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Traveling wave solutions of advection–diffusion equations with nonlinear diffusion

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Abstract

We study the existence of particular traveling wave solutions of a nonlinear parabolic degenerate diffusion equation with a shear flow. Under some assumptions we prove that such solutions exist at least for propagation speeds $c \in]c_*, +\infty[$, where $c_* > 0$ is explicitly computed but may not be optimal. We also prove that a free boundary hypersurface separates a region where $u = 0$ and a region where $u > 0$, and that this free boundary can be globally parametrized as a Lipschitz continuous graph under some additional non-degeneracy hypothesis; we investigate solutions which are, in the region $u > 0$, planar and linear at infinity in the propagation direction, with slope equal to the propagation speed.

Résumé

Nous étudions l'existence d'une classe particulière de solutions d'ondes pour une équation non linéaire parabolique dégénérée en présence d'un écoulement cisaillé. Sous certaines hypothèses nous prouvons que ces solutions existent au moins pour des vitesses de propagation $c \in]c^*, +\infty[$, où $c^* > 0$ est une vitesse critique calculée explicitement en fonction de l'écoulement mais peut-être pas optimale. Nous prouvons également qu'une hypersurface de frontière libre sépare une zone $u = 0$ d'une zone $u > 0$ et que, sous une hypothèse supplémentaire de non-dégénérescence, cette frontière peut être globalement paramétrée comme un graphe lipschitzien. Nous nous intéressons à des solutions qui, à l'infini dans la direction de propagation, sont planes et linéaires avec pente égale à la vitesse.

1. Introduction

Consider the advection–diffusion equation

$$\partial_t T - \nabla \cdot (\lambda \nabla T) + \nabla \cdot (VT) = 0, \qquad (t, X) \in \mathbb{R}^+ \times \mathbb{R}^d, \tag{1.1}$$

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where $T \geqslant 0$ is temperature, $\lambda \geqslant 0$ the diffusion coefficient and $V = V(x_1, \dots, x_d) \in \mathbb{R}^d$ a prescribed flow. In the context of high temperature hydrodynamics, the heat conductivity λ cannot be assumed to be constant as for the usual heat equation, but is rather of the form $\lambda = \lambda(T) = \lambda_0 T^m$ for some constant $\lambda_0 > 0$ and conductivity exponent $m > 0$ depending on the model, see [17]. We will consider here the case $m \neq 1$. In Physics of Plasmas and particularly in the context of Inertial Confinement Fusion, the dominant mechanism of heat transfer is the so-called electronic Spitzer heat diffusivity, corresponding to $m = 5/2$ in the formula above.

Suitably rescaling one may set $\lambda_0 = m + 1$, yielding the nonlinear parabolic equation

$$\partial_t T - \Delta\big(T^{m+1}\big) + \nabla \cdot (VT) = 0. \tag{1.2}$$

When temperature takes negligible values, say $T = \varepsilon \to 0$, then the diffusion coefficient $\lambda(T) = \lambda_0 T^m$ may vanish and the equation becomes degenerate. As a result free boundaries may arise. We are interested here in traveling waves with such free boundaries $\Gamma := \partial\{T > 0\} \neq \emptyset$, and in addition $T \to +\infty$ in the propagation direction.

When $V \equiv 0$ (1.2) is usually called the Porous Medium Equation – PME in short –

$$\partial_t T - \Delta\big(T^{m+1}\big) = 0 \tag{PME}$$

and has been widely studied in the literature. We refer the reader to the book [16] for general references on this topic and to [1–3] for well-posedness of the Cauchy problem and regularity questions. As for most of the free boundary scenarios, we do not expect smooth solutions to exist, since along the free boundary a gradient discontinuity may occur: a main difficulty is to develop a suitable notion of viscosity and/or weak solutions; see [13] for a general theory of viscosity solutions and [6] in the particular case of the PME, [16] for weak solutions.

The question of parametrization, time evolution and regularity of the free boundary for (PME) is not trivial. It has been studied in detail in [8–10]. When the flow is potential $V = \nabla \Phi$ (1.2) has recently been studied in [15], where the authors investigate the long time asymptotics of the free boundary for compactly supported solutions.

We consider here a two-dimensional periodic incompressible shear flow

$$V(x, y) = \begin{pmatrix} \alpha(y) \\ 0 \end{pmatrix}, \qquad \alpha(y + 1) = \alpha(y)$$

for a sufficiently smooth $\alpha(y)$, which we normalize to be mean-zero $\int_0^1 \alpha(y)\,\mathrm{d}y = 0$. In this setting (1.2) becomes the following advection–diffusion equation

$$\partial_t T - \Delta\big(T^{m+1}\big) + \alpha(y)\partial_x T = 0 \tag{AD-E}$$

with 1-periodic boundary conditions in the y direction.

Remark 1. For simplicity and concreteness all our results will be stated here in two dimensions $(x, y) \in \mathbb{R} \times \mathbb{R}$, but may easily be extended to the d-dimensional case $(x, y) \in \mathbb{R} \times \mathbb{R}^{d-1}$. All the proofs below will indeed apply to the letter by considering separately each component of ∇_y instead of simply ∂_y.

For physically relevant temperature $T \geqslant 0$ it is standard to use the so-called pressure variable $u = \frac{m+1}{m} T^m$, which satisfies

$$\partial_t u - m u \Delta u + \alpha(y)\partial_x u = |\nabla u|^2. \tag{1.3}$$

Remark 2. When $m = 1$ pressure u is proportional to temperature T, and this particular case will not be considered here.

Looking for wave solutions $u(t, x, y) = p(x + ct, y)$ yields the following equation for the wave profiles $p(x, y)$

$$-mp\Delta p + (c + \alpha)p_x = |\nabla p|^2, \quad (x, y) \in \mathbb{R} \times \mathbb{T}^1. \tag{1.4}$$

In the case of a trivial flow $\alpha \equiv 0$ it is well known [16] that for any prescribed propagation speed $c > 0$ there exists a corresponding planar viscosity solution given by

$$p(x, y) = p_c(x) = c[x]^+, \tag{1.5}$$

where $[.]^+$ denotes the positive part. This profile is trivial for $x \leqslant 0$ and linear for $x > 0$, with slope exactly equal to the speed c. The free boundary $\Gamma = \{x = 0\}$ moves in the original frame with constant speed $x(t) = -ct + cst$, and the slope at infinity therefore fully determines the propagation.

In this particular case the free boundary is non-degenerate, $\nabla p = (c, 0) \neq 0$ in the "hot" region $p > 0$. The differential equation satisfied by the free boundary Γ in the general case was specified in [10], where the authors also show that if the initial free boundary is non-degenerate then it starts to move immediately with normal velocity $V = -\nabla p|_\Gamma$.

In presence of a nontrivial flow $\alpha \neq 0$ a natural question to ask is whether (AD-E) can be considered as a perturbation of (PME). More specifically we are interested here in the following questions: (i) do y-periodic traveling waves behaving linearly at infinity and possessing free boundaries still exist? (ii) If so for which propagation speeds $c > 0$, and is it still true that the slope at infinity equals the speed c? (iii) Is the free boundary wrinkled by the flow and how can we parametrize it? (iv) Is the free boundary non-degenerate and what is its regularity?

We answer the first three questions, at least for propagation speeds large enough.

Main Theorem 1. *Let $c_* := -\min \alpha > 0$: for any $c > c_*$ there exists a nontrivial traveling wave profile, which is a continuous viscosity solution $p(x, y) \geqslant 0$ of* (1.4) *on the infinite cylinder. This profile satisfies*

1. *if $D^+ := \{p > 0\}$ denotes the positive set, we have $D^+ \neq \emptyset$ and $p|_{D^+} \in \mathcal{C}^\infty(D^+)$,*
2. *p is globally Lipschitz,*
3. *p is planar and linear $p(x, y) \sim cx$ uniformly in y when $x \to +\infty$.*

Moreover there exists a free boundary $\Gamma = \partial(D^+) \neq \emptyset$ which can be parametrized as follows: there exists an upper semi-continuous function $I(y)$ such that $p(x, y) > 0 \Leftrightarrow x > I(y)$. Further:

- *If y_0 is a continuity point of I then $\Gamma \cap \{y = y_0\} = (I(y_0), y_0)$.*
- *If y_0 is a discontinuity point then $\Gamma \cap \{y = y_0\} = [\underline{I}(y_0), I(y_0)] \times \{y = y_0\}$, where $\underline{I}(y_0) := \liminf_{y \to y_0} I(y)$.*

The question (iv) is still open. The non-degeneracy of the pressure at the free boundary $\nabla p|_\Gamma \neq 0$ and the free boundary regularity are closely related. For the PME it was discussed in [8–10]. We have, however, a partial answer under some strong non-degeneracy assumption.

Proposition 1.1. *With the same hypotheses as in Main Theorem 1, assume the additional non-degeneracy condition $p_x \geqslant a > 0$ holds in D^+ for some constant a. Then the function $I(y)$ defined in Main Theorem 1 is Lipschitz on the torus, and the free boundary $\Gamma = \partial\{p > 0\}$ is the graph $x = I(y)$.*

Remark 3. The condition of linear growth at infinity is natural because it mimics the planar traveling wave (1.5) for the PME. Let us also point out, a posteriori, that this linearity appears very naturally in our proof, see Section 5.

Recalling that we normalized $\int_{\mathbb{T}^1} \alpha(y)\,\mathrm{d}y = 0$, we will always assume in the following that the propagation speed $c > 0$ is large enough such that

$$0 < c_0 \leqslant c + \alpha \leqslant c_1 \tag{1.6}$$

for some constants c_0, c_1. This is indeed consistent with $c > c_* = -\min \alpha > 0$ in the Main Theorem 1.

The method of proof of Main Theorem 1 is standard. We refer the reader to [4] for a general review of this method and to [8] for the special case of the PME. The proof relies on a simple observation: if $p \geqslant \delta > 0$ (1.4) is uniformly elliptic; we shall refer in the sequel to any solution $p \geqslant \delta > 0$ as a δ-solution. The main steps are the following.

We first regularize (1.4) by considering its δ-solutions with $\delta \ll 1$ on finite cylinders $[-L, L] \times \mathbb{T}^1$, $L \gg 1$ with suitable boundary conditions. In Section 2 we solve this regularized uniformly elliptic problem, and derive monotonicity estimates of its solution in the x direction. In Section 3 we take the limit $L \to +\infty$ for fixed $\delta > 0$, and establish

Theorem 1.1. *For any $\delta > 0$ small enough there exists a smooth δ-solution $p \geqslant \delta$ on the infinite cylinder such that $\lim_{x \to -\infty} p(x, y) = \delta$ and $p(x, y) \underset{x \to +\infty}{\sim} cx$ uniformly in y.*

The linear behavior will be actually proved in Section 5 for the final viscosity solution, but the proof can be easily adapted for the δ-solutions. We complete the proof of parts 1 and 2 of Theorem 1 in Section 4 by taking the degenerate limit $\delta \searrow 0$. Section 4 also contains the analysis of the free boundary and the proof of Proposition 1.1. In Section 5 we investigate the linear growth and planar behavior at infinity. We refine part 3 of Theorem 1 as

Theorem 1.2. *Both for the viscosity solution of Main Theorem 1 and the δ-solution of Theorem 1.1, the following holds when $x \to +\infty$:*

1. $p(x,y) \sim cx$, $p_x(x,y) \sim c$ *and* $p_y(x,y) \to 0$ *uniformly in* y.
2. *If* $1 < m \notin \mathbb{N}$ *and* $N := [m]$, *there exist* $q_1, \dots, q_N$ *and* $q^* \in \mathbb{R}$ *such that*

$$p(x,y) = cx + x\left(q_1 x^{-\frac{1}{m}} + \dots + q_N x^{-\frac{N}{m}}\right) + q^* + o(1).$$

The second part is novel compared to the PME, for which the planar wave is exactly linear $p = cx$ at infinity. Once again, we prove this statement for the final viscosity solution, but the proof extends to the δ-solutions. Section 6 is finally devoted to uniqueness of the wave profiles, and we establish

Theorem 1.3. *For conductivity exponents $m > 1$ the δ-solutions of Theorem 1.1 are unique up to x-translations.*

2. δ-solutions on finite domains

Here we solve (1.4) on truncated cylinders $D_L = [-L, L] \times \mathbb{T}^1$, $L \gg 1$ with a uniform ellipticity condition $p \geqslant \delta > 0$. We show in this section that this ellipticity can be achieved by setting suitable boundary conditions, and we therefore consider the following problem

$$0 < \delta < A < B, \quad \begin{cases} -mp\Delta p + (c+\alpha)p_x = |\nabla p|^2 & (D_L), \\ p = A & (x = -L), \\ p = B & (x = +L), \end{cases} \tag{2.1}$$

where the constants A and B are specified later.

We will show that any solution of (2.1) must satisfy $p_x > 0$, and therefore $p \geqslant A > 0$ on D_L. Thus (2.1) is uniformly elliptic. We prove this x-monotonicity of p using the following nonlinear comparison principle, which relies on the celebrated Sliding Method of Berestycki and Nirenberg [5].

Let $a < b$, $\Omega =]a, b[\times \mathbb{T}^1$ and for any function $f \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ define the nonlinear differential operator

$$\Phi(f) := -mf\Delta f + (c+\alpha)f_x - |\nabla f|^2. \tag{2.2}$$

Theorem 2.1 *(Comparison Principle). If $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ satisfy $u, v > 0$ in $\overline{\Omega}$ and*

$$\forall (x,y) \in \Omega \quad \begin{cases} u(a,y) < u(x,y) < u(b,y), \\ v(a,y) < v(x,y) < v(b,y) \end{cases} \tag{2.3}$$

then

$$\left.\begin{array}{ll} \Phi(u) \geqslant 0 & (\Omega) \\ 0 \geqslant \Phi(v) & (\Omega) \\ u \geqslant v & (\partial\Omega) \end{array}\right\} \quad \Rightarrow \quad u \geqslant v \quad (\overline{\Omega}).$$

This leads to the following definition for sub and supersolutions: a function $p^+ \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ (resp. p^-) is a supersolution (resp. subsolution) if there holds $\Phi(p^+) \geqslant 0$ in Ω (resp. $\Phi(p^-) \leqslant 0$).

Proof of Theorem 2.1. The proof is strongly inspired from [5]. Arguing by contradiction, assume $u(x_0, y_0) < v(x_0, y_0)$ holds for some $(x_0, y_0) \in \Omega$: taking advantage of the invariance under x-shifts of (2.2) we will compare u with suitable translates of v and obtain a contradiction with the classical Minimum Principle.

For $\tau \in]0, b-a[$ let us denote $v^\tau(x,y) = v(x-\tau, y)$ the τ-translation of v to the right, which is defined (at least) on $\Omega^\tau =]a+\tau, b[\times \mathbb{T}^1$. Hypothesis (2.3) and $u \geqslant v$ on $\partial\Omega$ imply $u(b,y) > u(a,y) \geqslant v(a,y)$, hence $u > v^\tau$ in

$\overline{\Omega^\tau}$ at least for large τ. Slowly sliding back v to the left, $u(x_0, y_0) < v(x_0, y_0)$ shows that there exists a first critical translation $\tau^* \in]0, b-a[$ such that u and v^* have a contact point $(x_c, y_c) \in \overline{\Omega^*}$. By continuity we have $u \geqslant v^*$ in $\overline{\Omega^*}$, and (2.3) easily shows that this contact point is in fact an interior one (namely $u > v^*$ in $\partial\Omega^*$). Invariance under x-shifts yields $\Phi(v^*) = \Phi(v)^* \leqslant 0 \leqslant \Phi(u)$, and subtracting $\Phi(u) - \Phi(v^*) \geqslant 0$ we see that $z = u - v^* \geqslant 0$ satisfies

$$\mathcal{L}z := -mu\Delta z + \left[(c+\alpha)z_x - \left(\nabla u + \nabla v^{\tau*}\right)\cdot\nabla z\right] - \left(m\Delta v^{\tau*}\right)z \geqslant 0 \quad \text{in } \Omega^*.$$

This finally contradicts the classical Minimum Principle and the interior minimum point $z^*(x_c, y_c) = 0$ (the above inequality is indeed uniformly elliptic since we assumed $u > 0$ in $\overline{\Omega} \supset \overline{\Omega^*}$). □

Condition (2.3) may seem quite restrictive at first glance, as it requires u, v to lie strictly between their boundary values along lines $y = cst$: the following proposition ensures that this is *a priori* true for any (positive) classical solution of problem (2.1). In fact we will prove later a much stronger statement, namely that solutions are increasing in x.

Proposition 2.1. *Any positive solution $p \in \mathcal{C}^2(D_L) \cap \mathcal{C}(\overline{D_L})$ of problem (2.1) satisfies*

$$\forall (x, y) \in D_L \quad A \equiv p(-L, y) < p(x, y) < p(L, y) \equiv B.$$

Proof. Assume p is such a solution: since $p > 0$ on the (compact) cylinder $[-L, L] \times \mathbb{T}^1$ we may simply consider $-mp\Delta p + (c+\alpha)p_x - |\nabla p|^2 = 0$ as a uniformly elliptic equation $\mathcal{L}p = 0$ with no zero-th order term. The classical weak Comparison Principles therefore imply $A = \min_{\partial D_L} p \leqslant p \leqslant \max_{\partial D_L} p = B$ on $\overline{D_L}$, and the classical strong Comparison Principles moreover ensure that the inequalities are strict in D_L. □

Corollary 2.1. *There exists at most one solution $p \in \mathcal{C}^2(D_L) \cap \mathcal{C}(\overline{D_L})$ of problem (2.1).*

We will show below that this solution in fact exists.

Proof of Corollary 2.1. Assume $p_1 \neq p_2$ are two different solutions. Then $\Phi(p_1) = \Phi(p_2) = 0$ and by previous proposition p_1, p_2 satisfy condition (2.3): Theorem 2.1 yields $p_i \geqslant p_j$ and therefore $p_1 = p_2$. □

We construct now two different types of planar sub and supersolutions adapted to our problem on finite cylinders.

An elementary computation shows that a planar affine function $p^+(x, y) = A^+x + B^+$ is a supersolution (resp. $p^-(x, y) = A^-x + B^-$ is a subsolution) if and only if $0 + (c+\alpha)A^+ \geqslant (A^+)^2$ (resp. A^-, $\leqslant$). Due to hypothesis (1.6) this condition is satisfied as soon as $0 \leqslant A^+ \leqslant c_0$ (resp. $A^- \geqslant c_1$ or $A^- \leqslant 0$): any affine function with positive slope $A^+ \leqslant c_0$ (resp. $A^- \geqslant c_1$) is hence a supersolution (resp. subsolution).

We will also use some additional planar sub and supersolutions built as follows: we claim that for any $x_0 \in \mathbb{R}$, $M > \delta > 0$ and $C > 0$ the following one-dimensional boundary value problem

$$u_C(x): \quad \begin{cases} -muu'' + Cu' = (u')^2, \\ u(-\infty) = \delta, \\ u(x_0) = M \end{cases}$$

has a unique solution, which will be either a sub or supersolution depending on the choice of C. This solution is moreover increasing and convex in x, as shown below together with existence.

Using the Sliding Method we see that any such solution u_C must be increasing in x, and we may therefore implicitly set $u'(x) = F(u) > 0$. Hence considering u as a variable (instead of x) leads to solving

$$u > \delta: \quad -mu\frac{\mathrm{d}F}{\mathrm{d}u}(u) + C = F(u), \qquad F(\delta) = u'(-\infty) = 0,$$

which has a unique solution $F = F_C(u) = C[1 - (\frac{\delta}{u})^{1/m}] > 0$ defined for all $u > \delta$ and satisfying in addition $F_C(+\infty) = C$ (condition $u(x_0) = M$ will be taken care of later). Back to the x variable, the Implicit Functions Theorem yields a corresponding $u_C(x)$ satisfying $u_C(-\infty) = \delta$, $u'_C > 0$ and blowing linearly as $u'_C(+\infty) = C$. This solution is defined up to x-shifts, and the additional pinning condition $u_C(x_0) = M$ finally ensures uniqueness. Convexity is a simple consequence of $u''(x) = F(u)\frac{\mathrm{d}F}{\mathrm{d}u}(u) > 0$.

One then easily computes for $0 < C \leqslant c_0$ (hence $c + \alpha(y) \geqslant C$)

$$\Phi(u_C) \stackrel{\text{def}}{=} -mu_C \Delta u_C + (c+\alpha)u_C' - \left(u_C'\right)^2 \geqslant -mu_C u_C'' + Cu_C' - \left(u_C'\right)^2 = 0,$$

and u_C is therefore a supersolution. A similar computation shows that if $C \geqslant c_1$ (hence $c + \alpha(y) \leqslant C$) then u_C is a subsolution $\Phi(u_C) \leqslant 0$. This allows to build planar sub and supersolutions tailored to (2.1) and taking into account boundary conditions as follows: let $\delta > 0$ be a small elliptic regularization parameter, and define

$$p^+(x,y) := u_{c_0}(x), \qquad \begin{cases} -muu'' + c_0 u' = \left(u'\right)^2, \\ u(-\infty) = \delta, \\ u(0) = 1. \end{cases} \tag{2.4}$$

If

$$B := p^+(L,y) = u_{c_0}(L) \quad (= cst), \tag{2.5}$$

similarly define

$$p^-(x,y) := u_{c_1}(x), \qquad \begin{cases} -muu'' + c_1 u' = \left(u'\right)^2, \\ u(-\infty) = \delta, \\ u(L) = B. \end{cases} \tag{2.6}$$

As discussed above p^-, p^+ are planar sub and supersolutions on $D_L =]-L, +L[\times \mathbb{T}^1$. They satisfy all the hypotheses of our comparison Theorem 2.1, and therefore $p^- \leqslant p^+$. We prove below that, choosing

$$A := \frac{p^+(-L,y) + p^-(-L,y)}{2} = \frac{u_{c_0}(-L) + u_{c_1}(-L)}{2} \quad (= cst), \tag{2.7}$$

there exists exactly one solution p of problem (2.1) lying between p^- and p^+.

Theorem 2.2. *Fix $\delta > 0$ small enough: for $L > 0$ large enough and A, B defined by (2.7), (2.5) there exists a unique positive classical solution $p \in \mathcal{C}^2(D_L) \cap \mathcal{C}^1(\overline{D_L})$ of (2.1). Moreover, it satisfies $p^-(x) \leqslant p(x,y) \leqslant p^+(x)$ on $\overline{D_L}$ and $p \in \mathcal{C}^\infty(D_L)$.*

Proof. Uniqueness is given by Corollary 2.1. It was shown in [11] that if there exist *strict* sub and supersolutions $p^- < p^+$ then there is a classical solution p satisfying $p^- \leqslant p \leqslant p^+$. Note, however, that we set $C = c_0, c_1$ in (2.4)–(2.6). This corresponds to non-strict inequalities $\Phi(p^+) \geqslant 0$ and $\Phi(p^-) \leqslant 0$, meaning that these particular sub and supersolutions are not strict. It is however easy to approximate $p^\pm$ by *strict* sub and supersolutions $p^\pm_\varepsilon$, where $p^+_\varepsilon > p^+$ and $p^-_\varepsilon < p^-$ are such that $p^\pm_\varepsilon \to p^\pm$ uniformly on D_L when $\varepsilon \searrow 0$; this can be done setting $c_0 - \varepsilon$ instead of c_0 and $c_1 + \varepsilon$ instead of c_1 in (2.4)–(2.6) (and also suitably modifying boundary conditions).

All the hypotheses of Theorem 1 in [11] are easily checked here, and we conclude that there exists at least one solution $p_\varepsilon \in \mathcal{C}^{2,\alpha}(D_L) \cap \mathcal{C}^1(\overline{D_L})$ such that $p^-_\varepsilon \leqslant p_\varepsilon \leqslant p^+_\varepsilon$ on $\overline{D_L}$ and satisfying boundary conditions $p_\varepsilon(-L,y) = A$, $p_\varepsilon(+L,y) = B$. By uniqueness (Corollary 2.1) this solution is independent of ε, i.e. $p_\varepsilon = p$; taking the limit $\varepsilon \searrow 0$ yields $p^- \leqslant p \leqslant p^+$ on $\overline{D_L}$, and p is finally smooth inside D_L by standard elliptic regularity. □

As we let $L \to \infty$ in the next section, we need monotonicity of p in the x direction as well as an estimate on p_x uniformly in L, the size of the cylinder D_L.

Proposition 2.2. *The unique solution $p(x,y)$ of (2.1) satisfies*

$$0 < p_x \leqslant c_1 \tag{2.8}$$

on $\overline{D_L}$, where $c_1 > 0$ is the upper bound for the flow given in (1.6).

Proof. $p \in \mathcal{C}^\infty(D_L) \cap \mathcal{C}^1(\overline{D_L})$ is smooth enough to differentiate (1.4) with respect to x, and $q := p_x \in \mathcal{C}^\infty(D_L) \cap \mathcal{C}(\overline{D_L})$ satisfies

$$-mp\Delta q + \left[(c+\alpha)q_x - 2\nabla p \cdot \nabla q\right] - (m\Delta p)q = 0. \tag{2.9}$$

We first prove the upper estimate $q \leqslant c_1$. Since we set $p(-L, y) = A < B = p(L, y)$ there exists at least a point inside D_L where $q = p_x > 0$; any possible maximum interior point therefore satisfies $q > 0$, and of course $\nabla q = 0$, $\Delta q \leqslant 0$. Using (2.9) we compute at such a positive interior maximum point $(m\Delta p)q = -mp\Delta q \geqslant 0$, hence $m\Delta p \geqslant 0$ and $mp\Delta p \geqslant 0$. The original equation (1.4) satisfied by $p > 0$ yields now

$$0 \leqslant mp\Delta p = (c+\alpha)p_x - |\nabla p|^2 \quad \Rightarrow \quad (p_x)^2 \leqslant |\nabla p|^2 \leqslant (c+\alpha)p_x,$$

and since $q = p_x > 0$ at this maximum point we obtain $q = p_x \leqslant (c+\alpha) \leqslant c_1$.

We just controlled any potential maximum value for p_x inside the cylinder, and we control next p_x on the boundaries using sub and supersolutions as barriers for p. Recall that we set *flat* boundary values $p(-L, y) = A$ and $p(L, y) = B$.

On the right side we use the previous subsolution $p^-(x)$ as a barrier from below: recalling that $p^-_x \leqslant c_1$ and that p^- and p agree at $x = L$ we obtain $p_x(L, y) \leqslant p^-_x(L) \leqslant c_1$.

On the left we use a new planar supersolution as a barrier from above: let $\overline{p}(x)$ be the unique affine function connecting $\overline{p}(-L) = A$ and $\overline{p}(L) = B$, hence with slope $s = \frac{B-A}{2L}$. Using (2.4)–(2.6) it is easy to see that $B \sim c_0 L$ and $A \sim \delta$ when $L \to +\infty$ for fixed δ. As a consequence $s \sim c_0/2 < c_0$ for L large enough, and $\overline{p}$ is indeed a supersolution (see discussion above for affine sub and supersolutions). Since p agrees with $\overline{p}$ on both boundaries our comparison Theorem 2.1 ensures that $p \leqslant \overline{p}$ on D_L, thus $p_x \leqslant s \leqslant c_0 \leqslant c_1$ on the left boundary.

Monotonicity $q = p_x > 0$ inside D_L is a classical consequence of the Sliding Method [5]: non-monotonicity would indeed yield an interior contact point between p and some suitable x-shift $p(.+\tau)$, thus contradicting the usual strong Comparison Principle. Strict monotonicity $p_x > 0$ at the (flat) boundaries is finally an immediate consequence of Hopf Lemma for $-mp\Delta p + (c+\alpha)p_x - |\nabla p|^2 = 0$, seen as a linear elliptic equation $\mathcal{L}p = 0$ with no zero-th order coefficient (by construction p attains its strict minimum at any point on the left boundary and its strict maximum at any point on the right one). □

Proposition 2.3 *(Uniform pinning). There exist large constants $K > 0$, $K_1 \approx K - \sqrt{K}$ and $K_2 \approx K + \sqrt{K}$ such that, for any L large and any δ small enough, there exists $x^* = x^*(L, \delta) \in]0, L[$ such that*

1. $\lim_{L\to+\infty}(L - x^*) = +\infty$,
2. $K_1 \leqslant p(x^*, y) \leqslant K_2$.

The constants K, K_1, K_2 depend on the upper bound c_1 in (1.6) and $m > 0$, but not on L or δ.

Proof. The idea is as follows. When x increases from $-L$ to L the map $x \mapsto \int_{\mathbb{T}^1} p(x, y)\,\mathrm{d}y$ increases from $A \sim \delta \leqslant 1$ to $B \sim c_0 L \gg 1$. For fixed large K and L large enough this integral has to take on the value K at least for some $x \in]-L, L[$. The equation for p then allows to control the y-oscillations of p along this line by $\mathcal{O}(\sqrt{K})$. If K is chosen large enough these oscillations are small compared to the average, and p along this line will therefore be of the same order $\mathcal{O}(K)$ than its average $\int p\,\mathrm{d}y$. This will be our pinning line $x = x^*$ (up to some further small translation).

- Choose a large constant $K > 1$, and for $x \in [-L, L]$ define $F(x) := \int_{\mathbb{T}^1} p(x, y)\,\mathrm{d}y$: since $p(0, y) \leqslant p^+(0) = 1$ we have that $F(0) < K$. By convexity p^- lies above its tangent plane $t_L(x)$ at $x = L$, and we recall that we had set $p^-(L) = p(L, y) = p^+(L) = B$. For L large and δ small $t_L(x) = K$ has a unique solution $x = x_K$ given by $x_K = L + \frac{K-B}{p^-_x(L)}$, and $F(x_K) \geqslant p^-(x_K) \geqslant t_L(x_K) = K$. Remarking that F is increasing ($p_x > 0$), there exists a unique $x^*_K(L, \delta) \in]0, x_K]$ such that

$$F\left(x^*_K\right) = \int_{\mathbb{T}^1} p\left(x^*_K, y\right)\mathrm{d}y = K.$$

  Manipulating (2.4)–(2.6) it is easy to check that for K, δ fixed and $L \to +\infty$ there holds

$$\left.\begin{array}{l} B = p^+(L) \sim c_0 L \\ p^-_x(L) \sim c_1 \end{array}\right\} \quad \Rightarrow \quad x_K = L + \frac{K-B}{p^-_x(L)} \sim \left(1 - \frac{c_0}{c_1}\right)L;$$

  as a consequence of (1.6) the line $x = x^*_K(\delta, L)$ stays away from both boundaries.

- Let us now slide the whole picture to the left by setting $\tilde{p}(x,y)=p(x+x_K^*,y)$, so that $x=x_K^*$ corresponds in this new frame to $x=0$; for simplicity of notation we will use $p(x,y)$ instead of $\tilde{p}(x,y)$ below. The corresponding domain still grows in both directions when $L\to+\infty$, and by definition of x_K^* we have that

$$\int_{\mathbb{T}^1} p(0,y)\,\mathrm{d}y=K.$$

We claim now that there exists a constant C, depending only on $m\neq 1$ and the upper bound for the flow c_1, such that

$$\forall x>0,\qquad \iint_{[0,x]\times\mathbb{T}^1} |\nabla p|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant C(K+x). \tag{2.10}$$

Indeed, integrating by parts the Laplacian term in $-mp\Delta p+(c+\alpha)p_x=|\nabla p|^2$ over a subdomain $\Omega=[0,x]\times\mathbb{T}^1$ and combining the resulting $|\nabla p|^2$ term with the one on the right-hand side yields

$$(m-1)\iint_{\Omega}|\nabla p|^2\,\mathrm{d}x\,\mathrm{d}y+m\int_{\mathbb{T}^1} pp_x(0,y)\,\mathrm{d}y-m\int_{\mathbb{T}^1} pp_x(x,y)\,\mathrm{d}y+\iint_{\Omega''}(c+\alpha)p_x\,\mathrm{d}x\,\mathrm{d}y=0. \tag{2.11}$$

1. If $m-1>0$ we use $m\int_{\mathbb{T}^1} pp_x(0,y)\,\mathrm{d}y\geqslant 0$ and $\iint_{\Omega}(c+\alpha)p_x\,\mathrm{d}x\,\mathrm{d}y\geqslant 0$ in (2.11). This leads to $(m-1)\iint_{\Omega}|\nabla p|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant m\int_{\mathbb{T}^1} pp_x(x,y)\,\mathrm{d}y$, and since $0<p_x\leqslant c_1$

$$\iint_{\Omega}|\nabla p|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant\frac{mc_1}{m-1}\int_{\mathbb{T}^1} p(x,y)\,\mathrm{d}y.$$

Our monotonicity estimate $0<p_x\leqslant c_1$ again yields

$$\int_{\mathbb{T}^1} p(x,y)\,\mathrm{d}y=\int_{\mathbb{T}^1} p(0,y)\,\mathrm{d}y+\iint_{\Omega} p_x\,\mathrm{d}x\,\mathrm{d}y\leqslant K+c_1x,$$

and together with the previous inequality

$$\forall x>0,\qquad \iint_{[0,x]\times\mathbb{T}^1}|\nabla p|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant\frac{mc_1}{m-1}(K+c_1x)\leqslant C(K+x).$$

2. If $0<m<1$ we use $pp_x(x,y)>0$ in (2.11) to obtain

$$(1-m)\iint_{\Omega}|\nabla p|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant m\int_{\mathbb{T}^1} pp_x(0,y)\,\mathrm{d}y+\iint_{\Omega}(c+\alpha)p_x\,\mathrm{d}x\,\mathrm{d}y.$$

Since $0<p_x\leqslant c_1$ and $0<c+\alpha\leqslant c_1$ this leads to

$$\begin{aligned}\iint_{\Omega}|\nabla p|^2\,\mathrm{d}x\,\mathrm{d}y&\leqslant\frac{mc_1}{1-m}\int_{\mathbb{T}^1} p(0,y)\,\mathrm{d}y+\frac{1}{1-m}\iint_{\Omega} c_1^2\,\mathrm{d}x\,\mathrm{d}y\\&\leqslant C(K+x)\end{aligned}$$

with $C=\frac{1}{1-m}\max(mc_1,c_1^2)$ depending only on m and c_1.

- In the spirit of [12] we control now the oscillations $O(x)=|\max_{y\in\mathbb{T}^1} p(x,y)-\min_{y\in\mathbb{T}^1} p(x,y)|$ in the y direction: by Cauchy–Schwarz inequality we have that

$$O^2(x)\leqslant\left(\int_{\mathbb{T}^1}\left|p_y(x,y)\right|\mathrm{d}y\right)^2\leqslant\int_{\mathbb{T}^1}\left|p_y(x,y)\right|^2\mathrm{d}y\leqslant\int_{\mathbb{T}^1}|\nabla p|^2(x,y)\,\mathrm{d}y,$$

and integrating from $x=0$ to $x=1$ with (2.10) leads to

$$(1-0)\min_{x\in[0,1]} O^2(x) \leqslant \int_0^1 O^2(x)\,\mathrm{d}x \leqslant \iint_{[0,1]\times\mathbb{T}^1} |\nabla p|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant C(K+1).$$

Let now $x^* \in [0,1]$ be any point where $O^2(x)$ attains its minimum on this interval; along the particular line $x = x^*$ the last inequality yields

$$O(x^*) \leqslant \sqrt{C(K+1)} \tag{2.12}$$

and these oscillations are therefore controlled uniformly in L (C depends only on m and c_1). Moreover, $x^* \in [0,1]$ and $0 < p_x \leqslant c_1$ control p in average from below and from above

$$K = \int_{\mathbb{T}^1} p(0,y)\,\mathrm{d}y \leqslant \int_{\mathbb{T}^1} p(x^*,y)\,\mathrm{d}y \leqslant K + c_1 x^* \leqslant K + c_1. \tag{2.13}$$

- For K large enough but fixed (2.12), (2.13) imply $0 < K_1 \leqslant p(x^*,y) \leqslant K_2$ as desired, with $K_1 \approx K - \mathcal{O}(\sqrt{K})$ and $K_2 \approx K + \mathcal{O}(\sqrt{K})$ up to constants depending only on c_1 and m. Finally $x^* \in [0,1]$ may depend on L, δ, c_1 (and actually does) but stays far enough from both boundaries, so that the new translated domain still grows to infinity in both directions when $L \to +\infty$. □

3. δ-solutions on the infinite cylinder

From now on we will work in the translated frame $D_L =]-L-x^*, L-x^*[\times\mathbb{T}^1$, where $x^* = x^*(L,\delta)$ is defined as in Proposition 2.3 above. Since the domain depends on L, the solution depends on L as well. We emphasize that by writing $p = p^L$ ($\delta > 0$ is fixed so we may just omit the dependence on δ), and let also set $D = \mathbb{R}\times\mathbb{T}^1$ to be the infinite cylinder.

Theorem 3.1. *Up to a subsequence we have $p^L \to p$ in $\mathcal{C}^2_{\mathrm{loc}}(D)$ when $L \to +\infty$, where $p \in \mathcal{C}^\infty(D)$ is a classical solution of $-mp\Delta p + (c+\alpha)p_x = |\nabla p|^2$. This limit p satisfies*

1. $0 \leqslant p_x \leqslant c_1$,
2. $p \geqslant \delta$,
3. *p is nontrivial*: $K_1 \leqslant p(0,y) \leqslant K_2$

where K_1, K_2 are the pinning constants in Proposition 2.3.

Proof. Using interior L^q elliptic regularity arguments for fixed $q > d = 2$ we will obtain $W^{3,q}$ estimates on p^L, and this will allow to retrieve the strong convergence $p^L \to p$ in $\mathcal{C}^2_{\mathrm{loc}}$.

The most difficult term to estimate is $|\nabla p|^2$. We handle it using a different unknown which appears very naturally in the original setting (AD-E), namely

$$w := \frac{m^2}{m+1} p^{\frac{m+1}{m}} = m\left(\frac{m+1}{m}\right)^{\frac{1}{m}} T^{m+1}. \tag{3.1}$$

An easy computation shows that this new unknown satisfies on D_L a classical Poisson equation

$$\Delta w^L = f^L, \tag{3.2}$$

where the non-homogeneous part

$$f^L := (c+\alpha)\left(p^L\right)^{\frac{1}{m}-1} p^L_x \tag{3.3}$$

involves only p^L and p^L_x, on which we have local L^∞ control uniformly in L. Indeed, p^L is pinned at $x=0$ by $K_1 \leqslant p^L(0,y) \leqslant K_2$ and cannot grow too fast in the x direction because of $0 \leqslant p^L_x \leqslant c_1$.

If $m<1$ the exponent $\frac{1}{m}-1$ in (3.3) is positive and we control f^L uniformly in L on any compact set. However, if $m>1$, this exponent is negative and we need to bound p_L away from zero uniformly in L. For $\delta>0$ fixed this is easy since we constructed $p^L \geqslant p^- > \delta > 0$, but this will be a problem later when taking the limit $\delta \to 0$ (see next section).

As a consequence, for any fixed $q>d=2$, f^L is in L^q on any bounded subset $\Omega \subset D$ and we control

$$\|f^L\|_{L^q(\Omega)} \leqslant C$$

uniformly in L (C may of course depend on Ω, q and δ). Since w^L is defined as a positive power of p^L and p^L is locally controlled in the L^∞ norm uniformly in L the same holds for w^L,

$$\|w^L\|_{L^q(\Omega)} \leqslant C.$$

Let $\Omega =]-a,a[\times \mathbb{T}^1 \subset D$ and $K=\overline{\Omega}$; let also $\Omega_2 =]-2a,2a[\times \mathbb{T}^1$ and $\Omega_3 =]-3a,3a[\times \mathbb{T}^1$ so that $\Omega \Subset \Omega_2 \Subset \Omega_3$. By interior L^q elliptic regularity for strong solutions (the version we use here is [14, Theorem 9.11, p. 235]) there exists a constant C depending only on Ω_2, Ω_3 and q such that

$$\|w^L\|_{W^{2,q}(\Omega_2)} \leqslant C\left(\|w^L\|_{L^q(\Omega_3)} + \|f^L\|_{L^q(\Omega_3)}\right).$$

As discussed above we control w^L and f^L, hence

$$\|w^L\|_{W^{2,q}(\Omega_2)} \leqslant C \tag{3.4}$$

for some $C>0$ depending only on Ω_3, Ω_2 and q.

The next step is using (3.1)–(3.3) to express f^L only in terms of w^L

$$f^L = \frac{c+\alpha}{m+1}\left(w^L\right)^{-\frac{m}{m+1}} w^L_x.$$

Expressing ∇f^L only in terms of w^L, ∇w^L and $D^2 w^L$ (which are controlled by $\|w^L\|_{W^{2,q}(\Omega_2)}$), using the lower bound $p^L \geqslant \delta > 0$ and uniform control on p^L, (3.4) implies that

$$\|\nabla f^L\|_{L^q(\Omega_2)} \leqslant C$$

for some C depending only on the size a of Ω. Differentiating (3.2) implies

$$\Delta\left(\partial_i w^L\right) = \partial_i f^L, \quad i=1,2.$$

Repeating the previous L^q interior regularity argument on $\Omega \Subset \Omega_2$ yields

$$\|\partial_i w^L\|_{W^{2,q}(\Omega)} \leqslant C\left(\|\partial_i w^L\|_{L^q(\Omega_2)} + \|\partial_i f^L\|_{L^q(\Omega_2)}\right) \leqslant C,$$

and our previous estimate for ∇f^L together with (3.4) finally yields the higher estimate

$$\|w^L\|_{W^{3,q}(\Omega)} \leqslant C.$$

The set $K=\overline{\Omega}=[-a,a]\times\mathbb{T}^1$ is bounded and the exponent q was chosen larger than the dimension $d=2$. Thus compactness of the Sobolev embedding $W^{3,q}(\Omega) \Subset \mathcal{C}^2(K)$ implies, up to a subsequence, that

$$w^L \xrightarrow{\mathcal{C}^2(K)} w$$

when $L \to +\infty$. By the diagonal extraction of a subsequence we can assume that the limit w does not depend on the compact K. It means $w^L \to w$ in $\mathcal{C}^2_{\mathrm{loc}}$ on the infinite cylinder D. The algebraic relation (3.1) and $p^L \geqslant \delta > 0$ imply that

$$p^L \xrightarrow{\mathcal{C}^2_{\mathrm{loc}}(D)} p.$$

It implies that we can take the pointwise limit in the nonlinear equation. The limit p solves therefore the same equation $-mp\Delta p + (c+\alpha)p_x = |\nabla p|^2$ on the infinite cylinder.

The remaining estimates are easily obtained by taking the limit in $0 \leqslant p^L_x \leqslant c_1$, $\delta < p^- \leqslant p^L$ and in the pinning Proposition 2.3. Lastly, p is smooth by classical elliptic regularity. □

Proposition 3.1. *We have* $\lim_{x\to-\infty} p(x,y)=\delta$ *uniformly in* y.

Proof. The previous lower barrier $\delta < p^L$ on D_L immediately passes to the limit $L\to+\infty$, and

$$\forall (x,y)\in D,\quad p\geqslant \delta. \tag{3.5}$$

In order to estimate p from above let us go back to the untranslated frame $x\in[-L,L]$ and remark that by definition p^+ does not depend on L, see (2.4). An easy computation shows that $p^+(-L)\to\delta$ when $L\to+\infty$. The subsolution p^- actually depends on L through boundary condition, see (2.6), but using the monotonicity $\partial_x p^->0$ it is quite easy to prove that $p^-(-L)\sim\delta$ when $L\to+\infty$. The left boundary condition consequently reads

$$p^L(-L,y)=A=\frac{p^+(-L)+p^-(-L)}{2}\underset{L\to+\infty}{\sim}\delta.$$

However, the limit $\lim_{x\to-\infty} p(x,y)\overset{??}{=}\lim_{L\to+\infty} p^L(-L,y)$ is not clear because the convergence $p^L\to p$ is only local on compact sets (and also because we translated from one frame to another).

In order to circumvent this technical difficulty we move back to the translated frame and build for $x\in\,]-L-x^*,0[\,\times\mathbb{T}^1$ a family of planar supersolutions $\overline{p}_\varepsilon(x)$ independent of L such that $\overline{p}_\varepsilon(-\infty)=\delta+\varepsilon$. The construction is the following: fix $\varepsilon>0$ and define $\overline{p}_\varepsilon(x)$ as the unique solution of Cauchy problem

$$\overline{p}_\varepsilon(x):\quad \begin{cases} -muu''+c_0u'=(u')^2,\\ u(0)=2K_2,\\ u(-\infty)=\delta+\varepsilon, \end{cases} \tag{3.6}$$

where K_2 is the constant in Proposition 2.3 such that $p^L(0,y)\leqslant K_2$. As already computed the setting $C=c_0\leqslant c+\alpha$ in (3.6) implies that $\overline{p}_\varepsilon$ is a supersolution. By monotonicity both p^L and $\overline{p}_\varepsilon$ satisfy condition (2.3), and for L large and δ,ε small it is easy to check that $p\leqslant\overline{p}_\varepsilon$ on the boundaries $x=-L-x^*,0$: Theorem 2.1 guarantees that

$$\forall (x,y)\in\,\big]-L-x^*,0\big[\,\times\mathbb{T}^1,\quad p^L\leqslant\overline{p}_\varepsilon.$$

For δ,ε fixed, $\overline{p}_\varepsilon$ is independent of L: taking the limit $L\to+\infty$ yields

$$\forall (x,y)\in\,]-\infty,0[\,\times\mathbb{T}^1,\quad p(x,y)\leqslant\overline{p}_\varepsilon(x). \tag{3.7}$$

Taking now the limit $\varepsilon\to0$ in (3.6), it is easy to show that $\overline{p}_\varepsilon(x)\to\overline{p}(x)$ uniformly on $]-\infty,0]$, where $\overline{p}$ is the solution of the same Cauchy problem as $\overline{p}_\varepsilon$ – except for $\overline{p}(-\infty)=\delta$ instead of $\overline{p}_\varepsilon(-\infty)=\delta+\varepsilon$ – and satisfies $\lim_{x\to-\infty}\overline{p}(x)=\delta$. Combining the limit $\varepsilon\to0$ in (3.7) with the lower barrier (3.5) we finally obtain

$$\forall (x,y)\in\,]-\infty,0]\times\mathbb{T}^1,\quad \delta\leqslant p(x,y)\leqslant\underbrace{\overline{p}(x)}_{\to\delta}$$

as desired. □

Remark 4. The proof above actually implies a stronger statement than $\lim_{x\to-\infty}p(x,y)=\delta$, namely $\delta\leqslant p\leqslant\overline{p}$ for $x\to-\infty$: just working on the ODE $-m\overline{p}\,\overline{p}''+c_0\overline{p}'=(\overline{p}')^2$ it is straightforward to obtain the exponential decay $|p-\delta|=\mathcal{O}(e^{c_0x/m\delta})$. The exponential rate $c_0/m\delta$ degenerates when $\delta\searrow0$, which is consistent with the fact that a free boundary appears in this limit (see next section).

As stated in Theorem 3.1 the limit p is non-decreasing in the x direction (as a limit of increasing functions p^L). We establish below the strict monotonicity.

Proposition 3.2. $p_x>0$ *on the infinite cylinder.*

Proof. The argument is very similar to the proof of Proposition 2.2: differentiating the equation for p with respect to x yields a linear uniformly elliptic equation satisfied by $q=p_x\geqslant0$. The classical Minimum Principle implies that either $q>0$ everywhere or $q\equiv0$, and latter would contradict $p(-\infty,y)=\delta<K_2\leqslant p(0,y)$. □

4. Limit $\delta \to 0$ and the free boundary

In the previous section we constructed for any small $\delta > 0$ a nontrivial solution $p = \lim_{L\to+\infty} p^L$ of $-mp\Delta p + (c+\alpha)p_x = |\nabla p|^2$ on the infinite cylinder $D = \mathbb{R} \times \mathbb{T}^1$, satisfying the uniform ellipticity condition $p > \delta > 0$. Let us now write $p = p^\delta$ in order to stress the dependence on δ. The next step is now to take the limit $\delta \searrow 0$ (δ is an elliptic regularization parameter), yielding the desired viscosity solution.

For $\delta > 0$ let $E_\delta \subset \mathcal{C}^0(D)$ be the set of δ-solutions, which we recall are positive smooth solutions p^δ satisfying

1. $\lim_{x\to-\infty} p^\delta(x, y) = \delta$ uniformly in y,
2. $p^\delta(x, y) \sim cx$ uniformly in y at positive infinity.

According to our comparison Theorem 2.1 these must satisfy $p^\delta \geqslant \delta > 0$, in which case the equation becomes uniformly elliptic, and δ should therefore be seen here as a regularization parameter (this is very close to the vanishing viscosity method, but the equation itself is not modified). We define viscosity solutions as

Definition 1. A function $p \in \mathcal{C}^0(D)$ is a viscosity solution if there exist a sequence $\delta_n \searrow 0$ and δ_n-solutions $p^n \in E_{\delta_n}$ such that $\lim_{\delta_n\searrow 0} p^n = p$ in $\mathcal{C}^0_{\mathrm{loc}}(D)$.

This is not the standard definition: any viscosity solution p will however turn to be $\mathcal{C}^\infty$ on its positive set $D^+ = \{p > 0\}$ (see proof of Theorem 4.1 below), which is not clear with the usual definition (in addition to being a difficult question, see e.g. [7]). The delicate issue in the proof of existence is of course the loss of ellipticity when $\delta \searrow 0$.

If $p^\delta \geqslant \delta$ denotes the δ-solution constructed in Section 3, we will bound p^δ and ∇p^δ uniformly in δ. Extracting a (discrete) sequence $\delta_n \searrow 0$ Arzelà–Ascoli Theorem will therefore yield convergence $p^n \to p$ in $\mathcal{C}^0_{\mathrm{loc}}$, hence existence of a viscosity solution in the sense of Definition 1. At this stage we have pinned $0 < K_1 \leqslant p^\delta(0, y) \leqslant K_2$, and $0 < p^\delta_x \leqslant c_1$ holds on the infinite cylinder: we therefore control p^δ and p^δ_x uniformly on any compact set, but we still have no control at all on p^δ_y:

Proposition 4.1. *For any $a \geqslant 0$ there exists $C_a > 0$ such that, for any small $\delta > 0$,*

$$x \leqslant a \quad \Rightarrow \quad \left|p^\delta_y(x, y)\right| \leqslant C_a.$$

Proof. We will first establish this estimate for p^L on finite domains $[-L - x^*, a] \times \mathbb{T}^1$ by controlling $q = p^L_y$ at the boundaries and estimating the value of any possible interior extremal point. Taking the limit $L \to +\infty$ will then yield the desired estimate for $p^\delta = \lim_{L\to+\infty} p^L$.

- Fix $a \geqslant 0$ and choose L large: uniform pinning $K_1 \leqslant p^L(0, y) \leqslant K_2$ and monotonicity $0 < p^L_x \leqslant c_1$ allow to control p^L uniformly in δ, L from above and away from zero on any small compact set $K = [a - \varepsilon, a + \varepsilon] \times \mathbb{T}^1$. As in the proof of Theorem 3.1, applying an L^q interior elliptic regularity for $w = \frac{m^2}{m+1} p^{\frac{m+1}{m}}$ on the slightly larger set $\Omega_2 =]a - 2\varepsilon, a + 2\varepsilon[\times \mathbb{T}^1 \Supset \Omega := \mathring{K}$ we obtain
$$\left\|w^L\right\|_{W^{2,q}(\Omega)} \leqslant C\left(\left\|w^L\right\|_{L^q(\Omega_2)} + \left\|f^L\right\|_{L^q(\Omega_2)}\right) \leqslant C_a \quad \Rightarrow \quad \left\|p^L\right\|_{\mathcal{C}^1(K)} \leqslant C_a$$
for some constant C_a depending only on Ω, Ω_2 and $q > 2$ fixed, hence on a (it is here important that p^L is bounded away from zero uniformly in δ on Ω_2, see proof of Theorem 3.1 for details). In particular we have
$$\left|p^L_y(a, y)\right| \leqslant C_a. \tag{4.1}$$
Differentiating now (1.4) with respect to y we see that $q^L := p^L_y$ satisfies the linear elliptic equation
$$-mp^L\Delta q^L + \left[(c+\alpha)q^L_x - 2\nabla p^L \cdot \nabla q^L\right] - \left(m\Delta p^L\right)q^L = -\alpha_y p^L_x. \tag{4.2}$$
Let $\Omega_a =]-L - x^*, a[\times \mathbb{T}^1$: on the left $x = -L - x^*$ we had a flat boundary condition $p^L(-x * -L, y) = cst$ so that $q^L(-L - x^*, y) = 0$, and on the right boundary $x = a$ (4.1) holds. We therefore control $|q^L| = |p^L_y| \leqslant C_a$ on the boundaries uniformly in L.

- In order to control p_y^L inside Ω_a we remark that any interior maximum point satisfies $q > 0$ (unless by periodicity $p_y^L \equiv 0$, which is impossible if the flow $\alpha(y)$ is nontrivial), and of course $\Delta q^L \leqslant 0$ and $\nabla q^L = 0$. At such a maximum point (4.2) immediately yields

$$-\left(m\Delta p^L\right)q^L \leqslant -\alpha_y p_x^L;$$

using now $-mp^L\Delta p^L = |\nabla p^L|^2 - (c+\alpha)p_x^L$ and taking advantage of $0 < p_x < c_1$ and $c+\alpha \leqslant c_1$

$$\begin{aligned}\left(q^L\right)^3 - c_1^2 q^L &= \left[\left(p_y^L\right)^2 - c_1^2\right]q^L \leqslant \left[\left|\nabla p^L\right|^2 - (c+\alpha)p_x^L\right]q^L \\ &= -\left(mp^L\Delta p^L\right)q^L \leqslant -p^L\alpha_y p_x^L \leqslant C_a\end{aligned}$$

since by x-monotonicity (and pinning) we have bounds $|p^L|, |p_x^L| \leqslant C_a$ for $x \leqslant a$ ($|\alpha_y|$ is obviously also bounded independently of any parameter). This third order polynomial inequality in q^L immediately controls any possible positive interior maximum point $\max_{(x,y)\in\Omega_a} q^L(x,y) \leqslant C_a$ uniformly in L, δ. A similar computation controls q^L at any potential negative minimum point $\min_{(x,y)\in\Omega_a} q^L(x,y) \geqslant -C_a$, and combining with the previous boundary estimates yields

$$(x,y) \in \left[-L-x^*, a\right] \times \mathbb{T}^1 \quad \Rightarrow \quad \left|p_y^L(x,y)\right| \leqslant C_a. \tag{4.3}$$

Theorem 3.1 ensures that the convergence $p^L \to p^\delta$ holds in $\mathcal{C}^2_{\mathrm{loc}}(D)$: taking the limit $L \to +\infty$ in (4.3) finally yields the desired estimate for p^δ. □

We can now state our main convergence result:

Theorem 4.1. *For any (discrete) sequence $\delta_n \searrow 0$ there exist a subsequence δ_{n_k} and a nontrivial continuous function $p \geqslant 0$ such that $p^{\delta_{n_k}} \to p$ in $\mathcal{C}^0_{\mathrm{loc}}(D)$. Further, if $D^+ := \{p > 0\}$ denotes the non-empty positive set, the following holds:*

1. *p is Lipschitz on any subdomain $]-\infty, a] \times \mathbb{T}^1$, with a Lipschitz constant possibly depending on a.*
2. *p is a viscosity solution (in the sense of Definition 1) on the cylinder, and a smooth classical solution on D^+.*
3. *$0 < p_x \leqslant c_1$ on D^+.*
4. *p has a non-empty free boundary $\Gamma = \partial D^+ \neq \emptyset$ and there exists an upper semi-continuous function $I(y)$ with finite amplitude $x_1 \leqslant I(y) \leqslant x_2$ such that $p(x,y) > 0 \Leftrightarrow x > I(y)$.*
5. *If $y_0 \in \mathbb{T}^1$ is a point of discontinuity for I then $\Gamma \cap \{y = y_0\} = [\underline{I}(y_0), I(y_0)] \times \{y = y_0\}$, where $\underline{I}(y_0) := \liminf_{y\to y_0} I(y)$.*

Proof. Let δ_n be any sequence decreasing to zero and p^{δ_n} be the δ_n-solution previously constructed. The pinning $K_1 \leqslant p^\delta(0,y) \leqslant K_2$ and monotonicity $0 < p_x^\delta \leqslant c_1$ control p^{δ_n} on any fixed compact set $K = [-a,a] \times \mathbb{T}^1$ uniformly in δ_n, and on this compact set p_y^n is moreover bounded by Proposition 4.1. Arzelà–Ascoli Theorem guarantees, up to extraction, that $p^{n_k} \to p$ uniformly on K for some $p \in \mathcal{C}(K)$. By diagonal extraction we may assume that this limit is independent of the set K, which means local uniform convergence

$$p^{n_k} \xrightarrow{\mathcal{C}^0_{\mathrm{loc}}(D)} p.$$

p is nonnegative as a limit of positive functions, and nontrivial since for example we had pinned $0 < K_1 \leqslant p^\delta(0,y)$. Since the discrete extraction procedure is now complete we simply write $p^\delta = p^{n_k}$ in the following for the sake of simplicity, with a clear abuse of notations since δ was so far a *continuous* variable (see discussion at the beginning of Section 6).

1. Proposition 4.1 and monotonicity $0 < p_x^\delta \leqslant c_1$ yield Lipschitz estimates for p^δ on $]-\infty, a] \times \mathbb{T}^1$ uniformly in δ, for some Lipschitz constant C_a independent of δ: this obviously passes to the limit $\delta \searrow 0$, and p is therefore Lipschitz on any half-cylinder $]-\infty, a] \times \mathbb{T}^1$.

2. $p^\delta \in \mathcal{C}^\infty(D)$ was a classical solution of $-mp\Delta p + (c+\alpha)p_x = |\nabla p|^2$ on the infinite cylinder: according to Definition 1 we need to check that p^δ grows as cx when $x \to +\infty$. This is true, but the proof is long and technical: we will actually prove in Section 5 that the limit p itself grows linearly, and the proof of the linear growth for the final viscosity solution carries out for δ-solutions. We therefore temporarily admit here the linear growth for both δ and viscosity solutions (again, this will be investigated in detail in Section 5); as a consequence p is immediately a viscosity solution on the whole cylinder in the sense of Definition 1.

Remark 5. Regardless of this linear growth issue, the limit p is a viscosity solution in the classical sense as a consequence of usual stability theorems (see e.g. [13, §6]). This is just the classical construction by vanishing viscosity: the uniform ellipticity $p^\delta \geqslant \delta > 0$ degenerates when $\delta \searrow 0$.

In order to prove the above convergence $p^\delta \to p$ we could not apply the same local L^q interior elliptic regularity argument as in the proof of Theorem 3.1, mainly because we needed to bound p^L away from zero (cf. the negative p^L exponents $\frac{1}{m} - 1$ for the non-homogeneous term in (3.2)). This is of course impossible on the whole cylinder uniformly in δ because the equation degenerates when $\delta \searrow 0$ (this is indeed consistent with $p \equiv 0$ to the left of the free boundary, as claimed in our statement).

This strategy is however still efficient on the positive set $D^+ = \{p > 0\}$: indeed for any fixed compact subset $K \subset D^+$ we know *a priori* that the limit p is positive, and therefore so is p^δ uniformly in $\delta \searrow 0$. This allows to bound p^δ away from zero uniformly in δ on any compact set $K \subset D^+$ as

$$p^\delta\big|_K \geqslant C_K > 0,$$

where C_K depends only on K. The interior L^q regularity argument in the proof of Theorem 3.1 then applies to the letter, and

$$p^\delta \to \tilde{p} \quad \text{in } \mathcal{C}^2_{\text{loc}}\big(D^+\big).$$

The limit $\tilde{p} \in \mathcal{C}^2(D^+)$ is moreover a classical solution on D^+, and smooth by standard elliptic regularity. The previous $\mathcal{C}^0_{\text{loc}}(D)$ convergence finally implies that $p|_{D^+} = \tilde{p} \in \mathcal{C}^\infty(D^+)$ is a classical solution on D^+.

3. Convergence $p^\delta \to p$ is strong enough on D^+ in order to pass to the limit in $0 < p^\delta_x \leqslant c_1$, so that $0 \leqslant p_x \leqslant c_1$ on D^+. Strict monotonicity is obtained just as for the δ-solutions: differentiating the equation for p with respect to x yields and elliptic equation $Lq = 0$ satisfied by $q = p_x \geqslant 0$ on D^+ (where $p > 0$ is smooth). Applying the Minimum Principle shows that either $q > 0$ or $q \equiv 0$, and the latter is impossible due item 4 of our statement proved below: since p vanishes far enough to the left and takes positive values $K_1 \leqslant p(0, y)$ it must increase at least somewhere inside D^+.
4. In order to show the existence of the free boundary $\Gamma = \partial\{p > 0\} \neq \emptyset$ we build new suitable planar sub and supersolutions $p^{\delta,-}(x)$, $p^{\delta,+}(x)$ for p^δ as follows: defining $p^{\delta,-}$, $p^{\delta,+}$ to be the unique planar solutions of the following Cauchy problems

$$p^{\delta,-}(x): \quad \begin{cases} -muu'' + c_1 u' = \big(u'\big)^2, \\ u(-\infty) = \dfrac{\delta}{2}, \\ u(0) = K_1, \end{cases} \qquad p^{\delta,+}(x): \quad \begin{cases} -muu'' + c_0 u' = \big(u'\big)^2, \\ u(-\infty) = 2\delta, \\ u(0) = K_2, \end{cases}$$

which satisfy of course $\Phi(p^{\delta,-}) \leqslant \Phi(p^\delta) = 0 \leqslant \Phi(p^{\delta,+})$. Let us moreover recall from Proposition 3.1 that $\lim_{x\to-\infty} p^\delta(x, y) = \delta$ uniformly in y, so that $p^{\delta,-} < p^\delta < p^{\delta,+}$ when $x \to -\infty$. On the right boundary we set $p^{\delta,-}(0) = K_1 \leqslant p^\delta(0, y) \leqslant K_2 = p^{\delta,+}(0)$: applying Theorem 2.1 on $]-\infty, 0] \times \mathbb{T}^1$ yields

$$x \leqslant 0 \quad \Rightarrow \quad p^{\delta,-}(x) \leqslant p^\delta(x, y) \leqslant p^{\delta,+}(x) \tag{4.4}$$

(note that $p^{\delta,-}_x, p^{\delta,+}_x, p^\delta_x > 0$ so that condition (2.3) does hold). When $\delta \searrow 0$ one can prove that

$$p^{\delta,-}(x) \to p^-(x) := [K_1 + c_1 x]^+, \qquad p^{\delta,+}(x) \to p^+(x) := [K_2 + c_0 x]^+$$

uniformly on $\mathbb{R}^-$, where $[.]^+$ denotes the positive part. Taking the limit $\delta \to 0$ in (4.4) yields

$$x \leqslant 0 \quad \Rightarrow \quad p^-(x) \leqslant p(x, y) \leqslant p^+(x).$$

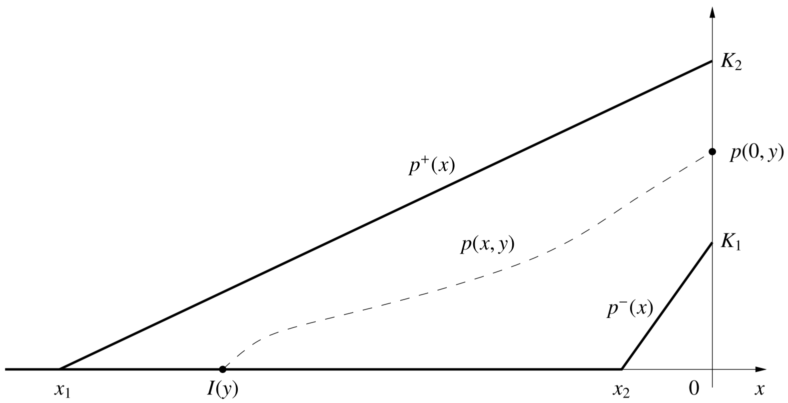

Fig. 1. Existence and width of the free boundary.

In particular

$$x < x_1 := -\frac{K_2}{c_0} \quad \Rightarrow \quad p(x, y) \leqslant p^+(x) = 0,$$
$$x > x_2 := -\frac{K_1}{c_1} \quad \Rightarrow \quad p(x, y) \geqslant p^-(x) > 0,$$

and p has a nontrivial interface of finite longitudinal size $\Gamma = \partial\{p > 0\} \subset \{x_1 \leqslant x \leqslant x_2\}$ as pictured in Fig. 1. For any $y \in \mathbb{T}^1$ the quantity

$$I(y) := \inf\bigl(x \in \mathbb{R},\ p(x, y) > 0\bigr) \tag{4.5}$$

is well defined by monotonicity, and also by definition $p(x, y) > 0 \Leftrightarrow x > I(y)$. This function $I(.)$ is upper semi-continuous, since its hypograph

$$\bigl\{(x, y),\ x \leqslant I(y)\bigr\} = \bigl\{(x, y),\ p(x, y) \leqslant 0\bigr\} = \bigl\{(x, y),\ p(x, y) = 0\bigr\} = D \setminus D^+$$

is a closed set (D^+ being open by continuity). It has moreover finite amplitude $x_1 \leqslant I(y) \leqslant x_2$ by construction.

5. Assume that $y_0 \in \mathbb{T}^1$ is a point of discontinuity, that is to say $\underline{I}(y_0) := \liminf_{y \to y_0} < I(y_0)$ (remember that I is upper semi-continuous); we prove by double inclusion that, if $\Gamma = \partial\{p > 0\}$, then $\Gamma \cap \{y = y_0\} = [\underline{I}(y_0), I(y_0)] \times \{y = y_0\}$. We write for simplicity $\Gamma_0 := \Gamma \cap \{y = y_0\}$, and let us point out that by definition $p(x, y_0) = 0$ holds for $x \leqslant I(y_0)$.
   - $\boxed{\Gamma_0 \subset [\underline{I}(y_0), I(y_0)] \times \{y = y_0\}}$ Let $(x_0, y_0) \in \Gamma_0$: by continuity we have $p(x_0, y_0) = 0$, and (4.5) implies $x_0 \leqslant I(y_0)$. Assume now by contradiction that $x_0 < \underline{I}(y_0)$: by definition of $\Gamma = \partial D^+$ there exists a sequence $(x_n, y_n) \to (x_0, y_0)$ such that $p(x_n, y_n) > 0$. But by definition of I we have $p(x_n, y_n) > 0 \Leftrightarrow x_n > I(y_n)$ and therefore $x_0 \geqslant \liminf_{y \to y_0} I(y) = \underline{I}(y_0)$.
   - $\boxed{\Gamma_0 \supset [\underline{I}(y_0), I(y_0)] \times \{y = y_0\}}$ Choose any point $(x_0, y_0) \in [\underline{I}(y_0), I(y_0)] \times \{y = y_0\}$: since $p(x_0, y_0) = 0$ and $\Gamma = \overline{D^+}/D^+$ we only need to build a sequence $(x_n, y_n) \to (x_0, y_0)$ such that $(x_n, y_n) \in D^+$. Let $y_n \to y_0$ be any sequence such that $I(y_n) \to \underline{I}(y_0)$. If $x_0 = \underline{I}(y_0)$ define $x_n := I(y_n) + 1/n$: we have that $x_n > I(y_n) \Rightarrow p(x_n, y_n) > 0$ hence $(x_n, y_n) \in D^+$, and clearly $(x_n, y_n) \to (\underline{I}(y_0), y_0)$. If now $x_0 > \underline{I}(y_0)$, define $x_n := x_0$; for n large enough we have again $x_n > I(y_n)$ hence $(x_n, y_n) \in D^+$, and $(x_n, y_n) \to (x_0, y_0)$. □

We prove now the Lipschitz regularity stated in Proposition 1.1:

Proof of Proposition 1.1. Under the non-degeneracy hypothesis $p_x|_{D^+} \geqslant a > 0$ we prove that the graph of $I(y)$ can be obtained as the uniform limit of the ε-levelset of p when $\varepsilon \searrow 0$, and that these levelsets are Lipschitz uniformly in ε.

Let us recall that $p|_{D^+} \in \mathcal{C}^\infty(D^+)$: strict monotonicity in x and the Implicit Functions Theorem show that, for any $\varepsilon > 0$, the ε-levelset Γ_ε of p can be globally parametrized as a smooth graph

$$p(x,y) = \varepsilon \quad \Leftrightarrow \quad x = I_\varepsilon(y),$$

where $I_\varepsilon \in \mathcal{C}^\infty(\mathbb{T}^1)$. Moreover $\frac{\mathrm{d}I_\varepsilon}{\mathrm{d}y} = -\frac{p_y}{p_x}|_{\Gamma_\varepsilon}$, and the non-degeneracy combined with Proposition 4.1 implies

$$\left|\frac{\mathrm{d}I_\varepsilon}{\mathrm{d}y}\right| \leqslant C$$

for some constant C independent of ε. By Arzelà–Ascoli Theorem we can assume, up to extraction, that $I_\varepsilon(.)$ converges to some $J(.)$ uniformly on $\mathbb{T}^1$. This limit is of course Lipschitz, and we show below that $J(y) = I(y)$, where I is defined as in Theorem 4.1 ($p(x,y) > 0 \Leftrightarrow x > I(y)$).

By continuity we have that $p(I_\varepsilon(y), y) = \varepsilon \Rightarrow p(J(y), y) = 0$. Fix $x_0 > J(y)$ and let ε be small enough: integrating $p_x \geqslant a > 0$ from $x = I_\varepsilon(y) < x_0$ to $x = x_0$ leads to $p(x_0, y) \geqslant \varepsilon + a(x_0 - I_\varepsilon(y))$. Taking the limit $\varepsilon \searrow 0$ yields $p(x_0, y) \geqslant a(x_0 - J(y)) > 0$, and therefore

$$\left.\begin{array}{l} p(J(y), y) = 0 \\ x_0 > J(y) \Rightarrow p(x_0, y) > 0 \end{array}\right\} \quad \Rightarrow \quad J(y) = \inf\bigl(x,\ p(x,y) > 0\bigr) = I(y)$$

by definition (4.5) of I. Thus $I = J$ is Lipschitz, and by continuity $\Gamma = \{x = I(y)\}$. □

Proposition 4.2. *The corresponding temperature variable* $v = (\frac{m}{m+1}p)^{\frac{1}{m}} \in \mathcal{C}(D)$ *solves the original equation* $\Delta(v^{m+1}) = (c+\alpha)v_x$ *in the weak sense on the cylinder*: *for any test function* $\Psi \in \mathcal{D}(D)$ *with compact support* K *there holds*

$$\iint_K v^{m+1}\Delta\Psi \,\mathrm{d}x\,\mathrm{d}y + \iint_K (c+\alpha)v\Psi_x \,\mathrm{d}x\,\mathrm{d}y = 0.$$

Proof. We denote by v^L and v^δ the temperature variables corresponding to our two successive approximations p^L and p^δ. Let $\Psi \in \mathcal{D}$ be any such test function with compact support $K \subset D$: let us recall that the finite cylinder grows in both directions, and consequently $K \subset D_L$ for L large enough. $p^L > 0$ was a smooth solution of $-mp\Delta p + (c+\alpha)p_x = |\nabla p|^2$ so that v^L was a smooth solution of $\Delta(v^{m+1}) - (c+\alpha)v_x = 0$, and therefore

$$\iint_K \bigl(v^L\bigr)^{m+1}\Delta\Psi \,\mathrm{d}x\,\mathrm{d}y + \iint_K (c+\alpha)v^L\Psi_x \,\mathrm{d}x\,\mathrm{d}y = 0.$$

When $L \to +\infty$ the $\mathcal{C}^2_{\mathrm{loc}}(D)$ convergence $p^L \to p^\delta$ implies the $\mathcal{C}^0_{\mathrm{loc}}(D)$ convergence $v^L \to v^\delta$, hence

$$\iint_K \bigl(v^\delta\bigr)^{m+1}\Delta\Psi \,\mathrm{d}x\,\mathrm{d}y + \iint_K \bigl(v^\delta\bigr)(c+\alpha)v\Psi_x \,\mathrm{d}x\,\mathrm{d}y = 0.$$

Using the $\mathcal{C}^0_{\mathrm{loc}}(D)$ convergence $p^\delta \to p$ the integrals above finally pass to the limit $\delta \to 0$. □

5. Behavior at infinity

We prove in this section that the behavior at infinity is not perturbed by the shear flow, compared to the classical PME traveling wave $p(x,y) = c[x - x_0]^+$. As mentioned above the results of this section are established directly for the final viscosity solution $p = \lim p^\delta$, but easily extend to the δ-solutions.

Theorem 5.1. $p(x,y)$ *is planar and* x*-linear at infinity, with slope exactly equal to the speed*:

$$p_x(x,y) \sim c, \qquad p_y(x,y) \to 0, \qquad p(x,y) \sim cx$$

uniformly in y *when* $x \to +\infty$.

We start by showing that $p(x,y)$ grows at least and at most linearly for two different slopes; using a Lipschitz scaling under which the equation is invariant, we will deduce that p is exactly linear and that its slope is given by its speed $c>0$. This will be done by proving that in the limit of an infinite zoom-out $(x,y)\to(X,Y)$ the scaled solution $P(X,Y)$ converges to a weak solution of the usual PME ($\alpha\equiv 0$) which has a flat free boundary $X=0$ and is in-between two hyperplanes. By uniqueness for such weak solutions of the PME our solution will agree with the classical planar traveling wave $P(X,Y)=[cX]^+$, hence the slope for $p(x,y)$ at infinity.

5.1. Minimal growth

Since $p_x\leqslant c_1$ we have an upper bound $p\leqslant K_2+c_1x\leqslant(c_1+\varepsilon)x$ for $x\gg 1$; we show in this section that we also have a similar linear lower bound:

Theorem 5.2. *There exists $\underline{C}>0$ such that*

$$x\geqslant 0\quad\Rightarrow\quad p(x,y)\geqslant \underline{C}x.$$

Let us recall that we have pinned

$$K_1\leqslant p(0,y)\leqslant K_2,\qquad K\leqslant\int_{\mathbb{T}^1}p(0,y)\,\mathrm{d}y\leqslant K+C$$

where $K_1\geqslant K-C\sqrt{K}$ and $K_2\leqslant K+C\sqrt{K}$. The constants C above depend only on $m>0$ and the upper bound for the flow $c_1\geqslant c+\alpha(y)$, and $K>0$ can be chosen as large as required (see proof of Proposition 2.3 for details).

We will denote by

$$O(x)=\max_{y\in\mathbb{T}^1}p(x,y)-\min_{y\in\mathbb{T}^1}p(x,y)$$

the oscillations in the y direction, which is a relevant quantity that we will need to control.

Lemma 5.1. *There exist a constant $C>0$ and a sequence $(x_n)_{n\geqslant 0}\in[n,n+1]$ such that*

$$O(x_n)\leqslant C\sqrt{\int_{\mathbb{T}^1}p(n+1,y)\,\mathrm{d}y}.$$

Proof. Integrating by parts $-mp\Delta p+(c+\alpha)p_x=|\nabla p|^2$ over $K_n=[n,n+1]\times\mathbb{T}^1$ we obtain

$$(m-1)\iint_{K_n}|\nabla p|^2\,\mathrm{d}x\,\mathrm{d}y+m\int_{\mathbb{T}^1}pp_x(n,y)\,\mathrm{d}y-m\int_{\mathbb{T}^1}pp_x(n+1,y)\,\mathrm{d}y+\iint_{K_n}(c+\alpha)p_x\,\mathrm{d}x\,\mathrm{d}y=0. \tag{5.1}$$

We distinguish again $m<1$ and $m>1$:

1. If $m<1$ we use $pp_x(n+1,y)>0$, $0<c+\alpha\leqslant c_1$ and $0<p_x\leqslant c_1$ in (5.1) to obtain

$$(1-m)\iint_{K_n}|\nabla p|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant m\int_{\mathbb{T}^1}pp_x(n,y)\,\mathrm{d}y+\iint_{K_n}(c+\alpha)p_x\,\mathrm{d}x\,\mathrm{d}y\leqslant mc_1\int_{\mathbb{T}^1}p(n,y)\,\mathrm{d}y+c_1^2.$$

   Choosing K large enough we can assume by monotonicity that $c_1^2\leqslant mc_1\int p(n,y)\,\mathrm{d}y$, and therefore

$$\iint_{K_n}|\nabla p|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant\frac{2mc_1}{1-m}\int_{\mathbb{T}^1}p(n,y)\,\mathrm{d}y.$$

   If $x_n\in[n,n+1]$ is any point where $O(x)$ attains its minimum on this interval, then

$$O(x_n) \leqslant \int_n^{n+1} O(x)\,dx \leqslant \int_n^{n+1} \left(\int_{\mathbb{T}^1} \left| p_y(x,y) \right| dy \right) dx \leqslant C \sqrt{\iint_{K_n} |\nabla p|^2\,dy}.$$

Using our previous estimate and monotonicity we finally obtain

$$O(x_n) \leqslant C \sqrt{\int_{\mathbb{T}^1} p(n,y)\,dy} \leqslant C \sqrt{\int_{\mathbb{T}^1} p(n+1,y)\,dy},$$

where C depends only on m and c_1.

2. If $m > 1$ we use $pp_x(n,y) > 0$, $(c+\alpha)p_x > 0$ and $p_x \leqslant c_1$ in (5.1), yielding

$$(m-1) \iint_{K_n} |\nabla p|^2\,dx\,dy \leqslant m \int_{\mathbb{T}^1} pp_x(n+1,y)\,dy \leqslant mc_1 \int_{\mathbb{T}^1} p(n+1,y)\,dy.$$

The rest of the computation is similar to the case $m < 1$. □

Corollary 5.1. *There exists $C > 0$ such that*

$$x \geqslant 0 \quad \Rightarrow \quad O(x) \leqslant C \sqrt{\int_{\mathbb{T}^1} p(x,y)\,dy}.$$

Proof. Since $0 < p_x \leqslant c_1$ the function $O(x)$ is clearly $2c_1$-Lipschitz, and by Lemma 5.1 ensures that

$$O(x) \leqslant O(x_n) + 2c_1 \leqslant C \sqrt{\int_{\mathbb{T}^1} p(n+1,y)\,dy} + 2c_1$$

for any $x \in [n, n+1]$. By monotonicity we can moreover assume that $2c_1 \leqslant C\sqrt{\int p(n+1,y)\,dy}$ if K is chosen large enough, and therefore

$$O(x) \leqslant C \sqrt{\int_{\mathbb{T}^1} p(n+1,y)\,dy}.$$

For the same reason we can also assume that

$$\int p(n+1,y)\,dy \leqslant \int p(x,y)\,dy + c_1 \leqslant C \int p(x,y)\,dy,$$

and combining with the previous inequality yields the desired result. □

Proposition 5.1. *For any $x \geqslant 0$ we have that*

$$\frac{d}{dx} \left(\int_{\mathbb{T}^1} p^{\frac{m+1}{m}}(x,y)\,dy \right) = \frac{m+1}{m} \int_{\mathbb{T}^1} \big(c + \alpha(y)\big) p^{\frac{1}{m}}(x,y)\,dy.$$

Proof. We establish this equality for the uniformly elliptic solution $p^\delta \geqslant \delta$ up to a constant C_δ, with $C_\delta \to 0$ when $\delta \to 0$. The equation for p^δ can be written in the divergence form

$$\nabla \cdot \left(\left(p^\delta\right)^{\frac{1}{m}} \nabla p^\delta \right) = \left((c+\alpha)\left(p^\delta\right)^{\frac{1}{m}} \right)_x,$$

and integrating by parts over $\Omega = [x_1, x_2] \times \mathbb{T}^1$ yields

$$\int_{\mathbb{T}^1} (p^\delta)^{\frac{1}{m}} p_x^\delta(x_2, y)\,\mathrm{d}y - \int_{\mathbb{T}^1} (p^\delta)^{\frac{1}{m}} p_x^\delta(x_1, y)\,\mathrm{d}y = \int_{\mathbb{T}^1} (c+\alpha)(p^\delta)^{\frac{1}{m}}(x_2, y)\,\mathrm{d}y - \int_{\mathbb{T}^1} (c+\alpha)(p^\delta)^{\frac{1}{m}}(x_1, y)\,\mathrm{d}y$$

for any $x_1 < x_2$. As a consequence, the quantity

$$F(x) := \int_{\mathbb{T}^1} (p^\delta)^{\frac{1}{m}} p_x^\delta(x, y)\,\mathrm{d}y - \int_{\mathbb{T}^1} (c+\alpha)(p^\delta)^{\frac{1}{m}}(x, y)\,\mathrm{d}y \equiv C_\delta \tag{5.2}$$

is constant. Let us recall from Proposition 3.1 that $p^\delta(-\infty, y) = \delta$ uniformly in y, and also the uniform bounds $c_0 \leqslant c + \alpha \leqslant c_1$ and $0 < p_x^\delta \leqslant c_1$: taking the limit $x \to -\infty$ in (5.2) leads to $C_\delta = \mathcal{O}(\delta^{\frac{1}{m}})$.

Fix any $x > 0$: the strong C^1_{loc} convergence $p^\delta \to p$ on $D^+ = \{p > 0\}$ is strong enough to take the limit by δ in (5.2), which reads

$$\int_{\mathbb{T}^1} p^{\frac{1}{m}} p_x(x, y)\,\mathrm{d}y - \int_{\mathbb{T}^1} (c+\alpha) p^{\frac{1}{m}}(x, y)\,\mathrm{d}y = 0.$$

Finally, p is smooth for $x > 0$ (because $p > 0$), and the last equality above easily yields the desired differential equation. □

We can now prove the claimed minimal growth:

Proof of Theorem 5.2. Define $f(x) := \int_{\mathbb{T}^1} p^{\frac{m+1}{m}}(x, y)\,\mathrm{d}y$: Proposition 5.1 reads

$$f'(x) = \frac{m+1}{m} \int_{\mathbb{T}^1} (c+\alpha) p^{\frac{1}{m}}\,\mathrm{d}y \tag{5.3}$$

for $x > 0$. By monotonicity $\int_{\mathbb{T}^1} p(x, y)\,\mathrm{d}y \geqslant \int_{\mathbb{T}^1} p(0, y)\,\mathrm{d}y = K$, and by Corollary 5.1 we control the oscillations $O(x) \leqslant C\sqrt{\int p(x, y)\,\mathrm{d}y}$. Choosing K large enough the oscillations of p are small compared to its average along *any* line $x = cst \geqslant 0$, hence

$$\int_{\mathbb{T}^1} (c+\alpha) p^{\frac{1}{m}}\,\mathrm{d}y \geqslant c_0 \int_{\mathbb{T}^1} p^{\frac{1}{m}}\,\mathrm{d}y \geqslant C\left(\int_{\mathbb{T}^1} p^{\frac{m+1}{m}}\,\mathrm{d}y\right)^{\frac{1}{m+1}} = Cf^{\frac{1}{m+1}}(x).$$

This estimate combined with (5.3) leads to $f'(x) \geqslant Cf^{\frac{1}{m+1}}(x)$, and integration yields

$$f^{\frac{m}{m+1}}(x) \geqslant Cx.$$

Finally, since we control the oscillations of p,

$$p(x, y) \geqslant C \int_{\mathbb{T}^1} p(x, y)\,\mathrm{d}y \geqslant C\left(\int_{\mathbb{T}^1} p^{\frac{m+1}{m}}(x, y)\,\mathrm{d}y\right)^{\frac{m}{m+1}} \geqslant Cf^{\frac{m}{m+1}}(x) \geqslant \underline{C}x. \qquad \square$$

5.2. Proof of Theorem 5.1

We start by estimating how fast p becomes planar at infinity:

Proposition 5.2. *Let as before* $O(x) := \max_{y\in\mathbb{T}^1} p(x, y) - \min_{y\in\mathbb{T}^1} p(x, y)$; *there exists* $C > 0$ *such that when* $x \to +\infty$

$$O(x) \leqslant \frac{C}{x}.$$

Proof. For x large enough we know that $p(x,y)>0$ is smooth; $w:=\frac{m^2}{m+1}p^{\frac{m+1}{m}}$ is therefore smooth, and satisfies as before

$$\Delta(w)=f, \qquad f=(c+\alpha)p^{\frac{1}{m}-1}p_x.$$

We will first show that the y-oscillations of w cannot blow too fast when $x\to+\infty$, and then deduce the desired planar behavior for p.

The Fourier series

$$w(x,y)=\sum_{n\in\mathbb{Z}}w_n(x)e^{2i\pi ny}$$

is at least pointwise convergent, and for $n\neq 0$ we have that

$$-w_n''(x)+4\pi^2n^2w_n(x)=f_n(x), \qquad f_n(x):=-\!\!\!\!\!\!\int_{\mathbb{T}^1} f(x,y)e^{-2i\pi ny}\,\mathrm{d}y. \tag{5.4}$$

The oscillations of w in the y direction are completely described by its Fourier coefficients $w_n(x)$ for $n\neq 0$, in which case (5.4) is strongly coercive. This coercivity will allow to control how fast $w_n(x)$ may grow when $x\to+\infty$, and therefore how much w can oscillate.

Since p is at least and at most linear and p_x, $c+\alpha$ are bounded we control

$$|f_n|(x)\leqslant Cx^{\frac{1}{m}-1} \tag{5.5}$$

uniformly in n. Moreover, taking real and imaginary parts of (5.4), we may assume that $w_n(x)$, $f_n(x)$ are real and that $n=|n|\geqslant 0$.

- We claim that there exists $C>0$ such that, for any $n\neq 0$ and $x\to+\infty$, there holds

  $$\left|w_n(x)\right|\leqslant \frac{C}{n^2}x^{\frac{1}{m}-1}. \tag{5.6}$$

  Indeed, since $0\leqslant w=\frac{m^2}{m+1}p^{\frac{m+1}{m}}\leqslant Cx^{\frac{m+1}{m}}$, we have that

  $$|w_n|^2(x)\leqslant \left\|w(x,.)\right\|^2_{L^2(\mathbb{T}^1)}\leqslant Cx^{2\frac{m+1}{m}}.$$

  As a consequence w_n cannot have a component on the homogeneous solution $e^{+2\pi nx}$ of (5.4) for $n\neq 0$, and it is then easy to see that it is explicitly given by

  $$w_n(x)=e^{-2\pi n(x-x_0)}w_n(x_0)+e^{-2\pi nx}\int_{x_0}^{x}e^{4\pi nz}\left(\int_z^{+\infty}e^{-2\pi nt}f_n(t)\,\mathrm{d}t\right)\mathrm{d}z. \tag{5.7}$$

  Our claim (5.6) is then easily obtained manipulating this explicit formula, the computations involving several integrations by parts and the fact that $w_n(x_0)$ is rapidly decreasing in n (since $w(x_0,.)\in\mathcal{C}^\infty(\mathbb{T}^1)$).
- As a consequence of (5.6), the series

  $$w^\perp(x,y):=w(x,y)-\!\!\!\!\!\!\int_{\mathbb{T}^1}w(x,y)\,\mathrm{d}y=\sum_{n\neq 0}w_n(x)e^{2i\pi ny}$$

  is uniformly convergent and $|w^\perp(x,y)|\leqslant Cx^{\frac{1}{m}-1}$. This clearly bounds the oscillations of w when $x\to+\infty$ by

  $$\max_{y\in\mathbb{T}^1}w(x,y)-\min_{y\in\mathbb{T}^1}w(x,y)\leqslant 2\left\|w^\perp(x,.)\right\|_{L^\infty(\mathbb{T}^1)}\leqslant Cx^{\frac{1}{m}-1}. \tag{5.8}$$

Translating the oscillations of w in terms of those of $p=Cw^{\frac{m}{m+1}}$ leads to

$$\begin{aligned}O(x)&=C\Big(\max_{y\in\mathbb{T}^1}w^{\frac{m}{m+1}}(x,y)-\min_{y\in\mathbb{T}^1}w^{\frac{m}{m+1}}(x,y)\Big)\\&\leqslant C\Big(\min_{y\in\mathbb{T}^1}w(x,y)\Big)^{\frac{m}{m+1}-1}\Big[\max_{y\in\mathbb{T}^1}w(x,y)-\min_{y\in\mathbb{T}^1}w(x,y)\Big].\end{aligned}$$

Since $w=\frac{m^2}{m+1}p^{\frac{m+1}{m}}\geqslant Cx^{\frac{m+1}{m}}$ and $\frac{m}{m+1}-1=-\frac{1}{m+1}$, estimate (5.8) finally implies that

$$O(x)\leqslant C\big(x^{\frac{m+1}{m}}\big)^{-\frac{1}{m+1}}\times Cx^{\frac{1}{m}-1}=\frac{C}{x}. \qquad \square$$

For any $\varepsilon>0$ let us introduce the Lipschitz scaling

$$P^\varepsilon(X,Y)=\varepsilon p(x,y), \qquad (x,y)=\frac{1}{\varepsilon}(X,Y);$$

when $\varepsilon\searrow 0$ this corresponds to zooming out on the whole picture. Uppercase letters will denote below the "fast" variables and functions, whereas lowercase will denote the "slow" ones. Since we want to zoom out it will be more convenient to consider below the cylinder $D=\mathbb{R}\times\mathbb{T}^1$ as a plane $\mathbb{R}^2$ with a 1-periodicity condition for p in the y direction, corresponding to a plane with ε-periodicity in Y for P^ε.

The proof of Theorem 5.1 relies on three key points: the first one is that the equation is invariant under this scaling. The second one is that, since the shear flow $\alpha(y)$ is 1-periodic with mean-zero, the corresponding flow $A^\varepsilon(Y)=\alpha(Y/\varepsilon)$ is ε-periodic with mean-zero in Y: Riemann–Lebesgue Theorem guarantees that $A^\varepsilon\rightharpoonup 0$ in a weak sense when $\varepsilon\to 0$, so that any limiting profile $P=\lim P^\varepsilon$ will not "see the flow" and thus satisfy the usual PME $-mP\Delta P+(c+0)P_X=|\nabla P|^2$. Finally, Proposition 5.2 guarantees that the oscillations of p in the y direction decrease at infinity: zooming out, the limit P will therefore be planar, $P_Y\equiv 0$.

In the limit of this infinite zoom-out the scaled profile indeed converges:

Proposition 5.3. *Up to a subsequence we have $P^\varepsilon(X,Y)\to P(X,Y)$ when $\varepsilon\searrow 0$. The convergence is uniform on $\mathbb{R}^-\times\mathbb{R}$ and $\mathcal{C}^1_{\mathrm{loc}}$ on $\mathbb{R}^{+*}\times\mathbb{R}$. Further:*

1. *P is continuous on the whole plane and $P\equiv 0$ for $X\leqslant 0$.*
2. *$0<\underline{C}X\leqslant P(X,Y)\leqslant c_1X$ for $X>0$, where $\underline{C}>0$ is the constant in Theorem 5.2 and $c_1\geqslant c+\alpha(y)$ is the upper bound for the flow.*

Proof. We pinned the original solution p such that $0\leqslant p(x,y)\leqslant K_2$ for $x\leqslant 0$, and this immediately implies that $P^\varepsilon=\varepsilon p\leqslant\varepsilon K_2\to 0$ uniformly on the closed left half-plane $X\leqslant 0$. On the right half-plane $0<P^\varepsilon_X(X,Y)=p_x(x,y)\leqslant c_1$ bounds P^ε from above as

$$P^\varepsilon(X,Y)\leqslant P^\varepsilon(0,Y)+c_1X\leqslant K_2\varepsilon+c_1X, \tag{5.9}$$

and Theorem 5.2 bounds P^ε away from zero

$$P^\varepsilon(X,Y)=\varepsilon p(X/\varepsilon,Y/\varepsilon)\geqslant\underline{C}X. \tag{5.10}$$

Let us recall that p is a smooth classical solution on $D^+=\{p>0\}\supset\mathbb{R}^+\times\mathbb{T}^1$: for $\varepsilon>0$ the rescaled profile P^ε is therefore a smooth classical solution of the rescaled equation

$$-mP^\varepsilon\Delta_{X,Y}P^\varepsilon+\big[c+A^\varepsilon(Y)\big]P^\varepsilon_X=\big|\nabla_{X,Y}P^\varepsilon\big|^2, \qquad A^\varepsilon(Y)=\alpha(Y/\varepsilon),$$

at least for $X>0$.

Using our previous interior elliptic L^q regularity argument for

$$W^\varepsilon:=\frac{m^2}{m+1}\big(P^\varepsilon\big)^{1+\frac{1}{m}}, \qquad F^\varepsilon:=\big(c+A^\varepsilon\big)\big(P^\varepsilon\big)^{\frac{1}{m}-1}P^\varepsilon_X, \qquad \Delta W^\varepsilon=F^\varepsilon,$$

we obtain as before an estimate

$$\big\|W^\varepsilon\big\|_{W^{2,q}(\mathcal{B}_1)}\leqslant C$$

uniformly in ε on any ball $\mathcal{B}_1\subset\mathbb{R}^{+*}\times\mathbb{R}$ of radius 1 and for $q>d=2$ (see proof of Theorem 3.1 for details). It is here important that P^ε is bounded away from zero uniformly in ε for $X>0$, see again proof of Theorem 3.1 (in particular the case $m<1$). By compactness $W^{2,q}\Subset\mathcal{C}^1$ on bounded balls ($q>d=2$) and moving the center of the ball $\mathcal{B}_1$ we may assume, up to extraction of a subsequence, that

$$W^\varepsilon\to W \quad \text{in } \mathcal{C}^1_{\mathrm{loc}}\big(\mathbb{R}^{+*}\times\mathbb{R}\big).$$

Since we took care to step out of the zero set uniformly in ε, this convergence easily translates into

$$P^\varepsilon \xrightarrow{\mathcal{C}^1_{\mathrm{loc}}(\mathbb{R}^{+*}\times\mathbb{R})} P,$$

and P is continuous on $\mathbb{R}^{+*}\times\mathbb{R}$ as a locally uniform limit of continuous functions. Taking the limit $\varepsilon\to 0$ for $X>0$ in $\underline{C}X \leqslant P^\varepsilon(X,Y)\leqslant K_2\varepsilon + c_1X$ we obtain

$$X>0 \quad\Rightarrow\quad \underline{C}X\leqslant P(X,Y)\leqslant c_1X$$

as claimed, which gives as a by product the continuity along $X=0$ (let us recall that $P\equiv 0$ on the left half-plane). □

Remark 6. No higher regularity can be obtained with this interior elliptic regularity argument: $\mathcal{C}^2$ convergence would require for example $W^{3,q}$ estimates, involving $\nabla_{(X,Y)}F^\varepsilon$ which contains the singular derivative $\partial_Y A^\varepsilon = \frac{1}{\varepsilon}\partial_y\alpha$.

As usual we need to determine the limiting equation satisfied by the limiting profile in some sense:

Proposition 5.4. *The limiting function P solves the PME*

$$-mP\Delta_{(X,Y)}P + cP_X = |\nabla_{(X,Y)}P|^2$$

in the weak sense on the whole plane.

Proof. By definition of solutions we want to prove that, for any test function $\Phi(X,Y)$ with compact support $K\subset\mathbb{R}^2$, the corresponding temperature $V(X,Y) := (\frac{m}{m+1}P(X,Y))^{\frac{1}{m}}$ satisfies

$$I := \iint\limits_K V^{m+1}\Delta\Phi\,\mathrm{d}X\,\mathrm{d}Y + \iint\limits_K cV\Phi_X\,\mathrm{d}X\,\mathrm{d}Y = 0$$

(note that the shear flow $A^\varepsilon(Y)\leftrightarrow\alpha(y)$ disappeared in the advection term). Let us recall from Proposition 4.2 that p was a weak solution on the whole plane, and that the equation is invariant under Lipschitz scaling: for any $\varepsilon>0$ the scaled temperature V^ε therefore satisfies

$$I(\varepsilon) := \iint\limits_K \left(V^\varepsilon\right)^{m+1}\Delta\Phi\,\mathrm{d}X\,\mathrm{d}Y + \iint\limits_K \left(c + A^\varepsilon\right)V^\varepsilon\Phi_X\,\mathrm{d}X\,\mathrm{d}Y = 0; \tag{5.11}$$

the problem is as usual to take the limit in this formulation.

- If $K\subset\mathbb{R}^{-*}\times\mathbb{R}$ this limit is straightforward: $(c+A^\varepsilon)$ is uniformly bounded ($c_0\leqslant c+A^\varepsilon\leqslant c_1$), and according to Proposition 5.3 $V^\varepsilon = (\frac{m}{m+1}P^\varepsilon)^{\frac{1}{m}}\to 0$ uniformly on K.
- If $K\subset\mathbb{R}^{+*}\times\mathbb{R}$ the limit V is positive so there is no such trivial convergence; it is convenient to split (5.11) in three parts $I = I_1 + I_1 + I_3 = 0$, with

$$I_1(\varepsilon) := \iint\limits_K \left(V^\varepsilon\right)^{m+1}\Delta\Phi\ \mathrm{d}X\ \mathrm{d}Y,$$

$$I_2(\varepsilon) := c\iint\limits_K V^\varepsilon\Phi_X\,\mathrm{d}X\,\mathrm{d}Y,$$

$$I_3(\varepsilon) := \iint\limits_K A^\varepsilon V^\varepsilon\Phi_X\,\mathrm{d}X\,\mathrm{d}Y.$$

The $\mathcal{C}^0_{\mathrm{loc}}$ convergence $P^\varepsilon\to P$ shows that I_1 and I_2 immediately pass to the limit. To deal with I_3 we compute with Fubini Theorem

$$I_3(\varepsilon) = \iint\limits_K A^\varepsilon V^\varepsilon\Phi_X\,\mathrm{d}X\,\mathrm{d}Y = \int\limits_{\mathbb{R}} A^\varepsilon(Y)\underbrace{\left(\int\limits_{\mathbb{R}} V^\varepsilon(X,Y)\Phi_X(X,Y)\,\mathrm{d}X\right)}_{:=\Psi^\varepsilon(Y)}\mathrm{d}Y;$$

since Φ has compact support and $V^\varepsilon \to V$ uniformly on K we deduce that $\Psi^\varepsilon(Y) \to \Psi(Y)$ uniformly on $\mathbb{R}$. Ψ^ε and Ψ have both compact support: the convergence $\Psi^\varepsilon \to \Psi$ therefore also holds in $L^1(\mathbb{R})$, and by Riemann–Lebesgue Theorem $A^\varepsilon \rightharpoonup 0$ weakly in $L^1(\mathbb{R})$ (let us recall that $A^\varepsilon(Y)$ is ε-periodic with mean-zero). $I_3(\varepsilon)$ is therefore a dual pairing $I_3(\varepsilon) = \langle A^\varepsilon, \Psi^\varepsilon\rangle_{(L'_1, L1)}$ of a weakly converging sequence with a strongly convergent one: hence the limit $I_3(\varepsilon) \to 0$.

- If $K \cap \{X = 0\} \neq \emptyset$ the convergence is more delicate because K crosses the free boundary and we do not have uniform convergence $V^\varepsilon \to V$ on K; however since $P(0, Y) = 0$ both V and V^ε have to be small on a neighborhood of $K \cap \{X = 0\}$. For small $r > 0$ we prove that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \leqslant \varepsilon_0$ there holds $|I - I(\varepsilon)| \leqslant r$.

  For $\eta > 0$ to be chosen later let us define the partition

$$K = \underbrace{\big(K \cap \{X < -\eta\}\big)}_{:=K^-} \cup \underbrace{\big(K \cap \big\{|X| \leqslant \eta\big\}\big)}_{:=K^\eta} \cup \underbrace{\big(K \cap \{X > +\eta\}\big)}_{:=K^+};$$

  K^η is a striped η-neighborhood of $K \cap \{X = 0\}$. On $K^\pm$ we already proved that I_1, I_2, I_3 converge: we only have to cope with the contribution from K^η, and it is clearly enough to prove separately

$$\begin{aligned}
&\iint_{K^\eta} \big|\big(V^\varepsilon\big)^{m+1} - V^{m+1}\big|.|\Delta\Phi|\,\mathrm{d}X\,\mathrm{d}Y \leqslant \frac{r}{3},\\
&c\iint_{K^\eta} \big|V^\varepsilon - V\big|.|\Phi_X|\,\mathrm{d}X\,\mathrm{d}Y \leqslant \frac{r}{3},\\
&\iint_{K^\eta} \big|A^\varepsilon\big|.\big|V^\varepsilon - V\big|.|\Phi_X|\,\mathrm{d}X\,\mathrm{d}Y \leqslant \frac{r}{3}.
\end{aligned} \tag{5.12}$$

  Let us recall the previous bounds for the pressure variables P and P^ε derived from the scaling and the Lipschitz estimate in the X direction:

$$-\eta \leqslant X \leqslant 0: \quad \begin{cases} 0 \leqslant P^\varepsilon \leqslant K_2\varepsilon, \\ P \equiv 0, \end{cases}$$

$$0 \leqslant X \leqslant \eta: \quad \begin{cases} 0 \leqslant P^\varepsilon(X, Y) \leqslant P^\varepsilon(0, Y) + c_1 X \leqslant K_2\varepsilon + c_1 X, \\ P \leqslant c_1 X. \end{cases}$$

  Choosing η and ε small, any positive power of the pressures P^ε, P can clearly be made as small as required on K^η; this is also true for any positive power of the corresponding temperatures V^ε, V (being themselves positive powers of the pressure), and all the terms $|\Delta\Phi|$, $|\Phi_x|$, $|A^\varepsilon|$ are bounded uniformly in ε: we complete the proof using the celebrated triangular inequality in the integrals (5.12). □

We can now finally prove Theorem 5.1:

Proof of Theorem 5.1. By Proposition 5.3 and up to extraction we have that $P^\varepsilon \to P$ uniformly on $\mathbb{R}^- \times \mathbb{R}$ and locally in $\mathcal{C}^1(\mathbb{R}^{+*} \times \mathbb{R})$; the corresponding temperature $V \geqslant 0$ is moreover a weak solution of the stationary PME $-\Delta_{X,Y}(V^{m+1}) + cV_X = 0$ (previous proposition). A flat free boundary $X = 0$ separates $P \equiv 0$ to the left from $P > 0$ to the right, where the pressure profile is in-between two hyperplanes $\underline{C}X \leqslant P(X, Y) \leqslant c_1 X$. According to Proposition 5.2 the transversal y-oscillations of p are bounded on the whole cylinder: according to our scaling this means that the Y-oscillations of $P^\varepsilon = \varepsilon p$ are of order ε, thus $P(X, Y) = P(X)$ only in the limit $\varepsilon \searrow 0$.

It is well known that there exists only one such *planar* solution of the PME, which is the standard planar traveling wave

$$P(X, Y) = [cX]^+.$$

Since the limit is unique the whole sequence actually converges, $\lim_{\varepsilon\to 0} P^\varepsilon = P$. For any $x_\varepsilon = \frac{1}{\varepsilon} \to +\infty$ the $\mathcal{C}^0_{\mathrm{loc}}$ convergence $P^\varepsilon(X, Y) \to [cX]^+$ on $\mathbb{R}^{+*} \times \mathbb{R}$ then implies

$$\begin{aligned}\max_{y\in\mathbb{T}^1}\left|p(x_\varepsilon,y)-cx_\varepsilon\right| &= \max_{Y\in[0,\varepsilon]}\left|\frac{1}{\varepsilon}P^\varepsilon(\varepsilon x_\varepsilon,Y)-cx_\varepsilon\right| \\ &= x_\varepsilon \max_{Y\in[0,\varepsilon]}\left|P^\varepsilon(1,Y)-P(1,Y)\right| = o(x_\varepsilon),\end{aligned}$$

which means precisely $p(x,y)\sim cx$ when $x\to+\infty$ (uniformly in y) as in our statement. Using now the stronger $\mathcal{C}^1_{\mathrm{loc}}$ convergence for $X>0$ we finally obtain

$$\max_{y\in\mathbb{T}^1}\left|p_x(x_\varepsilon,y)-c\right| = \max_{Y\in[0,\varepsilon]}\left|P^\varepsilon_X(1,Y)-P_X(1,Y)\right| = \underset{\varepsilon\to0}{o}(1) \quad\Rightarrow\quad p_x\sim c,$$

$$\max_{y\in\mathbb{T}^1}\left|p_y(x_\varepsilon,y)-0\right| = \max_{Y\in[0,\varepsilon]}\left|P^\varepsilon_Y(1,Y)-P_Y(1,Y)\right| = \underset{\varepsilon\to0}{o}(1) \quad\Rightarrow\quad p_y\to 0$$

as claimed. □

5.3. *Asymptotic expansion at infinity*

We have shown that $p(x,y)\sim cx$ uniformly in y when $x\to+\infty$. In this section we strengthen this estimate and derive and asymptotic expansion

$$p(x,y)=cx+q(x,y)$$

with $W^{1,\infty}$ estimates on q as $x\to+\infty$.

For any function $f(x,y)$ periodic in the y direction, we denote the average (the projection onto constants in $L^2(\mathbb{T}^1)$) by

$$\langle f\rangle(x) := \int_{\mathbb{T}^1} f(x,y)\,\mathrm{d}y.$$

The orthogonal projection onto functions with mean-zero is denoted by

$$f^\perp(x,y) := f(x,y)-\langle f\rangle(x).$$

The x derivative commutes with both these projectors, $\frac{\mathrm{d}}{\mathrm{d}x}\langle f\rangle=\langle f_x\rangle$ and $(f_x)^\perp=(f^\perp)_x$. The ansatz $p(x,y)=cx+q(x,y)$ gives

$$\langle p\rangle(x)=cx+\langle q\rangle(x), \qquad p^\perp(x,y)=q^\perp(x,y),$$

and q, $\langle q\rangle$, $q^\perp$ are $o(x)$. The main result of this section is

Theorem 5.3. *When $x\to+\infty$, we have that*:

1. *For any $m\neq1$ the correction $q(x,y)$ becomes planar*: *there exists $C>0$ such that*
$$\left|q^\perp\right|(x,y)+\left|\nabla q^\perp\right|(x,y)\leqslant\frac{C}{x}.$$
2. *Assume in addition that $1<m\notin\mathbb{N}^*$, and let $N=[m]$*: *there exist a finite sequence $q_1,\dots,q_N\in\mathbb{R}$ and some $q^*\in\mathbb{R}$ such that*
$$q(x,y)=x\left(q_1x^{-\frac{1}{m}}+q_2x^{-\frac{2}{m}}+\dots+q_Nx^{-\frac{N}{m}}\right)+q^*+o(1).$$

The orthogonal projection $p^\perp(x,y)$ is controlled by the oscillations in the y direction $|p^\perp(x,y)|\leqslant O(x)=\max_{y\in\mathbb{T}^1}p(x,y)-\min_{y\in\mathbb{T}^1}p(x,y)$, and Proposition 5.2 therefore implies that

$$\left|q^\perp\right|(x,y)=\left|p^\perp\right|(x,y)\leqslant\frac{C}{x} \tag{5.13}$$

when $x\to+\infty$.

We prove the first estimate of the theorem as a separate proposition.

Proposition 5.5. *There exists $C > 0$ such that*

$$\left|q^{\perp}(x, y)\right| + \left|\nabla q^{\perp}(x, y)\right| \leqslant \frac{C}{x}.$$

Let us stress that this statement holds for any m, although we will specifically consider $m > 1$ in the sequel.

Proof of Proposition 5.5. By (5.13) we already control $|q^{\perp}|$, and it is enough to control its gradient. The equation for p reads

$$\Delta p = \frac{(c+\alpha)p_x}{mp} - \frac{|\nabla p|^2}{mp}, \tag{5.14}$$

and when $x \to +\infty$ we know that $\nabla p \to (c, 0)$ and $p \sim cx$ uniformly in y: as a consequence $|\Delta p| \leqslant \frac{C}{x}$. Averaging in y yields $|\langle p\rangle''| \leqslant \frac{C}{x}$, and therefore

$$\left|\Delta\left(q^{\perp}\right)\right| = \left|\Delta\left(p^{\perp}\right)\right| = \left|\Delta p - \langle p\rangle''\right| \leqslant \frac{C}{x}.$$

Choose now x_0 large and $y_0 \in \mathbb{T}^1$, and denote by $\mathcal{B}_1$ the ball of radius 1 centered at (x_0, y_0). As discussed above there exists $C > 0$ such that, if x_0 is chosen large enough,

$$(x, y) \in \mathcal{B}_1 \quad \Rightarrow \quad \begin{cases} \left|q^{\perp}\right|(x, y) \leqslant \dfrac{C}{x_0}, \\ \left|\Delta q^{\perp}\right|(x, y) \leqslant \dfrac{C}{x_0}. \end{cases}$$

The constants above depend on the radius of the ball $R = 1$ but not on its center. Finally, the classical elliptic theory for Poisson equation on a ball controls the gradient at the center by $|\nabla q^{\perp}|(x_0, y_0) \leqslant C(\|q^{\perp}\|_{L^\infty(\mathcal{B}_1)} + \|\Delta q^{\perp}\|_{L^\infty(\mathcal{B}_1)})$, with C depending only on the radius of the ball. □

As a corollary, we have that

Lemma 5.2. *If $m > 1$, there exists $\lambda \in \mathbb{R}$ such that*

$$\langle q\rangle'(x) = \frac{\lambda}{(cx + \langle q\rangle)^{\frac{1}{m}}} + \mathcal{O}\left(\frac{1}{x^2}\right) \tag{5.15}$$

holds when $x \to +\infty$.

This technical result will later allow to establish the asymptotic expansion $q = x(\dots)$ stated in Theorem 5.3.

Proof of Lemma 5.2. Eq. (5.14) with $p(x, y) = cx + q(x, y)$ leads to

$$m\Delta q = \frac{(\alpha - c)q_x}{cx + q} - \frac{|\nabla q|^2}{cx + q} + \frac{c\alpha}{cx + q}. \tag{5.16}$$

By Proposition 5.5 we control $|q^{\perp}| = \mathcal{O}(1/x)$, and it is easy to expand

$$\frac{1}{cx + q} = \frac{1}{cx + \langle q\rangle + q^{\perp}} = \frac{1}{cx + \langle q\rangle}\left(1 - \frac{q^{\perp}}{cx + \langle q\rangle} + \mathcal{O}\left(\frac{1}{x^4}\right)\right).$$

This expansion allows to estimate separately the three terms in the right-hand side of (5.16), and in particular their average in y.

- The first one is

$$A(x,y) := \frac{1}{cx+q}(\alpha - c)q_x = -\frac{c}{cx+\langle q\rangle}\langle q\rangle' + \frac{(\alpha q^\perp)_x}{cx+\langle q\rangle} - \frac{\langle q\rangle'}{(cx+\langle q\rangle)^2}(\alpha q^\perp) + \underbrace{\frac{\langle q\rangle'}{cx+\langle q\rangle}\alpha - \frac{c}{cx+\langle q\rangle}(q_x)^\perp}_{\text{purely orthogonal}} + \underbrace{\mathcal{O}\left(\frac{1}{x^3}\right)}_{\text{lower order}}.$$

Averaging in y then yields

$$\langle A\rangle(x) = -\frac{c}{cx+\langle q\rangle}\langle q\rangle' + \frac{\langle \alpha q^\perp\rangle'}{cx+\langle q\rangle} - \frac{\langle q\rangle'}{(cx+\langle q\rangle)^2}\langle \alpha q^\perp\rangle + \mathcal{O}\left(\frac{1}{x^3}\right). \tag{5.17}$$

- We expand the second one as

$$\begin{aligned} B(x,y) &:= \frac{1}{cx+q}|\nabla q|^2 = \frac{1+\mathcal{O}(\frac{1}{x^2})}{cx+\langle q\rangle}\left[\left(\langle q\rangle'\right)^2 + \left|\nabla q^\perp\right|^2 + 2\langle q\rangle' q_x^\perp\right] \\ &= \frac{\langle q\rangle'}{cx+\langle q\rangle}\langle q\rangle' + \underbrace{\frac{2\langle q\rangle'}{cx+\langle q\rangle}(q_x)^\perp}_{\text{purely orthogonal}} + \underbrace{\mathcal{O}\left(\frac{1}{x^3}\right)}_{\text{lower order}}, \end{aligned}$$

and averaging leads to

$$\langle B\rangle(x) = \frac{\langle q\rangle'}{cx+\langle q\rangle}\langle q\rangle' + \mathcal{O}\left(\frac{1}{x^3}\right). \tag{5.18}$$

- The last term is

$$C(x,y) := \frac{1}{cx+q}c\alpha = -\frac{c}{(cx+\langle q\rangle)^2}\left(\alpha q^\perp\right) + \underbrace{\frac{c}{cx+\langle q\rangle}\alpha}_{\text{purely orthogonal}} + \underbrace{\mathcal{O}\left(\frac{1}{x^5}\right)}_{\text{lower order}},$$

and finally

$$\langle C\rangle(x) = -\frac{c}{(cx+\langle q\rangle)^2}\langle \alpha q^\perp\rangle + \mathcal{O}\left(\frac{1}{x^5}\right). \tag{5.19}$$

Averaging (5.16) in y reads $m\langle q\rangle''(x) = \langle A\rangle(x) - \langle B\rangle(x) + \langle C\rangle(x)$: taking advantage of (5.17)–(5.19) and rearranging, we obtain

$$m\langle q\rangle'' + \frac{c+\langle q\rangle'}{cx+\langle q\rangle}\langle q\rangle' = \left(\frac{\langle \alpha q^\perp\rangle}{cx+\langle q\rangle}\right)' + \mathcal{O}\left(\frac{1}{x^3}\right).$$

Multiplying by the integrating factor $(cx+\langle q\rangle)^{\frac{1}{m}}$ yields

$$\left((cx+\langle q\rangle)^{\frac{1}{m}}\langle q\rangle'\right)' = \frac{(cx+\langle q\rangle)^{\frac{1}{m}}}{m}\left(\frac{\langle \alpha q^\perp\rangle}{cx+\langle q\rangle}\right)' + \mathcal{O}\left(x^{\frac{1}{m}-3}\right). \tag{5.20}$$

If $f(x) := \frac{(cx+\langle q\rangle)^{\frac{1}{m}}}{m}(\frac{\langle \alpha q^\perp\rangle}{cx+\langle q\rangle})'$ denotes the first term in the right-hand side above, an integration by parts combined with $|q^\perp| \leqslant C/x \Rightarrow |\langle \alpha q^\perp\rangle| \leqslant C/x$ allows to show that f is integrable at infinity and that

$$\int_x^{+\infty} f(z)\,\mathrm{d}z = \mathcal{O}\left(x^{\frac{1}{m}-2}\right).$$

This is precisely where we used the technical assumption $m > 1$: otherwise this term may not be integrable at infinity.

Eq. (5.20) can therefore be integrated from x to $+\infty$: there is a $\lambda \in \mathbb{R}$ such that

$$\left(cx + \langle q\rangle\right)^{\frac{1}{m}} \langle q\rangle' - \lambda = -\int_x^{+\infty} \left[f(z) + \mathcal{O}\left(z^{\frac{1}{m}-3}\right)\right] \mathrm{d}z = \mathcal{O}\left(x^{\frac{1}{m}-2}\right),$$

and we conclude the proof dividing by $(cx + \langle q\rangle)^{\frac{1}{m}} \sim Cx^{\frac{1}{m}}$. □

We finally prove Theorem 5.3.

Proof of Theorem 5.3. The first item is stated in Proposition 5.5. Regarding the second item, let us recall that $q = \langle q\rangle + q^{\perp}$ and that $|q^{\perp}| + |\nabla q^{\perp}| \leqslant C/x$: our statement is actually that the asymptotic expansion holds for $\langle q\rangle$ instead of q, since the transversal part $|q^{\perp}|$ is negligible when $x \to +\infty$.

Let us recall from Lemma 5.2 that $\langle q\rangle(x)$ satisfies

$$\langle q\rangle' = \frac{\lambda}{(cx + \langle q\rangle)^{\frac{1}{m}}} + \mathcal{O}\left(\frac{1}{x^2}\right) \tag{5.21}$$

for some $\lambda \in \mathbb{R}$. If $\lambda = 0$ then $\langle q\rangle'$ is integrable and our statement immediately holds with $q_1 = \cdots = q_N = 0$.

If $\lambda \neq 0$ (5.21) with $cx + \langle q\rangle \sim cx$ yields $\langle q\rangle' \sim \lambda_1/x^{\frac{1}{m}}$, which is not integrable if $m > 1$: integrating therefore yields $\langle q\rangle \sim q_1 x^{1-\frac{1}{m}}$. Injecting this equivalent into (5.21) leads to

$$\langle q\rangle' = \frac{\lambda}{(cx + q_1 x^{1-\frac{1}{m}} + o(x^{1-\frac{1}{m}}))^{\frac{1}{m}}} + \mathcal{O}\left(\frac{1}{x^2}\right).$$

Expanding the quotient in Taylor series at order two in powers of $x^{-\frac{1}{m}}$ yields now

$$\langle q\rangle' = \lambda_1 x^{-\frac{1}{m}} + \lambda_2 x^{-\frac{2}{m}} + o\left(x^{-\frac{2}{m}}\right) + \mathcal{O}\left(\frac{1}{x^2}\right),$$

and integrating

$$\langle q\rangle = x\left(q_1 x^{-\frac{1}{m}} + q_2 x^{-\frac{2}{m}}\right) + o\left(x^{-\frac{2}{m}}\right).$$

Injecting again into (5.21) yields the next order, and so forth: by induction one shows that

$$\begin{aligned} &\langle q\rangle = x\left(q_1 x^{-\frac{1}{m}} + \cdots + q_{k-1} x^{-\frac{k-1}{m}}\right) + o\left(x^{1-\frac{k-1}{m}}\right) \\ &\qquad\qquad\Downarrow \\ &\langle q\rangle' = \lambda_1 x^{-\frac{1}{m}} + \cdots + \lambda_k x^{-\frac{k}{m}} + o\left(x^{-\frac{k}{m}}\right) + O\left(x^{-2}\right). \end{aligned} \tag{5.22}$$

- As long as $k \leqslant N = [m] < m$ the last term $\lambda_k x^{-\frac{k}{m}}$ in the expansion of $\langle q\rangle'$ above is not integrable, and we may continue the induction
$$\langle q\rangle' = \lambda_1 x^{-\frac{1}{m}} + \lambda_k x^{-\frac{k}{m}} + o\left(x^{-\frac{k}{m}}\right) + O\left(x^{-2}\right) \quad \Rightarrow \quad \langle q\rangle = x\left(q_1 x^{-\frac{1}{m}} + \cdots + q_k x^{-\frac{k}{m}}\right) + o\left(x^{1-\frac{k}{m}}\right).$$
- If now $k = N + 1 = [m] + 1 > m$, the terms $\lambda_k x^{-\frac{k}{m}} + o(x^{-\frac{k}{m}}) + O(x^{-2})$ in (5.22) are integrable: integrating one last time we obtain as desired
$$\langle q\rangle = x\left(q_1 x^{-\frac{1}{m}} + \cdots + q_N x^{-\frac{N}{m}}\right) + q^* + o(1),$$
where q^* is the constant of integration. □

Remark 7. Let us stress that the condition $m \notin \mathbb{N}$ is purely technical. If $m = [m] = N$ is integer we may obtain at some point $\langle q\rangle' = \lambda_1 x^{-\frac{1}{m}} + \cdots + \frac{\lambda_N}{x} + \cdots$ in the induction above. This would yield of course a logarithmic term, which would have to be properly taken into account. An asymptotic expansion could be obtained nonetheless, but the resulting computations would be long and not very insightful.

6. Uniqueness

In this section we prove that, given $m > 1$ and $\delta > 0$, wave profiles of δ-solutions are unique up to x-translations. The asymptotic expansion at infinity $p = cx + q_1 x^{1-\frac{1}{m}} + \cdots$ in Theorem 5.3, which was established only for $m > 1$, is deeply involved in the proof and this is the reason why we only consider here $m > 1$.

Since we defined viscosity solutions as limits of *sequences* of δ_n-solutions when $\delta_n \searrow 0$, uniqueness of viscosity solutions (again up to shifts) does not immediately follow since the limit may unfortunately depend on the sequence δ_n. It is however our belief that, suitably shifting in the x direction, one may pin the (continuous) family of $(p^\delta)_{\delta>0}$ so that it is decreasing when $\delta \searrow 0$. Monotonicity would then of course imply uniqueness of viscosity solutions, but also be an interesting result by itself. In order to keep this paper in a reasonable length we will not address this issue here, but let us point out that the above pinning would be straightforward if the coefficients in the asymptotic expansion at infinity were independent of δ (which we believe holds).

Let us stress that all the results in Section 5 were stated for the final viscosity solution $p = \lim p^\delta$, but easily extend to the δ-solutions for $\delta > 0$. Through this whole section we fix δ and denote by p, p_1, p_2 any (smooth) δ-solutions in order to keep our notations light.

The main result of this section is

Theorem 6.1. *The δ-solutions are unique up to finite x-translation.*

Let us start with some technical statements:

Proposition 6.1. *Any δ-solution has an asymptotic expansion*

$$p(x, y) = cx + x\left(q_1 x^{-\frac{1}{m}} + \cdots + q_N x^{-\frac{N}{m}}\right) + q^* + o(1)$$

uniformly in y when $x \to +\infty$, where $q_1, \ldots, q_N, q^* \in \mathbb{R}$ and $1 \leqslant N = [m] < m$.

Remark 8. The coefficients q_i, q^* and the remainder $o(1)$ above may depend of course on δ, which is fixed here.

Proof of Proposition 6.1. We may proceed exactly as we did for the final viscosity solution, see Section 5 and in particular the proof of Theorem 5.3. □

The following holds at negative infinity, where we recall that $p(-\infty, y) = \delta > 0$ uniformly in y.

Lemma 6.1. *We have that*

$$|\nabla p| \to 0, \qquad \left|D^2 p\right| \to 0$$

uniformly in y when $x \to -\infty$.

Proof. Let $w := \frac{m^2}{m+1} p^{\frac{m+1}{m}}$ and $f := (c + \alpha) p^{\frac{1}{m}-1} p_x$, and recall that the Poisson equation

$$\Delta w = f$$

holds in the whole cylinder. Taking advantage of $p(-\infty, y) = \delta > 0$ (w thus being uniformly bounded away from zero and locally uniformly from above) we may safely apply our previous interior elliptic regularity argument on $\Omega_n :=]-n, -n+1[\times \mathbb{T}^1$ ($n \in \mathbb{N}$) to show that

$$\|p\|_{W^{3,q}(\Omega_n)} \leqslant C$$

for some constant $C > 0$ and $q > d = 2$ both independent of n.

Setting

$$\Omega :=]0, 1[\times \mathbb{T}^1, \qquad p^n(x, y) := p(x - n, y),$$

the previous estimate reads

$$\left\|p^n\right\|_{W^{3,q}(\Omega)} \leqslant C.$$

By compactness $W^{3,q}(\Omega) \Subset \mathcal{C}^2(\overline{\Omega})$ we may extract a subsequence $p^{n_k} \to p^\infty$ in $\mathcal{C}^2(\overline{\Omega})$. Since $p(-\infty, y) = \delta$, the limit $p^\infty(x, y) = p(-\infty, y) = cst = \delta$ is unique: standard separation arguments show that the whole sequence converges

$$p^n \to \delta \quad \text{in } \mathcal{C}^2(\overline{\Omega}),$$

which immediately implies our statement. □

Proposition 6.2. *The coefficients $q_1, \ldots, q_N$ in the asymptotic expansion are unique.*

Proof. Let p_1 and p_2 be two different δ-solutions, thus satisfying

$$p_1 = cx + x\left(q_{1,1}x^{-\frac{1}{m}} + \cdots + q_{1,N}x^{-\frac{N}{m}}\right) + q_1^* + o(1),$$
$$p_2 = cx + x\left(q_{2,1}x^{-\frac{1}{m}} + \cdots + q_{2,N}x^{-\frac{N}{m}}\right) + q_2^* + o(1)$$

when $x \to +\infty$ for some constants $q_{i,k}, q_i^* \in \mathbb{R}$, $i = 1, 2$, $k = 1, \ldots, N$ and $N = [m]$.

Assume by contradiction that $q_{1,1} > q_{2,1}$: we will first slide p_2 far enough to the right so that $p_2 < p_1$ on the whole cylinder. Slowly sliding p_2 back to the left we will obtain a contact point between p_1 and a translate of p_2, thus contradicting the classical Maximum Principle.

In fact, Lemma 6.1 allows to pin p_1 such that, for $x \leqslant 0$, there holds

1. $\delta \leqslant p_1(x, y) \leqslant p_1(0, y) \leqslant 2\delta$,
2. $|\nabla p_1|$ and $|\Delta p_1|$ are small.

This can be done suitably sliding since $p_1 \to \delta$, $|\nabla p_1| \to 0$ and $|\Delta p_1| \to 0$ when $x \to -\infty$. In this proof p_1 will be fixed once and for all, and we will only slide p_2 with respect to p_1 (for the sake of clarity p_2 denotes below any translation).

- Since we assumed that $q_{1,1} > q_{2,1}$ we have

  $$[p_1 - p_2](+\infty, y) = +\infty$$

  for any (finite) translation p_2. Using $\partial_x p_i > 0$ we may therefore slide p_2 far enough to the right so that

  $$x \geqslant 0 \quad \Rightarrow \quad p_1(x, y) > p_2(x, y).$$

  We claim that, applying a suitable comparison principle, we may assume that $p_1 > p_2$ also holds for $x < 0$. In order to see this, define $z := p_1 - p_2$ and subtract the equation for p_2 from the equation for p_1 to obtain

  $$\mathcal{L}[z] := -mp_2\Delta z + \left[(c + \alpha)z_x - (\nabla p_1 + \nabla p_2) \cdot \nabla z\right] - (m\Delta p_1)z = 0. \tag{6.1}$$

  Testing $\bar{z}(x) := e^{\lambda x}$ as a supersolution for some $\lambda > 0$, an elementary computation leads to

  $$\mathcal{L}[\bar{z}] = e^{\lambda x}\left(-mp_2\lambda^2 + (c + \alpha)\lambda - (\partial_x p_1 + \partial_x p_2)\lambda - m\Delta p_1\right).$$

  Sliding p_2 far enough to the right we have, for $x \leqslant 0$, that $p_2 \sim \delta$ and that $\partial_x p_2$ is negligible. Since we also pinned $|\nabla p_1|$ and $|\Delta p_1|$ to be small, the main contribution in the parenthesis of the right-hand side above comes from the first two terms. Choosing $\lambda > 0$ small enough, it is clearly possible to satisfy

  $$-mp_2\lambda^2 + (c + \alpha)\lambda \gtrapprox -m\delta\lambda^2 + c_0\lambda > 0 \quad \text{if } x < 0$$

  (choose for example $\lambda = c_0/2m\delta$), and therefore

  $$x < 0 \quad \Rightarrow \quad \mathcal{L}[\bar{z}] > 0.$$

  Setting $z := w\bar{z}$, the new variable w satisfies this time an elliptic equation

  $$\tilde{\mathcal{L}}[w] = 0,$$

  where $\tilde{\mathcal{L}}$ is uniformly elliptic, has positive zero-th order coefficient $\mathcal{L}[\bar{z}] > 0$, and therefore satisfies the Minimum Principle.

On the right boundary $x = 0$ we may assume that $p_1(0, y) > p_2(0, y)$ (once again sliding p_2 far enough to the right), and therefore $w(0, y) > 0$. At negative infinity we had exponential convergence $|p_i(x, y) - \delta| \leqslant Ce^{\frac{c_0}{\delta m}x}$, and we choose the supersolution $\bar{z} = e^{\lambda x}$ to decay slowly ($\lambda > 0$ was chosen small enough, for example $\lambda = c_0/2m\delta$), hence $|z| = |p_1 - p_2| \leqslant Ce^{\frac{c_0}{m\delta}x} \ll |\bar{z}|$ and $w(-\infty, y) = 0$. The Minimum Principle applied to $\tilde{L}[w] = 0$ finally shows that $w(x, y) > 0$ for $x < 0$, and therefore $p_1 > p_2$ on the whole cylinder $D = \mathbb{R} \times \mathbb{T}^1$ if p_2 is slided far enough to the right.

- Slowly sliding back to the left we obtain a first critical translation p_2^*, after which we cannot keep translating to the left without breaking $p_1 \geqslant p_2$ (this critical translation exists because sliding p_2 far enough to the left the two solutions must cross at some point). By continuity we have that $z^* = p_1 - p_2^* \geqslant 0$, and we claim that there exists a contact point $(x_0, y_0) \in D$ such that $z^*(x_0, y_0) = 0$. Temporarily admitting this, we obtain a contradiction as follows: $z^* \geqslant 0$ satisfies (6.1), which is uniformly elliptic with bounded zero-th order coefficient, and attains a minimum point $z^* = 0$ in D. The Minimum Principle shows that $z^* \equiv 0$, thus contradicting $q_{1,1} > q_{2,1} \Rightarrow z(+\infty, y) = +\infty$.
  In order to obtain such a contact point, assume by contradiction that $p_1 > p_2^*$ on the whole cylinder: condition $q_{1,1} > q_{2,1}$ shows that $p_1 - p_2^* \geqslant C_a > 0$ on any sub-cylinder $x \geqslant -a$ for any $a > 0$ large and some constant C_a. We may then slide p_2 slightly further to the left in such a way that $p_1 > p_2$ if $x \geqslant a$. Repeating the above comparison argument for $x \leqslant a$, we see that $p_1 > p_2$ also holds to the left, hence on the whole cylinder. This contradicts the fact that p_2^* was a critical translation.

We just proved that $q_{1,1} > q_{2,1}$ cannot hold, and by symmetry $p_1 \leftrightarrow p_2$ we obtain

$$q_{1,1} = q_{2,1}.$$

We may now repeat the very same argument to show that $q_{1,2} = q_{2,2}$, and so forth ($q_{1,k} = q_{2,k} \Rightarrow q_{1,k+1} = q_{2,k+1}$). □

We can now prove uniqueness of the δ-solutions:

Proof of Theorem 6.1. Let p_1, p_2 be two δ-solutions; we pin as before p_1 once and for all, and only slide p_2. Let us stress that both solutions have now the same coefficients $q_1, \ldots, q_N$ in their asymptotic expansion at infinity

$$i = 1, 2, \quad x \to +\infty: \quad p_i(x, y) = cx + x\bigl(q_1 x^{-\frac{1}{m}} + \cdots + q_N x^{-\frac{N}{m}}\bigr) + q_i^* + o_i(1),$$

except maybe for the last two terms (the lower order $q_i^* + o_i(1)$). We recall that $z := p_1 - p_2$ satisfies (6.1) of the form $L[z] = 0$, and remark the following: for any τ-translation $p_2(x - \tau, y)$, uniqueness of the coefficients $q_1, \ldots, q_N$ shows that

$$p_1(x, y) - p_2(x - \tau, y) \underset{+\infty}{=} c\tau + q_1^* - q_2^* + o(1). \tag{6.2}$$

This means that, depending on the translation, only two scenarios are possible at infinity: either $[p_1 - p_2](+\infty, y) = 0$, or $[p_1 - p_2](+\infty, y) = cst \neq 0$.

We showed previously that sliding p_2 far enough to the right $p_1 > p_2$ must hold, and that slowly sliding back to the left there exists a first critical translation $\overline{p}_2$ such that $p_1 \geqslant \overline{p}_2$. Similarly translating p_2 far enough to the left we have that $p_1 < p_2$, and there exists a first critical translation $\underline{p}_2$ coming from the left side such that $p_1 \leqslant \underline{p}_2$.

1. If there exists a contact point $p_1(x_0, y_0) = \overline{p}_2(x_0, y_0)$ then $\bar{z} = p_1 - \overline{p}_2$ is nonnegative (because $\overline{p}_2$ is a critical translation coming from the right side), satisfies an elliptic equation $\mathcal{L}[\bar{z}] = 0$ with bounded zero-th order coefficient, and attains an interior minimum point $\bar{z}(x_0, y_0) = 0$. The classical Minimum Principle shows that $\bar{z} \equiv 0$, meaning precisely that p_1 can be deduced from p_2 by translation.
   We may therefore assume that no such contact point exists, and the only possible scenario is therefore that $[p_1 - \overline{p}_2](+\infty, y) = 0$ (otherwise $[p_1 - \overline{p}_2](+\infty, y) = cst > 0$ according to (6.2) and we could slide p_2 a little further to the left as in the proof of Proposition 6.2, thus contradicting the fact that $\overline{p}_2$ is critical).
2. Similarly arguing for $\underline{z}$, we may assume that $[p_1 - \underline{p}_2](+\infty, y) = 0$.
3. As a consequence $[\overline{p}_2 - \underline{p}_2](+\infty, y) = 0$, and therefore $\overline{p}_2 = \underline{p}_2$. We conclude recalling that we constructed $\overline{p}_2 \leqslant p_1 \leqslant \underline{p}_2$, hence $p_1 = \overline{p}_2 = \underline{p}_2$. □

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