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On the analysis of traveling waves to a nonlinear flux limited reaction–diffusion equation ☆

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Abstract

In this paper we study the existence and qualitative properties of traveling waves associated with a nonlinear flux limited partial differential equation coupled to a Fisher–Kolmogorov–Petrovskii–Piskunov type reaction term. We prove the existence and uniqueness of finite speed moving fronts of C^2 classical regularity, but also the existence of discontinuous entropy traveling wave solutions.

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1. Introduction and main results

The aim of this paper is to analyze the existence of traveling waves associated with a heterogeneous nonlinear diffusion partial differential equation coupled to a reaction term of Fisher–Kolmogorov–Petrovskii–Piskunov type. The nonlinear diffusion term has been motivated in different contexts and from different points of view (see the pioneering work [31]). Also, it has been deduced in the Monge–Kantorovich's optimal mass transport framework, where it is usually called the relativistic heat equation [16], or in astrophysics [28]. The existence and uniqueness of entropy solutions for the nonlinear parabolic flux diffusion was proved in [4], while in [5] the finite speed of propagation was analyzed. The resulting reaction–flux-limited–diffusion system exhibits new properties compared with the classical reaction term coupled to the linear diffusion equation, such as the existence of singular traveling waves. This opens new research perspectives regarding its application to the biological or traffic flow frameworks.

Reaction–diffusion systems consist in mathematical models describing the dynamics of the concentration of one or more populations distributed in space under the influence of two processes: Local reactions, in which the populations

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interact with each other; and diffusion, which provokes the populations to spread out in space. In the reaction–diffusion context, the notion of population must be understood in a wide sense, say particles or concentrations in chemical processes, cells, morphogens, among others or more recently in computer science or complex systems (see for instance [18,20,23,26,27,29,32,34,36]). Much attention has been paid on these systems within a broad variety of applications, starting from linear diffusion of type

$$\frac{\partial u}{\partial t} = \nu \Delta u + f(u), \qquad u(t=0,x) = u_0(x), \tag{1}$$

where ν is the diffusion coefficient and f represents the reaction term. Cooperative behavior has often been modeled from diffusive coupling of nonlinear elements, and reaction–diffusion equations provide the prototypical description of such systems. These systems have also attracted the researcher attention as archetypal models for pattern formation. This is particularly connected with the study of traveling waves, i.e. solutions of the type $u(t,x) = u(x - \sigma t)$.

In many applications (particularly in complex systems) reaction–diffusion equations often provide a natural mathematical description of these dynamical networks, whose elements are coupled through diffusion in many instances. The correct description of reaction–diffusion phenomena requires a detailed knowledge of the interactions between individuals and groups of individuals. This line of research motivates the study of nonlinear cooperative behavior in complex systems [9,10], which is a subject interconnected with reaction–diffusion systems through hydrodynamical limits of kinetic equations. There is a wide literature raising the universality of application of reaction–diffusion systems. Nevertheless, there are limitations to the reaction–diffusion description.

In biochemical networks constituted by small cellular geometries, a macroscopic reaction–diffusion model may be inappropriate. In some circumstances the coupling among elements is not diffusive or the diffusive processes are nonlinear, which will strongly influence the dynamical behavior of the network. In [33] it is proposed a nonlinear degenerate density-dependent diffusion motivated by the fact that there are biological (mating, attracting and repelling substances, overcrowding, spatial distribution of food, social behavior, etc.) and physical (light, temperature, humidity, etc.) factors entailing that the probability is no longer a space-symmetric function, i.e. it looses the homogeneity, so that linear diffusion does not constitute a good approach. This heterogeneity property of the diffusion operator comes from the heterogeneous character of the equation and/or from the underlying domain (we also refer to [11–14]). The same sort of problems concerning the universality in the applicability occur when a mean-field interaction among particles is not available (see for example [10]) or when particles are dilute or large with respect to the vessel or the media where they are moving [6,32]. In these cases, the linear diffusion approach might not be the most appropriate. From a modeling viewpoint, the above processes seem to require the incorporation of one or various phenomena not included within linear diffusion theory, such as the finite speed of propagation of matter or the existence of nonsmooth densities (singular traveling waves), for example. The mathematical argument justifying that even in case that the solution has not compact support, its size (invoking mass or concentration according to the case dealt with) is very small out of some ball with large radius, might be unrealistic. Indeed, in several applications in biology (morphogenesis) [1,12,17,36,35], social sciences [9] or traffic flow [15], this kind of situations (solutions with large queues) can activate other processes. This is the case, for example, of the biochemical processes inside the cells whose activation depends on the time of exposure as well as on the received concentration of morphogen (see [1,17]). Then, exploring or modeling new nonlinear transport/diffusion phenomena is an interesting subject not only from the viewpoint of applications, but also from a mathematical perspective.

Motivated by the above considerations, the objective of this paper consists in analyzing the existence of traveling waves for the following one-dimensional, nonlinear flux limited reaction–diffusion equation

$$\frac{\partial u}{\partial t} = \nu \partial_x \left(\frac{u \partial_x u}{\sqrt{|u|^2 + \frac{\nu^2}{c^2} |\partial_x u|^2}} \right) + f(u), \qquad u(t=0,x) = u_0(x), \tag{2}$$

where ν is the viscosity and c is a constant velocity related to the internal properties of the particles. There are various reasons that motivate this choice for the nonlinear diffusion term. First of all, the solutions to this system have finite speed of propagation as opposite to the linear heat equation, i.e. for a compactly supported initial data, the growth velocity of the support of the solution is bounded by c (see [2]). Furthermore, Eq. (2) is an extension of the heat equation in the following sense: Rewrite the heat equation as

$$\frac{\partial u}{\partial t} = \nu \frac{\partial}{\partial x} \left[u \frac{\partial}{\partial x} \ln u \right] = \nu \frac{\partial}{\partial x} [uv], \tag{3}$$

where v is a microscopic velocity, in such a way that it can be seen as a transport kinetic equation. The velocity v is determined by the entropy of the system, $S(u) = u \ln u$, and by the concentration u, via the following formula

$$v = \frac{\partial}{\partial x}\left(\frac{S(u)}{u}\right). \tag{4}$$

Note that $\frac{S(u)}{u} = \ln u$ is known as the chemical potential.

We propose to modify the form of the flux in (3) by considering a new microscopic velocity averaged with respect to the line element associated with the motion of the particle, so that the new velocity is given by $\frac{\partial}{\partial v}\sqrt{1+|v|^2} = \frac{v}{\sqrt{1+|v|^2}}$, that is

$$\tilde{v} = \frac{\partial_x(S(u)/u)}{\sqrt{1+[\partial_x(S(u)/u)]^2}}, \tag{5}$$

which leads us to (2). This implies that the chemical potential is now finite, that is not the case for the linear heat equation. Thus, the velocity for which the concentration or density u is transported depends on the entropy of the system (determining its disorder) as well as on its density under an appropriate measure.

For the reaction term we will consider a canonical model of Fisher [21] or KPP [25] (for Kolmogorov, Petrovsky and Piskunov) type, to be denoted FKPP from now on. For the linear diffusion case, the properties associated with this system are well understood in the homogeneous framework (see for example [7,8,21,25]).

The term $f(u)$ is written as $uK(u)$, where K is known in a biological setting as the growth rate of the population. The main hypotheses about the FKPP reaction term $K \in C^1([0,1])$ are typically written as

$$\text{(i)}\quad K(1) = 0, \qquad \text{(ii)}\quad K'(s) < 0, \quad s \in (0,1]. \tag{6}$$

These assumptions have some consequences on $f(u)$, such us $f(0) = f(1) = 0$, $f'(1) < 0$, $f'(0) > 0$, $f > 0$ in $(0,1)$. Hypothesis (i) in (6) is a normalization property of the carrying capacity, while (ii) represents a saturation of the media when the concentration is increasing. Typical examples of such nonlinearities are $K(s) = k(1-s)$ or $K(s) = k(1-s^2)$, where $k = K(0) = f'(0)$ is a constant related to the growth rate of the (biological) particles, usually called intrinsic growth rate. In [21,25] it was proved that, under the above assumptions, there is a threshold value $\sigma^* = 2\sqrt{\nu k}$ for the speed σ associated with the linear diffusion system (1). Namely, no fronts exist for $\sigma < \sigma^*$, and there is a unique front (up to space or time shifts) for all $\sigma \geqslant \sigma^*$.

The existence and uniqueness theory for solutions to the flux limited reaction–diffusion equation (2) was done in [2] (see also the references therein for a complete study of the relativistic heat equation). The natural concept of solution for this problem implies the use of Kruzkov's entropy solutions. In fact, in [2] it is proved that for any initial datum $0 \leqslant u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, there exists a unique entropy solution u of (2) in $[0,T) \times \mathbb{R}^N$, for every $T > 0$, such that $u(t=0) = u_0$. These solutions live in a subspace of Bounded Variation functions. Moreover, if $u(t)$, $\bar{u}(t)$ are the entropy solutions corresponding to the initial data $u_0, \bar{u}_0 \in (L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))^+$, respectively, then

$$\left\|u(t) - \bar{u}(t)\right\|_{L^1(\mathbb{R}^N)} \leqslant e^{t\|f\|_{Lip}} \|u_0 - \bar{u}_0\|_{L^1(\mathbb{R}^N)}, \quad \forall t \geqslant 0,$$

where $\|f\|_{Lip}$ denotes the Lipschitz constant for f in $[0,1]$. The existence of entropy solutions with initial data in L^∞ was extended in [2, Proposition 3.14].

One of the most important differences between the linear and the nonlinear diffusion models (1) and (2) emerges when traveling waves are studied. A traveling wave is a solution having a constant profile which moves with constant speed, i.e. a solution of the form $u(t,x) = u(\xi)$ with $\xi = x - \sigma t$, for some constant σ. The function u is usually called the wave profile and the constant σ is the wave speed. Let us give a simple example, that may be useful to illustrate the results obtained in this paper for Eq. (2), by means of the following simplified reaction–flux-limited–diffusion equation

$$\partial_t u = \partial_x\left(u \frac{\partial_x u}{|\partial_x u|}\right) + u(1-u), \tag{7}$$

which allows to compute explicit traveling waves. Indeed, the equation satisfied by a decreasing wave front profile $u(\xi) = u(x - \sigma t)$ is

$$-\sigma \dot{u} = -\dot{u} + u(1-u).$$

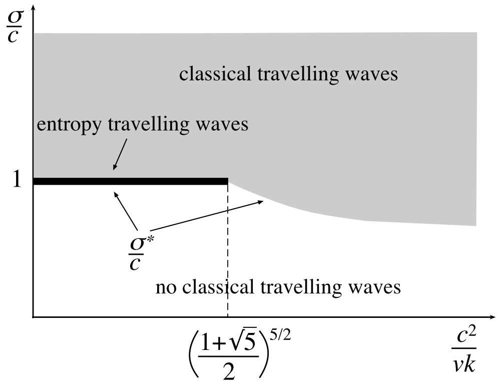

Fig. 1. Different regions determining the existence of classical or entropy traveling waves. The non-constant part of the curve defining σ^* stems from a cubic function involved in the planar dynamical system associated with the traveling wave equation (see (24)–(26) below).

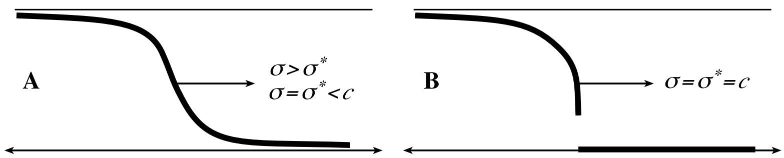

Fig. 2. Different types of traveling waves: (A) classical and (B) entropy wave fronts.

Then, it can be easily proved the existence of a unique, global classical solution given by

$$u_\sigma(\xi) = \frac{1}{e^{\frac{1}{\sigma-1}\xi} + 1}, \quad \xi \in \mathbb{R},$$

only if $\sigma > 1$ (up to space or time shifts). Furthermore, $u = \chi_{\{\xi<0\}}$ stands for the traveling wave profile of an entropy solution to Eq. (7) with $\sigma = 1$, where χ_A denotes the characteristic function of the set A. Let us observe how regular and discontinuous solutions coexist for this simplified model. To see the details of the above results the interested reader can consult [2].

As in the previous case, singular profiles are found when looking for traveling waves of Eq. (2), which to a certain extent constitute the equivalent notion of shock waves in hyperbolic models for traffic flow. On the other hand, there exist significant differences regarding σ between the traveling waves of the nonlinear reaction–diffusion equation (2) and those of Eq. (1).

In this paper, we look for a particular kind of traveling waves called *wave fronts*, determined by a decreasing wave profile $u \in (0, 1)$ such that $\lim_{\xi\to-\infty} u(\xi) = 1$ and $\lim_{\xi\to\infty} u(\xi) = 0$, satisfying (2) in a sense to be specified later. According to the degenerate character of the flux limiter at $u \equiv 0$, we split the analysis of wave fronts into two steps. For the positive part $u(\xi) > 0$, $\forall \xi \in (-\infty, \xi_0)$, we impose that $u \in C^2$ solves the equation in a classical sense. Thus, if $\xi_0 = \infty$ we will have a *classical solution* verifying the equation everywhere in the domain of definition. If $\xi_0 < \infty$, we will see that the null extension of the positive part can be an entropy solution under certain conditions, these solutions being *discontinuous*. The entropy criterium is necessary within this problem, since it selects traveling waves of discontinuous type.

Our main result is the following (see Figs. 1 and 2).

Theorem 1.1. *Given $\sigma^* \leqslant c$, depending on ν, c, and $k = K(0)$, there exists a wave front fulfilling one of the following properties*:

(i) *If the wave speed satisfies $\sigma > \sigma^*$ or $\sigma = \sigma^* < c$, then the traveling wave is a classical solution to* (2).
(ii) *If the wave speed satisfies $\sigma = \sigma^* = c$, then the traveling wave is a discontinuous entropy solution to* (2).

Remark 1. The existence of nonclassical traveling waves solutions in the case $\sigma < \sigma^*$ is an open problem. Also, the existence of other kind of traveling waves such as those with pulses or soliton-type shapes constitutes an interesting problem to be explored (see for example [30] or [19] in another context).

In Section 2 we will determine σ^* in terms of the parameters ν, c, and k. The analytical theory dealing with the existence of a solution-set-structure follows from the associated asymptotic initial value problem satisfied by the traveling wave profile. This problem is framed in the theory of planar dynamical systems, where the wave speed σ is a parameter.

Another fundamental property of Eq. (1) concerns the asymptotic speed of spreading and was established in [8]: If $u_0 \geqslant 0$ is a continuous, compactly supported function in $\mathbb{R}^N$ and $u_0 \not\equiv 0$, then the solution $u(t,x)$ with initial data $u(t=0,x) = u_0(x)$ spreads out with speed σ^* in all directions as $t \to +\infty$, i.e. $\max_{|x|\leqslant \sigma t} |u(t,x) - 1| \to 0$ for each $\sigma \in [0, \sigma^*)$, and $\max_{|x|\geqslant \sigma t} u(t,x) \to 0$ for each $\sigma > \sigma^*$. A similar result is expected in our context via the control of the entropy solution in the set $\{x > \sigma t\}$ by means of an exponential function with negative exponent (see Proposition 3.4 below).

The paper is organized as follows. In Section 2 we introduce the asymptotic second-order initial value problem associated with the detection of traveling wave solutions, as well as the equivalent first-order planar system linked to the traveling waves equation. Then, we deal with the existence and uniqueness of regular traveling waves. Finally, Section 3 is devoted to analyze the singular wave profiles that can be identified as entropy solutions.

2. An equivalent problem for classical traveling waves

As mentioned before, the aim of this section is to analyze the wave front solutions to Eq. (2).

2.1. Traveling wave equations

The existence of a regular traveling wave $u(x - \sigma t)$ solving Eq. (2) leads to the problem consisting of finding a solution to the following equation

$$\nu\left(\frac{uu'}{\sqrt{|u|^2 + \frac{\nu^2}{c^2}|u'|^2}}\right)' + \sigma u' + f(u) = 0, \tag{8}$$

which is defined on $(-\infty, \xi_0)$ and satisfies

$$\lim_{\xi\to-\infty} u(\xi) = 1 \tag{9}$$

and

$$u'(\xi) < 0 \quad \text{for any } \xi \in (-\infty, \xi_0). \tag{10}$$

The constant σ is a further unknown of the problem. Let us analyze this asymptotic initial value problem where $f(u) = uK(u)$ and K fulfills (6). The following result contributes to the determination of the asymptotic value of the derivative of u.

Lemma 2.1. *Let $u : (-\infty, \xi_0) \to (0,1)$ be a solution of (8) that satisfies (9)–(10). Then,*

$$\lim_{\xi\to-\infty} u'(\xi) = 0. \tag{11}$$

Proof. Take $\xi_n \to -\infty$ with $\xi_n < \xi_0$. For any fixed $n \in \mathbb{N}$ we use the mean value theorem in the interval $[\xi_n - 1, \xi_n]$ to obtain the existence of a sequence $s_n \in [\xi_n - 1, \xi_n]$ satisfying

$$u'(s_n) = u(\xi_n) - u(\xi_n - 1) \to 0.$$

Then, we integrate (8) over $[s_n, \xi_n]$ and analyze the terms of the following identity

$$\int_{s_n}^{\xi_n} \nu\left(\frac{u(\delta)u'(\delta)}{\sqrt{|u(\delta)|^2 + \frac{\nu^2}{c^2}|u'(\delta)|^2}}\right)' d\delta + \int_{s_n}^{\xi_n} \sigma u'(\delta)\, d\delta + \int_{s_n}^{\xi_n} f\big(u(\delta)\big)\, d\delta = 0.$$

The third term satisfies

$$\int_{s_n}^{\xi_n} f\big(u(\delta)\big)\,d\delta \to 0,$$

as the interval is bounded and the integrand converges uniformly to zero. The second term clearly (Leibnitz's rule) equals

$$\sigma\big(u(\xi_n) - u(s_n)\big),$$

that tends to zero because of (9). Finally, the first term takes the form

$$\nu \frac{u(\xi_n)u'(\xi_n)}{\sqrt{|u(\xi_n)|^2 + \frac{\nu^2}{c^2}|u'(\xi_n)|^2}} - \nu \frac{u(s_n)u'(s_n)}{\sqrt{|u(s_n)|^2 + \frac{\nu^2}{c^2}|u'(s_n)|^2}},$$

which also converges to zero as n goes to infinity. Therefore, using that $u'(s_n) \to 0$ we have

$$\nu \frac{u(\xi_n)u'(\xi_n)}{\sqrt{|u(\xi_n)|^2 + \frac{\nu^2}{c^2}|u'(\xi_n)|^2}} \to 0.$$

Now, thanks to (10) one gets

$$\frac{u(\xi_n)u'(\xi_n)}{\sqrt{|u(\xi_n)|^2 + \frac{\nu^2}{c^2}|u'(\xi_n)|^2}} = \frac{-1}{\sqrt{\frac{1}{|u'(\xi_n)|^2} + \frac{\nu^2}{c^2}\frac{1}{|u(\xi_n)|^2}}},$$

hence

$$\frac{1}{|u'(\xi_n)|^2} + \frac{\nu^2}{c^2}\frac{1}{|u(\xi_n)|^2} \to \infty.$$

As the second term tends to $\frac{\nu^2}{c^2}$, then $\frac{1}{|u'(\xi_n)|^2} \to \infty$, and thus $u'(\xi_n) \to 0$. As consequence, we have shown that $u'(\xi_n) \to 0$ for any $\xi_n \to -\infty$. This proves (11). □

In a classical framework, looking for traveling wave solutions is equivalent to finding heteroclinic trajectories of a planar system of ODEs which arises after transforming the original problem into traveling wave coordinates (see [21,25,33]). By applying the same ideas when searching for traveling waves of Eq. (2), one is led to a system that is non-unique in the sense that it strongly depends upon the change of variables chosen. For example, the usual change $r = u'$ is not adequate to remove the singularity. Hence, a more detailed analysis of the phase diagram of the planar system of ODEs is required. To this aim, define

$$r(\xi) = -\frac{\nu}{c}\frac{u'(\xi)}{\sqrt{|u(\xi)|^2 + \frac{\nu^2}{c^2}|u'(\xi)|^2}}, \tag{12}$$

where u is a positive solution of (8), (9), (10). Then, (u, r) satisfies the first-order differential system

$$\left.\begin{aligned} u' &= -\frac{c}{\nu}\frac{ur}{\sqrt{1-r^2}}, \\ r' &= \frac{c}{\nu}\frac{r(r-\frac{\sigma}{c})}{\sqrt{1-r^2}} + \frac{1}{c}K(u). \end{aligned}\right\} \tag{13}$$

By using that $u' < 0$, (12) yields $r \in (0, 1)$. Also, Lemma 2.1 implies $\lim_{\xi\to-\infty} r(\xi) = 0$. As consequence, the problem consisting of finding a maximal solution to (8)–(10) is equivalent to searching for a solution $(u, r) : (-\infty, \xi_0) \to (0, 1)^2$ to (13), maximal in $(0, 1)^2$, that satisfies

$$\lim_{\xi\to-\infty} u(\xi) = 1, \qquad \lim_{\xi\to-\infty} r(\xi) = 0. \tag{14}$$

We now analyze the equilibrium points of the system (13), given by $(1,0)$ and $(0,r^*)$, where $r^* \in (0,1)$ is a root of

$$\frac{c}{\nu}\frac{r(r-\frac{\sigma}{c})}{\sqrt{1-r^2}} + \frac{1}{c}k = 0, \tag{15}$$

with $k = K(0) = f'(0)$. The existence of equilibrium points $(u,r) = (0,r^*)$ will determine the behavior of the solution to (13)–(14), and consequently of the solution to (8)–(10). More precisely, the following result is obtained.

Proposition 2.1. *There always exists a solution u to Eq.* (8) *that satisfies* (9) *and* (10)*. This solution is unique* (*up to a translation*) *and fulfills the following properties*:

(i) *If there exist no roots* $r^* \in (0,1)$ *of* (15)*, then the existence interval for u can be extended to* $(-\infty,\xi_0)$*, with* $\xi_0 < \infty$*, and*

$$\lim_{\xi\to\xi_0} u(\xi) > 0, \qquad \lim_{\xi\to\xi_0} u'(\xi) = -\infty. \tag{16}$$

(ii) *If there exist roots of* (15)*, then* $\xi_0 = \infty$ *and u satisfies*

$$\lim_{\xi\to\infty} u(\xi) = 0. \tag{17}$$

As a consequence, this solution is maximal in $\mathbb{R}\times(-1,1)$ *and is located in* $(0,1)^2$.

To prove Proposition 2.1 we will need two preliminary results describing some properties of r and u.

Lemma 2.2. *Let* $-\infty < \xi_0 \leqslant \infty$ *and* $(u,r) : (-\infty,\xi_0) \to (0,1)^2$ *be a solution to* (13) *that satisfies* (14)*. Then,* $r'(\xi) > 0$*. The same holds true for any extension of* (u,r)*. In particular, the maximal solution* (u_M, r_M) *associated with* (u,r) *remains in* $(0,1)^2$ *and verifies* $r'_M(\xi) > 0$.

We will give the proof of this result at the end of this section by analyzing in detail the zeros of r' in (13) and describing the phase diagram associated with (13)–(14).

The following result concerns with the strict positivity of u.

Lemma 2.3. *Let* $(u,r) : (\xi_1,\xi_0) \to (0,1)^2$ *be a solution to* (13)*, where* $-\infty \leqslant \xi_1 < \xi_0 \leqslant \infty$ *are such that*

$$\lim_{\xi\to\xi_0} r(\xi) = 1, \quad r'(\xi) > 0.$$

Then

$$\lim_{\xi\to\xi_0} u(\xi) > 0.$$

Proof. We will follow a *reductio ad absurdum* argument by considering a particular solution $(\bar{u},\bar{r})$ to the system (13) such that $\lim_{\xi\to\xi_0} \bar{r}(\xi) = 1$, $\bar{r}'(\xi) > 0$, and $\lim_{\xi\to\xi_0} \bar{u}(\xi) = 0$. This solution allows to define $\tilde{u}(r) := \bar{u}(\bar{r}^{-1}(r))$ in an interval $(1-\varepsilon,1)$ that satisfies

$$z' = \frac{-zr}{r(r-\frac{\sigma}{c}) + \frac{\nu}{c^2}K(\tilde{u}(r))\sqrt{1-r^2}}, \quad z(1) = 0.$$

If $\frac{\sigma}{c} \neq 1$, this equation is locally Lipschitz-continuous in z and the point $(1,0)$ is regular. Then, by using the uniqueness of the initial value problem z must vanish identically, which is a contradiction. If $\frac{\sigma}{c} = 1$, then the differential equation is singular. However, $\tilde{u}$ is a solution of the differential equation

$$z' = -z\frac{h(r)}{\sqrt{1-r}}$$

with

$$h(r) = \frac{r}{-r\sqrt{1-r} + \frac{\nu}{c^2}K(\tilde{u}(r))\sqrt{1+r}}.$$

The term $\frac{h(r)}{\sqrt{1-r}}$ is singular but improperly integrable, thus the associated differential equation has again a unique solution as deduced from standard ODEs theory. □

We are now in a position to prove Proposition 2.1.

2.2. *Proof of Proposition 2.1*

A local analysis of (13) gives the following Jacobian matrix in (u, r)

$$J[u,r] = \begin{pmatrix} -\frac{c}{\nu}\frac{r}{\sqrt{1-r^2}} & -\frac{c}{\nu}\frac{u}{(1-r^2)^{3/2}} \\ \frac{K'(u)}{c} & -\frac{c}{\nu}\frac{\frac{\sigma}{c}-2r+r^3}{(1-r^2)^{3/2}} \end{pmatrix}.$$

Clearly,

$$J[1,0] = \begin{pmatrix} 0 & -\frac{c}{\nu} \\ \frac{K'(1)}{c} & -\frac{\sigma}{\nu} \end{pmatrix}$$

has two eigenvalues $\lambda_- < 0 < \lambda_+$ (because $K'(1) < 0$) which are given by $\lambda_\pm = -\frac{\sigma}{2\nu} \pm \sqrt{(\frac{\sigma}{2\nu})^2 - \frac{K'(1)}{\nu}}$. The local unstable manifold theorem (see [22,24]) then guarantees the existence of a curve γ for which the corresponding solution satisfies (14). As the slope of the eigenvector associated with λ_+ is negative (see Remark 2, where it is explicitly computed), only one branch of $\gamma - \{(1,0)\}$ is locally contained in $(0,1)^2$. Let us take γ maximal (in the sense of forward prolongation) in $(0,1)^2$. Then, there exist solutions of (13) satisfying (14). Uniqueness (up to time translation) comes up from the local uniqueness of the branch γ. Now, Lemmata 2.2 and 2.3 can be applied.

From the fact that u' has opposite sign to r, we can deduce that u satisfies (9) and (10). According to the existence or not of roots to Eq. (15) we will prove the statements (ii) or (i) of Proposition 2.1, respectively. Let us choose $(u,r) : (-\infty, \xi_0) \to (0,1)^2$ to be a particular solution of (13) satisfying (14). Then, Lemma 2.2 implies that the following limit exists:

$$\lim_{\xi\to\xi_0} r(\xi) = r_L.$$

Let us prove that r_L is a lower bound for any (possible) root r^* of (15), i.e. $r_L \leqslant r^*$. As a matter of fact, if $r(\bar{\xi}) = r^*$ for $\bar{\xi} \in (-\infty, \xi_0)$, then (6) leads to

$$r'(\bar{\xi}) = \frac{c}{\nu}\frac{r(\bar{\xi})(r(\bar{\xi}) - \frac{\sigma}{c})}{\sqrt{1-r^2(\bar{\xi})}} + \frac{1}{c}K\big(u(\bar{\xi})\big) < \frac{c}{\nu}\frac{r(\bar{\xi})(r(\bar{\xi}) - \frac{\sigma}{c})}{\sqrt{1-r^2(\bar{\xi})}} + \frac{1}{c}k = 0,$$

which contradicts Lemma 2.2. We now discuss the case in which there exists a root r^* of (15). Then, we have $u < 1$ and $r(\xi) < r^*$ for any $\xi \in (-\infty, \xi_0)$. Thus, $0 < r(\xi) < r_L < 1$ and the pair $(u(\xi), r(\xi))$ lives in a compact set for ξ near ξ_0, away from $r = 0$, $r = 1$, and also maximal in $\mathbb{R} \times (-1, 1)$. Global continuation theorems imply $\xi_0 = \infty$.

To prove (17) we observe that

$$\lim_{\xi\to\infty}\frac{u'(\xi)}{u(\xi)} = -\frac{c}{\nu}\lim_{\xi\to\infty}\frac{r(\xi)}{\sqrt{1-(r(\xi))^2}} = -\frac{c}{\nu}\frac{r_L}{\sqrt{1-r_L^2}} < 0. \tag{18}$$

Hence, we can use a Gronwall-type estimate in an interval $(\xi^*, +\infty)$, with ξ^* large enough so that $u'(\xi) \leqslant -\alpha u(\xi)$ holds, where α is a positive constant and $\xi > \xi^*$.

In case that Eq. (15) has no roots, we first prove that $r_L = 1$. Arguing again by *reductio ad absurdum* (that is, assuming $r_L < 1$), we can use a similar argument as in the previous case by using r_L instead of r^*. In this way, we find that $\xi_0 = +\infty$ (because $\lim_{\xi\to\xi_0} u(\xi) \neq 0$ except if $\xi_0 = \infty$) along with (17). On the other hand, since r has a limit as ξ goes to $+\infty$, it is clear that $r'(\xi_n) \to 0$ up to a subsequence. Using this fact in the second equation of (13) we obtain that r_L is a root of (15), which contradicts our assumption. Hence, $r_L = 1$ holds and the first equation of (13) leads to

$$\lim_{\xi \to \xi_0} \frac{u'(\xi)}{u(\xi)} = -\infty. \tag{19}$$

Now, we use Lemma 2.3 to show the first part of (16). There only remains to prove that $\xi_0 < \infty$. This can be achieved by means of another *reductio ad absurdum* argument. Actually, if $\xi_0 = +\infty$ we get a sequence ξ_n for which $u'(\xi_n) \to 0$, which contradicts (19). □

Remark 2. It is possible to follow the track of the solution to (13) starting from the point $(u, r) = (0, 1)$. Denote $r = \tilde{r}(u)$ the smallest root of

$$\frac{1}{K(u)} \frac{c^2}{\nu} \left(\frac{\sigma}{c} - \tilde{r}(u) \right) = \frac{\sqrt{1 - (\tilde{r}(u))^2}}{\tilde{r}(u)}, \quad u \in (0, 1).$$

The eigenfunction associated with the eigenvalue $\lambda_+ = -\frac{\sigma}{2\nu} + \sqrt{(\frac{\sigma}{2\nu})^2 - \frac{K'(1)}{\nu}}$, defined at the beginning of the proof of Proposition 2.1, determines the local unstable manifold and is defined by $(c\frac{\sigma + \sqrt{-4K'(1)\nu + \sigma^2}}{2K'(1)\nu}, 1)$. On the other hand, it is easy to check that the following identity

$$\lim_{u \to 1} \tilde{r}(u) = \frac{\nu}{c\sigma} K'(1)$$

holds. Then, $(1, \frac{\nu}{c\sigma} K'(1))$ is the tangent vector to the solution curve $r = \tilde{r}(u)$. Comparing the slopes of the above vectors each other leads to the following unrestricted inequality

$$\frac{2K'(1)\nu}{c(\sigma + \sqrt{-4K'(1)\nu + \sigma^2})} > \frac{\nu}{c\sigma} K'(1).$$

Therefore, the curve $r = \tilde{r}(u)$ starting at $u = 1$ verifies that $r'|_{u=1} < 0$.

2.3. *Existence of roots to Eq. (15)*

To conclude the proof we describe the existence of roots to Eq. (15) in function of the parameters σ, c, ν and $k = K(0)$. This problem is equivalent to find the zeros of the equation

$$\frac{c^2}{\nu k} \left(\frac{\sigma}{c} - r \right) = g(r), \quad r \in (0, 1), \tag{20}$$

where g is defined as

$$g(r) = \frac{\sqrt{1 - r^2}}{r},$$

which is a decreasing function with a pole at $r = 0$. The left-hand side is a decreasing linear function that touches the r-axis at $\frac{\sigma}{c}$ with slope $-\frac{1}{k}\frac{c^2}{\nu}$. Then, when

$$\frac{\sigma}{c} > 1 \tag{21}$$

there exists at least one root of Eq. (20) (see the first two cases in Fig. 3). Define $\tilde{r}$ as the smallest root of (20) in $(0, 1)$.

Let us now focus our attention on the case

$$\frac{\sigma}{c} \leqslant 1. \tag{22}$$

Now, the existence of roots to Eq. (20) depends on $\frac{\sigma}{c}$ as well as on the slope $-\frac{c^2}{\nu}\frac{1}{k}$ of the straight line in the left-hand side of (20). Let us prove that, for a range of values of $\frac{\sigma}{c}$ determined by $m = \frac{c^2}{\nu}\frac{1}{k}$, there exists a root of (20). Note that

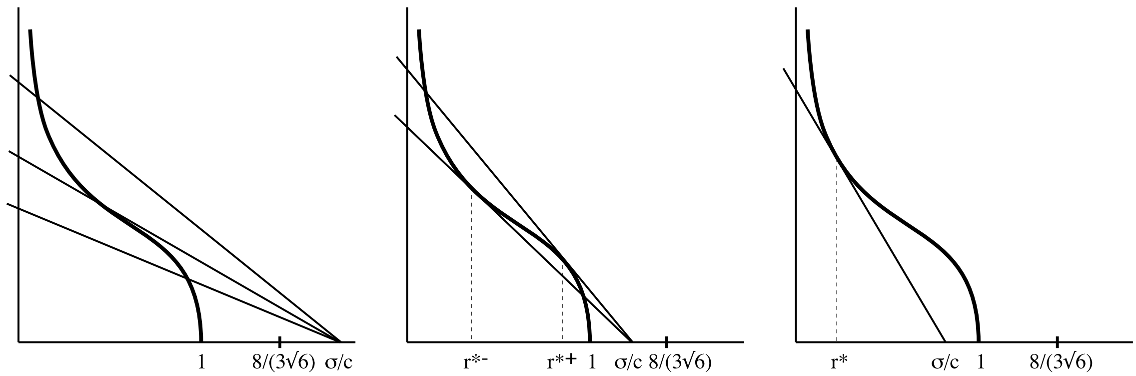

Fig. 3. The curved lines represent the function $g(r)$, while the straight lines do the job for $\frac{c^2}{\nu K(u)}(\frac{\sigma}{c} - r)$, for different values u.

$g'(r)$ has a unique maximum in $(0,1)$ verifying $g'(r) \leqslant -\frac{3\sqrt{3}}{2} = g'(\sqrt{2/3})$ and $\lim_{r\to 0} g'(r) = \lim_{r\to 1} g'(r) = -\infty$. Then, if $-m \leqslant -\frac{3\sqrt{3}}{2}$, we can claim that there exist roots in $(0,1)$ to the equation

$$g'(r) = -m. \tag{23}$$

In fact, when the inequality is strict, i.e. $-m < -\frac{3\sqrt{3}}{2}$, there are two roots in $(0,1)$ while there is only one if the equality is fulfilled (see Fig. 3). Denote now $\tilde{r}$ the smallest real root of (20), $\tilde{r} \in (0, \sqrt{2/3})$, and consider the intersection $\tilde{\delta}$ of the tangent to g at $\tilde{r}$ with the abscissa, which has the expression

$$\tilde{\delta} = \tilde{\delta}(m) = \tilde{r} - \frac{g(\tilde{r})}{g'(\tilde{r})} = 2\tilde{r} - \tilde{r}^3. \tag{24}$$

Clearly, we have that for any $\frac{\sigma}{c} \geqslant \tilde{\delta}(m)$ Eq. (20), with $m = \frac{c^2}{\nu}\frac{1}{k}$, has at least one root in $(0,1)$. To analyze the case $\frac{\sigma}{c} < 1$ we will check the range of values of m for which $\tilde{\delta}(m) \leqslant 1$. By using (24) we deduce that $\tilde{\delta}(m) \leqslant 1$ if and only if $\tilde{r} \leqslant \frac{\sqrt{5}-1}{2}$ or, according to (23),

$$m \geqslant \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{5}{2}}. \tag{25}$$

In conclusion, under condition (25) there exists a root of (20) in $(0,1)$, for every $\frac{\sigma}{c} \geqslant \tilde{\delta}(m)$.

Define $\sigma^*(m)$ as follows

$$\frac{\sigma^*(m)}{c} = \begin{cases} \tilde{\delta}(m), & \text{if } m \geqslant (\frac{1+\sqrt{5}}{2})^{\frac{5}{2}}, \\ 1, & \text{otherwise.} \end{cases} \tag{26}$$

Then, we have proved the following

Proposition 2.2. *There exists a solution to Eq.* (15) *in* $r \in (0,1)$ *if and only if* $\sigma > \sigma^*$ *or* $\sigma = \sigma^* < c$, *where* σ^* *is defined by* (26).

As consequence, combining Propositions 2.2 and 2.1 allows to deduce the existence of a classical solution (as established in Theorem 1.1).

2.4. *Proof of Lemma 2.2*

In order to prove Lemma 2.2, let us provide a description of the positive invariant set associated with the flux defined by the planar system (13). The values (u, r) for which $r' = 0$ are defined by the equation

$$K(u) = -\frac{c^2}{\nu}\frac{r(r - \frac{\sigma}{c})}{\sqrt{1-r^2}}. \tag{27}$$

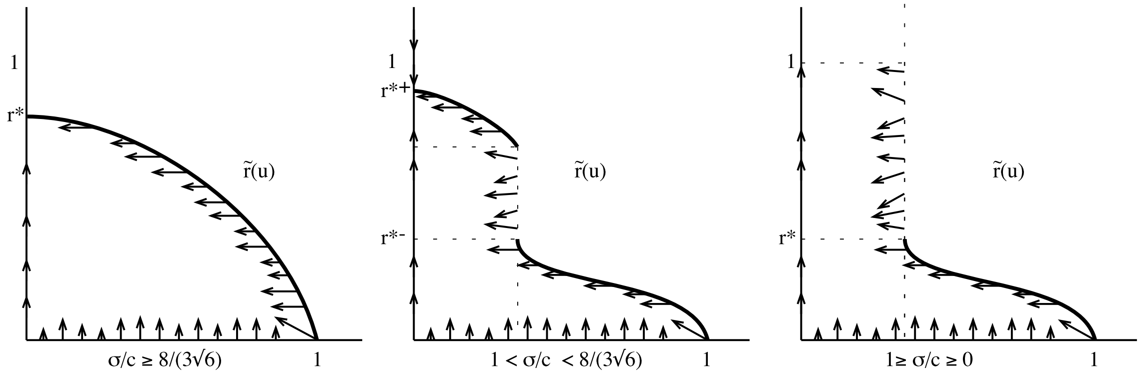

Fig. 4. Description of the positive invariant regions S in terms of the curves $\tilde{r}(u)$.

The roots of this equation can be equivalently obtained as the intersections between $g(r) = \frac{\sqrt{1-r^2}}{r}$ and the straight line $-\frac{c^2}{K(u)\nu}(r - \frac{\sigma}{c})$. The straight line is determined by the point $(\frac{\sigma}{c}, 0)$ and the slope $-\frac{c^2}{K(u)\nu}$, where only the latter depends on u. Using (6), we have that the slope is a decreasing function of u verifying

$$-\infty < -\frac{c^2}{K(u)\nu} \leqslant -\frac{c^2}{K(0)\nu} = -\frac{c^2}{k\nu}, \quad u \in [0, 1).$$

Our purpose now is to describe the function $\tilde{r}(u)$, which is defined by the smallest root of (27) for σ, c and ν fixed. We will prove that he number of these roots as well as their existence depend on the value $\frac{\sigma}{c}$. Simple calculus gives that the tangent to g at $(\frac{\sigma}{c}, 0)$ satisfies

$$r(2 - r^2) = -\frac{g(r)}{g'(r)} + r = \frac{\sigma}{c}.$$

The maximum value of the function $r(2 - r^2)$, reached at $\sqrt{2/3}$, is $8/(3\sqrt{6})$. The value assumed by $\frac{\sigma}{c}$, as compared to 1 and $8/(3\sqrt{6})$, will determine our casuistry. In Fig. 3 the curved lines describe the function $g(r)$, while the straight lines represent the function $\frac{1}{K(u)}\frac{c^2}{\nu}(\frac{\sigma}{c} - r)$.

In the first case (left-hand side in Fig. 3), $\frac{\sigma}{c} \geqslant 8/(3\sqrt{6})$, the straight lines have a unique intersection with the curve $g(r)$, and consequently $\tilde{r}(u)$ is uniquely determined and is a decreasing function. The second case (central picture in Fig. 3) corresponds to $1 < \frac{\sigma}{c} < 8/(3\sqrt{6})$. It is a simple matter to check that again $\tilde{r}(u)$ is uniquely determined and is a decreasing function with the shape given in Fig. 4 in terms of the two critical values r^{*+} and r^{*-}. Finally, the third case $0 \leqslant \frac{\sigma}{c} \leqslant 1$ is represented by the picture in the right-hand side of Fig. 3. The function $\tilde{r}(u)$ has the same monotonicity and well-definition properties that in the previous cases, but now the critical value r^* determines the range of definition.

Let us now prove that the region

$$S = \left((u, r) \in (0, 1)^2, \begin{cases} 0 < r < \tilde{r}(u), & \text{if } \tilde{r}(u) \text{ is defined,} \\ 0 < r < 1, & \text{otherwise} \end{cases} \right) \tag{28}$$

is positively invariant. In order to show the positive invariance of S we will describe the flux at the boundary. First, we observe that the segment $\{(u, r),\ 0 \leqslant r < 1,\ u = 0\}$ at the left-hand side of the square $(0, 1)^2$ is invariant, which prevents the solutions to escape through it. Every point of the segment $\{(u, r),\ 0 < u < 1,\ r = 0\}$ at the bottom of the square $(0, 1)^2$ has a strict incoming flux because the vector field is vertical through this segment. The arrow at the corner $(u, r) = (1, 0)$ corresponds to the discussion carried out in Remark 2. The solid lines in Fig. 4 correspond to the curves $\tilde{r}(u)$ and satisfy that the vertical components of the flux vanish because $r' = 0$ while $u' < 0$. The dashed lines corresponding to the slopes in the curves $\tilde{r}(u)$ are also incoming points since $u' < 0$ there. In Fig. 4 we have plotted the phase diagram (slope field) of the planar system (13), $(u, r) : (-\infty, \xi_0) \to (0, 1)^2$ with boundary conditions (14) and (17). Therefore, we have proved that if there exists $\bar{\xi}$ such that $(u(\bar{\xi}), r(\bar{\xi})) \in S$, then $(u(\xi), r(\xi)) \in S$ for any $\xi \geqslant \bar{\xi}$.

The proof concludes once we prove the existence of a sequence $\bar{\xi}_n \to -\infty$ such that $(u(\bar{\xi}_n), r(\bar{\xi}_n)) \in S$. Using (14), we can deduce the existence of a sequence $\bar{\xi}_n \to -\infty$ for which $r'(\bar{\xi}_n) > 0$. Now, we observe that the graphic of $\tilde{r}(u)$ splits $(0,1) \times (0, r^*)$ into two components characterized by $r' > 0$ or $r' < 0$. Since $(u(\bar{\xi}_n), r(\bar{\xi}_n)) \to (1,0)$, it is clear that $(u(\bar{\xi}_n), r(\bar{\xi}_n)) \in S \cap (0,1) \times (0, r^*)$ for n large enough. □

3. Entropy solutions and consequences

In this section we deal with the study of discontinuous traveling waves. To the authors knowledge, there is no previous literature reporting on the existence of singular traveling waves. In this case it is necessary to use the notion of entropy solution for this equation, which has been introduced in [2].

The main result of this section is the following

Theorem 3.2. *Assume $\sigma = \sigma^* = c$. Then, there exists a discontinuous entropy traveling wave that solves Eq.* (2)*.*

The existence of entropy traveling wave solutions if $\sigma < \sigma^*$ is an open problem.

Define

$$v(t,x) = \begin{cases} u(x - \sigma t), & x - \sigma t < \xi_0, \\ 0, & \text{otherwise}, \end{cases} \tag{29}$$

where $\sigma \leqslant \sigma^*$, $\xi_0 < \infty$, and $u : (-\infty, \xi_0) \to (0,1)$ is a solution to Eq. (8) given by Proposition 2.1. The conditions stated in (16) then imply that v is discontinuous.

It is by no means trivial to prove that some of these functions v are entropy solutions. This follows from the next two results.

Lemma 3.4. *Any solution to Eq.* (8) *satisfying* (9)–(10) *is log-concave in* $(-\infty, \xi_0)$.

Proof. To see that $\log(v(\xi))$ is concave, it is enough to prove that $\frac{v'(\xi)}{v(\xi)}$ is decreasing. Using (13) we have

$$\frac{v'(\xi)}{v(\xi)} = -\frac{c}{\nu} \frac{r(\xi)}{\sqrt{1 - r(\xi)^2}}.$$

The result follows from Lemma 2.2, since the function $r \to \frac{r}{\sqrt{1-r^2}}$, $r \in (0,1)$, is strictly increasing. □

The following proposition allows to characterize the entropy solutions. The proof follows the same lines of Proposition 6.6 in [3], where a similar result was obtained in the case of compactly supported solutions for a version of the equation without the reaction FKPP term. Thus, combining Theorem 3.4 and Proposition 6.6 in [3] along with the null flux at infinity for non-compactly supported solutions and Proposition 3.15 in [2], we have

Proposition 3.3. *Let $v : [0,T) \times \mathbb{R} \to [0,1)$ and $\Omega = \operatorname{supp}(v(0,\cdot))$ be such that the following properties hold for any $t \in [0,T)$:*

(i) $\operatorname{supp}(v(t,\cdot)) = \overline{\Omega_t}$, *where* $\Omega_t = \Omega + B(0, ct)$.
(ii) $v \in C^2(\Omega_t)$ *and solves Eq.* (2).
(iii) $v(t,x)$ *has a vertical contact angle at the boundary of* Ω_t*, for any* $t \in (0,T)$.
(iv) $v(t,x)$ *is log-concave in* Ω_t.

Then, v is an entropy solution.

This result allows to select an entropy solution v from those defined by (29). Properties (ii) and (iv) of Proposition 3.3 are satisfied by any v, but only when $\sigma = \sigma^* = c$ the statement (i) holds, i.e. $\operatorname{supp}(v) = \overline{\Omega}(t)$. Moreover, we conclude the proof of Theorem 3.2 by proving that, in this case, v has a vertical contact angle at the boundary of $\Omega(t)$, and therefore (iii) is also satisfied.

The following result can be deduced directly from Proposition 2.1. We actually give here a more explicit description of the vertical angle near ξ_0.

Lemma 3.5. *Let u be a discontinuous traveling wave for $\sigma = \sigma^* = c$. Then, the vertical angle near ξ_0 is of order $(\xi_0 - \xi)^{-\frac{1}{2}}$.*

Proof. Our starting point is system (13). By using Lemma 2.3 we can assure, when $\sigma \leqslant \sigma^*$, that there exist a constant $\alpha_\sigma > 0$ and ξ_0 such that $u(\xi_0) = \alpha_\sigma$ and $r(\xi_0) = 1$. In the case $\sigma = \sigma^* = c$, (13) leads to

$$r' = \frac{1}{c}K(u) - \frac{c}{\nu} r \frac{\sqrt{1-r}}{\sqrt{1+r}}.$$

Clearly $r'(\xi_0) = \frac{1}{c}K(\alpha_\sigma) < \infty$. A Taylor expansion of $r(\xi)$ around ξ_0 leads to $r(\xi) = 1 + \frac{1}{c}K(\alpha_\sigma)(\xi - \xi_0) + O((\xi - \xi_0)^2)$. Now, combining this expression with the equation for u and integrating between ξ_0 and ξ, $0 < \xi_0 - \xi \ll 1$, we obtain

$$-\log\bigl(u(\xi_0)\bigr) + \log\bigl(u(\xi)\bigr) = \frac{c}{\nu}\frac{2}{(2\frac{1}{c}K(\alpha_\sigma))^{\frac{1}{2}}}(\xi_0 - \xi)^{\frac{1}{2}} - \frac{c}{\nu}\left(\frac{K(\alpha_\sigma)}{2c}\right)(\xi_0 - \xi)^{\frac{3}{2}}.$$

Neglecting higher-order terms we find $u(\xi) = \alpha_\sigma e^{\frac{c}{\nu}\frac{2}{(2\frac{1}{c}K(\alpha_\sigma))^{\frac{1}{2}}}(\xi_0 - \xi)^{\frac{1}{2}}}$, or

$$u(\xi) = \alpha_\sigma + \alpha_\sigma \frac{c}{\nu}\frac{2}{(2\frac{1}{c}K(\alpha_\sigma))^{\frac{1}{2}}}(\xi_0 - \xi)^{\frac{1}{2}}, \quad \text{for } 0 < \xi_0 - \xi \ll 1,$$

after Taylor expansion. □

Remark 3. Since classical solutions are in particular entropy solutions, the existence of traveling waves for $\sigma \geqslant \sigma^*$ is now completed. The existence of an entropy solution for $\sigma < \sigma^*$ is an open question. We can only claim that the corresponding function v, defined by (29), is not an entropy solution. This follows from Theorem 3.9 of [2], which establishes that the support of any log-concave solution moves with speed c while the support of $v(t,\cdot)$ moves with speed $\sigma < c$.

Remark 4. The existence of other traveling waves than wave fronts is also an open question. It can be proved that no more classical (C^2) wave fronts exist. The authors' conjecture is that no more entropy traveling wave solutions will exist, but it is likely to be a much harder problem.

To conclude this section we propose an application of the traveling wave solutions with $\sigma^* < c$ that allows to bound entropy solutions.

Proposition 3.4. *Let $u_0 : \mathbb{R} \to [0, 1)$ be a measurable function with compact support and* $\operatorname{ess\,sup}(u_0) < 1$. *Let also $u(t,x)$ be an entropy solution to Eq. (2) with initial data u_0. Then*

$$\operatorname{ess\,sup}_{x\in\mathbb{R}}\bigl(u(t,x)\bigr) < 1,$$

and for any $c > \sigma > \sigma^*$ there exist positive constants α and β (not depending upon σ) such that

$$\operatorname{ess\,sup}_{|x|>\sigma t} u(t,x) \leqslant \alpha e^{-\beta(\sigma-\sigma^*)t}.$$

In addition, if $\sigma > c$ we have

$$\operatorname{ess\,sup}_{|x|>\sigma t} u(t,x) = 0$$

for large values of t.

Proof. Let $v^*(t,x) = u^*(x - \sigma^* t)$ be a C^2 traveling wave solution to Eq. (2) defined by Theorem 1.1. Then, we can take a translation of u^*, still denoted u^* for simplicity, such that $u^*(\xi) \geqslant u_0(\xi)$. A comparison principle for entropy solutions (see Theorem 3.8 in [2]) leads to

$$u(t,x) \leqslant u^*\bigl(x - \sigma^* t\bigr), \quad \text{a.e. } (t,x) \in \mathbb{R}^2.$$

On the other hand, for a classical traveling wave there exist positive constants α and β such that

$$u(\xi) \leqslant \alpha e^{-\beta\xi}, \quad \xi \in \mathbb{R}.$$

This upper estimate is a consequence of the facts that u^* is uniformly bounded and that $\lim_{\xi\to\infty} \frac{(u^*(\xi))'}{u(\xi)}$ is strictly negative as pointed out in (18). Hence, we find

$$u(t,x) \leqslant u^*\big(x - \sigma^* t\big) \leqslant \alpha e^{-\beta(x-\sigma^* t)}, \quad \text{a.e. } (t,x) \in \mathbb{R}^2. \tag{30}$$

Assuming now that $x > \sigma t$, we deduce from (30) the inequality

$$u(t,x) \leqslant \alpha e^{-\beta(\sigma-\sigma^*)t}, \quad \text{a.e. } (t,x) \in \mathbb{R}^2,\ x > \sigma t. \tag{31}$$

In the case $x < -\sigma t$ we can argue in a similar way by using a classical traveling wave $\tilde{u}^*(\sigma^* t - x)$ such that $u_0(\xi) < \tilde{u}^*(-\xi)$.

The second assertion follows by a comparison argument with the singular traveling wave defined in (29). □

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