and we evaluate the result at $t = 0$. We obtain

$$\Delta_g \partial_t u_0 + \lambda_0 \partial_t u_0 = -\partial_t \lambda_0 u_0 \tag{9}$$

in Ω_0. We multiply (9) by u_0, and (8), evaluated at $t = 0$, by $\partial_t u_0$, subtract the results and integrate it over Ω_0 to get

$$\begin{aligned}
\partial_t\lambda_0 \int_{\Omega_0} u_0^2\,\mathrm{dvol}_g &= \int_{M\setminus\Omega_0} (\partial_t u_0 \Delta_g u_0 - u_0 \Delta_g \partial_t u_0)\,\mathrm{dvol}_g \\
&= \int_{\partial M \cup \partial\Omega_0} \big(\partial_t u_0 g(\nabla u_0, \nu_0) - u_0 g(\nabla\partial_t u_0, \nu_0)\big)\,\mathrm{dvol}_g
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial\Omega_0} \big(\partial_t u_0 g(\nabla u_0, \nu_0) - u_0 g(\nabla \partial_t u_0, \nu_0)\big)\, \mathrm{dvol}_g \\
&\quad + \int_{\partial M} \big(\partial_t u_0 g(\nabla u_0, \nu_0) - u_0 g(\nabla \partial_t u_0, \nu_0)\big)\, \mathrm{dvol}_g \\
&= - \int_{\partial\Omega_0} \big(g(\nabla u_0, \nu_0)\big)^2 g(\Xi, \nu_0)\, \mathrm{dvol}_g,
\end{aligned}
$$

where we have used (5), (6) or (7), the fact that $u_0 = 0$ on $\partial\Omega_0$, and the fact that $u_0 = 0$ or $g(\nabla u_0, \nu_0) = 0$ on ∂M. The result follows at once from the fact that u_0 is normalized to have $L^2(\Omega_0)$-norm equal to 1. Observe that in the previous argument ∂M can be empty. □

This result allows us to characterize extremal domains for the first eigenvalue of the Laplace–Beltrami operator under our particular 0 mixed boundary conditions, and states the problem of finding extremal domains into the solvability of an over-determined elliptic problem. The proof of the following proposition is a direct consequence of the previous result and we do not report it (see also Proposition 2.2 in [16]).

Proposition 2.2. *Given a smooth domain Ω_0 contained in the interior of M, the domain $M\backslash\overline{\Omega}_0$ is extremal if and only if there exists a constant λ_0 and a positive function u_0 (if $\partial M \neq \emptyset$ we take a* 0 *Dirichlet* (*CASE* 1) *or a* 0 *Neumann* (*CASE* 2) *boundary condition on ∂M) such that*

$$
\begin{cases}
\Delta_g u_0 + \lambda_0 u_0 = 0 & \text{in } M\backslash\overline{\Omega}_0, \\
u_0 = 0 & \text{on } \partial\Omega_0, \\
g(\nabla u_0, \nu_0) = \text{constant} & \text{on } \partial\Omega_0,
\end{cases}
\tag{10}
$$

where ν_0 is the normal vector field about $\partial\Omega_0$ pointing into Ω_0.

Therefore, in order to find extremal domains, it is enough to find a domain $M\backslash\overline{\Omega}_0$ (regular enough) for which the over-determined problem (10) has a nontrivial positive solution. In this paper we will solve this problem to find domains $M\backslash\overline{\Omega}_0$ whose volume is close to the volume of the compact manifold M.

3. Rephrasing the problem

Given a point $p \in M$ we denote by $E_1, \dots, E_n$ an orthonormal basis of the tangent plane to M at p. Geodesic normal coordinates $x := (x^1, \dots, x^n) \in \mathbb{R}^n$ at p are defined by

$$
X(x) := \mathrm{Exp}_p^g\left(\sum_{j=1}^n x^j E_j\right).
$$

We recall the Taylor expansion of the coefficients g_{ij} of the metric X^*g in these coordinates.

Proposition 3.1. *At the point of coordinate x, the following expansion holds*

$$
g_{ij} = \delta_{ij} + \frac{1}{3}\sum_{k,\ell} R_{ikj\ell} x^k x^\ell + \frac{1}{6}\sum_{k,\ell,m} R_{ikjl,m} x^k x^\ell x^m + \mathcal{O}\big(|x|^4\big). \tag{11}
$$

Here R is the curvature tensor of g and

$$
\begin{aligned}
R_{ikj\ell} &= g\big(R(E_i, E_k)E_j, E_\ell\big), \\
R_{ikj\ell,m} &= g\big(\nabla_{E_m} R(E_i, E_k)E_j, E_\ell\big)
\end{aligned}
$$

are evaluated at the point p.

The proof of this proposition can be found in [23] or also in [19].

It will be convenient to identify $\mathbb{R}^n$ with T_pM (the tangent space at p) and S^{n-1} with the unit sphere in T_pM. If $x := (x^1, \ldots, x^n) \in \mathbb{R}^n$, we set

$$\Theta(x) := \sum_{i=1}^{n} x^i E_i \in T_pM.$$

Given a continuous function $f : S^{n-1} \mapsto (0, \infty)$ whose L^∞-norm is small (say less than the cut locus of p) we define

$$B_f^g(p) := \left\{ \operatorname{Exp}_p\big(\Theta(x)\big) \colon x \in \mathbb{R}^n,\ 0 \leqslant |x| \leqslant f\big(x/|x|\big) \right\}.$$

The superscript g is meant to remind the reader that this definition depends on the metric.

Our aim is to show that, for all $\epsilon > 0$ small enough, we can find a point $p \in M$ and a function $v : S^{n-1} \to \mathbb{R}$ such that

$$\operatorname{Vol} B^g_{\epsilon(1+v)}(p) = \epsilon^n \operatorname{Vol} \mathring{B}_1$$

where $\mathring{B}_1$ is the unit (closed) Euclidean ball, and the over-determined problem

$$\begin{cases} \Delta_g \phi + \lambda \phi = 0 & \text{in } M \setminus B^g_{\epsilon(1+v)}(p), \\ \phi = 0 & \text{on } \partial B^g_{\epsilon(1+v)}(p), \\ g(\nabla \phi, \nu) = \text{constant} & \text{on } \partial B^g_{\epsilon(1+v)}(p) \end{cases} \tag{12}$$

with 0 Dirichlet (CASE 1) or 0 Neumann (CASE 2) boundary condition on ∂M if $\partial M \neq \emptyset$, has a positive solution, where ν is the normal vector about $\partial B^g_{\epsilon(1+v)}(p)$ pointing into $B^g_{\epsilon(1+v)}(p)$.

Observe that, considering the dilated metric $\bar{g} := \epsilon^{-2} g$, the above problem is equivalent to finding a point $p \in M$ and a function $v : S^{n-1} \to \mathbb{R}$ such that

$$\operatorname{Vol} B^{\bar{g}}_{1+v}(p) = \operatorname{Vol} \mathring{B}_1$$

and for which the over-determined problem

$$\begin{cases} \Delta_{\bar{g}} \bar{\phi} + \bar{\lambda} \bar{\phi} = 0 & \text{in } M \backslash B^{\bar{g}}_{1+v}(p), \\ \bar{\phi} = 0 & \text{on } \partial B^{\bar{g}}_{1+v}(p), \\ \bar{g}(\nabla \bar{\phi}, \bar{\nu}) = \text{constant} & \text{on } \partial B^{\bar{g}}_{1+v}(p) \end{cases}$$

with 0 Dirichlet (CASE 1) or 0 Neumann (CASE 2) boundary condition on ∂M if $\partial M \neq \emptyset$, has a positive solution, where $\bar{\nu}$ is the normal vector field about $\partial B^{\bar{g}}_{1+v}(p)$ in the metric $\bar{g}$. We can simply consider

$$\phi = \bar{\phi}$$

(naturally it will not have the norm equal to 1, but depending on ϵ) and

$$\lambda = \epsilon^{-2} \bar{\lambda}.$$

In what it follows we will consider sometimes the metric g and sometimes the metric $\bar{g}$, in order to simplify the computations we will meet.

4. The first eigenfunction outside a small ball

The positive solution of the problem

$$\begin{cases} \Delta_g \phi_\epsilon + \lambda_\epsilon \phi_\epsilon = 0 & \text{in } M \setminus B^g_\epsilon(p), \\ \phi_\epsilon = 0 & \text{on } \partial B^g_\epsilon(p) \end{cases} \tag{13}$$

with 0 Dirichlet (CASE 1) or 0 Neumann (CASE 2) condition on ∂M if $\partial M \neq \emptyset$, normalized to have $L^2(M \setminus B^g_\epsilon(p))$-norm equal to 1, a priori is not known.

Let $p \in M$, let c_n be a constant, and let Γ_p be a Green function over M with respect to the point p defined by

$$-(\Delta_g + \lambda_0)\Gamma_p = c_n\big(\delta_p - \phi_0(p)\phi_0\big) \quad \text{in } M \tag{14}$$

with 0 Dirichlet boundary condition (for CASE 1) or 0 Neumann boundary condition (for CASE 2) at ∂M if $\partial M \neq \emptyset$, and normalization

$$\int_M \Gamma_p \phi_0 \,\mathrm{dvol}_g = 0,$$

where δ_p is the Dirac distribution for the manifold M with metric g at the point p. We remark that Γ_p exists because

$$\int_M \big[\delta_p - \phi_0(p)\phi_0\big]\phi_0 \,\mathrm{dvol}_g = 0.$$

It is easy to check that for each dimension n of the manifold it is possible to choose the constant c_n in order to have the following expansions of Γ_p in a neighborhood of the point p in the geodesic normal coordinates x (see [2]):

$$\begin{aligned} &\text{for } n = 2: \quad \Gamma_p(x) = \log|x| + o\big(\log|x|\big), \\ &\text{for } n \geqslant 3: \quad \Gamma_p(x) = |x|^{2-n} + o\big(|x|^{2-n}\big). \end{aligned} \tag{15}$$

For our problem it will be very useful to consider weighted Hölder spaces $\mathcal{C}^{k,\alpha}_\delta(M \setminus \{p\})$, $\delta \in \mathbb{R}$, defined as the spaces of functions in $\mathcal{C}^{k,\alpha}(M \setminus \{p\})$ such that, in the normal geodesic coordinates x around p,

$$\begin{aligned} \|u\|_{\mathcal{C}^{k,\alpha}_\delta(M\setminus\{p\})} :={}& \sup_{\mathring{B}_{R_0}} |x|^{-\delta}|u| + \sup_{\mathring{B}_{R_0}} |x|^{1-\delta}|\nabla u| + \sup_{\mathring{B}_{R_0}} |x|^{2-\delta}\big|\nabla^2 u\big| + \cdots \\ &+ \sup_{\mathring{B}_{R_0}} |x|^{k-\delta}\big|\nabla^k u\big| + \sup_{0<R\leqslant R_0} \sup_{x,y\in \mathring{B}_R\setminus \mathring{B}_{R/2}} R^{k+\alpha-\delta}\left|\frac{\nabla^k u(x) - \nabla^k u(y)}{|x-y|^\alpha}\right| < \infty \end{aligned}$$

where R_0 is a small positive constant chosen in order to have the existence of the local coordinates in $B^g_{R_0}(p)$. For a clear exposition of the basic facts and properties of such weighted Hölder spaces and the theory of elliptic operator between weighted Hölder spaces we remind to Chapter 2 of [15] (see also [13,12,14]).

Let us consider $\varphi \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$, where m is meant to point out that functions have 0 (Euclidean) average over S^{n-1}, and let H_φ be a bounded harmonic extension of φ to $\mathbb{R}^n \setminus \mathring{B}_1$:

$$\begin{cases} \Delta_{\mathring{g}} H_\varphi = 0 & \text{in } \mathbb{R}^n \setminus \mathring{B}_1, \\ H_\varphi = \varphi & \text{on } \partial \mathring{B}_1 \end{cases} \tag{16}$$

where $\mathring{g}$ is the Euclidean metric and we identified $\partial \mathring{B}_1$ with S^{n-1}. We have

Lemma 4.1. *The following estimate holds*

$$\big\|H_\varphi(x)\big\|_{\mathcal{C}^{2,\alpha}_{1-n}(\mathbb{R}^n\setminus\mathring{B}_1)} \leqslant c\|\varphi\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}$$

for some positive constant c. In particular

$$\lim_{|x|\to+\infty} H_\varphi(x) = 0.$$

Proof. Let us consider

$$\varphi = \sum_{j=1}^{\infty} \varphi_j$$

the eigenfunction decomposition of φ, i.e.

$$\Delta_{S^{n-1}}\varphi_j = -j(n-2+j)\varphi_j. \tag{17}$$

It is easy to check that

$$H_\varphi(x) = \sum_{j=1}^{\infty} |x|^{2-n-j} \varphi_j\big(x/|x|\big)$$

is the solution of (16). Let us fix $|x|$. We have

$$\big|H_\varphi(x)\big| \leqslant \sum_{j=1}^{\infty} |x|^{2-n-j} \big|\varphi_j\big(x/|x|\big)\big| = |x|^{1-n}\big|\varphi_1\big(x/|x|\big)\big| + \sum_{j=2}^{\infty} |x|^{2-n-j}\big|\varphi_j\big(x/|x|\big)\big|. \tag{18}$$

Now, we estimate $\|\varphi_j\|_{L^\infty(S^{n-1})}$. From (17) we have

$$\|\varphi_j\|_{W^{2k,2}(S^{n-1})} \leqslant c j^k (n-2+j)^k \|\varphi_j\|_{L^2(S^{n-1})}$$

and by the Sobolev Embedding Theorem we have that $W^{2k,2}(S^{n-1}) \subseteq L^\infty(S^{n-1})$ when $4k > n-1$. We conclude that there exists a positive number $P(n)$ depending only on the dimension n such that

$$\|\varphi_j\|_{L^\infty(S^{n-1})} \leqslant c j^{P(n)} \|\varphi_j\|_{L^2(S^{n-1})}.$$

Moreover

$$\|\varphi_j\|^2_{L^2(S^{n-1})} \leqslant \|\varphi\|^2_{L^2(S^{n-1})} \leqslant \mathrm{Vol}_{\mathring{g}}\big(S^{n-1}\big) \|\varphi\|^2_{L^\infty(S^{n-1})}$$

and we can conclude that there exists a constant c such that

$$\|\varphi_j\|_{L^\infty(S^{n-1})} \leqslant c j^{P(n)} \|\varphi\|_{L^\infty(S^{n-1})}.$$

From (18) we get

$$\big|H_\varphi(x)\big| \leqslant c|x|^{1-n} \|\varphi\|_{L^\infty(S^{n-1})} \left(1 + \sum_{j=2}^{\infty} |x|^{1-j} j^{P(n)}\right).$$

It is easy to check that for $|x| \geqslant 2$

$$\sum_{j=2}^{\infty} |x|^{1-j} j^{P(n)} < \infty$$

and this allows us to conclude that for $|x| \geqslant 2$ there exists a constant c such that

$$\big|H_\varphi(x)\big| \leqslant c|x|^{1-n} \|\varphi\|_{L^\infty(S^{n-1})}. \tag{19}$$

By the maximum principle this inequality is valid also for $1 \leqslant |x| \leqslant 2$. Standard elliptic estimates apply to give also

$$\big|\nabla H_\varphi(x)\big| \leqslant c|x|^{-n} \|\varphi\|_{L^\infty(S^{n-1})}. \tag{20}$$

Finally, (19) and (20) give the following estimate

$$\big\|H_\varphi(x)\big\|_{\mathcal{C}^{2,\alpha}_{1-n}(\mathbb{R}^n \setminus \mathring{B}_1)} \leqslant c\|\varphi\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}$$

for some constant c. From (19) it is clear that

$$\lim_{|x| \to +\infty} H_\varphi(x) = 0.$$

This completes the proof of the lemma. □

Let us define a continuous extension of H_φ to $\mathbb{R}^n$ in this way:

$$\tilde{H}_\varphi(x) = \begin{cases} 0 & \text{for } |x| \leqslant \frac{1}{2}, \\ (2|x|-1) H_\varphi(\frac{x}{|x|}) & \text{for } \frac{1}{2} \leqslant |x| \leqslant 1, \\ H_\varphi(x) & \text{for } \mathbb{R}^n \setminus \mathring{B}_1 \end{cases}$$

and let us denote

$$H_{\varphi,\epsilon}(x) = H_\varphi\left(\frac{x}{\epsilon}\right)$$

and

$$\tilde{H}_{\varphi,\epsilon}(x) = \tilde{H}_\varphi\left(\frac{x}{\epsilon}\right).$$

Let χ be a cutoff function defined in M, identically equal to 1 for $|x| \leqslant R_0$ (where x are the normal geodesic coordinates at p) and identically equal to 0 in $M \setminus B^g_{2R_0}(p)$.

The main result of this section is the following:

Proposition 4.2. *Let $n \geqslant 3$ and $\delta \in (2-n, \min\{4-n, 0\})$. For all ϵ small enough there exists $(\Lambda_\epsilon, \varphi_\epsilon, w_\epsilon)$ in a neighborhood of $(0,0,0)$ in $\mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathcal{C}^{2,\alpha}_\delta(M \setminus \{p\})$ such that the function*

$$\phi_\epsilon = \phi_0 - \epsilon^{n-2}\big(\phi_0(p) + \Lambda_\epsilon\big)\Gamma_p + w_\epsilon + \chi\tilde{H}_{\varphi_\epsilon,\epsilon} \tag{21}$$

(considered in $M \setminus B^g_\epsilon(p)$), is a positive solution of (13) where

$$\lambda = \lambda_0 + c_n\phi_0(p)^2\epsilon^{n-2} + \mathcal{O}\big(\epsilon^{n-1}\big). \tag{22}$$

Moreover the following estimations hold:

- *If ϕ_0 is not a constant function (CASE 1) then there exists a positive constant c such that*

  $$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c\epsilon \quad \text{and} \quad \|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\delta(M\setminus\{p\})} \leqslant c\big(\epsilon^{2n-4} + \epsilon^n + \epsilon^{3-\delta}\big).$$

- *If ϕ_0 is a constant function (CASE 2) then there exists a positive constant c such that*

  $$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c\epsilon \quad \text{if } n = 3,$$
  $$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c\epsilon^\beta, \quad \forall\beta < 2 \text{ if } n = 4,$$
  $$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c\epsilon^2 \quad \text{if } n \geqslant 5$$

  and

  $$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\delta(M\setminus\{p\})} \leqslant c\epsilon^2 \quad \text{if } n = 3,$$
  $$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\delta(M\setminus\{p\})} \leqslant c\epsilon^4 \quad \text{if } n = 4,$$
  $$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\delta(M\setminus\{p\})} \leqslant c\big(\epsilon^{1+n} + \epsilon^{4-\delta}\big) \quad \text{if } n \geqslant 5.$$

Proof. First we prove that

$$\lambda - \lambda_0 = \mathcal{O}\big(\epsilon^{n-2}\big). \tag{23}$$

By definition

$$\lambda = \min_{u \in H^1_0(M\setminus B^g_\epsilon(p))} \frac{\int_{M\setminus B^g_\epsilon(p)} |\nabla^g u|^2 \,\mathrm{dvol}_g}{\int_{M\setminus B^g_\epsilon(p)} u^2 \,\mathrm{dvol}_g}. \tag{24}$$

Let us consider a sequence of functions $u_j \in H^1_0(M \setminus B^g_\epsilon(p))$ converging to the function

$$u_*(x) = \begin{cases} (\frac{|x|}{\epsilon} - 1)\phi_0(\frac{2\epsilon x}{|x|}) & \text{in } B^g_{2\epsilon}(p) \setminus B^g_\epsilon(p), \\ \phi_0 & \text{in } M \setminus B^g_{2\epsilon}(p). \end{cases}$$

It is easy to check that

$$\int_{M\setminus B^g_\epsilon(p)} u_*^2 \,\mathrm{dvol}_g = \int_M \phi_0^2 \,\mathrm{dvol}_g + \mathcal{O}\big(\epsilon^n\big)$$

while

$$\int_{M\setminus B^g_\epsilon(p)} |\nabla u_*|^2 \,\mathrm{dvol}_g = \int_M |\nabla \phi_0|^2 \,\mathrm{dvol}_g + \mathcal{O}\big(\epsilon^{n-2}\big).$$

From the last two relations and (24) we have

$$\lambda = \lambda_0 + \epsilon^{n-2}\mu \tag{25}$$

where $\mu = \mathcal{O}(1)$, and then (23).

Define

$$\phi_\epsilon = \phi_0 - \epsilon^{n-2}\big(\phi_0(p) + \Lambda\big)\Gamma_p + w + \chi\,\tilde{H}_{\varphi,\epsilon}$$

for some $(\Lambda, \varphi, w) \in \mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathcal{C}^{2,\alpha}(M \setminus \{p\})$. Then ϕ_ϵ satisfies the first equation of (13) in $M \setminus B^g_\epsilon(p)$, with λ as in (25), if and only if

$$\begin{aligned}&\big(\Delta_g + \lambda_0 + \epsilon^{n-2}\mu\big)w + \epsilon^{n-2}\big[\mu - c_n\phi_0(p)\big(\phi_0(p) + \Lambda\big)\big]\phi_0 + H_{\varphi,\epsilon}\Delta_g\chi + \chi\,\Delta_g H_{\varphi,\epsilon}\\ &\quad + 2\nabla^g H_{\varphi,\epsilon}\nabla^g\chi - \epsilon^{2n-4}\mu\big(\phi_0(p) + \Lambda\big)\Gamma_p + \big(\lambda_0 + \epsilon^{n-2}\mu\big)\chi\,H_{\varphi,\epsilon} = 0\end{aligned} \tag{26}$$

in $M \setminus B^g_\epsilon(p)$. This equation can be considered in $M \setminus \{p\}$ if we replace $H_{\varphi,\epsilon}$ by $\tilde{H}_{\varphi,\epsilon}$, and $\Delta_g H_{\varphi,\epsilon}$ by a continuous extension $\widetilde{\Delta_g H}_{\varphi,\epsilon}$ of $\Delta_g H_{\varphi,\epsilon}$ (such continuous extension can be defined in the same way of $\tilde{H}_{\varphi,\epsilon}$ as a continuous extension of the function $H_{\varphi,\epsilon}$). Remark that the term $\nabla^g H_{\varphi,\epsilon}\nabla^g\chi$ is 0 in a neighborhood of $\partial B^g_\epsilon(p)$, then it can be extended to 0 in $B^g_\epsilon(p)$.

We need the following:

Lemma 4.3. *Let $n \geqslant 3$. The operator*

$$\big(\Delta_g + \lambda_0 + \epsilon^{n-2}\mu\big) : \mathcal{C}^{2,\alpha}_{\delta,\perp,0}\big(M \setminus \{p\}\big) \to \mathcal{C}^{0,\alpha}_{\delta-2,\perp}\big(M \setminus \{p\}\big),$$

where the subscript $\perp$ is meant to point out that functions are L^2-orthogonal to ϕ_0 and the subscript 0 is meant to point out that functions satisfy the 0 Dirichlet (in CASE 1) or 0 Neumann (in CASE 2) boundary condition on ∂M if $\partial M \neq \emptyset$, is an isomorphism for $\delta \in (2-n, 0)$ and ϵ small enough.

Proof. Let $\delta \in (2-n, 0)$ and $n \geqslant 3$. For all $f \in \mathcal{C}^{0,\alpha}_{\delta-2}(\mathring{B}_1 \setminus \{0\})$ there exists one and only one solution $u \in \mathcal{C}^{2,\alpha}_{\delta}(\mathring{B}_1 \setminus \{0\})$ of

$$\begin{cases}\Delta_{\mathring{g}} u = f & \text{in } \mathring{B}_1 \setminus \{0\},\\ u = 0 & \text{on } \partial\mathring{B}_1.\end{cases} \tag{27}$$

The proof of this fact can be found in [15] or in [13]. Take the normal geodesic coordinates in $B^g_{R_0}(p)$ (keep in mind that R_0 is small), and let $f \in \mathcal{C}^{0,\alpha}_{\delta-2}(M \setminus \{p\})$. Considering the dilated metric $R_0^{-2}g$, the parameterization of $B^g_{R_0}(p)$ given by

$$Y(y) := \mathrm{Exp}^g_p\bigg(R_0\sum_i y^i E_i\bigg)$$

and the ball $\mathring{B}_1$ endowed with the metric $\breve{g} = Y^*(R_0^{-2}g)$, the existence and the unicity of a solution of the problem

$$\begin{cases}(\Delta_g + \lambda_0)u = f & \text{in } B^g_{R_0} \setminus \{p\},\\ u = 0 & \text{on } \partial B^g_{R_0}\end{cases}$$

are equivalent to the existence and the unicity of a solution of the problem

$$\begin{cases} \big(\Delta_{\breve{g}} + R_0^2 \lambda_0\big) u = Y^* f & \text{in } \mathring{B}_1 \setminus \{0\}, \\ u = 0 & \text{on } \partial \mathring{B}_1. \end{cases}$$

Considering that the difference between the coefficients of the metric $\breve{g}$ and the metric $\mathring{g}$ can be estimated by a constant times R_0^2 (see Proposition 3.1), the operator $\Delta_{\breve{g}} + R_0^2 \lambda_0$ is a small perturbation of the operator $\Delta_{\mathring{g}}$ when R_0 is small. We conclude that there exists a positive R_0 (small enough) such that, when $\delta \in (2-n, 0)$ and $n \geqslant 3$, for all $f \in \mathcal{C}^{0,\alpha}_{\delta-2}(M \setminus \{p\})$ there exists a unique solution $u \in \mathcal{C}^{2,\alpha}_{\delta}(B^g_{R_0} \setminus \{p\})$ of

$$\begin{cases} (\Delta_g + \lambda_0) u = f & \text{in } B^g_{R_0} \setminus \{p\}, \\ u = 0 & \text{on } \partial B^g_{R_0}. \end{cases}$$

Now, consider the solution of

$$(\Delta_g + \lambda_0) v = f - (\Delta_g + \lambda_0)(\tilde{\chi} u) \tag{28}$$

with 0 Dirichlet boundary condition at ∂M, where $\tilde{\chi}$ is a cutoff function equal to 1 for $|x| \leqslant R_0/2$ and equal to 0 for $|x| \geqslant R_0$. We remark that this equation is well defined in M, because f and $(\Delta_g + \lambda_0)(\tilde{\chi} u)$ have the same singularity at p. Moreover, if f is L^2-orthogonal to ϕ_0, then $f - (\Delta_g + \lambda_0)(\tilde{\chi} u)$ is L^2-orthogonal to ϕ_0. Hence, there exists a unique solution $v \in \mathcal{C}^{2,\alpha}_{\perp,0}(M)$ to (28), and we have

$$(\Delta_g + \lambda_0)(\tilde{\chi} u + v) = f$$

in $M \setminus \{p\}$, with 0 Dirichlet condition at ∂M. Obviously $w = \tilde{\chi} u + v \in \mathcal{C}^{2,\alpha}_{\delta,\perp}(M \setminus \{p\})$. We conclude that for $\delta \in (2-n, 0)$ and $n \geqslant 3$ and for all $f \in \mathcal{C}^{0,\alpha}_{\delta-2,\perp}(M \setminus \{p\})$ there exists a unique solution $w \in \mathcal{C}^{2,\alpha}_{\delta,\perp}(M \setminus \{p\})$ of

$$(\Delta_g + \lambda_0) w = f$$

in $M \setminus \{p\}$ with 0 Dirichlet condition at ∂M. This result is still true for the operator $\Delta_g + \lambda_0 + \epsilon^{n-2}\mu$ when ϵ is small enough. The proof does not change if we consider the 0 Neumann boundary condition on ∂M instead of the 0 Dirichlet boundary condition. This completes the proof of the lemma. □

In order to simplify the notation we define

$$\begin{aligned} A &:= \epsilon^{n-2}\big[\mu - c_n \phi_0(p)\big(\phi_0(p) + \Lambda\big)\big]\phi_0, \\ B &:= \tilde{H}_{\varphi,\epsilon} \Delta_g \chi + \chi \widetilde{\Delta_g H}_{\varphi,\epsilon} + 2\nabla^g H_{\varphi,\epsilon} \nabla^g \chi, \\ C &:= -\epsilon^{2n-4}\mu\big(\phi_0(p) + \Lambda\big)\Gamma_p, \\ D &:= \big(\lambda_0 + \epsilon^{n-2}\mu\big)\chi \tilde{H}_{\varphi,\epsilon}. \end{aligned}$$

We remark that $\Gamma_p \in \mathcal{C}^{0,\alpha}_{\delta-2}(M \setminus \{p\})$ if $\delta < 4-n$. Eq. (26), extended to $M \setminus \{p\}$, becomes

$$\big(\Delta_g + \lambda_0 + \epsilon^{n-2}\mu\big) w = -(A + B + C + D).$$

By Lemma 4.3, if we choose μ in order to verify

$$\int_M (A + B + C + D)\phi_0 = 0, \tag{29}$$

there exists a solution $w(\epsilon, \Lambda, \varphi) \in \mathcal{C}^{2,\alpha}_{\delta,\perp,0}(M \setminus \{p\})$ to Eq. (26) with

$$\delta \in \big(2-n, \min\{0, 4-n\}\big),$$

for all $\Lambda \in \mathbb{R}$, for all $\varphi \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$, and for all ϵ small enough, and then

$$\phi_\epsilon = \phi_0 + \epsilon^{n-2}\big(\phi_0(p) + \Lambda\big)\Gamma_p + w(\epsilon, \Lambda, \varphi) + \chi H_{\varphi,\epsilon} \tag{30}$$

satisfies the first equation of (13) in $M\backslash B^g_\epsilon(p)$. From (29) we get

$$\mu = \frac{\epsilon^{n-2} c_n \phi_0(p)(\phi_0(p)+\Lambda) - \int_M B\phi_0 - \lambda_0 \int_M \chi \tilde{H}_{\varphi,\epsilon}\phi_0}{\epsilon^{n-2}(1+\int_M \chi \tilde{H}_{\varphi,\epsilon}\phi_0)}.$$

It is easy to check that

$$\int_M B\phi_0 \leqslant c\epsilon^{n-1}\|\varphi\|_{L^\infty(S^{n-1})}$$

and

$$\int_M \chi \tilde{H}_{\varphi,\epsilon}\phi_0 \leqslant c\epsilon^{n-1}\|\varphi\|_{L^\infty(S^{n-1})}$$

from which it follows that

$$\mu = c_n\phi_0(p)\big(\phi_0(p)+\Lambda\big) + \mathcal{O}(\epsilon)\|\varphi\|_{L^\infty(S^{n-1})}. \tag{31}$$

Then, using Lemma 4.1, we have the following estimations:

- $\|A\|_{\mathcal{C}^{0,\alpha}_{\delta-2}(M\backslash\{p\})} \leqslant c\epsilon^{n-1}\|\varphi\|_{L^\infty(S^{n-1})}$.
- $\|B\|_{\mathcal{C}^{0,\alpha}_{\delta-2}(M\backslash\{p\})} \leqslant c(\epsilon^{n-1}+\epsilon^{2-\delta})\|\varphi\|_{L^\infty(S^{n-1})}$.
- $\|C\|_{\mathcal{C}^{0,\alpha}_{\delta-2}(M\backslash\{p\})} \leqslant c\epsilon^{2n-4}$.
- $\|D\|_{\mathcal{C}^{0,\alpha}_{\delta-2}(M\backslash\{p\})} \leqslant c(\epsilon^{2-\delta}+\epsilon^{n-1})\|\varphi\|_{L^\infty(S^{n-1})}$.

In particular we get

$$\|A+B+C+D\|_{\mathcal{C}^{0,\alpha}_{\delta-2}(M\backslash\{p\})} \leqslant c\big(\epsilon^{2n-4}+\epsilon^{n-1}\|\varphi\|_{L^\infty(S^{n-1})}+\epsilon^{2-\delta}\|\varphi\|_{L^\infty(S^{n-1})}\big)$$

and then

$$\|w\|_{\mathcal{C}^{2,\alpha}_{\delta}(M\backslash\{p\})} \leqslant c\big(\epsilon^{2n-4}+\epsilon^{n-1}\|\varphi\|_{L^\infty(S^{n-1})}+\epsilon^{2-\delta}\|\varphi\|_{L^\infty(S^{n-1})}\big).$$

We have proved the following:

First intermediate result. *Let $\delta \in (2-n, 4-n)$. For all $\Lambda \in \mathbb{R}$, for all $\varphi \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$, for all ϵ small enough, there exists a function $w(\epsilon,\Lambda,\varphi) \in \mathcal{C}^{2,\alpha}_{\delta,\perp,0}(M\setminus\{p\})$ such that ϕ_ϵ defined in (30) is a positive solution of the first equation of (13). Moreover there exists a positive constant c such that*

$$\|w\|_{\mathcal{C}^{2,\alpha}_{\delta}(M\backslash\{p\})} \leqslant c\big(\epsilon^{2n-4}+\epsilon^{n-1}\|\varphi\|_{L^\infty(S^{n-1})}+\epsilon^{2-\delta}\|\varphi\|_{L^\infty(S^{n-1})}\big). \tag{32}$$

We consider now the second equation of (13). Define

$$N(\epsilon,\Lambda,\varphi) := \big[\phi_0(\epsilon y) - \epsilon^{n-2}\big(\phi_0(p)+\Lambda\big)\Gamma(\epsilon y) + \big(w(\epsilon,\Lambda,\varphi)\big)(\epsilon y) + \varphi(y)\big]_{y\in S^{n-1}}.$$

We remark that N represents the boundary value of ϕ_ϵ, is well defined in a neighborhood of $(0,0,0)$ in $[0,+\infty)\times\mathbb{R}\times\mathcal{C}^{2,\alpha}_m(S^{n-1})$, and takes its values in $\mathcal{C}^{2,\alpha}(S^{n-1})$. It is easy to compute the differential of N with respect to Λ and φ at $(0,0,0)$:

$$\big(\partial_\Lambda N(0,0,0)\big)(\tilde{\Lambda}) = -\tilde{\Lambda},$$
$$\big(\partial_\varphi N(0,0,0)\big)(\tilde{\varphi}) = \tilde{\varphi}.$$

From the estimation of the function w it follows that

$$\begin{aligned}\|w\|_{L^\infty(\partial B^g_\epsilon(p))} &\leqslant \epsilon^\delta \|w\|_{\mathcal{C}^{2,\alpha}_\delta(M\setminus\{p\})}\\ &\leqslant c\big(\epsilon^{2n-4+\delta} + \epsilon^{n-1+\delta}\|\varphi\|_{L^\infty(S^{n-1})} + \epsilon^2\|\varphi\|_{L^\infty(S^{n-1})}\big).\end{aligned}$$

Then we can estimate $N(\epsilon,0,0)$:

$$\big\|N(\epsilon,0,0)\big\|_{L^\infty(S^{n-1})} \leqslant \big\|\phi_0(\epsilon x) - \epsilon^{n-2}\phi_0(p)\Gamma(\epsilon x)\big\|_{L^\infty(S^{n-1})} + \big\|\big(w(\epsilon,0,0)\big)(\epsilon x)\big\|_{L^\infty(S^{n-1})}.$$

Here we have again to distinguish two cases, according to the behavior of the function ϕ_0. If ϕ_0 is not a constant function (CASE 1) we have (using the expansion (15) of Γ_p)

$$\big\|N(\epsilon,0,0)\big\|_{L^\infty(S^{n-1})} \leqslant c\epsilon.$$

The same estimate is obtained if ϕ_0 is a constant function (CASE 2) and $n=3$. In CASE 2 and $n=4$ we get

$$\big\|N(\epsilon,0,0)\big\|_{L^\infty(S^{n-1})} \leqslant c\epsilon^\beta$$

$\forall\beta<2$ and when $n\geqslant 5$:

$$\big\|N(\epsilon,0,0)\big\|_{L^\infty(S^{n-1})} \leqslant c\epsilon^2.$$

The implicit function theorem applies to give

Second intermediate result. *Let $\delta\in(2-n,\min\{4-n,0\})$, and ϵ be small enough. Then there exists $(\Lambda_\epsilon,\varphi_\epsilon)$ in a neighborhood of $(0,0)$ in $\mathbb{R}\times\mathcal{C}^{2,\alpha}_m(S^{n-1})$ such that $N(\epsilon,\Lambda_\epsilon,\varphi_\epsilon)=0$ (i.e. ϕ_ϵ defined in (30), with $\Lambda=\Lambda_\epsilon$ and $\varphi=\varphi_\epsilon$, is a positive solution of (13)). Moreover the following estimations hold*:

- *If ϕ_0 is not a constant function* (*CASE* 1) *then*

$$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c\epsilon.$$

- *If ϕ_0 is a constant function* (*CASE* 2) *then*

$$\begin{aligned}&|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c\epsilon \quad \textit{if } n=3,\\ &|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c\epsilon^\beta, \quad \forall\beta<2 \textit{ if } n=4,\\ &|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c\epsilon^2 \quad \textit{if } n\geqslant 5.\end{aligned}$$

From the first and second intermediate results, we get the following existence result: for all ϵ small enough there exists $(\Lambda_\epsilon,\varphi_\epsilon,w_\epsilon)$ in a neighborhood of $(0,0,0)$ in $\mathbb{R}\times\mathcal{C}^{2,\alpha}_m(S^{n-1})\times\mathcal{C}^{2,\alpha}_\delta(M\setminus\{p\})$ such that (21), considered in $M\setminus B^g_\epsilon(p)$, is a positive solution of (13). Expansion (22) follows from (25) and (31). Moreover:

- If ϕ_0 is not a constant function (CASE 1) then there exists a positive constant c such that

$$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c\epsilon$$

and from (32)

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\delta(M\setminus\{p\})} \leqslant c\big(\epsilon^{2n-4} + \epsilon^n + \epsilon^{3-\delta}\big).$$

- If ϕ_0 is a constant function (CASE 2) then there exists a positive constant c such that

$$\begin{aligned}&|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c\epsilon \quad \text{if } n=3,\\ &|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c\epsilon^\beta, \quad \forall\beta<2 \text{ if } n=4,\\ &|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c\epsilon^2 \quad \text{if } n\geqslant 5\end{aligned}$$

and from (32)

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\delta(M\setminus\{p\})} \leqslant c\epsilon^2 \quad \text{if } n=3,$$
$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\delta(M\setminus\{p\})} \leqslant c\epsilon^4 \quad \text{if } n=4,$$
$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\delta(M\setminus\{p\})} \leqslant c\big(\epsilon^{1+n}+\epsilon^{4-\delta}\big) \quad \text{if } n\geqslant 5.$$

This completes the proof of the result. □

For the case $n=2$ we can adapt the proof of the previous proposition, obtaining

Proposition 4.4. *Let $n=2$ and $\delta\in(0,1)$. For all ϵ small enough there exists $(\Lambda_\epsilon,\varphi_\epsilon,w_\epsilon)$ in a neighborhood of $(0,0,0)$ in $\mathbb{R}\times\mathcal{C}^{2,\alpha}_m(S^{n-1})\times(\tilde\chi\mathbb{R}\oplus\mathcal{C}^{2,\alpha}_\delta(M\setminus\{p\}))$, where $\tilde\chi$ is some cutoff function equal to 1 in a neighborhood of the point p, such that the function*

$$\phi_\epsilon = \phi_0 - (\log\epsilon)^{-1}\big(\phi_0(p)+\Lambda_\epsilon\big)\Gamma_p + w_\epsilon + \chi\tilde H_{\varphi_\epsilon,\epsilon} \tag{33}$$

considered in $M\setminus B^g_\epsilon(p)$, is a positive solution of (13) where

$$\lambda = \lambda_0 + c_n\phi_0(p)^2(\log\epsilon)^{-1} + o\big((\log\epsilon)^{-1}\big). \tag{34}$$

Moreover the following estimations hold: there exists a positive constant c such that

$$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c(\log\epsilon)^{-1} \quad \textit{and} \quad \|w_\epsilon\|_{\tilde\chi\mathbb{R}\oplus\mathcal{C}^{2,\alpha}_\delta(M\setminus\{p\})} \leqslant c(\log\epsilon)^{-2}.$$

Proof. We will follow the proof of the previous proposition, adapting it to the case of dimension 2. Take a sequence of functions $u_j\in H^1_0(M\setminus B^g_\epsilon(p))$ converging to the function

$$u_*(x) = \begin{cases} (\log\frac{1}{\sqrt\epsilon})^{-1}\log\frac{|x|}{\epsilon}\cdot\phi_0(\frac{\sqrt\epsilon x}{|x|}) & \text{in } B^g_{\sqrt\epsilon}(p)\setminus B^g_\epsilon(p),\\ \phi_0(x) & \text{in } M\setminus B^g_{\sqrt\epsilon}(p).\end{cases}$$

It is easy to check that

$$\int_{M\setminus B^g_\epsilon(p)} u_*^2\,\mathrm{dvol}_g = \int_M \phi_0^2\,\mathrm{dvol}_g + \mathcal{O}(\epsilon)$$

while

$$\int_{M\setminus B^g_\epsilon(p)} |\nabla u_*|^2\,\mathrm{dvol}_g = \int_M |\nabla\phi_0|^2\,\mathrm{dvol}_g + \mathcal{O}\big((\log\epsilon)^{-1}\big).$$

Using (24) we have

$$\lambda-\lambda_0 = \mathcal{O}\big((\log\epsilon)^{-1}\big).$$

Define

$$\phi_\epsilon = \phi_0 - (\log\epsilon)^{-1}\big(\phi_0(p)+\Lambda\big)\Gamma_p + w + \chi\tilde H_{\varphi,\epsilon}$$

for some $(\Lambda,\varphi,w)\in\mathbb{R}\times\mathcal{C}^{2,\alpha}_m(S^{n-1})\times\mathcal{C}^{2,\alpha}(M\setminus\{p\})$. Then ϕ_ϵ satisfies the first equation of (13), with $\lambda=\lambda_0+(\log\epsilon)^{-1}\mu$, if and only if

$$\begin{aligned}&\big(\Delta_g+\lambda_0+(\log\epsilon)^{-1}\mu\big)w + (\log\epsilon)^{-1}\big[\mu - c_n\phi_0(p)\big(\phi_0(p)+\Lambda\big)\big]\phi_0 + \tilde H_{\varphi,\epsilon}\Delta_g\chi + \chi\Delta_g\tilde H_{\varphi,\epsilon}\\ &\quad + 2\nabla^g\tilde H_{\varphi,\epsilon}\nabla^g\chi - (\log\epsilon)^{-2}\mu\big(\phi_0(p)+\Lambda\big)\Gamma_p + \big(\lambda_0+(\log\epsilon)^{-1}\mu\big)\chi\tilde H_{\varphi,\epsilon} = 0\end{aligned} \tag{35}$$

in $M\setminus B^g_\epsilon(p)$. This equation can be considered, after extension of functions as in Eq. (26), over $M\setminus\{p\}$.

Let $\tilde\chi$ be some cutoff function on the manifold M identically equal to 1 in $B^g_{R_0}(p)$ and identically equal to 0 in $M\backslash B^g_{R_0}(p)$, and $\delta\in(0,1)$. The operator

$$(\Delta_g + \lambda_0) : \mathbb{R}\tilde{\chi} \oplus \mathcal{C}^{2,\alpha}_{\delta,\perp,0}\big(M \setminus \{p\}\big) \to \mathcal{C}^{0,\alpha}_{\delta-2,\perp}\big(M \setminus \{p\}\big),$$

where the subscript $\perp$ is meant to point out that functions are L^2-orthogonal to ϕ_0 and the subscript 0 is meant to point out that functions satisfy the 0 Dirichlet (in CASE 1) or 0 Neumann (in CASE 2) boundary condition on ∂M if $\partial M \neq \emptyset$, is an isomorphism. The same result holds for the operator

$$\big(\Delta_g + \lambda_0 + (\log\epsilon)^{-1}\mu\big) : \mathbb{R}\tilde{\chi} \oplus \mathcal{C}^{2,\alpha}_{\delta,\perp,0}\big(M \setminus \{p\}\big) \to \mathcal{C}^{0,\alpha}_{\delta-2,\perp}\big(M \setminus \{p\}\big)$$

if ϵ is small enough. The proof of these facts is very similar to the proof of Lemma 4.3 (for other details see [15]).

In order to simplify the notation we define

$$A := (\log\epsilon)^{-1}\big[\mu - c_n\phi_0(p)\big(\phi_0(p) + \Lambda\big)\big]\phi_0,$$
$$B := \tilde{H}_{\varphi,\epsilon}\Delta_g\chi + \chi\Delta_g\tilde{H}_{\varphi,\epsilon} + 2\nabla^g\tilde{H}_{\varphi,\epsilon}\nabla^g\chi,$$
$$C := -(\log\epsilon)^{-2}\mu\big(\phi_0(p) + \Lambda\big)\Gamma_p,$$
$$D := \big(\lambda_0 + (\log\epsilon)^{-1}\mu\big)\chi\tilde{H}_{\varphi,\epsilon}.$$

We remark that $\Gamma_p \in \mathcal{C}^{0,\alpha}_{\delta-2}(M \setminus \{p\})$ when $\delta \in (0,1)$. Eq. (35) becomes

$$\big(\Delta_g + \lambda_0 + (\log\epsilon)^{-1}\mu\big)w = -(A + B + C + D).$$

If we choose μ in order to verify

$$\int_M (A + B + C + D)\phi_0 = 0 \tag{36}$$

there exists a solution $w(\epsilon,\Lambda,\varphi) = w^{(1)} + w^{(2)} \in \tilde{\chi}\mathbb{R} \oplus \mathcal{C}^{2,\alpha}_{\delta,\perp,0}(M \setminus \{p\})$ of Eq. (35) for $\delta \in (0,1)$, for all $\Lambda \in \mathbb{R}$, for all $\varphi \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$, and for all ϵ small enough, and then

$$\phi_\epsilon = \phi_0 - (\log\epsilon)^{-1}\big(\phi_0(p) + \Lambda\big)\Gamma_p + w(\epsilon,\Lambda,\varphi) + \chi H_{\varphi,\epsilon} \tag{37}$$

satisfies the first equation of (13). From (36) we get

$$\mu = \frac{(\log\epsilon)^{-1}c_n\phi_0(p)(\phi_0(p) + \Lambda) - \int_M B\phi_0 - \lambda_0\int_M \chi\tilde{H}_{\varphi,\epsilon}\phi_0}{(\log\epsilon)^{-1}(1 + \int_M \chi\tilde{H}_{\varphi,\epsilon}\phi_0)}.$$

It is easy to check that

$$\int_M B\phi_0 \leqslant c\epsilon\|\varphi\|_{L^\infty(S^{n-1})}$$

and

$$\int_M \chi\tilde{H}_{\varphi,\epsilon}\phi_0 \leqslant c\epsilon\|\varphi\|_{L^\infty(S^{n-1})}.$$

Hence

$$\mu = c_n\phi_0(p)\big(\phi_0(p) + \Lambda\big) + \mathcal{O}(\epsilon\log\epsilon)\|\varphi\|_{L^\infty(S^{n-1})}. \tag{38}$$

We want now to give some estimations on the function w. By the previous facts and Lemma 4.1 we have the following estimations:

- $\|A\|_{\mathcal{C}^{0,\alpha}_{\delta-2}(M\setminus\{p\})} \leqslant c\epsilon\|\varphi\|_{L^\infty(S^{n-1})}$.
- $\|B\|_{\mathcal{C}^{0,\alpha}_{\delta-2}(M\setminus\{p\})} \leqslant c\epsilon\|\varphi\|_{L^\infty(S^{n-1})}$.
- $\|C\|_{\mathcal{C}^{0,\alpha}_{\delta-2}(M\setminus\{p\})} \leqslant c(\log\epsilon)^{-2}$.
- $\|D\|_{\mathcal{C}^{0,\alpha}_{\delta-2}(M\setminus\{p\})} \leqslant c\epsilon\|\varphi\|_{L^\infty(S^{n-1})}$.

In particular we get

$$\|A+B+C+D\|_{\mathcal{C}^{0,\alpha}_{\delta-2}(M\setminus\{p\})} \leqslant c\big((\log\epsilon)^{-2}+\epsilon\|\varphi\|_{L^\infty(S^{n-1})}\big)$$

where we used the fact that for ϵ small enough and $\delta\in(0,1)$ we have $\epsilon^{2-\delta}<\epsilon$. This gives us an estimation on the function w:

$$\big|w^{(1)}\big|+\big\|w^{(2)}\big\|_{\mathcal{C}^{2,\alpha}_{\delta}(M\setminus\{p\})} \leqslant c\big((\log\epsilon)^{-2}+\epsilon\|\varphi\|_{L^\infty(S^{n-1})}\big).$$

We proved the following:

First intermediate result. *Let $n=2$ and $\delta\in(0,1)$. For all $\Lambda\in\mathbb{R}$, for all $\varphi\in\mathcal{C}^{2,\alpha}_m(S^{n-1})$, for all ϵ small enough, there exists a function $w(\epsilon,\Lambda,\varphi)=w^{(1)}+w^{(2)}\in\tilde{\chi}\mathbb{R}\oplus\mathcal{C}^{2,\alpha}_{\delta,\perp,0}(M\setminus\{p\})$ such that (37) is a positive solution of the first equation of (13). Moreover there exists a positive constant c such that*

$$\big|w^{(1)}\big|+\big\|w^{(2)}\big\|_{\mathcal{C}^{2,\alpha}_{\delta}(M\setminus\{p\})} \leqslant c\big((\log\epsilon)^{-2}+\epsilon\|\varphi\|_{L^\infty(S^{n-1})}\big).$$

Now, consider to the second equation of (13), i.e. the boundary condition. Define

$$N(\epsilon,\Lambda,\varphi):=\big[\phi_0(\epsilon y)-(\log\epsilon)^{-1}\big(\phi_0(p)+\Lambda\big)\Gamma(\epsilon y)+\big(w(\epsilon,\Lambda,\varphi)\big)(\epsilon y)+\varphi(y)\big]_{y\in S^{n-1}}.$$

We remark that N represents the boundary value of ϕ_ϵ, it is well defined in a neighborhood of $(0,0,0)$ in $[0,+\infty)\times\mathbb{R}\times\mathcal{C}^{2,\alpha}_m(S^{n-1})$, and takes its values in $\mathcal{C}^{2,\alpha}(S^{n-1})$. The differential of N with respect to Λ and φ at $(0,0,0)$ is

$$\big(\partial_\Lambda N(0,0,0)\big)(\tilde{\Lambda})=-\tilde{\Lambda},$$
$$\big(\partial_\varphi N(0,0,0)\big)(\tilde{\varphi})=\tilde{\varphi}.$$

The previous estimations give us

$$\|w\|_{L^\infty(\partial B^g_\epsilon(p))} \leqslant c\big((\log\epsilon)^{-2}+\epsilon\|\varphi\|_{L^\infty(S^{n-1})}\big).$$

For $N(\epsilon,0,0)$ we have

$$\big\|N(\epsilon,0,0)\big\|_{L^\infty(S^{n-1})} \leqslant \big\|\phi_0(\epsilon y)-(\log\epsilon)^{-1}\phi_0(p)\Gamma(\epsilon y)\big\|_{L^\infty(S^{n-1})}+\big\|\big(w(\epsilon,0,0)\big)(\epsilon y)\big\|_{L^\infty(S^{n-1})}$$

and then

$$\big\|N(\epsilon,0,0)\big\|_{L^\infty(S^{n-1})} \leqslant c(\log\epsilon)^{-1}.$$

The implicit function theorem applies to give

Second intermediate result. *Let $n=2$, $\delta\in(0,1)$ and ϵ be small enough. Then there exists $(\Lambda_\epsilon,\varphi_\epsilon)$ in a neighborhood of $(0,0)$ is $\mathbb{R}\times\mathcal{C}^{2,\alpha}_m(S^{n-1})$ such that $N(\epsilon,\Lambda_\epsilon,\varphi_\epsilon)=0$. Moreover the following estimation holds*

$$|\Lambda_\epsilon|+\|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c(\log\epsilon)^{-1}.$$

The statement of the proposition follows immediately from the two intermediate results and (38). □

In order to simplify the next computations, in dimension 2 we will consider the following function as our positive solution of (13):

$$\phi_\epsilon=\log\epsilon\big[\phi_0-(\log\epsilon)^{-1}\big(\phi_0(p)+\Lambda_\epsilon\big)\Gamma_p+w_\epsilon+\chi H_{\varphi_\epsilon,\epsilon}\big].$$

Remark that this function, considered in the coordinates $y=\epsilon x$, converges near p, in a sense to be made precise, to the function $-\phi_0(p)\log|y|$ when ϵ tends to 0.

5. Perturbing the complement of a ball

The following result follows from the implicit function theorem.

Proposition 5.1. *Given a point $p \in M$, there exists $\epsilon_0 > 0$ and for all $\epsilon \in (0, \epsilon_0)$ and all function $\bar{v} \in C^{2,\alpha}(S^{n-1})$ satisfying*

$$\|\bar{v}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant \epsilon_0,$$

and

$$\int_{S^{n-1}} \bar{v}\, \mathrm{dvol}_{\mathring{g}} = 0,$$

there exists a unique positive function $\phi = \phi(\epsilon, p, \bar{v}) \in \mathcal{C}^{2,\alpha}(M \backslash B^g_{\epsilon(1+v)}(p))$, a constant $\lambda = \lambda(\epsilon, p, \bar{v}) \in \mathbb{R}$ and a constant $v_0 = v_0(\epsilon, p, \bar{v}) \in \mathbb{R}$ such that

$$\mathrm{Vol}_g\big(B^g_{\epsilon(1+v)}(p)\big) = \mathrm{Vol}_{\mathring{g}}(\mathring{B}_\epsilon)$$

where $v := v_0 + \bar{v}$ and ϕ is a solution of the problem

$$\begin{cases} \Delta_g \phi + \lambda \phi = 0 & \text{in } M \setminus B^g_{\epsilon(1+v)}(p), \\ \phi = 0 & \text{on } \partial B^g_{\epsilon(1+v)}(p) \end{cases} \tag{39}$$

which is normalized by setting

$$\int_{M \backslash B^g_{\epsilon(1+v)}(p)} \phi^2\, \mathrm{dvol}_g = 1. \tag{40}$$

In addition ϕ, λ and v_0 depend smoothly on the function $\bar{v}$ and the parameter ϵ.

Proof. We begin by proving that given a point $p \in M$, there exists $\epsilon_0 > 0$ and for all $\epsilon \in (0, \epsilon_0)$ and all function $\bar{v} \in C^{2,\alpha}(S^{n-1})$ satisfying

$$\|\bar{v}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant \epsilon_0,$$

and

$$\int_{S^{n-1}} \bar{v}\, \mathrm{dvol}_{\mathring{g}} = 0,$$

there exists a unique constant $v_0 = v_0(\epsilon, p, \bar{v}) \in \mathbb{R}$ such that

$$\mathrm{Vol}_g\big(B^g_{\epsilon(1+v)}(p)\big) = \mathrm{Vol}_{\mathring{g}}(\mathring{B}_\epsilon) = \epsilon^n\, \mathrm{Vol}_{\mathring{g}}(\mathring{B}_1) \tag{41}$$

where $v := v_0 + \bar{v}$. Define the dilated metric $\bar{g} = \epsilon^{-2} g$. Instead of working on a domain depending on the function $v = v_0 + \bar{v}$, it will be more convenient to work on a fixed domain

$$\mathring{B}_1 := \big\{ y \in \mathbb{R}^n \colon |y| < 1 \big\},$$

endowed with a metric depending on the function v. This can be achieved by considering the parameterization of $B^g_{\epsilon(1+v)}(p) = B^{\bar{g}}_{(1+v)}(p)$ given by

$$Y(y) := \mathrm{Exp}^{\bar{g}}_p \left(\left(1 + v_0 + \bar{\chi}(y) \left(\bar{v} \left(\frac{y}{|y|} \right) \right) \right) \sum_i y^i E_i \right)$$

where $\bar{\chi}$ is a cutoff function identically equal to 0 when $|y| \leqslant 1/2$ and identically equal to 1 when $|y| \geqslant 3/4$.

Hence (using the result of Proposition 3.1) the coordinates we consider from now on are $y \in \mathring{B}_1$ and in these coordinates the metric $\hat{g} := Y^* \bar{g}$ can be written as

$$\hat{g} = (1 + v_0)^2 \left(\mathring{g} + \sum_{i,j} C^{ij} \, dy_i \, dy_j \right),$$

where the coefficients $C^{ij} \in \mathcal{C}^{1,\alpha}(\mathring{B}_\epsilon)$ are functions of y depending on ϵ, $v = v_0 + \bar{v}$ and the first partial derivatives of v. Moreover, $C^{ij} \equiv 0$ when $\epsilon = 0$ and $\bar{v} = 0$. Observe that

$$(\epsilon, v_0, \bar{v}) \mapsto C^{ij}(\epsilon, v)$$

are smooth maps. Condition (41), when ϵ is small enough and not zero, is equivalent to

$$\mathrm{Vol}_{\hat{g}}(\mathring{B}_1) = \mathrm{Vol}_{\mathring{g}}(\mathring{B}_1)$$

that makes sense also for $\epsilon = 0$. When $\epsilon = 0$ and $\bar{v} \equiv 0$, the metric $\hat{g} = (1 + v_0)^2 \mathring{g}$ is nothing but the Euclidean metric. We define

$$N(\epsilon, \bar{v}, v_0) := \mathrm{Vol}_{\hat{g}}(\mathring{B}_1) - \mathrm{Vol}_{\mathring{g}}(\mathring{B}_1).$$

Observe that N depends on the choice of the point $p \in M$. We have

$$N(0, 0, 0) = 0.$$

It should be clear that the mapping N is a smooth map from a neighborhood of $(0, 0, 0)$ in $[0, \infty) \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathbb{R}$ into a neighborhood of 0 in $\mathbb{R}$.

We claim that the partial differential of N with respect to v_0, computed at $(0, 0, 0, 0)$, is given by

$$\partial_{v_0} N(0, 0, 0) = n \, \mathrm{Vol}_{\mathring{g}}(\mathring{B}_1).$$

Indeed, this time we have $\hat{g} = (1 + v_0)^2 \mathring{g}$ since $\bar{v} \equiv 0$ and $\epsilon = 0$ and hence

$$N(0, 0, v_0) = \big((1 + v_0)^n - 1 \big) \, \mathrm{Vol}_{\mathring{g}}(\mathring{B}_1).$$

So we get

$$\partial_{v_0} N(0, 0, 0) = n \, \mathrm{Vol}_{\mathring{g}}(\mathring{B}_1)$$

and the claim follows at once.

Hence the partial differential of N with respect to v_0, computed at $(0, 0, 0)$ is precisely invertible from $\mathbb{R}$ into $\mathbb{R}$ and the implicit function theorem ensures, for all $(\epsilon, \bar{v})$ in a neighborhood of $(0, 0)$ in $[0, \infty) \times \mathcal{C}^{2,\alpha}_m(S^{n-1})$, the existence of a (unique) $v_0 \in \mathbb{R}$ such that $N(\epsilon, \bar{v}, v_0) = 0$. When $v_0 = 0$, we can estimate

$$\hat{g}_{ij} = \delta_{ij} + \mathcal{O}(\epsilon^2),$$

hence

$$N(\epsilon, 0, 0) = \mathcal{O}(\epsilon^2).$$

The implicit function theorem immediately implies that the solution of

$$N(\epsilon, 0, v_0) = 0$$

satisfies

$$\big| v_0(\epsilon, p, 0) \big| \leqslant c \epsilon^2.$$

The fact that v_0 depends smoothly on the parameter ϵ and the function $\bar{v}$ is standard.

We have now, for all $0 < \epsilon < \epsilon_0$ and all function $\bar{v}$ of mean 0, a function $v = v(\epsilon, p, \bar{v}) \in \mathcal{C}^{2,\alpha}(S^{n-1})$ such that

$$\mathrm{Vol}_g \big(B^g_{\epsilon(1+v)}(p) \big) = \mathrm{Vol}_{\mathring{g}}(\mathring{B}_1).$$

Then it is easy to find a solution (ϕ, λ) to the problem (39) and to multiply it by a constant in order to verify (40). The fact that ϕ and λ depend smoothly on the parameter ϵ and the function $\bar{v}$ is standard. $\square$

We will denote the function $\phi = \phi(\epsilon, p, \bar{v})$ as $\phi_{\epsilon,\bar{v}}$, without noting the dependence on the point p. The same for eigenvalues: $\lambda = \lambda_{\epsilon,\bar{v}}$, and $\bar{\lambda} = \bar{\lambda}_{\epsilon,\bar{v}} = \epsilon^2 \lambda_{\epsilon,\bar{v}}$. When $n = 2$, we take $\phi_{\epsilon,\bar{v}} = \log \epsilon \cdot \phi(\epsilon, p, \bar{v})$. Denote

$$\hat{\phi} = \hat{\phi}_{\epsilon,\bar{v}} = Y^* \phi_{\epsilon,\bar{v}}$$

in a neighborhood of $\partial B^g_\epsilon(p)$. We will use such a notation through all the paper: for a general f considered in a neighborhood of $\partial B^g_\epsilon(p)$ we denote

$$\hat{f} = Y^* f.$$

We define the operator F:

$$F(p, \epsilon, \bar{v}) = \hat{g}(\nabla\hat{\phi}, \hat{\nu})|_{\partial \mathring{B}_1} - \frac{1}{\mathrm{Vol}_{\hat{g}}(\partial \mathring{B}_1)} \int_{\partial \mathring{B}_1} \hat{g}(\nabla\hat{\phi}, \hat{\nu})\, \mathrm{dvol}_{\hat{g}},$$

where $\hat{\nu}$ denotes the unit normal vector field about $\partial \mathring{B}_1$ with respect to the metric $\hat{g}$, and (ϕ, v_0) is the solution of (39) provided by Proposition 5.1. Recall that $v = v_0 + \bar{v}$. It is clear that F is well defined from a neighborhood of $M \times (0, 0)$ in $M \times (0, \infty) \times \mathcal{C}^{2,\alpha}_m(S^{n-1})$ into $\mathcal{C}^{1,\alpha}_m(S^{n-1})$. But F can be defined also for $\epsilon = 0$. In fact, from Propositions 4.2 and 4.4 we have that the first eigenfunction $\phi_{\epsilon,0}$ over $M \setminus B^g_{\epsilon(1+v_0)}(p)$ is given by

$$\phi_{\epsilon,0} = \phi_0 - \epsilon^{n-2}(1 + v_0)^{n-2}\big(\phi_0(p) + \Lambda_\epsilon\big)\Gamma_p + w_\epsilon + \chi H_{\varphi_\epsilon,\epsilon} \quad \text{if } n \geqslant 3,$$
$$\phi_{\epsilon,0} = \log\big(\epsilon(1 + v_0)\big)\big[\phi_0 - \big(\log\big(\epsilon(1 + v_0)\big)\big)^{-1}\big(\phi_0(p) + \Lambda_\epsilon\big)\Gamma_p + w_\epsilon + \chi H_{\varphi_\epsilon,\epsilon}\big] \quad \text{if } n = 2$$

where $v_0 = v_0(p, \epsilon, 0) = \mathcal{O}(\epsilon^2)$, for some $(\Lambda_\epsilon, w_\epsilon, \varphi_\epsilon) \in \mathbb{R} \times C^{2,\alpha}(M \setminus B^{\bar{g}}_{1+v_0}(p)) \times C^{2,\alpha}_m(S^{n-1})$, where the estimations of Propositions 4.2 and 4.4 hold because $v_0 = \mathcal{O}(\epsilon^2)$. If we consider these expressions only in a neighborhood of $\partial B^g_{\epsilon(1+v_0)}(p)$ and the parameterization Y given in the proof of Proposition 5.1 with coordinates y in a neighborhood of $\partial \mathring{B}_1$, it is easy to see that the function $\hat{\phi}_0 = Y^*\phi_0$ is equal to the constant function $\hat{\phi}_0 = \phi_0(p)$ when $\epsilon = 0$ and then, by the expansion of the function Γ_p and the estimations on $(\Lambda_\epsilon, w_\epsilon, \varphi_\epsilon)$, we have that when $\epsilon = 0$ the function $\hat{\phi}_{\epsilon,0}(y)$ is equal to

$$\phi_1(y) = \begin{cases} \phi_0(p)(1 - |y|^{2-n}) & \text{if } n \geqslant 3, \\ \phi_0(p) \log|y| & \text{if } n = 2. \end{cases}$$

In a neighborhood of $\partial \mathring{B}_1$ the metric $\hat{g}$ converges, for $\epsilon = 0$, to the Euclidean metric, and then $F(p, 0, 0)$ is the normal derivative of ϕ_1 at $\partial \mathring{B}_1$ minus its Euclidean mean, hence equal to 0. Similarly, we can define $F(p, 0, \bar{v})$. When v is small enough, $\phi_{\epsilon,\bar{v}}$ is close to $\phi_{\epsilon,0}$, and if we consider it only in a neighborhood of $\partial B^g_{\epsilon(1+v)}(p)$ and the parameterization Y given in the proof of Proposition 5.1 with coordinates y in a neighborhood of $\partial \mathring{B}_1$, when $\epsilon = 0$ the function $\hat{\phi}_{\epsilon,\bar{v}}(y)$ converges to the harmonic function on $\mathbb{R}^n \setminus \mathring{B}_{1+v}$ which has 0 boundary condition on $\partial \mathring{B}_{1+v}$ and is asymptotic to $\phi_0(p)$ at infinity for $n \geqslant 3$ and to $\phi_0(p) \log|y|$ for $n = 2$. The fact that F depends smoothly on the function $\bar{v}$ is standard.

In summary, F is well defined from a neighborhood of $M \times (0, 0)$ in $M \times [0, \infty) \times \mathcal{C}^{2,\alpha}_m(S^{n-1})$ into $\mathcal{C}^{1,\alpha}_m(S^{n-1})$, and can be differentiated with respect to $\bar{v}$. Moreover $F(p, 0, 0) = 0$.

Our aim is to find $(p, \epsilon, \bar{v})$ such that $F(p, \epsilon, \bar{v}) = 0$. Observe that in this case ϕ will be the solution of problem (12).

6. Some estimates

Let us consider the normal geodesic coordinates x around p. Using the result of Proposition 3.1 it is easy to show that

$$g^{ij} = \delta_{ij} - \frac{1}{3}R_{ikj\ell}x^k x^\ell - \frac{1}{6}R_{ikj\ell,m}x^k x^\ell x^m + \mathcal{O}\big(|x|^4\big),$$
$$\log|g| = \frac{1}{3}R_{k\ell}x^k x^\ell + \frac{1}{6}R_{k\ell,m}x^k x^\ell x^m + \mathcal{O}\big(|x|^4\big) \tag{42}$$

where

$$R_{k\ell} = \sum_{i=1}^n R_{iki\ell} \quad \text{and} \quad R_{k\ell,m} = \sum_{i=1}^n R_{iki\ell,m}.$$

A straightforward calculation allows us to obtain the expansion of Γ_p. Recall that

$$\Delta_g := \sum_{i,j} g^{ij}\partial_{x_i}\partial_{x_j} + \sum_{i,j}\partial_{x_i}g^{ij}\partial_{x_j} + \frac{1}{2}\sum_{i,j}g^{ij}\partial_{x_i}\log|g|\partial_{x_j}. \tag{43}$$

The function Γ_p is defined by (14). Then locally $\Gamma_p = G_1 + G_2$ where G_1 is locally a solution of

$$-(\Delta_g + \lambda_0)G_1 = c_n\delta_p$$

and G_2 is locally a solution of

$$-(\Delta_g + \lambda_0)G_2 = -c_n\phi_0(p)\phi_0.$$

Clearly, in the normal geodesic coordinates near p we have that

$$G_2 = a_n + b_n\cdot x + \mathcal{O}\big(|x|^2\big)$$

where a_n is a constant and b_n is an n-dimensional vector. For the function G_1 it is possible to obtain an expansion near p starting from the solution G of

$$-\Delta_{\mathring{g}}G = -2\pi\delta_0 \quad \text{for } n = 2,$$
$$-\Delta_{\mathring{g}}G = (n-2)\omega_{n-1}\delta_0 \quad \text{for } n \geqslant 3$$

where ω_{n-1} is the Euclidean volume of S^{n-1}, and recall that $\mathring{g}$ is the Euclidean metric. It is well known that $G(x) = |x|^{2-n}$ for $n \geqslant 3$ and $G(x) = \ln|x|$ for $n = 2$. Considering formulas (11), (42) and (43), we obtain for $n \geqslant 5$:

$$\begin{aligned}\Gamma_p(x) = |x|^{2-n} &+ \left(\frac{2-n}{18}R_{ikj\ell}x^i x^k x^j x^\ell|x|^{-n} - \frac{1}{12}R_{j\ell}x^j x^\ell|x|^{2-n} + \frac{\mathrm{Scal}(p) - 6\lambda_0}{12(4-n)}|x|^{4-n}\right)\\ &+ \left(\frac{2-n}{48}R_{ikj\ell,t}x^i x^k x^j x^\ell x^t|x|^{-n} + \frac{1}{36}R_{\cdot kj\ell,\cdot}x^k x^j x^\ell|x|^{2-n}\right.\\ &\left.- \frac{1}{24}R_{j\ell,t}x^j x^\ell x^t|x|^{2-n} + \frac{\mathrm{Scal}_{,t}}{24(4-n)}x^t|x|^{4-n}\right) + a_n + \mathcal{O}\big(|x|^{6-n}\big).\end{aligned} \tag{44}$$

When $n = 4$ we have

$$\begin{aligned}\Gamma_p(x) = |x|^{-2} &+ \left(-\frac{1}{9}R_{ikj\ell}x^i x^k x^j x^\ell|x|^{-4} - \frac{1}{12}R_{j\ell}x^j x^\ell|x|^{-2} + \frac{\mathrm{Scal}(p) - 6\lambda_0}{12}\log|x|\right)\\ &+ \left(-\frac{1}{24}R_{ikj\ell,t}x^i x^k x^j x^\ell x^t|x|^{-4} + \frac{1}{36}R_{\cdot kj\ell,\cdot}x^k x^j x^\ell|x|^{-2}\right.\\ &\left.- \frac{1}{24}R_{j\ell,t}x^j x^\ell x^t|x|^{-2} + \frac{\mathrm{Scal}_{,t}}{24}x^t\log|x|\right) + +a_4 + b_4\cdot x + \mathcal{O}\big(|x|^\alpha\big),\end{aligned} \tag{45}$$

for all $\alpha < 2$. In the above expressions we used the notation

$$R_{\cdot kj\ell,\cdot} := \sum_{i=1}^n R_{ikj\ell,i}.$$

To obtain such formulas, we used the symmetries of the Riemann tensor ($-R_{kij\ell} = R_{ikj\ell} = R_{j\ell ik}$, for every i, k, j, ℓ), the facts that $R_{iijl} = 0$, $R_{iijl,t} = 0$, $R_{i\ell} = R_{\ell i}$, $R_{ikji,t} = R_{ikij,t}$ (because for geodesic normal coordinates the Christoffel symbols vanish at the origin), the definition of the scalar curvature $\sum_i R_{ii} = \mathrm{Scal}(p)$ and the second Bianchi identity

$$\sum_j R_{tj,j} = \sum_j R_{jt,j} = \frac{1}{2}\,\mathrm{Scal}_{,t}\,.$$

Remark 6.1. If $n \geqslant 6$, the regular part of the Green function Γ_p is completely included in the neglected term $\mathcal{O}(|x|^{6-n})$, see formula (44). At order $2-n$ the function looks like the standard Green function. From order $3-n$ to $5-n$, terms depend only on the local geometry of the manifold near p, and global geometry appears only at terms of order $6-n$ or bigger, and we will see in the following sections that such terms can be neglected in ours computations. If $n = 5$, the situation is a little bit different. In fact the regular part of the Green function Γ_p is not completely included in the neglected term $\mathcal{O}(|x|^{6-n}) = \mathcal{O}(|x|)$, but the only term of the regular part of the Green function Γ_p not included in the term $\mathcal{O}(|x|)$ is a term of order 0, i.e. the constant a_5, see formula (44). As we will see, such a constant can be neglected in ours computations. If $n = 4$ the regular part of the Green function Γ_p also is not completely included in the neglected term $\mathcal{O}(|x|^{\alpha})$, $\alpha < 2$, but the only terms of such regular part not included are terms of order 0 and 1, i.e. $a_4 + b_4 \cdot x$, see formula (45). As we will see, also in this case such terms can be neglected in ours computations.

The fact that in our next computations the regular part of the Green function Γ_p can be neglected for $n \geqslant 4$ is a crucial ingredient, and by this fact we will obtain that in CASE 2 only the local geometry of the manifold plays a rôle.

For $n = 2$ and $n = 3$ we are not able to state a result in CASE 2 exactly because, following our approach, in such dimensions the regular part of the Green function Γ_p cannot be neglected. This is the reason for which in this section we do not give the expansion of Γ_p for the dimensions 2 and 3 (it would be completely unuseful).

The main result of this section is

Proposition 6.2. *In CASE* 1 (*i.e. when ϕ_0 is not constant*) *there exists a constant $c > 0$ such that, for all $p \in M$ and all $\epsilon \geqslant 0$ small enough we have*

$$\big\|F(p,\epsilon,0)\big\|_{\mathcal{C}^{1,\alpha}} \leqslant c\epsilon \quad \textit{if } n \geqslant 3,$$
$$\big\|F(p,\epsilon,0)\big\|_{\mathcal{C}^{1,\alpha}} \leqslant c\epsilon\log\epsilon \quad \textit{if } n = 2.$$

Moreover there exists a constant C_n (depending only on n), such that for all $a \in \mathbb{R}^n$ the following estimates hold

$$\bigg|\int_{S^{n-1}} \mathring{g}(a,\cdot)F(p,\epsilon,0)\,\mathrm{dvol}_{\mathring{g}} - C_n\epsilon\, g\big(\nabla\phi_0(p), \Theta(a)\big)\bigg| \leqslant c\epsilon^2\|a\| \quad \textit{if } n \geqslant 3,$$

$$\bigg|\int_{S^{n-1}} \mathring{g}(a,\cdot)F(p,\epsilon,0)\,\mathrm{dvol}_{\mathring{g}} - C_n\epsilon\log\epsilon\, g\big(\nabla\phi_0(p), \Theta(a)\big)\bigg| \leqslant c\epsilon^2\log\epsilon\|a\| \quad \textit{if } n = 2.$$

In CASE 2 (*i.e. when ϕ_0 is a constant function*) *there exists a constant $c > 0$ such that, for all $p \in M$ and all $\epsilon \geqslant 0$ small enough we have*

$$\big\|F(p,\epsilon,0)\big\|_{\mathcal{C}^{1,\alpha}} \leqslant c\epsilon^2 \quad \textit{if } n \geqslant 5,$$
$$\big\|F(p,\epsilon,0)\big\|_{\mathcal{C}^{1,\alpha}} \leqslant c\epsilon^2\log\epsilon \quad \textit{if } n = 4.$$

Moreover there exists a constant C_n (depending only on n), such that for all $a \in \mathbb{R}^n$ the following estimates hold

$$\bigg|\int_{S^{n-1}} \mathring{g}(a,\cdot)F(p,\epsilon,0)\,\mathrm{dvol}_{\mathring{g}} - C_n\epsilon^3 g\big(\nabla\,\mathrm{Scal}(p), \Theta(a)\big)\bigg| \leqslant c\epsilon^4\|a\| \quad \textit{if } n \geqslant 5,$$

$$\bigg|\int_{S^{n-1}} \mathring{g}(a,\cdot)F(p,\epsilon,0)\,\mathrm{dvol}_{\mathring{g}} - C_n\epsilon^3\log\epsilon\, g\big(\nabla\,\mathrm{Scal}(p), \Theta(a)\big)\bigg| \leqslant c\epsilon^3\|a\| \quad \textit{if } n = 4.$$

Proof. Let ϵ be small enough, and $\bar{v}=0$. We know that $v_0=\mathcal{O}(\epsilon^2)$, then from Proposition 4.2 it follows that for all ϵ small enough there exists $(\Lambda_\epsilon,\varphi_\epsilon,w_\epsilon)$ in a neighborhood of $(0,0,0)$ in $\mathbb{R}\times\mathcal{C}^{2,\alpha}_m(S^{n-1})\times\mathcal{C}^{2,\alpha}(M\setminus B^g_\epsilon(p))$ such that the first eigenfunction of $-\Delta_g$ over the complement of $B^g_{\epsilon(1+v_0)}(p)$ with 0 Dirichlet condition at $\partial B^g_{\epsilon(1+v_0)}(p)$ is given by

$$\phi_{\epsilon,0}=\phi_0-\epsilon^{n-2}(1+v_0)^{n-2}\big(\phi_0(p)+\Lambda_\epsilon\big)\Gamma_p+w_\epsilon+\chi H_{\varphi_\epsilon,\epsilon}$$

if $n\geqslant 3$, and by

$$\phi_{\epsilon,0}=\log\big(\epsilon(1+v_0)\big)\big[\phi_0-\big(\log\epsilon(1+v_0)\big)^{-1}\big(\phi_0(p)+\Lambda_\epsilon\big)\Gamma_p+w_\epsilon+\chi H_{\varphi_\epsilon,\epsilon}\big]$$

if $n=2$, where estimates given in Propositions 4.2 and 4.4 hold because $v_0=\mathcal{O}(\epsilon^2)$.

From the expression of $\phi_{\epsilon,0}$ it follows that in CASE 1 we have

$$\begin{aligned}\int_{S^{n-1}}\mathring{g}(a,\cdot)\hat{g}(\nabla\hat{\phi}_{\epsilon,0},\hat{\nu})|_{\partial\mathring{B}_1}\,\mathrm{dvol}_{\mathring{g}}&=\big(1+\mathcal{O}(\epsilon)\big)\int_{S^{n-1}}\mathring{g}(a,\cdot)\frac{\partial\hat{\phi}_{\epsilon,0}}{\partial|y|}\bigg|_{\partial\mathring{B}_1}\mathrm{dvol}_{\mathring{g}}\\&=\epsilon\big(1+\mathcal{O}(\epsilon)\big)\int_{S^{n-1}}\mathring{g}(a,\cdot)\frac{\partial\phi_{\epsilon,0}}{\partial|x|}\bigg|_{\partial\mathring{B}_\epsilon}\mathrm{dvol}_{\mathring{g}}\\&=\epsilon\big(1+\mathcal{O}(\epsilon)\big)\bigg[\int_{S^{n-1}}\mathring{g}(a,\cdot)\frac{\partial\phi_0}{\partial|x|}\bigg|_{\partial\mathring{B}_\epsilon}\mathrm{dvol}_{\mathring{g}}+\mathcal{O}(\epsilon)\bigg]\\&=C_n\epsilon g\big(\nabla\phi_0(p),\Theta(a)\big)+\mathcal{O}\big(\epsilon^2\big)\end{aligned}$$

for $n\geqslant 3$, and

$$\int_{S^{n-1}}\mathring{g}(a,\cdot)\hat{g}(\nabla\hat{\phi}_{\epsilon,0},\hat{\nu})|_{\partial\mathring{B}_1}\,\mathrm{dvol}_{\mathring{g}}=C_2\epsilon\log\epsilon g\big(\nabla\phi_0(p),\Theta(a)\big)+\mathcal{O}\big(\epsilon^2\log\epsilon\big)$$

for $n=2$, where

$$C_n=\int_{S^{n-1}}\big(x^1\big)^2\,\mathrm{dvol}_{\mathring{g}}=\frac{1}{n}\mathrm{Vol}_{\mathring{g}}\big(S^{n-1}\big).$$

All the estimates for CASE 1 follow at once from this computation together with the fact that, when $\bar{v}\equiv 0$, the unit normal vector $\hat{\nu}$ about the boundary is given by $(1+v_0)|y|(1+\mathcal{O}(\epsilon))$ because the metric $\hat{g}$ near p is the Euclidean metric multiplied by $(1+v_0)^2$ and perturbed by some $\mathcal{O}(\epsilon^2)$ terms.

For CASE 2 the situation is much more complex. In fact, if ϕ_0 is constant we have

$$\int_{S^{n-1}}\mathring{g}(a,\cdot)\hat{g}(\nabla\hat{\phi}_0,\hat{\nu})|_{\partial\mathring{B}_1}\,\mathrm{dvol}_{\mathring{g}}=0.$$

Let us compute now

$$\epsilon^{n-2}(1+v_0)^{n-2}\big(\phi_0(p)+\Lambda_\epsilon\big)\int_{S^{n-1}}\mathring{g}(a,\cdot)\hat{g}(\nabla\hat{\Gamma}_p,\hat{\nu})|_{\partial\mathring{B}_1}\,\mathrm{dvol}_{\mathring{g}}.$$

We remark that the previous term is equal to

$$\big(1+\mathcal{O}(\epsilon)\big)\epsilon^{n-2}\phi_0(p)\int_{S^{n-1}}\mathring{g}(\cdot,a)\frac{\partial\hat{\Gamma}_p}{\partial r}\,\mathrm{dvol}_{\mathring{g}}.$$

For this reason we will compute

$$\epsilon^{n-2}\phi_0(p)\int_{S^{n-1}}\mathring{g}(\cdot,a)\frac{\partial\hat{\Gamma}_p}{\partial r}\,\mathrm{dvol}_{\mathring{g}}.$$

Recall that

$$\hat{\Gamma}_p(y) = \Gamma_p\big(\epsilon(1+v_0)y\big)$$

in a neighborhood of $\partial \mathring{B}_1$, then from (44) and (45) (keeping in mind that $v_0 = \mathcal{O}(\epsilon^2)$) we obtain easily the expression of $\hat{\Gamma}_p(y)$ in power of ϵ. Observe that, in the expansion of $\hat{\Gamma}_p$, terms which contain an even number of coordinates, such as $y^i y^j y^k y^\ell$ or $y^j y^\ell$ etc. do not contribute to the result since, once derived with respect to r they still contain an even number of coordinates, and multiplied then by $\mathring{g}(y,a)$, their average over S^{n-1} is 0. Then, considering only terms which contain an odd number of coordinates we have for $n \geqslant 5$:

$$\begin{aligned}
&\epsilon^{n-2} \int_{S^{n-1}} \mathring{g}(y,a) \frac{\partial \hat{\Gamma}_p}{\partial r} \,\mathrm{dvol}_{\mathring{g}} \\
&\quad = \epsilon^3 a_\sigma \Bigg[\int_{S^{n-1}} y^\sigma \cdot \frac{y^\tau}{|y|} \cdot \frac{\partial}{\partial y^\tau} \bigg(\frac{2-n}{48} R_{ikj\ell,t} y^i y^k y^j y^\ell y^t |y|^{-n} + \frac{1}{36} R_{\cdot kj\ell,\cdot} y^k y^j y^\ell |y|^{2-n} \\
&\qquad + \frac{\mathrm{Scal}_{,t}}{64(4-n)} y^t |y|^{4-n} - \frac{1}{24} R_{j\ell,t} y^j y^\ell y^t |y|^{2-n} \bigg) \,\mathrm{dvol}_{\mathring{g}} \Bigg] + \mathcal{O}\big(\epsilon^4\big) \\
&\quad = \epsilon^3 (5-n) a_\sigma \Bigg[\int_{S^{n-1}} y^\sigma \bigg(\frac{2-n}{48} R_{ikj\ell,t} y^i y^k y^j y^\ell y^t + \frac{1}{36} R_{\cdot kj\ell,\cdot} y^k y^j y^\ell \\
&\qquad + \frac{\mathrm{Scal}_{,t}}{24(4-n)} y^t - \frac{1}{24} R_{j\ell,t} y^j y^\ell y^t \bigg) \,\mathrm{dvol}_{\mathring{g}} \Bigg] + \mathcal{O}\big(\epsilon^4\big).
\end{aligned}$$

We make use of the identities in Appendix A to conclude that there exists a constant $C_n^{(1)}$ such that

$$\epsilon^{n-2} \phi_0(p) \int_{S^{n-1}} \mathring{g}(y,a) \frac{\partial \hat{\Gamma}_p}{\partial r}(y) = C_n^{(1)} \epsilon^3 g\big(\nabla \mathrm{Scal}(p), \Theta(a)\big) + \mathcal{O}\big(\epsilon^4\big), \tag{46}$$

where we have

$$C_n^{(1)} = \frac{5-n}{4n} \mathrm{Vol}_{\mathring{g}}\big(S^{n-1}\big) \bigg[-\frac{1}{3(n+2)} + \frac{1}{6(4-n)} \bigg] \phi_0(p).$$

For $n = 4$ we have

$$\begin{aligned}
\epsilon^{n-2} \int_{S^{n-1}} \mathring{g}(y,a) \frac{\partial \hat{\Gamma}_p}{\partial r} \,\mathrm{dvol}_{\mathring{g}} &= \epsilon^3 \log \epsilon \, a_\sigma \Bigg[\int_{S^3} y^\sigma \cdot \frac{y^\tau}{|y|} \cdot \frac{\partial}{\partial y^\tau} \bigg(\frac{\mathrm{Scal}_{,t}}{24} y^t \log |y| \bigg) \,\mathrm{dvol}_{\mathring{g}} \Bigg] + \mathcal{O}\big(\epsilon^3\big) \\
&= \frac{1}{96} \mathrm{Vol}_{\mathring{g}}\big(S^3\big) \epsilon^3 \log \epsilon \, g\big(\nabla \mathrm{Scal}(p), \Theta(a)\big) + \mathcal{O}\big(\epsilon^3\big)
\end{aligned} \tag{47}$$

and then we set $C_4^{(1)} = \frac{1}{96} \mathrm{Vol}_{\mathring{g}}(S^3)$.

The last term we have to compute is

$$\int_{S^{n-1}} \mathring{g}(a,\cdot) \hat{g}\big(\nabla(\hat{w}_\epsilon + \hat{H}_{\varphi_\epsilon,\epsilon}), \hat{\nu}\big)\big|_{\partial \mathring{B}_1} \,\mathrm{dvol}_{\mathring{g}}.$$

As before we have

$$\int_{S^{n-1}} \mathring{g}(a,\cdot) \hat{g}\big(\nabla(\hat{w}_\epsilon + \hat{H}_{\varphi_\epsilon,\epsilon}), \hat{\nu}\big)\big|_{\partial \mathring{B}_1} \,\mathrm{dvol}_{\mathring{g}} = \big(1 + \mathcal{O}(\epsilon)\big) \int_{S^{n-1}} \mathring{g}(a,\cdot) \frac{\partial(\hat{w}_\epsilon + \hat{H}_{\varphi_\epsilon,\epsilon})}{\partial r}\bigg|_{\partial \mathring{B}_1} \,\mathrm{dvol}_{\mathring{g}}.$$

In Proposition 4.2 we proved that in CASE 2

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\delta(M \setminus \{p\})} \leqslant c\epsilon^4$$

for $n=4$ and

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\delta(M\setminus\{p\})} \leqslant c\big(\epsilon^{1+n}+\epsilon^{4-\delta}\big)$$

for $n \geqslant 5$. Hence

$$\|\nabla \hat{w}_\epsilon\|_{L^\infty(\partial \mathring{B}_1)} \leqslant c\epsilon^{4+\delta}$$

for $n=4$ and

$$\|\nabla \hat{w}_\epsilon\|_{L^\infty(\partial \mathring{B}_1)} \leqslant c\big(\epsilon^{1+n+\delta}+\epsilon^{4}\big)$$

for $n \geqslant 5$ (keep in mind that we are estimating the gradient of the dilated function $\hat{w}_\epsilon$). Remember that $\delta \in (2-n, 4-n)$ because $n \geqslant 4$. It follows that we can choose δ in order to have

$$\int_{S^{n-1}} \mathring{g}(a,\cdot)\frac{\partial \hat{w}_\epsilon}{\partial r}\bigg|_{\partial \mathring{B}_1} \mathrm{dvol}_{\mathring{g}} = \mathcal{O}\big(\epsilon^\beta\big)$$

with $\beta=4$ for $n \geqslant 5$ and $\beta=3$ for $n=4$. Let us consider now $\hat{H}_{\varphi_\epsilon,\epsilon}$. We do not know the expression of $\hat{H}_{\varphi_\epsilon,\epsilon}$ in a neighborhood of $\partial \mathring{B}_1$, but we can know its value on $\partial \mathring{B}_1$. From the equality $\hat{\phi}_\epsilon=0$ on $\partial \mathring{B}_1$, using the estimate on the function $\hat{w}_\epsilon$, we have that

$$\begin{aligned}\hat{H}_{\varphi,\epsilon} = {} & -\phi_0(p)+(1+v_0)^{n-2}\big(\phi_0(p)+\Lambda_\epsilon\big)\bigg[\epsilon^2\bigg(\frac{2-n}{18}R_{ikj\ell}y^iy^ky^jy^\ell-\frac{1}{12}R_{j\ell}y^jy^\ell+\frac{\mathrm{Scal}(p)-6\lambda_0}{12(4-n)}\bigg)\\ & +\epsilon^3\bigg(\frac{2-n}{48}R_{ikj\ell,t}y^iy^ky^jy^\ell y^t+\frac{1}{36}R_{\cdot kj\ell,\cdot}y^ky^jy^\ell-\frac{1}{24}R_{j\ell,t}y^jy^\ell y^t+\frac{\mathrm{Scal}_{,t}}{24(4-n)}y^t\bigg)\bigg]+\mathcal{O}\big(\epsilon^4\big),\end{aligned}$$

on $\partial \mathring{B}_1$, for $n \geqslant 5$. For $n=4$ we have

$$\begin{aligned}\hat{H}_{\varphi,\epsilon} = {} & -\phi_0(p)+(1+v_0)^{n-2}\big(\phi_0(p)+\Lambda_\epsilon\big)\bigg[\epsilon^2\log\epsilon\frac{\mathrm{Scal}(p)-6\lambda_0}{12}\\ & +\epsilon^2\bigg(-\frac{1}{9}R_{ikj\ell}y^iy^ky^jy^\ell-\frac{1}{12}R_{j\ell}y^jy^\ell\bigg)+\epsilon^3\log\epsilon\frac{\mathrm{Scal}_{,t}}{24}y^t\bigg]+\mathcal{O}\big(\epsilon^3\big),\end{aligned}$$

on $\partial \mathring{B}_1$. Let us define a harmonic extension of $\mathring{g}(y,a)$ to $\mathbb{R}^n\setminus \mathring{B}_1$:

$$\begin{cases}\Delta_{\mathring{g}}G_a=0 & \text{in } \mathbb{R}^n\setminus \mathring{B}_1,\\ G_a=\mathring{g}(y,a) & \text{on } \partial \mathring{B}_1.\end{cases}$$

It is easy to check that

$$G_a(y)=|y|^{-n}\mathring{g}(y,a).$$

We observe that functions G_a and $\hat{H}_{\varphi_\epsilon,\epsilon}$ converge by Lemma 4.1 to 0 when $|y|\to+\infty$. Then

$$\begin{aligned}\int_{S^{n-1}} \mathring{g}(a,\cdot)\frac{\partial \hat{H}_{\varphi_\epsilon,\epsilon}}{\partial r}\bigg|_{\partial \mathring{B}_1} \mathrm{dvol}_{\mathring{g}} &= \int_{S^{n-1}} \hat{H}_{\varphi_\epsilon,\epsilon}\frac{\partial G_a}{\partial r}\bigg|_{\partial \mathring{B}_1} \mathrm{dvol}_{\mathring{g}}\\ &= (1-n)\int_{S^{n-1}} \hat{H}_{\varphi_\epsilon,\epsilon}\mathring{g}(y,a)\,\mathrm{dvol}_{\mathring{g}}.\end{aligned}$$

Using the expansion of the value of $\hat{H}_{\varphi_\epsilon,\epsilon}$ on $\partial \mathring{B}_1$, and the identities in Appendix A, we conclude that there exists a constant $C_n^{(2)}$ such that

$$\int_{S^{n-1}} \mathring{g}(y,a)\hat{H}_{\varphi_\epsilon,\epsilon}\,\mathrm{dvol}_{\mathring{g}} = C_n^{(2)}\epsilon^3 g\big(\nabla\,\mathrm{Scal}(p),\Theta(a)\big)+\mathcal{O}\big(\epsilon^4\big), \tag{48}$$

where

$$C_n^{(2)} = \frac{1}{4n}\mathrm{Vol}_{\mathring{g}}\big(S^{n-1}\big)\left[-\frac{1}{3(n+2)} + \frac{1}{6(4-n)}\right]\phi_0(p)$$

for $n \geqslant 5$, and for $n = 4$

$$\int\limits_{S^{n-1}} \mathring{g}(y,a)\hat{H}_{\varphi_\epsilon,\epsilon}\,\mathrm{dvol}_{\mathring{g}} = C_4^{(2)}\epsilon^3 \log\epsilon\, g\big(\nabla\,\mathrm{Scal}(p), \Theta(a)\big) + \mathcal{O}\big(\epsilon^3\big), \tag{49}$$

with

$$C_4^{(2)} = \frac{1}{96}\,\mathrm{Vol}_{\mathring{g}}\big(S^3\big).$$

Summarizing, we conclude that in CASE 2

$$\big\|F(p,\epsilon,0)\big\|_{\mathcal{C}^{1,\alpha}} = \mathcal{O}\big(\epsilon^2\big)$$

and from (46), (47), (48) and (49) we have that there exists a constant C_n depending only on n, such that for all $a \in \mathbb{R}^n$ the following estimates hold: for $n \geqslant 5$

$$\left|\,\int\limits_{S^{n-1}} \mathring{g}(a,\cdot)F(p,\epsilon,0)\,\mathrm{dvol}_{\mathring{g}} - C_n\epsilon^3 g\big(\nabla\,\mathrm{Scal}(p), \Theta(a)\big)\right| \leqslant c\epsilon^4\|a\|$$

where

$$C_n = \frac{6-2n}{n}\mathrm{Vol}_{\mathring{g}}\big(S^{n-1}\big)\left[-\frac{1}{3(n+2)} + \frac{1}{6(4-n)}\right]\phi_0(p)$$

and for $n = 4$

$$\left|\,\int\limits_{S^{n-1}} \mathring{g}(a,\cdot)F(p,\epsilon,0)\,\mathrm{dvol}_{\mathring{g}} - C_4\epsilon^3 \log\epsilon\, g\big(\nabla\,\mathrm{Scal}(p), \Theta(a)\big)\right| \leqslant c\epsilon^3\|a\|,$$

where

$$C_4 = -\frac{1}{48}\mathrm{Vol}_{\mathring{g}}\big(S^3\big)\phi_0(p).$$

Remark that $C_n \neq 0$ for all $n \geqslant 4$. This completes the proof of the result. □

Remark 6.3. According to Remark 6.1, the regular part of the Green function Γ_p does not play a rôle in our computations. The proof of Proposition 6.2 shows that in CASE 2 when we compute the normal derivative of the first eigenfunction of the Laplace–Beltrami operator on the complement of a small ball, the first term of the Green function Γ_p playing a rôle is the term of order $5-n$ if $n \geqslant 5$ and the term equivalent (up to a constant) to $|x|\log|x|$ if $n = 4$. If $n \geqslant 6$ or $n = 4$ such term comes totally from the local geometry of the manifold. For $n = 5$ such term contains the constant a_5 coming from the regular part of the Green function Γ_p, but such a constant disappears when we differentiate.

7. Linearizing the operator F

Our next task will be to understand the structure of L_0, the operator obtained by linearizing F with respect to $\bar{v}$ at $\epsilon = 0$ and $\bar{v} = 0$. We will see that this operator is a first order elliptic operator which does not depend on the point p.

Recall the definition of ϕ_1 in $\mathbb{R}^n \setminus \{0\}$

$$\phi_1(y) = \begin{cases} \phi_0(p)\big(1-|y|^{2-n}\big) & \text{if } n \geqslant 3, \\ \phi_0(p)\log|y| & \text{if } n = 2. \end{cases}$$

For all $\bar{v} \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$ let ψ be the (unique) bounded solution of

$$\begin{cases} \Delta_{\mathring{g}} \psi = 0 & \text{in } \mathbb{R}^n \setminus \mathring{B}_1, \\ \psi = -\partial_r \phi_1 \bar{v} & \text{on } \partial \mathring{B}_1 \end{cases} \tag{50}$$

where $r = |y|$. By Lemma 4.1, $|\psi(y)| \to 0$ when $|y| \to \infty$. We define

$$H(\bar{v}) := \left(\partial_r \psi + \partial_r^2 \phi_1 \bar{v}\right)\big|_{\partial \mathring{B}_1}. \tag{51}$$

We will need the following result:

Proposition 7.1. *The operator*

$$H : \mathcal{C}^{2,\alpha}_m\left(S^{n-1}\right) \to \mathcal{C}^{1,\alpha}_m\left(S^{n-1}\right)$$

defined in (51) is a self-adjoint, first order elliptic operator. The kernel of H is given by V_1, the eigenspace of $-\Delta_{S^{n-1}}$ associated to the eigenvalue $n-1$. Moreover there exists $c > 0$ such that

$$\|w\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant c \left\| H(w) \right\|_{\mathcal{C}^{1,\alpha}(S^{n-1})},$$

provided w is $L^2(S^{n-1})$-orthogonal to $V_0 \oplus V_1$, where V_0 is the eigenspace associated to constant functions.

Proof. The fact that H is a first order elliptic operator is standard since it is the sum of the Dirichlet-to-Neumann operator for $\Delta_{\mathring{g}}$ and a constant times the identity. In particular, elliptic estimates yield

$$\left\| H(w) \right\|_{\mathcal{C}^{1,\alpha}(S^{n-1})} \leqslant c \|w\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}.$$

The fact that the operator H is (formally) self-adjoint is easy. Let ψ_1 (resp. ψ_2) be the solution of (50) corresponding to the function w_1 (resp. w_2). We compute

$$\begin{aligned} &\partial_r \phi_1(1) \int\limits_{\partial \mathring{B}_1} \left(H(w_1) w_2 - w_1 H(w_2)\right) \mathrm{dvol}_{\mathring{g}} \\ &\quad = \partial_r \phi_1(1) \int\limits_{\partial \mathring{B}_1} (\partial_r \psi_1 w_2 - \partial_r \psi_2 w_1)\, \mathrm{dvol}_{\mathring{g}} \\ &\quad = \int\limits_{\partial \mathring{B}_1} (\psi_1 \partial_r \psi_2 - \psi_2 \partial_r \psi_1)\, \mathrm{dvol}_{\mathring{g}} \\ &\quad = \lim_{R \to \infty} \Bigg[\int\limits_{\mathring{B}_R \setminus \mathring{B}_1} (\psi_1 \Delta_{\mathring{g}} \psi_2 - \psi_2 \Delta_{\mathring{g}} \psi_1)\, \mathrm{dvol}_{\mathring{g}} - \int\limits_{\partial \mathring{B}_R} (\psi_1 \partial_r \psi_2 - \psi_2 \partial_r \psi_1)\, \mathrm{dvol}_{\mathring{g}} \Bigg] \\ &\quad = 0. \end{aligned}$$

Let us consider

$$w = \sum_{j \geqslant 1} w_j$$

the eigenfunction decomposition of w, as in (17). Namely $w_j \in V_j$, the eigenspace associated to the eigenvalue $j(n-2+j)$. Let ψ_j be the bounded solution of

$$\begin{cases} \Delta_{\mathring{g}} \psi_j = 0 & \text{in } \mathbb{R}^n \setminus \mathring{B}_1, \\ \psi_j = -\partial_r \phi_1 w_j & \text{on } \partial \mathring{B}_1 \end{cases} \tag{52}$$

i.e.

$$\psi_j(y) = -|y|^{2-n-j} w_j\left(y/|y|\right) \partial_r \phi_1|_{\partial \mathring{B}_1}.$$

Then

$$H(w) = \sum_j \partial_r \psi_j + \partial_r^2 \phi_1\big|_{\partial \mathring{B}_1} w = \sum_j \big[-(2-n-j)\partial_r \phi_1|_{\partial \mathring{B}_1} + \partial_r^2 \phi_1\big|_{\partial \mathring{B}_1}\big] w_j.$$

With this alternative formula, it is clear that H preserves the eigenspaces V_j and in particular, H maps into the space of functions whose mean over S^{n-1} is 0. Moreover, it is easy to see that V_1 is the only kernel of the operator. In fact,

$$\partial_r \phi_1|_{\partial \mathring{B}_1} = \begin{cases} -(2-n)\phi_0(p) & \text{if } n \geqslant 3, \\ \phi_0(p) & \text{if } n = 2 \end{cases}$$

and

$$\partial_r^2 \phi_1|_{\partial \mathring{B}_1} = \begin{cases} -(2-n)(1-n)\phi_0(p) & \text{if } n \geqslant 3, \\ -\phi_0(p) & \text{if } n = 2 \end{cases}$$

and then $H(w_j) = 0$ if and only if $j = 1$. This completes the proof of the result. □

The main result of this section is the following:

Proposition 7.2. *The operator L_0 is equal to H.*

Proof. By definition, the operator L_0 is the linear operator obtained by linearizing F with respect to $\bar{v}$ at $\epsilon = 0$ and $\bar{v} = 0$. In other words, we have

$$L_0(\bar{w}) = \lim_{s \to 0} \frac{F(p, 0, s\bar{w}) - F(p, 0, 0)}{s}.$$

We know that $F(p, 0, 0) = 0$. Our next step is to compute $F(p, 0, s\bar{w})$, and for this we have to study $F(p, \epsilon, s\bar{w})$. Writing $\bar{v} = s\bar{w}$, we can consider a parameterization Y of $B_{2\epsilon}^g(p)$ given by the following expression:

$$Y(y) := \mathrm{Exp}_p^{\bar{g}}\left(\left(1 + \chi_1(y) v_0 + s\chi_2(y)\left(\bar{w}\left(\frac{y}{|y|}\right)\right)\right) \sum_i y^i E_i\right)$$

where $\bar{g}$ is the dilated metric $\epsilon^{-2} g$, y belongs to the Euclidean ball $\mathring{B}_2$ of radius 2 centered at 0, χ_1 is a cutoff function identically equal to 1 when $0 < |y| \leqslant 4/3$ and identically equal to 0 when $5/3 \leqslant |y| \leqslant 2$, χ_2 is a cutoff function identically equal to 1 when $3/4 \leqslant |y| \leqslant 4/3$ and identically equal to 0 when $0 < |y| \leqslant 1/2$ and $5/3 \leqslant |y| \leqslant 2$, and $v_0 = v_0(p, \epsilon, s\bar{w})$. We set

$$\hat{g} := Y^* \bar{g}$$

over $\mathring{B}_2$. Remark that $\hat{g}$ is an extension of the metric $\hat{g}$ defined on $\mathring{B}_1$ in Section 5. We remark that $\hat{\phi}_{\epsilon,0} := Y^* \phi_{\epsilon,0}$ is a solution on $\mathring{B}_2 \backslash \mathring{B}_1$ of

$$\Delta_{\hat{g}} \hat{\phi}_{\epsilon,0} + \hat{\lambda}_{\epsilon,0} \hat{\phi}_{\epsilon,0} = 0$$

where $\hat{\lambda}_{\epsilon,0} = \bar{\lambda}_{\epsilon,0} = \epsilon^2 \lambda_{\epsilon,0}$. If we set $\bar{\phi}_{\epsilon,0}(y) = \phi_{\epsilon,0}(\epsilon y)$ in a neighborhood of $\partial \mathring{B}_1$, where $x = \epsilon y$ are the normal geodesic coordinates around p defined in Section 3, we have

$$\hat{\phi}_{\epsilon,0}(y) = \bar{\phi}_{\epsilon,0}\big(\big(1 + v_0 + s\bar{w}(y)\big) y\big) \tag{53}$$

on $\partial \mathring{B}_1$. Writing the first eigenfunction of $-\Delta_{\bar{g}}$ on $M \backslash B_{1+v}^{\bar{g}}(p)$ as $\phi = \phi_{\epsilon,0} + \psi$ and $\bar{\lambda} = \bar{\lambda}_{\epsilon,0} + \tau$, we find that

$$\begin{cases} (\Delta_{\bar{g}} + \bar{\lambda}_{\epsilon,0})\psi + \tau\psi + \tau\phi_{\epsilon,0} = 0 & \text{in } M \setminus B_{1+v}^{\bar{g}}(p), \\ \psi = -\phi_{\epsilon,0} & \text{on } \partial B_{1+v}^{\bar{g}}(p) \end{cases} \tag{54}$$

where we can normalize as

$$\int_{M\setminus B^{\bar g}_{1+v}(p)} (\phi_{\epsilon,0}+\psi)^2\,\mathrm{dvol}_{\bar g} = \int_{M\setminus B^{\bar g}_{1+v_0}(p)} \phi_{\epsilon,0}^2\,\mathrm{dvol}_{\bar g} \tag{55}$$

(the v_0 in the second integral is evaluated at $\bar v = 0$) and we have the condition on the volume of the domain

$$\mathrm{Vol}_{\hat g}(\mathring{B}_1) = \mathrm{Vol}_{\mathring{g}}(\mathring{B}_1). \tag{56}$$

Obviously ψ, τ and v_0 are smooth functions of s. When $s=0$, we have $\phi=\phi_{\epsilon,0}$, $\bar\lambda = \bar\lambda_{\epsilon,0}$ and $v_0 = \mathcal{O}(\epsilon^2)$. Therefore, ψ and τ vanish when $s=0$. We set

$$\dot\psi = \partial_s\psi|_{s=0}, \qquad \dot\tau = \partial_s\tau|_{s=0}, \quad \text{and} \quad \dot v_0 = \partial_s v_0|_{s=0}.$$

Differentiating (54) with respect to s and evaluating the result at $s=0$, we obtain

$$\begin{cases} (\Delta_{\bar g} + \bar\lambda_{\epsilon,0})\dot\psi + \dot\tau\phi_{\epsilon,0} = 0 & \text{in } M\setminus B^{\bar g}_{1+v_0}(p), \\ \dot\psi = -\bar g(\nabla\phi_{\epsilon,0},\bar\nu)(\dot v_0 + \bar w) & \text{on } \partial B^{\bar g}_{1+v_0}(p) \end{cases} \tag{57}$$

where v_0 is evaluated at $s=0$. Observe that the second equation of (57) follows from (53).

Differentiating (55) with respect to s and evaluating the result at $s=0$, we obtain that ψ is L^2-orthogonal to $\phi_{\epsilon,0}$ on $M\setminus B^{\bar g}_{1+v_0}(p)$. Hence

$$\phi = \phi_{\epsilon,0} + s\dot\psi + \mathcal{O}(s^2)$$

where $\dot\psi$ is the solution of (57) L^2-orthogonal to $\phi_{\epsilon,0}$. Differentiating (56) with respect to s and evaluating the result at $s=0$, we obtain

$$\int_{S^{n-1}} (\dot v_0 + \bar w)\,\mathrm{dvol}_{\hat g} = 0$$

where the metric $\hat g$ is evaluated at $s=0$. Since the discrepancy between the metric $\hat g$ and the Euclidean metric $\mathring{g}$ at $\partial\mathring{B}_1$ can be estimated by a constant times ϵ^2 when $s=0$, and the Euclidean average of $\bar w$ is 0, we get that

$$\dot v_0 = \mathcal{O}(\epsilon^2).$$

Moreover we know that for $s=0$ we have $v_0 = \mathcal{O}(\epsilon^2)$. From the expansion of v_0 with respect to s we get

$$v_0 = \mathcal{O}(\epsilon^2) + \mathcal{O}(s^2).$$

Now, in $\mathring{B}_{4/3}\setminus\mathring{B}_1$, we have

$$\begin{aligned}\hat\phi(y) &= \bar\phi_{\epsilon,0}\big((1+v_0+s\bar w(y/|y|))y\big) + s\dot\psi(y) + \mathcal{O}(s^2) \\ &= \bar\phi_{\epsilon,0}\big((1+v_0(0))y\big) + s\big(\mathring{g}(\nabla\bar\phi_{\epsilon,0}\big((1+v_0(0))y\big),(\dot v_0 + \bar w(y/|y|))y) + \dot\psi\big) + \mathcal{O}(s^2)\end{aligned}$$

where we denoted $v_0(p,\epsilon,0) = v_0|_{s=0} = v_0(0)$. To complete the proof of the result, it suffices to compute the normal derivative of the function $\hat\phi$ when the normal is computed with respect to the metric $\hat g$. We use polar coordinates $y = rz$ where $r>0$ and $z\in S^{n-1}$. Then the metric $\hat g$ can be expanded in $\mathring{B}_{4/3}\setminus\mathring{B}_{3/4}$ as

$$\hat g = (1+v_0+s\bar w)^2\,dr^2 + 2s(1+v_0+s\bar w)r\,d\bar w\,dr + r^2(1+v_0+s\bar w)^2\mathring{h} + s^2r^2\,d\bar w^2 + \mathcal{O}(\epsilon^2)$$

where $\mathring{h}$ is the metric on S^{n-1} induced by the Euclidean metric. It follows from this expression, together with the estimation of v_0, that the unit normal vector field to $\partial\mathring{B}_1$ for the metric $\hat g$ is given by

$$\hat\nu = \big((1+s\bar w)^{-1} + \mathcal{O}(s^2)\big)\partial_r + \mathcal{O}(s)\partial_{z_j} + \mathcal{O}(\epsilon^2).$$

Using this, we conclude that

$$\hat g(\nabla\hat\phi,\hat\nu) = \partial_r\bar\phi_{\epsilon,0}(y) + \mathcal{O}(s)\partial_{z_j}\bar\phi_{\epsilon,0}(y) + s\big(\bar w\partial_r^2\bar\phi_{\epsilon,0}(y) + \partial_r\dot\psi\big) + \mathcal{O}(s^2) + \mathcal{O}(\epsilon^2) \tag{58}$$

on $\partial \mathring{B}_1$. When $\epsilon = 0$ we have $\bar{\phi}_{\epsilon,0}(y) = \phi_1(y)$. It follows that $F(p, 0, s\bar{w})$, up to terms of order $\mathcal{O}(s^2)$, is given by

$$\partial_r \phi_1|_{\partial \mathring{B}_1} + s\bar{w}\partial_r^2 \phi_1|_{\partial \mathring{B}_1} + s \lim_{\epsilon \to 0} \partial_r \dot{\psi}|_{\partial \mathring{B}_1}$$

minus its Euclidean mean. We need now the following:

Lemma 7.3. *Evaluate v_0 at $s = 0$. Let $\delta \in (2 - n, 0)$ if $n \geqslant 3$ and $\delta \in (0, 1)$ if $n = 2$. Let H_φ be the function defined in Section 4. For all ϵ small enough there exists a constant $\dot{\tau}$ and $(K_\epsilon, \varphi_\epsilon, \eta_\epsilon)$ in a neighborhood of $(0, 0, 0)$ in $\mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathcal{C}^{2,\alpha}_\delta(M \setminus \{p\})$ such that the function*

$$\dot{\psi} = K_\epsilon + \chi(\psi + H_{\varphi_\epsilon}) + \eta_\epsilon \tag{59}$$

defined in $M \setminus B^{\bar{g}}_{1+v_0}(p)$, is the solution of (57) L^2-orthogonal to $\phi_{\epsilon,0}$, where χ is a cutoff function equal to 1 in $B^{\bar{g}}_{R_0/\epsilon}(p)$ and equal to 0 out of $B^{\bar{g}}_{2R_0/\epsilon}(p)$ and ψ is defined by (50). Moreover the following estimations hold

$$|K_\epsilon| \leqslant c(\epsilon^2 + \epsilon^{n-1}), \qquad \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c(\epsilon^2 + \epsilon^{n-1}) \quad \text{and} \quad \|\eta_\epsilon\|_{\mathcal{C}^{2,\alpha}_\delta(M\setminus\{p\})} \leqslant c(\epsilon^2 + \epsilon^{n-1}).$$

Proof. Define

$$\dot{\psi} = K + \chi(\psi + H_\varphi) + \eta \tag{60}$$

for some $(K, \varphi, \eta) \in \mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathcal{C}^{2,\alpha}(M \setminus \{p\})$, where χ is a cutoff function equal to 1 in $B^{\bar{g}}_{R_0/\epsilon}(p)$ and equal to 0 out of $B^{\bar{g}}_{2R_0/\epsilon}(p)$. Then $\dot{\psi}$ satisfies the first equation of (57), if and only if

$$\begin{aligned}(\Delta_{\bar{g}} + \bar{\lambda}_{\epsilon,0})\eta = {} & -\psi\Delta_{\bar{g}}\chi - \chi\Delta_{\bar{g}}\psi - 2\nabla^{\bar{g}}\psi\nabla^{\bar{g}}\chi - H_\varphi\Delta_{\bar{g}}\chi - \chi\Delta_{\bar{g}}H_\varphi - 2\nabla^{\bar{g}}H_\varphi\nabla^{\bar{g}}\chi \\ & - \bar{\lambda}_{\epsilon,0}\chi(\psi + H_\varphi) - \bar{\lambda}_{\epsilon,0}K - \dot{\tau}\phi_{\epsilon,0}.\end{aligned} \tag{61}$$

We say that $f \in \mathcal{C}^{2,\alpha}_\delta(M \setminus B^{\bar{g}}_{1+v_0}(p))$ if f is the restriction to $M \setminus B^{\bar{g}}_{1+v_0}(p)$ of a function in $\mathcal{C}^{2,\alpha}_\delta(M \setminus \{p\})$. For $n \geqslant 3$ and $\delta \in (2 - n, 0)$, the operator

$$(\Delta_{\bar{g}} + \bar{\lambda}_{\epsilon,0}) : \mathcal{C}^{2,\alpha}_{\delta,\perp,0}(M \setminus B^{\bar{g}}_{1+v_0}(p)) \to \mathcal{C}^{0,\alpha}_{\delta-2,\perp}(M \setminus B^{\bar{g}}_{1+v_0}(p)),$$

where the subscript $\perp$ is meant to point out that functions are L^2-orthogonal to $\phi_{\epsilon,0}$, and the subscript 0 is meant to point out that functions satisfy the 0 Dirichlet (CASE 1) or 0 Neumann (CASE 2) condition on ∂M and the 0 Dirichlet condition on $\partial B^{\bar{g}}_{1+v_0}(p)$, is an isomorphism. For $n = 2$ and $\delta \in (0, 1)$ the same result holds for the operator

$$(\Delta_{\bar{g}} + \bar{\lambda}_{\epsilon,0}) : \tilde{\chi}\mathbb{R} \oplus \mathcal{C}^{2,\alpha}_{\delta,\perp,0}(M \setminus B^{\bar{g}}_{1+v_0}(p)) \to \mathcal{C}^{0,\alpha}_{\delta-2,\perp}(M \setminus B^{\bar{g}}_{1+v_0}(p)),$$

where $\tilde{\chi}$ is a cutoff function equal to 1 in a neighborhood of the origin. See Section 4, or [15], for more details.

To simplify the notation define

$$\begin{aligned}A &:= -\psi\Delta_{\bar{g}}\chi - 2\nabla^{\bar{g}}\psi\nabla^{\bar{g}}\chi - H_\varphi\Delta_{\bar{g}}\chi - 2\nabla^{\bar{g}}H_\varphi\nabla^{\bar{g}}\chi, \\ B &:= -\chi\Delta_{\bar{g}}\psi - \bar{\lambda}_{\epsilon,0}\chi(\psi + H_\varphi) - \chi\Delta_{\bar{g}}H_\varphi - \bar{\lambda}_{\epsilon,0}K, \\ C &:= -\dot{\tau}\phi_{\epsilon,0}.\end{aligned}$$

Eq. (61) becomes

$$(\Delta_{\bar{g}} + \bar{\lambda}_{\epsilon,0})\eta = A + B + C.$$

By the last result, if we chose $\dot{\tau}$ in order to verify

$$\int_{M\setminus B^{\bar{g}}_{1+v_0}(p)} (A + B + C)\phi_{\epsilon,0} = 0 \tag{62}$$

there exists a solution $\eta = \eta(\epsilon, K, \varphi) \in \mathcal{C}^{2,\alpha}_{\delta,\perp,0}(M \setminus B^{\bar{g}}_{1+v_0}(p))$ (or $\tilde{\chi}\mathbb{R} \oplus \mathcal{C}^{2,\alpha}_{\delta,\perp,0}(M \setminus B^{\bar{g}}_{1+v_0}(p))$ if $n = 2$) of Eq. (61) for all ϵ small enough, for all constant K and all function φ, and then

$$\dot{\psi} = K + \chi(\psi + H_\varphi) + \eta$$

satisfies the first equation of (57).

We want now to give some estimations on the function η. By Lemma 4.1 we have the following estimations:

- $\|A\|_{\mathcal{C}^{0,\alpha}_{\delta-2}(M\setminus B^{\bar{g}}_{1+v_0}(p))} \leqslant c\epsilon^{n-1}(1 + \|\varphi\|_{L^\infty(S^{n-1})})$.
- $\|B\|_{\mathcal{C}^{0,\alpha}_{\delta-2}(M\setminus B^{\bar{g}}_{1+v_0}(p))} \leqslant c\epsilon^{2}(1 + \|\varphi\|_{L^\infty(S^{n-1})})$.

In particular we get that

$$\dot{\tau} \leqslant c\big(\epsilon^2 + \epsilon^{n-1}\big)(1 + \|\varphi\|_{L^\infty(S^{n-1})})$$

and then

$$\|A + B + C\|_{\mathcal{C}^{0,\alpha}_{\delta-2}(M\setminus B^{\bar{g}}_{1+v_0}(p))} \leqslant c\big(\epsilon^2 + \epsilon^{n-1}\big)\big(1 + \|\varphi\|_{L^\infty(S^{n-1})}\big).$$

This gives an estimation on the function η:

$$\|\eta\|_{\mathcal{C}^{2,\alpha}_{\delta}(M\setminus B^{\bar{g}}_{1+v_0}(p))} \leqslant c\big(\epsilon^2 + \epsilon^{n-1}\big)\big(1 + \|\varphi\|_{L^\infty(S^{n-1})}\big).$$

Summarizing, we have proved the following: for all $\varphi \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$, for all constant K, for all ϵ small enough, there exists a function $\eta = \eta(\epsilon, K, \varphi) \in \mathcal{C}^{2,\alpha}_{\delta,\perp,0}(M \setminus B^{\bar{g}}_{1+v_0}(p))$ (or $\tilde{\chi}\mathbb{R} \oplus \mathcal{C}^{2,\alpha}_{\delta}(M \setminus B^{\bar{g}}_{1+v_0}(p))$ if $n = 2$) such that (60) is a positive solution of the first equation of (57). Moreover there exists a positive constant c such that

$$\|\eta\|_{\mathcal{C}^{2,\alpha}_{\delta}(M\setminus B^{\bar{g}}_{1+v_0}(p))} \leqslant c\big(\epsilon^2 + \epsilon^{n-1}\big)\big(1 + \|\varphi\|_{L^\infty(S^{n-1})}\big).$$

Now, consider the second equation of (57). Let us define

$$Z(\epsilon, K, \varphi) := \big[K + \chi(y)\big(\psi(y) + H_\varphi(y)\big) + \eta(\epsilon, \varphi)(y)\big]_{y\in S^{n-1}}.$$

We remark that Z, that represents the boundary value of (60) with $\eta = \eta(\epsilon, K, \varphi)$, is well defined in a neighborhood of $(0, 0, 0)$ in $[0, +\infty) \times \mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1})$, and takes its values in $\mathcal{C}^{2,\alpha}(S^{n-1})$. It is easy to compute the differential of Z with respect to K and φ at $(0, 0, 0)$:

$$\big(\partial_\varphi Z(0,0,0)\big)(\tilde{K}) = \tilde{K},$$
$$\big(\partial_\varphi Z(0,0,0)\big)(\tilde{\varphi}) = \tilde{\varphi}.$$

We can estimate $Z(\epsilon, 0, 0)$:

$$\big\|Z(\epsilon,0,0) + \partial_r\phi_1\bar{w}\big\|_{L^\infty(S^{n-1})} \leqslant c\big(\epsilon^2 + \epsilon^{n-1}\big)$$

and then

$$\big\|Z(\epsilon,0,0) + \bar{g}(\nabla\phi_{\epsilon,0}, \bar{\nu})(\dot{v}_0 + \bar{w})\big\|_{L^\infty(S^{n-1})} \leqslant c\big(\epsilon^2 + \epsilon^{n-1}\big).$$

The implicit function theorem applies to give the following result: if ϵ is small enough, there exists $(K_\epsilon, \varphi_\epsilon)$ in a neighborhood of $(0, 0)$ in $\mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1})$ such that (60) is a positive solution of (57). Moreover the following estimations hold

$$|K_\epsilon| \leqslant c\big(\epsilon^2 + \epsilon^{n-1}\big) \quad \text{and} \quad \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c\big(\epsilon^2 + \epsilon^{n-1}\big).$$

Summarizing, we get the following existence result: for all ϵ small enough there exists a constant $\dot{\tau}$ and $(K_\epsilon, \varphi_\epsilon, \eta_\epsilon)$ in a neighborhood of $(0, 0, 0)$ in $\mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathcal{C}^{2,\alpha}_{\delta}(M \setminus B^{\bar{g}}_{1+v_0}(p))$ (or $\mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \tilde{\chi}\mathbb{R} \oplus \mathcal{C}^{2,\alpha}_{\delta}(M \setminus B^{\bar{g}}_{1+v_0}(p))$ if $n = 2$) such that the function

$$\dot{\psi} = K_\epsilon + \chi(\psi + H_{\varphi_\epsilon}) + \eta_\epsilon$$

defined in $M \setminus B^{\bar g}_{1+v_0}(p)$, is solution of (57). Moreover

$$|K_\epsilon| \leqslant c\big(\epsilon^2 + \epsilon^{n-1}\big), \qquad \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leqslant c\big(\epsilon^2 + \epsilon^{n-1}\big)$$

and

$$\|\eta_\epsilon\|_{\mathcal{C}^{2,\alpha}_\delta(M\setminus B^{\bar g}_{1+v_0}(p))} \leqslant c\big(\epsilon^2 + \epsilon^{n-1}\big).$$

The last norm is that of $\tilde\chi\mathbb{R} \oplus \mathcal{C}^{2,\alpha}_\delta(M \setminus B^{\bar g}_{1+v_0}(p))$ if $n = 2$. This completes the proof of the lemma. $\square$

Using the previous lemma, we have that for ϵ small enough

$$\partial_r \dot\psi|_{\partial \mathring{B}_1} = \partial_r \psi|_{\partial \mathring{B}_1} + \mathcal{O}\big(\epsilon^2\big)$$

for $n \geqslant 3$, and

$$\partial_r \dot\psi|_{\partial \mathring{B}_1} = \partial_r \psi|_{\partial \mathring{B}_1} + \mathcal{O}(\epsilon)$$

for $n = 2$. The statement of Proposition 7.2 follows at once from the fact that $\partial_r\phi_1$ is constant while the term $\bar w\partial_r^2\phi_1 + \partial_r\psi$ has mean 0 on the boundary $\partial\mathring{B}_1$. $\square$

Denote by L_ϵ the linearization of F with respect to $\bar v$, computed at the point $(p,\epsilon,0)$. It is easy to check

Lemma 7.4. *There exists a constant $c > 0$ such that, for all $\epsilon > 0$ small enough we have the estimates*

$$\begin{aligned}
&\big\|(L_\epsilon - L_0)\bar v\big\|_{\mathcal{C}^{1,\alpha}} \leqslant c\epsilon\|\bar v\|_{\mathcal{C}^{2,\alpha}} \quad \text{in CASE 1 and } n \geqslant 3,\\
&\big\|(L_\epsilon - L_0)\bar v\big\|_{\mathcal{C}^{1,\alpha}} \leqslant c\epsilon\log\epsilon\|\bar v\|_{\mathcal{C}^{2,\alpha}} \quad \text{in CASE 1 and } n = 2,\\
&\big\|(L_\epsilon - L_0)\bar v\big\|_{\mathcal{C}^{1,\alpha}} \leqslant c\epsilon^2\|\bar v\|_{\mathcal{C}^{2,\alpha}} \quad \text{in CASE 2 and } n \geqslant 5,\\
&\big\|(L_\epsilon - L_0)\bar v\big\|_{\mathcal{C}^{1,\alpha}} \leqslant c\epsilon^2\log\epsilon\|\bar v\|_{\mathcal{C}^{2,\alpha}} \quad \text{in CASE 2 and } n = 4.
\end{aligned}$$

Proof. L_ϵ and L_0 are first order differential operators. We already know the expression of L_0. We have

$$L_\epsilon(\bar w) = \lim_{s\to 0}\frac{F(p,\epsilon,s\bar w) - F(p,\epsilon,0)}{s}.$$

$F(p,\epsilon,s\bar w)$ is given by (58) minus its mean, in the metric $\hat g$. $F(p,\epsilon,0)$, up to terms of order $\mathcal{O}(\epsilon^2)$, is given by $\partial_r\bar\phi_{\epsilon,0}(y)$ at $\partial\mathring{B}_1$ minus its the mean, in the metric $\hat g$ evaluated at $s = 0$. The proof of the lemma follows at once from Proposition 6.2 and Lemma 7.3. $\square$

8. Proof of the main result

We shall now prove that, for $\epsilon > 0$ small enough, it is possible to solve the equation

$$F(p,\epsilon,\bar v) = 0.$$

Unfortunately, we will not be able to solve this equation at once. Instead, we first prove

Proposition 8.1. *There exists $\epsilon_0 > 0$ such that, for all $\epsilon \in [0,\epsilon_0]$ and for all $p \in M$, there exists a unique function $\bar v = \bar v(p,\epsilon)$ and a vector $a = a(p,\epsilon) \in \mathbb{R}^n$ such that*

$$F(p,\epsilon,\bar v) + \mathring{g}(a,\cdot) = 0.$$

The function $\bar v$ and the vector a depend smoothly on p and ϵ and we have

$$|a| + \|\bar{v}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant c\epsilon \quad \textit{in CASE 1 and } n \geqslant 3,$$
$$|a| + \|\bar{v}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant c\epsilon \log\epsilon \quad \textit{in CASE 1 and } n = 2,$$
$$|a| + \|\bar{v}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant c\epsilon^2 \quad \textit{in CASE 2 and } n \geqslant 5,$$
$$|a| + \|\bar{v}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leqslant c\epsilon^2 \log\epsilon \quad \textit{in CASE 2 and } n = 4.$$

Proof. We fix $p \in M$ and define

$$\bar{F}(p,\epsilon,\bar{v},a) := F(p,\epsilon,\bar{v}) + \mathring{g}(a,\cdot).$$

It is easy to check that $\bar{F}$ is a smooth map from a neighborhood of $(p,0,0,0)$ in $M \times [0,\infty) \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathbb{R}^n$ into a neighborhood of 0 in $\mathcal{C}^{1,\alpha}(S^{n-1})$. Moreover

$$\bar{F}(p,0,0,0) = 0$$

and the differential of $\bar{F}$ with respect to $\bar{v}$, computed at $(p,0,0,0)$ is given by H. Finally the image of the linear map $a \mapsto \mathring{g}(a,\cdot)$ is just the vector space V_1. By Proposition 7.1, the implicit function theorem applies to get the existence of $\bar{v}$ and a, smoothly depending on p and ϵ such that $\bar{F}(p,\epsilon,\bar{v},a) = 0$. The estimates for $\bar{v}$ and a follow at once from Proposition 6.2. □

Proof of Theorem 1.3. In view of the result of the previous proposition, it is enough to show that, provided that ϵ is small enough, it is possible to choose a good point $p \in M$ such that $a(p,\epsilon) = 0$. We claim that there exists a constant $\tilde{C}_n > 0$ (only depending on n) such that

$$\Theta\big(a(p,\epsilon)\big) = -\epsilon\tilde{C}_n\nabla^g\phi_0(p) + \mathcal{O}\big(\epsilon^2\big) \quad \text{in CASE 1 and } n \geqslant 3,$$
$$\Theta\big(a(p,\epsilon)\big) = -\epsilon\log\epsilon\,\tilde{C}_n\nabla^g\phi_0(p) + \mathcal{O}(\epsilon) \quad \text{in CASE 1 and } n = 2,$$
$$\Theta\big(a(p,\epsilon)\big) = -\epsilon^3\tilde{C}_n\nabla^g\,\mathrm{Scal}(p) + \mathcal{O}\big(\epsilon^4\big) \quad \text{in CASE 2 and } n \geqslant 5,$$
$$\Theta\big(a(p,\epsilon)\big) = -\epsilon^3\log\epsilon\,\tilde{C}_n\nabla^g\,\mathrm{Scal}(p) + \mathcal{O}\big(\epsilon^3\big) \quad \text{in CASE 2 and } n = 4.$$

For all $b \in \mathbb{R}^n$ we compute

$$\begin{aligned}
\int_{S^{n-1}} \mathring{g}(a,\cdot)\mathring{g}(b,\cdot)\,\mathrm{dvol}_{\mathring{g}} &= -\int_{S^{n-1}} F(p,\epsilon,\bar{v})\mathring{g}(b,\cdot)\,\mathrm{dvol}_{\mathring{g}} \\
&= -\int_{S^{n-1}} \big(F(p,\epsilon,0) + L_0\bar{v}\big)\mathring{g}(b,\cdot)\,\mathrm{dvol}_{\mathring{g}} \\
&\quad - \int_{S^{n-1}} \big(F(p,\epsilon,\bar{v}) - F(p,\epsilon,0) - L_\epsilon\bar{v}\big)\mathring{g}(b,\cdot)\,\mathrm{dvol}_{\mathring{g}} \\
&\quad - \int_{S^{n-1}} (L_\epsilon - L_0)\bar{v}\,\mathring{g}(b,\cdot)\,\mathrm{dvol}_{\mathring{g}}.
\end{aligned}$$

Now, we use the fact that $\bar{v}$ is $L^2(S^{n-1})$-orthogonal to linear functions and hence so is $L_0\bar{v}$. Therefore,

$$\int_{S^{n-1}} L_0\bar{v}\,\mathring{g}(b,\cdot)\,\mathrm{dvol}_{\mathring{g}} = 0.$$

Using the fact that

$$\bar{v} = \mathcal{O}(\epsilon) \quad \text{in CASE 1 and } n \geqslant 3,$$
$$\bar{v} = \mathcal{O}(\epsilon\log\epsilon) \quad \text{in CASE 1 and } n = 2,$$
$$\bar{v} = \mathcal{O}\big(\epsilon^2\big) \quad \text{in CASE 2 and } n \geqslant 5,$$
$$\bar{v} = \mathcal{O}\big(\epsilon^2\log\epsilon\big) \quad \text{in CASE 2 and } n = 4$$

we get

$$F(p,\epsilon,\bar v) - F(p,\epsilon,0) - L_\epsilon \bar v = \mathcal{O}(\epsilon^2) \quad \text{in CASE 1 and } n \geqslant 3,$$
$$F(p,\epsilon,\bar v) - F(p,\epsilon,0) - L_\epsilon \bar v = \mathcal{O}(\epsilon^2 (\log\epsilon)^2) \quad \text{in CASE 1 and } n = 2,$$
$$F(p,\epsilon,\bar v) - F(p,\epsilon,0) - L_\epsilon \bar v = \mathcal{O}(\epsilon^4) \quad \text{in CASE 2 and } n \geqslant 5,$$
$$F(p,\epsilon,\bar v) - F(p,\epsilon,0) - L_\epsilon \bar v = \mathcal{O}(\epsilon^4 (\log\epsilon)^2) \quad \text{in CASE 2 and } n = 4.$$

Similarly, from Proposition 7.4 we have

$$(L_\epsilon - L_0)\bar v = \mathcal{O}(\epsilon^2) \quad \text{in CASE 1 and } n \geqslant 3,$$
$$(L_\epsilon - L_0)\bar v = \mathcal{O}(\epsilon^2 (\log\epsilon)^2) \quad \text{in CASE 1 and } n = 2,$$
$$(L_\epsilon - L_0)\bar v = \mathcal{O}(\epsilon^4) \quad \text{in CASE 2 and } n \geqslant 5,$$
$$(L_\epsilon - L_0)\bar v = \mathcal{O}(\epsilon^4 (\log\epsilon)^2) \quad \text{in CASE 2 and } n = 4.$$

The claim then follows from Proposition 6.2 and the fact that

$$\int_{S^{n-1}} \mathring{g}(a,\cdot)\mathring{g}(b,\cdot)\,\mathrm{dvol}_{\mathring{g}} = g(\Theta(a),\Theta(b)) \int_{S^{n-1}} (x_1)^2\,\mathrm{dvol}_{\mathring{g}} = \frac{1}{n}\mathrm{Vol}_{\mathring{g}}(S^{n-1})\, g(\Theta(a),\Theta(b)).$$

Now if we assume that p_0 is a nondegenerate critical point of the function ϕ_0 (CASE 1) or a nondegenerate critical point of the scalar curvature (CASE 2), we can apply once more the implicit function theorem to solve the equations

$$G(\epsilon,p) := \epsilon^{-1}\Theta(a(p,\epsilon)) = 0 \quad \text{in CASE 1 and } n \geqslant 3,$$
$$G(\epsilon,p) := (\epsilon\log\epsilon)^{-1}\Theta(a(p,\epsilon)) = 0 \quad \text{in CASE 1 and } n = 2,$$
$$G(\epsilon,p) := \epsilon^{-3}\Theta(a(p,\epsilon)) = 0 \quad \text{in CASE 2 and } n \geqslant 5,$$
$$G(\epsilon,p) := \epsilon^{-3}(\log\epsilon)^{-1}\Theta(a(p,\epsilon)) = 0 \quad \text{in CASE 2 and } n = 4.$$

It should be clear that G depends smoothly on $\epsilon \in [0,\epsilon_0)$ and $p \in M$. Moreover we have

$$G(0,p) = -\tilde{C}_n \nabla^g \phi_0(p)$$

in CASE 1 and

$$G(0,p) = -\tilde{C}_n \nabla^g \mathrm{Scal}(p)$$

in CASE 2. Hence, under the hypothesis on p_0, we have $G(0,p_0) = 0$ in both cases. By assumption the differential of G with respect to p, computed at p_0 is invertible. Therefore, for all ϵ small enough there exists p_ϵ close to p_0 such that

$$\Theta(a(p_\epsilon,\epsilon)) = 0.$$

In addition we have

$$\mathrm{dist}(p_0,p_\epsilon) \leqslant c\epsilon.$$

This completes the proof of Theorem 1.3. □

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Appendix A

We recall here some geometrical formulas, used in Section 6, and its proofs.

Lemma A.1. *For all $\sigma = 1, \ldots, n$, we have the following equalities*:

(1) $\sum_{i,j,k,\ell,m} \int_{S^{n-1}} R_{ikj\ell,m} x^i x^j x^k x^\ell x^m x^\sigma \,\mathrm{dvol}_{\mathring{g}} = 0$.
(2) $\sum_{j,k,\ell} \int_{S^{n-1}} R_{\cdot kj\ell,\cdot} x^j x^k x^\ell x^\sigma \,\mathrm{dvol}_{\mathring{g}} = 0$.
(3) $\sum_{i,\ell,m} \int_{S^{n-1}} R_{i\ell,m} x^i x^\ell x^m x^\sigma \,\mathrm{dvol}_{\mathring{g}} = \frac{2}{n(n+2)} \mathrm{Vol}_{\mathring{g}}(S^{n-1})\,\mathrm{Scal}_{,\sigma}$.
(4) $\sum_t \int_{S^{n-1}} \mathrm{Scal}_{,t} x^t x^\sigma \,\mathrm{dvol}_{\mathring{g}} = \frac{1}{n} \mathrm{Vol}_{\mathring{g}}(S^{n-1}) \mathrm{Scal}_{,\sigma}$.

Proof. For the proof of the first three equalities, see [16]. Let us prove the forth. We have that $\int_{S^{n-1}} \mathrm{Scal}_{,t} x^t x^\sigma \,\mathrm{dvol}_{\mathring{g}} = 0$ unless the indices t and σ are equal. Then

$$\sum_t \int_{S^{n-1}} \mathrm{Scal}_{,t} x^t x^\sigma \,\mathrm{dvol}_{\mathring{g}} = \mathrm{Scal}_{,\sigma} \int_{S^{n-1}} \left(x^\sigma\right)^2 \mathrm{dvol}_{\mathring{g}} = \frac{1}{n} \mathrm{Vol}_{\mathring{g}}\left(S^{n-1}\right) \mathrm{Scal}_{,\sigma}. \qquad \square$$

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