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Minimizers of the Willmore functional with a small area constraint

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Abstract

We show the existence of a smooth spherical surface minimizing the Willmore functional subject to an area constraint in a compact Riemannian three-manifold, provided the area is small enough. Moreover, we partially classify complete surfaces of Willmore type with positive mean curvature in Riemannian three-manifolds.

Keywords: Willmore functional; Minimizers; Direct method

1. Introduction

For a three-dimensional complete Riemannian manifold (M, g) and an immersion $f : \Sigma \to M$ the Willmore functional is defined by

$$\mathcal{W}(f) = \frac{1}{2} \int_{\Sigma} H^2 \, \mathrm{d}\mu,$$

where H is the induced mean curvature and μ the induced area measure. In the following we let A resp. $\mathring{A}$ be the second fundamental form resp. the trace-free second fundamental form of the immersion f. Critical points of $\mathcal{W}$ are called Willmore surfaces and they are solutions of the Euler–Lagrange equation

$$\Delta H + H|\mathring{A}|^2 + H \operatorname{Ric}(\nu, \nu) = 0,$$

where Ric denotes the Ricci curvature of (M, g) and ν is the normal vector to Σ in M.

In the literature other possible definitions of the Willmore functional for immersions in a Riemannian manifold were considered, for example:

$$\int_{\Sigma} |A|^2 \, \mathrm{d}\mu, \qquad \int_{\Sigma} |\mathring{A}|^2 \, \mathrm{d}\mu, \quad \text{or} \quad \int_{\Sigma} \left(H^2 + \kappa^M\right) \mathrm{d}\mu.$$

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Here κ^M denotes the sectional curvature of M. In a curved ambient manifold the Gauss equation and the Gauss–Bonnet theorem yield

$$\mathcal{W}(\Sigma) = \frac{1}{2}\int_{\Sigma} |A|^2 \, \mathrm{d}\mu + \int_{\Sigma} G(\nu, \nu) \, \mathrm{d}\mu + 2\pi\big(1 - q(\Sigma)\big)$$

where $q(\Sigma)$ is the genus of Σ, $G = \mathrm{Ric} - \frac{1}{2}\mathrm{Sc}\, g$ is the Einstein tensor, Ric denotes the Ricci curvature and Sc the scalar curvature of (M, g). Hence, the functionals above differ by lower order terms involving the curvature of (M, g). In particular, if $(M, g) = (\mathbb{R}^3, \delta)$ the only difference is a multiple of $q(\Sigma)$.

In this paper we are interested in surfaces of Willmore type, i.e. critical points of $\mathcal{W}$ subject to an area constraint. These surfaces are solutions of the Euler–Lagrange equation

$$\Delta H + H|\mathring{A}|^2 + H\,\mathrm{Ric}(\nu, \nu) + \lambda H = 0,$$

with the Lagrange parameter λ. In [9] we studied spherical surfaces of Willmore type with positive mean curvature in small geodesic balls. Assuming a certain lower bound on the Lagrange parameter, we showed that such surfaces can only concentrate at critical points of the scalar curvature of M. This paper establishes the existence of minimizers of $\mathcal{W}$ with fixed small area and verifies that the minimizers satisfy the assumptions of [9]. Our main result is as follows:

Theorem 1.1. *Let (M, g) be a compact, closed Riemannian manifold. Then there exists a constant $a_0 > 0$ such that for all $a \in (0, a_0)$ there is a smooth spherical surface Σ_a of positive mean curvature that minimizes the Willmore functional among all immersed surfaces with area a.*

For any sequence $a_i \to 0$ there is a subsequence $a_{i'}$ such that $\Sigma_{a_{i'}}$ is asymptotic to a geodesic sphere centered at a point $p \in M$ where Sc *attains its maximum.*

The existence of a $W^{2,2} \cap W^{1,\infty}$ conformal immersion minimizing $\mathcal{W}$ with prescribed small area was recently and independently obtained by Chen and Li [2].

Kuwert and Schätzle [8] constructed smooth minimizers in a conformal class of the Willmore functional in $\mathbb{R}^3$. Existence was recently generalized to arbitrary co-dimension by Kuwert and Li [6] in the class $W^{2,2} \cap W^{1,\infty}$ and by Rivière [14].

The existence of a smooth minimizer of the Willmore functional in $\mathbb{R}^n$ with prescribed genus was first proved by Simon [16] under a Douglas-type condition, which was established by Bauer and Kuwert [1] (see also [14] for a new proof of this result). Recently, Schygulla [15] suitably modified the arguments of Simon in order to prove the existence of a minimizing Willmore sphere in $\mathbb{R}^3$ with prescribed isoperimetric ratio.

Under suitable curvature assumptions, Kuwert, Mondino, and Schygulla [7] recently showed the existence of smooth spherical minimizers of the functionals $\int_\Sigma |A|^2 \, \mathrm{d}\mu$ and $\int_\Sigma (1 + |H|^2) \, \mathrm{d}\mu$ in Riemannian manifolds. The role of the curvature assumptions is to ensure a uniform bound on the area of surfaces in a minimizing sequence.

Some other existence and non-existence results of critical points of $\mathcal{W}$ by Mondino can be found in [11,12].

Previously, in a joint work with F. Schulze [10], we proved the existence of a foliation of the end of an asymptotically flat manifold with positive mass by spherical surfaces of Willmore type. For the existence result we studied perturbed geodesic spheres and we used the implicit function theorem to find suitable deformations of these spheres.

In this paper we use the direct method of the calculus of variations in order to construct the minimizer of $\mathcal{W}$. A major difficulty is the invariance of $\mathcal{W}$ under diffeomorphisms. We overcome this problem by showing that spherical surfaces with small enough area have small diameter since the Willmore energy is a priori close to 8π. Hence the surface is contained in a small geodesic ball around a point $p \in M$ where the metric g is a small perturbation of the Euclidean metric. Thus we can apply a result of De Lellis and Müller [3,4] which gives the existence of a $W^{2,2} \cap W^{1,\infty}$ conformal parametrization $F : S^2 \to \Sigma \subset \mathbb{R}^3$ of Σ. Therefore, instead of studying minimizing sequences of immersions $f_k : \Sigma_k \to M$ of $\mathcal{W}$, we consider minimizing sequences of parametrizations $F_k \in W^{2,2} \cap W^{1,\infty}(S^2, \mathbb{R}^3)$. Within this class we are able to show the existence of a minimizer of $\mathcal{W}$.

In order to show the higher regularity of the minimizer in $W^{2,2} \cap W^{1,\infty}$ we suitably modify the arguments of [7] and [15]. In our situation their arguments heavily simplify since the smallness of the area rules out bad points for the minimizing sequence and we can use the fact that there exists a limiting parametrization.

As the last step we show that the minimizers we construct satisfy the assumptions of the main result in [9] and hence we conclude that the minimizing surfaces have to concentrate around a maximal point of the scalar curvature of M as the area tends to zero.

In the following we give a brief outline of the paper. In Section 2 we review manifolds with bounded geometry and show that the diameter estimates of [16] extend to these ambient manifolds. Moreover we show that by a scaling argument one can adjust the area of a surface without changing the Willmore energy too much. This fact will be crucial in the proof of the smoothness of the minimizer of $\mathcal{W}$ with prescribed area.

In Section 3 we study the Willmore functional for immersions $F \in W^{2,2} \cap W^{1,\infty}(S^2, \mathbb{R}^3)$ and we show the lower semi-continuity and the differentiability of the functional.

In Section 4 we construct the smooth minimizer and hence prove the existence part of Theorem 1.1 using the methods described above.

In Section 5 we prove an integral estimate and estimate for the Lagrange parameter of the minimizer. These estimates then allow us to apply the results of [9].

In Appendix A we derive a variant of the stability inequality of minimal surfaces for surfaces of Willmore type with positive mean curvature. Using the methods of [5] we are then able to partially classify complete surfaces of Willmore type with positive mean curvature in Riemannian manifolds. As a corollary we obtain that the only complete surfaces of this type in $\mathbb{R}^3$ are round spheres.

2. Preliminaries

2.1. Manifolds with bounded geometry

In this section we recall some basic properties of manifolds with bounded geometry. The main point is that such manifolds have uniformly controlled normal coordinates.

Definition 2.1. Let (M, g) be a complete Riemannian manifold. We say that (M, g) has bounded geometry, if there exists a constant $0 < C_B < \infty$ such that for each $p \in M$ we have $\operatorname{inj}(M, g, p) \geqslant C_B^{-1}$ and if the Riemann tensor and its first derivative are bounded $|\operatorname{Rm}| + |\nabla \operatorname{Rm}| \leqslant C_B$.

In the following we use B_r to denote a Euclidean ball centered at the origin of radius r and $\mathcal{B}_r(p) \subset M$ to denote a geodesic ball of radius r centered at $p \in M$.

Remark 2.2. If (M, g) has bounded geometry with constant C_B, then there exist constants $h_0 < \infty$ and $\rho_0 > 0$, depending only on C_B, such that for every $p \in M$, we can introduce normal coordinates $\phi : B_{\rho_0} \to \mathcal{B}_{\rho_0}(p)$ for the metric g such that in these coordinates the metric g satisfies

$$g = g^E + h,$$

with

$$\sup_{B_{\rho_0}} \left(|x|^{-2}|h| + |x|^{-1}|\partial h| + \left|\partial^2 h\right|\right) \leqslant h_0.$$

Here g^E denotes the Euclidean metric induced by the normal coordinates, $|x|$ denotes the Euclidean distance to p and ∂ the connection of g^E.

If (M, g) is compact or asymptotically flat, then it is of bounded geometry for some constant C_B.

2.2. Area, diameter and the Willmore energy

To proceed, we need a generalization of Lemma 1.1 from [16] to general ambient manifolds. Although this is straightforward, we present the proof here to show where the non-flat ambient geometry has to be taken into account. For ease of presentation, we split this into three separate statements.

Lemma 2.3. *(Cf. [9, Lemma 2.2].) Let $g = g^E + h$ on B_{ρ_0} be given, such that*

$$\sup_{B_{\rho_0}} \big(|x|^{-2}|h| + |x|^{-1}|\partial h|\big) \leqslant h_0.$$

Then there exists a purely numerical constant $c > 0$ such that if $\rho_1 := \min\{\rho_0, \frac{1}{c\sqrt{h_0}}\}$, then for all surfaces $\Sigma \subset B_r$ with $r \in (0, \rho_1)$ we have that

$$|\Sigma| \leqslant r^2 \mathcal{W}(\Sigma).$$

The following proposition is very similar to the calculations in [16, Section 1]. We add only minor modifications in order to deal with the non-flat background.

Proposition 2.4. *Let $g = g^E + h$ on B_{ρ_0} be given, such that*

$$\sup_{B_{\rho_0}} \big(|x|^{-2}|h| + |x|^{-1}|\partial h|\big) \leqslant h_0.$$

Then there exists a purely numerical constant c, such that for every smooth surface $\Sigma \subset B_{\rho_0}$ with $\partial\Sigma \subset \partial B_{\rho_0}$ and $0 \in \Sigma$ we have that

$$\pi \leqslant c\big(\big(1 + h_0 r^2\big) r^{-2}|\Sigma_r| + \mathcal{W}(\Sigma_r)\big)$$

for all $r \leqslant \rho$. Here $\Sigma_r := \Sigma \cap B_r$.

Proof. In B_{ρ_0} consider the position vector field x. We denote by c a constant that is purely numerical, but which may change from line to line. For all surfaces Σ as in the statement we have

$$|\operatorname{div}_\Sigma x - 2| \leqslant c h_0 |x|^2$$

in view of the assumption on h. Furthermore, we calculate that in B_{ρ_0}

$$\mathrm{d}|x| = \frac{x}{|x|}.$$

In particular, away from the origin, we have

$$\operatorname{div}_\Sigma\big(|x|^{-2}x\big) = |x|^{-2}\operatorname{div}_\Sigma x - 2|x|^{-3}\mathrm{d}|x|\big(x^T\big).$$

Thus

$$\big|\operatorname{div}_\Sigma\big(|x|^{-2}x\big) - 2|x|^{-4}\big|x^\perp\big|^2\big| \leqslant c h_0, \tag{1}$$

where $x^\perp$ denotes the projection of x onto the normal bundle of Σ.

Choose $0 < s < r < \rho_0$ such that Σ intersects ∂B_r and ∂B_s transversely (note that the set of radii satisfying this condition is dense in $(0, \rho_0)$). Let

$$\Sigma_{s,r} := \Sigma \cap (B_r \setminus \bar{B}_s)$$

and integrate Eq. (1) on $\Sigma_{s,r}$ to obtain

$$\bigg|\int_{\Sigma_{s,r}} \operatorname{div}_\Sigma\big(|x|^{-2}x\big)\,\mathrm{d}\mu - \int_{\Sigma_{s,r}} 2|x|^{-4}|x^\perp|^2\,\mathrm{d}\mu\bigg| \leqslant c h_0 |\Sigma_{s,r}|. \tag{2}$$

Using Stokes, we infer that

$$\int_{\Sigma_{s,r}} \operatorname{div}_\Sigma\big(|x|^{-2}x\big)\,\mathrm{d}\mu = \int_{\Sigma_{s,r}} H|x|^{-2}\langle x, \nu\rangle\,\mathrm{d}\mu - \int_{\Sigma\cap\partial B_s} |x|^{-2}\langle x, \eta\rangle\,\mathrm{d}\sigma + \int_{\Sigma\cap\partial B_r} |x|^{-2}\langle x, \eta\rangle\,\mathrm{d}\sigma. \tag{3}$$

Here ν is the normal vector of Σ and η denotes the co-normal of $\partial B_s \cap \Sigma$ and $\partial B_r \cap \Sigma$ in Σ respectively. We chose the orientation so that η points in direction of ∇r. To proceed, we note that

$$\int_{\Sigma\cap\partial B_\sigma} |x|^{-2}\langle x,\eta\rangle\,\mathrm{d}\sigma = \sigma^{-2}\int_{\Sigma\cap\partial B_\sigma}\langle x,\eta\rangle\,\mathrm{d}\sigma = \sigma^{-2}\int_{\Sigma_\sigma}\operatorname{div}_\Sigma x^T\,\mathrm{d}\sigma,$$

so that

$$\left|\int_{\Sigma\cap\partial B_\sigma}|x|^{-2}\langle x,\eta\rangle\,\mathrm{d}\sigma - 2\sigma^{-2}|\Sigma_\sigma| + \sigma^{-2}\int_{\Sigma_\sigma}H\langle x^\perp,\nu\rangle\,\mathrm{d}\mu\right| \leqslant ch_0|\Sigma_\sigma|. \tag{4}$$

Furthermore

$$\int_{\Sigma_{s,r}}|x|^{-4}\left|x^\perp\right|^2 - \frac{1}{2}H|x|^{-2}\langle x,\nu\rangle\,\mathrm{d}\mu = \int_{\Sigma_{s,r}}\left||x|^{-2}x^\perp - \frac{1}{4}H\nu\right|^2 - \frac{1}{16}H^2\,\mathrm{d}\mu. \tag{5}$$

Inserting (3)–(5) into (2), we infer that

$$\begin{aligned}
&\int_{\Sigma_{s,r}}\left||x|^{-2}x^\perp - \frac{1}{4}H\nu\right|^2\mathrm{d}\mu - \frac{1}{2}s^{-2}\int_{\Sigma_s}H\langle x,\nu\rangle\,\mathrm{d}\mu + s^{-2}|\Sigma_s| \\
&\quad\leqslant r^{-2}|\Sigma_r| + \frac{1}{8}\mathcal{W}(\Sigma_{s,r}) - \frac{1}{2}r^{-2}\int_{\Sigma_r}H\langle x,\nu\rangle\,\mathrm{d}\mu + ch_0|\Sigma_r|.
\end{aligned}$$

Since Σ is smooth at the origin, we can let $s\to 0$ and drop the square term on the left to obtain

$$\pi \leqslant r^{-2}|\Sigma_r| + \frac{1}{8}\mathcal{W}(\Sigma_r) - \frac{1}{2}r^{-2}\int_{\Sigma_r}H\langle x,\nu\rangle\,\mathrm{d}\mu + ch_0|\Sigma_r|.$$

Using Cauchy–Schwarz on the third term and recalling that $r<\rho_0$ we infer the estimate

$$\pi \leqslant c\big(\big(1+h_0r^2\big)r^{-2}|\Sigma_r| + \mathcal{W}(\Sigma_r)\big) \tag{6}$$

for a purely numerical constant c. Since the values of r for which (6) holds are dense in $(0,\rho_0]$ we arrive at the claimed estimate by approximation. □

The next lemma shows that the diameter of a surface Σ contained in a Riemannian manifold (M,g) is bounded in terms of its area and its Willmore energy. We define

$$\operatorname{diam}(\Sigma) := \max\big\{d_{(M,g)}(p,q):\ p,q\in\Sigma\big\}$$

to be the extrinsic diameter of Σ. Here $d_{(M,g)}(p,q)$ denotes the geodesic distance of p and q in the ambient manifold M.

Lemma 2.5. *Let (M,g) be a manifold with C_B-bounded geometry. Then there exists a constant C depending only on C_B such that for all smooth connected surfaces Σ we have*

$$\operatorname{diam}(\Sigma) \leqslant C\big(|\Sigma|^{1/2}\mathcal{W}(\Sigma)^{1/2} + |\Sigma|\big).$$

Proof. Let h_0 and ρ_0 be as in Remark 2.2. Choose $p,q\in\Sigma$ such that $d := d_{(M,g)}(p,q) = \operatorname{diam}(\Sigma)$. Assume for now that $r\in(0,\frac{d}{2})$ is chosen such that $r<\rho_0$.

Let N be the largest integer smaller than d/r and let $p_0 = p$. For $j = 1,\dots,N-1$ we choose $p_j\in\Sigma$ at distance $(j+\frac{1}{2})r$ to p_0, which is possible since Σ is connected. Then the geodesic balls $\mathcal{B}_{r/2}(p_j)$ are pairwise disjoint for $j = 0,\dots,N-1$. Using Proposition 2.4 with p_j as center and summing over j yields

$$N\pi \leqslant c\big(\mathcal{W}(\Sigma) + \big(1+h_0r^2\big)r^{-2}|\Sigma|\big). \tag{7}$$

With this in mind, we let

$$r := \min\left\{\frac{\rho_1}{2}, \frac{1}{4}\sqrt{\frac{|\Sigma|}{\mathcal{W}(\Sigma)}}\right\},$$

where $\rho_1 = \min\{\rho_0, \frac{1}{c\sqrt{h_0}}\}$ is such that Lemma 2.3 applies with ρ_0 and h_0 as above.

We have to check that $r < d/2$. Assume for the contrary that $d/2 \leqslant r$. Then in particular $d \leqslant \rho_1$ and Lemma 2.3 implies that

$$\sqrt{\frac{|\Sigma|}{\mathcal{W}(\Sigma)}} \leqslant d \leqslant 2r \leqslant \frac{1}{2}\sqrt{\frac{|\Sigma|}{\mathcal{W}(\Sigma)}},$$

a contradiction. Hence $r < d/2$ and thus $N \geqslant \frac{d}{2r}$. Revisiting (7) thus yields

$$d \leqslant c\big(r\mathcal{W}(\Sigma) + \big(1 + h_0 r^2\big) r^{-1}|\Sigma|\big)$$

and since

$$r^{-1} \leqslant \max\left\{\frac{2}{\rho_1}, 4\sqrt{\frac{\mathcal{W}(\Sigma)}{|\Sigma|}}\right\}$$

we find that

$$\begin{aligned} d &\leqslant c\big(1 + r^2 h_0\big)\sqrt{|\Sigma|\mathcal{W}(\Sigma)} + c/\rho_1|\Sigma| \\ &\leqslant c\big(1 + r^2 h_0\big)\sqrt{|\Sigma|\mathcal{W}(\Sigma)} + c\big(\rho_0^{-1} + \sqrt{h_0}\big)|\Sigma|. \end{aligned}$$

This yields the claimed estimate. □

2.3. Area adjustment by scaling

Lemma 2.6. *Let (M, g) be a manifold with C_B-bounded geometry. Then there exists $\rho_1 > 0$ with the following property. Let $r \in (0, \rho_1)$, $p \in M$, and let x be the position vector field with respect to geodesic normal coordinates in $\mathcal{B}_r(p)$. Denote by $\Phi : \mathcal{B}_{r/4}(p) \times (-\infty, 2) \to \mathcal{B}_r(p)$ the flow associated to x. Then for every $a \in \mathbb{R}$ and $\Sigma \subset \mathcal{B}_{r/4}(p)$ with $|\Sigma| \in (\frac{a}{2}, \frac{3a}{2})$ there exists $t_0 \in \mathbb{R}$ with $|\Phi_{t_0}(\Sigma)| = a$ and $|t_0| \leqslant 2\frac{||\Sigma|-a|}{a}$.*

Proof. Let ρ_0 and h_0 as in Remark 2.2. We choose $\rho_1 \in (0, \rho_0)$ such that

$$|\nabla x - \mathrm{Id}| \leqslant \frac{1}{2}$$

on $\mathcal{B}_{\rho_1}(p)$ for all $p \in M$. Note that then ρ_1 depends only on C_B.

Let $r \in (0, \rho_1)$ and $\Sigma' \subset \mathcal{B}_r(p)$ be an arbitrary surface with $|\Sigma'| \geqslant a/2$. Then we calculate that

$$\frac{d}{dt}\bigg|_{t=0}\big|\Sigma'\big| = \int_{\Sigma'} H\langle x, \nu\rangle\,\mathrm{d}\mu = \int_{\Sigma'} \mathrm{div}_{\Sigma'}\, x\,\mathrm{d}\mu \geqslant \big|\Sigma'\big| \geqslant \frac{a}{2}.$$

If we consider Σ as in the statement of the lemma, we can apply this estimate to $\Sigma_t = \Phi_t(\Sigma)$ as long as $\Sigma_t \in B_r(p)$ and $|\Sigma_t| \in (\frac{a}{2}, \frac{3a}{2})$. In particular the area of Σ_t is a continuous and strictly increasing function of t. In addition we have that

$$\begin{aligned} &\big|\Sigma_{t^+}\big| \geqslant a \quad \text{for } t^+ := \max\left\{0, 2\frac{a - |\Sigma|}{a}\right\}, \quad \text{and} \\ &\big|\Sigma_{t^-}\big| \leqslant a \quad \text{for } t^- := \min\left\{0, 2\frac{|\Sigma| - a}{a}\right\}. \end{aligned}$$

This yields the claim. □

Lemma 2.7. *Let (M, g) be a manifold with C_B-bounded geometry and let ρ_1, be as in Lemma 2.6. There exists a constant C depending only on C_B with the following property. Let $r \in (0, \rho_1)$ and let $\Sigma \subset \mathcal{B}_{r/2}(p)$. Then*

$$\frac{d}{dt}\bigg|_{t=0} \int_{\Sigma_t} |A_t|^2 \,\mathrm{d}\mu_t \leqslant C|\Sigma|^{1/2}\bigg(\int_\Sigma H^2 \,\mathrm{d}\mu\bigg)^{1/2} + Cr\int_\Sigma |A|^2 \,\mathrm{d}\mu + C(1+r)|\Sigma|.$$

Here we use the notation of Lemma 2.6, so that $\Sigma_t = \Phi_t(\Sigma)$, A_t denotes the second fundamental form of Σ_t and $\mathrm{d}\mu_t$ its induced measure.

Proof. We use the Gauss equation to write

$$\int_\Sigma |A|^2 \,\mathrm{d}\mu = \int_\Sigma H^2 \,\mathrm{d}\mu - 2\int_\Sigma G(\nu,\nu)\,\mathrm{d}\mu - 4\pi\big(1 - g(\Sigma)\big)$$

where $g(\Sigma)$ is the genus of Σ and $G(\nu,\nu) = \mathrm{Ric} - \frac{1}{2}\,\mathrm{Sc}\, g$. Therefore we have

$$\frac{d}{dt}\bigg|_{t=0} \int_{\Sigma_t} |A_t|^2 \,\mathrm{d}\mu_t = 2\delta_{\langle x,\nu\rangle}\mathcal{W}(\Sigma) - \delta_{\langle x,\nu\rangle}\mathcal{V}(\Sigma)$$

where

$$\mathcal{V}(\Sigma) = 2\int_\Sigma G(\nu,\nu)\,\mathrm{d}\mu.$$

We estimate the variations of $\mathcal{W}$ and $\mathcal{V}$ separately. We have that

$$|\nabla x - \mathrm{Id}\,| \leqslant C|x|^2 \quad \text{and} \quad |\nabla^2 x| \leqslant C \tag{8}$$

where C is a constant depending only on C_B.

Consider the function $\langle x,\nu\rangle$ on Σ. A calculation shows that with respect to an adapted ON-Frame $\{e_1, e_2, e_3 = \nu\}$, we have

$$\begin{aligned}\Delta\langle x,\nu\rangle &= \langle \nabla_{e_i,e_i} x, \nu\rangle - H\langle \nabla_\nu x, \nu\rangle + 2\langle \nabla_{e_i} x, e_k\rangle A_{ik} - \langle x,\nu\rangle|A|^2 + \langle x, e_k\rangle \nabla_i A_{ik} \\ &= H - \langle x,\nu\rangle|A|^2 + \langle x, e_k\rangle\nabla_{e_k} H + O(1) + O(r) * A.\end{aligned}$$

In the last equality we used the Codazzi equation to rewrite $\operatorname{div} A = \nabla H + \mathrm{Ric}(\nu,\cdot)$ together with the fact that $x = O(r)$. In addition we used (8) and use the notation $O(1)$ for terms which are bounded by a constant C and $O(r) * A$ for terms bounded by $Cr|A|$. Here as usual C depends only on C_B.

We calculate the variation of the Willmore functional with respect to scaling:

$$\begin{aligned}\delta_{\langle x,\nu\rangle}\mathcal{W} &= \int_\Sigma \langle x,\nu\rangle\big(\Delta H + H|\mathring{A}|^2 + H\,\mathrm{Ric}(\nu,\nu)\big)\,\mathrm{d}\mu \\ &= \int_\Sigma H\Delta\langle x,\nu\rangle + \langle x,\nu\rangle H|\mathring{A}|^2 + \langle x,\nu\rangle H\,\mathrm{Ric}(\nu,\nu)\,\mathrm{d}\mu \\ &= \int_\Sigma H^2 - \frac{1}{2}H^3\langle x,\nu\rangle + H\nabla_k H\langle x, e_k\rangle + H\big(O(1) + O(r) * A\big)\,\mathrm{d}\mu.\end{aligned} \tag{9}$$

Integrate by parts the third term on the right to calculate further

$$\begin{aligned}\int_\Sigma H\nabla_k H\langle x, e_k\rangle\,\mathrm{d}\mu &= \int_\Sigma \frac{1}{2}\nabla_k\big(H^2\big)\langle x, e_k\rangle\,\mathrm{d}\mu = -\frac{1}{2}\int_\Sigma H^2\nabla_k\langle x, e_k\rangle\,\mathrm{d}\mu \\ &= -\frac{1}{2}\int_\Sigma H^2 \operatorname{div}_\Sigma x - H^3\langle x,\nu\rangle\,\mathrm{d}\mu.\end{aligned}$$

Inserting this into Eq. (9) and taking into account (8) yields that

$$\delta_{\langle x,\nu\rangle}\mathcal{W} = \int_\Sigma H\big(O(1) + O(r) * A\big)\,\mathrm{d}\mu.$$

This can be estimated as follows:

$$|\delta_{\langle x,\nu\rangle}\mathcal{W}| \leqslant C|\Sigma|^{1/2}\bigg(\int_\Sigma H^2\,\mathrm{d}\mu\bigg)^{1/2} + Cr\int_\Sigma |A|^2\,\mathrm{d}\mu.$$

We proceed with the variation of $\mathcal{V}(\Sigma)$. From [10, (75)] we obtain that

$$\frac{1}{2}\delta_{\langle x,\nu\rangle}\mathcal{V}(\Sigma) = \int_\Sigma \langle x,\nu\rangle\big({}^M\nabla_\nu G(\nu,\nu) + HG(\nu,\nu)\big) - 2G\big(\nu,\nabla\big(\langle x,\nu\rangle\big)\big)\,\mathrm{d}\mu.$$

Straightforward estimates show that

$$\big|\delta_{\langle x,\nu\rangle}\mathcal{V}(\Sigma)\big| \leqslant C(1+r)|\Sigma| + Cr|\Sigma|^{1/2}\bigg(\int_\Sigma |A|^2\,\mathrm{d}\mu\bigg)^{1/2}.$$

This implies the claim. □

Lemma 2.8. *Let (M,g) be a manifold with C_B-bounded geometry and let ρ_1, be as in Lemma 2.6. For every constant C_0 there exists a constant C_1 with the following properties. If $r\in(0,\rho_1)$, $a\in(0,C_0r^2)$ and $\Sigma\subset B_{r/4}(p)$ with $|\Sigma|\in(\frac{a}{2},\frac{3a}{2})$ then there exists a surface $\Sigma'\subset\mathcal{B}_r(p)$ with $|\Sigma'|=a$ and*

$$\bigg|\int_{\Sigma'} |A|^2\,\mathrm{d}\mu - \int_\Sigma |A|^2\,\mathrm{d}\mu\bigg| \leqslant C_1 r\frac{||\Sigma|-a|}{a}\bigg(1 + \int_\Sigma |A|^2\,\mathrm{d}\mu\bigg).$$

Proof. Let Σ be as in the statement. Using Lemma 2.6 we can find $\Sigma'\subset\mathcal{B}_r(p)$ with $|\Sigma'|=a$ in the form $\Sigma'=\Phi_{t_0}(\Sigma)$, where Φ_t is as in Lemma 2.6. In addition, we have that $|t_0|\leqslant 2\frac{||\Sigma|-a|}{a}\leqslant 2$.

To analyze the amount the second fundamental form has changed, we assume for definiteness that $t_0>0$. From Lemma 2.7 we find that for all $\Sigma_t=\Phi_t(\Sigma)$ with $t\in[0,t_0]$ we have that

$$\frac{d}{dt}\bigg(1+\int_{\Sigma_t}|A_t|^2\,\mathrm{d}\mu_t\bigg) \leqslant Cr\bigg(1+\int_{\Sigma_t}|A_t|^2\,\mathrm{d}\mu_t\bigg),$$

where the constant C only depends on C_B, C_0 and ρ_1. In particular, we used that the area of all Σ_t is bounded by $\frac{3C_0}{2}r^2\leqslant\frac{3C_0}{2}\rho_1 r$. Integrating this ordinary differential inequality on $[0,t_0]$ and using the fact that $|t_0|\leqslant 2$ is a priori bounded, we arrive at the claimed estimate. □

3. Analytical aspects of the Willmore functional

In this section we consider the Willmore functional in the space of parametrizations which are in a subset of $W^{2,2}(S^2,\mathbb{R}^3)\cap W^{1,\infty}(S^2,\mathbb{R}^3)$. We assume that $\mathbb{R}^3$ is equipped with a smooth metric g of which we assume that with respect to standard coordinates all components and derivatives up to second order thereof are bounded. When we refer to coordinates on $\mathbb{R}^3$ we use Greek indices, and when referring to coordinates on S^2 we use Latin indices.

We define the space

$$B := \big\{F\in W^{2,2}\big(S^2,\mathbb{R}^3\big)\cap W^{1,\infty}\big(S^2,\mathbb{R}^3\big)\;\big|\;\bar g,\bar g^{\#}\in L^\infty\big(S^2\big)\cap W^{1,2}\big(S^2\big)\big\}$$

where we denote by $\bar g$ the pull-back metric F^*g on S^2 and by $\bar g^{\#}$ its inverse. The function spaces and all tensor norms are defined with respect to the standard metric on the sphere and the standard metric g^E on $\mathbb{R}^3$.

3.1. Definition of the Willmore functional

First we establish that the Willmore functional

$$\mathcal{W}(F)=\frac{1}{2}\int\limits_{F(S^2)} H^2\,\mathrm{d}\mu$$

is well defined for $F\in B$. Denote by h_{ij} the second fundamental form of $F(S^2)$, by $\bar{\Gamma}$ the Christoffel symbols of $\bar{g}$ and by Γ the Christoffel symbols of the ambient metric g. The Weingarten equation gives

$$F^\alpha_{ij}=-h_{ij}\nu^\alpha+\bar{\Gamma}^m_{ij}F^\alpha_m-\Gamma^\alpha_{\beta\gamma}F^\beta_iF^\gamma_j. \tag{10}$$

Here we use the shorthand notation

$$F^\alpha_i=\frac{\partial F^\alpha}{\partial x^i}\quad\text{and}\quad F^\alpha_{ij}=\frac{\partial^2F^\alpha}{\partial x^i\partial x^j}.$$

In fact we can take Eq. (10) as the definition of the second fundamental form. Note that since $\bar{g},\bar{g}^\#\in W^{1,2}(S^2)$ we have that $\bar{\Gamma}\in L^2(S^2)$ so that the second fundamental form is in $L^2(S^2)$. Taking the trace of (10) gives

$$H=-\left(\bar{g}^{ij}F^\alpha_{ij}+\left(\Gamma^\alpha_{\beta\gamma}\circ F\right)F^\beta_iF^\gamma_j\right)(g_{\alpha\delta}\circ F)\nu^\delta. \tag{11}$$

In particular, we can write the Willmore functional as

$$\mathcal{W}(F)=\frac{1}{2}\int\limits_{S^2}\left[\left(\bar{g}^{ij}F^\alpha_{ij}+\left(\Gamma^\alpha_{\beta\gamma}\circ F\right)F^\beta_iF^\gamma_j\right)(g_{\alpha\delta}\circ F)\nu^\delta\right]^2\sqrt{|\bar{g}|}\,\mathrm{d}x \tag{12}$$

where $|\bar{g}|=\det(\bar{g})$ and $\mathrm{d}x$ denotes the standard volume element on S^2. Clearly $\mathcal{W}$ is continuous on B where we equip B with the topology induced by convergence of F in $W^{2,2}(S^2,\mathbb{R}^3)\cap W^{1,\infty}(S^2,\mathbb{R}^3)$ and $\bar{g}$ and $\bar{g}^\#$ in $W^{1,2}\cap L^\infty$.

3.2. Lower semi-continuity

In this section we show lower semi-continuity of $\mathcal{W}$ in B with respect to weak convergence.

Proposition 3.1. *Assume that $F_k, F\in B$ are parametrizations of surfaces Σ_k and Σ and that g is a smooth metric on $\mathbb{R}^3$ such that all coefficients of g with respect to standard coordinates on $\mathbb{R}^3$ and all their derivatives are bounded. If*

$$\begin{cases} F_k\rightharpoonup F & \text{weakly in } W^{2,2}\left(S^2,\mathbb{R}^3,g^E\right),\\ F_k\overset{*}{\rightharpoonup}F & \text{weakly-* in } W^{1,\infty}\left(S^2,\mathbb{R}^3,g^E\right),\\ \bar{g}_k\overset{*}{\rightharpoonup}\bar{g} & \text{weakly-* in } L^\infty\left(S^2\right) \text{ and}\\ \bar{g}^\#_k\overset{*}{\rightharpoonup}\bar{g}^\# & \text{weakly-* in } L^\infty\left(S^2\right), \end{cases} \tag{13}$$

where $(\bar{g}_k)_{ij}:=g(\frac{\partial F_k}{\partial x^i},\frac{\partial F_k}{\partial x_j})$ are the coefficients of the induced metric on Σ_k, pulled back to S^2, then

$$\mathcal{W}(F)\leqslant\liminf_{k\to\infty}\mathcal{W}(F_k).$$

Proof. Let $|\bar{g}_k|=\det(g_k)_{ij}$, and let ν_k be the normal of Σ_k with respect to g. The above convergence (13) implies that for any $1<q<\infty$ we have

$$\begin{aligned} &(\bar{g}_k)_{ij}\to\bar{g}_{ij}\quad\text{in } L^q\left(S^2\right), &&(\bar{g}_k)^{ij}\to\bar{g}^{ij}\quad\text{in } L^q\left(S^2\right),\\ &\nu_k\to\nu\quad\text{in } L^q\left(S^2\right), &&\nu_k\overset{*}{\rightharpoonup}\nu\quad\text{in } L^\infty\left(S^2\right),\\ &|\bar{g}_k|\to|\bar{g}|\quad\text{in } L^q\left(S^2\right), &&|\bar{g}_k|\overset{*}{\rightharpoonup}|\bar{g}|\quad\text{in } L^\infty\left(S^2\right). \end{aligned} \tag{14}$$

From Eq. (12) we find that

$$\mathcal{W}(F_k) = \frac{1}{2}\int_{S^2} \big(\bar{g}_k^{ij}\big((F_k)_{ij}^{\alpha} + \Gamma^{\alpha}_{\beta\gamma} \circ F_k (F_k)_i^{\beta}(F_k)_j^{\gamma}\big)(g \circ F_k)_{\alpha\beta}\nu_k^{\beta}\big)^2 \sqrt{|g_k|}\,\mathrm{d}x.$$

We split this expression into two parts

$$\mathcal{W}(F_k) = \mathcal{W}_1(F_k) + \mathcal{W}_2(F_k)$$

with

$$\mathcal{W}_1(F_k) = \frac{1}{2}\int_{S^2} \left(\bar{g}_k^{ij}\frac{\partial^2 F_k^{\alpha}}{\partial x^i \partial x^j}(g \circ F_k)_{\alpha\beta}\nu_k^{\beta}\right)^2 \sqrt{|g_k|}\,\mathrm{d}x$$

and

$$\begin{aligned}\mathcal{W}_2(F_k) = \frac{1}{2}\int_{S^2} &\big(2\bar{g}_k^{ij}(F_k)_{ij}^{\alpha}(g_{\alpha\beta} \circ F_k)\nu_k^{\beta}\bar{g}_k^{ab}\big(\Gamma^{\delta}_{\varepsilon\mu} \circ F_k\big)(F_k)_a^{\varepsilon}(F_k)_b^{\mu}(g_{\delta\rho} \circ F_k)\nu_k^{\rho} \\ &+ \big(\bar{g}_k^{ij}\Gamma^{\alpha}_{\beta\gamma} \circ F_k (F_k)_i^{\beta}(F_k)_j^{\gamma}(g_{\alpha\beta} \circ F_k)\nu_k^{\beta}\big)^2\big)\sqrt{|g_k|}\,\mathrm{d}x.\end{aligned}$$

By the Sobolev embedding and smoothness of g it follows that $g_{\alpha\beta} \circ F_k$ converges in $L^{\infty}(S^2)$ to $g_{\alpha\beta} \circ F$ and that $\Gamma^{\alpha}_{\beta\gamma} \circ F_k$ converges in $L^{\infty}(S^2)$ to $\Gamma^{\alpha}_{\beta\gamma} \circ F$. In view of the convergence (14) it thus follows that $\mathcal{W}_2$ is continuous with respect to the convergence in (13):

$$\mathcal{W}_2(F_k) \to \mathcal{W}_2(F) \quad \text{for } k \to \infty.$$

To analyze $\mathcal{W}_1(F_k)$ we let ϕ be the integrand in the definition of $\mathcal{W}_1(F_k)$:

$$\mathcal{W}_1(F_k) = \frac{1}{2}\int_{S^2} \phi\big(\bar{g}_k^{ij}, (g_{\alpha\beta} \circ F_k), \nu_k, \sqrt{|g_k|}, (F_k)_{ij}^{\alpha}\big)\,\mathrm{d}x.$$

Then ϕ is smooth with respect to all variables, $\phi \geqslant 0$, and ϕ is convex with respect to the last set of variables $(F_k)_{ij}^{\alpha}$ as it is the concatenation of the following three maps: The linear (in $(F_k)_{ij}^{\alpha}$) map

$$\big(\bar{g}_k^{ij}, (g_{\alpha\beta} \circ F_k), \nu_k, \sqrt{|g_k|}, (F_k)_{ij}^{\alpha}\big) \mapsto \bar{g}_k^{ij}(F_k)_{ij}^{\alpha}(g_{\alpha\beta} \circ F_k)\nu_k^{\beta},$$

the convex map $\xi \mapsto \xi^2$ for $\xi \in \mathbb{R}$ and the linear multiplication by $\sqrt{|g_k|}$. Lower semi-continuity then follows as in the proof of [17, Theorem 1.6]. □

3.3. Differentiability

Given a map $F \in B$ and a smooth vector field $X \in \mathcal{X}(\mathbb{R}^3)$ we have that $X \circ F$ is in $W^{2,2} \cap W^{1,\infty}(S^2, \mathbb{R}^3, g^E)$. Furthermore, for small enough $\varepsilon > 0$ the map $F_t := F + t(X \circ F)$ is in B for all $t \in (-\varepsilon, \varepsilon)$ since the inverse of the metric $\bar{g}_t$ induced by F_t is a smooth function of $\bar{g}_t$.

We can thus calculate the variation of $\mathcal{W}$ in direction of X

$$\delta_X \mathcal{W}(F) = \frac{\partial}{\partial t}\bigg|_{t=0} \mathcal{W}(F_t).$$

A fairly long but standard calculation shows that $\mathcal{W}$ is indeed differentiable and that its variation in direction X is given by

$$\begin{aligned}\delta_X \mathcal{W}(F) = \int_{S^2} &-H\bar{g}^{ij}g\big(\nabla^2_{F_i,F_j}X, \nu\big) - 2H\bar{g}^{ik}\bar{g}^{jl}h_{ij}g(\nabla_{F_i}X, F_j) \\ &+ H^2 g(\nabla_{\nu}X, \nu) - H\,\mathrm{Ric}(X, \nu) + \frac{1}{2}H^2\,\mathrm{div}^T X\,\mathrm{d}\mu\end{aligned}$$

where Ric denotes the Ricci curvature of g, ∇ the connection of g and ∇^2 the second covariant derivative of vector fields with respect to g. There is also a formulation of the Euler–Lagrange equation for variations $X \in W^{2,2} \cap W^{1,\infty}(S^2, \mathbb{R}^3)$ which are not induced by a smooth ambient variation. To consider such vector fields, derivatives of X have to be calculated with the pull-back $\nabla^* = F^*\nabla$ of the Levi-Civita connection of g. This affects only the way in which the second derivatives are calculated. We obtain the following expression:

$$\delta_X \mathcal{W}(F) = \int_{S^2} -H\bar{g}^{ij} g\big((\nabla^*)^2_{F_i,F_j} X, \nu\big) - 2H\bar{g}^{ik}\bar{g}^{jl} h_{ij} g\big(\nabla^*_{F_i} X, F_j\big) - H \operatorname{Ric}(X,\nu) + \frac{1}{2} H^2 \operatorname{div}^T X \,\mathrm{d}\mu.$$

Note that if F is $C^4(S^2, \mathbb{R}^3)$, we can integrate by parts the terms involving derivatives of X and write the variation in a more familiar form:

$$\delta_X \mathcal{W}(F) = \int_{S^2} g(X,\nu)\big(\Delta H + H|\mathring{A}|^2 + H \operatorname{Ric}(\nu,\nu)\big)\,\mathrm{d}\mu.$$

4. Direct minimization

In this section we construct minimizers for the Willmore functional subject to a small area constraint by direct minimization. We assume that (M, g) is compact without boundary.

Fix a point $p \in M$. For $r < \operatorname{inj}(M, g, p)$ we consider the geodesic spheres $S_r(p)$. By [11] these surfaces satisfy

$$\mathcal{W}\big(S_r(p)\big) = 8\pi - \frac{4\pi r^2}{3} \operatorname{Sc}(p) + O\big(r^3\big). \tag{15}$$

In particular, for a given $\varepsilon > 0$ there exists a constant $0 < a_0 = a_0(\varepsilon)$ such that for $|S_r(p)| \leqslant a_0$ we have $\mathcal{W}(S_r) \leqslant 8\pi + \varepsilon$.

Fix some $a \in (0, a_0)$. We consider a minimizing sequence for $\mathcal{W}$ of surfaces Σ_k with $|\Sigma_k| = a$. By comparison with geodesic spheres, we can assume that $\mathcal{W}(\Sigma_k) \leqslant 8\pi + \varepsilon$. Thus, in view of Lemma 2.5 there exists a constant C such that

$$\operatorname{diam}(\Sigma_k) \leqslant C\big(a_0^{1/2} + a_0\big)$$

for all k. By choosing a_0 small enough, we can ensure that $4\operatorname{diam}(\Sigma_k) < \operatorname{inj}(M, g)$ uniformly. By compactness of M, by choosing a_0 small enough, and by passing to a subsequence if necessary, we can assume that all the Σ_k are contained in $\mathcal{B}_{\rho_0/16}(p)$ for some suitable $p \in M$, where ρ_0 is as in Remark 2.2. We decorate all geometric quantities on Σ_k by the subscript k, i.e. $H_k, \nu_k, \ldots$.

By the Gauss equation, we have

$$\mathcal{W}(\Sigma_k) = 8\pi + \int_{\Sigma_k} |\mathring{A}_k|^2 \,\mathrm{d}\mu_k + \int_{\Sigma_k} G(\nu_k, \nu_k)\,\mathrm{d}\mu_k.$$

By assumption we have $\mathcal{W}(\Sigma_k) \leqslant 8\pi + \varepsilon$. Since the curvature of (M, g) is bounded, we can estimate the last term by Ca_0. Thus we obtain the estimate

$$\|\mathring{A}_k\|^2_{L^2(\Sigma_k)} \leqslant \varepsilon + Ca_0.$$

From $|A_k|^2 = |\mathring{A}_k|^2 + \frac{1}{2}H_k^2$, we also get

$$\|A_k\|^2_{L^2(\Sigma_k)} \leqslant 8\pi + 2\varepsilon + Ca_0.$$

In the following proposition we show that we can pass this sequence to a (weak) limit, and that $\mathcal{W}$ is lower semi-continuous under this limit.

Proposition 4.1. *Let (M, g) be compact without boundary. Then there exists $\varepsilon > 0$, depending only on the geometry of M, such that the following holds. If Σ_k is a sequence of immersed surfaces with*

$$|\Sigma_k| = a < \varepsilon \quad \text{and} \quad \mathcal{W}(\Sigma_k) < 8\pi + \varepsilon \tag{16}$$

then there exists a family of parametrizations

$$F_k : S^2 \to \Sigma_k$$

such that the F_k converge weakly in $W^{2,2}$ and weakly- in $W^{1,\infty}$ to a limiting parametrization*

$$F : S^2 \to \Sigma \subset (M, g)$$

such that $|\Sigma| = a$ and

$$\mathcal{W}(\Sigma) \leqslant \liminf_{k\to\infty} \mathcal{W}(\Sigma_k).$$

Proof. Let (ρ_0, h_0) be as in Remark 2.2. Using the reasoning prior to the statement of this proposition, we can assume without loss of generality, that $\Sigma_k \subset \mathcal{B}_{\rho_0}(p)$ for some $p \in M$. Introducing normal coordinates $x : B_{\rho_0} \to \mathcal{B}_{\rho_0}(p)$ we can pull-back the Σ_k for all k and the metric g to B_{ρ_0} and g has the form

$$g = g^E + h$$

with h as in Remark 2.2. Assuming that a_0 is small enough, it is easy to see that Eq. (16) implies that Σ_k satisfies

$$\mathcal{W}^E(\Sigma_k) < 8\pi + 2\varepsilon$$

where $\mathcal{W}^E$ denotes the Willmore functional computed with respect to the Euclidean background metric g^E. Via the Gauss equation with respect to the Euclidean background, this implies that

$$\|\mathring{A}_k^E\|_{L^2(\Sigma_k, g^e)} < 2\varepsilon$$

on Σ. The estimates of De Lellis and Müller [3,4] imply that there exist conformal (with respect to the Euclidean background) parametrizations $F_k : S^2 \to \Sigma_k$ which are uniformly bounded in $W^{2,2}(S^2, \mathbb{R}^3) \cap W^{1,\infty}(S^2, \mathbb{R}^3)$. Moreover, the metric $\bar{g}_k$ is conformal to the standard metric on S^2 with conformal factor u_k which satisfies $\|u_k\|_{C^0(S^2)} \leqslant C\varepsilon$. Thus there is a subsequence of the F_k which we relabel to F_k, such that for any given $1 \leqslant p < \infty$ we have

$$\begin{cases} F_k \rightharpoonup F & \text{weakly in } W^{2,2}(S^2, \mathbb{R}^3, g^E), \\ F_k \overset{*}{\rightharpoonup} F & \text{weakly-* in } W^{1,\infty}(S^2, \mathbb{R}^3, g^E), \\ F_k \to F & \text{in } W^{1,p}(S^2, \mathbb{R}^3, g^E), \\ \bar{g}_k \overset{*}{\rightharpoonup} \bar{g} & \text{weakly-* in } L^\infty(S^2) \text{ and} \\ \bar{g}_k^\# \overset{*}{\rightharpoonup} \bar{g}^\# & \text{weakly-* in } L^\infty(S^2), \end{cases} \tag{17}$$

for a function $F \in W^{1,\infty} \cap W^{2,2}(S^2, \mathbb{R}^3)$ and a metric $\bar{g} \in L^\infty(S^2)$. Denote $\Sigma = F(S^2)$, then this convergence implies that $|\Sigma| = \lim_{k\to\infty} |\Sigma_k| = a$. Proposition 3.1 implies the lower semi-continuity of $\mathcal{W}$ with respect to the convergence in Eq. (17). □

In the following we show that the limiting parametrization $F \in W^{2,2} \cap W^{1,\infty}(S^2, M)$ of Σ of the minimizing sequence F_k for the Willmore functional is smooth. This follows from suitable modifications of a recent result of Kuwert, Mondino and Schygulla [7] on the existence of smooth spheres minimizing $\int_\Sigma |A|^2\,\mathrm{d}\mu$ resp. $\int_\Sigma (|H|^2 + 1)\,\mathrm{d}\mu$ in Riemannian manifolds satisfying suitable curvature conditions and a result of Schygulla [15] on the existence of a minimizing Willmore sphere with prescribed isoperimetric ratio in $\mathbb{R}^3$. Both of these results rely on the fundamental existence result (and especially the approximate graphical decomposition lemma) of Simon [16].

In the following we indicate how to modify the arguments of Section 3 in [7] in order to handle the present situation. First of all we note that because of the above proposition we get that the Radon measures on M

$$\mu_k(E) = \int_{F_k^{-1}(E)} \mathrm{d}\mu_{g_k}(y) \quad \text{and} \quad \alpha_k(E) = \int_{F_k^{-1}(E)} |A_k|^2\,\mathrm{d}\mu_{g_k},$$

where g_k is the metric on S^2 induced by F_k, converge weakly to limiting Radon measures μ and α. Note that for $\mathcal{W} < 8\pi + \varepsilon$ the monotonicity formula implies that the density of μ is one on its support. We have that

$$\mu(E) = \int\limits_{F^{-1}(E)} \mathrm{d}\mu_g(y)$$

is the induced measure of the limiting immersion $F : S^2 \to \mathbb{R}^3$.

We let $\delta > 0$ and we define δ-bad points by

$$B^\delta = \bigl\{\xi \in \operatorname{spt}\mu \bigm| \alpha\bigl(\{\xi\}\bigr) \geqslant \delta^2\bigr\}.$$

Note that the δ here plays the role that ε_0 has in [7].

We remark that for any $\delta > 0$ we can choose $\varepsilon \leqslant C\delta^2$ for a suitable constant C depending only on C_B so that $B^\delta = \emptyset$. This can be seen as follows:

Assume that $B^\delta \supset \{p_1, \dots, p_l\}$ and choose a radius $\rho > 0$ such that $B_\rho(p_i) \cap B_\rho(p_j) = \emptyset$ for all $1 \leqslant i, j \leqslant l$. Then we have

$$\begin{aligned}
8\pi + \varepsilon &> \lim_{k\to\infty} \mathcal{W}(\Sigma_k) \\
&\geqslant \lim_{k\to\infty} \sum_{i=1}^{l} \mathcal{W}\bigl(\Sigma_k, B_\rho(p_i)\bigr) + \mathcal{W}\left(\Sigma_k, \Sigma_k \Big\backslash \bigcup_{i=1}^{l} B_\rho(p_i)\right) \\
&\geqslant \mathcal{W}\left(\Sigma, \Sigma \Big\backslash \bigcup_{i=1}^{l} B_\rho(p_i)\right) + l\delta^2,
\end{aligned}$$

where $\mathcal{W}(\Sigma, E) = \frac{1}{2}\int_{\Sigma\cap E} |H|^2 \,\mathrm{d}\mu$. Since $H \in L^2(\Sigma)$ we can choose ρ so small that $\sum_{i=1}^{l} \mathcal{W}(\Sigma, B_\rho(p_i)) \leqslant \frac{1}{2} l\delta^2$ and hence we get

$$8\pi + \varepsilon > \mathcal{W}(\Sigma) + \frac{1}{2} l\delta^2 \geqslant 8\pi - C\varepsilon + \frac{1}{2} l\delta^2,$$

where C only depends on C_B and not on the sequence Σ_k. This implies that $l = 0$, provided $\varepsilon \leqslant (4(C+1))^{-1}\delta^2$.

Choosing $\delta = \delta_0$ from the approximate graphical decomposition lemma (in the form of Lemma 3.4 in [7]) and choosing ε according to the above reasoning, we conclude that locally $\operatorname{spt}\mu_k$ can be written as a multivalued graph away from a small set of pimples.

The key result in order to get the regularity is a power-decay result for the second fundamental form (see e.g. Lemma 3.6 in [7]). Once we have this estimate, we can follow the rest of the argument of [7] in order to conclude that $\operatorname{spt}\mu$ can locally be written as $C^{1,\alpha} \cap W^{2,2}$ graphs. Note that in our case we already ruled out the existence of bad points and we also know that the limiting measure μ is coming from the limiting immersion F. After having obtained this preliminary regularity result, we can express $\mathcal{W}$ in terms of the graph functions and since F is a minimizer of $\mathcal{W}$ subject to the area constraint, we conclude that graph functions solve the weak Euler–Lagrange equation. Using the difference quotient technique as in [16] we finally get that $\operatorname{spt}\mu$ and hence F are smooth (in our case we get an additional lower order term coming from the Lagrange parameter but this doesn't affect the very general argument of Simon).

In order to get the power-decay result for A_k we follow closely the arguments of Lemma 3.6 in [7] and Lemma 5 in [15]. More precisely we use the same replacement procedures for the graph functions u_k on balls of radii $\gamma \in (\varrho/16, \varrho/32)$ as in the above mentioned lemmas in order to get a comparison surface $\tilde{\Sigma}_k$ which satisfies

$$\bigl| |\tilde{\Sigma}_k| - a \bigr| \leqslant c\varrho^2.$$

Using Lemma 2.8 (note that $\Sigma_k \subset B_{\rho/16}(p)$ and hence by construction we can assume that $\tilde{\Sigma}_k \subset B_{\rho/4}(p)$) we conclude that there exists a surface Σ'_k with $|\Sigma'_k| = a$ and

$$\int\limits_{\Sigma'_k} |A'_k|^2 \,\mathrm{d}\mu'_k \leqslant \int\limits_{\tilde{\Sigma}_k} |\tilde{A}_k|^2 \,\mathrm{d}\tilde{\mu}_k + C\varrho^2 a^{-1}.$$

These last two estimates allow us to use Σ'_k as a comparison sequence to the minimizing sequence Σ_k and once we obtained this fact we can follow the rest of the argument of Lemma 3.6 in [7] word by word in order to get the desired power-decay.

Thus we have proved:

Theorem 4.2. *Let (M, g) be a compact, closed Riemannian manifold. Then there exists a constant $a_0 > 0$ such that for all $a \in (0, a_0)$ there is a smooth surface Σ_a that minimizes the Willmore functional among all immersed surfaces with area a.*

5. The geometry of critical points

In this section we consider smooth solutions to the Euler–Lagrange equation of the Willmore functional subject to an area constraint, that is surfaces Σ on which we have

$$\Delta H + H|\mathring{A}|^2 + H\,\mathrm{Ric}(\nu, \nu) + H\lambda = 0. \tag{18}$$

We show that if the area is small and the Willmore energy is close to that of the Euclidean sphere, Σ is very close to a geodesic sphere. This can be used to conclude via the main result in [9], that Σ is close to a critical point of the scalar curvature.

Proposition 5.1. *Assume that (M, g) has C_B-bounded geometry. Then there exist constants $C < \infty$ and $\varepsilon > 0$, depending only on C_B such that the following holds. If $\Sigma \subset (M, g)$ is a connected immersion and:*

1. *Σ satisfies Eq. (18),*
2. *$\lambda \leqslant \varepsilon|\Sigma|^{-1}$,*
3. *$\mathcal{W}(\Sigma) \leqslant 8\pi + \varepsilon$, and*
4. *$|\Sigma| \leqslant \varepsilon$.*

Then Σ satisfies the following estimate:

$$\int_\Sigma \left|\nabla^2 H\right|^2 + H^2|\nabla H|^2 + H^4|\mathring{A}|^2 \,\mathrm{d}\mu \leqslant C.$$

For the proof we need the Bochner identity in the following form:

Lemma 5.2. *Let (M, g) be a manifold and $\Sigma \subset M$ a smooth, compact, immersed 2-surface. Then for all $f \in C^\infty(\Sigma)$ we have that:*

$$\int_\Sigma \left|\nabla^2 f\right|^2 \mathrm{d}\mu = \int_\Sigma (\Delta f)^2 + |\nabla f|^2\left(\frac{1}{2}|\mathring{A}|^2 - \frac{1}{4}H^2 - \frac{1}{2}\,\mathrm{Sc} + \mathrm{Ric}(\nu, \nu)\right)\mathrm{d}\mu.$$

Here Sc *and* Ric *denote the scalar and Ricci curvature of (M, g).*

Proof. The Bochner identity states that

$$\int_\Sigma \left|\nabla^2 f\right|^2 \mathrm{d}\mu = \int_\Sigma (\Delta f)^2 - {}^\Sigma\mathrm{Rc}(\nabla f, \nabla f)\,\mathrm{d}\mu$$

where ${}^\Sigma\mathrm{Rc}$ denotes the intrinsic Ricci curvature of Σ. Since ${}^\Sigma\mathrm{Rc}(\nabla f, \nabla f) = \frac{1}{2}{}^\Sigma\mathrm{Sc}|\nabla f|^2$ we can use the Gauss equation

$$\frac{1}{2}{}^\Sigma\mathrm{Sc} = \frac{1}{2}\,\mathrm{Sc} - \mathrm{Ric}(\nu, \nu) + \frac{1}{4}H^4 - \frac{1}{2}|\mathring{A}|^2$$

to infer the claim. □

Proof of Proposition 5.1. By the integrated Gauss equation we have that

$$\mathcal{W}(\Sigma) = 8\pi + \int_\Sigma |\mathring{A}|^2\,\mathrm{d}\mu + 2\int_\Sigma G(\nu, \nu)\,\mathrm{d}\mu$$

so that by the assumptions on the area and $\mathcal{W}$ we obtain

$$\|\mathring{A}\|_{L^2}^2 \leqslant C\varepsilon.$$

Thus by choosing ε small we may later on assume that $\|\mathring{A}\|_{L^2}^2$ is as small as we desire.

Furthermore, since $|\Sigma|$ is small, and $\mathcal{W}$ is uniformly bounded, in view of Lemma 2.5, we may also assume that $\Sigma \subset \mathcal{B}_r(p)$ for some $p \in M$ and $r \leqslant C|\Sigma|^{1/2}$. Thus, we can use the Michael–Simon–Sobolev inequality as in [9, Lemma 2.3] with a uniform constant on Σ, that is a constant that is at most double the one in Euclidean space, provided ε is small enough.

Multiply Eq. (18) by ΔH and integrate. Using Young's inequality, integration by parts of the term including λ and since Ric is bounded, we obtain:

$$\int_\Sigma \frac{1}{2}(\Delta H)^2 - \lambda|\nabla H|^2\,\mathrm{d}\mu \leqslant C\int_\Sigma H^2|\mathring{A}|^4 + H^2\,\mathrm{d}\mu.$$

Thus

$$\int_\Sigma \frac{1}{2}(\Delta H)^2\,\mathrm{d}\mu \leqslant C + \varepsilon|\Sigma|^{-1}\int_\Sigma |\nabla H|^2\,\mathrm{d}\mu + C\int_\Sigma H^2|\mathring{A}|^4\,\mathrm{d}\mu. \tag{19}$$

By the Bochner identity from Lemma 5.2 we infer the estimate

$$\int_\Sigma \left|\nabla^2 H\right|^2 + H^2|\nabla H|^2\,\mathrm{d}\mu \leqslant C\int_\Sigma (\Delta H)^2 + \left(1+|\mathring{A}|^2\right)|\nabla H|^2\,\mathrm{d}\mu. \tag{20}$$

The Michael–Simon–Sobolev inequality implies that

$$\int_\Sigma |\nabla H|^2\,\mathrm{d}\mu \leqslant C|\Sigma|\int_\Sigma \left|\nabla^2 H\right|^2 + H^2|\nabla H|^2\,\mathrm{d}\mu. \tag{21}$$

Inserting this and Eq. (19) into (20) yields

$$\begin{aligned}\int_\Sigma \left|\nabla^2 H\right|^2 + H^2|\nabla H|^2\,\mathrm{d}\mu &\leqslant C + C\int_\Sigma |\mathring{A}|^2|\nabla H|^2 + H^2|\mathring{A}|^4\,\mathrm{d}\mu \\ &\quad + C\left(|\Sigma| + \varepsilon\right)\int_\Sigma \left|\nabla^2 H\right|^2 + H^2|\nabla H|^2\,\mathrm{d}\mu.\end{aligned}$$

Thus, if $|\Sigma|$ and ε are small enough, we can absorb part of the right hand side to the left and we infer that

$$\int_\Sigma \left|\nabla^2 H\right|^2 + H^2|\nabla H|^2\,\mathrm{d}\mu \leqslant C + C\int_\Sigma H^2|\mathring{A}|^4 + |\mathring{A}|^2|\nabla H|^2\,\mathrm{d}\mu. \tag{22}$$

To proceed, recall the Simons identity, which implies that on an arbitrary immersed surface we have

$$-\mathring{A}^{ij}\Delta\mathring{A}_{ij} + \frac{1}{2}H^2|\mathring{A}|^2 = -\left\langle \mathring{A}, \nabla^2 H\right\rangle + |\mathring{A}|^4 + |\mathring{A}|^2\operatorname{Ric}(\nu,\nu) - 2\mathring{A}^{ij}\mathring{A}^l_j\operatorname{Ric}_{il} - 2\langle \mathring{A}, \nabla\omega\rangle. \tag{23}$$

Here ω is the one-form $\omega = \operatorname{Ric}(\nu,\cdot)^T$ where the superscript ^{T} denotes projection to the tangential space of Σ. Multiply (23) by H^2 and integrate. Integration by parts and the Codazzi equation $\operatorname{div}\mathring{A} = \frac{1}{2}\nabla H + \omega$ yields

$$\begin{aligned}&\int_\Sigma H^2|\nabla\mathring{A}|^2 + 2H\nabla_k H\mathring{A}^{ij}\nabla_k\mathring{A}_{ij} + \frac{1}{2}H^4|\mathring{A}|^2\,\mathrm{d}\mu \\ &\quad\leqslant \int_\Sigma \left\langle \operatorname{div}\left(H^2\mathring{A}\right), \nabla H + 2\omega\right\rangle\mathrm{d}\mu + C + C\int_\Sigma H^2|\mathring{A}|^4\,\mathrm{d}\mu.\end{aligned} \tag{24}$$

Estimate

$$\left|\int_\Sigma 2H\nabla_k H\mathring{A}^{ij}\nabla_k\mathring{A}_{ij}\,\mathrm{d}\mu\right| \leqslant \frac{1}{4}\int_\Sigma H^2|\nabla\mathring{A}|^2\,\mathrm{d}\mu + 4\int_\Sigma |\mathring{A}|^2|\nabla H|^2\,\mathrm{d}\mu. \tag{25}$$

Using the Codazzi equation to infer $\operatorname{div}\mathring{A} = \frac{1}{2}\nabla H + \omega$ we calculate further

$$\begin{aligned}\int_\Sigma \langle\operatorname{div}(H^2\mathring{A}), \nabla H + 2\omega\rangle\,\mathrm{d}\mu &= \int_\Sigma 2H\mathring{A}(\nabla H, \nabla H + 2\omega) + H^2\langle\operatorname{div}\mathring{A}, \nabla H + 2\omega\rangle\,\mathrm{d}\mu \\ &= \int_\Sigma 2H\mathring{A}(\nabla H, \nabla H + 2\omega) + \frac{1}{2}H^2|\nabla H + 2\omega|^2\,\mathrm{d}\mu \\ &\leqslant C + C\int_\Sigma H^2|\nabla H|^2 + |\mathring{A}|^2|\nabla H|^2\,\mathrm{d}\mu. \end{aligned}\tag{26}$$

Combining Eqs. (24), (25) and (26) yields

$$\int_\Sigma H^2|\nabla A|^2 + H^4|\mathring{A}|^2\,\mathrm{d}\mu \leqslant C + C\int_\Sigma H^2|\nabla H|^2 + |\mathring{A}|^2|\nabla H|^2 + H^2|\mathring{A}|^4\,\mathrm{d}\mu. \tag{27}$$

In view of (22) we finally infer

$$\int_\Sigma |\nabla^2 H|^2 + |A|^2|\nabla H|^2 + H^2|\nabla A|^2 + H^4|\mathring{A}|^2\,\mathrm{d}\mu \leqslant C + C\int_\Sigma H^2|\mathring{A}|^4 + |\mathring{A}|^2|\nabla H|^2\,\mathrm{d}\mu. \tag{28}$$

To proceed, we apply the Michael–Simon–Sobolev inequality and estimate

$$\begin{aligned}\int_\Sigma H^2|\mathring{A}|^4\,\mathrm{d}\mu &\leqslant C\left(\int_\Sigma |\nabla H||\mathring{A}|^2 + |H||\mathring{A}||\nabla\mathring{A}| + H^2|\mathring{A}|^2\,\mathrm{d}\mu\right)^2 \\ &\leqslant \left(\int_\Sigma |\mathring{A}|^2\,\mathrm{d}\mu\right)\left(\int_\Sigma |\mathring{A}|^2|\nabla H|^2 + H^2|\nabla A|^2 + H^4|\mathring{A}|^2\,\mathrm{d}\mu\right). \end{aligned}\tag{29}$$

This shows that if ε and hence $\|\mathring{A}\|^2_{L^2}$ is small, the first term on the right of Eq. (28) can be absorbed to the left.

The second term on the right of (28) requires a little more work. By the Michael–Simon–Sobolev inequality we have

$$\begin{aligned}\int_\Sigma |\mathring{A}|^2|\nabla H|^2\,\mathrm{d}\mu &\leqslant C\int_\Sigma |\nabla\mathring{A}||\nabla H| + |\mathring{A}||\nabla^2 H| + |H||\mathring{A}||\nabla H|\,\mathrm{d}\mu \\ &\leqslant C\left(\int_\Sigma |\mathring{A}|^2\,\mathrm{d}\mu\right)\left(\int_\Sigma |\nabla^2 H|^2 + H^2|\nabla H|^2\,\mathrm{d}\mu\right) + C\int_\Sigma |\nabla A|^2\,\mathrm{d}\mu. \end{aligned}\tag{30}$$

The first term on the right is of the same type as before. To estimate the second integrate the Simons identity (23) to obtain

$$\int_\Sigma |\nabla\mathring{A}|^2 + \frac{1}{2}H^2|\mathring{A}|^2\,\mathrm{d}\mu \leqslant C\int_\Sigma |\mathring{A}|^2 - \langle\mathring{A}, \nabla^2 H\rangle - 2\langle\mathring{A}, \nabla\omega\rangle + C|\mathring{A}|^4\,\mathrm{d}\mu. \tag{31}$$

The Michael–Simon–Sobolev inequality implies that

$$\int_\Sigma |\mathring{A}|^4\,\mathrm{d}\mu \leqslant \left(\int_\Sigma |\mathring{A}|^2\,\mathrm{d}\mu\right)\left(\int_\Sigma |\nabla\mathring{A}|^2 + H^2|\mathring{A}|^2\,\mathrm{d}\mu\right),$$

so that the last term on the right of (31) can be absorbed to the left. Calculate further that

$$-2\int_\Sigma \langle \mathring{A}, \nabla\omega\rangle\,\mathrm{d}\mu = 2\int_\Sigma \langle \operatorname{div}\mathring{A}, \omega\rangle\,\mathrm{d}\mu$$

so that by Young's inequality

$$\left|2\int_\Sigma \langle \mathring{A}, \nabla\omega\rangle\,\mathrm{d}\mu\right| \leqslant \frac{1}{2}\int_\Sigma |\nabla\mathring{A}|^2\,\mathrm{d}\mu + C\int_\Sigma |\omega|^2\,\mathrm{d}\mu.$$

Thus the second term on the right of Eq. (31) can also be absorbed to the left. Using Hölder's inequality we infer the estimate

$$\int_\Sigma |\nabla\mathring{A}|^2 + H^2|\mathring{A}|^2\,\mathrm{d}\mu \leqslant C\biggl(|\Sigma| + \int_\Sigma |\mathring{A}|^2\,\mathrm{d}\mu\biggr) + C\biggl(\int_\Sigma |\mathring{A}|^2\,\mathrm{d}\mu\biggr)\biggl(\int_\Sigma \left|\nabla^2 H\right|^2\mathrm{d}\mu\biggr). \tag{32}$$

Since the Codazzi equation implies $\nabla H = 2\operatorname{div}\mathring{A} - 2\omega$ we furthermore get

$$\int_\Sigma |\nabla H|^2\,\mathrm{d}\mu \leqslant C\varepsilon + C\int_\Sigma |\nabla\mathring{A}|^2\,\mathrm{d}\mu.$$

Thus the previous equation implies that

$$\int_\Sigma |\nabla A|^2\,\mathrm{d}\mu \leqslant C\varepsilon + C\biggl(\int_\Sigma |\mathring{A}|^2\,\mathrm{d}\mu\biggr)\biggl(\int_\Sigma \left|\nabla^2 H\right|^2\mathrm{d}\mu\biggr). \tag{33}$$

Substituting this into Eq. (30) yields that

$$\int_\Sigma |\mathring{A}|^2|\nabla H|^2\,\mathrm{d}\mu \leqslant C\varepsilon + C\varepsilon\biggl(\int_\Sigma \left|\nabla^2 H\right|^2 + H^2|\nabla H|^2\,\mathrm{d}\mu\biggr).$$

Thus we have shown that all but the constant term on the right of Eq. (22) can be absorbed to the left, provided ε is small enough. □

We wish to complement these estimates by an estimate for λ.

Proposition 5.3. *Let (M, g) be a Riemannian manifold with C_B-bounded geometry. Let Σ be a surface satisfying (18) for some $\lambda \in \mathbb{R}$. Let ρ_0 be as in Remark 2.2 and assume that $\Sigma \subset \mathcal{B}_r(p)$ for some $p \in M$ and $0 < r < \rho_0$. Then we have the estimate*

$$|\lambda| \leqslant C|\Sigma|^{-1}\biggl(|\Sigma|^{1/2} + r\int_\Sigma |A|^2\,\mathrm{d}\mu\biggr).$$

Proof. The proof is based on the fact that the Willmore functional is scale invariant with respect to the Euclidean metric. Note that if Σ is of Willmore type we have for all variations with normal velocity f that

$$\delta_f\mathcal{W}(\Sigma) = \lambda\delta_f\mathcal{A}(\Sigma)$$

where $\mathcal{A}$ denotes the area functional. This implies that if $\delta_f\mathcal{A} \neq 0$, we can write

$$\lambda = \delta_f\mathcal{W}/\delta_f\mathcal{A}.$$

In Euclidean space we can choose $f = \langle x, \nu\rangle$ to be the normal velocity corresponding to scaling and infer that $\delta_f\mathcal{W} = 0$ whereas $\delta_f\mathcal{A} = 2|\Sigma|$ so that in combination we get that $\lambda = 0$.

In the situation as in the statement, this reasoning still works although with some error terms. Let $\Sigma \subset \mathcal{B}_r(p)$ as in the statement of the proposition. Let x denote the position vector field on $\mathcal{B}_{\rho_0}(p)$ with respect to normal coordinates on $\mathcal{B}_{\rho_0}(p)$. Since $\nabla_{\frac{\partial}{\partial x^i}}(x^j\frac{\partial}{\partial x^j}) = \delta_i^j\frac{\partial}{\partial x^j} + x^j\Gamma_{ij}^k\frac{\partial}{\partial x^k}$ it follows that

$$|\nabla x - \mathrm{Id}| \leqslant Cr^2 \quad\text{and}\quad \left|\nabla^2 x\right| \leqslant C. \tag{34}$$

The first step is to estimate the variation of area with respect to a normal variation corresponding to scaling in normal coordinates. That is, we use $f = \langle x, \nu\rangle$ as a normal variation for Σ. This yields

$$\delta_f \mathcal{A}(\Sigma) = \int_\Sigma H\langle x, \nu\rangle \,\mathrm{d}\mu = \int_\Sigma \operatorname{div}_\Sigma x \,\mathrm{d}\mu.$$

Since $|\operatorname{div}_\Sigma x - 2| \leqslant C|x|^2$ by (34) and the fact that $\Sigma \subset \mathcal{B}_r(p)$, we thus find that, provided r is small enough,

$$\delta_{\langle x,\nu\rangle}\mathcal{A} \geqslant |\Sigma|. \tag{35}$$

As in the proof of Lemma 2.7 we can estimate that

$$|\delta_{\langle x,\nu\rangle}\mathcal{W}| \leqslant C|\Sigma|^{1/2}\biggl(\int_\Sigma H^2 \,\mathrm{d}\mu\biggr)^{1/2} + Cr\int_\Sigma |A|^2 \,\mathrm{d}\mu.$$

In combination with Eq. (35), this yields the claimed estimate for λ. $\square$

The main result of this section is a straightforward consequence of the combination of Propositions 5.1 and 5.3:

Theorem 5.4. *Given a Riemannian manifold (M, g) with C_B-bounded geometry there exist constants $C < \infty$ and $\varepsilon > 0$ depending only on C_B such that the following holds.*

Assume that $\Sigma \subset (M, g)$ is a connected immersion that satisfies the following conditions:

1. *Σ satisfies Eq.* (18)*,*
2. *$\mathcal{W}(\Sigma) \leqslant 8\pi + \varepsilon$, and*
3. *$|\Sigma| \leqslant \varepsilon$.*

Then Σ satisfies the following estimate:

$$\int_\Sigma \bigl|\nabla^2 H\bigr|^2 + H^2|\nabla H|^2 + H^4|\mathring{A}|^2 \,\mathrm{d}\mu \leqslant C.$$

Corollary 5.5. *Assume Σ is as in Theorem 5.4. Then we have the following estimates*:

$$\|\mathring{A}\|_{L^2(\Sigma)} \leqslant C|\Sigma| \quad \textit{and} \quad \|H - 2/R\|_{L^\infty(\Sigma)} \leqslant C|\Sigma|^{1/2}$$

where R is such that $|\Sigma| = 4\pi R^2$. In particular, if the area of Σ is small enough we have that $H > 0$.

Proof. In order to show the first estimate we apply the Hölder and Michael–Simon–Sobolev inequality to get

$$\int_\Sigma |\mathring{A}|^2 \,\mathrm{d}\mu \leqslant C|\Sigma| \int_\Sigma \bigl(|\nabla\mathring{A}|^2 + H^2|\mathring{A}|^2\bigr) \,\mathrm{d}\mu.$$

Combining this estimate with (32) and Theorem 5.4 yields

$$\int_\Sigma |\mathring{A}|^2 \,\mathrm{d}\mu \leqslant C|\Sigma|,$$

and inserting this new information once more into (32) gives the desired result.

To see the second estimate, note that by the estimates of De Lellis and Müller [3, Theorem 1.1] we have

$$\bigl\|H^e - 2/R^e\bigr\|_{L^2(\Sigma)} \leqslant C\bigl\|\mathring{A}^e\bigr\|_{L^2(\Sigma)},$$

where we denote geometric objects which are calculated with respect to the Euclidean metric by a superscript e. In the following we assume that $\Sigma \subset \mathcal{B}_r(p)$ with $r \leqslant C|\Sigma|^{1/2}$ (see Lemma 2.5).

From [9, Lemmas 2.1, 3.3] we have that

$$\|H - H^e\|_{L^2(\Sigma)} \leqslant C|\Sigma|$$

and

$$\left||\Sigma| - |\Sigma|^e\right| \leqslant C|\Sigma|^2$$

which implies that

$$\left|R - R^e\right| \leqslant C|\Sigma|^{3/2}.$$

Combining the last four estimates with [9, Lemma 2.5] we conclude for $|\Sigma|$ small enough

$$\|H - 2/R\|_{L^2(\Sigma)} \leqslant C|\Sigma|. \tag{36}$$

From [9, Lemma 3.7] we furthermore infer that

$$\|H - 2/R\|^4_{L^\infty(\Sigma)} \leqslant \|H - 2/R\|^2_{L^2(\Sigma)} \int\limits_\Sigma \left|\nabla^2 H\right|^2 + H^4|H - 2/R|^2 \,\mathrm{d}\mu. \tag{37}$$

In order to estimate the right hand side we note that

$$|H|^2 \leqslant \left(|H - 2/R| + 2/R\right)^2 \leqslant 2|H - 2/R|^2 + 8/R^2$$

and thus

$$\begin{aligned} H^4|H - 2/R|^2 &\leqslant 2H^2|H - 2/R|^4 + 8H^2R^{-2}|H - 2/R|^2 \\ &\leqslant 2H^2|H - 2/R|^4 + 16R^{-2}|H - 2/R|^4 + 64R^{-4}|H - 2/R|^2 \end{aligned}$$

so that in view of (36)

$$\int\limits_\Sigma H^4|H - 2/R|^2 \,\mathrm{d}\mu \leqslant C\|H - 2/R\|^4_{L^\infty(\Sigma)} + C.$$

This estimates the second term in the integral in (37) whereas Theorem 5.4 estimates the first one. In combination we arrive at

$$\|H - 2/R\|^4_{L^\infty(\Sigma)} \leqslant C|\Sigma|^2\left(C + C\|H - 2/R\|^4_{L^\infty(\Sigma)}\right).$$

The second term on the right can be absorbed to the left if $|\Sigma|$ is small enough. This yields the second estimate. □

Corollary 5.6. *Let (M, g) be a compact, closed Riemannian manifold. Let Σ_a be the surfaces from Theorem 4.2.*

For any sequence $a_i \to 0$ there is a subsequence $a_{i'}$ such that $\Sigma_{a_{i'}}$ is asymptotic in $W^{2,2}$ to a geodesic sphere $\mathcal{S}_{r_{i'}}(p_{i'})$ where $r_{i'} = \sqrt{a_{i'}/4\pi}$ and $p_{i'} \to p \in M$ and Sc attains its maximum at p.

Proof. If a is small enough, we know from Lemma 2.5 that there exists a point $q_a \in M$ such that $\Sigma_a \subset \mathcal{B}_{r_0}(q_a)$ with $2r_0$ a lower bound for the injectivity radius of (M, g).

We also know that Theorem 5.4 applies to Σ_a. Hence [3, Theorem 1.1] implies that for $r_a := \sqrt{a/4\pi}$ there exists $p_a \in \mathcal{B}_{r_0}(q_a)$ such that Σ_a can be parameterized over the Euclidean sphere $S_a := S_{r_a}(p_a)$ in normal coordinates centered at q_a by $\psi_a : S_a := S_{r_a}(p_a) \to \Sigma_a$ that is conformal with respect to the Euclidean metric in normal coordinates centered at q_a such that

$$\begin{aligned} &a^{-1}\|\psi_a - \mathrm{id}_{S_a}\|_{L^2(S_{r_a}(p_a))} + a^{-1/2}\left\|d(\psi_a - \mathrm{id}_{S_a})\right\|_{L^2(S_{r_a}(p_a))} + \left\|d^2(\psi_a - \mathrm{id}_{S_a})\right\|_{L^2(S_{r_a}(p_a))} \\ &\quad \leqslant C\|\mathring{A}\|_{L^2(\Sigma_a)} \leqslant Ca. \end{aligned}$$

This implies in particular that Σ_a is $W^{2,2}$ close to S_a which in turn is C^2-close to the geodesic sphere $\mathcal{S}_a := \mathcal{S}_{r_a}(p_a)$ with respect to g centered at p_a. In particular Σ_a is $W^{2,2}$-close to $\mathcal{S}_a$.

Since the estimates from Theorem 5.4 are sufficient to carry out the analysis in [9], we obtain that $|\nabla \operatorname{Sc}(p_a)| \to 0$ as $a \to 0$.

Since M is compact, there exists a subsequence $(a_{i'})$ of the sequence $a_i \to 0$ and a point $p \in M$ so that $p_{a_{i'}} \to p$. By continuity we have $\nabla \operatorname{Sc}(p) = 0$ and p is a critical point of the scalar curvature. Moreover, from [9, Theorem 5.1] we obtain the expansion

$$\left| \mathcal{W}(\Sigma_a) - 8\pi + \frac{|\Sigma_a|}{3} \operatorname{Sc}(p) \right| \leqslant C a^{3/2}.$$

Comparing this expansion to the one for geodesic spheres, Eq. (15), we find that the scalar curvature of (M, g) attains its maximum at p or the $\Sigma_{a_{i'}}$ cannot be optimal if $a_{i'}$ is small enough. □

Note added in proof

Two month after the submission of this paper, the authors became aware of a paper by Mondino and Rivière [13]. As a special case of their results, Mondino and Rivière also obtain the existence and regularity of smooth minimizers of $\mathcal{W}(f)$ with fixed small area.

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Appendix A. Complete surfaces of Willmore type in Riemannian manifolds

In this section we use the methods developed in [5] to classify complete surfaces of Willmore type with positive mean curvature in Riemannian manifolds.

We start by recalling the Gauss equation

$$^{\Sigma}\operatorname{Sc} = \operatorname{Sc} - 2\operatorname{Ric}(\nu, \nu) + \frac{1}{2}H^2 - |\mathring{A}|^2 \tag{A.1}$$

and the Euler–Lagrange equation satisfied by surfaces of Willmore type

$$\Delta H + H|\mathring{A}|^2 + H\operatorname{Ric}(\nu, \nu) + \lambda H = 0. \tag{A.2}$$

Here $\lambda \in \mathbb{R}$ is the Lagrange multiplier. Letting $f \in C^1_c(\Sigma)$ and multiplying (A.2) with $f^2 H^{-1}$ we get after integrating by parts

$$\int_\Sigma f^2\big(|\mathring{A}|^2 + \operatorname{Ric}(\nu, \nu) + \lambda + |\nabla \log H|^2\big)\,\mathrm{d}\mu = 2\int_\Sigma f\langle \nabla f, \nabla \log H\rangle\,\mathrm{d}\mu.$$

Using Young's inequality we conclude

$$\int_\Sigma f^2\big(|\mathring{A}|^2 + \operatorname{Ric}(\nu, \nu) + \lambda\big)\,\mathrm{d}\mu \leqslant \int_\Sigma |\nabla f|^2\,\mathrm{d}\mu.$$

Replacing the Ricci curvature on the left hand side by inserting (A.1) we finally get the following lemma.

Lemma A.1. *Let Σ be a surface of Willmore type with positive mean curvature. Then we have for any $f \in C^1_c(\Sigma)$*

$$\int_\Sigma f^2\left(\frac{1}{2}|\mathring{A}|^2 + \frac{1}{4}H^2 + \frac{1}{2}\operatorname{Sc} - \frac{1}{2}{}^{\Sigma}\operatorname{Sc} + \lambda\right)\mathrm{d}\mu \leqslant \int_\Sigma |\nabla f|^2\,\mathrm{d}\mu. \tag{A.3}$$

In particular, if $\lambda \geqslant -\frac{1}{2}\operatorname{Sc}$ we have

$$\int_\Sigma f^2\left(\frac{1}{2}|\mathring{A}|^2 + \frac{1}{4}H^2 - \frac{1}{2}{}^{\Sigma}\operatorname{Sc}\right)\mathrm{d}\mu \leqslant \int_\Sigma |\nabla f|^2\,\mathrm{d}\mu. \tag{A.4}$$

These inequalities are similar to the stability inequality for minimal surfaces. Indeed they allow us to classify surfaces of Willmore type with positive mean curvature. We directly get the following corollary:

Corollary A.2. *Let $\Sigma \subset M$ be a compact surface of Willmore type with positive mean curvature and let $\lambda \geqslant -\frac{1}{2}\,\mathrm{Sc}$. Then Σ is a topological sphere.*

Proof. In this situation we can insert $f \equiv 1$ into (A.4) and with the help of the Gauss–Bonnet theorem we get

$$0 \leqslant \int_\Sigma \frac{1}{2}|\mathring{A}|^2 + \frac{1}{4}H^2\,\mathrm{d}\mu \leqslant 4\pi\big(1 - q(\Sigma)\big), \tag{A.5}$$

where $q(\Sigma)$ is the genus of Σ. But if Σ is a torus then we conclude from the above inequality that $H \equiv 0$ which contradicts the assumption of the corollary. This finishes the proof. □

Lemma A.3. *Let $\Sigma \subset M$ be a non-compact, complete surface of Willmore type with positive mean curvature and let $\lambda \geqslant -\frac{1}{2}\,\mathrm{Sc}$. Then Σ is either conformally equivalent to the plane or to a cylinder. In the latter case Σ has infinite absolute total curvature.*

Proof. We follow closely the proof of Theorem 3 in [5]. Assume that the universal covering space of Σ is $B_1(0)$. Defining $q = \frac{1}{2}{}^\Sigma\mathrm{Sc} - \frac{1}{2}|\mathring{A}|^2 - \frac{1}{4}H^2$ and using (A.4) we see that we can apply Lemma 1 of [5] and we get a positive solution g on Σ of the equation

$$\Delta g - \frac{1}{2}{}^\Sigma\mathrm{Sc}\,g + \left(\frac{1}{2}|\mathring{A}|^2 + \frac{1}{4}H^2\right)g = 0.$$

This solution can be lifted to $B_1(0)$ and by Corollary 3 in [5] this is a contradiction since $\frac{1}{2}{}^\Sigma\mathrm{Sc} = K$ and $\frac{1}{2}|\mathring{A}|^2 + \frac{1}{4}H^2 \geqslant 0$.

Hence the covering space of Σ is $\mathbb{C}$ and this shows that Σ is either a plane or a cylinder. If Σ is a cylinder with finite absolute total curvature, then we can continue arguing as in Theorem 3 of [5] and we conclude

$$\int_\Sigma |\mathring{A}|^2 + \frac{1}{2}H^2\,\mathrm{d}\mu \leqslant \int_\Sigma {}^\Sigma\mathrm{Sc}\,\mathrm{d}\mu \leqslant 0.$$

This contradicts the assumption $H > 0$ and finishes the proof of the lemma. □

For $M = \mathbb{R}^3$ and $\lambda = 0$ we have the following theorem.

Theorem A.4. *Let $\Sigma \subset \mathbb{R}^3$ be a complete Willmore surface with positive mean curvature. Then Σ is a round sphere.*

Proof. Combining Corollary A.2 and Lemma A.3 we conclude that Σ is either a topological sphere or its universal cover is $\mathbb{C}$. If Σ is a topological sphere we conclude from (A.5) that

$$\int_\Sigma |\mathring{A}|^2 + \frac{1}{2}H^2\,\mathrm{d}\mu \leqslant 8\pi.$$

Since on the other hand $\int_\Sigma \frac{1}{2}H^2\,\mathrm{d}\mu \geqslant 8\pi$ for all closed surfaces $\Sigma \subset \mathbb{R}^3$ we find that Σ is umbilic and hence a round sphere.

Next we rule out the case that the universal cover of Σ is $\mathbb{C}$. The Gauss equation (A.1) yields that

$$\frac{1}{2}{}^\Sigma\mathrm{Sc} = \frac{1}{4}H^2 - \frac{1}{2}|\mathring{A}|^2.$$

Inserting this into (A.4) we have for every $f \in C^1_c(\Sigma)$

$$\int_\Sigma f^2|\mathring{A}|^2\,\mathrm{d}\mu \leqslant \int_\Sigma |\nabla f|^2\,\mathrm{d}\mu.$$

Hence, defining $q = -|\mathring{A}|^2$, we can apply Theorem 1 of [5] and get a positive solution g of

$$\Delta g + |\mathring{A}|^2 g = 0$$

on Σ and by lifting also on $\mathbb{C}$. Hence we have a positive super-harmonic function g on $\mathbb{C}$ which must be constant. This implies that $\mathring{A} \equiv 0$ and therefore Σ is a flat plane, which contradicts our assumptions. $\square$

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