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Compactness of immersions with local Lipschitz representation

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Abstract

We consider immersions admitting uniform representations as a λ-Lipschitz graph. In codimension 1, we show compactness for such immersions for arbitrary fixed $\lambda < \infty$ and uniformly bounded volume. The same result is shown in arbitrary codimension for $\lambda \leqslant \frac{1}{4}$.

1. Introduction

In [14] J. Langer investigated compactness of immersed surfaces in $\mathbb{R}^3$ admitting uniform bounds on the second fundamental form and the area of the surfaces. For a given sequence $f^i : \Sigma^i \to \mathbb{R}^3$, there exist, after passing to a subsequence, a limit surface $f : \Sigma \to \mathbb{R}^3$ and diffeomorphisms $\phi^i : \Sigma \to \Sigma^i$, such that $f^i \circ \phi^i$ converges in the C^1-topology to f. In particular, up to diffeomorphism, there are only finitely many manifolds admitting such an immersion. The finiteness of topological types was generalized by K. Corlette in [6] to immersions of arbitrary dimension and codimension. Moreover, the compactness theorem was generalized by S. Delladio in [7] to hypersurfaces of arbitrary dimension. The general case, that is compactness in arbitrary dimension and codimension, was proved by the author in [4].

The proof strongly relies on a fundamental principle which we like to describe in the following. A simple consequence of the implicit function theorem says that any immersion can locally be written as the graph of a function $u : B_r \to \mathbb{R}^k$ over the affine tangent space. Moreover, for a given $\lambda > 0$ we can choose $r > 0$ small enough such that $\|Du\|_{C^0(B_r)} \leqslant \lambda$. If this is possible at any point of the immersion with the same radius r, we call f an (r, λ)-immersion.

Using the Sobolev embedding it can be shown that a uniform L^p-bound for the second fundamental form with p greater than the dimension implies that for any $\lambda > 0$ there is an $r > 0$ such that every immersion is an (r, λ)-immersion.

Inspired by this result, it is a natural generalization to investigate compactness properties also for (r, λ)-immersions with fixed r and λ; this is the topic of the present paper. In the proof of the theorem of Langer it is essential that λ can be chosen very small. Then, using the local graph representation over B_r, all immersions are close to each other and nearly flat. These properties are used repeatedly, for example for the construction of the diffeomorphism ϕ^i.

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Here, we would first like to show compactness of (r,λ)-immersions in codimension 1 for any fixed λ. We do not require any smallness assumption for λ. Moreover, we do not only consider immersions with graph representations over the affine tangent space, but also over other appropriately chosen m-spaces. Let $\mathfrak{F}^1(r,\lambda)$ be the set of C^1-immersions $f: M^m \to \mathbb{R}^{m+1}$ with $0 \in f(M)$, which may locally be written over an m-space as the graph of a λ-Lipschitz function $u: B_r \to \mathbb{R}$ (the precise definitions of all notations used in this paper are given in Section 2). Here all manifolds are assumed to be compact. Moreover, let $\mathfrak{F}^1_{\mathcal{V}}(r,\lambda)$ be the set of immersions in $\mathfrak{F}^1(r,\lambda)$ with $\mathrm{vol}(M) \leqslant \mathcal{V}$. Similarly, we define the set $\mathfrak{F}^0(r,\lambda)$ by replacing C^1-immersions in $\mathfrak{F}^1(r,\lambda)$ by Lipschitz functions. We obtain the following compactness result:

Theorem 1.1 *(Compactness of (r,λ)-immersions in codimension one). The set $\mathfrak{F}^1_{\mathcal{V}}(r,\lambda)$ is relatively compact in $\mathfrak{F}^0(r,\lambda)$ in the following sense*:

Let $f^i: M^i \to \mathbb{R}^{m+1}$ be a sequence in $\mathfrak{F}^1_{\mathcal{V}}(r,\lambda)$. Then, after passing to a subsequence, there exist an $f: M \to \mathbb{R}^{m+1}$ in $\mathfrak{F}^0(r,\lambda)$ and a sequence of diffeomorphisms $\phi^i: M \to M^i$, such that $f^i \circ \phi^i$ is uniformly Lipschitz bounded and converges uniformly to f.

Here the Lipschitz bound for $f^i \circ \phi^i$ is shown with respect to the local representations of some finite atlas of M. For these representations, we obtain a Lipschitz constant L depending only on λ. As an immediate consequence of Theorem 1.1 we deduce the following corollary:

Corollary 1.2. *There are only finitely many manifolds in $\mathfrak{F}^1_{\mathcal{V}}(r,\lambda)$ up to diffeomorphism.*

The situation is slightly different when considering (r,λ)-immersions in arbitrary codimension. For the construction of the diffeomorphisms ϕ^i one uses a kind of projection in an averaged normal direction ν. In higher codimension, the averaged normal ν cannot be constructed as in the case of hypersurfaces. We will give an alternative construction involving a Riemannian center of mass. However, for doing so we have to assume here that λ is not too large. Let $\mathfrak{F}^1_{\mathcal{V}}(r,\lambda)$ and $\mathfrak{F}^0(r,\lambda)$ be defined as above, but this time for functions with values in $\mathbb{R}^{m+k}$ for a fixed k. We obtain the following theorem:

Theorem 1.3 *(Compactness of (r,λ)-immersions in arbitrary codimension). Let $\lambda \leqslant \frac{1}{4}$. Then $\mathfrak{F}^1_{\mathcal{V}}(r,\lambda)$ is relatively compact in $\mathfrak{F}^0(r,\lambda)$ in the sense of Theorem 1.1.*

As in Corollary 1.2, we deduce for $\lambda \leqslant \frac{1}{4}$ that there are only finitely many manifolds in $\mathfrak{F}^1_{\mathcal{V}}(r,\lambda)$ up to diffeomorphism. Surely, the bound $\lambda \leqslant \frac{1}{4}$ is not optimal; at the end of Section 6 we will discuss some possibilities how to prove the theorem for bigger Lipschitz constant.

In [14] and [4] any sequence of immersions with L^p-bounded second fundamental form, $p > m$, is shown to be also a sequence of (r,λ)-immersions (for some fixed r and λ). The same conclusion holds in many other situations, where the geometric data (such as curvature bounds) ensure uniform graph representations with control over the slope of the graphs. Hence it seems natural to unearth the compactness of (r,λ)-immersions as a theorem on its own. In any general situation, where compactness of immersions is desired (e.g. when considering convergence of geometric flows), only the condition of Definition 2.2 in Section 2 has to be verified. If in addition some bound for higher derivatives of the graph functions is known (or for instance a $C^{0,\alpha}$-bound for Du), with methods as in [4] one easily derives additional properties of the limit, such as higher order differentiability or curvature bounds. Hence, Theorems 1.1 and 1.3 can be seen as the most general kind of compactness theorem in this context.

2. Definitions and preliminaries

We begin with some general notations: For $n = m + k$ let $G_{n,m}$ denote the Grassmannian of (non-oriented) m-dimensional subspaces of $\mathbb{R}^n$. Unless stated otherwise let B_ϱ denote the open ball in $\mathbb{R}^m$ of radius $\varrho > 0$ centered at the origin.

Now let M be an m-dimensional manifold without boundary and $f: M \to \mathbb{R}^n$ a C^1-immersion. Let $q \in M$ and let T_qM be the tangent space at q. Identifying vectors $X \in T_qM$ with $f_*X \in T_{f(q)}\mathbb{R}^n$, we may consider T_qM as an

m-dimensional subspace of $\mathbb{R}^n$. Let $(T_qM)^\perp$ denote the orthogonal complement of T_qM in $\mathbb{R}^n$, that is

$$\mathbb{R}^n = T_qM \oplus (T_qM)^\perp$$

and $(T_qM)^\perp$ is perpendicular to T_qM. We may define the tangent-space field

$$\begin{aligned} \tau_f : M &\to G_{n,m}, \\ q &\mapsto T_qM, \end{aligned} \tag{2.1}$$

and the normal-space field

$$\begin{aligned} \nu_f : M &\to G_{n,k}, \\ q &\mapsto (T_qM)^\perp. \end{aligned} \tag{2.2}$$

2.1. *The notion of an* (r, λ)*-immersion*

We call a mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ a *Euclidean isometry*, if there is a rotation $R \in \mathbb{SO}(n)$ and a translation $T \in \mathbb{R}^n$, such that $A(x) = Rx + T$ for all $x \in \mathbb{R}^n$.

For a given point $q \in M$ let $A_q : \mathbb{R}^n \to \mathbb{R}^n$ be a Euclidean isometry, which maps the origin to $f(q)$, and the subspace $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^k$ onto $f(q) + \tau_f(q)$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^m$ be the standard projection onto the first m coordinates.

Finally let $U_{r,q} \subset M$ be the q-component of the set $(\pi \circ A_q^{-1} \circ f)^{-1}(B_r)$. Although the isometry A_q is not uniquely determined, the set $U_{r,q}$ does not depend on the choice of A_q.

We come to the central definition (as first defined in [14]):

Definition 2.1. An immersion f is called an (r, λ)-immersion, if for each point $q \in M$ the set $A_q^{-1} \circ f(U_{r,q})$ is the graph of a differentiable function $u : B_r \to \mathbb{R}^k$ with $\|Du\|_{C^0(B_r)} \leqslant \lambda$.

Here, for any $x \in B_r$ we have $Du(x) \in \mathbb{R}^{k\times m}$. In order to define the C^0-norm for Du, we have to fix a matrix norm for $Du(x)$. Of course all norms on $\mathbb{R}^{k\times m}$ are equivalent, therefore our results are true for any norm (possibly up to multiplication by some positive constant). Let us agree upon

$$\|A\| = \left(\sum_{j=1}^{m} |a_j|^2 \right)^{\frac{1}{2}}$$

for $A = (a_1, \ldots, a_m) \in \mathbb{R}^{k\times m}$. For this norm we have $\|A\|_{\mathrm{op}} \leqslant \|A\|$ for any $A \in \mathbb{R}^{k\times m}$ and the operator norm $\|\cdot\|_{\mathrm{op}}$. Hence the bound $\|Du\|_{C^0(B_r)} \leqslant \lambda$ directly implies that u is λ-Lipschitz. Moreover the norm $\|Du\|_{C^0(B_r)}$ does not depend on the choice of the isometry A_q.

2.2. *The notion of a generalized* (r, λ)*-immersion*

For any (r, λ)-immersion $f : M \to \mathbb{R}^n$ and any $q \in M$, we have a local graph representation over the affine tangent space $f(q) + \tau_f(q)$. It is natural to extend this definition to immersions with local graph representations over other appropriately chosen m-spaces in $\mathbb{R}^n$.

For a given $q \in M$ and a given m-space $E \in G_{n,m}$ let $A_{q,E} : \mathbb{R}^n \to \mathbb{R}^n$ be a Euclidean isometry, which maps the origin to $f(q)$, and the subspace $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^k$ onto $f(q) + E$.

Let $U_{r,q}^E \subset M$ be the q-component of the set $(\pi \circ A_{q,E}^{-1} \circ f)^{-1}(B_r)$. Again the isometry $A_{q,E}$ is not uniquely determined but the set $U_{r,q}^E$ does not depend on the choice of $A_{q,E}$.

Definition 2.2. An immersion f is called a generalized (r, λ)-immersion, if for each point $q \in M$ there is an $E = E(q) \in G_{n,m}$, such that the set $A_{q,E}^{-1} \circ f(U_{r,q}^E)$ is the graph of a differentiable function $u : B_r \to \mathbb{R}^k$ with $\|Du\|_{C^0(B_r)} \leqslant \lambda$.

Obviously every (r,λ)-immersion is a generalized (r,λ)-immersion, as we can choose $E(q)=\tau_f(q)$ for any $q\in M$.

For fixed dimension m and codimension k we denote by $\mathfrak{F}^1(r,\lambda)$ the set of generalized (r,λ)-immersions $f:M\to\mathbb{R}^{m+k}$ with $0\in f(M)$, where M is any compact m-manifold without boundary. For $\mathcal{V}>0$ we denote by $\mathfrak{F}^1_{\mathcal{V}}(r,\lambda)$ the set of all immersions in $\mathfrak{F}^1(r,\lambda)$ with $\mathrm{vol}(M)\leqslant\mathcal{V}$. Here the volume of M is measured with respect to the volume measure induced by the metric f^*g_{eucl}. Note that M is *not* fixed in these sets (in order to obtain a set in a strict set theoretical sense one may consider every manifold as embedded in $\mathbb{R}^N$ for an $N=N(m)$). The condition $0\in f(M)$ can be weakened in many applications to $f(M)\cap K\neq\emptyset$ for a compact set $K\subset\mathbb{R}^{m+k}$.

The notion of a generalized (r,λ)-immersion has one major advantage: As the definition does not make use of the existence of a tangent space, it allows us to define similar notions for functions into $\mathbb{R}^n$ which are not immersed. For a given $E\in G_{n,m}$ the set $U^E_{r,q}$ can be defined for *any* continuous function $f:M\to\mathbb{R}^n$. Moreover the condition $\|Du\|_{C^0(B_r)}\leqslant\lambda$ in the smooth case corresponds to a Lipschitz bound of the function u. Hence the following definition can be seen as the natural generalization to continuous functions:

Definition 2.3. A continuous function f is called an (r,λ)-function, if for each point $q\in M$ there is an $E=E(q)\in G_{n,m}$, such that the set $A^{-1}_{q,E}\circ f(U^E_{r,q})$ is the graph of a Lipschitz continuous function $u:B_r\to\mathbb{R}^k$ with Lipschitz constant λ.

We additionally assume here, that E can be chosen such that f is injective on $U^E_{r,q}$. This property is *not* implied by the preceding definition, if one reads the latter word for word.

We shall always consider (r,λ)-functions defined on compact topological manifolds (without boundary). Using the local Lipschitz graph representation, any such manifold can be endowed with an atlas with bi-Lipschitz change of coordinates. If the Lipschitz constant of the graphs is sufficiently small (and hence the coordinate changes are almost isometric with bi-Lipschitz constant close to 1), by the results in [13] there exists even a smooth atlas. In our case, the limit manifold both in Theorems 1.1 and 1.3 will be smooth.

Finally, we define the set $\mathfrak{F}^0(r,\lambda)$ by replacing generalized (r,λ)-immersions in $\mathfrak{F}^1(r,\lambda)$ by (r,λ)-functions.

2.3. Geometry of Grassmann manifolds

For $k,n\in\mathbb{N}$ with $0<k<n$ let $G_{n,k}$ again be the set of (non-oriented) k-dimensional subspaces of $\mathbb{R}^n$.

The set $G_{n,k}$ may be endowed with the structure of a differentiable $k(n-k)$-dimensional manifold, see e.g. [15]. Moreover there is a Riemannian metric g on $G_{n,k}$ being invariant under the action of $\mathbb{O}(n)$ in $\mathbb{R}^n$. It is unique up to multiplication by a positive constant (and — again up to multiplication by a positive constant — the only metric being invariant under the action of $\mathbb{SO}(n)$ in $\mathbb{R}^n$ except for the case $G_{4,2}$). For more details we refer the reader to [16].

In general, if (M,g) is a Riemannian manifold, the induced distance on M is defined by

$$d(p,q)=\inf\bigl\{L(\gamma)\bigm|\gamma:[a,b]\to M\text{ piecewise smooth curve with }\gamma(a)=p,\ \gamma(b)=q\bigr\}.\tag{2.3}$$

Here $L(\gamma):=\int_a^b|\frac{d\gamma}{dt}(t)|\,dt$ denotes the length of γ. If M is complete, by the theorem of Hopf–Rinow any two points $p,q\in M$ can be joined by a geodesic of length $d(p,q)$. This applies to the Grassmannian as $G_{n,k}$ is complete.

Now suppose that $E,G\in G_{n,k}$ are two close k-planes; this means that the projection of each onto the other is non-degenerate. Applying a transformation to principal axes, there are orthonormal bases $\{v_1,\dots,v_k\}$ of E and $\{w_1,\dots,w_k\}$ of G such that

$$\langle v_i,w_j\rangle=\delta_{ij}\cos\theta_i\quad\text{with }\theta_i\in\left[0,\frac{\pi}{2}\right)$$

for $1\leqslant i,j\leqslant k$. For given k-spaces E and G, the $\theta_1,\dots,\theta_k$ are uniquely determined (up to the order) and called the *principal angles* between E and G. Under all metrics on $G_{n,k}$ being invariant under the action of $\mathbb{O}(n)$, there is exactly one metric g with

$$d(E,G)=\left(\sum_{i=1}^k\theta_i^2\right)^{\frac{1}{2}}$$

for all close k-planes E and G, where d denotes the distance corresponding to g, and $\theta_1, \dots, \theta_k$ the principal angles between E and G as defined above; see [2] and the references given there. We shall always use this distinguished metric.

We will need the following estimate for the sectional curvatures of a Grassmannian:

Lemma 2.4. *Let* $\max\{k, n-k\} \geqslant 2$. *Let* $K(\cdot,\cdot)$ *denote the sectional curvature of* $G_{n,k}$ *and let* $X, Y \in T_P G_{n,k}$ *be linearly independent tangent vectors for a* $P \in G_{n,k}$. *Then*

$$0 \leqslant K(X,Y) \leqslant 2.$$

Proof. For $\min\{k, n-k\} = 1$ all sectional curvatures are constant with $K(X,Y) = 1$. For a proof see [16, p. 351]. For $\min\{k, n-k\} \geqslant 2$ we have $0 \leqslant K(X,Y) \leqslant 2$ by [17, Theorem 3]. □

The injectivity radius of $G_{n,k}$ is $\frac{\pi}{2}$ (see [2, p. 53]). A subset U of a Riemannian manifold (M, g) is said to be *convex,* if and only if for each $p, q \in U$ the shortest geodesic from p to q is unique in M and lies entirely in U. For the Grassmannian $G_{n,k}$, any open Riemannian ball $B_\varrho(P)$ around $P \in G_{n,k}$ with $\varrho < \frac{\pi}{4}$ is convex; see [8, p. 228].

2.4. The Riemannian center of mass

The well-known Euclidean center of mass may be generalized to a *Riemannian center of mass* on Riemannian manifolds. This was introduced by K. Grove and H. Karcher in [9]. A simplified treatment is given in [13]. See also [11]. We like to give a short sketch of this concept.

Let (M, g) be a complete Riemannian manifold with induced distance d as in (2.3). Let μ be a probability measure on M, i.e. a nonnegative measure with

$$\mu(M) = \int_M d\mu = 1.$$

Let q be a point in M and $B_\varrho = B_\varrho(q)$ a convex open ball of radius ϱ around q in M. Suppose

$$\operatorname{spt}\mu \subset B_\varrho,$$

where $\operatorname{spt}\mu$ denotes the support of μ. We define a function

$$P : \overline{B}_\varrho \to \mathbb{R},$$
$$P(p) = \int_M d(p,x)^2 \, d\mu(x).$$

Definition 2.5. A $q \in \overline{B}_\varrho$ is called a center of mass for μ if

$$P(q) = \inf_{p \in \overline{B}_\varrho} \int_M d(p,x)^2 \, d\mu(x).$$

The following theorem asserts the existence and uniqueness of a center of mass:

Theorem 2.6. *If the sectional curvatures of* M *in* B_ϱ *are at most* κ *with* $0 < \kappa < \infty$ *and if* ϱ *is small enough such that* $\varrho < \frac{1}{4}\pi\kappa^{-1/2}$, *then* P *is a strictly convex function on* B_ϱ *and has a unique minimum point in* $\overline{B}_\varrho$ *which lies in* B_ϱ *and is the unique center of mass for* μ.

Proof. See [13, Theorem 1.2] and the following pages there. □

In the preceding theorem, we do not require the bound κ to be attained; in particular all sectional curvatures are also allowed to be less than or equal to 0. The same applies to the following lemma:

Lemma 2.7. *Assume that the sectional curvatures of M in B_ϱ are at most κ with $0<\kappa<\infty$ and $\varrho<\frac{1}{4}\pi\kappa^{-1/2}$. Let μ_1,μ_2 be two probability measures on M with $\operatorname{spt}\mu_1\subset B_\varrho$, $\operatorname{spt}\mu_2\subset B_\varrho$ with centers of mass q_1,q_2 respectively. Then for a universal constant $C=C(\kappa,\varrho)<\infty$*

$$d(q_1,q_2)\leqslant C\int\limits_M d(q_2,x)\,d|\mu_1-\mu_2|(x),$$

where $|\mu_1-\mu_2|$ denotes the total variation measure of the signed measure $\mu_1-\mu_2$.

Proof. Let $P_i(p)=\frac{1}{2}\int_M d(p,x)^2\,d\mu_i(x)$ for $i=1,2$. By Theorem 1.5.1 in [13], with

$$C=C(\kappa,\varrho):=1+\left(\kappa^{1/2}\varrho\right)^{-1}\tan\left(2\kappa^{1/2}\varrho\right), \tag{2.4}$$

we have for all $y\in B_\varrho$ the estimate

$$d(q_1,y)\leqslant C\left|\operatorname{grad}P_1(y)\right|.$$

Using $\operatorname{spt}\mu_i\subset B_\varrho$, by Theorem 1.2 in [13] we have

$$\operatorname{grad}P_i(y)=-\int\limits_{B_\varrho}\exp_y^{-1}(x)\,d\mu_i(x), \tag{2.5}$$

where $\exp_y^{-1}:B_\varrho\to T_yM$ is considered as a vector valued function.

Moreover, as q_2 is a center of mass,

$$\operatorname{grad}P_2(q_2)=0.$$

Then by the arguments of [11, Lemma 4.8.7] (where manifolds of nonpositive sectional curvature are considered), we have

$$\begin{aligned}d(q_1,q_2)&\leqslant C\left|\operatorname{grad}P_1(q_2)\right|\\&=C\left|\int\limits_{B_\varrho}\exp_{q_2}^{-1}(x)\,d\mu_1(x)\right|\\&=C\left|\int\limits_{B_\varrho}\exp_{q_2}^{-1}(x)\,d\mu_1(x)-\int\limits_{B_\varrho}\exp_{q_2}^{-1}(x)\,d\mu_2(x)\right|\\&\leqslant C\int\limits_M d(q_2,x)\,d|\mu_1-\mu_2|(x),\end{aligned}$$

where we used $|\exp_{q_2}^{-1}(x)|=d(q_2,x)$ and $\operatorname{spt}\mu_i\subset B_\varrho$ in the last line. $\square$

2.5. Basics for the proof

We like to fix some further notation and to deduce some basic facts that are needed in the proof.

First of all let us simplify the notation. For a given (r,λ)-immersion $f:M\to\mathbb{R}^{m+1}$ and for every $q\in M$ we can choose an $E_q\in G_{m+1,m}$ with the properties of Definition 2.2. This yields a mapping $\mathcal{E}:M\to G_{m+1,m}$, $q\mapsto E_q$. For every (r,λ)-immersion we choose and fix such a mapping $\mathcal{E}$. So every given (r,λ)-immersion f can be thought of as a pair $(f,\mathcal{E})$, even if $\mathcal{E}$ is not explicitly mentioned in the notation. With $A_{q,\mathcal{E}(q)}$ and $U_{r,q}^{\mathcal{E}(q)}$ as in Definition 2.2, we set

$$A_q:=A_{q,\mathcal{E}(q)}$$

and for $0<\varrho\leqslant r$

$$U_{\varrho,q}:=U_{\varrho,q}^{\mathcal{E}(q)}.$$

However, all properties shown below for $U_{\varrho,q}$ are true for any admissible choice of $\mathcal{E}$.

As an analogue to Lemma 3.1 in [14] we obtain the following statement, where f is assumed to be a *generalized* (r, λ)-immersion here:

Lemma 2.8. *Let $f : M \to \mathbb{R}^{m+1}$ be an (r, λ)-immersion and $p, q \in M$.*

a) *If $0 < \varrho \leqslant r$ and $p \in U_{\varrho,q}$, then $|f(q) - f(p)| < (1 + \lambda)\varrho$.*
b) *If $0 < \varrho \leqslant r$ and $\delta = [3(1 + \lambda)]^{-1}\varrho$ and $U_{\delta,q} \cap U_{\delta,p} \neq \emptyset$, then $U_{\delta,p} \subset U_{\varrho,q}$.*

Proof. a) Pass to the graph representation, use the bound on the C^0-norm of the derivative of the graph and the triangular inequality.

b) Let $x \in U_{\delta,p}$ and $y \in U_{\delta,q} \cap U_{\delta,p}$. With $\varphi_q := \pi \circ A_q^{-1} \circ f$ we have

$$\begin{aligned}\left|\varphi_q(x)\right| &\leqslant \left|f(x) - f(q)\right| \\ &\leqslant \left|f(x) - f(p)\right| + \left|f(p) - f(y)\right| + \left|f(y) - f(q)\right| \\ &< 3(1 + \lambda)\delta \\ &= \varrho.\end{aligned}$$

Hence $U_{\delta,p} \subset \varphi_q^{-1}(B_\varrho)$. But $U_{\delta,p} \cup U_{\delta,q}$ is a connected set containing q, hence included in the q-component of $\varphi_q^{-1}(B_\varrho)$, that is in $U_{\varrho,q}$. We conclude $U_{\delta,p} \subset U_{\varrho,q}$. □

Now let $r, \lambda > 0$ be given. For $l \in \mathbb{N}_0$ define $\delta_l := [3(1 + \lambda)]^{-l} r$. For an (r, λ)-immersion $f : M \to \mathbb{R}^{m+1}$, by Lemma 2.8 b) we have the following important property:

$$\text{If } p, q \in M \text{ and } U_{\delta_{l+1},q} \cap U_{\delta_{l+1},p} \neq \emptyset, \text{ then } U_{\delta_{l+1},p} \subset U_{\delta_l,q}. \tag{2.6}$$

If $f : M \to \mathbb{R}^{m+1}$ is an (r, λ)-immersion and $p \in M$, we may use the local graph representation to conclude that the set $f(U_{r,p})$ is homeomorphic to the ball B_r. Hence we may choose a continuous unit normal $\nu_p : U_{r,p} \to \mathbb{S}^m$ with respect to $f|U_{r,p}$. If $q \in M$ is another point and $\nu_q : U_{r,q} \to \mathbb{S}^m$ a continuous unit normal on $U_{r,q}$, we note that ν_p and ν_q do not necessarily coincide on $U_{r,p} \cap U_{r,q}$. However, we have the following statement:

Lemma 2.9. *Let $f : M \to \mathbb{R}^{m+1}$ be an (r, λ)-immersion and $p, q \in M$. Let $\nu_p : U_{\delta_1,p} \to \mathbb{S}^m$, $\nu_q : U_{\delta_1,q} \to \mathbb{S}^m$ be continuous unit normals. Suppose $U_{\delta_1,p} \cap U_{\delta_1,q} \neq \emptyset$. Then exactly one of the following two statements is true*:

- *$\nu_p(x) = \nu_q(x)$ for every $x \in U_{\delta_1,p} \cap U_{\delta_1,q}$,*
- *$\nu_p(x) = -\nu_q(x)$ for every $x \in U_{\delta_1,p} \cap U_{\delta_1,q}$.*

Proof. Choose a $\xi \in U_{\delta_1,p} \cap U_{\delta_1,q}$. First suppose that $\nu_p(\xi) = \nu_q(\xi)$. As $U_{r,p}$ is homeomorphic to B_r and connected, there are exactly two continuous unit normals on $U_{r,p}$. Let ν be the one with $\nu(\xi) = \nu_p(\xi)$. Let $W = \{x \in U_{\delta_1,p}\colon \nu(x) = \nu_p(x)\}$. Then W is a nonempty subset of the connected set $U_{\delta_1,p}$. Moreover W is easily seen to be open and closed in $U_{\delta_1,p}$. Therefore $W = U_{\delta_1,p}$ and $\nu_p = \nu$ on $U_{\delta_1,p}$. As $U_{\delta_1,q} \subset U_{r,p}$ by (2.6), the preceding argumentation can also be applied to ν_q. With $\nu(\xi) = \nu_p(\xi) = \nu_q(\xi)$ we conclude $\nu_q = \nu$ on $U_{\delta_1,q}$. Hence $\nu_p = \nu = \nu_q$ on $U_{\delta_1,p} \cap U_{\delta_1,q}$, as in the claim above. If $\nu_p(\xi) = -\nu_q(\xi)$, a similar argument yields $\nu_p = -\nu_q$ on $U_{\delta_1,p} \cap U_{\delta_1,q}$. □

Remark 2.10. The statement of the preceding lemma might seem to be obvious at first sight. However one can think of a Möbius strip covered by two open sets U and V, each of which is homeomorphic to B_r, such that $U \cap V$ has exactly two components. If we choose continuous unit normals ν_1, ν_2 on U, V respectively, we have $\nu_1 = \nu_2$ on one of the components, and $\nu_1 = -\nu_2$ on the other. Such a behavior of the normals is excluded by Lemma 2.9, irrespective whether $U_{\delta_1,p} \cap U_{\delta_1,q}$ is connected or not.

We need the notion of a δ-net:

Definition 2.11. Let $Q = \{q_1, \ldots, q_s\}$ be a finite set of points in M and let $0 < \delta < r$. We say that Q is a δ-net for f, if $M = \bigcup_{j=1}^{s} U_{\delta,q_j}$.

Note that every δ-net is also a δ'-net if $0<\delta<\delta'<r$.

The following statement is a bit stronger than Lemma 3.2 in [14]. It bounds the number of elements in a δ-net by an argumentation similar to that in the proof of Vitali's covering theorem. Simultaneously, similarly to Besicovitch's covering theorem, it gives a bound (which does not depend on the volume) how often any fixed point in M is covered by the net. More precisely, we have the following lemma:

Lemma 2.12. *For $l\in\mathbb{N}$, every (r,λ)-immersion on a compact m-manifold M admits a δ_l-net Q with*

$$|Q|\leqslant \delta_{l+1}^{-m}\operatorname{vol}(M),$$
$$\bigl|\{q\in Q\colon\ p\in U_{\delta_2,q}\}\bigr|\leqslant\bigl[3(1+\lambda)\bigr]^{(l+1)m}\quad \textit{for every fixed } p\in M.$$

Proof. Let $q_1\in M$ be an arbitrary point. Assume we have found points $\{q_1,\dots,q_\nu\}$ in M with the property $U_{\delta_{l+1},q_j}\cap U_{\delta_{l+1},q_k}=\emptyset$ for $j\neq k$. Suppose $U_{\delta_l,q_1}\cup\dots\cup U_{\delta_l,q_\nu}$ does not cover M. Then choose a point $q_{\nu+1}$ from the complement. Then $U_{\delta_{l+1},q_k}\cap U_{\delta_{l+1},q_{\nu+1}}=\emptyset$ for $k\leqslant\nu$, as otherwise $U_{\delta_{l+1},q_{\nu+1}}\subset U_{\delta_l,q_k}$ by (2.6). As

$$\begin{aligned}\operatorname{vol}(M)&\geqslant\sum_{j=1}^{s}\operatorname{vol}(U_{\delta_{l+1},q_j})\\&\geqslant\sum_{j=1}^{s}\mathcal{L}^m(B_{\delta_{l+1}})\\&\geqslant s\delta_{l+1}^m,\end{aligned}$$

this procedure yields after at most $\delta_{l+1}^{-m}\operatorname{vol}(M)$ steps a cover.

For the second relation let $p\in M$. Let $Q=\{q_1,\dots,q_s\}$ be the net that we found above. Moreover let $Z(p)=\{q\in Q\colon\ p\in U_{\delta_2,q}\}$. By Lemma 2.8 b) we have

$$\bigcup_{q\in Z(p)}U_{\delta_2,q}\subset U_{\delta_1,p}.$$

Hence we may estimate as above

$$\begin{aligned}\operatorname{vol}(U_{\delta_1,p})&\geqslant\sum_{q\in Z(p)}\operatorname{vol}(U_{\delta_{l+1},q})\\&\geqslant\bigl|Z(p)\bigr|\delta_{l+1}^m\mathcal{L}^m(B_1).\end{aligned}\tag{2.7}$$

As the immersion is an (r,λ)-immersion, we have

$$\operatorname{vol}(U_{\delta_1,p})\leqslant(1+\lambda)^m\delta_1^m\mathcal{L}^m(B_1).\tag{2.8}$$

Combining (2.7) and (2.8), we estimate

$$\begin{aligned}\bigl|Z(p)\bigr|&\leqslant(1+\lambda)^m\delta_1^m\delta_{l+1}^{-m}\\&=3^{lm}(1+\lambda)^{(l+1)m},\end{aligned}$$

which implies the statement. $\square$

We would like to emphasize that the second estimate in the preceding lemma does not depend on the volume $\operatorname{vol}(M)$. This will be necessary in order to obtain estimates for Lipschitz constants and for angles between different spaces depending only on λ but not on $\operatorname{vol}(M)$.

Definition 2.13. Let $f:M\to\mathbb{R}^{m+1}$ be an (r,λ)-immersion. Let $l\in\mathbb{N}$ and let $Q=\{q_1,\dots,q_s\}$ be a δ_l-net for f. For $\iota\in\{0,1,\dots,l\}$ and $j\in\{1,\dots,s\}$ we define

$$Z_\iota(j):=\{1\leqslant k\leqslant s\colon\ U_{\delta_\iota,q_j}\cap U_{\delta_\iota,q_k}\neq\emptyset\}.$$

For $\nu_1, \nu_2 \in \mathbb{R}^{m+1}\backslash\{0\}$ let $\sphericalangle(\nu_1, \nu_2)$ denote the non-oriented angle between ν_1 and ν_2, that is

$$0 \leqslant \sphericalangle(\nu_1, \nu_2) \leqslant \pi,$$
$$\sphericalangle(\nu_1, \nu_2) = \arccos \frac{\langle \nu_1, \nu_2 \rangle}{|\nu_1||\nu_2|}.$$

We consider the metric space $(\mathbb{S}^m, d)$, where $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ is the m-dimensional unit sphere and d the intrinsic metric on $\mathbb{S}^m$, that is

$$d(\cdot,\cdot) = \sphericalangle(\cdot,\cdot). \tag{2.9}$$

For $A \subset \mathbb{S}^m$ and $x \in \mathbb{S}^m$ let $\operatorname{dist}(x, A) = \inf\{d(x, y)\colon\ y \in A\}$. For $\varrho > 0$ let $B_\varrho(A) = \{x \in \mathbb{S}^m\colon\ \operatorname{dist}(x, A) < \varrho\}$. Moreover let $\mathcal{S} \subset \mathcal{P}(\mathbb{S}^m)$ denote the set of closed nonempty subsets of $\mathbb{S}^m$. We denote by $d_{\mathcal{H}}$ the Hausdorff metric on $\mathcal{S}$, given by

$$d_{\mathcal{H}} : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_{\geqslant 0},$$
$$(S_1, S_2) \mapsto \inf\bigl\{\varrho > 0\colon\ S_1 \subset B_\varrho(S_2),\ S_2 \subset B_\varrho(S_1)\bigr\}.$$

We will need the following well-known version of the theorem of Arzelà–Ascoli for the Hausdorff metric (see [1, p. 125]):

Lemma 2.14. *Let (X, d) be a compact metric space and $\mathcal{A}$ the set of closed nonempty subsets of X. Then $(\mathcal{A}, d_{\mathcal{H}})$ is compact, i.e. every sequence in $\mathcal{A}$ has a subsequence that converges to an element in $\mathcal{A}$.*

We will have to estimate the size of some tubular neighborhoods. To do this we need to introduce some more notation. Suppose we are given $\varrho > 0$ and $u \in C^1(\overline{B}_\varrho)$ with $\|Du\|_{C^0(B_\varrho)} \leqslant \lambda$. Moreover let $T \in C^1(\overline{B}_\varrho, \mathbb{R}^{m+1})$ be a mapping with $|T(x)| = 1$ for all $x \in B_\varrho$. Suppose that T is L-Lipschitz for an L with $0 < L < \infty$. Let $\omega : \overline{B}_\varrho \to G_{m+1,1}$, $q \mapsto \operatorname{span}\{T(q)\}$. Finally, let $\nu : B_\varrho \to \mathbb{S}^m$ be a continuous unit normal with respect to the graph $x \mapsto (x, u(x))$. We consider a vector bundle E over B_ϱ, given by

$$E = \bigl\{(x, y) \in B_\varrho \times \mathbb{R}^{m+1}\colon\ y \in \omega(x)\bigr\}.$$

For $\varepsilon > 0$ let

$$E^\varepsilon = \bigl\{(x, y) \in E\colon\ |y| < \varepsilon\bigr\} \subset E.$$

Moreover we define a mapping

$$F : E \to \mathbb{R}^{m+1},$$
$$(x, y) \mapsto \bigl(x, u(x)\bigr) + y, \tag{2.10}$$

where $y \in \omega(x)$.

Lemma 2.15 *(Size of tubular neighborhoods). Let $\gamma < \frac{\pi}{2}$ and $\varepsilon = \frac{1}{L}\cos\gamma$. With the notation as above, assume that*

$$\sphericalangle\bigl(T(p), \nu(q)\bigr) \leqslant \gamma \quad \text{for every } p, q \in B_\varrho. \tag{2.11}$$

Then the following are true:

a) *The mapping $F|E^\varepsilon$ is a diffeomorphism onto an open neighborhood of $\{(x, u(x)) \in \mathbb{R}^m \times \mathbb{R}\colon\ x \in B_\varrho\}$.*
b) *Let $\sigma := \min\{\frac{\varrho}{2}\cos\gamma, \frac{\cos^2\gamma}{2L(1+\lambda)}\}$. Then*

$$B_\sigma\bigl(\bigl\{\bigl(x, u(x)\bigr) \in \mathbb{R}^m \times \mathbb{R}\colon\ x \in \overline{B}_{\frac{\varrho}{2}}\bigr\}\bigr) \subset F\bigl(E^\varepsilon\bigr),$$

where $B_\sigma(A) = \{x \in \mathbb{R}^{m+1}\colon\ \operatorname{dist}(x, A) < \sigma\}$ for $A \subset \mathbb{R}^{m+1}$ with dist *the Euclidean distance.*

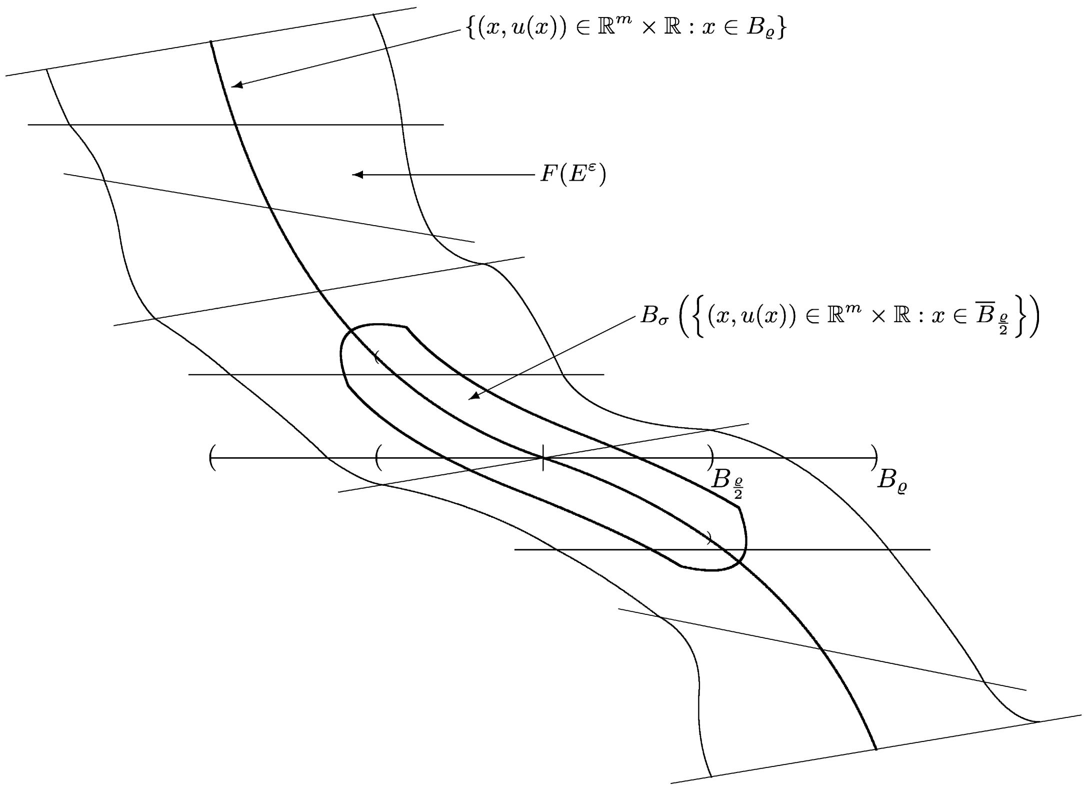

Fig. 2.1. Tubular neighborhood around a graph.

The trivial but long proof is carried out in detail in Appendix A. (See Fig. 2.1.)

Finally we like to define a metric for graph systems. First of all let

$$\mathfrak{G}^s = \left\{(A_j, u_j)_{j=1}^s \colon\ A_j : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1} \text{ is a Euclidean isometry, } u_j \in C^1(\overline{B}_r)\right\}.$$

Every Euclidean isometry $A : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ splits uniquely into a rotation $R \in \mathbb{SO}(m+1)$ and a translation $T \in \mathbb{R}^{m+1}$. If $\|\cdot\|$ denotes the operator norm and if $\Gamma = (A_j, u_j)_{j=1}^s \in \mathfrak{G}^s$, $\tilde{\Gamma} = (\tilde{A}_j, \tilde{u}_j)_{j=1}^s \in \mathfrak{G}^s$, we set

$$\begin{aligned}
&\mathfrak{d}(\cdot,\cdot) : \mathfrak{G}^s \times \mathfrak{G}^s \to \mathbb{R},\\
&\mathfrak{d}(\Gamma, \tilde{\Gamma}) = \sum_{j=1}^{s} \left(\|R_j - \tilde{R}_j\| + |T_j - \tilde{T}_j| + \|u_j - \tilde{u}_j\|_{C^0(B_r)}\right).
\end{aligned} \tag{2.12}$$

This makes $(\mathfrak{G}^s, \mathfrak{d})$ a metric space.

3. Transversality and tubular neighborhoods

In this section we like to construct lines in $\mathbb{R}^{m+1}$, that intersect each (appropriately restricted) immersion f^i transversally — even in the case, that the Lipschitz constant λ of the graph functions is large. This yields local tubular neighborhoods around f^i and is the crucial step in the proof.

Let $r > 0$ and $\lambda, \mathcal{V} < \infty$. Let $f^i : M^i \to \mathbb{R}^{m+1}$ be a sequence of (r, λ)-immersions as in Theorem 1.1. By Lemma 2.12 we can choose δ_5-nets $Q^i = \{q_1^i, \ldots, q_{s^i}^i\}$ for M^i with $s^i \leqslant \delta_6^{-m} \operatorname{vol}(M^i)$ and with

$$\left|\left\{q \in Q^i \colon\ p \in U^i_{\delta_2, q}\right\}\right| \leqslant \left[3(1+\lambda)\right]^{6m} \quad \text{for every fixed } p \in M^i. \tag{3.1}$$

As $\operatorname{vol}(M^i) \leqslant \mathcal{V}$, we may pass to a subsequence such that each net has exactly s points for a fixed $s \in \mathbb{N}$.

For every $i \in \mathbb{N}$, $\iota \in \{0, 1, \dots, 5\}$ and $j \in \{1, \dots, s\}$ we have

$$Z_\iota^i(j) \subset \mathcal{P}\big(\{1, \dots, s\}\big).$$

Hence, by successively passing to subsequences, we may assume that $Z_\iota^i(j)$ does not depend on i. Denote it by $Z_\iota(j)$.

To simplify the notation, for $0 < \varrho \leqslant r$ we set $U_{\varrho,j}^i := U_{\varrho,q_j^i}^i$.

Moreover, we choose for every $i \in \mathbb{N}$ and every $j \in \{1, \dots, s\}$ a continuous unit normal $\nu_j^i : U_{r,j}^i \to \mathbb{S}^m$ with respect to $f^i | U_{r,j}^i$. Let these normal mappings be fixed from now on.

We set

$$S_j^i := \overline{\nu_j^i\big(U_{\delta_1,j}^i\big)} \subset \mathbb{S}^m,$$

where the closure is taken with respect to the metric defined in (2.9).

For each fixed j, this yields a sequence $(S_j^i)_{i\in\mathbb{N}}$ in $\mathcal{S}$. By Lemma 2.14, passing to a further subsequence (if need be), we can assume

$$S_j^i \to S_j' \quad \text{in } (\mathcal{S}, d_{\mathcal{H}}) \text{ as } i \to \infty$$

for each fixed $j \in \{1, \dots, s\}$, where $S_j' \in \mathcal{S}$. In particular for every j

$$\big(S_j^i\big)_{i\in\mathbb{N}} \quad \text{is a Cauchy sequence in } (\mathcal{S}, d_{\mathcal{H}}). \tag{3.2}$$

By (3.2) we may choose another subsequence such that for every j

$$d_{\mathcal{H}}\big(S_j^k, S_j^l\big) < \frac{\pi}{4} - \frac{1}{2}\arctan\lambda \quad \text{for all } k, l \in \mathbb{N}. \tag{3.3}$$

To each $q_j^i \in Q^i$ we may assign a neighborhood $U_{r,j}^i$, a Euclidean isometry A_j^i and a differentiable function $u_j^i : B_r \to \mathbb{R}$ as in Definition 2.2. This yields the corresponding graph systems $\Gamma^i = (A_j^i, u_j^i)_{j=1}^s \in \mathfrak{G}^s$. As $\|Du_j^i\|_{C^0(B_r)} \leqslant \lambda$ and as $f^i(M^i)$ is uniformly bounded, a subsequence of $(\Gamma^i)_{i\in\mathbb{N}}$ converges in $(\mathfrak{G}^s, \mathfrak{d})$. In particular

$$\big(\Gamma^i\big)_{i\in\mathbb{N}} \quad \text{is a Cauchy sequence in } \big(\mathfrak{G}^s, \mathfrak{d}\big). \tag{3.4}$$

Let constants L, γ and σ be defined by

$$L := \big[3(1+\lambda)\big]^{6m+4} r^{-1}, \tag{3.5}$$

$$\gamma := \frac{\pi}{4} + \frac{1}{2}\arctan\lambda, \tag{3.6}$$

$$\sigma := \frac{\cos^2\gamma}{2L(1+\lambda)}. \tag{3.7}$$

By (3.4) we may pass to another subsequence such that

$$\mathfrak{d}\big(\Gamma^k, \Gamma^l\big) < \big[3(1+\lambda)(1+r)\big]^{-1}\sigma \quad \text{for all } k, l \in \mathbb{N}. \tag{3.8}$$

For $i = 1$ we sometimes suppress the index 1 and write for instance q_j and u_j instead of q_j^1 and u_j^1. For the immersion f^1, let $\mathcal{E}^1 : M^1 \to G_{m+1,m}$ be a mapping as explained above. We set $E_j := \mathcal{E}^1(q_j^1) \in G_{m+1,m}$ (this means E_j is an m-space for the point $q_j^1 \in M^1$ as in Definition 2.2).

Our next task is to find a mapping $\omega : M^1 \to G_{m+1,1}$, which defines the direction in which we project from $f^1(M^1)$ onto $f^i(M^i)$ in order to construct diffeomorphisms $\phi^i : M^1 \to M^i$. First we would like to give a local construction. In Lemma 3.5 we will show that ω is even globally well defined. The construction is similar to that in [14], but more involved.

We choose a C^∞-function $g : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ with the following properties:

- $g(t) = 1$ for $t < \frac{\delta_1}{r}$,
- $0 \leqslant g(t) \leqslant 1$ for $t \in [\frac{\delta_1}{r}, 1]$,

- $g(t) = 0$ for $t > 1$,
- $-2 \leqslant g'(t) \leqslant 0$ for all $t > 0$.

We note that $\frac{\delta_1}{r} = [3(1+\lambda)]^{-1} \leqslant \frac{1}{3}$, hence such a function g exists.

Let

$$\begin{aligned} Z : M^1 &\to \mathcal{P}(\{1, \dots, s\}), \\ q &\mapsto \{1 \leqslant k \leqslant s \colon q \in U^1_{\delta_2,k}\}. \end{aligned}$$

By (3.1) we have

$$\left|Z(q)\right| \leqslant \left[3(1+\lambda)\right]^{6m} \quad \text{for every } q \in M^1. \tag{3.9}$$

For every $k \in \{1, \dots, s\}$ we choose a unit vector w_k that is perpendicular to the subspace E_k defined above. Let these vectors $w_1, \dots, w_s$ be fixed from now on.

Now let $j \in \{1, \dots, s\}$, $q \in U^1_{\delta_3,j}$ and $k \in Z(q)$. Lemma 2.8 b) yields

$$U^1_{\delta_1,j} \subset U^1_{r,k}.$$

In particular $f^1(U^1_{\delta_1,j})$ is the graph of a λ-Lipschitz function on a subset of E_k. This implies

$$\text{either} \quad \sphericalangle\big(w_k, \nu^1_j(q_j)\big) \leqslant \arctan\lambda \quad \text{or} \quad \sphericalangle\big(-w_k, \nu^1_j(q_j)\big) \leqslant \arctan\lambda. \tag{3.10}$$

Set

$$\nu_k := \begin{cases} w_k, & \text{if } \sphericalangle(w_k, \nu^1_j(q_j)) \leqslant \arctan\lambda, \\ -w_k, & \text{otherwise.} \end{cases} \tag{3.11}$$

If we replace the point q_j by any other point $p \in U^1_{\delta_1,j}$, the relation (3.10) will still be true. As ν^1_j is continuous and $U^1_{\delta_1,j}$ is connected, we easily conclude

$$\sphericalangle\big(\nu_k, \nu^1_j(p)\big) \leqslant \arctan\lambda \quad \text{for every } p \in U^1_{\delta_1,j}, \tag{3.12}$$

where ν_k is the fixed vector defined in (3.11). We finally define a function

$$\begin{aligned} S : U^1_{\delta_3,j} &\to \mathbb{R}^{m+1}, \\ q &\mapsto \sum_{k \in Z(q)} g\left(\frac{|f^1(q) - f^1(q_k)|}{\delta_2}\right)\nu_k. \end{aligned}$$

Lemma 3.1. *The following inequalities hold*:

a) $|S(q)| \geqslant (1+\lambda)^{-1}$ *for every* $q \in U^1_{\delta_3,j}$.
b) $\sphericalangle(S(q), \nu^i_j(p)) \leqslant \frac{\pi}{4} + \frac{1}{2}\arctan\lambda$ *for every* $q \in U^1_{\delta_3,j}$ *and every* $p \in U^i_{\delta_1,j}$.

Proof. a) Let $q \in U^1_{\delta_3,j}$. As Q^1 is a δ_4-net for f^1, there is a $k \in \{1, \dots, s\}$ with $q \in U^1_{\delta_4,k}$. By Lemma 2.8 a) we have $|f^1(q) - f^1(q_k)| < \delta_3$, hence

$$\frac{|f^1(q) - f^1(q_k)|}{\delta_2} < \frac{\delta_3}{\delta_2} = \frac{\delta_1}{r}.$$

By the definition of g this yields

$$g\left(\frac{|f^1(q) - f^1(q_k)|}{\delta_2}\right) = 1.$$

Now let $l \in Z(q)$. By (3.12) we have $\sphericalangle(\nu_l, \nu^1_j(q)) \leqslant \arctan\lambda$. Hence

$$\begin{aligned}
\langle \nu_l, \nu_j^1(q)\rangle &= |\nu_l|\,|\nu_j^1(q)|\cos\bigl(\sphericalangle(\nu_l, \nu_j^1(q))\bigr)\\
&\geqslant \cos(\arctan\lambda)\\
&= \bigl(1+\lambda^2\bigr)^{-\frac{1}{2}}\\
&\geqslant (1+\lambda)^{-1}.
\end{aligned}$$

We note that $q \in U^1_{\delta_4,k}$ in particular implies $k \in Z(q)$. Finally we estimate

$$\begin{aligned}
|S(q)| &\geqslant \langle S(q), \nu_j^1(q)\rangle\\
&= g\left(\frac{|f^1(q)-f^1(q_k)|}{\delta_2}\right)\langle \nu_k, \nu_j^1(q)\rangle + \sum_{l\in Z(q)\setminus\{k\}} g\left(\frac{|f^1(q)-f^1(q_l)|}{\delta_2}\right)\langle \nu_l, \nu_j^1(q)\rangle\\
&\geqslant \left(1+\sum_{l\in Z(q)\setminus\{k\}} g\left(\frac{|f^1(q)-f^1(q_l)|}{\delta_2}\right)\right)(1+\lambda)^{-1}\\
&\geqslant (1+\lambda)^{-1}.
\end{aligned}$$

b) Let $q \in U^1_{\delta_3,j}$ and $p \in U^i_{\delta_1,j}$. By (3.3) there is a $p' \in U^1_{\delta_1,j}$ with

$$\sphericalangle\bigl(\nu_j^1(p'), \nu_j^i(p)\bigr) \leqslant \frac{\pi}{4} - \frac{1}{2}\arctan\lambda. \tag{3.13}$$

By (3.12), every ν_k with $k \in Z(q)$ lies in the cone

$$C = \bigl\{v \in \mathbb{R}^{m+1}\setminus\{0\}\colon \sphericalangle\bigl(v, \nu_j^1(p')\bigr) \leqslant \arctan\lambda\bigr\}.$$

By the definition of S, also the non-zero vector $S(q)$ lies in C, i.e.

$$\sphericalangle\bigl(S(q), \nu_j^1(p')\bigr) \leqslant \arctan\lambda. \tag{3.14}$$

Using the triangular inequality, we conclude by (3.13) and (3.14) that

$$\sphericalangle\bigl(S(q), \nu_j^i(p)\bigr) \leqslant \frac{\pi}{4} + \frac{1}{2}\arctan\lambda. \qquad \square$$

By Lemma 3.1 a) the mapping S does not vanish on $U^1_{\delta_3,j}$. We define T by normalizing S, that is

$$\begin{aligned}
T : U^1_{\delta_3,j} &\to \mathbb{R}^{m+1},\\
q &\mapsto \frac{S(q)}{|S(q)|}.
\end{aligned}$$

Identifying $U^1_{\delta_3,j}$ with B_{δ_3} by means of the diffeomorphism $\pi\circ A_j^{-1}\circ f^1 : U^1_{\delta_3,j} \to B_{\delta_3}$, we may consider T and S as mappings defined on the ball B_{δ_3}. We show, that T considered as mapping on B_{δ_3} is Lipschitz with respect to the Euclidean norm:

Lemma 3.2. *The mapping $T : B_{\delta_3} \to \mathbb{R}^{m+1}$ is L-Lipschitz with $L = [3(1+\lambda)]^{6m+4}r^{-1}$.*

Proof. Let $x, y \in B_{\delta_3}$. Then there are unique $p, q \in U^1_{\delta_3,j}$ with $\pi\circ A_j^{-1}\circ f^1(p) = x$, $\pi\circ A_j^{-1}\circ f^1(q) = y$.

Let $k \in Z(p)\setminus Z(q)$. Then $p \in U^1_{\delta_3,j}\cap U^1_{\delta_2,k}$. Lemma 2.8 b) implies $U^1_{\delta_3,j} \subset U^1_{\delta_1,k}$, so in particular $q \in U^1_{\delta_1,k}$. Now assume $|f^1(q)-f^1(q_k)| < \delta_2$. With $\varphi_k = \pi\circ A_k^{-1}\circ f^1$ this implies $\varphi_k(q) \in B_{\delta_2}$. Hence $q \in U^1_{\delta_1,k}\cap\varphi_k^{-1}(B_{\delta_2}) = U^1_{\delta_2,k}$. But this contradicts $k \notin Z(q)$. Therefore $|f^1(q)-f^1(q_k)| \geqslant \delta_2$ and hence $g(\frac{|f^1(q)-f^1(q_k)|}{\delta_2}) = 0$ by the definition of g.

The same argument shows $g(\frac{|f^1(p)-f^1(q_l)|}{\delta_2}) = 0$ for all $l \in Z(q)\setminus Z(p)$.

Using the preceding considerations, $\|g'\|_{C^0(\mathbb{R}_{\geqslant 0})} \leqslant 2$ and $|Z(p)| \leqslant [3(1+\lambda)]^{6m}$, $|Z(q)| \leqslant [3(1+\lambda)]^{6m}$, we estimate as follows:

$$\begin{aligned}
\left|S(x)-S(y)\right| &= \left|\sum_{k\in Z(p)} g\left(\frac{|f^1(p)-f^1(q_k)|}{\delta_2}\right)\nu_k - \sum_{l\in Z(q)} g\left(\frac{|f^1(q)-f^1(q_l)|}{\delta_2}\right)\nu_l\right| \\
&= \left|\sum_{k\in Z(p)\cup Z(q)} \left[g\left(\frac{|f^1(p)-f^1(q_k)|}{\delta_2}\right) - g\left(\frac{|f^1(q)-f^1(q_k)|}{\delta_2}\right)\right]\nu_k\right| \\
&\leqslant \sum_{k\in Z(p)\cup Z(q)} \|g'\|_{C^0(\mathbb{R}_{\geqslant 0})}\left|\frac{|f^1(p)-f^1(q_k)|}{\delta_2} - \frac{|f^1(q)-f^1(q_k)|}{\delta_2}\right| \\
&\leqslant \sum_{k\in Z(p)\cup Z(q)} \frac{2}{\delta_2}\left|f^1(p)-f^1(q)\right| \\
&\leqslant 4\left[3(1+\lambda)\right]^{6m+2} r^{-1}\left|\left(x,u_j(x)\right)-\left(y,u_j(y)\right)\right| \\
&\leqslant 4\left[3(1+\lambda)\right]^{6m+2} r^{-1}(1+\lambda)|x-y|.
\end{aligned}$$

By Lemma 3.1 a) we have $|S(z)| \geqslant (1+\lambda)^{-1}$ for every $z \in U^1_{\delta_3,j}$. Hence

$$\begin{aligned}
\left|T(x)-T(y)\right| = \left|\frac{S(x)}{|S(x)|} - \frac{S(y)}{|S(y)|}\right| &\leqslant 4\left[3(1+\lambda)\right]^{6m+2}(1+\lambda)^2 r^{-1}|x-y| \\
&\leqslant \left[3(1+\lambda)\right]^{6m+4} r^{-1}|x-y|. \qquad \square
\end{aligned}$$

Remark 3.3. Of course, T is also Lipschitz as a mapping on $U^1_{\delta_3,j}$ with respect to the metric induced by f^1. The estimate of the Lipschitz constant gets even better in this case. Moreover, we note that in the preceding lemma L depends on r. However, we will see that the Lipschitz constant of $f^i \circ \phi^i$ does not depend on r in the end.

We set

$$\begin{aligned}
\omega : U^1_{\delta_3,j} &\to G_{m+1,1}, \\
q &\mapsto \operatorname{span}\{S(q)\},
\end{aligned}$$

which is well defined as $S(q) \neq 0$ by Lemma 3.1 a).

We like to explain how ω locally forms a tubular neighborhood around f^1:

For that we consider the mapping

$$\begin{aligned}
g_k : U^1_{\delta_2,k} &\to \mathbb{R}, \\
q &\mapsto g\left(\frac{|f^1(q)-f^1(q_k)|}{\delta_2}\right).
\end{aligned}$$

As g is smooth and $g(t)=0$ for $t \geqslant 1$, it is easily seen that g_k can be extended to a smooth function $\bar{g}_k : M^1 \to \mathbb{R}$ by setting $\bar{g}_k = 0$ outside $U^1_{\delta_2,k}$. This implies that $S : U^1_{\delta_3,j} \to \mathbb{R}^{m+1}$ is differentiable, even if the sum in the definition of S depends on $Z(q)$. Hence also $T = \frac{S}{|S|}$ is differentiable. Moreover Lemma 3.2 says that T is L-Lipschitz with $L = [3(1+\lambda)]^{6m+4} r^{-1}$ and by Lemma 3.1 b) we have

$$\sphericalangle\left(T(p), \nu^1_j(q)\right) \leqslant \frac{\pi}{4} + \frac{1}{2}\arctan\lambda \quad \text{for all } p, q \in U^1_{\delta_3,j}.$$

Finally, after a rotation and a translation, $f(U^1_{\delta_3,j})$ may be written as the graph of a C^1-function $u^1_j : B_{\delta_3} \to \mathbb{R}$. Let us introduce some more notation:

We consider a vector bundle $\hat{E}_j$ over $U^1_{\delta_3,j}$, given by

$$\hat{E}_j = \left\{(x,y) \in U^1_{\delta_3,j} \times \mathbb{R}^{m+1} \colon\ y \in \omega(x)\right\}$$

with bundle projection $\hat{\pi}$. We may identify the zero section of $\hat{E}_j$ with $U^1_{\delta_3,j}$. For $\varepsilon > 0$ let

$$E_j^\varepsilon = \{(x, y) \in \hat{E}_j\colon |y| < \varepsilon\} \subset \hat{E}_j.$$

Finally we define a mapping

$$\begin{aligned} F_j : \hat{E}_j &\to \mathbb{R}^{m+1}, \\ (x, y) &\mapsto f(x) + y, \end{aligned} \tag{3.15}$$

where $y \in \omega(x)$.

Lemma 3.4. *Let $\varepsilon = \frac{1}{L}\cos\gamma$, where L and γ are as in (3.5), (3.6). Then the following are true:*

- *$F_j|E_j^\varepsilon$ is a diffeomorphism onto an open neighborhood of $f^1(U^1_{\delta_3,j})$,*
- *$F_j|U^1_{\delta_3,j} = f^1|U^1_{\delta_3,j}$,*
- *for each fibre $\hat{E}_q = \hat{\pi}^{-1}(q)$ it holds $F_j(\hat{E}_q) = \omega(q)$.*

Moreover for $\sigma = \frac{\cos^2\gamma}{2L(1+\lambda)}$ we have the inclusion

$$B_\sigma\big(f^1\big(U^1_{\delta_4,j}\big)\big) \subset F_j\big(E_j^\varepsilon\big).$$

Proof. This is just a reformulation of Lemma 2.15. Note that

$$\frac{\cos\gamma}{L(1+\lambda)} < \big[3(1+\lambda)\big]^{-3} r = \delta_3,$$

hence $\sigma = \min\{\frac{\delta_3}{2}\cos\gamma, \frac{\cos^2\gamma}{2L(1+\lambda)}\} = \frac{\cos^2\gamma}{2L(1+\lambda)}$. □

Up to this point we have constructed for each $j \in \{1, \ldots, s\}$ a tubular neighborhood locally around $f(U^1_{\delta_3,j})$. Since the mapping S depends on j, we should write more accurately S_j instead of S. In the same way we should write ω_j instead of ω. However, we can show that ω is globally well defined. More precisely we have the following lemma:

Lemma 3.5. *Let $j, k \in \{1, \ldots, s\}$. Then*

$$\omega_j = \omega_k \quad \text{on } U^1_{\delta_3,j} \cap U^1_{\delta_3,k}.$$

In particular there is a smooth mapping $\omega : M^1 \to G_{m+1,1}$ with $\omega|U^1_{\delta_3,j} = \omega_j$ for each $j \in \{1, \ldots, s\}$.

Proof. Let $j, k \in \{1, \ldots, s\}$. For $q \in U^1_{\delta_3,j} \cap U^1_{\delta_3,k}$ we show that either $S_j(q) = S_k(q)$ or $S_j(q) = -S_k(q)$, which implies the statement.

Let $q \in U^1_{\delta_3,j} \cap U^1_{\delta_3,k}$ and $l \in Z(q)$. Lemma 2.8 b) implies

$$\begin{aligned} U^1_{\delta_1,j} &\subset U^1_{r,l}, \\ U^1_{\delta_1,k} &\subset U^1_{r,l}. \end{aligned}$$

As in (3.10) we conclude

$$\big(\text{either } \sphericalangle\big(w_l, \nu^1_j(q_j)\big) \leqslant \arctan\lambda \text{ or } \sphericalangle\big(-w_l, \nu^1_j(q_j)\big) \leqslant \arctan\lambda\big)$$

and

$$\big(\text{either } \sphericalangle\big(w_l, \nu^1_k(q_k)\big) \leqslant \arctan\lambda \text{ or } \sphericalangle\big(-w_l, \nu^1_k(q_k)\big) \leqslant \arctan\lambda\big).$$

We define vectors as in (3.11), the first time depending on j, the second time on k:

$$\nu_{j,l} := \begin{cases} w_l, & \text{if } \sphericalangle(w_l, \nu^1_j(q_j)) \leqslant \arctan\lambda, \\ -w_l, & \text{otherwise}, \end{cases}$$

$$\nu_{k,l} := \begin{cases} w_l, & \text{if } \sphericalangle(w_l, \nu^1_k(q_k)) \leqslant \arctan\lambda, \\ -w_l, & \text{otherwise}. \end{cases}$$

Then

$$S_j(q)=\sum_{l\in Z(q)} g\left(\frac{|f^1(q)-f^1(q_l)|}{\delta_2}\right)\nu_{j,l}$$

and

$$S_k(q)=\sum_{l\in Z(q)} g\left(\frac{|f^1(q)-f^1(q_l)|}{\delta_2}\right)\nu_{k,l}.$$

By Lemma 2.9, we have $\nu_j^1=\nu_k^1$ on $U^1_{\delta_1,j}\cap U^1_{\delta_1,k}$, or $\nu_j^1=-\nu_k^1$ on $U^1_{\delta_1,j}\cap U^1_{\delta_1,k}$. Let us first assume

$$\nu_j^1=\nu_k^1 \quad \text{on } U^1_{\delta_1,j}\cap U^1_{\delta_1,k}. \tag{3.16}$$

Since $q\in U^1_{\delta_3,j}\cap U^1_{\delta_3,k}$, we conclude with Lemma 2.8 b)

$$U^1_{\delta_3,j}\subset U^1_{\delta_2,k}, \qquad U^1_{\delta_3,k}\subset U^1_{\delta_2,j},$$

in particular

$$\{q_j,q_k\}\subset U^1_{\delta_3,j}\cup U^1_{\delta_3,k}\subset U^1_{\delta_1,j}\cap U^1_{\delta_1,k}. \tag{3.17}$$

By (3.12) together with (3.17) we have

$$\sphericalangle\big(\nu_{j,l},\nu_j^1(q_k)\big)\leqslant \arctan\lambda, \tag{3.18}$$

by (3.16), (3.17) and (3.18) moreover

$$\sphericalangle\big(\nu_{j,l},\nu_k^1(q_k)\big)\leqslant \arctan\lambda. \tag{3.19}$$

We already know that $\nu_{j,l}=\nu_{k,l}$ or $\nu_{j,l}=-\nu_{k,l}$, thus (3.19) allows us to conclude that

$$\nu_{j,l}=\nu_{k,l}.$$

Since this is true for all $l\in Z(q)$, we conclude $S_j(q)=S_k(q)$ and hence $\omega_j(q)=\omega_k(q)$.

If $\nu_j^1=-\nu_k^1$ on $U^1_{\delta_1,j}\cap U^1_{\delta_1,k}$, one similarly concludes $\nu_{j,l}=-\nu_{k,l}$ for all $l\in Z(q)$. This implies $S_j(q)=-S_k(q)$ and hence again $\omega_j(q)=\omega_k(q)$. □

4. Intersection points and definition of ϕ^i

In this section we like to show that for $p\in M^1$ the line $f^1(p)+\omega(p)$ intersects each appropriately restricted immersion $f^i(M^i)$ in exactly one point. Using this, we are able to give a definition of the mappings $\phi^i: M^1\to M^i$. Each ϕ^i will be shown to be a diffeomorphism. Moreover, it will be shown that $f^i\circ\phi^i$ is uniformly Lipschitz bounded.

Lemma 4.1. *For $p\in U^1_{\delta_3,j}$ the line $f^1(p)+\omega(p)$ intersects the set $f^i(U^i_{\delta_1,j})$ in exactly one point. This point lies in $f^i(U^i_{\delta_2,j})$.*

Proof. Let $p\in U^1_{\delta_3,j}$. First we show that $f^1(p)+\omega(p)$ intersects $f^i(U^i_{\delta_2,j})$. By Lemma 3.1 b) we have

$$\sphericalangle\big(T(p),\nu_j^i(q)\big)\leqslant\frac{\pi}{4}+\frac{1}{2}\arctan\lambda \quad \text{for every } q\in U^i_{\delta_1,j}. \tag{4.1}$$

Let $G=\{(x,y)\in U^i_{\delta_2,j}\times\mathbb{R}^{m+1}\colon\ y\in\omega(p)\}$. We note here that $\omega(p)$ does not depend on x. Let the function F be defined by

$$\begin{aligned}&F:G\to\mathbb{R}^{m+1},\\ &(x,y)\mapsto f(x)+y,\end{aligned} \tag{4.2}$$

where $y \in \omega(p)$. With arguments as in Lemma 3.4, using (4.1) and the fact that $\omega(p)$ is constant, we conclude that $F(G)$ forms a tubular neighborhood around $f^i(U^i_{\delta_2,j})$, and moreover $B_\sigma(f^i(U^i_{\delta_3,j})) \subset F(G)$ with σ as in (3.7).

We would like to show that $f^1(U^1_{\delta_3,j}) \subset B_\sigma(f^i(U^i_{\delta_3,j}))$. For that let $p' \in U^1_{\delta_3,j}$. Then there is a unique $x \in B_{\delta_3}$ with $f^1(p') = A^1_j(x, u^1_j(x))$. Moreover there is a unique $q' \in U^i_{\delta_3,j}$ with $f^i(q') = A^i_j(x, u^i_j(x))$. We estimate

$$\begin{aligned}
\left|f^i(q') - f^1(p')\right| &= \left|A^i_j\left(x, u^i_j(x)\right) - A^1_j\left(x, u^1_j(x)\right)\right| \\
&= \left|R^i_j\left(x, u^i_j(x)\right) + T^i_j - R^1_j\left(x, u^1_j(x)\right) - T^1_j\right| \\
&\leqslant \left|R^i_j\left(x, u^i_j(x)\right) - R^i_j\left(x, u^1_j(x)\right)\right| + \left|R^i_j\left(x, u^1_j(x)\right) - R^1_j\left(x, u^1_j(x)\right)\right| + \left|T^i_j - T^1_j\right| \\
&= \left|R^i_j\left(\left(x, u^i_j(x)\right) - \left(x, u^1_j(x)\right)\right)\right| + \left|\left(R^i_j - R^1_j\right)\left(x, u^1_j(x)\right)\right| + \left|T^i_j - T^1_j\right| \\
&\leqslant \left|u^i_j(x) - u^1_j(x)\right| + \left\|R^i_j - R^1_j\right\|\left|\left(x, u^1_j(x)\right)\right| + \left|T^i_j - T^1_j\right| \\
&< \frac{\sigma}{3} + \frac{\sigma}{3} + \frac{\sigma}{3} \\
&= \sigma,
\end{aligned}$$

where in the sixth line we used $|(x, u^1_j(x))| \leqslant (1+\lambda)r$ and $\mathfrak{d}(\Gamma^1, \Gamma^i) < [3(1+\lambda)(1+r)]^{-1}\sigma$ which follows from (3.8). Hence $f^1(U^1_{\delta_3,j}) \subset B_\sigma(f^i(U^i_{\delta_3,j}))$, i.e. $f^1(U^1_{\delta_3,j})$ lies within the tubular neighborhood defined above. But this means that there is a $q \in U^i_{\delta_2,j}$ such that $f^1(p) + \omega(p)$ equals $f^i(q) + \omega(p)$. Hence $f^1(p) + \omega(p)$ intersects $f^i(U^i_{\delta_2,j})$ in the point $f^i(q)$.

It remains to show that $f^1(p) + \omega(p)$ intersects $f^i(U^i_{\delta_1,j})$ in not more than one point. By (4.1) we have $\sphericalangle(T(p), \nu^i_j(q)) < \frac{\pi}{2}$ for every $q \in U^i_{\delta_1,j}$. By the definition of ω this implies $\mathbb{R}^{m+1} = \tau_{f^i}(q) \oplus \omega(p)$ for every $q \in U^i_{\delta_1,j}$. As f^i is an (r,λ)-immersion, we conclude by Lemma A.1 in Appendix A that $f^1(p) + \omega(p)$ intersects $f^i(U^i_{\delta_1,j})$ in at most one point. □

The following lemma will be needed in order to show that the mappings ϕ^i are well defined:

Lemma 4.2. *Let $p \in U^1_{\delta_3,j} \cap U^1_{\delta_3,k}$. Let S_1 be the intersection point of $f^1(p) + \omega(p)$ with $f^i(U^i_{\delta_1,j})$, and S_2 the intersection point of $f^1(p) + \omega(p)$ with $f^i(U^i_{\delta_1,k})$. Finally let $\sigma_1 \in U^i_{\delta_1,j}$ with $f^i(\sigma_1) = S_1$, and $\sigma_2 \in U^i_{\delta_1,k}$ with $f^i(\sigma_2) = S_2$. Then $\sigma_1 = \sigma_2$.*

Proof. By Lemma 4.1 we have $S_2 \in f^i(U^i_{\delta_2,k})$, hence $\sigma_2 \in U^i_{\delta_2,k}$. As $p \in U^1_{\delta_3,j} \cap U^1_{\delta_3,k}$, we have in particular $U^1_{\delta_2,j} \cap U^1_{\delta_2,k} \neq \emptyset$ and hence $U^1_{\delta_2,k} \subset U^1_{\delta_1,j}$ by Lemma 2.8 b). By Lemma 4.1 the set $f^1(p) + \omega(p)$ has exactly one point of intersection with $f^i(U^i_{\delta_1,j})$. We conclude $\sigma_1 = \sigma_2$. □

Now we are able to define the mappings $\phi^i : M^1 \to M^i$. Let $p \in M^1$. Then $p \in U^1_{\delta_3,j}$ for some j. The line $f^1(p) + \omega(p)$ intersects $f^i(U^i_{\delta_1,j})$ in exactly one point S_p. Moreover there is exactly one point $\sigma_p \in U^i_{\delta_1,j}$ with $f^i(\sigma_p) = S_p$. We set $\phi^i(p) := \sigma_p$. The mappings ϕ^i are well defined by Lemma 4.2. Clearly we have $f^i \circ \phi^i(p) = S_p$.

We like to show that each ϕ^i is a diffeomorphism. For that we follow in parts the argumentation of [4]:

Lemma 4.3. *Each of the mappings $\phi^i : M^1 \to M^i$ is surjective.*

Proof. Let $q \in M^i$. As Q^i is a δ_4-net for f^i, there is a $j \in \{1, \dots, s\}$ with $q \in U^i_{\delta_4,j}$. By Lemma 3.4, for $\varepsilon = \frac{1}{L}\cos\gamma$ the set $F(E^\varepsilon_j)$ forms a tubular neighborhood around $f^1(U^1_{\delta_3,j})$, and moreover $B_\sigma(f^1(U^1_{\delta_4,j})) \subset F_j(E^\varepsilon_j)$ with σ as in (3.7). With (3.8) and an estimation completely analogous to that in the proof of Lemma 4.1, one shows $f^i(U^i_{\delta_4,j}) \subset B_\sigma(f^1(U^1_{\delta_4,j}))$. Hence, for every $q \in U^i_{\delta_4,j}$ there is a $p \in U^1_{\delta_3,j}$ with $f^i(q) \in f^1(p) + \omega(p)$. By the definition of ϕ^i this yields $\phi^i(p) = q$. □

Lemma 4.4. *Each of the mappings $\phi^i : M^1 \to M^i$ is injective.*

Proof. First we note that for every $j \in \{1, \dots, s\}$ we have $\phi^i(U^1_{\delta_5,j}) \subset U^i_{\delta_4,j}$. This is shown by the same arguments as in Lemma 4.1. Moreover, by the proof of Lemma 4.3, we know that $f^i(U^i_{\delta_4,j}) \subset F_j(E^\varepsilon_j)$. Using that Q^1 is a δ_5-net for f^1, we conclude $f^i \circ \phi^i(x) \in F_j(\hat{E}_x \cap E^\varepsilon_j)$ for every $x \in U^1_{\delta_3,j}$ (where $\hat{E}_x = \hat{\pi}^{-1}(x)$). As $F_j|E^\varepsilon_j$ is a diffeomorphism, we conclude that ϕ^i is injective on $U^1_{\delta_3,j}$.

For showing global injectivity, let $x, y \in M^1$ with $x \neq y$. As Q^1 is a δ_5-net for f^1, there are j, k with $x \in U^1_{\delta_5,j} \subset U^1_{\delta_4,j}$, $y \in U^1_{\delta_5,k} \subset U^1_{\delta_4,k}$.

Case 1. $U^1_{\delta_4,j} \cap U^1_{\delta_4,k} = \emptyset$.

By the considerations at the beginning of this proof, we have $\phi^i(x) \in U^i_{\delta_4,j}$, $\phi^i(y) \in U^i_{\delta_4,k}$. As $U^1_{\delta_4,j} \cap U^1_{\delta_4,k} = \emptyset$, we also have $U^i_{\delta_4,j} \cap U^i_{\delta_4,k} = \emptyset$. This implies $\phi^i(x) \neq \phi^i(y)$.

Case 2. $U^1_{\delta_4,j} \cap U^1_{\delta_4,k} \neq \emptyset$.

By Lemma 2.8 b) we have $U^1_{\delta_4,k} \subset U^1_{\delta_3,j}$. By the considerations of above, ϕ^i is injective on $U^1_{\delta_3,j}$. Again we conclude $\phi^i(x) \neq \phi^i(y)$. $\square$

Corollary 4.5. *Each mapping $\phi^i : M^1 \to M^i$ is a diffeomorphism.*

Proof. As in Lemma 4.4 we have $f^i \circ \phi^i(x) \in F_j(\hat{E}_x \cap E^\varepsilon_j)$ for every $x \in U^1_{\delta_3,j}$. Using a trivialization of the trivial bundle $\hat{E}_j$, one easily concludes that $f^i \circ \phi^i : M^1 \to \mathbb{R}^{m+1}$ is an immersion (see also [4]). Moreover, the mapping ϕ^i is surjective by Lemma 4.3, and injective by Lemma 4.4. We conclude that ϕ^i is a diffeomorphism. $\square$

Finally we would like to prove that the reparametrizations $f^i \circ \phi^i$ are uniformly Lipschitz bounded. As above, for $j \in \{1, \dots, s\}$ we can consider $f^i \circ \phi^i|U^1_{\delta_3,j}$ also as a mapping defined on B_{δ_3}. This mapping shall be denoted by $\hat{f}^i : B_{\delta_3} \to \mathbb{R}^{m+1}$.

Lemma 4.6. *Let $j \in \{1, \dots, s\}$. Let $\hat{f}^i : B_{\delta_3} \to \mathbb{R}^{m+1}$ be the local representation of $f^i \circ \phi^i|U^1_{\delta_3,j}$ as explained above. Then $\hat{f}^i$ is Λ-Lipschitz for a finite constant $\Lambda = \Lambda(\lambda)$.*

Proof. Let $x, y \in B_{\delta_3}$. Then there are unique $\mu_1, \mu_2 \in \mathbb{R}$ such that

$$\hat{f}^i(x) = \big(x, u^1_j(x)\big) + \mu_1 T(x), \qquad \hat{f}^i(y) = \big(y, u^1_j(y)\big) + \mu_2 T(y).$$

By the construction of the mappings ϕ^i we have $|\mu_1|, |\mu_2| < \varepsilon$, where $\varepsilon = \frac{1}{L}\cos\gamma < r$. Let $E \in G_{m+1,m}$ be the m-space perpendicular to $T(x)$. We define an affine subspace $\tilde{E} := (x, u^1_j(x)) + E$. Let $\tilde{\pi} : \mathbb{R}^{m+1} \to \tilde{E}$ denote the orthogonal projection onto $\tilde{E}$. As

$$\tilde{\pi}\big(\big(x, u^1_j(x)\big) + \mu T(x)\big) = \big(x, u^1_j(x)\big)$$

for any $\mu \in \mathbb{R}$, we may estimate as follows:

$$\begin{aligned}
\big|\tilde{\pi}\big(\hat{f}^i(x)\big) - \tilde{\pi}\big(\hat{f}^i(y)\big)\big| &= \big|\tilde{\pi}\big(\big(x, u^1_j(x)\big) + \mu_1 T(x)\big) - \tilde{\pi}\big(\big(y, u^1_j(y)\big) + \mu_2 T(y)\big)\big| \\
&= \big|\tilde{\pi}\big(\big(x, u^1_j(x)\big) + \mu_2 T(x)\big) - \tilde{\pi}\big(\big(y, u^1_j(y)\big) + \mu_2 T(y)\big)\big| \\
&\leqslant \big|\big(x, u^1_j(x)\big) - \big(y, u^1_j(y)\big) + \mu_2\big(T(x) - T(y)\big)\big| \\
&\leqslant |x - y| + \big|u^1_j(x) - u^1_j(y)\big| + r\big|T(x) - T(y)\big| \\
&\leqslant (1 + \lambda + rL)|x - y|.
\end{aligned} \tag{4.3}$$

By Lemma A.1 together with Lemma 3.1 b), the set $f^i(U^i_{\delta_1,j})$ is the graph of a function $\tilde{u}$ on an open subset U of $\tilde{E}$. In the same manner, $f^i(U^i_{\delta_2,j})$ is the graph of the same function restricted to a subset $V \Subset U$. Again by Lemma 3.1 b), on convex subsets of U the function $\tilde{u}$ is λ'-Lipschitz with $\lambda' = \tan\gamma$, where γ is as in (3.6). Let $\varrho > 0$ be small enough, such that $B_\varrho(\xi) \subset U$ for any $\xi \in V$ (where here $B_\varrho(\xi)$ denotes an open ball in $\tilde{E}$).

Now assume $|x-y| < \frac{\varrho}{1+\lambda+rL}$. By Lemma 4.1 we have $\hat{f}^i(z) \in f^i(U^i_{\delta_2,j})$ for any $z \in B_{\delta_3}$. Hence by (4.3) the points $\tilde{\pi}(\hat{f}^i(x))$ and $\tilde{\pi}(\hat{f}^i(y))$ lie both in the convex subset $B_\varrho(\tilde{\pi}(\hat{f}^i(x)))$ of U. We conclude

$$\begin{aligned} \left|\hat{f}^i(x) - \hat{f}^i(y)\right| &= \left|\left(\tilde{\pi}\left(\hat{f}^i(x)\right), \tilde{u}\left(\tilde{\pi}\left(\hat{f}^i(x)\right)\right)\right) - \left(\tilde{\pi}\left(\hat{f}^i(y)\right), \tilde{u}\left(\tilde{\pi}\left(\hat{f}^i(y)\right)\right)\right)\right| \\ &\leqslant (1+\tan\gamma)(1+\lambda+rL)|x-y|. \end{aligned} \tag{4.4}$$

If $x, y \in B_{\delta_3}$ are arbitrary points, let $N \in \mathbb{N}$ with $N > \frac{1+\lambda+rL}{\varrho}|x-y|$. We define $x_\iota = x + \iota\frac{y-x}{N} \in B_{\delta_3}$ for $\iota = 0, \ldots, N$. Then, using a telescoping sum and (4.4), we have

$$\begin{aligned} \left|\hat{f}^i(x) - \hat{f}^i(y)\right| &\leqslant \sum_{\iota=0}^{N-1}\left|\hat{f}^i(x_\iota) - \hat{f}^i(x_{\iota+1})\right| \\ &\leqslant (1+\tan\gamma)(1+\lambda+rL)|x-y|. \end{aligned}$$

By the definitions of L and γ, the quantities rL and γ depend only on λ. Hence $\hat{f}^i$ is Λ-Lipschitz with $\Lambda = \Lambda(\lambda) = (1+\tan\gamma)(1+\lambda+rL)$. $\square$

Remark 4.7. If we choose some of the constants more carefully, we can give a better bound for Λ in the preceding lemma. Choosing the right hand side in (3.3) extremely small, we can replace γ by a number $\tilde{\gamma}$ which is slightly greater than $\arctan\lambda$. Moreover, we can choose ε with $|\mu_1|, |\mu_2| < \varepsilon$ so small, that the term εL can almost be neglected. With these constants, we finally obtain $\Lambda = (1+\tan\tilde{\gamma})(1+\lambda+\varepsilon L) < 2(1+\lambda)^2$. In particular, Λ does not depend on the dimension m here, although L depends on m.

Finally, by Lemma 4.6, we may pass to a subsequence such that $f^i \circ \phi^i$ converges uniformly to a limit function $f: M^1 \to \mathbb{R}^{m+1}$. As limit manifold we define $M := M^1$. Thus the limit manifold is a compact differentiable m-manifold.

5. The limit function lies in $\mathfrak{F}^0(r, \lambda)$

Up to this point we have found a subsequence and diffeomorphisms $\phi^i: M^1 \to M^i$, such that $f^i \circ \phi^i$ is uniformly Lipschitz bounded and converges uniformly to an $f: M^1 \to \mathbb{R}^{m+1}$. In this section we will show that the limit function f lies in $\mathfrak{F}^0(r, \lambda)$.

For that we have to show, that for each point $q \in M^1$ there is an $E = E(q) \in G_{m+1,m}$, such that f is injective on $U^E_{r,q}$ and the set $A^{-1}_{q,E} \circ f(U^E_{r,q})$ is the graph of a Lipschitz continuous function $u: B_r \to \mathbb{R}$ with Lipschitz constant λ.

So let $q \in M^1$. Let $q^i = \phi^i(q) \in M^i$. As each f^i is an (r, λ)-immersion, there are $E^i \in G_{m+1,m}$ such that for each i the set $(A^i_{q^i,E^i})^{-1} \circ f^i(U^{E^i}_{r,q^i})$ is the graph of a differentiable function $u^i: B_r \to \mathbb{R}$ with $\|Du^i\|_{C^0(B_r)} \leqslant \lambda$.

Passing to another subsequence, we may assume

$$\begin{aligned} &u^i \to u \quad \text{uniformly}, \\ &E^i \to E \quad \text{for the metric } d \text{ defined in (2.3)}, \end{aligned}$$

as $i \to \infty$, where $u: B_r \to \mathbb{R}$ and $E \in G_{m+1,m}$. In particular, u is Lipschitz continuous with Lipschitz constant λ.

Let $A_{q,E}$ be a Euclidean isometry, which maps the origin to $f(q)$, and the subspace $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}$ onto $f(q) + E$. Then we have in any case $A_{q,E}(\{(x, u(x)): x \in B_r\}) \subset f(M^1)$.

To finish the proof, we show that f is injective on $U^E_{r,q}$ and that $A^{-1}_{q,E} \circ f(U^E_{r,q})$ is the graph of the function u. This is true, if and only if for every ϱ with $0 < \varrho < r$ the function f is injective on $U^E_{\varrho,q}$ and the set $A^{-1}_{q,E} \circ f(U^E_{\varrho,q})$ is the graph of the function $u|B_\varrho$.

We first show the graph property. Let a ϱ with $0<\varrho<r$ be given. Let $\varepsilon>0$ with $\varepsilon<\min\{\varrho, r-\varrho\}$. Moreover, let $U^i_\varrho \subset M^1$ be the q-component of the set $(\pi \circ A^{-1}_{q^i,E^i} \circ f^i \circ \phi^i)^{-1}(B_\varrho)$. Again, $U^E_{\varrho,q} \subset M^1$ is the q-component of $(\pi \circ A^{-1}_{q,E} \circ f)^{-1}(B_\varrho)$. By the definition of U^i_ϱ, we have $A^{-1}_{q^i,E^i} \circ f^i \circ \phi^i(U^i_\varrho) = \{(x, u^i(x))\colon x \in B_\varrho\}$. As $A^{-1}_{q^i,E^i} \circ f^i \circ \phi^i \to A^{-1}_{q,E} \circ f$ uniformly, we conclude by the definitions of U^i_ϱ and $U^E_{\varrho,q}$ that $U^i_{\varrho-\varepsilon} \subset U^E_{\varrho,q} \subset U^i_{\varrho+\varepsilon}$ for i sufficiently large, in particular

$$\big\{\big(x, u^i(x)\big)\colon x \in B_{\varrho-\varepsilon}\big\} \subset A^{-1}_{q^i,E^i} \circ f^i \circ \phi^i\big(U^E_{\varrho,q}\big) \subset \big\{\big(x, u^i(x)\big)\colon x \in B_{\varrho+\varepsilon}\big\}.$$

Letting $i \to \infty$, we obtain

$$\big\{\big(x, u(x)\big)\colon x \in B_{\varrho-\varepsilon}\big\} \subset A^{-1}_{q,E} \circ f\big(U^E_{\varrho,q}\big) \subset \big\{\big(x, u(x)\big)\colon x \in B_{\varrho+\varepsilon}\big\}.$$

As this is true for every $\varepsilon>0$ with $\varepsilon<\min\{\varrho, r-\varrho\}$, we conclude by the definition of $U^E_{\varrho,q}$ that $A^{-1}_{q,E} \circ f(U^E_{\varrho,q}) = \{(x,u(x))\colon x \in B_\varrho\}$. This is the desired graph property.

Similarly, one shows that f is injective on $U^E_{\varrho,q}$. We have $f(x) = \lim_{i\to\infty} f^i \circ \phi^i(x)$ for all $x \in U^E_{\varrho,q}$, and moreover $U^E_{\varrho,q} \subset U^i_{\varrho+\varepsilon}$ for i sufficiently large. The functions $f^i \circ \phi^i$ are injective on $U^i_{\varrho+\varepsilon}$ and it holds $A^{-1}_{q^i,E^i} \circ f^i \circ \phi^i(U^i_{\varrho+\varepsilon}) = \{(x, u^i(x))\colon x \in B_{\varrho+\varepsilon}\}$. Using $A_{q^i,E^i} \to A_{q,E}$, one easily concludes that $A^{-1}_{q,E} \circ f$ and hence also f is injective on $U^E_{\varrho,q}$.

This shows that the limit function f lies in $\mathfrak{F}^0(r,\lambda)$.

6. Compactness in higher codimension

In the final section we want to prove Theorem 1.3, that is compactness of (r,λ)-immersions in higher codimension with $\lambda \leqslant \frac{1}{4}$. Our main task here is to give an analogous construction of the averaged normal projection for arbitrary codimension. For that we shall use a Riemannian center of mass, which was introduced in Section 2.

So let f^i be a sequence as in Theorem 1.3 with $\lambda \leqslant \frac{1}{4}$. For all objects of the preceding sections that are defined also in arbitrary codimension, we shall use precisely the same notation. We note that Lemmas 2.8 and 2.12 are true also in higher codimension. For $q \in M^1$ we set

$$\lambda^q_j := g\bigg(\frac{|f^1(q) - f^1(q_j)|}{\delta_2}\bigg).$$

As in the proof of Lemma 3.1 a) we conclude that there is a $k \in Z(q)$ with $\lambda^q_k = 1$. For each $j \in \{1,\dots,s\}$ let $N_j \in G_{n,k}$ be the k-space perpendicular to E_j. We define for each $q \in M^1$ a probability measure μ_q on $G_{n,k}$ by

$$\mu_q = \bigg(\sum_{j\in Z(q)} \lambda^q_j\bigg)^{-1} \sum_{j\in Z(q)} \lambda^q_j \delta_{N_j},$$

where δ_N denotes the Dirac measure on $G_{n,k}$ supported at $N \in G_{n,k}$.

Moreover, let

$$\begin{aligned} \nu: M^1 &\to G_{n,k}, \\ q &\mapsto \big(T_q M^1\big)^\perp \end{aligned}$$

be the normal-space field of f^1 as defined in (2.2). Now consider

$$\begin{aligned} &P: \overline{B}_{\frac{\pi}{6}}\big(\nu(q)\big) \to \mathbb{R}, \\ &P(p) = \int\limits_{G_{n,k}} d(p,x)^2\, d\mu_q(x), \end{aligned}$$

where $\overline{B}_{\frac{\pi}{6}}(\nu(q)) \subset G_{n,k}$ is the closed ball of radius $\frac{\pi}{6}$ around $\nu(q)$. Here the radius is measured with respect to the canonical distance d on $G_{n,k}$ as defined in (2.3).

Lemma 6.1. *For every $q \in M^1$ it holds* $\operatorname{spt}\mu_q \subset B_{\frac{\pi}{12}}(\nu(q))$.

Proof. By the definition of μ_q it is sufficient to show that N_j lies in $B_{\frac{\pi}{12}}(\nu(q))$ for every $j \in Z(q)$. So let $j \in Z(q)$. By the definition of $Z(q)$ we have $q \in U^1_{\delta_2,j}$. We deduce that N_j is the graph of a linear function h over $\nu(q)$ with $\|Dh\| = (\sum_{i=1}^k |\partial_i h|^2)^{\frac{1}{2}} \leqslant \lambda \leqslant \frac{1}{4}$. Let $\theta_1, \dots, \theta_k$ be the principal angles between N_j and $\nu(q)$. After a suitable rotation we may assume that $\tan\theta_i = |\partial_i h|$ for every $i \in \{1, \dots, k\}$. Using $\theta \leqslant \tan\theta$ for $\theta \in [0, \frac{\pi}{2})$, we estimate $d(N_j, \nu(q)) = (\sum_{i=1}^k \theta_i^2)^{\frac{1}{2}} \leqslant (\sum_{i=1}^k (\tan\theta_i)^2)^{\frac{1}{2}} = (\sum_{i=1}^k |\partial_i h|^2)^{\frac{1}{2}} \leqslant \lambda \leqslant \frac{1}{4} < \frac{\pi}{12}$. Hence N_j lies in $B_{\frac{\pi}{12}}(\nu(q))$. $\square$

In particular we have $\operatorname{spt}\mu_q \subset B_{\frac{\pi}{6}}(\nu(q))$. Hence we conclude by Lemma 2.4 and Theorem 2.6, that there is exactly one center of mass $N(q) \in B_{\frac{\pi}{6}}(\nu(q)) \subset G_{n,k}$ for μ_q. In this way we may define a mapping

$$N : M^1 \to G_{n,k}, \qquad q \mapsto N(q).$$

An important property of the averaged normal N constructed in this way is its differentiability. It is needed in order to obtain diffeomorphisms $\phi^i : M^1 \to M^i$. We will show that N is in C^{k} if the function f^1 is in C^{k} (here we denote by k the degree of differentiability, and by k the codimension). First, for functions defined on manifolds, we need the following variation of the implicit function theorem:

Lemma 6.2. *Let M be a smooth m-manifold, (N, g) be a smooth Riemannian n-manifold and $f : M \times N \to \mathbb{R}$ be a mapping. For every fixed $x \in M$, assume that*

$$h_x : N \to \mathbb{R}, \quad h_x = f(x, \cdot)$$

is in $C^2(N)$ and is strictly convex. Let $\mathrm{k} \geqslant 1$ be an integer. Denoting by $\operatorname{grad} h_x$ the gradient of the fixed function h_x defined above, assume that

$$H : M \times N \to TN, \quad (x, y) \mapsto \operatorname{grad} h_x(y)$$

is in $C^{\mathrm{k}}(M \times N, TN)$. Let $(x_0, y_0) \in M \times N$ be a point with $H(x_0, y_0) = 0 \in T_{y_0}N$.

Then there are open neighborhoods $U \subset M$ of x_0 and $V \subset N$ of y_0, and moreover a function $F \in C^{\mathrm{k}}(U, V)$, such that $\{(x, y) \in U \times V\colon\ H(x, y) = 0 \in T_y N\} = \{(x, F(x))\colon\ x \in U\}$.

Proof. Let $\varphi_1 : U_1 \to \varphi(U_1)$ be a coordinate chart of M with $x_0 \in U_1$, and let $\varphi_2 : V_1 \to \varphi_2(V_1)$ be a coordinate chart of N with $y_0 \in V_1$. For fixed $x \in M$, in the local coordinates φ_2 we have

$$\operatorname{grad} h_x = \sum_{i,j=1}^n g^{ij} \partial_j h_x \partial_i, \tag{6.1}$$

and, with the corresponding Christoffel symbols $\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^n g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$, the components of the Hessian $D^2_{ij} h_x = \partial_i \partial_j h_x - \sum_{k=1}^n \Gamma^k_{ij} \partial_k h_x$. If we assume φ_2 to be Riemannian normal coordinates centered in y_0, we obtain

$$D^2_{ij} h_x(y_0) = \partial_i \partial_j h_x(y_0). \tag{6.2}$$

Let us now consider the local representations of h_x and f in the coordinates φ_2 and $\varphi_1 \times \varphi_2$ respectively. We denote these representations simply by h_x and f again. Moreover, we identify x_0 and y_0 with $\varphi_1(x_0)$ and $\varphi_2(y_0)$ respectively. The condition on h_x to be strictly convex means that the Hessian $D^2 h_x$ is positive definite in every point. Hence, by (6.2), the Hessian matrix $D^2 h_x(y_0)$ of the local representation is positive definite, in particular

$$D^2 h_{x_0}(y_0) \quad \text{is invertible.} \tag{6.3}$$

The Jacobian Df may be considered as a mapping $Df : \Omega \to \mathbb{R}^{m+n}$, where $\Omega = \varphi_1(U_1) \times \varphi_2(V_1) \subset \mathbb{R}^m \times \mathbb{R}^n$. We write $Df = (D_x f, D_y f) \in \mathbb{R}^m \times \mathbb{R}^n$ and consider the mapping $D_y f : \Omega \to \mathbb{R}^n$. Similarly, for the Jacobian of $D_y f$, we write $D(D_y f) = (D_x(D_y f), D_y(D_y f)) \in \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times n}$. As $D_y f(x_0, y_0) = Dh_{x_0}(y_0)$ and as $H(x_0, y_0) = 0$, we conclude

$$D_y f(x_0, y_0) = 0. \tag{6.4}$$

Similarly, as $D_y(D_y f)(x_0, y_0) = D^2 h_{x_0}(y_0)$, we know by (6.3) that

$$D_y(D_y f)(x_0, y_0) \quad \text{is invertible.} \tag{6.5}$$

The assumption on H to be in C^{k} implies by (6.1) that also $D_y f : \Omega \to \mathbb{R}^n$ is in C^{k}. Hence we may use (6.4), (6.5) and apply the usual implicit function theorem to the function $D_y f$. From this we deduce the statement. □

Using the preceding lemma, we are able to deduce that the mapping N is differentiable:

Lemma 6.3. *Let $N : M^1 \to G_{n,k}$ be the averaged normal corresponding to $f^1 : M^1 \to \mathbb{R}^n$, as constructed above. Assume that $f^1 \in C^{\mathrm{k}}(M^1, \mathbb{R}^n)$ for a $\mathrm{k} \geqslant 1$. Then $N \in C^{\mathrm{k}}(M^1, G_{n,k})$.*

Proof. Let $q_0 \in M^1$ be a point. We show that N is C^{k} in a neighborhood of q_0. Let $W \subset M^1$ be an open neighborhood of q_0 with $\nu(W) \subset B_{\frac{\pi}{12}}(\nu(q_0))$. By Lemma 6.1 we have $\operatorname{spt}\mu_q \subset B_{\frac{\pi}{6}}(\nu(q_0))$ for every $q \in W$; this will be implicitly used in the following argumentation. Let

$$G : W \times B_{\frac{\pi}{6}}\big(\nu(q_0)\big) \to \mathbb{R},$$
$$(q, p) \mapsto \int_{G_{n,k}} d(p, x)^2 \, d\mu_q(x).$$

Moreover, for fixed $q \in W$ let $h_q : B_{\frac{\pi}{6}}(\nu(q_0)) \to \mathbb{R}$, $h_q := G(q, \cdot)$. By this definition, h_q is smooth on $B_{\frac{\pi}{6}}(\nu(q_0))$ and by Theorem 2.6 strictly convex. We denote by $\operatorname{grad} h_q$ the gradient of the fixed function h_q, and define

$$H : W \times B_{\frac{\pi}{6}}\big(\nu(q_0)\big) \to TB_{\frac{\pi}{6}}\big(\nu(q_0)\big), \quad (q, p) \mapsto \operatorname{grad} h_q(p).$$

With (2.5) and the definition of μ_q, we calculate

$$H(q, p) = -2\Bigg(\sum_{j\in Z(q)} \lambda_j^q\Bigg)^{-1} \sum_{j\in Z(q)} \lambda_j^q \exp_p^{-1}(N_j). \tag{6.6}$$

As $\lambda_j^q = g(\frac{|f^1(q) - f^1(q_j)|}{\delta_2})$ and by the definition of g, the mapping $q \mapsto \lambda_j^q$ is in C^{k} if f is in C^{k}. Moreover, as for every $j \in Z(q)$ the mapping $p \mapsto \exp_p^{-1}(N_j)$ is smooth, we conclude that H is in C^{k}. Note that g is smooth with $g(1) = 0$, hence H is C^{k} even if the sums in (6.6) depend on $Z(q)$.

As $N(q) \in B_{\frac{\pi}{6}}(\nu(q_0))$ is the center of mass for μ_q, we have $H(q, N(q)) = 0$ for every $q \in W$, in particular $H(q_0, N(q_0)) = 0$.

Now we are in a position to apply Lemma 6.2. We conclude that there are open neighborhoods $U \subset W$ of q_0, $V \subset B_{\frac{\pi}{6}}(\nu(q_0))$ of $N(q_0)$, and a function $F \in C^{\mathrm{k}}(U, V)$ with $\{(x, y) \in U \times V\colon\ H(x, y) = 0\} = \{(x, F(x))\colon\ x \in U\}$. By Theorem 2.6 we deduce, that N coincides with F on U. Hence N is in C^{k} on U. □

Remark 6.4. In particular, the preceding lemma shows that the averaged normal N can be used for the projection in the case of immersions with L^p-bounded second fundamental form, which was the case considered in [4]. For an (r, λ)-immersion $f \in C^{\mathrm{k}}$, the normal ν_f is in $C^{\mathrm{k}-1}$, while the averaged normal N is in C^{k}. In particular, the averaged normal of a C^1-immersion is differentiable and forms locally a tubular neighborhood around the immersion. Thus it is possible to construct diffeomorphisms $\phi^i : M^1 \to M^i$ using the averaged normal. However, if one likes to show convergence as in [14] and in [4], we require N even to be in C^2. For that purpose, an additional smoothing of f is unavoidable; this was also performed by Langer (see the first paragraph on p. 229 in [14], where a C^1-perturbation is made in order to smooth the immersion). On the other hand, a pure smoothing argument would not suffice to prove Theorems 1.1 and 1.3. As in general the limit is not even differentiable, one has to project from $f^{i_0}(M^{i_0})$ for a fixed and sufficiently large i_0. The averaged normal is needed then in order to estimate the size of the tubular neighborhood.

As in the case of codimension 1, we may consider the restriction of N to $U^1_{\delta_3,j}$ as a mapping defined on B_{δ_3}. As an analogue of Lemma 3.2 we show the following statement:

Lemma 6.5. *If we consider $G_{n,k}$ as a metric space with the geodesic distance d, the mapping $N : B_{\delta_3} \to G_{n,k}$ is L-Lipschitz with $L = 4^{12m+6}r^{-1}$.*

Proof. Let $x, y \in B_{\delta_3}$. Then there are unique $p, q \in U^1_{\delta_3,j}$ with $\pi \circ A_j^{-1} \circ f^1(p) = x$, $\pi \circ A_j^{-1} \circ f^1(q) = y$. By the arguments at the beginning of the proof of Lemma 3.2, one shows $\lambda^p_j = 0$ for $j \in Z(q) \setminus Z(p)$ and $\lambda^q_j = 0$ for $j \in Z(p) \setminus Z(q)$. Again as in Lemma 3.2, we estimate

$$\begin{aligned} \left|\lambda^p_j - \lambda^q_j\right| &\leqslant 36(1+\lambda)^3 r^{-1}|x-y| \\ &\leqslant 72r^{-1}|x-y|, \end{aligned} \tag{6.7}$$

where we used $\lambda \leqslant \frac{1}{4}$. Now we note that $\sum_{k\in Z(p)\cup Z(q)} \lambda^p_k \geqslant 1$ and $\sum_{k\in Z(p)\cup Z(q)} \lambda^q_k \geqslant 1$. Moreover $|Z(p) \cup Z(q)| \leqslant 2[3(1+\lambda)]^{6m} \leqslant 2 \cdot 4^{6m}$ for $\lambda \leqslant \frac{1}{4}$ by (3.9). Using all this, we obtain

$$\begin{aligned} \left|\left(\sum_{k\in Z(p)} \lambda^p_k\right)^{-1} - \left(\sum_{k\in Z(q)} \lambda^q_k\right)^{-1}\right| &\leqslant \sum_{k\in Z(p)\cup Z(q)} \left|\lambda^p_k - \lambda^q_k\right| \\ &\leqslant 9 \cdot 4^{6m+2} r^{-1}|x-y|. \end{aligned} \tag{6.8}$$

Using (6.7) and (6.8), one easily concludes

$$\left|\left(\sum_{k\in Z(p)} \lambda^p_k\right)^{-1} \lambda^p_j - \left(\sum_{k\in Z(q)} \lambda^q_k\right)^{-1} \lambda^q_j\right| \leqslant 10 \cdot 4^{6m+2} r^{-1}|x-y|. \tag{6.9}$$

Now assume $k \in Z(p)$. Then $U^1_{\delta_3,j} \cap U^1_{\delta_2,k} \neq \emptyset$, hence $U^1_{\delta_3,j} \subset U^1_{\delta_1,k}$ by Lemma 2.8 b). This implies $q \in U^1_{\delta_1,k}$. By a calculation as in Lemma 6.1 we deduce $N_k \in B_{\frac{\pi}{12}}(\nu(q))$. We conclude that both $\operatorname{spt}\mu_p$ and $\operatorname{spt}\mu_q$ are a subset of $B_{\frac{\pi}{12}}(\nu(q))$. This enables us to apply Lemma 2.7 with $\mu_1 = \mu_p$ and $\mu_2 = \mu_q$.

With Lemma 2.7, the definitions of μ_p and μ_q, and (6.9) we estimate

$$\begin{aligned} d\big(N(x), N(y)\big) &\leqslant C \int_{G_{n,k}} d\big(N(q), z\big)\, d|\mu_p - \mu_q|(z) \\ &= C \sum_{j\in Z(p)\cup Z(q)} d\big(N(q), N_j\big) \left|\left(\sum_{k\in Z(p)} \lambda^p_k\right)^{-1} \lambda^p_j - \left(\sum_{k\in Z(q)} \lambda^q_k\right)^{-1} \lambda^q_j\right| \\ &\leqslant C \cdot 10 \cdot 4^{6m+2} \sum_{j\in Z(p)\cup Z(q)} d\big(N(q), N_j\big) r^{-1}|x-y| \\ &\leqslant 4^{12m+6} r^{-1}|x-y|, \end{aligned}$$

where in the last line we used $d(N(q), N_j) < \frac{\pi}{6}$, $|Z(p) \cup Z(q)| \leqslant 2 \cdot 4^{6m}$ and $C = 1 + (\kappa^{1/2}\varrho)^{-1} \tan(2\kappa^{1/2}\varrho) < 16$ for $\kappa = 2$ and $\varrho = \frac{\pi}{6}$ by (2.4). $\square$

Now the remainder of the proof is analogous to the case of codimension 1. First we note that by the preceding lemma one easily derives an estimate for the size of the tubular neighborhood around f^1 formed by N. This is done by using elementary geometry in much the same way as in Appendix A (where the case of codimension 1 is considered); as we assumed λ to be small and hence N nearly to be perpendicular to f^1, it is even easier here as we can estimate rather roughly (and do not need an analogue of Lemma 3.1 b) for that). Moreover, we can show the existence and uniqueness of intersection points of $f^1(p) + N(p)$ with an appropriate restriction of $f^i(M^i)$ by the fixed point argument of [4]. To show surjectivity of ϕ^i one uses the estimate for the size of the tubular neighborhood and shows that $f^i(M^i)$ lies within this neighborhood. The rest of the proof is the same as in the case of codimension 1.

The question arises, whether compactness in higher codimension, that is Theorem 1.3, can also be shown for an arbitrary Lipschitz constant λ (as in the case of codimension 1). Surely, the bound $\lambda \leqslant \frac{1}{4}$ is not optimal. One could try to find the largest possible bound for λ, and — in the case that it is finite — to give a counterexample for immersions exceeding this bound. We would like to suggest two possibilities for extending the construction in this section to immersions with Lipschitz constant larger than the ones considered here: First, as proposed in the remark on p. 511 in [13], one could use another definition for the center of mass, which allows one to define centers in larger balls. The second is to find a center of mass not in a convex ball, but in a larger convex subset of $G_{n,k}$. Such kind of subsets of Grassmannians have been detected by J. Jost and Y.L. Xin in [12].

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Appendix A. Size of tubular neighborhoods

In this appendix we like to prove Lemma 2.15, that is we estimate the size of a tubular neighborhood around a graph depending on different quantities such as angles and Lipschitz constants. We shall use the notations introduced in the paragraph preceding Lemma 2.15. For a general treatise on the existence of tubular neighborhoods see [5] and [10]. Moreover, in Lemma A.1 we will show a result needed for proving that the projection in Section 4 has at most one point of intersection with an appropriate subset of $f^i(M^i)$.

Proof of Lemma 2.15. a) We like to start with the following *initial consideration:*

Let $q \in B_\varrho$. Let $f(x) = (x, u(x))$ and $\tau_f(q) \in G_{m+1,m}$ be the tangent space at q as in (2.1). In particular $\tau_f(q)$ is an m-space in $\mathbb{R}^{m+1}$ perpendicular to $\nu(q)$. Furthermore let $K \subset \tau_f(q)$ be a 1-dimensional subspace of $\tau_f(q)$. Let $p \in B_\varrho$ and let $\alpha \leqslant \frac{\pi}{2}$ be the smaller angle enclosed by the lines $\omega(p)$ and K. From (2.11) we deduce

$$\alpha \geqslant \frac{\pi}{2} - \gamma > 0. \tag{A.1}$$

Now let us come to the *main part* of the proof:

Let $x, y \in B_\varrho$ be points with $x \neq y$. Without loss of generality we may assume $x - y \in \mathbb{R}^1 \times \{0\} \subset \mathbb{R}^m$. By the mean value theorem there is a $z \in \{(1-t)x + ty\colon\ t \in (0,1)\} \subset B_\varrho$ with

$$\partial_1 u(z) = \frac{u(x) - u(y)}{x_1 - y_1},$$

where x_1, y_1 are the first coordinate of the vectors x, y respectively. Let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{R}^m$. We set

$$K := \operatorname{span}\bigl\{\bigl(e_1, \partial_1 u(z)\bigr)\bigr\} \subset \tau_f(z).$$

Let $\alpha \leqslant \frac{\pi}{2}$ be the smaller angle enclosed by the lines $\omega(y)$ and K. By (A.1) we have $\alpha \geqslant \frac{\pi}{2} - \gamma$. In particular the smaller angle between $\omega(y)$ and the line through $(x, u(x))$ and $(y, u(y))$ is greater than or equal to $\frac{\pi}{2} - \gamma$ (see Fig. A.1).

Let $\pi^\perp((x, u(x)))$ denote the orthogonal projection of $(x, u(x))$ onto $F(\{y\} \times \omega(y)) = (y, u(y)) + \omega(y)$. Then

$$\begin{aligned} \bigl|\bigl(x, u(x)\bigr) - \pi^\perp\bigl(\bigl(x, u(x)\bigr)\bigr)\bigr| &\geqslant \bigl|\bigl(x, u(x)\bigr) - \bigl(y, u(y)\bigr)\bigr| \sin\biggl(\frac{\pi}{2} - \gamma\biggr) \\ &\geqslant |x - y| \sin\biggl(\frac{\pi}{2} - \gamma\biggr) \\ &= |x - y| \cos\gamma. \end{aligned} \tag{A.2}$$

Now we distinguish two cases:

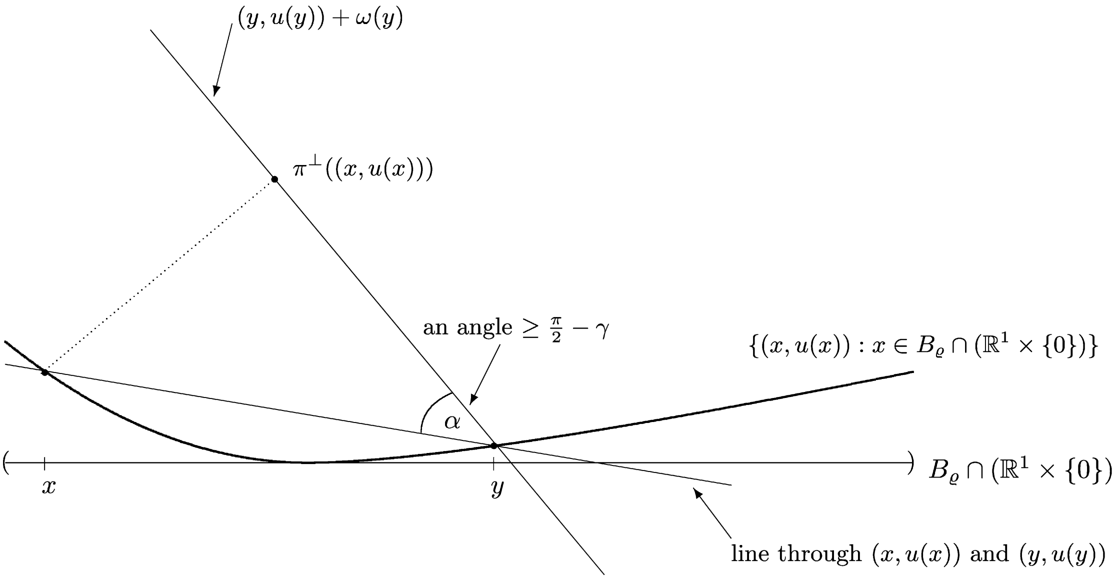

Fig. A.1. Calculation of the distance between $(x,u(x))$ and $\pi^\perp((x,u(x)))$. Note that, unlike the rest of the figure, the line $(y,u(y))+\omega(y)$ does not necessarily lie in the plane $(\mathbb{R}^1 \times \{0\}) \times \mathbb{R}^1 \subset \mathbb{R}^{m+1}$.

Case 1.

$$\big[\big(x,u(x)\big)+\omega(x)\big] \cap \big[\big(y,u(y)\big)+\omega(y)\big] = \emptyset. \tag{A.3}$$

In this case we do not need any further estimations.

Case 2.

$$\big[\big(x,u(x)\big)+\omega(x)\big] \cap \big[\big(y,u(y)\big)+\omega(y)\big] \neq \emptyset. \tag{A.4}$$

We now have to consider the following two subcases 2.*i* and 2.*ii:*

2.i. The case $|x-y| \leqslant \frac{1}{L}$.

Let $\theta = \sphericalangle(T(x),T(y))$. By the assumption (A.4) we have $\theta > 0$. Using $|T(x)| = |T(y)| = 1$, we estimate

$$\begin{aligned}\theta &= 2\arcsin\left(\frac{|T(x)-T(y)|}{2}\right)\\ &\leqslant 2\arcsin\left(\frac{L}{2}|x-y|\right)\\ &< \frac{\pi}{2}.\end{aligned} \tag{A.5}$$

Now let $\xi \in \mathbb{R}^{m+1}$ denote the intersection point of $(x,u(x))+\omega(x)$ with $(y,u(y))+\omega(y)$. (See Fig. A.2.)

Then, using (A.2) and (A.5),

$$\begin{aligned}\big|\big(x,u(x)\big)-\xi\big| &= \frac{|(x,u(x))-\pi^\perp((x,u(x)))|}{\sin\theta}\\ &\geqslant \frac{|x-y|\cos\gamma}{\sin(2\arcsin(\frac{L}{2}|x-y|))}\end{aligned}$$

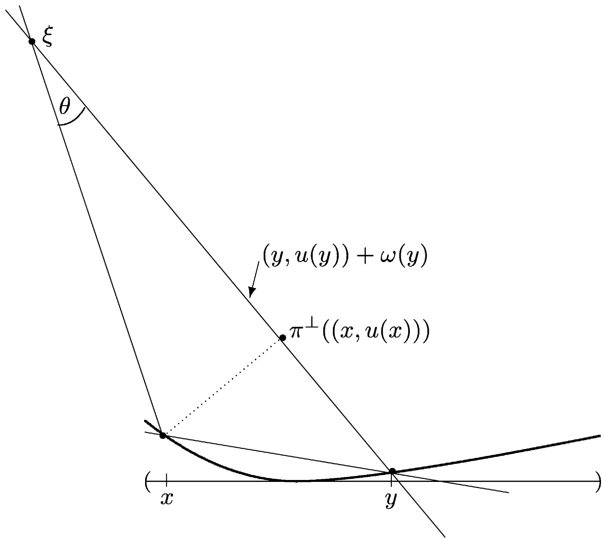

Fig. A.2. Calculation of the distance between $(x,u(x))$ and ξ. Again, $(y,u(y))+\omega(y)$ does not necessarily lie in $(\mathbb{R}^1\times\{0\})\times\mathbb{R}^1$.

$$= \frac{1}{L}\frac{\cos\gamma}{\sqrt{1-\frac{L^2}{4}|x-y|^2}}$$
$$> \frac{1}{L}\cos\gamma.$$

2.ii. The case $|x-y| > \frac{1}{L}$.

Let ξ be as in Case 2.*i*. Then (A.2) directly implies

$$\big|\big(x,u(x)\big)-\xi\big| > \frac{1}{L}\cos\gamma.$$

Let $\varepsilon = \frac{1}{L}\cos\gamma$. Summarizing Case 1, Case 2.*i* and 2.*ii*, we conclude that F is injective on E^ε. Applying well-known results from elementary differential topology, we deduce that $F|E^\varepsilon$ is a diffeomorphism onto an open neighborhood of $\{(x,u(x))\in\mathbb{R}^m\times\mathbb{R}\colon x\in B_\varrho\}$. This proves part a) of Lemma 2.15.

b) Let $\partial(F(E^\varepsilon))$ denote the boundary of $F(E^\varepsilon)$ in $\mathbb{R}^{m+1}$, where $\varepsilon=\frac{1}{L}\cos\gamma$ as in part a). Let $x\in\overline{B}_{\frac{\varrho}{2}}$. We have to show

$$\operatorname{dist}\big(\big(x,u(x)\big),\partial\big(F\big(E^\varepsilon\big)\big)\big)\geqslant\sigma$$

with $\sigma=\min\{\frac{\varrho}{2}\cos\gamma,\frac{\cos^2\gamma}{2L(1+\lambda)}\}$ as in Lemma 2.15 b).

So let $\zeta\in\partial(F(E^\varepsilon))\subset\mathbb{R}^{m+1}$. Then there are two cases:

Case 1. $\zeta=(y,u(y))+\vartheta$ for a $y\in B_\varrho$ and a $\vartheta\in\omega(y)$ with $|\vartheta|=\varepsilon$.

We distinguish two subcases 1.*i* and 1.*ii:*

1.i. The case $|x-y|\leqslant\frac{\cos\gamma}{L(1+\lambda+\cos\gamma)}$.

As u is λ-Lipschitz, we have

$$\big|\big(x,u(x)\big)-\big(y,u(y)\big)\big|\leqslant(1+\lambda)|x-y|$$
$$\leqslant\frac{(1+\lambda)\cos\gamma}{L(1+\lambda+\cos\gamma)}.$$

Then

$$\begin{aligned}\big|\big(x,u(x)\big)-\zeta\big| &\geqslant \big|\zeta-\big(y,u(y)\big)\big|-\big|\big(x,u(x)\big)-\big(y,u(y)\big)\big|\\ &\geqslant \frac{1}{L}\cos\gamma-\frac{(1+\lambda)\cos\gamma}{L(1+\lambda+\cos\gamma)}\\ &=\frac{\cos^2\gamma}{L(1+\lambda+\cos\gamma)}.\end{aligned}$$

1.*ii*. The case $|x-y|>\frac{\cos\gamma}{L(1+\lambda+\cos\gamma)}$.

Again let $\pi^{\perp}((x,u(x)))$ be the orthogonal projection of $(x,u(x))$ onto $(y,u(y))+\omega(y)$. With (A.2) we estimate

$$\begin{aligned}\big|\big(x,u(x)\big)-\zeta\big| &\geqslant \big|\big(x,u(x)\big)-\pi^{\perp}\big(\big(x,u(x)\big)\big)\big|\\ &\geqslant |x-y|\cos\gamma\\ &>\frac{\cos^2\gamma}{L(1+\lambda+\cos\gamma)}.\end{aligned}\tag{A.6}$$

Both in Case 1.*i* and Case 1.*ii* we have

$$\big|\big(x,u(x)\big)-\zeta\big|\geqslant\frac{\cos^2\gamma}{2L(1+\lambda)}.\tag{A.7}$$

Case 2. $\zeta=(z,u(z))+\upsilon$ for a $z\in\partial B_\varrho$ and $\upsilon\in\omega(z)$ with $|\upsilon|\leqslant\varepsilon$.

As $x\in\overline{B}_{\frac{\varrho}{2}}$ we have $|x-z|\geqslant\frac{\varrho}{2}$. Considering the orthogonal projection onto $(z,u(z))+\omega(z)$, we estimate as in (A.6)

$$\big|\big(x,u(x)\big)-\zeta\big|\geqslant\frac{\varrho}{2}\cos\gamma.\tag{A.8}$$

With (A.7) and (A.8) we have in any case

$$\big|\big(x,u(x)\big)-\zeta\big|\geqslant\min\left\{\frac{\varrho}{2}\cos\gamma,\frac{\cos^2\gamma}{2L(1+\lambda)}\right\}.$$

This proves part b) of Lemma 2.15. □

Lemma A.1. *Let $f:M^m\to\mathbb{R}^{m+1}$ be an (r,λ)-immersion, $q\in M$ and $0<\varrho\leqslant r$. Let $\omega\in G_{m+1,1}$ with $\mathbb{R}^{m+1}=\tau_f(p)\oplus\omega$ for all $p\in U_{\varrho,q}$. Then for every $x\in\mathbb{R}^{m+1}$ the set $x+\omega$ intersects $f(U_{\varrho,q})$ in at most one point.*

Proof. After a rotation and a translation we may assume $f(U_{\varrho,q})=\{(y,u(y)):\ y\in B_\varrho\}$ with a C^1-function $u:B_\varrho\to\mathbb{R}$. Suppose the assertion of the lemma is false. Then there is an $x\in\mathbb{R}^{m+1}$ such that $x+\omega$ intersects $f(U_{\varrho,q})$ in $(y,u(y))$ and in $(z,u(z))$ with $y\neq z$. We may assume $y-z\in\mathbb{R}^1\times\{0\}\subset\mathbb{R}^m$. With the same argument as in the paragraph after (A.1) we conclude that there is a $\xi\in\{(1-t)y+tz:\ t\in(0,1)\}\subset B_\varrho$ with $\omega=\operatorname{span}\{(e_1,\partial_1u(\xi))\}$. Moreover there is a unique $\zeta\in U_{\varrho,q}$ with $\tau_f(\zeta)=\operatorname{span}\{(e_1,\partial_1u(\xi)),\dots,(e_m,\partial_mu(\xi))\}$. Hence $\omega\subset\tau_f(\zeta)$. But this contradicts $\mathbb{R}^{m+1}=\tau_f(p)\oplus\omega$ for all $p\in U_{\varrho,q}$. □

References

[1] H.W. Alt, Lineare Funktionalanalysis, fourth edition, Springer, Berlin, Heidelberg, 2002.
[2] A.A. Borisenko, Yu.A. Nikolaevskii, Grassmann manifolds and the Grassmann image of submanifolds, Russian Math. Surveys 46 (2) (1991) 45–95.
[3] P. Breuning, Immersions with local Lipschitz representation, dissertation, Freiburg, 2011.
[4] P. Breuning, Immersions with bounded second fundamental form, arXiv:1201.4562v1, 2011, preprint.
[5] T. Bröcker, K. Jänich, Introduction to Differential Topology, Cambridge University Press, New York, 1982.
[6] K. Corlette, Immersions with bounded curvature, Geom. Dedicata 33 (1990) 153–161.

[7] S. Delladio, On hypersurfaces in R^{n+1} with integral bounds on curvature, J. Geom. Anal. 11 (2000) 17–41.
[8] D. Gromoll, W. Klingenberg, W. Meyer, Riemannsche Geometrie im Großen, Lecture Notes in Math., vol. 55, Springer, Berlin, Heidelberg, New York, 1968.
[9] K. Grove, H. Karcher, How to conjugate C^1-close group actions, Math. Z. 132 (1973) 11–20.
[10] M.W. Hirsch, Differential Topology, Grad. Texts in Math., vol. 33, Springer, New York, 1976.
[11] J. Jost, Riemannian Geometry and Geometric Analysis, third edition, Springer, Berlin, Heidelberg, 2002.
[12] J. Jost, Y.L. Xin, Bernstein type theorems for higher codimension, Calc. Var. Partial Differential Equations 9 (4) (1999) 277–296.
[13] H. Karcher, Riemannian center of mass and mollifier smoothing, Comm. Pure Appl. Math. 30 (1977) 509–541.
[14] J. Langer, A compactness theorem for surfaces with L_p-bounded second fundamental form, Math. Ann. 270 (1985) 223–234.
[15] J.M. Lee, Introduction to Smooth Manifolds, Grad. Texts in Math., vol. 218, Springer, New York, 2003.
[16] K. Leichtweiss, Zur Riemannschen Geometrie in Grassmannschen Mannigfaltigkeiten, Math. Z. 76 (1961) 334–366.
[17] Y.-C. Wong, Sectional curvatures of Grassmann manifolds, Proc. Natl. Acad. Sci. USA 60 (1968) 75–79.