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The IVP for the dispersion generalized Benjamin–Ono equation in weighted Sobolev spaces

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Abstract

We study the initial value problem associated to the dispersion generalized Benjamin–Ono equation. Our aim is to establish persistence properties of the solution flow in weighted Sobolev spaces and to deduce from them some sharp unique continuation properties of solutions to this equation. In particular, we shall establish optimal decay rate for the solutions of this model.

Résumé

Nous étudions le problème de Cauchy associé à l'équation de Benjamin–Ono avec dispersion généralisée. Notre objectif est d'établir les propriétés de persistance de la solution dans des espaces de Sobolev avec poids et d'en déduire quelques propriétés de prolongement unique pour ses solutions. En particulier, nous établirons un taux de décroissance optimal pour les solutions de ce modèle.

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1. Introduction

This work is concerned with the initial value problem (IVP) for the dispersion generalized Benjamin–Ono (DGBO) equation

$$\begin{cases} \partial_t u + D^{1+a}\partial_x u + u\partial_x u = 0, \quad t, x \in \mathbb{R},\ 0 < a < 1, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

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where D^s denotes the homogeneous derivative of order $s \in \mathbb{R}$,

$$D^s = (-\Delta)^{s/2} \quad \text{so} \quad D^s f = c_s\big(|\xi|^s \widehat{f}\,\big)^\vee, \quad \text{with } D^s = (\mathcal{H}\partial_x)^s \quad \text{if } n = 1,$$

where $\mathcal{H}$ denotes the Hilbert transform,

$$\mathcal{H}f(x) = \frac{1}{\pi}\text{p.v.}\left(\frac{1}{x} * f\right)(x) = \frac{1}{\pi}\lim_{\epsilon\downarrow 0}\int\limits_{|y|\geqslant\epsilon}\frac{f(x-y)}{y}\,dy = \big(-i\,\text{sgn}(\xi)\widehat{f}(\xi)\big)^\vee(x).$$

These equations model vorticity waves in the coastal zone, see [37] and references therein.

When $a = 1$ the equation in (1.1) becomes the famous Korteweg–de Vries (KdV) equation

$$\partial_t u - \partial_x^3 u + u\partial_x u = 0, \quad t, x \in \mathbb{R}, \tag{1.2}$$

and when $a = 0$ the equation in (1.1) agrees with the well-known Benjamin–Ono (BO) equation

$$\partial_t u + \mathcal{H}\partial_x^2 u + u\partial_x u = 0, \quad t, x \in \mathbb{R}. \tag{1.3}$$

Both the KdV and the BO equations originally arise as models in one-dimensional waves propagation (see [33,5,39]) and have widely been studied in many different contexts. They present several similarities: both possess infinite conserved quantities, define Hamiltonian systems, have multi soliton solutions and are completely integrable. The local well-posedness (LWP) and global well-posedness (GWP) of their associated IVP in the classical Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{R}$, have been extensively investigated.

In the case of the KdV equation this problem has been studied in [41,6,26,29,7,30,11], and finally [18] where global well-posedness was established for $s \geqslant -3/4$.

In the case of the BO equation the same well-posedness problem has been considered in [41,1,24,40,31,27,43,35,8], and [23] where global well-posedness was established for $s \geqslant 0$ (for further discussion we refer to [34]).

However, there are two remarkable differences between the existence theory for these two models. The first is the fact that one can give a local existence theory for the IVP associated to the KdV in $H^s(\mathbb{R})$ based only on the contraction principle. This cannot be done in the case of the BO. This is a consequence of the lack of smoothness of the application data–solution in the BO setting established in [37]. There it was proved that this map is not locally C^2. Actually, in [32] it was proved that this map is not even locally uniformly continuous.

The second remarkable difference between these equations is concerned with the persistent property of the solutions (i.e. if the data $u_0 \in X$, a function space, then the corresponding solution $u(\cdot)$ describes a continuous curve in X, $u \in C([-T,T]:X)$, $T > 0$) in weighted Sobolev spaces. In [26] it was shown that the KdV flow preserves the Schwartz class. However, it was first established by Iorio [24,25] that in general, polynomial type decay is not preserved by the BO flow. The results in [24,25] were recently extended to fractional order weighted Sobolev spaces in [15]. In order to present these results, we introduce the weighted Sobolev spaces

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2\big(|x|^{2r}\,dx\big), \quad s, r \in \mathbb{R}, \tag{1.4}$$

and

$$\dot{Z}_{s,r} = \big\{f \in H^s(\mathbb{R}) \cap L^2\big(|x|^{2r}\,dx\big):\ \widehat{f}(0) = 0\big\}, \quad s, r \in \mathbb{R}. \tag{1.5}$$

The well-posedness results for the IVP associated to the BO equation in weighted Sobolev spaces can be stated as:

Theorem A. *(See [15].)*

(i) *Let $s \geqslant 1$, $r \in [0,s]$, and $r < 5/2$. If $u_0 \in Z_{s,r}$, then the solution u of the IVP associated to the BO equation* (1.3) *satisfies that*

$$u \in C\big([0,\infty) : Z_{s,r}\big).$$

(ii) *For $s > 9/8$ ($s \geqslant 3/2$), $r \in [0,s]$, and $r < 5/2$ the IVP associated to the BO equation* (1.3) *is LWP (GWP resp.) in $Z_{s,r}$.*

(iii) *If $r \in [5/2, 7/2)$ and $r \leqslant s$, then the IVP* (1.3) *is GWP in $\dot{Z}_{s,r}$.*

Theorem B. *(See [15].) Let $u \in C([0,T] : Z_{2,2})$ be a solution of the IVP (1.3). If there exist two different times $t_1, t_2 \in [0,T]$ such that*

$$u(\cdot, t_j) \in Z_{5/2,5/2}, \quad j = 1, 2, \quad \text{then} \quad \widehat{u}_0(0) = 0 \quad \big(\text{so } u(\cdot,t) \in \dot{Z}_{5/2,5/2}\big). \tag{1.6}$$

Theorem C. *(See [15].) Let $u \in C([0,T] : \dot{Z}_{3,3})$ be a solution of the IVP (1.3). If there exist three different times $t_1, t_2, t_3 \in [0,T]$ such that*

$$u(\cdot, t_j) \in \dot{Z}_{7/2,7/2}, \quad j = 1, 2, 3, \quad \text{then} \quad u(x,t) \equiv 0. \tag{1.7}$$

We point out that Iorio's results correspond to the indexes $s \geqslant r = 2$ in Theorem A part (ii), $s \geqslant r = 3$ in Theorem A part (iii) and $s \geqslant r = 4$ in Theorem C.

Regarding the DGBO equation (1.1), we notice that for $a \in (0,1)$ the dispersive effect is stronger than the one for the BO equation but still too weak compared to that of the KdV equation. Indeed it was shown in [37] that for the IVP associated to the DGBO equation (1.1) the flow map data–solution from $H^s(\mathbb{R})$ to $C([0,T] : H^s(\mathbb{R}))$ fails to be locally C^2 at the origin for any $T > 0$ and any $s \in \mathbb{R}$ as in the case of the BO equation. Therefore, so far local well-posedness in classical Sobolev spaces $H^s(\mathbb{R})$ for (1.1) cannot be obtained by an argument based only on the contraction principle. Local well-posedness in classical Sobolev spaces for (1.1) has been studied in [29,20,36,19,21] where local well-posedness was established for $s \geqslant 0$.

Real solutions of the IVP (1.1) satisfy at least three conserved quantities:

$$I_1(u) = \int_{-\infty}^{\infty} u(x,t)\,dx, \qquad I_2(u) = \int_{-\infty}^{\infty} u^2(x,t)\,dx,$$
$$I_3(u) = \int_{-\infty}^{\infty} \left(\big|D^{\frac{1+a}{2}} u\big|^2 + \frac{u^3}{6} \right)(x,t)\,dx. \tag{1.8}$$

In particular, we have that the local results in [21] extend globally in time.

Concerning the form of the traveling wave solution of (1.1) it is convenient to consider

$$v(x,t) = -u(x,-t),$$

where $u(x,t)$ satisfies Eq. (1.1). Thus,

$$\partial_t v - D^{1+a}\partial_x v + v\partial_x v = 0, \quad t, x \in \mathbb{R},\ 0 \leqslant a \leqslant 1. \tag{1.9}$$

Traveling wave solutions of (1.9) are solutions of the form

$$v(x,t) = c^{1+a}\phi_a\big(c\big(x - c^{1+a}t\big)\big), \quad c > 0,$$

where ϕ_a is called the ground state, which is an even, positive, decreasing (for $x > 0$) function. In the case of the KdV equation ($a = 1$ in (1.9)) one has that

$$\phi_1(x) = \frac{3}{2}\operatorname{sech}^2\left(\frac{x}{2}\right),$$

whose uniqueness follows by elliptic theory.

In the case of the BO equation ($a = 0$ in (1.9)) one has that

$$\phi_0(x) = \frac{4}{1+x^2}, \tag{1.10}$$

whose uniqueness (up to symmetry of the equation) was established in [2].

In the case $a \in (0,1)$ in (1.9) the existence of the ground state was established in [44] by variational arguments. Recently, uniqueness of the ground state for $a \in (0,1)$ was established in [17]. However, no explicit formula is know for ϕ_a, $a \in (0,1)$. In [28] the following upper bound for the decay of the ground state was deduced

$$\phi_a(x) \leqslant \frac{c_a}{(1+x^2)^{1+a/2}}, \quad 0 < a < 1.$$

Thus, one has that for $a \in [0, 1)$ the ground state has a very mild decay in comparison with that for the KdV equation $a = 1$. Roughly speaking, this is a consequence of the non-smoothness of the symbol modeling the dispersive relation in (1.1) $\sigma_a(\xi) = |\xi|^{1+a}\xi$.

Our goal in this work is to extend the results in Theorems A–C for the DGBO equation (1.1), by proving persistent properties of solution of (1.1) in the weighted Sobolev spaces (1.4). This will lead us to obtain some optimal uniqueness properties of solutions of this equation as well as to establish what is the maximum rate of decay of a solution of (1.1).

In order to motivate our results we first recall the fact that for dispersive equations the decay of the data is preserved by the solution only if they have enough regularity. More precisely, persistence property of the solution $u = u(x, t)$ of the IVP (1.1) in the weighted Sobolev spaces $Z_{s,r}$ can only hold if $s \geqslant (1 + a)r$. This can be seen from the fact that the linear part of Eq. (1.1)

$$L = \partial_t + D^{1+a}\partial_x \quad \text{commutes with } \Gamma = x - (a + 2)t D^{1+a}. \tag{1.11}$$

Hence, it is natural to consider well-posedness in the weighted Sobolev spaces $Z_{s,r}$, $s \geqslant (1 + a)r$.

Let us state our main results:

Theorem 1.1.

(a) *Let $a \in (0, 1)$. If $u_0 \in Z_{s,r}$, then the solution u of the IVP (1.1) satisfies $u \in C([-T, T] : Z_{s,r})$ if either*
  (i) *$s \geqslant (1 + a)$ and $r \in (0, 1]$, or*
  (ii) *$s \geqslant 2(1 + a)$ and $r \in (1, 2]$, or*
  (iii) *$s \geqslant [(r + 1)^-](1 + a)$ and $2 < r < 5/2 + a$, with $[\cdot]$ denoting the integer part function.*

(b) *If $u_0 \in \dot{Z}_{s,r}$, then the solution u of the IVP (1.1) satisfies*

$$u \in C\big([-T, T] : \dot{Z}_{s,r}\big),$$

whenever
  (iv) *$s \geqslant [(r + 1)^-](1 + a)$ and $5/2 + a \leqslant r < 7/2 + a$.*

Theorem 1.2. *Let $u \in C([-T, T] : Z_{s,(5/2+a)^-})$ with*

$$T > 0 \quad \text{and} \quad s \geqslant (1 + a)(5/2 + a) + (1 - a)/2$$

be a solution of the IVP (1.1). If there exist two times $t_1, t_2 \in [-T, T]$, $t_1 \neq t_2$, such that

$$u(\cdot, t_j) \in Z_{s,5/2+a}, \quad j = 1, 2, \tag{1.12}$$

then

$$\widehat{u}(0, t) = \int u(x, t)\, dx = \int u_0(x)\, dx = \widehat{u}_0(0) = 0 \quad \text{for all } t \in [-T, T]. \tag{1.13}$$

Remarks.

(a) Theorem 1.2 shows that persistence in $Z_{s,r}$ with $r = (5/2 + a)^-$ is the best possible for general initial data. In fact, it shows that for data $u_0 \in Z_{s,r}$, $s \geqslant (1 + a)r + (1 - a)/2$, $r \geqslant 5/2 + a$ with $\widehat{u}_0(0) \neq 0$ the corresponding solution $u = u(x, t)$ verifies that

$$|x|^{(5/2+a)^-} u \in L^\infty\big([0, T] : L^2(\mathbb{R})\big), \quad T > 0,$$

but there does not exist a non-trivial solution u corresponding to data u_0 with $\widehat{u}_0(0) \neq 0$ such that

$$|x|^{5/2+a} u \in L^\infty\big(\big[0, T'\big] : L^2(\mathbb{R})\big), \quad \text{for some } T' > 0.$$

(b) The result in Theorem 1.1 for $s = 1 + a$ was established in [10].

Theorem 1.3. *Let $u \in C([-T, T] : Z_{s,(7/2+a)^-})$ with*

$$T > 0 \quad \text{and} \quad s \geqslant (1 + a)(7/2 + a) + \frac{1 - a}{2}$$

be a solution of the IVP (1.1)*. If there exist three different times* $t_1, t_2, t_3 \in [-T, T]$ *such that*

$$u(\cdot, t_j) \in \dot{Z}_{s,7/2+a}, \quad j = 1, 2, 3, \tag{1.14}$$

then

$$u \equiv 0.$$

Remarks.

(a) Theorem 1.3 shows that the decay $r = (7/2 + a)^-$ is the largest possible. More precisely, Theorem 1.1 part (b) tells us that there are non-trivial solutions $u = u(x, t)$ verifying

$$|x|^{(7/2+a)^-} u \in L^\infty\big([0, T] : L^2(\mathbb{R})\big), \quad T > 0,$$

and Theorem 1.3 guarantees that there does not exist a non-trivial solution such that

$$|x|^{7/2+a} u \in L^\infty\big(\big[0, T'\big] : L^2(\mathbb{R})\big), \quad \text{for some } T' > 0.$$

(b) We shall prove this result in the most general case $s = (1 + a)(7/2 + a) + \frac{1-a}{2}$. Also, we will carry out the details in the case $a \in [1/2, 1)$. It will be clear from our argument how to extend the result to the case $a \in (0, 1/2)$.

Theorem 1.4. *Let* $u \in C([-T, T] : Z_{s,(7/2+a)^-})$ *with*

$$T > 0 \quad \text{and} \quad s \geqslant (1 + a)(7/2 + a) + (1 - a)/2$$

be a solution of the IVP (1.1)*. If there exist* $t_1, t_2 \in [-T, T]$*,* $t_1 \neq t_2$*, such that*

$$u(\cdot, t_j) \in \dot{Z}_{s,7/2+a}, \quad j = 1, 2,$$

and

$$\int x u(x, t_1)\, dx = 0 \quad \text{or} \quad \int x u(x, t_2)\, dx = 0, \tag{1.15}$$

then

$$u \equiv 0.$$

Remark. Theorem 1.4 tells us that the conditions of Theorem 1.3 can be reduced to two times provided the first momentum of the solution u vanishes at one of them.

Theorem 1.5. *Let* $u \in C([-T, T] : Z_{s,(7/2+a)^-})$ *with*

$$T > 0 \quad \text{and} \quad s \geqslant (1 + a)\big(7/2 + [1 + 2a]/2\big) + (1 - a)/2$$

be a non-trivial solution of the IVP (1.1) *such that*

$$u_0 \in \dot{Z}_{s,\frac{7}{2}+\tilde{a}}, \quad \tilde{a} = [1 + 2a]/2, \quad \text{and} \quad \int_{-\infty}^{\infty} x u_0(x)\, dx \neq 0. \tag{1.16}$$

Then there exists $t^* \neq 0$ *with*

$$t^* = -\frac{4}{\|u_0\|_2^2} \int_{-\infty}^{\infty} x u_0(x)\, dx, \tag{1.17}$$

such that $u(t^*) \in \dot{Z}_{s,\frac{7}{2}+\tilde{a}}$.

Remarks.

(a) Notice that $\tilde{a} > a$, so Theorem 1.5 shows that the condition of Theorem 1.3 at two times is in general not sufficient to guarantee that $u \equiv 0$. So, in this regard Theorem 1.4 is optimal.
(b) The results in Theorems 1.3 and 1.5 present a striking difference with other unique continuation properties deduced for other dispersive models. Using the information at two different times, uniqueness results have been established for the generalized KdV equation in [13], for the semi-linear Schrödinger equation in [14], and for the Camassa–Holm model in [22]. Theorem 1.5 affirms that the uniqueness condition with the weight $|x|^{7/2+a}$ does not hold at two different times but Theorem 1.3 guarantees that it does at three times. Similar result for the Benjamin–Ono equation ($a = 0$ in (1.1)) was obtained in [16].

One can consider the IVP (1.1) with $a > 1$. In this case our results still hold, with the appropriate modification in the well-posedness in $H^s(\mathbb{R})$, if a is not an odd integer. In the case where a is an odd integer, one has solutions with exponential decay as in the case of the KdV equation ($a = 1$ in (1.1)).

Finally, we consider the generalization of the IVP (1.1) to higher nonlinearity

$$\begin{cases} \partial_t u + D^{1+a}\partial_x u + u^k \partial_x u = 0, \quad t, x \in \mathbb{R},\ k \in \mathbb{Z}^+, \\ u(x,0) = u_0(x). \end{cases} \tag{1.18}$$

In this case our positive results, Theorems 1.1–1.2, still hold (with the appropriate modification in the well-posedness in $H^s(\mathbb{R})$). Our unique continuation results (Theorems 1.3–1.4) can be extended to the case where k in (1.18) is odd. In this case one has that the time evolution of the first momentum of the solution is given by the formula

$$\int_{-\infty}^{\infty} xu(x,t)\,dx = \int_{-\infty}^{\infty} xu_0(x)\,dx + \frac{1}{k+1}\int_0^t \int_{-\infty}^{\infty} u^{k+1}(x,t)\,dx.$$

Thus, it is an increasing function. Hence, defining $t^* \neq 0$ as the solution of the equation

$$\int_0^{t^*} \int_{-\infty}^{\infty} xu(x,t)\,dx\,dt = 0, \tag{1.19}$$

one sees that there is at most one solution of (1.19) but its existence is not guaranteed. So the statements in Theorems 1.3–1.4 would have to be modified accordingly to this fact.

The rest of this paper is organized as follows: Section 2 contains some preliminary estimates to be used in the coming sections. Section 3 contains the proof of Theorem 1.1. Theorems 1.2, 1.3, 1.4, and 1.5 will be proven in Sections 4, 5, 6, and 7 respectively.

2. Preliminary estimates

We begin this section by introducing the notation needed in this work. We use $\|\cdot\|_{L^p}$ to denote the $L^p(\mathbb{R})$ norm. If necessary, we use subscript to inform which variable we are concerned with. The mixed norm $L^q_t L^r_x$ of $f = f(x,t)$ is defined as

$$\|f\|_{L^q_t L^r_x} = \left(\int \|f(\cdot,t)\|^q_{L^r_x}\,dt \right)^{1/q},$$

with the usual modifications when $q = \infty$ or $r = \infty$. The $L^r_x L^q_t$ norm is similarly defined.

We define the spatial Fourier transform of $f(x)$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx.$$

We shall also define J^s to be the Fourier multiplier with symbol $\langle\xi\rangle^s = (1+|\xi|^2)^{\frac{s}{2}}$. Thus, the norm in the Sobolev space $H^s(\mathbb{R})$ is given by

$$\|f\|_{s,2} \equiv \|J^s f\|_{L^2_x} = \|\langle\xi\rangle^s \widehat{f}\|_{L^2_\xi}.$$

A function $\chi \in C_0^\infty$, supp $\chi \subseteq [-2,2]$ and $\chi \equiv 1$ in $(-1,1)$ will appear several times in our arguments.

For $a \in (0,1)$ fixed we introduce F_j's as being

$$F_j(t,\xi,\widehat{u}_0) = \partial_\xi^j\big(e^{-it|\xi|^{1+a}\xi}\widehat{u}_0(\xi)\big), \tag{2.1}$$

for $j = 0,1,2,3,4$. Thus

$$\begin{aligned}
F_1(t,\xi,\widehat{u}_0) &= -(2+a)it|\xi|^{1+a}e^{-it|\xi|^{1+a}\xi}\widehat{u}_0(\xi) + e^{-it|\xi|^{1+a}\xi}\partial_\xi\widehat{u}_0(\xi),\\
F_2(t,\xi,\widehat{u}_0) &= e^{-it|\xi|^{1+a}\xi}\big(-it(2+a)(1+a)|\xi|^a\operatorname{sgn}(\xi)\widehat{u}_0(\xi)\\
&\quad - (2+a)^2t^2|\xi|^{2(a+1)}\widehat{u}_0(\xi) - 2it(2+a)|\xi|^{1+a}\partial_\xi\widehat{u}_0(\xi) + \partial_\xi^2\widehat{u}_0(\xi)\big)\\
&= (B_1+B_2+B_3+B_4)(t,\xi,\widehat{u}_0),\\
F_3(t,\xi,\widehat{u}_0) &= e^{-it|\xi|^{1+a}\xi}\big(-ita(1+a)(2+a)|\xi|^{a-1}\widehat{u}_0(\xi)\\
&\quad - 3t^2(2+a)^2(1+a)|\xi|^{2a+1}\operatorname{sgn}(\xi)\widehat{u}_0(\xi) + it^3(2+a)^3|\xi|^{3(1+a)}\widehat{u}_0\\
&\quad - 3it(2+a)(1+a)|\xi|^a\operatorname{sgn}(\xi)\partial_\xi\widehat{u}_0(\xi) - 3t^2(2+a)^2|\xi|^{2(1+a)}\partial_\xi\widehat{u}_0(\xi)\\
&\quad - 3it(2+a)|\xi|^{1+a}\partial_\xi^2\widehat{u}_0(\xi) + \partial_\xi^3\widehat{u}_0(\xi)\big)\\
&= (D_1+D_2+D_3+D_4+D_5+D_7)(t,\xi,\widehat{u}_0),\\
F_4(t,\xi,\widehat{u}_0) &= e^{-it|\xi|^{1+a}\xi}\big(-it(2+a)(1+a)a(a-1)|\xi|^{a-2}\operatorname{sgn}(\xi)\widehat{u}_0(\xi)\\
&\quad - t^2(2+a)^2(1+a)(7a+3)|\xi|^{2a}\widehat{u}_0(\xi) + 6it^3(2+a)^3(1+a)|\xi|^{3a+2}\operatorname{sgn}(\xi)\widehat{u}_0(\xi)\\
&\quad + t^4(2+a)^4|\xi|^{4(a+1)}(\xi)\widehat{u}_0(\xi) - 4ita(1+a)(2+a)|\xi|^{a-1}\partial_\xi\widehat{u}_0(\xi)\\
&\quad - 12t^2(2+a)^2(1+a)|\xi|^{2a+1}\operatorname{sgn}(\xi)\partial_\xi\widehat{u}_0(\xi)\\
&\quad + 4it^3(2+a)^3|\xi|^{3(1+a)}\partial_\xi\widehat{u}_0 - 6t^2(2+a)^2|\xi|^{2(1+a)}\partial_\xi^2\widehat{u}_0(\xi)\\
&\quad - 6it(2+a)(1+a)|\xi|^a\operatorname{sgn}(\xi)\partial_\xi^2\widehat{u}_0(\xi) - 4it(2+a)|\xi|^{1+a}\partial_\xi^3\widehat{u}_0(\xi) + \partial_\xi^4\widehat{u}_0(\xi)\big)\\
&= (E_1+\cdots+E_{11})(t,\xi,\widehat{u}_0).
\end{aligned}$$

The next two results will be essential in the analysis below.

The first one is an extension of the Calderón commutator theorem [9]:

Lemma 2.1. *Let $\mathcal{H}$ denote the Hilbert transform. Then for any $p \in (1,\infty)$ and any $l,m \in \mathbb{Z}^+ \cup \{0\}$ there exists $c = c(p;l;m) > 0$ such that*

$$\big\|\partial_x^l[\mathcal{H};\psi]\partial_x^m f\big\|_{L^p} \leqslant c\big\|\partial_x^{m+l}\psi\big\|_{L^\infty}\|f\|_{L^p}. \tag{2.2}$$

The proof follows by results in [4], for a different proof see [12, Lemma 3.1].

Proposition 2.2. *Let $\alpha \in [0,1)$, $\beta \in (0,1)$ with $\alpha+\beta \in [0,1]$. Then for any $p,q \in (1,\infty)$ and for any $\delta > 1/q$ there exists $c = c(\alpha;\beta;p;q;\delta) > 0$ such that*

$$\big\|D^\alpha\big[D^\beta;\psi\big]D^{1-(\alpha+\beta)}f\big\|_{L^p} \leqslant c\big\|J^\delta\partial_x\psi\big\|_{L^q}\|f\|_{L^p}, \tag{2.3}$$

where $J := (1-\partial_x^2)^{1/2}$.

See [12, Proposition 3.2].

Using the notation

$$W_a(t)f = \big(e^{-it|\xi|^{1+a}\xi}\widehat{f}\big)^\vee \tag{2.4}$$

we recall the following linear estimates:

Proposition 2.3 *(Smoothing effects and maximal function).*

(1) *Homogeneous*:

$$\left\| D^{(1+a)/2} W_a(t) f \right\|_{L_x^\infty L_T^2} \leqslant c_a \| f \|_2. \tag{2.5}$$

(2) *Nonhomogeneous and duality*:

$$\left\| D^{s+a/2+1/2} \int_0^t W_a(t-t') F(t')\,dt' \right\|_{L_x^\infty L_T^2} + \left\| D^s \int_0^t W_a(t-t') F(t')\,dt' \right\|_{L_T^\infty L_x^2} \leqslant T^{a/2} \left\| D^{s-1/2+a/2} F \right\|_{L_x^{2/(2-a)} L_T^2}. \tag{2.6}$$

(3) *Maximal function estimate*

$$\left\| W_a(t) f \right\|_{L_x^2 L_T^\infty} \leqslant c(1+T)^\rho \| f \|_{s,2} \tag{2.7}$$

where $\rho > 3/4$ *and* $s > (2+a)/4$.

Proof. For the proof of inequalities (2.5) and (2.7) see [29]. The inequality (2.6) follows by interpolation. □

Proposition 2.4.

(i) *Given* $\phi \in L^\infty(\mathbb{R})$, *with* $\partial_x^\alpha \phi \in L^2(\mathbb{R})$ *for* $\alpha = 1, 2$, *then for any* $\theta \in (0,1)$

$$\left\| [J^\theta; \phi] f \right\|_2 \leqslant c_{\theta,\phi} \| f \|_2. \tag{2.8}$$

(ii) *If* $\eta \in (0,1]$, *then*

$$\left\| J^\eta (fg) - f J^\eta g \right\|_2 \leqslant c \| \widehat{\partial_x f} \|_1 \| g \|_2. \tag{2.9}$$

Proof. We first prove (2.8). Since

$$\left([J^\theta; \phi] f\right)^\wedge(\xi) = \left(J^\theta(\phi f) - \phi J^\theta f\right)^\wedge(\xi) = \int \left(\left(1+\xi^2\right)^{\theta/2} - \left(1+\eta^2\right)^{\theta/2}\right) \widehat{\phi}(\xi-\eta) \widehat{f}(\eta)\,d\eta,$$

the mean value theorem leads to

$$\left|\left([J^\theta; \phi] f\right)^\wedge(\xi)\right| \leqslant c_\theta \int |\xi-\eta| \left|\widehat{\phi}(\xi-\eta)\right| \left|\widehat{f}(\eta)\right| d\eta = c_\theta \left(|\widehat{\partial_x \phi}| * |\widehat{f}|\right)(\xi).$$

Then by Young's inequality

$$\left\| [J^\theta; \phi] f \right\|_2 \leqslant c_\theta \| |\widehat{\partial_x \phi}| * |\widehat{f}| \|_2 \leqslant c_\theta \| \widehat{\partial_x \phi} \|_1 \| \widehat{f} \|_2 \leqslant c_\theta \| \partial_x \phi \|_{1,2} \| f \|_2 \leqslant c_{\theta,\phi} \| f \|_2.$$

To show (2.9) we notice that

$$\begin{aligned}
\left\| J^\eta(fg) - f J^\eta g \right\|_2 &\leqslant \left\| \int \left| \left(1+|\xi|^2\right)^{\eta/2} - \left(1+|\zeta|^2\right)^{\eta/2} \right| \left|\widehat{f}(\xi-\zeta)\right| \left|\widehat{g}(\zeta)\right| d\zeta \right\|_2 \\
&\leqslant \left\| \int |\xi-\zeta| \left|\widehat{f}(\xi-\zeta)\right| \left|\widehat{g}(\zeta)\right| d\zeta \right\|_2 \leqslant \left\| |\widehat{\partial_x f}| * |\widehat{g}| \right\| \\
&\leqslant \| \widehat{\partial_x f} \|_1 \| g \|_2. \qquad \square
\end{aligned}$$

Proposition 2.5. *Given* $\phi \in L^\infty(\mathbb{R})$, *with* $\partial_x^\alpha \phi \in L^2(\mathbb{R})$ *for* $\alpha = 1, 2$, *then for any* $\theta \in (0,1)$

$$\left\| J^\theta(\phi f) \right\|_2 \leqslant c_{\theta,\phi} \left\| J^\theta f \right\|_2. \tag{2.10}$$

Proof. We just need to write

$$J^\theta(\phi f) = [J^\theta, \phi]f + \phi J^\theta f$$

and use the hypotheses and Proposition 2.4 (2.8). □

We recall the following characterization of the $L_s^p(\mathbb{R}^n) = (1-\Delta)^{-s/2}L^p(\mathbb{R}^n)$ spaces given in [42] (see [3] for the case $p = 2$).

Theorem 2.6. *(See [42].) Let $b \in (0,1)$ and $2n/(n+2b) < p < \infty$. Then $f \in L_b^p(\mathbb{R}^n)$ if and only if*

$$\text{(a)} \quad f \in L^p(\mathbb{R}^n), \tag{2.11}$$

$$\text{(b)} \quad \mathcal{D}^b f(x) = \left(\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2b}}\,dy \right)^{1/2} \in L^p(\mathbb{R}^n), \tag{2.12}$$

with

$$\|f\|_{b,p} \equiv \|(1-\Delta)^b f\|_p = \|J^s f\|_p \simeq \|f\|_p + \|D^b f\|_p \simeq \|f\|_p + \|\mathcal{D}^b f\|_p. \tag{2.13}$$

For the proof of this theorem we refer the reader to [42]. One sees that from (2.12) for $p = 2$ and $b \in (0,1)$ one has

$$\|\mathcal{D}^b(fg)\|_2 \leqslant \|f\mathcal{D}^b g\|_2 + \|g\mathcal{D}^b f\|_2 \tag{2.14}$$

and

$$\|\mathcal{D}^b f\|_2 = c\|D^b f\|_2. \tag{2.15}$$

We shall use these estimates throughout in our arguments.

As an application of Theorem 2.6 we also have the following estimate:

Proposition 2.7. *Let $b \in (0,1)$. For any $t > 0$*

$$\mathcal{D}^b\left(e^{-it|x|^{1+a}x}\right) \leqslant c\left(|t|^{b/(2+a)} + |t|^b|x|^{(1+a)b}\right). \tag{2.16}$$

For the proof of Proposition 2.7 we refer to [38].

Also as a consequence of the estimate (2.14) one has the following interpolation inequality.

Lemma 2.8. *Let $\alpha, b > 0$. Assume that $J^\alpha f = (1-\Delta)^{\alpha/2} f \in L^2(\mathbb{R})$ and $\langle x\rangle^b f = (1+|x|^2)^{b/2} f \in L^2(\mathbb{R})$. Then for any $\theta \in (0,1)$*

$$\|J^{\theta\alpha}(\langle x\rangle^{(1-\theta)b} f)\|_2 \leqslant c\|\langle x\rangle^b f\|_2^{1-\theta}\|J^\alpha f\|_2^\theta. \tag{2.17}$$

Moreover, the inequality (2.17) is still valid with $\langle x\rangle_N^\theta$ in (3.2) instead of $\langle x\rangle$ with a constant c independent of N.

We refer to [15] for the proof of Lemma 2.8.

As a further direct consequence of Theorem 2.6 we deduce the following result. It will be useful in several of our arguments.

Proposition 2.9. *For any $\theta \in (0,1)$ and $\alpha > 0$,*

$$\mathcal{D}^\theta\left(|\xi|^\alpha \chi(\xi)\right)(\eta) \sim \begin{cases} c|\eta|^{\alpha-\theta} + c_1, & \alpha \neq \theta,\ |\eta| \ll 1, \\ c(-\ln|\eta|)^{\frac{1}{2}}, & \alpha = \theta,\ |\eta| \ll 1, \\ \frac{c}{|\eta|^{1/2+\theta}}, & |\eta| \gg 1, \end{cases} \tag{2.18}$$

with $\mathcal{D}^\theta(|\xi|^\alpha\chi(\xi))(\cdot)$ continuous in $\eta \in \mathbb{R} - \{0\}$.

In particular, one has that

$$\mathcal{D}^\theta\left(|\xi|^\alpha\chi(\xi)\right) \in L^2(\mathbb{R}) \quad \text{if and only if} \quad \theta < \alpha + 1/2.$$

Similar result holds for $\mathcal{D}^\theta(|\xi|^\alpha \operatorname{sgn}(\xi)\chi(\xi))(\eta)$.

Proof. We restrict ourselves to the case $\alpha \neq \theta$. First we consider the case $\mathcal{D}^\theta(|\xi|^\alpha \chi(\xi))$. It is easy to see that for $\eta \neq 0$, $\mathcal{D}^\theta(|\xi|^\alpha \chi(\xi))(\eta)$ is continuous in $\epsilon < |\eta| < 1/\epsilon$ for any $\epsilon > 0$.

Let us consider $|\eta| < 1/2$. Without loss of generality we assume that $\eta \in (0, 1/2)$. Thus

$$\begin{aligned}\big(\mathcal{D}^\theta\big(|\xi|^\alpha \chi(\xi)\big)(\eta)\big)^2 &= \int \frac{(|\xi+\eta|^\alpha \chi(\xi+\eta) - |\eta|^\alpha \chi(\eta))^2}{|\xi|^{1+2\theta}}\,d\xi \\ &\leqslant c\int_{-\eta/2}^{\eta/2} \frac{(|\xi+\eta|^\alpha - |\eta|^\alpha)^2}{|\xi|^{1+2\theta}}\,d\xi + c\int_{\eta/2}^{2} \frac{(|\xi+\eta|^\alpha + |\eta|^\alpha)^2}{|\xi|^{1+2\theta}}\,d\xi \\ &\equiv A_1 + A_2.\end{aligned}$$

To bound A_1 we use that

$$\big|(\xi+\eta)^\alpha - \eta^\alpha\big| \leqslant |\xi|\frac{c_\alpha}{\eta^{1-\alpha}} \quad \forall \xi\colon -\eta/2 < \xi < \eta/2.$$

Thus

$$A_1 \leqslant c\int_{-\eta/2}^{\eta/2} \frac{|\xi|^2}{\eta^{2(1-\alpha)}|\xi|^{1+2\theta}}\,d\xi \leqslant c\eta^{2(\alpha-\theta)}.$$

For A_2 we have that

$$\big|(\xi+\eta)^\alpha + \eta^\alpha\big| \leqslant c\xi^\alpha \quad \text{for } \xi\colon \eta/2 \leqslant \xi \leqslant 2.$$

So

$$A_2 \leqslant c\int_{\eta/2}^{2} \frac{\xi^{2\alpha}}{\xi^{1+2\theta}}\,d\xi \leqslant c\eta^{2(\alpha-\theta)} + c_1, \quad \text{if } \alpha \neq \theta,$$

where $c_1 = 0$ if $\theta > \alpha$.

Let us consider the case $|\eta| > 100$. Without loss of generality we assume $\eta > 100$. Then

$$\big(\mathcal{D}^\theta\big(|\xi|^\alpha \chi(\xi)\big)(\eta)\big)^2 = \int \frac{(|\xi+\eta|^\alpha \chi(\xi+\eta))^2}{|\xi|^{1+2\theta}}\,d\xi \leqslant c\int_{-2-\eta}^{2-\eta} \frac{d\xi}{|\xi|^{1+2\theta}} \leqslant \frac{c}{\eta^{1+2\theta}}.$$

Finally we consider $\mathcal{D}^\theta(|\xi|^\alpha \operatorname{sgn}(\xi)\chi(\xi))(\eta)$. We notice that the previous computation for $|\eta| > 100$ is similar, so we just need to consider the case $|\eta| < 1$. Assume without loss of generality that $0 < \eta < 1$.

Since

$$\mathcal{D}^\theta\big(|\xi|^\alpha \operatorname{sgn}(\xi)\chi(\xi)\big)(\xi) = \int_{-2-\eta}^{2-\eta} \frac{(|\xi+\eta|^\alpha \operatorname{sgn}(\xi+\eta)\chi(\xi+\eta) - |\eta|^\alpha \operatorname{sgn}(\eta)\chi(\eta))^2}{|\xi|^{1+2\theta}}\,d\xi.$$

The bound for $\xi + \eta > 0$ is similar to that given before, so we assume $\xi + \eta < 0$ ($\xi < -\eta$) and consider

$$\int_{-2-\eta}^{-\eta} \frac{(|\xi+\eta|^\alpha + |\eta|^\alpha)^2}{|\xi|^{1+2\theta}}\,d\xi = \int_{-2\eta}^{-\eta} \cdots + \int_{-2-\eta}^{-2\eta} \cdots = \tilde{A}_1 + \tilde{A}_2.$$

A familiar argument shows that

$$\tilde{A}_1 \leqslant c\frac{\eta^{2\alpha}}{\eta^{1+2\theta}}\eta = \eta^{2(\alpha-\theta)}$$

and

$$\tilde{A}_2 \leqslant \int_{-2-\eta}^{-2\eta} \frac{|\xi|^{2\alpha}}{|\xi|^{1+2\theta}}\,d\xi = \eta^{2(\alpha-\theta)} + c,$$

with $c = 0$ if $\theta > \alpha$. □

3. Proof of Theorem 1.1

Proposition 3.1. *If $u_0 \in H^{1+a}(\mathbb{R})$ and $|x|^\theta u_0 \in L^2(\mathbb{R})$, $\theta \in (0,1)$, then the solution u of the IVP (1.1) satisfies*

$$u \in C\big([0,T] : H^{1+a}(\mathbb{R}) \cap L^2\big(|x|^{2\theta}\big) \equiv Z_{1+a,\theta}\big), \tag{3.1}$$

where T is given by the local theory.

Proof. We use the differential equation and the local theory such that $u \in C([0,T] : H^{1+a}(\mathbb{R}))$ exists and is the limit of smooth solutions.

We define for $\theta \in (0,1)$

$$\langle x\rangle_N^\theta = \begin{cases} \langle x\rangle^\theta = (1+x^2)^{\theta/2}, & \text{if } |x| \leqslant N, \\ (2N)^\theta, & \text{if } |x| \geqslant 3N, \end{cases} \tag{3.2}$$

with $\langle x\rangle_N^\theta$ smooth, even, nondecreasing for $x \geqslant 0$.

We multiply the equation in (1.1) by $\langle x\rangle_N^{2\theta} u$ and integrate in the x-variable to get

$$\frac{1}{2}\frac{d}{dt}\int \big(\langle x\rangle_N^\theta u\big)^2\,dx + \underbrace{\int \langle x\rangle_N^\theta D^{1+a}\partial_x u \langle x\rangle_N^\theta u\,dx}_{A_1} + \underbrace{\int \langle x\rangle_N^{2\theta} u^2 \partial_x u\,dx}_{A_2} = 0.$$

To estimate A_2 we integrate by parts to get

$$A_2 = \frac{1}{3}\int \langle x\rangle_N^{2\theta}\partial_x\big(u^3\big)\,dx = -\frac{1}{3}\int \partial_x\big(\langle x\rangle_N^{2\theta}\big)u^3\,dx. \tag{3.3}$$

We shall use that

$$\partial_x\big(\langle x\rangle_N^{2\theta}\big) \leqslant c_\theta \langle x\rangle_N^{2\theta-1} \leqslant c_\theta \langle x\rangle_N^\theta, \tag{3.4}$$

since $\theta \in (0,1)$. Notice that c_θ is independent of N. Thus

$$A_2 \leqslant \big\|\langle x\rangle_N^\theta u\big\|_2 \|u\|_2 \|u\|_\infty. \tag{3.5}$$

Now we turn to A_1. We write

$$\langle x\rangle_N^\theta D^{1+a}\partial_x u = D^a\big(\langle x\rangle_N^\theta D\partial_x u\big) - \big[D^a; \langle x\rangle_N^\theta\big]D\partial_x u \equiv B_1 + B_2. \tag{3.6}$$

Using Proposition 2.2

$$\big\|\big[D^a; \langle x\rangle_N^\theta\big]D^{1-a} f\big\|_2 \leqslant c\big\|J^\delta \partial_x \langle x\rangle_N^\theta\big\|_q \|f\|_2, \tag{3.7}$$

with $\delta > 1/q$. Thus

$$\|B_2\|_2 = \big\|\big[D^a; \langle x\rangle_N^\theta\big]D\partial_x u\big\|_2 \leqslant c_\theta \big\|D^a \partial_x u\big\|_2, \tag{3.8}$$

with c_θ independent of N since $\partial_x \langle x\rangle_N^\theta$ is bounded independent of N. Here we are assuming that $\theta < 1$ such that $J^\delta \partial_x \langle x\rangle_N^\theta \in L^q$ for appropriate values of δ, q with $\delta > 1/q$. When $\theta = 1$, $\|J^\delta \partial_x \langle x\rangle_N^1\|_q$ is not bounded uniformly on N by a constant and we cannot do this.

Also observe that the bound $\|D^a \partial_x u\|_2$ is natural from the fact that the operator $\Gamma = x - (2+a)tD^{1+a}$ which commutes with $\partial_t + D^{1+a}\partial_x$.

Hence it remains to consider B_1 in (3.6). We write

$$B_1 = D^a\big(\langle x\rangle_N^\theta D\partial_x u\big) = D^a\partial_x\big(\langle x\rangle_N^\theta Du\big) - D^a\big(\big(\partial_x \langle x\rangle_N^\theta\big)Du\big) \equiv C_1 + C_2, \tag{3.9}$$

with

$$\begin{aligned}\|C_2\|_2 &= \big\|D^a\big(\big(\partial_x\langle x\rangle_N^\theta\big)Du\big)\big\|_2\\ &\leqslant \big\|\big[D^a;\partial_x\langle x\rangle_N^\theta\big]Du\big\|_2+\big\|\partial_x\langle x\rangle_N^\theta D^{1+a}u\big\|_2\\ &\underset{(3)}{\leqslant} c\big\|J^\delta\partial_x^2\langle x\rangle_N^\theta\big\|_q\big\|D^{1-a}u\big\|_2+c\big\|D^{1+a}u\big\|_2\\ &\leqslant c\big\|J^{1+a}u\big\|_2,\quad c\text{ independent of }N,\end{aligned}\tag{3.10}$$

where in (3) we have used again (3.7) (Proposition 2.2).

To estimate C_1 in (3.9) we write

$$C_1=D^a\partial_x\big(\langle x\rangle_N^\theta Du\big)=D^a\partial_x D\big(\langle x\rangle_N^\theta u\big)-D^a\partial_x\big[D;\langle x\rangle_N^\theta\big]u\equiv K_1+K_2.\tag{3.11}$$

Since $D=\mathcal{H}\partial_x$ one has

$$\begin{aligned}\big[D;\langle x\rangle_N^\theta\big]f &= D\big(\langle x\rangle_N^\theta f\big)-\langle x\rangle_N^\theta Df\\ &=\mathcal{H}\partial_x\big(\langle x\rangle_N^\theta f\big)-\langle x\rangle_N^\theta\mathcal{H}\partial_x f\\ &=\mathcal{H}\big(\big(\partial_x\langle x\rangle_N^\theta\big)f\big)-\big[\mathcal{H};\langle x\rangle_N^\theta\big]\partial_x f.\end{aligned}\tag{3.12}$$

Therefore

$$K_2=-D^a\partial_x\mathcal{H}\big(\big(\partial_x\langle x\rangle_N^\theta\big)u\big)+D^a\partial_x\big[\mathcal{H};\langle x\rangle_N^\theta\big]\partial_x u\equiv Q_1+Q_2.\tag{3.13}$$

To bound Q_2 we use the commutator estimate in Lemma 2.1

$$\big\|\partial^j[\mathcal{H};a]\partial^m f\big\|_2\leqslant c\big\|\partial^{j+m}a\big\|_\infty\|f\|_2,\tag{3.14}$$

and interpolation ($\|D^a f\|_2\leqslant\|f\|_2^{1-a}\|Df\|_2^a$) to get

$$\begin{aligned}\|Q_2\|_2 &= \big\|D^a\partial_x\big[\mathcal{H};\langle x\rangle_N^\theta\big]\partial_x u\big\|_2\\ &\leqslant \big\|D\partial_x\big[\mathcal{H};\langle x\rangle_N^\theta\big]\partial_x u\big\|_2^a\big\|\partial_x\big[\mathcal{H};\langle x\rangle_N^\theta\big]\partial_x u\big\|_2^{1-a}\\ &\leqslant \big\|\partial_x^2\big[\mathcal{H};\langle x\rangle_N^\theta\big]\partial_x u\big\|_2^a\big\|\partial_x\big[\mathcal{H};\langle x\rangle_N^\theta\big]\partial_x u\big\|_2^{1-a}\\ &\leqslant c\big(\big\|\partial_x^3\langle x\rangle_N^\theta\big\|_\infty+\big\|\partial_x^2\langle x\rangle_N^\theta\big\|_\infty\big)\|u\|_2.\end{aligned}\tag{3.15}$$

Using previous arguments we also have

$$\begin{aligned}\|Q_1\|_2 &= \big\|\mathcal{H}D^a\partial_x\big(\big(\partial_x\langle x\rangle_N^\theta\big)u\big)\big\|_2\\ &\leqslant \big\|D^a\big(\big(\partial_x^2\langle x\rangle_N^\theta\big)u\big)\big\|_2+\big\|D^a\big(\big(\partial_x\langle x\rangle_N^\theta\big)\partial_x u\big)\big\|_2\\ &\leqslant \big(\big\|D^a\partial_x^2\langle x\rangle_N^\theta\big\|_2\|u\|_\infty+\big\|\partial_x^2\langle x\rangle_N^\theta\big\|_\infty\big\|D^a u\big\|_2\big)\\ &\quad+\big\|\big[D^a;\partial_x\langle x\rangle_N^\theta\big]\partial_x u\big\|_2+\big\|\big(\partial_x\langle x\rangle_N^\theta\big)D^a\partial_x u\big\|_2\\ &\leqslant c\big\|J^{1+a}u\big\|_2.\end{aligned}\tag{3.16}$$

Finally, we turn to the term K_1 in (3.11), Parseval's identity yields

$$\int D^a\partial_x D\big(\langle x\rangle_N^\theta u\big)\langle x\rangle_N^\theta u=\int\partial_x D^{(1+a)/2}\big(\langle x\rangle_N^\theta u\big)D^{(1+a)/2}\big(\langle x\rangle_N^\theta u\big)\equiv 0.$$

Since from the local existence theory [19] we know that

$$\sup_{[0,T]}\big\|u(t)\big\|_{1+a,2}=\sup_{[0,T]}\big\|u(t)\big\|_{H^{1+a}}\leqslant c,\tag{3.17}$$

combining the above estimate and taking limit as $N\to\infty$ we obtain

$$\sup_{[0,T]}\big\|\langle x\rangle^\theta u(t)\big\|_2\leqslant\tilde c_\theta\quad\text{for all }\theta\in[0,1),\tag{3.18}$$

which yields the result.

We observe that the argument above shows that if in addition to $u_0\in H^{1+a}(\mathbb{R})\cap L^2(|x|^{2\theta})$, $\theta\in(0,1)$, $u_0\in H^{1+a+\alpha}(\mathbb{R})$ with $D^\alpha u_0\in L^2(|x|^{2\theta})$, for $\alpha>0$, then $D^\alpha u\in C([0,T]:H^{1+a}(\mathbb{R})\cap L^2(|x|^{2\theta}))$, $\alpha>0$. □

3.1. Case $s = 1 + a$, $r = 1$

Let $u_0 \in H^{1+a}(\mathbb{R}) \cap L^2(|x|^2\,dx)$. We observe that the persistence result in this case was already proved by Colliander, Kenig and Stafillani [10]. However, by convenience we present a different proof.

First notice that

$$\begin{aligned} xD^{1+a}\partial_x f &= \big(xD^{1+a}\partial_x f\big)^{\wedge\vee} = \big(i\partial_\xi\big(|\xi|^{1+a} i\xi \widehat{f}\,\big)\big)^\vee \\ &= \big(-(2+a)|\xi|^{1+a}\widehat{f} + |\xi|^{1+a} i\xi i\partial_\xi \widehat{f}\,\big)^\vee \\ &= -(2+a)D^{1+a}f + D^{1+a}\partial_x(xf). \end{aligned} \tag{3.19}$$

Hence if u satisfies

$$\partial_t u + D^{1+a}\partial_x u + u\partial_x u = 0, \tag{3.20}$$

then

$$\partial_t(xu) + D^{1+a}\partial_x(xu) - (2+a)D^{1+a}u + xu\partial_x u = 0. \tag{3.21}$$

In this case the standard energy estimate argument shows that

$$\frac{d}{dt}\|xu(t)\|_2^2 \leqslant c_a \big\|D^{1+a}u(t)\big\|_2 \big\|xu(t)\big\|_2 + \|u\|_\infty \big\|u(t)\big\|_2 \big\|xu(t)\big\|_2. \tag{3.22}$$

Since

$$\sup_{[0,T]} \big\|u(t)\big\|_{1+a,2} \leqslant c\big(a; T; \|u_0\|_{1+a}\big), \tag{3.23}$$

one has from (3.22) that

$$\sup_{[0,T]} \big\|xu(t)\big\|_{1+a,2} \leqslant c\big(a; T; \|u_0\|_{1+a,2}; \|xu_0\|_2\big). \tag{3.24}$$

Remark 3.2. By taking derivatives D^α in Eq. (3.20) and repeating the above argument we have that if in addition to $u_0 \in H^{1+a}(\mathbb{R}) \cap L^2(|x|^2\,dx)$ one has

$$D^\alpha u_0 \in H^{1+a}(\mathbb{R}), \qquad xD^\alpha u_0 \in L^2(\mathbb{R}), \qquad \text{for some } \alpha > 0, \tag{3.25}$$

then

$$\sup_{[0,T]} \big\|xD^\alpha u(t)\big\|_2 \leqslant c\big(T; a; \|u_0\|_{1+a+\alpha,2}; \|xu_0\|_2; \big\|xD^\alpha u_0\big\|_2\big). \tag{3.26}$$

3.2. Case $s = 2(1 + a)$, $r \in (1, 2)$

Let $u_0 \in H^{2(1+a)} \cap L^2(|x|^{2r}\,dx)$, $r = 1 + \theta$, $\theta \in (0, 1)$.

Reapplying the method for the weight $\langle x\rangle_N^{2\theta}$, $\theta \in (0, 1)$, multiplying Eq. (3.21) by $\langle x\rangle_N^{2\theta} xu$ and integrating the result, one gets

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \big(\langle x\rangle_N^\theta xu\big)^2\,dx + \int \langle x\rangle_N^\theta D^{1+a}\partial_x(xu)\langle x\rangle_N^\theta xu \\ &\quad - (2+a)\int \langle x\rangle_N^\theta D^{1+a}u\langle x\rangle_N^\theta xu + \int \langle x\rangle_N^\theta xu\partial_x\langle x\rangle_N^\theta xu \\ &= 0. \end{aligned} \tag{3.27}$$

From the previous analysis and Proposition 3.1, we only need to handle the last two terms in (3.27).

First we notice that

$$\int \langle x\rangle_N^\theta D^{1+a}u\langle x\rangle_N^\theta xu\,dx \leqslant \big\|\langle x\rangle_N^\theta D^{1+a}u\big\|_2 \big\|\langle x\rangle_N^\theta xu\big\|_2. \tag{3.28}$$

Next

$$\langle x\rangle_N^\theta D^{1+a} f = \underbrace{-\big[D^a;\langle x\rangle_N^\theta\big]D^{1-a}D^a f}_{C_1} + \underbrace{D^a\langle x\rangle_N^\theta Df}_{C_2},$$

and by the commutator estimate

$$\|C_1\|_2 \leqslant \big\|D^a f\big\|_2 \quad \text{uniformly in } N \text{ for } \theta\in(0,1).$$

On the other hand,

$$C_2 = D^a\langle x\rangle_N^\theta\partial_x\mathcal{H}f = D^a\partial_x\big(\langle x\rangle_N^\theta\mathcal{H}f\big) - D^a\big(\partial_x\langle x\rangle_N^\theta\mathcal{H}f\big) \equiv K_1 + K_2,$$

where

$$\begin{aligned}\|K_2\|_2 &= \big\|D^a\big(\partial_x\langle x\rangle_N^\theta\mathcal{H}f\big)\big\|_2 \leqslant \big\|J^a\big(\partial_x\langle x\rangle_N^\theta\mathcal{H}f\big)\big\|_2\\ &\leqslant \big\|\big[J^a;\partial_x\langle x\rangle_N^\theta\big]\mathcal{H}f\big\|_2 + \big\|\partial_x\langle x\rangle_N^\theta J^a\mathcal{H}f\big\|_2\\ &\leqslant \|f\|_2 + \big\|J^a f\big\|_2 \leqslant c\big\|J^a f\big\|_2.\end{aligned}$$

So we consider K_1,

$$K_1 = D^a\partial_x\big(\langle x\rangle_N^\theta\mathcal{H}f\big) = D^a\mathcal{H}\partial_x\big(\langle x\rangle_N^\theta f\big) + D^a\partial_x\big[\langle x\rangle_N^\theta;\mathcal{H}\big]f = K_{1,1} + K_{1,2}.$$

For $K_{1,2}$ we write

$$\begin{aligned}\|K_{1,2}\|_2 &= \big\|D^a\partial_x\big[\langle x\rangle_N^\theta;\mathcal{H}\big]f\big\|_2\\ &\leqslant \big\|D\partial_x\big[\langle x\rangle_N^\theta;\mathcal{H}\big]f\big\|_2^a\big\|\partial_x\big[\langle x\rangle_N^\theta;\mathcal{H}\big]f\big\|_2^{1-a}\\ &\leqslant \big\|\partial_x^2\langle x\rangle_N^\theta\big\|_\infty^a\|f\|_2^a\big\|\partial_x\langle x\rangle_N^\theta\big\|_\infty^{1-a}\|f\|_2^{1-a} \leqslant c_\theta\|f\|_2.\end{aligned}$$

Finally,

$$\begin{aligned}\|K_{1,1}\|_2 &= \big\|D^{1+a}\big(\langle x\rangle_N^\theta f\big)\big\|_2 \leqslant \big\|J^{1+a}\big(\langle x\rangle_N^\theta f\big)\big\|_2\\ &\leqslant \big\|J^{(1+a)(1+\theta)}f\big\|_2^{1/1+\theta}\big\|\langle x\rangle_N^{1+\theta}f\big\|_2^{\theta/1+\theta}\\ &\leqslant \big\|J^{2(1+a)}f\big\| + \big\|\langle x\rangle_N^\theta xf\big\|_2 + \|f\|_2 + \|xf\|_2,\end{aligned}$$

which completes the estimate for (3.28) if $s \geqslant 2(1+a)$.

For the nonlinear term coming from (3.21) we have that

$$\int\langle x\rangle_N^\theta xu\partial_x u\langle x\rangle_N^\theta xu \leqslant \big\|\langle x\rangle^\theta xu\big\|_2^2\|\partial_x u\|_\infty \leqslant \big\|\langle x\rangle^\theta xu\big\|_2^2\|u\|_{2(1+a),2}.$$

This proves that if $u_0\in H^{2(1+a)}(\mathbb{R})$ and $|x|^{1+\theta}u\in L^2(\mathbb{R})$, $\theta\in(0,1)$, then persistence holds in

$$H^{2(1+a)}\cap L^2\big(|x|^{2(1+\theta)}\,dx\big).$$

Remark 3.3. The above argument also shows that if in addition to $u_0\in H^{2(1+a)}(\mathbb{R})\cap L^2(|x|^{2(1+\theta)}\,dx)$ one has $D^\alpha u_0\in H^{2(1+a)}(\mathbb{R})\cap L^2(|x|^{2(1+\theta)}\,dx)$, $\alpha>0$, then $D^\alpha u\in C([0,T]:H^{2(1+a)}(\mathbb{R})\cap L^2(|x|^{2(1+\theta)}\,dx))$.

3.3. *Case $s=2(1+a)$, $r=2$*

We observe that an argument similar to that in [10] also gives the persistence of the solution to the IVP (1.1) in

$$u_0\in H^{2(1+a)}(\mathbb{R})\cap L^2\big(|x|^4\,dx\big). \tag{3.29}$$

In fact, using for

$$\begin{aligned}x^2D^{1+a}\partial_x f &= \big(x^2D^{1+a}\partial_x f\big)^{\wedge\vee} = \big(-\partial_\xi^2\big(|\xi|^{1+a}i\xi\widehat{f}\,\big)\big)^\vee\\ &= \big(-(2+a)(1+a)|\xi|^a i\,\mathrm{sgn}(\xi)\widehat{f} - 2(2+a)|\xi|^{1+a}\widehat{xf} + |\xi|^{1+a}i\xi\widehat{\big(x^2f\big)}\big)^\vee\\ &= -(2+a)(1+a)D^a\mathcal{H}f - 2(2+a)D^{1+a}(xf) + D^{1+a}\partial_x\big(x^2f\big),\end{aligned}$$

we get the equation for x^2u

$$\partial_t\big(x^2u\big)+D^{1+a}\partial_x\big(x^2u\big)-2(2+a)D^{1+a}(xu)-(1+a)(2+a)D^a\mathcal{H}u-x^2u\partial_xu=0,$$

for which a familiar argument also shows that

$$\sup_{[0,T]}\big\|x^2u(t)\big\|_2\leqslant c\big(T;a;\big\|x^2u_0\big\|_2;\|u_0\|_{2(1+a),2}\big). \tag{3.30}$$

As before we notice that if in addition to (3.29) one has that

$$D^\alpha u_0\in H^{2(1+a)}(\mathbb{R})\cap L^2\big(|x|^4\,dx\big)\equiv Z_{2(1+a),2},\quad \alpha>0, \tag{3.31}$$

then

$$\sup_{[0,T]}\big\|x^2D^\alpha u(t)\big\|_2\leqslant c\big(T;a;\big\|x^2u_0\big\|_2;\big\|x^2D^\alpha u_0\big\|_2;\|u_0\|_{2(1+a)+\alpha,2}\big). \tag{3.32}$$

3.4. *Case $s=s_r\equiv[(r+1)^-](1+a)$, $r\in(2,5/2+a)$*

We observe that the equation for xu

$$\partial_t(xu)+D^{1+a}\partial_x(xu)-(2+a)D^{1+a}u+(xu)\partial_xu=0, \tag{3.33}$$

and the previous argument for $|x|^l$, with $l\in(0,2)$ will provide the result if the contribution for the extra term in (3.33) $c_aD^{1+a}u$ can be handled, i.e. if for $l=1+\theta$

$$\sup_{[0,T]}\big\||x|^{1+\theta}D^{1+a}u(t)\big\|_2\leqslant M. \tag{3.34}$$

We claim that if $\theta\in(0,a+1/2)$, $u_0\in H^{s_r}(\mathbb{R})\cap L^2(|x|^{2+\theta})$, $s=r[(r+1)^-](1+a)$, we obtain (3.34) and hence the desired result. From Remark 3.2 it will suffice to have

$$\big\||x|^{1+\theta}D^{1+a}u_0\big\|_2\leqslant c\big(\big\||x|^{2+\theta}u_0\big\|_2;\big\|D^{(1+a)(1+\theta)}u_0\big\|_2\big), \tag{3.35}$$

with c independent of N.

Proof of (3.35). Using the identity

$$xD^{1+a}f=D^{1+a}(xf)+(1+a)D^a\mathcal{H}f, \tag{3.36}$$

we have to control the L^2 norms of the terms

$$K_1=|x|^\theta D^a\mathcal{H}u_0\quad\text{and}\quad K_2=|x|^\theta D^{1+a}(xu_0). \tag{3.37}$$

We can estimate K_1 as

$$\begin{aligned}\|K_1\|_2&\leqslant\big\|D_\xi^\theta\big(|\xi|^a\operatorname{sgn}(\xi)\chi(\xi)\widehat{u}_0(\xi)\big)\big\|_2\\&\quad+\big\|D_\xi^\theta\big(|\xi|^a\operatorname{sgn}(\xi)\big(1-\chi(\xi)\big)\widehat{u}_0(\xi)\big)\big\|_2=K_{1,1}+K_{1,2},\end{aligned} \tag{3.38}$$

where

$$\begin{aligned}K_{1,1}&\leqslant\big\|D_\xi^\theta\big(|\xi|^a\operatorname{sgn}(\xi)\chi(\xi)\widehat{u}_0(0)\big)\big\|_2+\Big\|D_\xi^\theta\Big(\underbrace{|\xi|^a\operatorname{sgn}(\xi)\chi(\xi)\big(\widehat{u}_0(\xi)-\widehat{u}_0(0)\big)}_{L(\xi)}\Big)\Big\|_2\\&=\tilde{K}_{1,1}+\big\|D_\xi^\theta L\big\|_2.\end{aligned}$$

Next we have that

$$\big\|D_\xi^\theta L\big|_2\leqslant\big\|L(\xi)\big\|_2^{1-\theta}\big\|\partial_\xi L(\xi)\big\|_2^\theta.$$

Then

$$\begin{aligned}\|\partial_\xi L\| &\leqslant c\big(\|\partial_\xi \widehat{u_0}\|_\infty + \big\|D_\xi^\theta \widehat{u_0}\big\|_2\big)\\ &\leqslant c\big(\|\widehat{xu_0}\|_\infty + \big\|\widehat{|x|^\theta u_0}\big\|_2\big)\\ &\leqslant c\big(\|xu_0\|_1 + \big\||x|^\theta u_0\big\|_2\big) \leqslant c\big\|\langle x\rangle^{\frac{3}{2}+} u_0\big\|_2,\end{aligned}$$

and

$$\tilde{K}_{1,1} < \infty$$

(by Proposition 2.9). □

Consider now $\|K_2\|_2$, we introduce the cutoff function χ to obtain

$$\begin{aligned}\|K_2\|_2 &= \big\||x|^\theta D^{1+a}(xu_0)\big\|_2\\ &= \big\|D_\xi^\theta\big(|\xi|^{1+a}(\widehat{xu_0})\big)\big\|_2\\ &\leqslant \big\|D_\xi^\theta\big(|\xi|^{1+a}\chi(\xi)(\widehat{xu_0})\big)\big\|_2 + \big\|D_\xi^\theta\big(|\xi|^{1+a}\big(1-\chi(\xi)\big)(\widehat{xu_0})\big)\big\|_2\\ &\leqslant K_{2,1} + K_{2,2}.\end{aligned}$$

Using Stein's derivative, Leibniz rule (2.14) and Proposition 2.9, we estimate $K_{2,1}$ as

$$\begin{aligned}K_{2,1} &= \big\|D_\xi^\theta\big(|\xi|^{1+a}\chi(\xi)(\widehat{xu_0})\big)\big\|_2\\ &\leqslant c\big\|\mathcal{D}_\xi^\theta\big(|\xi|^{1+a}\chi(\xi)\big)\big\|_\infty \|\widehat{xu_0}\|_2 + \big\||\xi|^{1+a}\chi(\xi)\big\|_\infty \big\|\mathcal{D}_\xi^\theta(\widehat{xu_0})\big\|_2\\ &\leqslant c_a\|xu_0\|_2 + c_a\big\|D_\xi^\theta(\widehat{xu_0})\big\|_2\\ &\leqslant c_a\big\|\langle x\rangle^{1+\theta}u_0\big\|_2.\end{aligned}$$

On the other hand, we notice that $\varphi_1(\xi) = \frac{|\xi|^{1+a}(1-\chi(\xi))}{\langle\xi\rangle^{1+a}}$ and $\varphi_2(\xi) = \frac{\xi}{\langle\xi\rangle}$ satisfy the hypothesis in Proposition 2.5, and hence it follows that

$$\begin{aligned}K_{2,2} &= \big\|D_\xi^\theta\big(|\xi|^{1+a}\big(1-\chi(\xi)\big)(\widehat{xu_0})\big)\big\|_2\\ &\leqslant \left\|J_\xi^\theta\left(\frac{|\xi|^{1+a}(1-\chi(\xi))}{\langle\xi\rangle^{1+a}}\langle\xi\rangle^{1+a}\widehat{xu_0}\right)\right\|_2\\ &\leqslant c\big\|J_\xi^\theta\big(\langle\xi\rangle^{1+a}\partial_\xi\widehat{u_0}\big)\big\|_2\\ &\leqslant c\big\|J_\xi^\theta\partial_\xi\big(\langle\xi\rangle^{1+a}\widehat{u_0}\big)\big\|_2 + \big\|J_\xi^\theta\big(\partial_\xi\big(\langle\xi\rangle^{1+a}\big)\widehat{u_0}\big)\big\|_2\\ &\leqslant c\big\|J_\xi^{1+\theta}\big(\langle\xi\rangle^{1+a}\widehat{u_0}\big)\big\|_2 + \left\|J_\xi^\theta\left(\frac{\xi}{\langle\xi\rangle}\langle\xi\rangle^a\widehat{u_0}\right)\right\|_2\\ &\leqslant c\big\|\langle x\rangle^{1+\theta}J^{1+a}u_0\big\|_2 + \big\|\langle x\rangle^\theta J^a u_0\big\|_2\\ &\leqslant c_a\big\|\langle x\rangle^{2+\theta}u_0\big\|_2^{1-1/(2+\theta)}\big\|J^{(2+\theta)(1+a)}u_0\big\|_2^{1/(2+\theta)} + c_a\big\|\langle x\rangle^{2+\theta}u_0\big\|_2^{\theta/(2+\theta)}\big\|J^{(2+\theta)\frac{a}{2}}u_0\big\|_2^{2/(2+\theta)},\end{aligned}$$

where in the last inequality we have applied complex interpolation ($u_0 \in H^{s_r}(\mathbb{R}) \cap L^2(|x|^{2r}\,dx)$ with $r = 2+\theta \in (2, 5/2+a)$).

3.5. Persistence property in $\dot{Z}_{s_r,r}$ with $s_r = [(r+1)^-](1+a)$ and $r \in [5/2+a, 7/2+a)$

To simplify the exposition we assume $r \in [3, 7/2+a)$.
We have established persistence in

$$Z_{s_r,r} \quad \text{with } r \in [2, 5/2+a), \tag{3.39}$$

for the equation (we are assuming $a \in (0, 1/2)$ for simplicity)

$$\partial_t u + D^{1+a}\partial_x u + u\partial_x u = 0, \tag{3.40}$$

and that xu satisfies the equation

$$\partial_t(xu) + D^{1+a}\partial_x(xu) - (2+a)D^{1+a}u + xu\partial_x u = 0. \tag{3.41}$$

Thus if we prove that $u_0 \in \dot{Z}_{s_r,r}$ with $r \in [5/2+a, 7/2+a)$, then the solution u satisfies

$$|x|^\alpha D^{1+a}u \in L^2(\mathbb{R}) \quad \text{for } \alpha = r-1 \in [2, 5/2+a), \tag{3.42}$$

the argument for proving the result in $Z_{s_r,r}$ as in (3.39) will provide the result.

Since $\widehat{u}(0,t) = \widehat{u}_0(0) = \int u_0(x)\,dx$ is preserved by the solution flow, it will suffice to show that if $u_0 \in \dot{Z}_{r(1+a),r}$, $r \in [5/2+a, 7/2+a)$, then $|x|^\alpha D^{1+a}u_0 \in L^2(\mathbb{R})$ for $\alpha = r-1$.

Since $\alpha \in [2, 5/2+a)$ with $a \in (0, 1/2)$ write $\alpha = 2+\theta$, $\theta \in (0,1)$, and use that

$$x^2 D^{1+a} f = -(1+a)aD^{a-1}f - 2(1+a)D^a\mathcal{H}(xf) + D^{1+a}\big(x^2 f\big). \tag{3.43}$$

Thus

$$\begin{aligned}|x|^{2+\theta}D^{1+a}u_0 &= -(1+a)a|x|^\theta D^{a-1}u_0 - 2(1+a)D^a\mathcal{H}(xu_0) + |x|^\theta D^{1+a}\big(x^2u_0\big)\\ &\equiv G_1 + G_2 + G_3.\end{aligned} \tag{3.44}$$

First we write G_1 using that

$$\begin{aligned}\big\||x|^\theta D^{a-1}u_0\big\|_2 &= \big\|D_\xi^\theta\big(|\xi|^{a-1}\widehat{u}_0\big)\big\|_2\\ &\leqslant \big\|D_\xi^\theta\big(|\xi|^{a-1}\chi(\xi)\widehat{u}_0\big)\big\|_2 + \big\|D_\xi^\theta\big(|\xi|^{a-1}\big(1-\chi(\xi)\big)\widehat{u}_0\big)\big\|_2.\end{aligned}$$

Now using that $\widehat{u}_0(0) = 0$, the Taylor expansion allows to write

$$\widehat{u}_0(\xi) = \xi\partial_\xi\widehat{u}_0(0) + \int_0^\xi (\xi-\zeta)\partial_\xi^2\widehat{u}(\zeta)\,d\zeta. \tag{3.45}$$

So

$$\begin{aligned}|\xi|^{a-1}\chi(\xi)\widehat{u}_0(\xi) &= |\xi|^a\operatorname{sgn}(\xi)\chi(\xi)\partial_\xi\widehat{u}_0(\xi) + \chi(\xi)|\xi|^{a-1}\int_0^\xi(\xi-\zeta)\partial_\xi^2\widehat{u}(\zeta)\,d\zeta\\ &\equiv |\xi|^a\operatorname{sgn}(\xi)\chi(\xi)\partial_\xi\widehat{u}_0(\xi) + Q_1\end{aligned}$$

and by Proposition 2.9

$$\begin{aligned}D_\xi^\theta\big(|\xi|^a\operatorname{sgn}(\xi)\chi(\xi)\partial_\xi\widehat{u}_0(\xi)\big) \in L^2(\mathbb{R}) \qquad &\text{if and only if} \quad \theta < a+1/2,\\ &\text{if and only if} \quad \alpha = 2+\theta < 5/2+a,\end{aligned}$$

which holds from our hypotheses, and

$$\begin{aligned}&\big\|D_\xi^\theta Q_1\big\|_2 \leqslant \|Q_1\|_2^{1-\theta}\|\partial_\xi Q_1\|_2^\theta,\\ &\|Q_1\|_2 \leqslant \big\|\chi(\xi)|\xi|^{a+1}\big\|\partial_\xi^2\widehat{u}_0\big\|_{L^\infty_{\{|\xi|<1\}}}\big\|_2 \leqslant \big\|\widehat{x^2u_0}\big\|_\infty \leqslant \big\|x^2u_0\big\|_1\\ &\qquad \leqslant c\big\|\langle x\rangle^{\frac{5}{2}+}u_0\big\|_2,\end{aligned}$$

and

$$\begin{aligned}\|\partial_\xi Q_1\|_2 \leqslant{}& \left\|\chi'(\xi)|\xi|^{a-1}\int_0^\xi(\xi-\zeta)\partial_\xi^2\widehat{u}_0(\zeta)\,d\zeta\right\|_2 + \left\|\chi(\xi)(a-1)|\xi|^{a-2}\int_0^\xi(\xi-\zeta)\partial_\xi^2\widehat{u}_0(\zeta)\,d\zeta\right\|_2\\ &+ \left\|\chi(\xi)|\xi|^{a-1}\int_0^\xi\partial_\xi^2\widehat{u}_0(\zeta)\,d\zeta\right\|_2\end{aligned}$$

$$\leqslant \|\partial_\xi^2\widehat{u}_0\|_\infty + c_a\big\|\chi(\xi)|\xi|^a\|\partial_\xi^2\widehat{u}_0\|_\infty\big\|_2$$
$$\leqslant c_a\|\partial_\xi^2\widehat{u}_0\|_\infty \leqslant c_a\|\widehat{x^2u_0}\|_\infty \leqslant c_a\|x^2u_0\|_{1,2}$$
$$\leqslant c_a\big(\|\langle x\rangle^2u_0\|_2 + \|J^3u_0\|_2^{1/3}\|\langle x\rangle^3u_0\|_2^{2/3}\big).$$

This provides the bound of G_1 in (3.44).

For G_2 we write

$$\begin{aligned}\big\||x|^\theta D^a\mathcal{H}(xu_0)\big\|_2 &= \big\|D_\xi^\theta\big(|\xi|^a\operatorname{sgn}(\xi)(\widehat{xu_0})\big)\big\|_2\\ &\leqslant \big\|D_\xi^\theta\big(|\xi|^a\operatorname{sgn}(\xi)\chi(\xi)(\widehat{xu_0})\big)\big\|_2 + \big\|D_\xi^\theta\big(|\xi|^a\operatorname{sgn}(\xi)\big(1-\chi(\xi)\big)(\widehat{xu_0})\big)\big\|_2\\ &\leqslant \|\widehat{xu_0}\|_\infty + \|\widehat{xu_0}\|_2 + \big\|J_\xi^\theta\big(\langle\xi\rangle^a\widehat{xu_0}\big)\big\|_2\\ &\leqslant \|xu_0\|_1 + \|xu_0\|_2 + \big\|J^{1+\theta}\big(\langle\xi\rangle^a\widehat{u}_0\big)\big\|_2 + \big\|J^\theta\big(\langle\xi\rangle^{a-1}\widehat{u}_0\big)\big\|_2\\ &\leqslant c\big\|\langle x\rangle^{\frac{3}{2}+}u_0\big\|_2 + \big\|\langle x\rangle^{1+\theta}J^au_0\big\|_2 + \big\|J^\theta u_0\big\|_2.\end{aligned}$$

To complete the estimate we use that

$$\big\|\langle x\rangle^{1+\theta}J^\theta u_0\big\|_2 \leqslant \big\|\langle x\rangle^3u_0\big\|_2^{(1+\theta)/3}\big\|J^mu_0\big\|_2^{(2-\theta)/3}$$

holds if $m(\frac{2-\theta}{3}) = a < r(1+a)(\frac{2-\theta}{3})$. Since $r \geqslant 3$ we just need $(1+a)(2-\theta) > a$, which holds since $\theta \in (0,1)$. This finishes the bound for G_2.

Finally, we consider G_3 in (3.44):

$$\begin{aligned}\big\||x|^\theta D^{1+a}\big(x^2u_0\big)\big\|_2 &= \big\|D_\xi^\theta\big(|\xi|^{1+a}\operatorname{sgn}(\xi)\partial_\xi^2\widehat{u}_0\big)\big\|_2\\ &\leqslant \big\|D_\xi^\theta\big(|\xi|^{1+a}\operatorname{sgn}(\xi)\chi(\xi)\partial_\xi^2\widehat{u}_0\big)\big\|_2 + \big\|D_\xi^\theta\big(|\xi|^{1+a}\operatorname{sgn}(\xi)\big(1-\chi(\xi)\big)\partial_\xi^2\widehat{u}_0\big)\big\|_2\\ &= A_1 + A_2.\end{aligned}$$

The Leibniz rule and Stein's derivatives give

$$A_1 \leqslant \big\|D_\xi^\theta\big(|\xi|^{1+a}\operatorname{sgn}(\xi)\chi(\xi)\big)\big\|_\infty\big\|\partial_\xi^2\widehat{u}_0\big\|_2 + \big\||\xi|^{1+a}\operatorname{sgn}(\xi)\chi(\xi)\big\|_\infty\big\|D_\xi^\theta\partial_\xi^2\widehat{u}_0\big\|_2 \leqslant \big\|\langle x\rangle^{2+\theta}u_0\big\|_2$$

and

$$\begin{aligned}A_2 &\leqslant \big\|D_\xi^\theta\big(\partial_\xi^2\big(|\xi|^{1+a}\operatorname{sgn}(\xi)\big(1-\chi(\xi)\big)\widehat{u}_0\big)\big)\big\|_2 + \big\|D_\xi^\theta\big(\partial_\xi\big(|\xi|^{1+a}\operatorname{sgn}(\xi)\big(1-\chi(\xi)\big)\partial_\xi\widehat{u}_0\big)\big)\big\|_2\\ &\quad + \big\|D_\xi^\theta\big(\partial_\xi^2\big(|\xi|^{1+a}\operatorname{sgn}(\xi)\big(1-\chi(\xi)\big)\widehat{u}_0\big)\big)\big\|_2\\ &\equiv K_1 + K_2 + K_3.\end{aligned}$$

Notice that

$$\partial_\xi\big(|\xi|^{1+a}\operatorname{sgn}(\xi)\big(1-\chi(\xi)\big)\big) = (1+a)|\xi|^a\operatorname{sgn}(\xi)\big(1-\chi(\xi)\big) + |\xi|^{1+a}\operatorname{sgn}(\xi)\chi'(\xi)$$

and

$$\partial_\xi^2\big(|\xi|^{1+a}\operatorname{sgn}(\xi)\big(1-\chi(\xi)\big)\big) = c|\xi|^{a-1}\operatorname{sgn}(\xi)\big(1-\chi(\xi)\big) + c|\xi|^a\operatorname{sgn}(\xi)\chi'(\xi) + |\xi|^{1+a}\operatorname{sgn}(\xi)\chi''(\xi).$$

So to bound K_3 we just need to consider

$$\begin{aligned}&\big\|D_\xi^\theta\big(|\xi|^{a-1}\operatorname{sgn}(\xi)\big(1-\chi(\xi)\big)\widehat{u}_0\big)\big\|_2\\ &\quad\leqslant \big\|D_\xi^\theta\big(|\xi|^{a-1}\operatorname{sgn}(\xi)\big(1-\chi(\xi)\big)\big)\big\|_\infty\|\widehat{u}_0\|_2 + \big\||\xi|^{a-1}\operatorname{sgn}(\xi)\big(1-\chi(\xi)\big)\big\|_\infty\big\|D_\xi^\theta\widehat{u}_0\big\|_2\\ &\quad\leqslant \|u_0\|_2 + \big\||x|^\theta u_0\big\|_2.\end{aligned}$$

To bound K_2 we just need to consider

$$\begin{aligned}\big\|D_\xi^\theta\big(|\xi|^a\operatorname{sgn}(\xi)\big(1-\chi(\xi)\big)\partial_\xi\widehat{u}_0\big)\big\|_2 &\leqslant \left\|D_\xi^\theta\left(\frac{|\xi|^a\operatorname{sgn}(\xi)(1-\chi(\xi))}{\langle\xi\rangle^a}\langle\xi\rangle^a\partial_\xi\widehat{u}_0\right)\right\|_2\\ &\leqslant \left\|J_\xi^\theta\left(\frac{|\xi|^a\operatorname{sgn}(\xi)(1-\chi(\xi))}{\langle\xi\rangle^a}\langle\xi\rangle^a\partial_\xi\widehat{u}_0\right)\right\|_2.\end{aligned}$$

Now notice that

$$\Phi(\xi)=\frac{|\xi|^a\operatorname{sgn}(\xi)(1-\chi(\xi))}{\langle\xi\rangle^a}\in L^\infty,\qquad \partial_\xi\Phi\cong\frac{1}{\langle\xi\rangle},\qquad \partial_\xi^2\Phi\cong\frac{1}{\langle\xi\rangle^2}\in L^2.$$

Thus Proposition 2.5 and interpolation (Lemma 2.8) yield

$$\begin{aligned}\left\|J_\xi^\theta\left(\frac{|\xi|^a\operatorname{sgn}(\xi)(1-\chi(\xi))}{\langle\xi\rangle^a}\langle\xi\rangle^a\partial_\xi\widehat{u}_0\right)\right\|_2 &\leqslant \left\|J_\xi^\theta\left(\langle\xi\rangle^a\partial_\xi\widehat{u}_0\right)\right\|_2\\ &\leqslant \left\|\partial_\xi J_\xi^\theta\left(\langle\xi\rangle^a\widehat{u}_0\right)\right\|_2+\left\|J_\xi^\theta\left(\langle\xi\rangle^{a-1}\widehat{u}_0\right)\right\|_2\\ &\leqslant \left\|J_\xi^{1+\theta}\left(\langle\xi\rangle^a\widehat{u}_0\right)\right\|_2+\left\|J_\xi^\theta\widehat{u}_0\right\|_2\\ &\leqslant \left\|\langle x\rangle^{1+\theta}J^a u_0\right\|_2+\left\||x|^\theta u_0\right\|_2.\end{aligned}$$

Finally to bound K_3 we just need to consider

$$\begin{aligned}\left\|D_\xi^\theta\left(|\xi|^{a-1}\operatorname{sgn}(\xi)\left(1-\chi(\xi)\right)\widehat{u}_0\right)\right\|_2 &\leqslant \left\|J_\xi^\theta\left(|\xi|^{a-1}\operatorname{sgn}(\xi)\left(1-\chi(\xi)\right)\widehat{u}_0\right)\right\|_2\\ &\leqslant \left\|J_\xi^\theta\widehat{u}_0\right\|_2\leqslant\left\|\langle x\rangle^\theta u_0\right\|_2,\end{aligned}$$

where we have used Proposition 2.5.

4. Proof of Theorem 1.2

Without loss of generality we assume that $t_1=0<t_2$.
We consider the case $0<a<1/2$ and hence $\frac{5}{2}+a=2+\alpha<3$.
From the hypothesis we have that $u\in C([-T,T]:Z_{(1+a)(\frac{5}{2}+a)+\frac{1-a}{2},\frac{5}{2}+a-\epsilon})$, for some $0<\epsilon<1$. Therefore

$$u\partial_x u\in C\left([-T,T]:Z_{(1+a)(\frac{5}{2}+a)-\frac{1+a}{2},4+2a-2\epsilon}\right)$$

and

$$u\partial_x u\in L^1\left([-T,T]:H^{s_0}(\mathbb{R})\right),\quad \text{with } s_0\in\left(0,(1+a)\left(\frac{5}{2}+a\right)\right).$$

The solution to the IVP (1.1) can be represented by Duhamel's formula

$$u(t)=W_a(t)u_0-\int_0^t W_a\left(t-t'\right)u\left(t'\right)\partial_x u\left(t'\right)dt',\tag{4.1}$$

or equivalently in Fourier space as

$$\widehat{u}(\xi,t)=e^{-it|\xi|^{1+a}\xi}\widehat{u}_0(\xi)-\frac{i}{2}\int_0^t e^{-i(t-t')|\xi|^{1+a}\xi}\xi\widehat{u}^2\left(\xi,t'\right)dt'.\tag{4.2}$$

With the notation introduced in (2.1), we have

$$\begin{aligned}\partial_\xi^2\widehat{u}(\xi,t)&=F_2(t,\xi,\widehat{u}_0)-\frac{i}{2}\int_0^t F_2\left(t-t',\xi,\xi\widehat{u}^2\left(\xi,t'\right)\right)\\ &=\sum_1^4 B_j(t,\xi,\widehat{u}_0)-\frac{i}{2}\int_0^t\sum_1^4 B_j\left(t-t',\xi,\xi\widehat{u}^2\right)dt'.\end{aligned}\tag{4.3}$$

We notice that for any $t\in\mathbb{R}$, and any $j=2,3,4$, we have

Claim 1. *Let* $\alpha = \frac{1}{2} + a \in (\frac{1}{2}, 1)$ *and* $j = 2, 3, 4$. *Then*

$$B_j(t, \xi, \widehat{u_0}) - \frac{i}{2}\int_0^t B_j\big(t - t', \xi, \xi\widehat{u}^2\big)\, dt' \in H^\alpha(\mathbb{R}), \tag{4.4}$$

for all $t \in \mathbb{R}$.

If we assume (4.4), it follows that

$$\partial_\xi^2\widehat{u}(\xi, t) \in H^\alpha(\mathbb{R}) \quad \text{if and only if} \quad B_1(t, \xi, \widehat{u_0}) - \frac{i}{2}\int_0^t B_1\big(t - t', \xi, \xi\widehat{u}^2\big)\, dt' \in H^\alpha(\mathbb{R}). \tag{4.5}$$

We split now B_1 as

$$\begin{aligned} B_1(t, \xi, \widehat{u_0}) &= c_1 t|\xi|^a \operatorname{sgn}(\xi) e^{-it|\xi|^{1+a}\xi}\widehat{u_0}(\xi) \\ &= c_1 t|\xi|^a \operatorname{sgn}(\xi) e^{-it|\xi|^{1+a}\xi}\widehat{u_0}(\xi)\big(\chi(\xi) + \big(1 - \chi(\xi)\big)\big) \\ &= B_{1,1} + B_{1,2}, \end{aligned} \tag{4.6}$$

where $c_1 = -i(2 + a)(1 + a)$. From the hypothesis, it is easy to check that for any $t \in \mathbb{R} - \{0\}$, $B_{1,2} \in H^1(\mathbb{R})$.

Next, we consider $B_{1,1}$

$$B_{1,1} = c_1 t|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\big(e^{-it|\xi|^{1+a}\xi} - 1\big)\widehat{u}_0(\xi) + c_1 t|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\widehat{u}_0(\xi) = \tilde{B}^1_{1,1} + \tilde{B}^2_{1,1}.$$

Once again, we can easily check that for any $t \in \mathbb{R} - \{0\}$, $\tilde{B}^1_{1,1} \in H^1(\mathbb{R})$.

We rewrite $\tilde{B}_{1,1}$ as:

$$\tilde{B}^2_{1,1} = c_1 t|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\big(\widehat{u}_0(\xi) - \widehat{u}_0(0)\big) + c_1 t|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\widehat{u}_0(0) \tag{4.7}$$

and notice that for any $t \in \mathbb{R} - \{0\}$, $c_1 t|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)(\widehat{u}_0(\xi) - \widehat{u}_0(0)) \in H^1(\mathbb{R})$.

Now, we apply the above argument for the inhomogeneous term

$$\int_0^t B_1\big(t - t', \xi, \xi\widehat{u}^2\big)\, dt',$$

to conclude that

$$\begin{aligned} &\left(B_1(t, \xi, \widehat{u_0}) - \frac{i}{2}\int_0^t B_1\big(t - t', \xi, \xi\widehat{u}^2\big)\, dt'\right) \\ &\quad - c_1\left(t|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\widehat{u}_0(0) - \frac{i}{2}\int_0^t \big(t - t'\big)|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\xi\widehat{u}^2\big(0, t'\big)\right) \in H^1(\mathbb{R}) \end{aligned} \tag{4.8}$$

and therefore from (4.5) and (4.8) we have that for any $t \in \mathbb{R} - \{0\}$

$$\partial_\xi^2\widehat{u}(\xi, t) \in H^\alpha(\mathbb{R}) \quad \text{if and only if}$$

$$t|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\widehat{u}_0(0) - \frac{i}{2}\int_0^t \big(t - t'\big)|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\xi\widehat{u}^2\big(0, t'\big) \in H^\alpha(\mathbb{R}).$$

Finally, we observe that for any $t \in \mathbb{R} - \{0\}$

$$\frac{i}{2}\int_0^t \big(t - t'\big)|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\xi\widehat{u}^2\big(0, t'\big) \in H^1(\mathbb{R}),$$

and hence

$$\partial_\xi^2 \widehat{u}(\xi,t) \in H^\alpha(\mathbb{R}) \quad \text{if and only if} \quad t|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\widehat{u}_0(0) \in H^\alpha(\mathbb{R}) \tag{4.9}$$

and since $\alpha = \frac{1}{2} + a$, it follows from Proposition 2.9, that (4.9) holds at $t = t_2$ if and only if $\widehat{u}_0(0) = 0$.

In order to complete the proof we go back to Claim 1.

Proof of Claim 1. We will only give the details in the case $j = 2$. We have

$$B_2(t,\xi,\widehat{u}_0) = c_2 e^{-it|\xi|^{1+a}\xi} t^2 |\xi|^{2(a+1)} \widehat{u}_0(\xi) \tag{4.10}$$

and therefore

$$\|B_2(t,\cdot,\widehat{u}_0)\|_2 \leqslant c_t \|D^{2(a+1)} u_0\|_2 \leqslant c_t \|u_0\|_{2(a+1),2} \tag{4.11}$$

and from Propositions 2.7, 2.5 and Lemma 2.8

$$\begin{aligned}
\|D^\alpha B_2(t,\cdot,\widehat{u}_0)\|_2 &\leqslant c_t\big(\|u_0\|_{2(a+1),2} + \big\||\xi|^{(2+\alpha)(2+a)}\widehat{u}_0\big\|_2 + \big\|\mathcal{D}_\xi^\alpha\big(|\xi|^{2(a+1)}\widehat{u}_0\big)\big\|_2\big) \\
&\leqslant c_t\left(\|u_0\|_{(2+\alpha)(a+1),2} + \left\|\mathcal{D}_\xi^\alpha\left(\frac{|\xi|^{2(a+1)}}{\langle\xi\rangle^{2(1+a)}}\langle\xi\rangle^{2(1+a)}\widehat{u}_0\right)\right\|_2\right) \\
&\leqslant c_t\big(\|u_0\|_{(5/2+a)(a+1),2} + \big\|J_\xi^\alpha\big(\langle\xi\rangle^{2(1+a)}\widehat{u}_0\big)\big\|_2\big) \\
&\leqslant c_t\big(\|u_0\|_{(5/2+a)(a+1),2} + \big\|\langle x\rangle^\alpha J^{2(1+a)}u_0\big\|_2\big) \\
&\leqslant c_t\big(\|u_0\|_{(5/2+a)(a+1),2} + \big\|\langle x\rangle^{5/2+a}u_0\big\|_2^{\frac{\alpha}{2+\alpha}}\big\|J^{(5/2+a)(a+1)}u_0\big\|_2^{\frac{2}{2+\alpha}}\big)
\end{aligned} \tag{4.12}$$

which are all finite since $u_0 \in Z_{(5/2+a)(a+1),5/2+a}$. □

5. Proof of Theorem 1.3

As it was remarked, we carry out the details for $a \in [1/2, 1)$. Thus, $7/2 + a = 4 + \alpha$ with $\alpha = a - 1/2 \in [0, 1/2)$. Again, we shall use the integral equation associated to the IVP (1.1)

$$u(t) = W_a(t)u_0 - \int_0^t W_a(t-t')u(t')\partial_x u(t')\,dt',$$

which in Fourier space reads as

$$\widehat{u}(\xi,t) = e^{-it|\xi|^{1+a}\xi}\widehat{u}_0(\xi) - \frac{i}{2}\int_0^t e^{-i(t-t')|\xi|^{1+a}\xi}\xi\widehat{u}^2(\xi,t')\,dt'. \tag{5.1}$$

With the notation introduced in (2.1), we have that

$$\begin{aligned}
\partial_\xi^4\widehat{u}(\xi,t) &= F_4(t,\xi,\widehat{u}_0) - \frac{i}{2}\int_0^t F_4\big(t-t',\xi,\xi\widehat{u}^2(\xi,t')\big) \\
&= \sum_1^{11} E_j(t,\xi,\widehat{u}_0) - \frac{i}{2}\int_0^t \sum_1^{11} E_j\big(t-t',\xi,\xi\widehat{u}^2\big)\,dt'.
\end{aligned} \tag{5.2}$$

By hypothesis we have that

$$u \in C\big([-T,T] : \dot{Z}_{(1+a)(\frac{7}{2}+a)+\frac{1-a}{2},\frac{7}{2}+a-\epsilon}\big), \quad \text{for some } 0 < \epsilon < 1. \tag{5.3}$$

Therefore

$$u\partial_x u \in C\big([-T,T] : Z_{(1+a)(\frac{7}{2}+a)-\frac{1+a}{2},6+2a-2\epsilon}\big), \tag{5.4}$$

and by using Proposition 2.3

$$u\partial_x u \in L^1\big([-T,T]:H^{s_0}(\mathbb{R})\big), \quad s_0 \in \big(0,(1+a)(7/2+a)\big). \tag{5.5}$$

In Fourier space these last two properties are

$$\widehat{u} \in C\big([-T,T]:Z_{\frac{7}{2}+a-\epsilon,(1+a)(\frac{7}{2}+a)+\frac{1-a}{2}}\big) \tag{5.6}$$

and

$$\xi\widehat{u} * \widehat{u} \in C\big([-T,T]:Z_{6+2a-2\epsilon,(1+a)(\frac{7}{2}+a)-\frac{1+a}{2}}\big). \tag{5.7}$$

Also for $j=1,2,3$ one has that

$$u(\cdot,t_j) \in \dot{Z}_{(7/2+a)(1+a),7/2+a} \quad \text{and so} \quad \widehat{u}(\cdot,t_j) \in \dot{Z}_{7/2+a,(7/2+a)(1+a)}. \tag{5.8}$$

We observe that from the equation in (1.1) it follows that

$$\frac{d}{dt}\int_{-\infty}^{\infty} xu(x,t)\,dx = \frac{1}{2}\big\|u(t)\big\|_2^2 = \frac{1}{2}\|u_0\|_2^2, \tag{5.9}$$

and hence

$$\int_{-\infty}^{\infty} xu(x,t)\,dx = \int_{-\infty}^{\infty} xu_0(x)\,dx + \frac{t}{2}\|u_0\|_2^2 \tag{5.10}$$

so the first momentum of a non-null solution is a strictly increasing function of t. If we prove that there exist $\tilde{t}_1 \in (t_1,t_2)$ and $\tilde{t}_2 \in (t_2,t_3)$ such that for $j=1,2$

$$\int_{-\infty}^{\infty} xu(x,\tilde{t}_j)\,dx = 0, \tag{5.11}$$

it will follow that $\|u_0\|_2 = 0$, and therefore $u \equiv 0$.

So we just need to show that using the hypothesis in (5.8) and (5.9) for $j=1,2$ there exists $\tilde{t}_1 \in (t_1,t_2)$ such that

$$\int_{-\infty}^{\infty} xu(x,\tilde{t}_1)\,dx = 0. \tag{5.12}$$

Without loss of generality we assume that $t_1 = 0 < t_2 < t_3$. Then, going back to Eq. (5.2) we observe that

$$\begin{aligned} E_1 = E_1(t,\xi,\widehat{u}_0) &= c_1 t|\xi|^{a-2}\operatorname{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\widehat{u}_0(\xi) \\ &= c_1 t|\xi|^{a-2}\operatorname{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\widehat{u}_0(\xi)\big(\chi(\xi)+1-\chi(\xi)\big) \\ &= E_{1,1}+E_{1,2}, \end{aligned} \tag{5.13}$$

with $E_{1,2} \in H^1(\mathbb{R})$ for any $t \in \mathbb{R}$. On the other hand, by using Taylor's formula and the fact that $\widehat{u}_0(0)=0$, we obtain

$$\widehat{u}_0(\xi) = \xi\,\partial_\xi\widehat{u}_0(0) + \int_0^{\xi}(\xi-\theta)\partial_\xi^2\widehat{u}(\theta)\,d\theta \equiv \xi\,\partial_\xi\widehat{u}_0(0) + R_2(\xi). \tag{5.14}$$

Therefore we can write $E_{1,1}$ as

$$\begin{aligned} E_{1,1}(t,\xi,\widehat{u}_0) &= c_1 t|\xi|^{a-2}\operatorname{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\chi(\xi)\big(\xi\,\partial_\xi\widehat{u}_0(0)+R_2(\xi)\big) \\ &= c_1 t|\xi|^{a-1}e^{-it|\xi|^{1+a}\xi}\chi(\xi)\partial_\xi\widehat{u}_0(0) + \tilde{R}_2(\xi,t). \end{aligned} \tag{5.15}$$

Let us see that for any $t \in \mathbb{R}$, $\tilde{R}_2(\xi,t) \in H^1(\mathbb{R})$. Thus

$$\begin{aligned}\|\tilde{R}_2(\cdot,t)\|_2 &\leqslant c_t \big\| |\xi|^{a-2}\chi(\xi)|\xi|^2 \|\partial_\xi^2\widehat{u}_0\|_\infty \big\|_2 \\ &\leqslant c_t \|\widehat{x^2 u_0}\|_\infty \leqslant c_t \|x^2 u_0\|_1 \\ &\leqslant c_t \|\langle x\rangle^{\frac{5}{2}+} u_0\|_2.\end{aligned}$$

Since $a \in (1/2, 1)$ (so $a-1 > -1/2$)

$$\begin{aligned}\|\partial_\xi \tilde{R}_2(.,t)\|_2 \leqslant c_t\big(&\big\||\xi|^{a-3}\chi(\xi)|\xi|^2\|\partial_\xi^2\widehat{u}_0\|_\infty\big\|_2 + \big\||\xi|^{a-2}\delta(0)\chi(\xi)|\xi|^2\|\partial_\xi^2\widehat{u}_0\|_\infty\big\|_2 \\ &+ \big\||\xi|^{2a+1}\chi(\xi)\|\partial_\xi^2\widehat{u}_0\|_\infty\big\|_2 + \big\||\xi|^{a-2}\chi(\xi)|\xi|\|\partial_\xi^2\widehat{u}_0\|_\infty\big\|_2\big) \\ \leqslant c_t\big(&\|\langle x\rangle^2 u_0\|_2 + \|\langle x\rangle^2\partial_x u_0\|_{1,2}\big).\end{aligned} \tag{5.16}$$

Next we observe that

$$\begin{aligned}&t|\xi|^{a-1}\operatorname{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\chi(\xi)\partial_\xi\widehat{u}_0(0) \\ &\quad = t|\xi|^{a-1}\operatorname{sgn}(\xi)\chi(\xi)\partial_\xi\widehat{u}_0(0)\big(1+\big(e^{-it|\xi|^{1+a}\xi}-1\big)\big) \\ &\quad = t|\xi|^{a-1}\operatorname{sgn}(\xi)\chi(\xi)\partial_\xi\widehat{u}_0(0) + Q_2(t,\xi)\end{aligned} \tag{5.17}$$

with

$$Q_2(t,\cdot) \in H^1(\mathbb{R}), \tag{5.18}$$

for all $t \in \mathbb{R}$, which follows by combining the estimate $|e^{i\tau}-1| \leqslant |\tau|$ and the fact that $a \in (1/2, 1)$.

Gathering the information from (5.13) to (5.18) we can conclude

$$E_1 - c_1 t|\xi|^{a-1}\operatorname{sgn}(\xi)\chi(\xi)\partial_\xi\widehat{u}_0(0) \in H^1(\mathbb{R}), \tag{5.19}$$

for all $t \in \mathbb{R}$, with $c_1 = -(2+a)(1+a)a(a-1)i$.

Combining the above argument and (5.4) we also conclude that

$$\int_0^t E_1\big(t-t',\xi,\xi\widehat{u}^2(\xi,t')\big)\,dt' - c_1\int_0^t \big(t-t'\big)|\xi|^{a-1}\chi(\xi)\partial_\xi\big(\xi\widehat{u}^2\big)\big(0,t'\big)\,dt' \in H^1(\mathbb{R})$$

for all $t \in \mathbb{R}$.

Now we shall rewrite the terms E_5's in (2.1) as

$$\begin{aligned}E_5 = E_5(t,\xi,\widehat{u}_0) &= c_5 t|\xi|^{a-1}\operatorname{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\partial_\xi\widehat{u}_0(\xi) \\ &= c_1 t|\xi|^{a-2}\operatorname{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\partial_\xi\widehat{u}_0(\xi)\big(\chi(\xi)+1-\chi(\xi)\big) \\ &= E_{5,1}+E_{5,2},\end{aligned} \tag{5.20}$$

with $E_{5,2} \in H^1(\mathbb{R})$ for any $t \in \mathbb{R}$. In fact

$$\begin{aligned}\|E_{5,2}\|_{1,2} &\leqslant c_t\big(\|D^{2a}(xu_0)\|_2 + \|\langle x\rangle^2 u_0\|_2\big) \\ &\leqslant c_t\big(\|xu_0\|_{2,2} + \|\langle x\rangle^2 u_0\|_2\big) \\ &\leqslant c_t\big(\|\partial_x u_0\|_2 + \|\langle x\rangle\partial_x^2 u_0\|_2 + \|\langle x\rangle^2 u_0\|_2\big).\end{aligned} \tag{5.21}$$

Also

$$\begin{aligned}E_{5,1} = {}& c_5 t|\xi|^{a-1}\operatorname{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\chi(\xi)\partial_\xi\widehat{u}_0(0) \\ &+ c_5 t|\xi|^{a-1}\operatorname{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\chi(\xi)\big(\partial_\xi\widehat{u}_0(\xi)-\partial_\xi\widehat{u}_0(0)\big).\end{aligned} \tag{5.22}$$

An argument similar to that one in (5.15)–(5.16) shows that

$$c_5 t|\xi|^{a-1}\operatorname{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\chi(\xi)\big(\partial_\xi\widehat{u}_0(\xi)-\partial_\xi\widehat{u}_0(0)\big) \in H^1(\mathbb{R}), \tag{5.23}$$

for all $t \in \mathbb{R}$. Now we consider

$$c_5 t|\xi|^{a-1}\operatorname{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\chi(\xi)\partial_\xi\widehat{u}_0(0)$$
$$= c_5 t|\xi|^{a-1}\operatorname{sgn}(\xi)\chi(\xi)\partial_\xi\widehat{u}_0(0) + c_5 t|\xi|^{a-1}\operatorname{sgn}(\xi)\big(e^{-it|\xi|^{1+a}\xi}-1\big)\chi(\xi)\partial_\xi\widehat{u}_0(0). \tag{5.24}$$

The arguments in (5.17) and (5.18) show that

$$t|\xi|^{a-1}\operatorname{sgn}(\xi)\big(e^{-it|\xi|^{1+a}\xi}-1\big)\chi(\xi)\partial_\xi\widehat{u}_0(0)\in H^1(\mathbb{R}), \tag{5.25}$$

for all $t\in\mathbb{R}$. Hence gathering the information from (5.20) to (5.24)

$$E_5(t,\xi,\widehat{u}_0) - c_5 t|\xi|^{a-1}\operatorname{sgn}(\xi)\chi(\xi)\partial_\xi\widehat{u}_0(0)\in H^1(\mathbb{R}) \tag{5.26}$$

for all $t\in\mathbb{R}$, with $c_5=-4i(2+a)(1+a)a$.

The above argument and (5.4) show that

$$\int_0^t E_5\big(t-t',\xi,\xi\widehat{u}^2(\xi,t')\big) - c_5\int_0^t (t-t')|\xi|^{a-1}\chi(\xi)\partial_\xi\big(\xi\widehat{u}^2\big)(0,t')\in H^1(\mathbb{R}),$$

for all $t\in\mathbb{R}$. We claim that for all $t\in\mathbb{R}$,

$$E_2(t,\cdot,\widehat{u}_0),\ E_3(t,\cdot,\widehat{u}_0)\in H^1(\mathbb{R}). \tag{5.27}$$

It suffices to consider E_3. So

$$\|E_3\|_2\leqslant c_t\|u_0\|_{3a+2,2} \tag{5.28}$$

and

$$\begin{aligned}
\|\partial_\xi E_3\|_2 &\leqslant c_t\big(\big\|\langle\xi\rangle^{4a+3}\widehat{u}_0\big\|_2+\big\|\langle\xi\rangle^{3a+2}\partial_\xi\widehat{u}_0\big\|_2\big)\\
&\leqslant c_t\big(\big\|\langle\xi\rangle^{4a+3}\widehat{u}_0\big\|_2+\big\|\partial_\xi\big(\langle\xi\rangle^{3a+2}\widehat{u}_0\big)\big\|_2\big)\\
&\leqslant c_t\big(\|u_0\|_{4a+3,2}+\big\|J_\xi\big(\langle\xi\rangle^{3a+2}\widehat{u}_0\big)\big\|_2\big)\\
&\leqslant c_t\big(\|u_0\|_{4a+3,2}+\big\|J_\xi^4\widehat{u}_0\big\|_2^{\frac14}\big\|\big(\langle\xi\rangle^{4a+\frac83}\widehat{u}_0\big)\big\|_2^{\frac34}\big)\\
&\leqslant c_t\big(\|u_0\|_{4a+3,2}+\big\|\langle x\rangle^4u_0\big\|_2^{\frac14}\big\|J^{4a+\frac83}u_0\big\|_2^{\frac34}\big).
\end{aligned} \tag{5.29}$$

Since $4a+\frac83\leqslant(\frac72+a)(1+a)$, and $4\leqslant\frac72+a$, the claim is proved.

A similar argument and (5.5) show that for $j=2,3$

$$\int_0^t E_j\big(t-t',\xi,\xi\widehat{u}^2(\xi,t')\big)\,dt'\in H^1(\mathbb{R}), \tag{5.30}$$

for all $t\in\mathbb{R}$. Let us see now that for $j=6,9$,

$$E_j(t,\cdot,\widehat{u}_0)\in H^1(\mathbb{R}), \tag{5.31}$$

for all $t\in\mathbb{R}$. In both cases the proof is similar so we just consider the case $j=9$. So

$$\begin{aligned}
\|E_9\|_2 &\leqslant c_t\big\|\langle\xi\rangle^a\partial_\xi^2\widehat{u}_0\big\|_2\\
&\leqslant c_t\big(\big\|\langle\xi\rangle^aJ_\xi^2\widehat{u}_0\big\|_2+\big\|\langle\xi\rangle^a\widehat{u}_0\big\|_2\big)\\
&\leqslant c_t\big(\big\|J^a\langle x\rangle^2u_0\big\|_2+\big\|J^au_0\big\|_2\big)
\end{aligned} \tag{5.32}$$

and

$$\begin{aligned}
\|\partial_\xi E_9\|_2 &\leqslant c_t\big(\big\||\xi|^{a-1}\partial_\xi^2\widehat{u}_0\big\|_2+\big\|\langle\xi\rangle^{1+2a}\partial_\xi^2\widehat{u}_0\big\|_2\big\|\langle\xi\rangle^a\partial_\xi^3\widehat{u}_0\big\|_2\big)\\
&\leqslant c_t\big(\big\|\partial_\xi^2\widehat{u}_0\big\|_\infty+\big\|\langle\xi\rangle^{1+2a}J_\xi^2\widehat{u}_0\big\|_2\big\|J_\xi^3\langle\xi\rangle^a\widehat{u}_0\big\|_2+\big\|\langle\xi\rangle^{1+2a}\widehat{u}_0\big\|_2\big)\\
&\leqslant c_t\big(\big\|x^2u_0\big\|_{1,2}+\big\|J^{2a+1}\langle x\rangle^2u_0\big\|_2+\big\|\langle x\rangle^3J^au_0\big\|_2+\|u_0\|_{2a+1,2}\big)\\
&\leqslant c_t\big(\big\|x^2u_0\big\|_2+\big\|J^{2a+1}\langle x\rangle^2u_0\big\|_2+\big\|\langle x\rangle^3J^au_0\big\|_2+\|u_0\|_{2a+1,2}\big),
\end{aligned} \tag{5.33}$$

which can be bounded by interpolation as well.

Also a similar argument and (5.5) show again that for $j = 6, 9$

$$\int_0^t E_j\big(t-t', \xi, \xi\widehat{u}^2(\xi, t')\big)\,dt' \in H^1(\mathbb{R}), \tag{5.34}$$

for all $t \in \mathbb{R}$. By hypotheses (5.3), (5.5) and (5.8) we have

Claim 2. *Let $\alpha = a - \frac{1}{2} \in (0, \frac{1}{2})$ and $j = 4, 7, 8$ and 11. Then*

$$E_j\big(t-t', \xi, \partial_\xi\big(\xi\widehat{u}^2\big)\big), \int_0^t E_j\big(t-t', \xi\widehat{u}^2\big)\big(\xi, t'\big)\,dt' \in H^\alpha(\mathbb{R}), \tag{5.35}$$

for all $t \in \mathbb{R}$.

Proof. Due to the form of the terms, by interpolation it suffices to consider the cases $j = 4$ and $j = 11$. Thus,

$$\|E_4\|_2 \leqslant c_t \big\| |\xi|^{4(1+a)}\widehat{u}_0 \big\|_2 \leqslant c_t \|u_0\|_{4(1+a),2} \tag{5.36}$$

and

$$\|E_{11}\|_2 \leqslant c_t \big\| \partial_\xi^4 \widehat{u}_0 \big\|_2 \leqslant c_t \big\| \langle\xi\rangle^4 u_0 \big\|_2, \tag{5.37}$$

and hence both quantities are finite. Now

$$\begin{aligned}
\big\| D_\xi^\alpha E_4 \big\|_2 &\leqslant c_t\big(\big\| |\xi|^{1+a}\widehat{u}_0 \big\|_2 + \big\| |\xi|^{(4+\alpha)(1+a)}\widehat{u}_0 \big\|_2 + \big\| \mathcal{D}_\xi^\alpha\big(|\xi|^{4(1+a)}\widehat{u}_0\big) \big\|_2 \big) \\
&\leqslant c_t\big(\big\| \langle\xi\rangle^{(4+\alpha)(1+a)}\widehat{u}_0 \big\|_2 + \big\| \mathcal{D}_\xi^\alpha\big(|\xi|^{4(1+a)}\widehat{u}_0\big) \big\|_2 \big) \\
&\leqslant c_t\big(\|u_0\|_{(4+\alpha)(1+a),2} + \big\| \mathcal{D}_\xi^\alpha\big(|\xi|^{4(1+a)}\widehat{u}_0\big) \big\|_2 \big),
\end{aligned}$$

but

$$\begin{aligned}
\big\| \mathcal{D}_\xi^\alpha\big(|\xi|^{4(1+a)}\widehat{u}_0\big) \big\|_2 &\leqslant \left\| \mathcal{D}_\xi^\alpha\left(\frac{|\xi|^{4(a+1)}}{\langle\xi\rangle^{4(1+a)}} \langle\xi\rangle^{4(1+a)}\widehat{u}_0 \right) \right\|_2 \\
&\leqslant c\big(\big\| \langle\xi\rangle^{4(1+a)}\widehat{u}_0 \big\|_2 + \big\| \mathcal{D}_\xi^\alpha\big(\langle\xi\rangle^{4(1+a)}\widehat{u}_0\big) \big\|_2 \big) \\
&\leqslant c\big(\big\| J^{4(1+a)}u_0 \big\|_2 + \big\| J_\xi^\alpha\big(\langle\xi\rangle^{4(1+a)}u_0\big) \big\|_2 \big) \\
&\leqslant c\big(\|u_0\|_{4(1+a),2} + \big\| \langle x\rangle^\alpha J^{4(1+a)}u_0 \big\|_2 \big) \\
&\leqslant c\big(\|u_0\|_{4(1+a),2} + \big\| \langle x\rangle^{4+\alpha}u_0 \big\|_2^{\frac{\alpha}{4+\alpha}} \big\| J^{(4+\alpha)(a+1)}u_0 \big\|_2^{\frac{4}{4+\alpha}} \big),
\end{aligned} \tag{5.38}$$

therefore

$$\big\| D_\xi^\alpha E_4 \big\|_2 \leqslant c\big(\|u_0\|_{(4+\alpha)(1+a),2} + \big\| \langle x\rangle^{4+\alpha}u_0 \big\|_2 \big), \tag{5.39}$$

which is finite by the hypothesis at $t_1 = 0$. Also

$$\begin{aligned}
\big\| D_\xi^\alpha E_{11} \big\|_2 &\leqslant c_t\big(\big\| \partial_\xi^4\widehat{u}_0 \big\|_2 + \big\| |\xi|^{\alpha(1+a)}\partial_\xi^4\widehat{u}_0 \big\|_2 + \big\| \mathcal{D}_\xi^\alpha\big(\partial_\xi^4\widehat{u}_0\big) \big\|_2 \big) \\
&\leqslant c_t\big(\big\| \langle x\rangle^4 u_0 \big\|_2 + \big\| \langle\xi\rangle^{\alpha(1+a)}\partial_\xi^4\widehat{u}_0 \big\|_2 + \big\| |x|^{4+\alpha}u_0 \big\|_2 \big) \\
&\leqslant c_t\big(\big\| J^{\alpha(1+a)}\big(x^4 u_0\big) \big\|_2 + \big\| \langle x\rangle^{4+\alpha}u_0 \big\|_2 \big) \\
&\leqslant c_t\big(\big\| J^{\alpha(1+a)}\big(\langle x\rangle^4 u_0\big) \big\|_2 + \big\| J^{\alpha(1+a)}\big(\langle x\rangle^2 u_0\big) \big\|_2 + \big\| J^{\alpha(1+a)}u_0 \big\|_2 + \big\| \langle x\rangle^{4+\alpha}u_0 \big\|_2 \big) \\
&\leqslant c_t\big(\big\| \langle x\rangle^{4+\alpha}u_0 \big\|_2^{\frac{4}{4+\alpha}} \big\| J^{(4+\alpha)(a+1)}u_0 \big\|_2^{\frac{\alpha}{4+\alpha}} + \big\| \langle x\rangle^4 u_0 \big\|_2^{\frac{1}{2}} \big\| J^{2\alpha(a+1)}u_0 \big\|_2^{\frac{1}{2}} \\
&\quad + \big\| J^{\alpha(1+a)}u_0 \big\|_2 + \big\| \langle x\rangle^{4+\alpha}u_0 \big\|_2 \big) \\
&\leqslant c_t\big(\|u_0\|_{(4+\alpha)(1+a),2} + \big\| \langle x\rangle^{4+\alpha}u_0 \big\|_2 \big).
\end{aligned} \tag{5.40}$$

Combining (5.36)–(5.40) and (5.5) it follows that

$$\int_0^t E_j\big(t-t',\xi,\xi\widehat{u}^2\big)\big(\xi,t'\big)\,dt' \in H^\alpha(\mathbb{R}), \tag{5.41}$$

for all $t\in\mathbb{R}$ for $j=4,11$, and consequently for $j=7,8,9$ as well.

Summing up, we can conclude that

$$D_\xi^\alpha \partial_\xi^4 \widehat{u}(\cdot,t)\in L^2(\mathbb{R}) \quad \text{if and only if}$$

$$D_\xi^\alpha\left(t|\xi|^{1-a}\chi(\xi)\partial_\xi\widehat{u}_0(0)-\int_0^t\big(t-t'\big)|\xi|^{1-a}\chi(\xi)\partial_\xi\widehat{u}^2\big(0,t'\big)\,dt'\right)\in L^2(\mathbb{R}), \tag{5.42}$$

for any fixed $t\in\mathbb{R}$. We observe that

$$\partial_\xi\widehat{u}_0(0)=\widehat{-ixu_0}(0)=-i\int xu_0(x)\,dx, \tag{5.43}$$

and by (5.9)

$$\begin{aligned}\partial_\xi\left(i\frac{\xi}{2}\widehat{u}^2\right)\big(0,t'\big)&=-\widehat{ixu\partial_x}u\big(0,t'\big)=-i\int xu\partial_x u\big(x,t'\big)\,dx\\&=\frac{i}{2}\big\|u\big(t'\big)\big\|_2^2=\frac{i}{2}\big\|u(0)\big\|_2^2=i\frac{d}{dt}\int xu(x,t)\,dx,\end{aligned} \tag{5.44}$$

and by integration by parts

$$\begin{aligned}&t_2\partial_\xi\widehat{u}_0(0)-\frac{i}{2}\int_0^{t_2}\big(t_2-t'\big)\partial_\xi\big(\xi\widehat{u}^2\big)\big(0,t'\big)\,dt'\\&\quad=-it_2\int xu_0\,dx-i\int_0^{t_2}\big(t_2-t'\big)\frac{d}{dt'}\int xu\big(x,t'\big)\,dx\,dt'\\&\quad=-it_2\int xu_0(x)\,dx-i\big(t_2-t'\big)\int xu\big(x,t'\big)\,dx\Big|_{t'=0}^{t'=t_2}-i\int_0^{t_2}\!\!\int xu\big(x,t'\big)\,dx\,dt'\\&\quad=-i\int_0^{t_2}\!\!\int xu\big(x,t'\big)\,dx\,dt'.\end{aligned} \tag{5.45}$$

Thus from our hypothesis at $t=t_2$, it follows that

$$D_\xi^\alpha\big(\chi(\xi)|\xi|^{a-1}\big)\int_0^{t_2}\!\!\int xu\big(x,t'\big)\,dx\,dt'\in L^2(\mathbb{R}), \tag{5.46}$$

but we recall that $\alpha=a-1/2$ and from Proposition 2.9

$$D_\xi^\alpha\big(\chi(\xi)|\xi|^{a-1}\big)\approx\mathcal{D}_\xi^\alpha\big(\chi(\xi)|\xi|^{\alpha-1/2}\big)\notin L^2(\mathbb{R}) \quad \text{if } \alpha\in(0,1).$$

Therefore for (5.46) to hold it is necessary that

$$\int_0^{t_2}\!\!\int xu\big(x,t'\big)\,dx\,dt'=0, \tag{5.47}$$

and since $F(t)=\int xu(x,t)\,dx$ is a continuous function, there exists $\tilde{t}_1\in(0,t_2)$ where $F(t)$ must vanish and this completes the proof. □

6. Proof of Theorem 1.4

Without loss of generality we can assume that

$$t_1 = 0 \quad \text{and} \quad \int x u_0(x)\,dx = 0. \tag{6.1}$$

Thus in this case, combining (5.43), (5.45)–(5.47) and (5.10), we have for $t_2 \neq 0$ that

$$\begin{aligned}
&\mathcal{D}_\xi^\alpha \partial_\xi^4 \widehat{u}(\cdot, t) \in L^2(\mathbb{R}), \quad \text{if and only if}\\
&\int_0^{t_2}\!\!\int x u(x, t')\,dx\,dt' = 0, \quad \text{if and only if}\\
&\int_0^{t_2} \frac{1}{2} t' \|u_0\|_2^2\,dt' = \frac{t_2^2}{4}\|u_0\|_2^2 = 0, \quad \text{if and only if}\\
&\|u_0\|_2^2 = 0 \quad \text{if and only if} \quad u_0 = 0.
\end{aligned} \tag{6.2}$$

7. Proof of Theorem 1.5

We shall consider only the case $a \in [1/2, 1)$, so that $\tilde{a} = 1$.

From the argument of the proof in Theorem 1.3, with $\alpha = 1/2$ in (5.36)–(5.40) and (5.5), we can conclude from our hypothesis $s \geqslant (\frac{7}{2} + a)(1 + a) + \frac{1-a}{2}$, that for $t \neq 0$

$$\begin{aligned}
&D_\xi^{1/2} \partial_\xi^4 \widehat{u}(\cdot, t) \in L^2(\mathbb{R}), \quad \text{if and only if}\\
&D_\xi^{1/2}\left(t|\xi|^{a-1}\chi(\xi)\partial_\xi \widehat{u}_0(0) - \int_0^t (t - t')|\xi|^{a-1}\chi(\xi)\partial_\xi\big(\xi\widehat{u}^2(0, t)\big)\,dt'\right) \in L^2(\mathbb{R}), \quad \text{if and only if}\\
&D_\xi^{1/2}\big(\chi(\xi)|\xi|^{a-1}\big)\int_0^t\!\!\int x u(x, t')\,dx\,dt' \in L^2(\mathbb{R}), \quad \text{if and only if}\\
&\int_0^t\!\!\int x u(x, t')\,dx\,dt' = \int_0^t \left(\int x u_0(x)\,dx + \frac{1}{2}t'\|u_0\|_2^2\right)dt' = t\left(\int x u_0(x)\,dx + \frac{1}{4}t\|u_0\|_2^2\right) = 0.
\end{aligned}$$

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