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Blow-up set for type I blowing up solutions for a semilinear heat equation

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Abstract

Let u be a type I blowing up solution of the Cauchy–Dirichlet problem for a semilinear heat equation,

$$\begin{cases} \partial_t u = \Delta u + u^p, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \varphi(x), & x \in \Omega, \end{cases} \tag{P}$$

where Ω is a (possibly unbounded) domain in $\mathbf{R}^N$, $N \geqslant 1$, and $p > 1$. We prove that, if $\varphi \in L^\infty(\Omega) \cap L^q(\Omega)$ for some $q \in [1, \infty)$, then the blow-up set of the solution u is bounded. Furthermore, we give a sufficient condition for type I blowing up solutions not to blow up on the boundary of the domain Ω. This enables us to prove that, if Ω is an annulus, then the radially symmetric solutions of (P) do not blow up on the boundary $\partial\Omega$.

1. Introduction

This paper concerns the blow-up problem for a semilinear heat equation,

$$\begin{cases} \partial_t u = \Delta u + u^p & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \text{ if } \partial\Omega \neq \emptyset, \\ u(x,0) = \varphi(x) \geqslant 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a (possibly unbounded) domain in $\mathbf{R}^N$, $N \geqslant 1$, $\partial_t = \partial/\partial t$, $p > 1$, $T > 0$, and $\varphi \in L^\infty(\Omega)$. Let T be the maximal existence time of the unique bounded solution u of (1.1). If $T < \infty$, then

$$\limsup_{t \to T} \|u(t)\|_{L^\infty(\Omega)} = \infty,$$

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and we call T the blow-up time of the solution u. The blow-up of u is said to be of type I if

$$\limsup_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(t)\|_{L^\infty(\Omega)}<\infty.$$

Furthermore, the blow-up of u is said to be of O.D.E. type if

$$\limsup_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(t)\|_{L^\infty(\Omega)}=\kappa \quad \text{with } \kappa=\left(\frac{1}{p-1}\right)^{1/(p-1)}.$$

If the blow-up of u is not of type I, then we say that the blow-up of u is of type II. We denote by $B(u)$ the blow-up set of the solution u, that is,

$$B(u)=\Big\{x\in\overline{\Omega}\colon \text{ there exists a sequence } \big\{(x_n,t_n)\big\}\subset\overline{\Omega}\times(0,T)$$
$$\text{such that } \lim_{n\to\infty}(x_n,t_n)=(x,T),\ \lim_{n\to\infty}u(x_n,t_n)=+\infty\Big\}.$$

We remark that $B(u)$ is a closed set in $\overline{\Omega}$.

The blow-up set for problem (1.1) has been studied intensively since the pioneering work due to Weissler [32]. See for example [1–3,6–27,31–35], and references therein. See also [30], which includes a good list of references in this topic. Among others, Friedman and McLeod [6] studied the blow-up set by using the comparison principle, and proved the following (see [6, Theorem 3.3]):

(a) If Ω is convex, then the boundary blow-up does not occur, that is, $B(u)\cap\partial\Omega=\emptyset$.

In [14–16], Giga and Kohn studied blow-up problem (1.1), and established a blow-up criterion for the solutions in the case where $(N-2)p<N+2$. This criterion implies the following:

(b) If Ω is a (possibly unbounded) convex domain and $(N-2)p<N+2$, then the blow-up set $B(u)$ is bounded provided that $\varphi\in H^1(\Omega)$;
(c) If Ω is strictly star-shaped about $a\in\partial\Omega$ and $(N-2)p<N+2$, then $a\notin B(u)$.

For assertion (b), see [16, Theorem 5.1, Remarks 5.2 and 5.4] and for assertion (c), see [16, Theorem 5.3]. Assertion (b) was also proved in [12] and [13] for the one dimensional case, with the initial function φ which deceases monotonically to 0 and which satisfies $0\leqslant\varphi(x)\leqslant C|x|^{-2/(p-1)}$ for some constant C. On the other hand, in [19], the second author of this paper and Mizoguchi proved that a blow-up criterion similar to that of [14–16] holds for type I blowing up solutions without the convexity of the domain Ω, and obtained the following:

(d) If Ω is a bounded smooth domain in $\mathbf{R}^N$ and $(N-2)p\leqslant N+2$, then type I blowing up solutions do not blow up on the boundary $\partial\Omega$.

Unfortunately, if Ω is not convex, then there are few results, except assertion (d), identifying whether the boundary blow-up occurs or not, and the following problem is still open as far as we know:

(P) Let Ω be an annulus in $\mathbf{R}^N$. Then does the radially symmetric solution of (1.1) blow up on the boundary $\partial\Omega$?

We remark that there exists a solution blowing up on the boundary of the domain for the equation

$$\partial_t u=u_{xx}+k\big(u^m\big)_x+u^{2m-1},$$

where $m>1$ and large enough $k>2/\sqrt{m}$ (see [4]).

In this paper we prove that the blow-up set of the solution u of (1.1) is bounded if the blow-up of the solution u is of type I and the initial function $\varphi\in L^\infty(\Omega)\cap L^q(\Omega)$ for some $q\in[1,\infty)$. Furthermore, we give a sufficient condition for the solution u not to blow up on the boundary of the domain Ω, and prove that, if Ω is annulus, then the radially symmetric solution does not blow up on the boundary $\partial\Omega$. In addition, we prove that, if Ω satisfies the exterior sphere condition and the solution u of (1.1) exhibits O.D.E. type blow-up, then the solution does not blow-up on the boundary $\partial\Omega$.

We introduce some notation. Let $B(x,r) = \{y \in \mathbf{R}^N:\ |y-x| < r\}$ for $x \in \mathbf{R}^N$ and $r > 0$. For any bounded continuous function f on $\overline{\Omega}$ and any constant η, we put

$$M(f,\eta) := \bigl\{x \in \overline{\Omega}:\ f(x) \geqslant \|f\|_{L^\infty(\Omega)} - \eta\bigr\}.$$

For any $\phi \in L^\infty(\mathbf{R}^N)$, let

$$\bigl(e^{t\Delta}\varphi\bigr)(x) := (4\pi t)^{-\frac{N}{2}} \int_{\mathbf{R}^N} e^{-\frac{|x-y|^2}{4t}} \phi(y)\,dy.$$

For any $\lambda > 0$, let ζ_λ be a solution of $\zeta' = \zeta^p$ with $\zeta(0) = \lambda$, that is,

$$\zeta_\lambda(t) := \kappa(S_\lambda - t)^{-\frac{1}{p-1}} \quad \text{with } S_\lambda = \frac{\lambda^{-(p-1)}}{p-1}. \tag{1.2}$$

Now we are ready to state the main results of this paper. The first theorem concerns the boundedness of the blow-up set for problem (1.1).

Theorem 1.1. *Let u be a solution of (1.1) which exhibits type I blow-up at $t = T$. If $\varphi \in L^\infty(\Omega) \cap L^q(\Omega)$ for some $q \in [1,\infty)$, then*

$$\sup_{x \in \Omega \setminus B(0,R),\ t \in (0,T)} \bigl|u(x,t)\bigr| < \infty$$

for some $R > 0$. In particular, the blow-up set $B(u)$ is bounded.

In the second theorem we give a result on the relationship between the location of the blow-up set and the level sets of the solution just before the blow-up time. Theorem 1.2 also gives a sufficient condition for type I blowing up solutions of (1.1) not to blow up on the boundary $\partial\Omega$.

Theorem 1.2. *Let u be a solution of (1.1) which exhibits type I blow-up at $t = T$. Assume*

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}+\frac{1}{2}} \bigl\|\nabla u(t)\bigr\|_{L^\infty(\Omega)} = 0. \tag{1.3}$$

Then the blow-up of u is of O.D.E. type. Furthermore, for any $\eta \in (0,\kappa)$, there exists a constant $T' \in (0,T)$ such that

$$B(u) \subset \bigcap_{T'<t<T} M\bigl((T-t)^{\frac{1}{p-1}} u(t), \eta\bigr). \tag{1.4}$$

In particular, the solution u does not blow up on the boundary $\partial\Omega$, that is, $B(u) \cap \partial\Omega = \emptyset$.

Here we remark that, if Ω is a smooth bounded domain and $(N-2)p < N+2$, then the blow-up of the solution is of type I and (1.3) holds (see Theorem 1.1 in [26]).

As an application of Theorem 1.2, we give the following result, which gives an affirmative answer to problem (P).

Corollary 1.1. *Let*

$$\Omega = \bigl\{x \in \mathbf{R}^N:\ a < |x| < b\bigr\}, \quad 0 < a < b < \infty.$$

Then the radially symmetric solution of (1.1) does not blow up on the boundary $\partial\Omega$.

Furthermore, we give the following theorem with the aid of Corollary 1.1.

Theorem 1.3. *Let Ω be a bounded domain in $\mathbf{R}^N$ satisfying the exterior sphere condition. Let u be a solution of (1.1) which exhibits O.D.E. type blow-up. Then the solution u does not blow up on the boundary $\partial\Omega$.*

In this paper we improve the arguments in [8], and give a blow-up criterion for the semilinear heat equations with small diffusion (see Proposition 2.1). This blow-up criterion enables us to study the location of the blow-up set for problem (1.1) by using the profile of the solution just before the blow-up time and to obtain Theorems 1.1 and 1.2. Furthermore, for the radially symmetric solutions of (1.1) in an annulus, we apply the arguments in [5] and [28] with the aid of [26,27,29], and obtain the blow-up estimates of the solution and its gradient. Then we can prove Corollary 1.1 with the aid of Theorem 1.2. In addition, we prove Theorem 1.3 by using Proposition 2.1 and Corollary 1.1.

The rest of this paper is organized as follows: In Section 2 we give some preliminary results on the blow-up problem (1.1). Section 3 is devoted to the proofs of Theorems 1.1, 1.2, and Corollary 1.1. In Section 4 we prove Theorem 1.3.

2. Preliminaries

In this section we give preliminary results on the blow-up problem for the semilinear heat equations. We first give a lemma on O.D.E. type blowing up solutions.

Lemma 2.1. *Assume the same conditions as in Theorem 1.2. Then the blow-up of the solution u is of O.D.E. type.*

Proof. We denote by T the blow-up time of the solution u of (1.1). Let $\epsilon > 0$ be a sufficiently small constant. Put

$$w_\epsilon(x,t) := \epsilon^{\frac{1}{p-1}} u(x, T-\epsilon+\epsilon t), \qquad \varphi_\epsilon(x) := \epsilon^{\frac{1}{p-1}} u(x, T-\epsilon). \tag{2.1}$$

Then w_ϵ blows up at $t = 1$ and satisfies

$$\begin{cases} \partial_t w_\epsilon = \epsilon \Delta w_\epsilon + w_\epsilon^p & \text{in } \Omega \times (0,1), \\ w_\epsilon(x,t) = 0 & \text{on } \partial\Omega \times (0,1), \\ w_\epsilon(x,0) = \varphi_\epsilon(x) & \text{in } \Omega. \end{cases} \tag{2.2}$$

By the comparison principle we see that

$$\big\| w_\epsilon(t) \big\|_{L^\infty(\Omega)} \leqslant \zeta_{\lambda_\epsilon}(t) \quad \text{for } 0 < t < S_{\lambda_\epsilon},$$

where $\lambda_\epsilon = \|\varphi_\epsilon\|_{L^\infty(\Omega)}$, and obtain $S_{\lambda_\epsilon} \leqslant 1$. This together with (1.2) implies

$$\|\varphi_\epsilon\|_{L^\infty(\Omega)} \geqslant \kappa. \tag{2.3}$$

Furthermore, since the blow-up of u is of type I, by (2.1) we can find a positive constant C such that

$$\|\varphi_\epsilon\|_{L^\infty(\Omega)} = \epsilon^{\frac{1}{p-1}} \big\| u(T-\epsilon) \big\|_{L^\infty(\Omega)} \leqslant \epsilon^{\frac{1}{p-1}} \cdot C\big(T-(T-\epsilon)\big)^{-\frac{1}{p-1}} \leqslant C. \tag{2.4}$$

On the other hand, by (1.3) and (2.1) we have

$$\begin{aligned} \lim_{\epsilon\to 0} \epsilon^{\frac{1}{2}} \|\nabla\varphi_\epsilon\|_{L^\infty(\Omega)} &= \lim_{\epsilon\to 0} \epsilon^{\frac{1}{p-1}+\frac{1}{2}} \big\| \nabla u(T-\epsilon) \big\|_{L^\infty(\Omega)} \\ &= \lim_{t\to T} (T-t)^{\frac{1}{p-1}+\frac{1}{2}} \big\| \nabla u(t) \big\|_{L^\infty(\Omega)} = 0. \end{aligned} \tag{2.5}$$

Then, by (2.3)–(2.5) we apply [7, Proposition 1] to problem (2.2), and obtain

$$\lim_{\epsilon\to 0} S_{\lambda_\epsilon} = 1.$$

This together with (1.2) yields $\lim_{\epsilon\to 0} \|\varphi_\epsilon\|_{L^\infty(\Omega)} = \kappa$, and we obtain

$$\lim_{t\to T} (T-t)^{\frac{1}{p-1}} \big\| u(t) \big\|_{L^\infty(\Omega)} = \lim_{\epsilon\to 0} \epsilon^{\frac{1}{p-1}} \big\| u(T-\epsilon) \big\|_{L^\infty(\Omega)} = \lim_{\epsilon\to 0} \|\varphi_\epsilon\|_{L^\infty(\Omega)} = \kappa.$$

Thus the blow-up of the solution u is of O.D.E. type, and Lemma 2.1 follows. □

Next we consider the blow-up problem for a semilinear heat equation with small diffusion. Let u_ϵ be a solution of

$$\begin{cases} \partial_t u = \epsilon\Delta u + u^p & \text{in } \Omega \times (0, T_\epsilon), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0, T_\epsilon) \text{ if } \partial\Omega \neq \emptyset, \\ u(x,0) = \varphi_\epsilon(x) \geqslant 0 & \text{in } \Omega, \end{cases} \tag{2.6}$$

where $N \geqslant 1$, Ω is a domain in $\mathbf{R}^N$, $p > 1$, $\epsilon > 0$, and $\varphi_\epsilon \in L^\infty(\Omega)$. Let T_ϵ and B_ϵ be the blow-up time and the blow-up set of the solution u_ϵ of problem (2.6), respectively. The rest of this section is devoted to the proof of the following proposition, which is the main ingredient of this paper and a modification of [8, Proposition 4.1].

Proposition 2.1. *Let u_ϵ be a solution of (2.6) with $T_\epsilon = 1$ such that*

$$\sup_{0<\epsilon<\epsilon_0} \sup_{0<t<1} (1-t)^{\frac{1}{p-1}} \left\| u_\epsilon(t) \right\|_{L^\infty(\Omega)} \leqslant C_* \tag{2.7}$$

for some $\epsilon_0 > 0$ and $C_* > 0$. Let Ω' be a domain such that $\Omega \subset \Omega'$ and $\{\tilde{\varphi}_\epsilon\}_{0<\epsilon<\epsilon_0}$ a family of functions belonging to $W^{1,\infty}(\Omega')$ such that

$$0 \leqslant \varphi_\epsilon \leqslant \tilde{\varphi}_\epsilon \quad \text{in } \Omega \text{ for all } \epsilon \in (0, \epsilon_0), \tag{2.8}$$

$$\sup_{0<\epsilon<\epsilon_0} \|\tilde{\varphi}_\epsilon\|_{L^\infty(\Omega')} < \infty. \tag{2.9}$$

Assume that there exists a constant $\eta > 0$ such that

$$\tilde{\varphi}_\epsilon(x) < \kappa - \eta \quad \text{on } \partial\Omega' \text{ if } \partial\Omega' \neq \emptyset. \tag{2.10}$$

Then, for any $\delta > 0$, there exist positive constants σ and ϵ_1 such that, if

$$\sup_{0<\epsilon<\epsilon_1} \epsilon^{\frac{1}{2}} \|\nabla\tilde{\varphi}_\epsilon\|_{L^\infty(\{x\in\overline{\Omega'}:\, \kappa-\eta\leqslant\tilde{\varphi}_\epsilon(x)\leqslant\kappa\})} \leqslant \sigma, \tag{2.11}$$

then there holds

$$B_\epsilon \subset \left\{ x \in \overline{\Omega}:\ \tilde{\varphi}_\epsilon(x) \geqslant \kappa - \delta \right\}, \quad 0 < \epsilon < \epsilon_1. \tag{2.12}$$

Here the constants σ and ϵ_1 are independent of the domain Ω.

Let $\delta > 0$. Let σ and ϵ_1 be sufficiently small positive constants to be chosen later, and assume (2.11). Let $\alpha \in (0, \min\{\kappa, \eta\}/10)$. For any $\epsilon \in (0, \epsilon_1)$, put

$$\varphi_\epsilon^*(x) = \begin{cases} \kappa - \alpha & \text{if } x \in \Omega' \text{ and } \tilde{\varphi}_\epsilon(x) \geqslant \kappa - \alpha, \\ \tilde{\varphi}_\epsilon(x) & \text{if } x \in \Omega' \text{ and } \kappa - 10\alpha \leqslant \tilde{\varphi}_\epsilon(x) \leqslant \kappa - \alpha, \\ \kappa - 10\alpha & \text{if } x \in \Omega' \text{ and } \tilde{\varphi}_\epsilon(x) \leqslant \kappa - 10\alpha, \\ \kappa - 10\alpha & \text{if } x \in \mathbf{R}^N \setminus \Omega'. \end{cases} \tag{2.13}$$

By (2.10) and (2.13) we see that $\varphi_\epsilon^* \in W^{1,\infty}(\mathbf{R}^N)$. Let β and γ be positive constants to be chosen later, and put

$$z(x,t) := \left(e^{\epsilon t\Delta}\varphi_\epsilon^*\right)(x), \tag{2.14}$$

$$w(t) := (\kappa - 3\alpha)^{-(p-1)} + \beta\sigma\left(1 - (1-t)^{\frac{1}{2}}\right), \tag{2.15}$$

and

$$f_\gamma(t) := e^{\gamma t}\left(e^{2(p-1)\gamma} - e^{(p-1)\gamma t}\right)^{-\frac{1}{p-1}}.$$

Here the function f_γ satisfies

$$f_\gamma'(t) = \gamma\left(f_\gamma(t) + f_\gamma(t)^p\right), \quad 0 < t < 2, \tag{2.16}$$

and there exists a positive constant c_γ, depending only on p and γ, such that

$$c_\gamma \leqslant \inf_{0<t<1} f_\gamma(t) < \sup_{0<t<1} f_\gamma(t) \leqslant c_\gamma^{-1}. \tag{2.17}$$

Furthermore, we define the following three functions v_1, v_2, and $\bar{v}$ by

$$v_1(x,t) := \big(z(x,t)^{-(p-1)} - (p-1)t\big)^{-\frac{1}{p-1}}, \tag{2.18}$$

$$v_2(x,t) := \big(z(x,t)^{-(p-1)} - w(t)\big)^{-\frac{1}{p-1}}, \tag{2.19}$$

$$\bar{v}(x,t) := v_1(x,t) + \sigma^{\frac{2}{p-1}} v_2(x,t)^2 + f_\gamma(t). \tag{2.20}$$

Then we prove the following proposition.

Lemma 2.2. *Assume the same conditions as in Proposition 2.1. Then, for any $\alpha \in (0, \min\{\kappa,\eta\}/10)$, there exist positive constants β_1, γ, σ, and ϵ_1 such that, if $\tilde{\varphi}_\epsilon$ satisfies (2.11), then the function $\bar{v}$ defined by (2.20) satisfies*

$$\partial_t \bar{v} \geqslant \epsilon \Delta \bar{v} + \bar{v}^p \quad \textit{in } E_\epsilon \tag{2.21}$$

for any $\beta \geqslant \beta_1$ and $\epsilon \in (0, \epsilon_1)$, where

$$E_\epsilon := \left\{(x,t) \in \mathbf{R}^N \times (0,1)\colon\ z(x,t)^{-(p-1)} - w(t) \geqslant \frac{1}{2} C_*^{-\frac{p-1}{2}} \sigma (1-t)^{\frac{1}{2}}\right\}. \tag{2.22}$$

Here C_* is the constant given in (2.7).

Proof. Let σ and ϵ_1 be positive constants to be chosen later, and assume (2.11). We first prove the following inequalities,

$$\kappa - 10\alpha \leqslant z(x,t) \leqslant \kappa - \alpha \quad \text{in } \mathbf{R}^N \times (0,\infty), \tag{2.23}$$

$$\big\|\nabla z(t)\big\|_{L^\infty(\mathbf{R}^N)} \leqslant \epsilon^{-\frac{1}{2}}\sigma \quad \text{in } (0,\infty), \tag{2.24}$$

$$v_1(x,t) \leqslant C \quad \text{in } \mathbf{R}^N \times (0,1), \tag{2.25}$$

$$\bar{v}^p - v_1^p \leqslant C\big(\sigma^{\frac{2}{p-1}} v_2^2 + \sigma^{\frac{2p}{p-1}} v_2^{2p} + f_\gamma + f_\gamma^p\big) \quad \text{in } E_\epsilon, \tag{2.26}$$

for all $\epsilon \in (0, \epsilon_1)$, where C is a positive constant, independent of β and γ. The inequality (2.23) easily follows from (2.13) and the comparison principle. By (2.11) and (2.13) we have

$$\sup_{t>0} \big\|\nabla z(t)\big\|_{L^\infty(\mathbf{R}^N)} \leqslant \big\|\nabla \varphi_\epsilon^*\big\|_{L^\infty(\mathbf{R}^N)} \leqslant \|\nabla \tilde{\varphi}_\epsilon\|_{L^\infty(\{x\in\overline{\Omega'}\colon\ \kappa-10\alpha\leqslant\tilde{\varphi}_\epsilon(x)\leqslant\kappa-\alpha\})} \leqslant \epsilon^{-\frac{1}{2}}\sigma,$$

and obtain the inequality (2.24). On the other hand, since

$$(\kappa-\alpha)^{-(p-1)} - (p-1) = (p-1)\big[\big(1-\kappa^{-1}\alpha\big)^{-(p-1)} - 1\big] > 0,$$

by (2.18) and (2.23) we have

$$v_1(x,t) \leqslant \big((\kappa-\alpha)^{-(p-1)} - (p-1)\big)^{-\frac{1}{p-1}} = \kappa\big[\big(1-\kappa^{-1}\alpha\big)^{-(p-1)} - 1\big]^{-\frac{1}{p-1}},$$

and obtain (2.25). The inequality (2.26) is obtained by the same argument as in (3.19) of [8], and we omit its details.

Next we prove (2.21) by using (2.23)–(2.26). Let β and γ be positive constants to be chosen later. By (2.16) and (2.20) we obtain

$$\begin{aligned}\partial_t \bar{v} - \big(\epsilon\Delta\bar{v} + \bar{v}^p\big) \geqslant{}& \frac{2}{p-1}\sigma^{\frac{2}{p-1}} w'(t) v_2^{p+1} + \gamma\big(f_\gamma(t) + f_\gamma(t)^p\big)\\ &- p\epsilon v_1^{2p-1} z^{-2p}|\nabla z|^2 - 2(p+1)\epsilon\sigma^{\frac{2}{p-1}} v_2^{2p} z^{-2p}|\nabla z|^2 - \big(\bar{v}^p - v_1^p\big)\end{aligned}$$

for all $(x,t) \in E_\epsilon$. Then, by (2.23)–(2.26) there exists a constant C_1, independent of β and γ, such that

$$\begin{aligned}\partial_t \bar{v} - \big(\epsilon\Delta\bar{v} + \bar{v}^p\big) \geqslant{}& \frac{2}{p-1}\sigma^{\frac{2}{p-1}} w'(t) v_2^{p+1} + \gamma\big(f_\gamma(t) + f_\gamma(t)^p\big)\\ &- C_1\sigma^2 - C_1\sigma^{\frac{2}{p-1}+2} v_2^{2p} - C_1\big(\sigma^{\frac{2}{p-1}} v_2^2 + \sigma^{\frac{2p}{p-1}} v_2^{2p} + f_\gamma + f_\gamma^p\big)\end{aligned} \tag{2.27}$$

for all $(x,t) \in E_\epsilon$. Let γ be a positive constant such that $\gamma \geqslant 3C_1$. By (2.17), taking a sufficiently small σ if necessary, we have

$$(\gamma - C_1)\big(f_\gamma(t) + f_\gamma(t)^p\big) - C_1\sigma^2 \geqslant 2C_1\big(c_\gamma + c_\gamma^p\big) - C_1\sigma^2 \geqslant C_1 c_\gamma.$$

This together with (2.15) and (2.27) implies that

$$\partial_t \bar{v} - \big(\epsilon\Delta\bar{v} + \bar{v}^p\big) \geqslant \frac{\beta}{p-1}\sigma^{\frac{p+1}{p-1}}(1-t)^{-\frac{1}{2}} v_2^{p+1} + C_1 c_\gamma - C_1\big(\sigma^{\frac{2}{p-1}} v_2^2 + 2\sigma^{\frac{2p}{p-1}} v_2^{2p}\big) \tag{2.28}$$

for all $(x,t) \in E_\epsilon$.

Let

$$\beta \geqslant \max\big\{8(p-1)C_1 C_*^{\frac{p-1}{2}}, c_\gamma^{-(p-1)}, 4C_1^2(p-1)^2\big\}. \tag{2.29}$$

By (2.19) and (2.22) we have

$$v_2(x,t)^{p-1} = \big(z(x,t)^{-(p-1)} - w(t)\big)^{-1} \leqslant 2C_*^{\frac{p-1}{2}}\sigma^{-1}(1-t)^{-\frac{1}{2}}, \quad (x,t) \in E_\epsilon,$$

and by (2.29) we obtain

$$\begin{aligned} 2C_1\sigma^{\frac{2p}{p-1}} v_2^{2p} &= 2C_1\sigma v_2^{p-1} \cdot \sigma^{\frac{p+1}{p-1}} v_2^{p+1} \\ &\leqslant 4C_1 C_*^{\frac{p-1}{2}}(1-t)^{-\frac{1}{2}}\sigma^{\frac{p+1}{p-1}} v_2^{p+1} \leqslant \frac{\beta}{2(p-1)}(1-t)^{-\frac{1}{2}}\sigma^{\frac{p+1}{p-1}} v_2^{p+1} \end{aligned} \tag{2.30}$$

for all $(x,t) \in E_\epsilon$. Therefore, by (2.28) and (2.30), we obtain

$$\partial_t \bar{v} - \big(\epsilon\Delta\bar{v} + \bar{v}^p\big) \geqslant \frac{\beta}{2(p-1)}\sigma^{\frac{p+1}{p-1}}(1-t)^{-\frac{1}{2}} v_2^{p+1} + C_1 c_\gamma - C_1\sigma^{\frac{2}{p-1}} v_2^2 \tag{2.31}$$

for all $(x,t) \in E_\epsilon$.

Put

$$E_{\epsilon,1} = \big\{(x,t) \in E_\epsilon\colon z(x,t)^{-(p-1)} - w(t) \geqslant \beta^{\frac{1}{2}}\sigma\big\}, \qquad E_{\epsilon,2} = E_\epsilon \setminus E_1.$$

By (2.19) and (2.29) we have

$$C_1\sigma^{\frac{2}{p-1}} v_2^2 \leqslant C_1\sigma^{\frac{2}{p-1}}\big(\beta^{\frac{1}{2}}\sigma\big)^{-\frac{2}{p-1}} = C_1\beta^{-\frac{1}{p-1}} \leqslant C_1 c_\gamma \tag{2.32}$$

for all $(x,t) \in E_{\epsilon,1}$. On the other hand, since

$$(1-t)^{-\frac{1}{2}} \geqslant 1, \qquad \sigma v_2^{p-1} \geqslant \sigma\big(\beta^{\frac{1}{2}}\sigma\big)^{-1} = \beta^{-\frac{1}{2}},$$

for all $(x,t) \in E_{\epsilon,2}$, by (2.29) we have

$$\begin{aligned} \frac{\beta}{2(p-1)}\sigma^{\frac{p+1}{p-1}}(1-t)^{-\frac{1}{2}} v_2^{p+1} &= \frac{\beta}{2(p-1)}(1-t)^{-\frac{1}{2}}\sigma v_2^{p-1} \cdot \sigma^{\frac{2}{p-1}} v_2^2 \\ &\geqslant \frac{\beta^{1/2}}{2(p-1)}\sigma^{\frac{2}{p-1}} v_2^2 \geqslant C_1\sigma^{\frac{2}{p-1}} v_2^2 \end{aligned}$$

for all $(x,t) \in E_{\epsilon,2}$. This together with (2.32) implies

$$\frac{\beta}{2(p-1)}\sigma^{\frac{p+1}{p-1}}(1-t)^{-\frac{1}{2}} v_2^{p+1} + C_1 c_\gamma \geqslant C_1\sigma^{\frac{2}{p-1}} v_2^2 \tag{2.33}$$

for all $(x,t) \in E_\epsilon$. Therefore, by (2.31) and (2.33) we have (2.21) for all $(x,t) \in E_\epsilon$. Thus Lemma 2.2 follows. □

Let β_1 be the constant given in Lemma 2.2, and put

$$\beta = \max\left\{\beta_1, \frac{C_*^{-(p-1)/2}}{2}\right\}. \tag{2.34}$$

Let χ be a C^∞ smooth function in $\mathbf{R}$ such that

$$\chi(z) = 1/4 \quad \text{for } z \leqslant 0, \qquad \chi(z) = z \quad \text{for } z \geqslant 1/2,\ 0 \leqslant \chi'(z) \leqslant 1 \text{ in } \mathbf{R},$$

and put

$$\bar{u}_\epsilon(x,t) = v_1(x,t) + C_*(1-t)^{-\frac{1}{p-1}} \chi\left(\frac{z(x,t)^{-(p-1)} - w(t)}{C_*^{-(p-1)/2}\sigma(1-t)^{1/2}}\right)^{-\frac{2}{p-1}} + f_\gamma(t). \tag{2.35}$$

This together with (2.20) and (2.22) implies that

$$\bar{u}_\epsilon(x,t) = \bar{v}(x,t) \quad \text{in } E_\epsilon. \tag{2.36}$$

Here we prove the following lemma.

Lemma 2.3. *Let $\bar{u}_\epsilon$ be the function defined in (2.35). Then*

$$\bar{u}_\epsilon(x,0) \geqslant \varphi_\epsilon(x), \quad x \in \Omega. \tag{2.37}$$

Proof. For any $x \in \Omega$ with $\tilde{\varphi}_\epsilon(x) \leqslant \kappa - 2\alpha$, by (2.8) and (2.13) we have

$$\bar{u}_\epsilon(x,0) \geqslant v_1(x,0) = \varphi_\epsilon^*(x) \geqslant \tilde{\varphi}_\epsilon(x) \geqslant \varphi_\epsilon(x). \tag{2.38}$$

On the other hand, for any $x \in \Omega$ with $\tilde{\varphi}_\epsilon(x) > \kappa - 2\alpha$, we have

$$z(x,0) = \varphi_\epsilon^*(x) > \kappa - 2\alpha.$$

Then, by (2.15) and (2.23) we have

$$z(x,0)^{-(p-1)} - w(0) < (\kappa - 2\alpha)^{-(p-1)} - (\kappa - 3\alpha)^{-(p-1)} \leqslant 0.$$

This together with (2.7) and (2.35) implies

$$\begin{aligned} \bar{u}_\epsilon(x,0) &\geqslant C_*\chi\left(\frac{z(x,0)^{-(p-1)} - w(0)}{C_*^{-(p-1)/2}\sigma}\right)^{-\frac{2}{p-1}} = 16^{\frac{1}{p-1}} C_* \\ &\geqslant C_* \geqslant u_\epsilon(x,0) = \varphi_\epsilon(x). \end{aligned} \tag{2.39}$$

Therefore, by (2.38) and (2.39) we have the inequality (2.37), and Lemma 2.3 follows. □

Now we are ready to complete the proof of Proposition 2.1.

Proof of Proposition 2.1. Let $h \in C^1(\mathbf{R})$ be such that

$$h(z) = -1 \quad \text{for } z \leqslant 1, \qquad h(z) = 1 \quad \text{for } z \geqslant 4,\ 0 \leqslant h'(z) \leqslant 1 \text{ in } \mathbf{R}.$$

By (2.7) we have

$$h\left(\frac{u_\epsilon(x,t)^{p-1}}{C_*^{p-1}(1-t)^{-1}}\right) = -1 \quad \text{in } \Omega \times (0,1),$$

and see that u_ϵ satisfies

$$\partial_t u_\epsilon = \epsilon \Delta u_\epsilon + u_\epsilon^p + \frac{1}{2}\left(h\left(\frac{u_\epsilon^{p-1}}{C_*^{p-1}(1-t)^{-1}}\right) + 1\right) G_\epsilon(x,t) \quad \text{in } \Omega \times (0,1), \tag{2.40}$$

where

$$G_\epsilon(x,t) = \partial_t \bar{u}_\epsilon - \left(\epsilon \Delta \bar{u}_\epsilon + \bar{u}_\epsilon^p\right).$$

On the other hand, by Lemma 2.2 and (2.36) we have

$$\partial_t \bar{u}_\epsilon \geqslant \epsilon \Delta \bar{u}_\epsilon + \bar{u}_\epsilon^p \quad \text{in } dE_\epsilon, \quad \text{that is,} \quad G_\epsilon \geqslant 0 \quad \text{in } E_\epsilon. \tag{2.41}$$

Furthermore, since

$$\chi\left(\frac{z_\epsilon(x,t)^{-(p-1)}-w(t)}{C_*^{-(p-1)/2}\sigma(1-t)^{1/2}}\right)\leqslant\frac{1}{2}\quad\text{in } \mathbf{R}^N\times[0,1)\setminus E_\epsilon,$$

we have

$$\bar u_\epsilon(x,t)\geqslant 4^{1/(p-1)}C_*(1-t)^{-1/(p-1)},\quad (x,t)\in\mathbf{R}^N\times[0,1)\setminus E_\epsilon,$$

and obtain

$$h\left(\frac{\bar u_\epsilon(x,t)^{p-1}}{C_*^{p-1}(1-t)^{-1}}\right)=1,\quad (x,t)\in\mathbf{R}^N\times[0,1)\setminus E_\epsilon. \tag{2.42}$$

Since $h\leqslant 1$, by (2.41) and (2.42) we have

$$\begin{aligned}&\partial_t\bar u_\epsilon-\left[\epsilon\Delta\bar u_\epsilon+\bar u_\epsilon^p+\frac{1}{2}\left(h\left(\frac{\bar u_\epsilon^{p-1}}{C_*^{p-1}(1-t)^{-1}}\right)+1\right)G_\epsilon(x,t)\right]\\&\quad=\frac{1}{2}\left(1-h\left(\frac{\bar u_\epsilon^{p-1}}{C_*^{p-1}(1-t)^{-1}}\right)\right)G_\epsilon(x,t)\geqslant 0\quad\text{in } \Omega\times(0,1).\end{aligned} \tag{2.43}$$

Therefore, by (2.37), (2.40), and (2.43) we apply the comparison principle to obtain

$$u_\epsilon(x,t)\leqslant\bar u_\epsilon(x,t)\quad\text{in } \Omega\times[0,1). \tag{2.44}$$

Without loss of generality we can assume that $\delta\in(0,\min\{\kappa,\eta\}/2)$, and let

$$\alpha=\delta/5\in\big(0,\min\{\kappa,\eta\}/10\big).$$

Let $0<\epsilon<\epsilon_1$ and $x_\epsilon\in\overline{\Omega}$ be such that $\tilde\varphi_\epsilon(x_\epsilon)<\kappa-\delta$. Then there exists a positive constant R, depending on ϵ and x_ϵ, such that

$$\tilde\varphi_\epsilon(x)<\kappa-\delta=\kappa-5\alpha,\quad x\in B(x_\epsilon,R)\cap\overline{\Omega}.$$

Then, by (2.13) we have

$$z(x,0)=\varphi_\epsilon^*(x)\leqslant\kappa-5\alpha \tag{2.45}$$

for all $x\in B(x_\epsilon,R)\cap\overline{\Omega}$. Furthermore, by [7, Lemma 1], taking sufficiently small σ and ϵ_1 if necessary, we have

$$\sup_{0<\epsilon<\epsilon_1}\sup_{0<t<1}\big\|z(t)-z(0)\big\|_{L^\infty(\mathbf{R}^N)}<\alpha.$$

This together with (2.45) implies that

$$z(x,t)\leqslant\kappa-4\alpha,\quad (x,t)\in\big(B(x_\epsilon,R)\cap\overline{\Omega}\big)\times[0,1), \tag{2.46}$$

for all $\epsilon\in(0,\epsilon_1)$. On the other hand, let C_1 be a positive constant such that

$$(\kappa-4\alpha)^{-(p-1)}-(\kappa-3\alpha)^{-(p-1)}\geqslant C_1. \tag{2.47}$$

Then, by (2.15), (2.34), (2.46), and (2.47), taking a sufficiently small σ if necessary, we obtain

$$\begin{aligned}z(x,t)^{-(p-1)}-w(t)&\geqslant(\kappa-4\alpha)^{-(p-1)}-\big[(\kappa-3\alpha)^{-(p-1)}+\beta\sigma\big(1-(1-t)^{\frac12}\big)\big]\\&\geqslant C_1-\beta\sigma+\beta\sigma(1-t)^{\frac12}\geqslant\frac{C_1}{2}+\frac12 C_*^{-\frac{p-1}{2}}\sigma(1-t)^{\frac12}\\&\geqslant\max\left\{\frac12 C_1,\frac12 C_*^{-\frac{p-1}{2}}\sigma(1-t)^{\frac12}\right\}\end{aligned} \tag{2.48}$$

for all $(x,t)\in(B(x_\epsilon,R)\cap\overline{\Omega})\times[0,1)$. This implies that $(B(x_\epsilon,R)\cap\overline{\Omega})\times[0,1)\subset E_\epsilon$ (see (2.22)). Therefore, by (2.17), (2.20), (2.25), (2.36), (2.44), and (2.48) we have

$$u_\epsilon(x,t)\leqslant\bar u_\epsilon(x,t)=\bar v(x,t)\leqslant v_1(x,t)+\sigma^{\frac{2}{p-1}}(C_1/2)^{-\frac{2}{p-1}}+c_\gamma^{-1}\leqslant C_2$$

for all $(x,t)\in(B(x_\epsilon,R)\cap\overline{\Omega})\times[0,1)$, where C_2 is a constant. This implies $x_\epsilon\notin B_\epsilon$. Therefore, by the arbitrariness of x_ϵ, we have (2.12) for all $\epsilon\in(0,\epsilon_1)$, and the proof of Proposition 2.1 is complete. □

3. Proof of Theorems 1.1 and 1.2

We prove Theorem 1.1 and Theorem 1.2 by using Proposition 2.1.

Proof of Theorem 1.1. Let ϵ_0 be a sufficiently small positive constant. Put

$$u_\epsilon(x,\tau) := \epsilon^{\frac{1}{p-1}} u(x, T-\epsilon+\epsilon\tau), \qquad \varphi_\epsilon(x) := \epsilon^{\frac{1}{p-1}} u(x, T-\epsilon), \qquad M_\epsilon := \sup_{0<t<T-\epsilon} \big\| u(t) \big\|_{L^\infty(\Omega)},$$

for all $\epsilon \in (0, \epsilon_0)$. Then u_ϵ satisfies

$$\begin{cases} \partial_\tau u_\epsilon = \epsilon \Delta u_\epsilon + u_\epsilon^p & \text{in } \Omega \times (0,1), \\ u_\epsilon(x,\tau) = 0 & \text{on } \partial\Omega \times (0,1), \\ u_\epsilon(x,0) = \varphi_\epsilon(x) & \text{in } \Omega, \end{cases} \tag{3.1}$$

and u_ϵ blows up at $\tau = 1$. This implies that $\|\varphi_\epsilon\|_{L^\infty(\Omega)} \geqslant \kappa$ (see (2.3)). Furthermore, since the blow-up of the solution u is of type I, we have

$$d_* := \sup_{0<\epsilon<\epsilon_0} \|\varphi_\epsilon\|_{L^\infty(\Omega)} < \infty. \tag{3.2}$$

On the other hand, letting $\varphi = 0$ outside Ω, we apply the comparison principle to obtain

$$0 \leqslant u(x,t) \leqslant e^{M_\epsilon^{p-1} t} \big(e^{t\Delta} \varphi \big)(x) \quad \text{in } \Omega \times (0, T-\epsilon). \tag{3.3}$$

Furthermore, since $\varphi \in L^q(\mathbf{R}^N)$, for any $\delta > 0$, we take a sufficiently large R so that

$$\int_{\mathbf{R}^N \setminus B(0,R)} \big|\varphi(y)\big|^q \, dy \leqslant \delta.$$

This together with the Hölder inequality implies that

$$\begin{aligned} \big| \big(e^{t\Delta} \varphi \big)(x) \big|^q &\leqslant (4\pi t)^{-\frac{N}{2}} \int_{\mathbf{R}^N} e^{-\frac{|x-y|^2}{4t}} \big|\varphi(y)\big|^q \, dy \\ &= (4\pi t)^{-\frac{N}{2}} \bigg(\int_{B(0,R)} + \int_{\mathbf{R}^N \setminus B(0,R)} \bigg) e^{-\frac{|x-y|^2}{4t}} \big|\varphi(y)\big|^q \, dy \\ &\leqslant (4\pi t)^{-\frac{N}{2}} e^{-\frac{(|x|-R)^2}{4t}} \|\varphi\|_{L^q(\mathbf{R}^N)}^q + (4\pi t)^{-\frac{N}{2}} \delta \end{aligned} \tag{3.4}$$

for all $x \in \mathbf{R}^N \setminus B(0,R)$. Therefore, since δ is arbitrary, by (3.3) and (3.4) we have

$$\lim_{L \to \infty} \big\| u(T-\epsilon) \big\|_{L^\infty(\Omega \setminus B(0,L))} = 0.$$

Then we can take a positive constant L_ϵ satisfying

$$0 \leqslant \varphi_\epsilon(x) \leqslant \kappa/2 \tag{3.5}$$

for all $x \in \Omega$ with $|x| \geqslant L_\epsilon$. For any $x \in \mathbf{R}^N$, we put

$$\tilde{\varphi}_\epsilon(x) = \begin{cases} \|\varphi_\epsilon\|_{L^\infty(\Omega)} & \text{if } |x| \leqslant L_\epsilon, \\ -(|x| - L_\epsilon) + \|\varphi_\epsilon\|_{L^\infty(\Omega)} & \text{if } L_\epsilon < |x| \leqslant L_\epsilon + \|\varphi_\epsilon\|_{L^\infty(\Omega)} - \kappa/2, \\ \kappa/2 & \text{if } |x| > L_\epsilon + \|\varphi_\epsilon\|_{L^\infty(\Omega)} - \kappa/2. \end{cases}$$

Then we have

$$\tilde{\varphi}_\epsilon \in W^{1,\infty}\big(\mathbf{R}^N\big), \qquad \|\varphi_\epsilon\|_{L^\infty(\Omega)} = \|\tilde{\varphi}_\epsilon\|_{L^\infty(\mathbf{R}^N)}, \qquad \|\nabla \tilde{\varphi}_\epsilon\|_{L^\infty(\mathbf{R}^N)} \leqslant 1, \tag{3.6}$$

and by (3.5) we obtain

$$\varphi_\epsilon(x) \leqslant \tilde{\varphi}_\epsilon(x) \quad \text{in } \Omega. \tag{3.7}$$

Therefore, by (3.2), (3.6), and (3.7) we apply Proposition 2.1 with $\delta = \kappa/4$, and obtain

$$B(u) = B(u_\epsilon) \subset \{x \in \Omega\colon \tilde{\varphi}_\epsilon(x) \geqslant 3\kappa/4\} \subset B\big(0, L_\epsilon + \|\varphi_\epsilon\|_{L^\infty(\Omega)} - \kappa/2\big)$$

for all sufficiently small $\epsilon > 0$. This means that $B(u)$ is bounded, and the proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. We use the same notation as in the proof of Theorem 1.1. By (1.3) we have

$$\lim_{\epsilon\to 0} \epsilon^{\frac{1}{2}} \|\nabla\varphi_\epsilon\|_{L^\infty(\Omega)} = 0.$$

Then, for any $\eta > 0$, we apply Proposition 2.1 with $\tilde{\varphi}_\epsilon = \varphi_\epsilon$ and $\Omega' = \Omega$ to u_ϵ, and have

$$B(u_\epsilon) \subset M(\varphi_\epsilon, \eta), \quad 0 < \epsilon < \epsilon_0, \tag{3.8}$$

for some $\epsilon_0 > 0$. Therefore, since $B(u) = B(u_\epsilon)$, by (3.8) we have

$$B(u) \subset \bigcap_{0<\epsilon<\epsilon_0} M\big(\epsilon^{\frac{1}{p-1}} u(T-\epsilon), \eta\big).$$

This implies (1.4). Furthermore, by Lemma 2.1, we see that the blow-up of the solution u is of O.D.E. type, and Theorem 1.2 follows. □

Next we prove Corollary 1.1 by using Theorem 1.2 with the aid of blow-up estimates of the solutions.

Proof of Corollary 1.1. Let $\Omega = \{a < |x| < b\}$ with $0 < a < b < \infty$. Let u be a radially symmetric solution of (1.1) blowing up at $t = T$. Then, due to Theorem 1.2, it suffices to prove

$$\sup_{0<t<T} (T-t)^{\frac{1}{p-1}} \big\|u(t)\big\|_{L^\infty(\Omega)} < \infty, \tag{3.9}$$

$$\lim_{t\to T} (T-t)^{\frac{1}{p-1}+\frac{1}{2}} \big\|\nabla u(t)\big\|_{L^\infty(\Omega)} = 0. \tag{3.10}$$

We first prove (3.9) by the same argument as in the proof of [5, Theorem 2.1]. For any $t \in (0, T)$, we put

$$M(t) := \|u\|_{L^\infty(\Omega\times(0,t))}, \qquad \lambda(t) := M(t)^{-\frac{p-1}{2}}.$$

Since $M(t)$ is a positive, continuous, and nondecreasing function on $(0, T)$ such that $M(t) \to \infty$ as $t \to T$, we can define $\tau(t)$ by

$$\tau(t) := \max\{\tau \in (0, T)\colon M(\tau) = 2M(t)\}, \quad 0 < t < T.$$

Then, similarly to [5], it suffices to prove that there exists a constant K such that

$$\lambda(t)^{-2}\big(\tau(t) - t\big) \leqslant K, \quad t \in (T/2, T). \tag{3.11}$$

We prove (3.11) by contradiction. Assume that there exists a sequence $\{t_j\}$ such that

$$\lim_{j\to\infty} \lambda(t_j)^{-2}\big(\tau(t_j) - t_j\big) = \infty.$$

For any $j = 1, 2, \ldots,$ we take a sequence $\{(r_j, \hat{t}_j)\} \subset [a, b] \times (0, t_j]$ satisfying

$$u(r_j, \hat{t}_j) \geqslant \frac{1}{2} M(t_j).$$

Put $\lambda_j = \lambda(t_j)$ and

$$v_j(\tau, s) := \lambda_j^{\frac{2}{p-1}} u\big(\lambda_j\tau + r_j, \lambda_j^2 s + \hat{t}_j\big) \quad \text{for } (\tau, s) \in I_j \times \big(-\lambda_j^{-2}\hat{t}_j, \lambda_j^{-2}(T - \hat{t}_j)\big),$$

where $I_j := \{\tau \in \mathbf{R}\colon \lambda_j\tau + r_j \in (a, b)\}$. Then v_j satisfies

$$\partial_s v_j = \partial_\tau^2 v_j + \lambda_j \frac{N-1}{r_j + \lambda_j\tau} \partial_\tau v_j + v_j^p \quad \text{in } I_j \times \big(-\lambda_j^{-2}\hat{t}_j, \lambda_j^{-2}(T - \hat{t}_j)\big).$$

Furthermore, we have

$$0 \leqslant v_j \leqslant 2 \quad \text{in } I_j \times \left(-\lambda_j^{-2}\hat{t}_j, \lambda_j^{-2}\left(\tau(t_j) - \hat{t}_j\right)\right], \qquad v_j(0,0) \geqslant \frac{1}{2}.$$

Since

$$0 < a \leqslant r_j \leqslant b, \qquad \lim_{j\to\infty} \lambda_j = 0, \qquad \lim_{j\to\infty} \lambda_j^{-2}\left(\tau(t_j) - \hat{t}_j\right) = \infty,$$

by the same argument as in [5] we see that there exist an unbounded open interval H with $0 \in \overline{H}$ and a subsequence $\{v_{j'}\}$ of $\{v_j\}$ such that $\{v_{j'}\}$ converges to some function v in $C^{2,1}_{loc}(\overline{H} \times (-\infty, \infty))$ and

$$\partial_s v = \partial_\tau^2 v + v^p \quad \text{in } H \times (-\infty, \infty), \tag{3.12}$$

$$0 \leqslant v \leqslant 2 \quad \text{in } H \times (-\infty, \infty), \tag{3.13}$$

$$v(\tau, s) = 0 \quad \text{in } (-\infty, \infty) \text{ if } \tau \in \partial H, \tag{3.14}$$

$$v(0,0) \geqslant \frac{1}{2}. \tag{3.15}$$

Then, by (3.12)–(3.14) we apply [29, Theorems A and 2.1] to obtain $v \equiv 0$ in $\overline{H} \times (-\infty, \infty)$. This contradicts (3.15). Therefore (3.11) holds, and we have (3.9).

Next we follow an argument in [28], and prove (3.10) by contradiction. Assume that there exist a positive constant m and a sequence $\{(r_n, t_n)\} \subset [a, b] \times (0, T)$ such that $t_n \to T$ as $n \to \infty$ and

$$M_n := (T - t_n)^{\frac{p+1}{2(p-1)}} \left|\partial_r u(r_n, t_n)\right| \geqslant m > 0, \quad n = 1, 2, \ldots.$$

Put

$$\mu_n = (T - t_n)^{\frac{1}{2}} M_n^{-\frac{p-1}{p+1}}, \qquad w_n(\tau, s) = \mu_n^{\frac{2}{p-1}} u\left(r_n + \mu_n\tau, t_n + \mu_n^2 s\right) \quad \text{in } I_n \times (-\alpha_n, 0],$$

where $I_n = \{\tau \in \mathbf{R}\colon \mu_n\tau + r_n \in (a, b)\}$ and $\alpha_n = \mu_n^{-2} t_n$. Then w_n satisfies

$$\partial_s w_n = \partial_\tau^2 w_n + \mu_n \frac{N-1}{r_n + \mu_n\tau} \partial_\tau w_n + w_n^p$$

in $I_n \times (-\alpha_n, 0]$. By (3.9) we have

$$\begin{aligned} \left|w_n(\tau, s)\right| &\leqslant C\mu_n^{\frac{2}{p-1}} \left(T - t_n - \mu_n^2 s\right)^{-\frac{1}{p-1}} \\ &= C\mu_n^{\frac{2}{p-1}} \left(T - t_n - (T - t_n) M_n^{-\frac{2(p-1)}{p+1}} s\right)^{-\frac{1}{p-1}} \\ &= C\left(M_n^{\frac{2(p-1)}{p+1}} - s\right)^{-\frac{1}{p-1}} \leqslant C\left(m^{\frac{2(p-1)}{p+1}} - s\right)^{-\frac{1}{p-1}} \leqslant C(-s)^{-\frac{1}{p-1}} \end{aligned}$$

for all $\tau \in I_n$ and $s \in (-\alpha_n, 0]$, where C is a constant. Then there exist an unbounded open interval I with $0 \in \bar{I}$ and a subsequence $\{w_{n'}\}$ of $\{w_n\}$ such that $\{w_{n'}\}$ converges to some function w in $C^{2,1}_{loc}(\bar{I} \times (-\infty, 0])$ and w satisfies

$$\partial_s w = \partial_\tau^2 w + w^p \quad \text{in } I \times (-\infty, 0], \qquad w(\tau, s) = 0 \quad \text{in } (-\infty, 0] \text{ if } \tau \in \partial I. \tag{3.16}$$

Therefore, by [27, Corollary 1] (see also [26, Corollary 1.6]) we have

$$w(\tau, s) \equiv 0 \quad \text{or} \quad w(\tau, s) = \kappa(T_0 - s)^{-1/(p-1)} \quad \text{for some } T_0 \geqslant 0.$$

On the other hand, since $|\partial_\tau w_n(0,0)| = 1$ for all n, we have $|(\nabla w)(0,0)| = 1$. This is a contradiction. Thus we have (3.10). Therefore we have (3.9) and (3.10), and the proof of Corollary 1.1 is complete. □

By Theorems 1.1 and 1.2 we can obtain the following result.

Theorem 3.1. *Let Ω be a (possibly unbounded) smooth domain in $\mathbf{R}^N$. Let u be a solution of (1.1) which exhibits type I blow-up at $t = T$. Assume*

$$\varphi \in L^\infty(\Omega) \cap L^q(\Omega) \quad \textit{for some } q \in [1, \infty), \qquad (N-2)p < N+2.$$

Then the blow-up set $B(u)$ is compact in Ω. In particular, $B(u) \cap \partial\Omega = \emptyset$.

Proof. By Theorem 1.1 we can find a positive constant R satisfying

$$\sup_{x\in\Omega\setminus B(0,R),\, t\in(0,T)} \left|u(x,t)\right| < \infty, \tag{3.17}$$

and obtain

$$B(u) \subset \overline{\Omega} \cap B(0,R). \tag{3.18}$$

Then, by (3.17) we apply the gradient estimates for parabolic equations to obtain

$$\left|\nabla u(x,t)\right| \leqslant C \tag{3.19}$$

for all $x \in \Omega \setminus B(0,R+1)$ and $t \in (0,T)$, where C is a constant. Furthermore, the solution u satisfies (1.3). Indeed, if not, there exist a positive constant m and a sequence $\{(x_n,t_n)\} \subset \overline{\Omega} \times (0,T)$ such that

$$M_n := (T-t_n)^{\frac{p+1}{2(p-1)}} \left|\nabla u(x_n,t_n)\right| \geqslant m > 0, \qquad n = 1,2,\ldots.$$

By (3.19) we can assume that $\{x_n\} \subset \Omega \cap B(0,R+1)$. Then, by using the similar argument as in the proof of [28, Theorem 2.1] with the aid of the Liouville type theorem (see [26] and [27]) we can obtain a contradiction (see also the proof of Corollary 1.1). Therefore, by Theorem 1.2 we have $B(u) \cap \partial\Omega = \emptyset$. This together with (3.18) implies that $B(u)$ is compact in Ω, and Theorem 3.1 follows. □

4. Proof of Theorem 1.3

In this section we prove Theorem 1.3 by using Proposition 2.1 and Corollary 1.1. In order to prove Theorem 1.3, we prepare the following lemma.

Lemma 4.1. *Let $\epsilon_0 > 0$ and $\{M_\epsilon\}_{0<\epsilon<\epsilon_0} \subset (0,\infty)$ be such that*

$$0 < \inf_{0<\epsilon<\epsilon_0} M_\epsilon \leqslant \sup_{0<\epsilon<\epsilon_0} M_\epsilon < \infty.$$

Let $\Omega = \{x \in \mathbf{R}^N\colon\ R_1 < |x| < R_2\}$ with $0 < R_1 < R_2 < \infty$. For any $\epsilon \in (0,\epsilon_0)$, let u_ϵ be the blowing up solution of

$$\begin{cases} \partial_t u = \epsilon\Delta u + u^p & \text{in } \Omega \times (0,T_\epsilon),\\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T_\epsilon),\\ u(x,0) = M_\epsilon & \text{in } \Omega, \end{cases}$$

where T_ϵ is the blow-up time of u_ϵ. Then there exists a constant $\epsilon_1 \in (0,\epsilon_0)$ such that

$$\sup_{0<\epsilon<\epsilon_1} \limsup_{t\to T_\epsilon} (T_\epsilon - t)^{\frac{1}{p-1}} \left\|u_\epsilon(t)\right\|_{L^\infty(\Omega)} < \infty, \tag{4.1}$$

$$\lim_{t\to T_\epsilon} \epsilon^{\frac12}(T_\epsilon - t)^{\frac{1}{p-1}+\frac12} \left\|\nabla u_\epsilon(t)\right\|_{L^\infty(\Omega)} = 0 \quad \textit{uniformly for } \epsilon \in (0,\epsilon_1). \tag{4.2}$$

Proof. We prove Lemma 4.1 by modifying the arguments in the proof of Corollary 1.1. We first prove (4.1). Let $\epsilon_1 \in (0,\epsilon_0)$ be a sufficiently small constant. Then, by [8, Proposition 2.1] we have

$$0 < \inf_{0<\epsilon<\epsilon_1} T_\epsilon \leqslant \sup_{0<\epsilon<\epsilon_1} T_\epsilon < \infty. \tag{4.3}$$

For any $t \in (0,T_\epsilon)$, put

$$M_\epsilon(t) := \|u_\epsilon\|_{L^\infty(\Omega\times(0,t))}, \qquad \lambda_\epsilon(t) := M_\epsilon(t)^{-\frac{p-1}{2}}.$$

Then, for any $t \in (0,T_\epsilon)$, we define $\tau_\epsilon(t)$ by

$$\tau_\epsilon(t) := \max\bigl\{\tau \in (0,T_\epsilon)\colon\ M_\epsilon(\tau) = 2M_\epsilon(t)\bigr\}.$$

Similarly to (3.11), we prove by contradiction that there exists a positive constant K such that

$$\lambda_\epsilon(t)^{-2}\bigl(\tau_\epsilon(t) - t\bigr) \leqslant K \tag{4.4}$$

for all $t \in (T_\epsilon/2, T_\epsilon)$ and all $\epsilon \in (0, \epsilon_1)$. Assume that there exist sequences $\{\epsilon_j\} \subset (0, \epsilon_1)$ and $\{t_j\} \subset (0, T_{\epsilon_j})$ such that

$$\lim_{j\to\infty} \epsilon_j = 0, \qquad \lim_{j\to\infty} \lambda_{\epsilon_j}(t_j)^{-2}\bigl(\tau_{\epsilon_j}(t_j) - t_j\bigr) = \infty.$$

For any $j = 1, 2, \dots,$ we can take a point $(r_j, \hat{t}_j) \in [R_1, R_2] \times (0, t_j]$ such that

$$u_{\epsilon_j}(r_j, \hat{t}_j) \geqslant \frac{1}{2} M_{\epsilon_j}(t_j).$$

Put $\lambda_j = \lambda_{\epsilon_j}(t_j)$ and

$$v_j(\tau, s) := \lambda_j^{\frac{2}{p-1}} u_{\epsilon_j}\bigl(\epsilon_j^{\frac{1}{2}} \lambda_j \tau + r_j, \lambda_j^2 s + \hat{t}_j\bigr) \quad \text{for } (\tau, s) \in I_j \times \bigl(-\lambda_j^{-2}\hat{t}_j, \lambda_j^{-2}(T_{\epsilon_j} - \hat{t}_j)\bigr),$$

where $I_j := \{\tau \in \mathbf{R}\colon\ \epsilon_j^{\frac{1}{2}} \lambda_j \tau + r_j \in (R_1, R_2)\}$. Then v_j satisfies

$$\partial_s v_j = \partial_\tau^2 v_j + \epsilon_j^{\frac{1}{2}} \lambda_j \frac{N-1}{r_j + \epsilon^{\frac{1}{2}} \lambda_j \tau} \partial_\tau v_j + v_j^p \quad \text{in } I_j \times \bigl(-\lambda_j^{-2}\hat{t}_j, \lambda_j^{-2}(T_{\epsilon_j} - \hat{t}_j)\bigr)$$

and

$$0 \leqslant v_j \leqslant 2 \quad \text{in } I_j \times \bigl(-\lambda_j^{-2}\hat{t}_j, \lambda_j^{-2}\bigl(\tau(t_j) - \hat{t}_j\bigr)\bigr], \qquad v_j(0,0) \geqslant \frac{1}{2}.$$

Then, by the similar argument as in the proof of (3.9) we obtain (3.12)–(3.15), which yield a contradiction. Therefore we have (4.4), which implies (4.1).

Next we prove (4.2) by contradiction. Assume that there exist sequences $\{\epsilon_n\} \subset (0, \epsilon_1)$ and $\{(r_n, t_n)\} \subset I \times (0, T_{\epsilon_n})$ and a positive constant m such that $\epsilon_n \to 0$, $|t_n - T_{\epsilon_n}| \to 0$ as $n \to \infty$, and

$$M_n := \epsilon_n^{\frac{1}{2}} (T_{\epsilon_n} - t_n)^{\frac{1}{p-1} + \frac{1}{2}} \bigl|\partial_r u_{\epsilon_n}(r_n, t_n)\bigr| \geqslant m > 0, \quad n = 1, 2, \dots.$$

Put

$$\mu_n := (T_{\epsilon_n} - t_n)^{\frac{1}{2}} M_n^{-\frac{p-1}{p+1}}, \qquad w_n(\tau, s) := \mu_n^{\frac{2}{p-1}} u_{\epsilon_n}\bigl(r_n + \epsilon_n^{\frac{1}{2}} \mu_n \tau, t_n + \mu_n^2 s\bigr) \quad \text{in } I_n \times (-\alpha_n, 0],$$

where $I_n = \{\tau \in \mathbf{R}\colon\ \epsilon_n^{\frac{1}{2}} \mu_n \tau + r_n \in (R_1, R_2)\}$ and $\alpha_n = \mu_n^{-2} t_n$. Then we have

$$\partial_s w_n = \partial_r^2 w_n + \epsilon_n^{\frac{1}{2}} \mu_n \frac{N-1}{r_n + \epsilon_n^{\frac{1}{2}} \mu_n \tau} \partial_r w_n + w_n^p$$

in $I_n \times (-\alpha_n, 0]$. On the other hand, by (4.1) we have

$$\begin{aligned} \bigl|w_n(\tau, s)\bigr| &\leqslant C \mu_n^{\frac{2}{p-1}} \bigl(T_{\epsilon_n} - t_n - \mu_n^2 s\bigr)^{-\frac{1}{p-1}} \\ &= C \mu_n^{\frac{2}{p-1}} \bigl(T_{\epsilon_n} - t_n - (T_{\epsilon_n} - t_n) M_n^{-\frac{2(p-1)}{p+1}} s\bigr)^{-\frac{1}{p-1}} \\ &= C \bigl(M_n^{\frac{2(p-1)}{p+1}} - s\bigr)^{-\frac{1}{p-1}} \leqslant C \bigl(m^{\frac{2(p-1)}{p+1}} - s\bigr)^{-\frac{1}{p-1}} \leqslant C(-s)^{-\frac{1}{p-1}} \end{aligned}$$

for all $(\tau, s) \in I_n \times (-\alpha_n, 0]$. Then, by the similar argument as in the proof of (3.10) we obtain (3.16), which yields a contradiction. Therefore we have (4.2), and the proof of Lemma 4.1 is complete. □

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. The proof is by contradiction. Let u be a solution of (1.1) which exhibits O.D.E. type blow-up at $t = T$. Assume that there exists a point

$$a \in B(u) \cap \partial\Omega. \tag{4.5}$$

Since Ω satisfies the exterior sphere condition, there exist a point $x_0 \in \mathbf{R}^N$ and positive constants R_1 and R_2 such that

$$a \in \partial B(x_0, R_1), \qquad B(x_0, R_1) \cap \Omega = \emptyset, \qquad \Omega \subset \Omega' := \bigl\{x \in \mathbf{R}^N\colon\ R_1 < |x - x_0| < R_2\bigr\}.$$

In what follows, we can assume, without loss of generality, that $x_0 = 0$. Let ϵ be a sufficiently small positive constant and put

$$u_\epsilon(x,\tau) := \epsilon^{\frac{1}{p-1}} u(x, T-\epsilon+\epsilon\tau), \qquad \varphi_\epsilon(x) := \epsilon^{\frac{1}{p-1}} u(x, T-\epsilon).$$

Then u_ϵ satisfies (3.1). Furthermore, since the blow-up of the solution u is of O.D.E. type, there holds

$$\lim_{\epsilon\to 0} \|\varphi_\epsilon\|_{L^\infty(\Omega)} = \lim_{t\to T} (T-t)^{\frac{1}{p-1}} \big\|u(t)\big\|_{L^\infty(\Omega)} = \kappa. \tag{4.6}$$

Let $v_\epsilon = v_\epsilon(x,\tau)$ be a radially symmetric blowing up solution of

$$\begin{cases} \partial_\tau v = \epsilon\Delta v + v^p & \text{in } \Omega' \times (0, T_\epsilon), \\ v(x,\tau) = 0 & \text{on } \partial\Omega' \times (0, T_\epsilon), \\ v(x,0) = \|\varphi_\epsilon\|_{L^\infty(\Omega)} & \text{in } \Omega', \end{cases} \tag{4.7}$$

where T_ϵ is the blow-up time of v_ϵ. Then the comparison principle together with (4.6) implies

$$0 \leqslant u_\epsilon \leqslant v_\epsilon \quad \text{in } \Omega \times (0, T_\epsilon), \qquad \frac{1}{2} < S_{\|\varphi_\epsilon\|_{L^\infty(\Omega)}} \leqslant T_\epsilon \leqslant 1, \tag{4.8}$$

for all sufficiently small $\epsilon > 0$. By (1.2), (4.6), and (4.8) we have

$$\nu_\epsilon := 2\max\{1 - T_\epsilon, \epsilon\} \to 0 \quad \text{as } \epsilon \to 0. \tag{4.9}$$

Furthermore, by Lemma 4.1 we can find a positive constant ϵ_1 such that

$$\sup_{0<\epsilon<\epsilon_1} \sup_{0<\tau<T_\epsilon} (T_\epsilon-\tau)^{\frac{1}{p-1}} \big\|v_\epsilon(\tau)\big\|_{L^\infty(\Omega')} < \infty, \tag{4.10}$$

$$\lim_{\tau\to T_\epsilon} \epsilon^{\frac{1}{2}} (T_\epsilon-\tau)^{\frac{1}{p-1}+\frac{1}{2}} \big\|\nabla v_\epsilon(\tau)\big\|_{L^\infty(\Omega')} = 0 \tag{4.11}$$

uniformly for all $\epsilon \in (0, \epsilon_1)$. Put

$$u_\epsilon^*(x,s) := \nu_\epsilon^{\frac{1}{p-1}} u_\epsilon(x, 1-\nu_\epsilon+\nu_\epsilon s), \qquad \varphi_\epsilon^*(x) := \nu_\epsilon^{\frac{1}{p-1}} u_\epsilon^*(x, 1-\nu_\epsilon), \qquad \tilde\varphi_\epsilon(x) := \nu_\epsilon^{\frac{1}{p-1}} v_\epsilon(x, 1-\nu_\epsilon).$$

Then $u_\epsilon^* = u_\epsilon^*(x,s)$ is a solution of

$$\begin{cases} \partial_s u = \epsilon\nu_\epsilon\Delta u + u^p & \text{in } \Omega \times (0,1), \\ u(x,s) = 0 & \text{on } \partial\Omega \times (0,1), \\ u(x,0) = \varphi_\epsilon^*(x) & \text{in } \Omega, \end{cases} \tag{4.12}$$

and blows up at $s = 1$. Furthermore, it holds

$$0 \leqslant \varphi_\epsilon^*(x) \leqslant \tilde\varphi_\epsilon(x) \quad \text{in } \Omega. \tag{4.13}$$

On the other hand, it follows from (4.9) that

$$T_\epsilon - (1-\nu_\epsilon) \geqslant \frac{\nu_\epsilon}{2},$$

and by (4.10) we have

$$\begin{aligned} \limsup_{\epsilon\to 0} \|\tilde\varphi_\epsilon\|_{L^\infty(\Omega')} &= \limsup_{\epsilon\to 0} \nu_\epsilon^{\frac{1}{p-1}} \big\|v_\epsilon(1-\nu_\epsilon)\big\|_{L^\infty(\Omega')} \\ &\leqslant \limsup_{\epsilon\to 0} 2^{\frac{1}{p-1}} \big(T_\epsilon - (1-\nu_\epsilon)\big)^{\frac{1}{p-1}} \big\|v_\epsilon(1-\nu_\epsilon)\big\|_{L^\infty(\Omega')} < \infty. \end{aligned} \tag{4.14}$$

Similarly, by (4.11) we have

$$\begin{aligned} \lim_{\epsilon\to 0} (\epsilon\nu_\epsilon)^{\frac{1}{2}} \|\nabla\tilde\varphi_\epsilon\|_{L^\infty(\Omega')} &= \lim_{\epsilon\to 0} \epsilon^{\frac{1}{2}} \nu_\epsilon^{\frac{1}{2}+\frac{1}{p-1}} \big\|\nabla v_\epsilon(1-\nu_\epsilon)\big\|_{L^\infty(\Omega')} \\ &\leqslant \lim_{\epsilon\to 0} 2^{\frac{1}{2}+\frac{1}{p-1}} \epsilon^{\frac{1}{2}} \big(T_\epsilon - (1-\nu_\epsilon)\big)^{\frac{1}{2}+\frac{1}{p-1}} \big\|\nabla v_\epsilon(1-\nu_\epsilon)\big\|_{L^\infty(\Omega')} = 0. \end{aligned} \tag{4.15}$$

Furthermore, since the blow-up of u is of type I, there exists a constant C such that

$$(1-s)^{\frac{1}{p-1}}\left\|u_\epsilon^*(s)\right\|_{L^\infty(\Omega)}=(1-s)^{\frac{1}{p-1}}\nu_\epsilon^{\frac{1}{p-1}}\epsilon^{\frac{1}{p-1}}\left\|u\big(T-\epsilon+\epsilon(1-\nu_\epsilon+\nu_\epsilon s)\big)\right\|_{L^\infty(\Omega)}$$
$$\leqslant C(1-s)^{\frac{1}{p-1}}\nu_\epsilon^{\frac{1}{p-1}}\epsilon^{\frac{1}{p-1}}\cdot\big(\epsilon\nu_\epsilon(1-s)\big)^{-\frac{1}{p-1}}=C \tag{4.16}$$

for all $s\in(0,1)$. Therefore, by (4.9), (4.13), (4.14), (4.15), and (4.16) we apply Proposition 2.1 to u_ϵ^*, which is a solution of problem (4.12), and obtain

$$B(u)\subset\big\{x\in\overline{\Omega'}:\ \tilde\varphi_\epsilon(x)\geqslant\kappa/2\big\} \tag{4.17}$$

for all sufficiently small $\epsilon>0$. Here we remark that the blow-up set of u_ϵ^* coincides with $B(u)$. On the other hand, since $a\in\partial\Omega'$, we have $\tilde\varphi_\epsilon=0$ at $x=a$ and

$$a\notin\big\{x\in\overline{\Omega'}:\ \tilde\varphi_\epsilon(x)\geqslant\kappa/2\big\}$$

for all sufficiently small $\epsilon>0$. This together with (4.17) implies $a\notin B(u)$. This contradicts (4.5), and Theorem 1.3 follows. □

References

[1] X.-Y. Chen, H. Matano, Convergence, asymptotic periodicity, and finite-point blow-up in one-dimensional semilinear heat equations, J. Differential Equations 78 (1989) 160–190.
[2] C. Cortazar, M. Elgueta, J.D. Rossi, The blow-up problem for a semilinear parabolic equation with a potential, J. Math. Anal. Appl. 335 (2007) 418–427.
[3] F. Dickstein, Blowup stability of solutions of the nonlinear heat equation with a large life span, J. Differential Equations 223 (2006) 303–328.
[4] M. Fila, M. Winkler, Single-point blow-up on the boundary where the zero Dirichlet boundary condition is imposed, J. Eur. Math. Soc. 10 (2008) 105–132.
[5] M. Fila, P. Souplet, The blow-up rate for semilinear parabolic problems on general domains, NoDEA Nonlinear Differential Equations Appl. 8 (2001) 473–480.
[6] A. Friedman, B. McLeod, Blow-up of positive solutions of semilinear heat equations, Indiana Univ. Math. J. 34 (1985) 425–447.
[7] Y. Fujishima, Location of the blow-up set for a superlinear heat equation with small diffusion, Differential Integral Equations 25 (2012) 759–786.
[8] Y. Fujishima, K. Ishige, Blow-up set for a semilinear heat equation with small diffusion, J. Differential Equations 249 (2010) 1056–1077.
[9] Y. Fujishima, K. Ishige, Blow-up for a semilinear parabolic equation with large diffusion on $\mathbf{R}^N$, J. Differential Equations 250 (2011) 2508–2543.
[10] Y. Fujishima, K. Ishige, Blow-up for a semilinear parabolic equation with large diffusion on $\mathbf{R}^N$. II, J. Differential Equations 252 (2012) 1835–1861.
[11] Y. Fujishima, K. Ishige, Blow-up set for a semilinear heat equation and pointedness of the initial data, Indiana Univ. Math. J., in press.
[12] V.A. Galaktionov, S.P. Kurdyumov, A.A. Samarskii, Asymptotic stability of invariant solutions of nonlinear equations of heat conduction with a source, Differential Equations 20 (1984) 461–476.
[13] V.A. Galaktionov, S.A. Posashkov, The equation $u_t=u_{xx}+u^\beta$. Localization, asymptotic behavior of unbounded solutions, Keldysh Inst. Appl. Math. Acad. Sci., USSR, preprint No. 97, 1985.
[14] Y. Giga, R.V. Kohn, Asymptotically self-similar blow-up of semilinear heat equations, Comm. Pure Appl. Math. 38 (1985) 297–319.
[15] Y. Giga, R.V. Kohn, Characterizing blowup using similarity variables, Indiana Univ. Math. J. 36 (1987) 1–40.
[16] Y. Giga, R.V. Kohn, Nondegeneracy of blowup for semilinear heat equations, Comm. Pure Appl. Math. 42 (1989) 845–884.
[17] Y. Giga, S. Matsui, S. Sasayama, Blow up rate for semilinear heat equations with subcritical nonlinearity, Indiana Univ. Math. J. 53 (2004) 483–514.
[18] K. Ishige, Blow-up time and blow-up set of the solutions for semilinear heat equations with large diffusion, Adv. Difference Equ. 8 (2002) 1003–1024.
[19] K. Ishige, N. Mizoguchi, Blow-up behavior for semilinear heat equations with boundary conditions, Differential Integral Equations 16 (2003) 663–690.
[20] K. Ishige, N. Mizoguchi, Location of blow-up set for a semilinear parabolic equation with large diffusion, Math. Ann. 327 (2003) 487–511.
[21] K. Ishige, H. Yagisita, Blow-up problems for a semilinear heat equation with large diffusion, J. Differential Equations 212 (2005) 114–128.
[22] H. Matano, F. Merle, Classification of type I and type II behaviors for a supercritical nonlinear heat equation, J. Funct. Anal. 256 (2009) 992–1064.
[23] H. Matano, F. Merle, Threshold and generic type I behaviors for a supercritical nonlinear heat equation, J. Funct. Anal. 261 (2011) 716–748.
[24] F. Merle, Solution of a nonlinear heat equation with arbitrarily given blow-up points, Comm. Pure Appl. Math. 45 (1992) 263–300.
[25] F. Merle, H. Zaag, Stability of the blow-up profile for equations of the type $u_t=\Delta u+|u|^{p-1}u$, Duke Math. J. 86 (1997) 143–195.
[26] F. Merle, H. Zaag, Optimal estimates for blowup rate and behavior for nonlinear heat equations, Comm. Pure Appl. Math. 51 (1998) 139–196.
[27] F. Merle, H. Zaag, A Liouville theorem for vector-valued nonlinear heat equations and applications, Math. Ann. 316 (2000) 103–137.

[28] N. Mizoguchi, Blowup rate of solutions for a semilinear heat equation with the Dirichlet boundary condition, Asymptot. Anal. 35 (2003) 91–112.
[29] P. Polàčik, P. Quittner, P. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems. Part II: parabolic equations, Indiana Univ. Math. J. 56 (2007) 879–908.
[30] P. Quittner, P. Souplet, Superlinear Parabolic Problems, Blow-up, Global Existence and Steady States, Birkhäuser Adv. Texts, Basler Lehrbücher Birkhäuser Verlag, Basel, 2007.
[31] J.J.L. Velázquez, Estimates on the $(n-1)$-dimensional Hausdorff measure of the blow-up set for a semilinear heat equation, Indiana Univ. Math. J. 42 (1993) 445–476.
[32] F.B. Weissler, Single point blow-up for a semilinear initial value problem, J. Differential Equations 55 (1984) 204–224.
[33] H. Yagisita, Blow-up profile of a solution for a nonlinear heat equation with small diffusion, J. Math. Soc. Japan 56 (2004) 993–1005.
[34] H. Zaag, On the regularity of the blow-up set for semilinear heat equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002) 505–542.
[35] H. Zaag, Determination of the curvature of the blow-up set and refined singular behavior for a semilinear heat equation, Duke Math. J. 133 (2006) 499–525.