Commutability of homogenization and linearization at identity in finite elasticity and applications

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Abstract

We prove under some general assumptions on elastic energy densities (namely, frame indifference, minimality at identity, non-degeneracy and existence of a quadratic expansion at identity) that homogenization and linearization commute at identity. This generalizes a recent result by S. Müller and the second author by dropping their assumption of periodicity. As a first application, we extend their Γ-convergence commutation diagram for linearization and homogenization to the stochastic setting under standard growth conditions. As a second application, we prove that the Γ-closure is local at identity for this class of energy densities.
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Résumé

Nous démontrons que linéarisation et homogénéisation commutent à l'identité sous des hypothèses générales sur la densité d'énergie élastique (à savoir indifférence matérielle, minimalité à l'identité, non-dégénérescence et existence d'un développement quadratique à l'identité). Ceci généralise un résultat récent de S. Müller et du second auteur au cas non-périodique. En particulier, nous étendons au cas de l'homogénéisation stochastique leur diagramme de commutation de la linéarisation et de l'homogénéisation au sens de la Γ-convergence. Par ailleurs, nous démontrons que la Γ-fermeture est locale à l'identité pour la classe de densités d'énergie non convexes considérée.
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1. Introduction

We study the commutability of linearization and homogenization at identity in finite elasticity. We consider an open bounded Lipschitz domain $D \subset \mathbb{R}^d$, and a family of integral functionals

$$\mathcal{I}_\varepsilon : H^1(D) \to [0, +\infty], \quad u \mapsto \int_D W_\varepsilon\big(x, \nabla u(x)\big)\, \mathrm{d}x$$

where $W_\varepsilon : D \times \mathbb{M}^d \to [0, +\infty]$ is a Borel function. As it is common in finite elasticity, we assume that W_ε is frame indifferent and minimal at identity. Moreover, we assume that W_ε is non-degenerate and admits a quadratic expansion at identity with quadratic term Q_ε; as a consequence, in situations when the deformation is close to a rigid-body motion, say when $|\nabla u - \mathrm{Id}| \sim h \ll 1$, we can accurately describe the functional $\mathcal{I}_\varepsilon$ (after scaling by h^{-2}) by the quadratic functional

$$\mathcal{E}_\varepsilon : H^1(D) \to [0, +\infty], \quad g \mapsto \int_D Q_\varepsilon\big(x, \nabla g(x)\big)\, \mathrm{d}x \quad \text{with } g(x) := h^{-1}\big(u(x) - x\big).$$

Since $Q_\varepsilon(\cdot, F)$ genuinely only depends on the symmetric part of the strain gradient F, the energy $\mathcal{E}_\varepsilon$ corresponds to linear elasticity. On the other hand, if $\mathcal{I}_\varepsilon$ has some specific structure in space rescaled by ε (think of periodicity for instance), we may expect a homogenization property to hold as ε vanishes, which justifies to replace the non-linear oscillating-in-space energy density $(x, F) \mapsto W_\varepsilon(x, F)$ by a nonlinear homogeneous-in-space energy density $F \mapsto W_{\mathrm{hom}}(F)$ (or more generally by an energy density $(x, F) \mapsto W^*(x, F)$ whose oscillations in x are independent of ε).

In this paper, we address the commutability of both limits (in h and ε), and prove that they indeed commute in the following sense: The quadratic expansion of the homogenized energy W_{hom} (resp. W^*) at identity coincides with the homogenization of the quadratic expansion Q_ε of the heterogeneous energy density at identity. In Theorem 2.1 we study functionals with standard growth and prove that the commutability holds (both, on the level of densities and on the level of the functionals), provided $\mathcal{I}_\varepsilon$ can be homogenized in the sense that $\mathcal{I}_\varepsilon$ $\Gamma(L^2)$-converges to a functional of the form

$$u \mapsto \int_D W^*(x, \nabla u)\, \mathrm{d}x.$$

In Theorems 2.4 and 2.5 we study unbounded energies and show that the commutability still holds provided both $\mathcal{I}_\varepsilon$ and $\mathcal{E}_\varepsilon$ can be homogenized, or at least at the level of the Γ-liminf and Γ-limsup if no assumption is made on $\mathcal{I}_\varepsilon$. These theorems cover in particular the case when $W_\varepsilon(x, F) = +\infty$ if $\det F \leqslant 0$—as it is desirable in finite elasticity. Our results generalize a recent work by S. Müller and the second author in [13] (see also [15]) by relaxing the periodicity assumption on W_ε (as well as the growth condition from above). In [13] the central object in the analysis is a multi-cell homogenization formula that allows in the periodic setting to compute the homogenized energy density W_{hom} by solving a sequence of periodic minimization problems on cubic domains invading $\mathbb{R}^d$. In [15] the commutability of homogenization and linearization (solely as a Γ-convergence statement on the level of the energies) has been extended in the periodic case to energy densities without growth condition from above by extensive use of two-scale convergence methods. In the general situation considered in the present contribution, both the multi-cell homogenization formula and two-scale convergence approaches do not apply. Instead, we study the asymptotic formula

$$W_D(F) := \lim_{\varepsilon \to 0} \bigg\{ \inf_{v \in H_0^1(D)} \mathcal{I}_\varepsilon(\varphi_F + v) \bigg\} \quad \big(\text{where } \varphi_F(x) := Fx\big)$$

which is well defined whenever $\mathcal{I}_\varepsilon$ Γ-converges and is equi-coercive. In Proposition 2.2 we establish a quadratic expansion at identity for W_D—which is the key insight in our analysis.

As a first application of Theorem 2.1, we show that linearization and stochastic homogenization commute at identity for energy densities which satisfy standard growth conditions (see Theorem 3.2). In a nutshell, what holds in [13] in the periodic setting is also proved to hold here in the stochastic setting. This shows that the arguments used by S. Müller and the second author in [13] are quite stable with respect to the structure assumption which ensures homogenization—at the core of the proof the quantitative rigidity estimate of [7].

As a second application of Theorem 2.1, we prove a "weak local property" of the Γ-closure of a class of integral functionals at identity. The problem of Γ-closure consists in characterizing all the energy densities which can be reached by Γ-convergence starting from a composite made of a finite number of constituents with prescribed

volume fraction. In particular, the Γ-closure is said to be local in some class of integrands if and only if any such "homogenized" energy density is the pointwise limit of a sequence of homogenized energy densities obtained by *periodic* homogenization. In the linear case, this property has been proved independently by Tartar in [18] and Lurie and Cherkaev in [11]. The corresponding locality property of the G-closure for monotone operators is due to Raitums in [16] (generalizing an unpublished work by Dal Maso and Kohn). Related results of locality of the Γ-closure in the class of convex integrands can be found in [1]. Yet, the local character of the Γ-closure is an open question in the class of quasiconvex nonconvex integrands satisfying standard growth conditions. We focus here on a smaller class. In particular, we consider energy densities which are frame indifferent, non-degenerate, minimal at identity, admit a quadratic Taylor expansion at identity, and satisfy standard growth conditions. Then, we show that for any $F \mapsto W^*(F)$ in the Γ-closure of this set, there exists a sequence of periodic energy densities whose homogenized energy densities have quadratic Taylor expansions arbitrary close to the Taylor expansion of W^* at identity (see Theorem 4.1). This can be seen as a weak version of the local character of the Γ-closure in this set at identity. Although quite restricted, this is the first such result for quasiconvex nonconvex energy densities.

This article is organized as follows: In Section 2 we state and prove our main theorem, the commutability of linearization and homogenization at identity. In Section 3 we apply this result to stochastic homogenization. The last section is dedicated to the local character of the Γ-closure at identity.

We will make use of the following notation throughout the text:

- $\mathbb{R}^+ := [0, +\infty)$ is the set of non-negative real numbers;
- d is the dimension;
- $\mathbb{M}^d$ denotes the space of $d \times d$ real matrices, and for all $F \in \mathbb{M}^d$, $\operatorname{sym} F = 1/2(F + F^T)$ is the associated symmetric matrix, and $\operatorname{skw} F = F - \operatorname{sym} F$ the associated skew-symmetric matrix;
- $\operatorname{SO}(d)$ is the set of rotations of $\mathbb{R}^d$;
- $\mathbb{T}_{\text{sym}}^d$ denotes the space of symmetric fourth order tensors on $\mathbb{R}^d$;
- D denotes an open bounded subset of $\mathbb{R}^d$ with a Lipschitz boundary (except for Theorems 2.4 and 2.5, and Proposition 2.2 in Section 2 where D is further assumed to be C^1);
- $U = (0, 1)^d$ is the unit cell;
- for all $F \in \mathbb{M}^d$, we define the function $\varphi_F : \mathbb{R}^d \to \mathbb{R}^d$ as $\varphi_F : x \mapsto Fx$;
- for all $p \in [1, +\infty]$, $L^p(D)$, $H^1(D)$, $W^{1,p}(D)$, $H_0^1(D)$, and $W_0^{1,p}(D)$ denote the standard Lebesgue, Hilbert and Sobolev spaces of maps from D to $\mathbb{R}^d$, and the associated subspaces of functions vanishing on the boundary ∂D (in the sense of traces);
- ε and h denote generic elements of vanishing families of positive numbers (ε) and (h), respectively;
- ρ (and ρ') denotes a modulus of approximation, i.e. ρ is an increasing function from $\mathbb{R}^+$ to $[0, +\infty]$ such that $\lim_{h \to 0} \rho(h) = 0$.

2. General commutability results

2.1. General framework

Throughout this article, we make the following assumptions on the energy densities.

Definition 1. For all $\alpha > 0$ and every modulus of approximation ρ, we denote by $\mathcal{W}_{\alpha,\rho}$ the set of Borel functions $W : \mathbb{M}^d \to [0, +\infty]$ which satisfy the following three properties:

(W1) W is frame indifferent, i.e.

$$W(RF) = W(F) \quad \text{for all } F \in \mathbb{M}^d, \ R \in \operatorname{SO}(d);$$

(W2) W is non-degenerate, i.e.

$$W(F) \geqslant \frac{1}{\alpha} \operatorname{dist}^2\big(F, \operatorname{SO}(d)\big) \quad \text{for all } F \in \mathbb{M}^d;$$

(W3) W is minimal at Id and admits the following quadratic expansion at Id:

$$\sup_{0<|G|\leqslant\delta}\frac{|W(\mathrm{Id}+G)-Q(G)|}{|G|^2}\leqslant\rho(\delta)\quad\text{for all }\delta>0,$$

where $Q:\mathbb{M}^d\to[0,\infty)$ is a quadratic form satisfying

$$0\leqslant Q(G)\leqslant\alpha|G|^2\quad\text{for all }G\in\mathbb{M}^d.$$

Remark 1. The energy densities of class $\mathcal{W}_{\alpha,\rho}$ describe elastic materials with a single, quadratic energy well at $\mathrm{SO}(d)$. The minimality condition in (W3) implies that the reference state $F=\mathrm{Id}$ is stress free. The combination of (W2) and (W3) might be interpreted as a generalization of Hooke's law to geometrically nonlinear material laws: For infinitesimal small strains we expect a linear stress-strain relation. Indeed, in view of condition (W3) the material law is sufficiently smooth to allow a linearization around the reference state; this is made precise in [13, Theorem 5.1] and [6] on the level of the associated energy functional. Let us remark that (W3) is typically satisfied if W is of class C^2 in a neighborhood of Id. Since in the homogenization of nonlinear materials buckling phenomena might occur even for deformations close to $\mathrm{SO}(d)$ (cf. [13, Section 7]), it is not clear whether C^2 regularity in a neighborhood of Id is stable by homogenization. However, as a by-product of our main result, we shall prove that (W3) is stable by homogenization.

For some results (in particular Theorem 2.1) we consider continuous energy densities that additionally satisfy standard growth conditions:

Definition 2. For all $p\in[1,+\infty)$ and $\alpha>0$, we denote by $\mathcal{W}_\alpha^p$ the set of continuous energy densities $W:\mathbb{M}^d\to\mathbb{R}$ which satisfy the following standard growth condition of order p:

(W4) $\forall F\in\mathbb{M}^d:\quad\dfrac{1}{\alpha}|F|^p-\alpha\leqslant W(F)\leqslant\alpha\big(|F|^p+1\big).$

In addition, we set $\mathcal{W}_{\alpha,\rho}^p:=\mathcal{W}_{\alpha,\rho}\cap\mathcal{W}_\alpha^p$ for every modulus of approximation ρ. Note that $\mathcal{W}_{\alpha,\rho}^p=\emptyset$ for $p<2$, and $\mathcal{W}_{\alpha,\rho}^p\neq\emptyset$ for $p\geqslant2$.

Remark 2. Let $W\in\mathcal{W}_{\alpha,\rho}$ and let Q denote the quadratic form associated with W through (W3). Because of (W1)–(W3) the quadratic form Q generically satisfies conditions that are common in linear elasticity; namely, the growth and ellipticity condition

(Q1) $\forall G\in\mathbb{M}^d:\quad\dfrac{1}{\alpha'}|\mathrm{sym}\,G|^2\leqslant Q(G)\leqslant\alpha'|G|^2$

for some positive constant α' that only depends on α, and

(Q2) $\forall G\in\mathbb{M}^d:\quad Q(\mathrm{skw}\,G)=0.$

The property (Q2) follows from a Taylor expansion of W at identity using (W3) and the fact that $W(F)$ depends only on $F^T F$ by (W1). The non-degeneracy condition (Q1) on the quadratic form is inherited from the non-degeneracy condition (W2).

Definition 3. We denote by $\mathcal{Q}_{\alpha'}$ the set of non-negative quadratic forms $Q:\mathbb{M}^d\to\mathbb{R}^+$ satisfying (Q1) and (Q2).

In this article we frequently consider mappings of the form $W(x,\nabla u(x))$ where W is a non-negative function defined on $D\times\mathbb{M}^d$ and $u\in W^{1,1}(D)$. To guarantee the measurability of such a composition we introduce the following definition:

Definition 4. Let $\mathcal{C}$ be one of the classes $\mathcal{W}_{\alpha,\rho}$, $\mathcal{W}_{\alpha}^{p}$, $\mathcal{W}_{\alpha,\rho}^{p}$ or $\mathcal{Q}_{\alpha}$, and $D \subset \mathbb{R}^{d}$ a Borel set. We denote by $\mathcal{C}(D \times \mathbb{M}^{d})$ the set of functions $W : D \times \mathbb{M}^{d} \to [0, +\infty]$ such that W is a Borel function (or equivalent to a Borel function) and $W(x, \cdot) \in \mathcal{C}$ for almost every $x \in D$.

Clearly, if $W : D \times \mathbb{M}^{d} \to \mathbb{R}^{+}$ satisfies $W(x, \cdot) \in \mathcal{W}_{\alpha,\rho}^{p}$ for almost every $x \in D$ and $W(\cdot, F)$ is measurable for all $F \in \mathbb{M}^{d}$, then W is a Carathéodory function and therefore $W \in \mathcal{W}_{\alpha,\rho}^{p}(D \times \mathbb{M}^{d})$.

Let us consider a family of energy densities $(W_{\varepsilon}) \subset \mathcal{W}_{\alpha,\rho}(D \times \mathbb{M}^{d})$. For almost every $x \in D$, we denote by $Q_{\varepsilon}(x, \cdot)$ the quadratic form associated with $W_{\varepsilon}(x, \cdot)$ through (W3); thus, Q_{ε} can be written as the pointwise limit

$$(x, G) \mapsto Q_{\varepsilon}(x, G) := \lim_{h \to 0} \frac{1}{h^{2}} W_{\varepsilon}(x, \mathrm{Id} + hG),$$

and therefore inherits the measurability properties of W_{ε}. We then define two families of integral functionals, namely $\mathcal{I}_{\varepsilon} : H^{1}(D) \to [0, +\infty]$ characterized by

$$\mathcal{I}_{\varepsilon}(u) := \int_{D} W_{\varepsilon}(x, \nabla u(x)) \, \mathrm{d}x, \tag{1}$$

and $\mathcal{E}_{\varepsilon} : H^{1}(D) \to [0, +\infty)$ characterized by

$$\mathcal{E}_{\varepsilon}(u) := \int_{D} Q_{\varepsilon}(x, \nabla u(x)) \, \mathrm{d}x. \tag{2}$$

2.2. Commutation result for energy density with p-growth

The main theorem of this paper is the following result, which generalizes [13, Theorems 1 and 2] to the non-periodic setting.

Theorem 2.1. *Let $2 \leqslant p < +\infty$, $(W_{\varepsilon}) \subset \mathcal{W}_{\alpha,\rho}^{p}(D \times \mathbb{M}^{d})$ be a family of energy densities and let $W^{*} \in \mathcal{W}_{\alpha}^{p}(D \times \mathbb{M}^{d})$. Assume that the associated family of energy functionals $\mathcal{I}_{\varepsilon}$ defined in (1) $\Gamma(L^{p})$-converges to the integral functional $\mathcal{I}^{*}$ on $W^{1,p}(D)$, defined by*

$$\mathcal{I}^{*}(u) := \int_{D} W^{*}(x, \nabla u(x)) \, \mathrm{d}x.$$

Then there exist positive constants α', α'', a modulus of approximation ρ' (all only depending on α and ρ), and $Q^{*} \in \mathcal{Q}_{\alpha''}(D \times \mathbb{M}^{d})$ such that the following properties hold:

(a) *$W^{*} \in \mathcal{W}_{\alpha',\rho'}^{p}(D \times \mathbb{M}^{d})$ and the expansion*

$$W^{*}(x, \mathrm{Id} + G) = Q^{*}(x, G) + o(|G|^{2})$$

holds for almost every $x \in D$ and for all $G \in \mathbb{M}^{d}$;

(b) *the energy functionals $\mathcal{E}_{\varepsilon}$ defined in (2) $\Gamma(L^{2})$-converge to $\mathcal{E}^{*} : H^{1}(D) \to [0, +\infty)$ defined by*

$$\mathcal{E}^{*}(u) := \int_{D} Q^{*}(x, \nabla u(x)) \, \mathrm{d}x;$$

(c) *the following diagram commutes*

$$
\begin{array}{ccc}
\mathcal{G}_{h,\varepsilon} & \xrightarrow{\ (1)\ } & \mathcal{E}_{\varepsilon} \\
{\scriptstyle(2)}\downarrow & & \downarrow{\scriptstyle(3)} \\
\mathcal{G}_{h}^{*} & \xrightarrow[\ (4)\ ]{} & \mathcal{E}^{*}
\end{array}
$$

where $\mathcal{G}_{h,\varepsilon}$ *and* $\mathcal{G}_h^*$ *denote the functionals from* $H_0^1(D)$ *to* $[0, +\infty]$ *defined as*

$$\mathcal{G}_{h,\varepsilon}(g) := \frac{1}{h^2}\mathcal{I}_\varepsilon(\varphi_{\mathrm{Id}} + hg), \qquad \mathcal{G}_h^*(g) := \frac{1}{h^2}\mathcal{I}^*(\varphi_{\mathrm{Id}} + hg);$$

and (1), (4), *and* (2), (3) *mean* Γ-*convergence in* $H_0^1(D)$ *with respect to the strong topology of* $L^2(D)$ *as* $h \to 0$ *and* $\varepsilon \to 0$, *respectively. Moreover, the families* $(\mathcal{I}_\varepsilon)$ *and* $(\mathcal{E}_\varepsilon)$ *are equi-coercive w.r.t. weak convergence in* $H_0^1(D)$.

Remark 3. Due to the compactness of integral functionals with standard p-growth conditions w.r.t. $\Gamma(L^p)$-convergence (see for instance [2, Theorem 12.5]), the assumptions on $\mathcal{I}_\varepsilon$ are always satisfied up to extraction of a subsequence.

Remark 4. If (W_ε) satisfies a growth condition from below of order $p \geqslant 2$ (uniformly in ε) then $\mathcal{I}_\varepsilon \equiv +\infty$ on $H^1(D) \setminus W^{1,p}(D)$ and it is natural to study the restricted functionals $\mathcal{I}_\varepsilon|_{W^{1,p}(D)}$ w.r.t. the strong topology in $L^p(D)$. In particular, $\mathcal{I}_\varepsilon|_{W^{1,p}(D)}$ is sequentially weakly lower semicontinuous in $W^{1,p}(D)$ if and only if it is lower semicontinuous w.r.t. strong convergence in $L^p(D)$. Note that due to condition (W2), (W_ε) generically satisfies a uniform growth condition from below of order $p = 2$.

Remark 5. As in [13], in Theorem 2.1 part (d), we can replace the function space $H_0^1(D)$ by the space

$$\mathcal{A}_\gamma := \{g \in H^1(D): g = 0 \text{ on } \gamma\},$$

where γ denotes a closed subset of ∂D with positive $(d-1)$-dimensional Hausdorff measure, and regular enough so that $\mathcal{A}_\gamma \cap W^{1,\infty}(D)$ is dense in $\mathcal{A}_\gamma$ (see [13, proof of Proposition 1] and [6] for details).

Theorem 2.1 follows from a result which is somewhat unrelated to homogenization, and establishes a quadratic expansion at Id for the asymptotic formula

$$W_D(F) := \lim_{\varepsilon \to 0}\left\{\inf_{v \in W_0^{1,p}(D)} \mathcal{I}_\varepsilon(\varphi_F + v)\right\} \tag{3}$$

if it exists.

Proposition 2.2. *Let* $2 \leqslant p < +\infty$, *let the domain* D *be* C^1, *and consider a family of energy densities* $(W_\varepsilon) \subset \mathcal{W}_{\alpha,\rho}(D \times \mathbb{M}^d)$. *Suppose that for all* $F \in \mathbb{M}^d$ *the limit* (3) *exists in* $[0, +\infty]$ *(where* $\mathcal{I}_\varepsilon$ *is as in* (1)*), and that the functionals* $\mathcal{E}_\varepsilon$ *defined in* (2) $\Gamma(L^2)$-*converge to a functional* $\mathcal{E}^*$ *on* $H^1(D)$. *Then there exist a constant* $\alpha' > 0$ *that only depends on* α, *and a modulus of approximation* ρ' *that additionally depends on* ρ *and on the geometry of* D, *such that* $\frac{1}{|D|}W_D \in \mathcal{W}_{\alpha',\rho'}$ *and*

$$\frac{|W_D(\mathrm{Id} + G) - \inf_{v \in H_0^1(D)} \mathcal{E}^*(\varphi_G + v)|}{|G|^2} \leqslant |D|\rho'(|G|) \tag{4}$$

for all $G \in \mathbb{M}^d$.

Remark 6. In the proof of Proposition 2.2 we make ρ' explicit:

$$\rho'(h) = C \max\left\{\rho(h + \sqrt{h})(1 + \alpha + \rho(h)) + h^{\frac{\mu}{4(2+\mu)}}(1 + \alpha + \rho(h))^{\frac{4+3\mu}{4(2+\mu)}}, h^{\frac{\mu}{2}} + \rho(h + \sqrt{h})(1 + h^{\frac{\mu}{2}})\right\}, \tag{5}$$

where the constants $C, \mu > 0$ only depend on α and on the geometry of D, i.e. C and μ are invariant under dilation, rotation, and translation of D. Note that for $h \ll 1$, (5) reduces to

$$\rho'(h) \sim C\left(\rho(\sqrt{h}) + h^{\frac{\mu}{4(2+\mu)}}\right).$$

Remark 7. The assumption on $\mathcal{E}_\varepsilon$ is no restriction. In particular, by the compactness of G-convergence (see for instance [10, Section 12.2]), we can always extract a subsequence of (ε) to which Proposition 2.2 applies.

In the proof of Proposition 2.2 we will make use of the following higher integrability and Lipschitz truncation result for minimizers of quadratic functionals:

Proposition 2.3. *Let $\alpha > 0$, $G \in \mathbb{M}^d$, and $Q \in \mathcal{Q}_\alpha(D \times \mathbb{M}^d)$. Set*

$$\mathcal{E}_G : H_0^1(D) \to [0, +\infty), \quad \mathcal{E}_G(g) := \int_D Q(x, G + \nabla g) \, dx.$$

(a) *The functional $\mathcal{E}_G$ admits a unique minimizer $g^* \in H_0^1(D)$, characterized by the Euler–Lagrange equation*

$$\int_D \langle \mathbb{L}(x)(G + \nabla g^*), \nabla \varphi \rangle \, dx = 0 \quad \text{for all } \varphi \in H_0^1(D)$$

where $\mathbb{L} \in L^\infty(D, \mathbb{T}_{\text{sym}}^d)$ is defined by

$$\langle \mathbb{L}(x)A, B \rangle = \frac{Q(x, A + B) - Q(x, A) - Q(x, B)}{2}$$

for all $A, B \in \mathbb{M}^d$ and almost every $x \in D$.

(b) *(Meyers' estimate). If in addition the domain D is C^1, then there exists a Meyers' exponent $\mu > 0$ and a positive constant C such that*

$$\left\| \nabla g^* \right\|_{L^{2+\mu}(D)}^{2+\mu} \leqslant C |D| |G|^{2+\mu}.$$

The exponent μ and the constant C only depend on α and on the geometry of the domain D.

(c) *(Lipschitz truncation). Let $\lambda > 0$. If in addition the domain D is C^1, then there exists a map $g \in W_0^{1,\infty}(D)$ such that*

$$\left| \nabla g(x) \right| \leqslant \lambda \quad \text{for a.e. } x \in D,$$

$$\mathcal{E}_G(g) - \mathcal{E}_G(g^*) \leqslant C \lambda^{-\mu} |D| |G|^{2+\mu},$$

where μ is the Meyers' exponent, and the constant C only depends on α and on the geometry of the domain D (in particular, it is independent of λ, G, and μ).

The first statement of Proposition 2.3 is standard and relies on Korn's inequality. The second statement is a higher integrability result for gradients in linear elasticity proved in [17]. This is the only place where we use the regularity of the domain. The third statement is essentially a combination of Meyers' estimate with a Lipschitz truncation argument from [7]. The constants C and μ only depend on the geometry of the domain in the sense that they can be chosen invariant under translation, rotation, and dilation of the domain. The proof of this statement is deferred to the end of this section.

Proof of Proposition 2.2. We divide the proof into three steps. In the first step, we introduce a quadratic form associated with W_D. The last two steps are dedicated to the proof of (4).

Step 1. Definition of the quadratic form Q_D.

By the assumptions on W_ε the associated quadratic form Q_ε belongs to $\mathcal{Q}_{\tilde{\alpha}'}(D \times \mathbb{M}^d)$, where $\tilde{\alpha}' > 0$ only depends on α. Remark 2 and Korn's inequality on D thus imply that the quadratic energies $\mathcal{E}_\varepsilon$ are equi-coercive functionals on $H_0^1(D)$, so that the associated elliptic operators are compact w.r.t. G-convergence (see for instance [10, Section 12.2]). In particular, this yields Γ-convergence of the energy functionals to an integral functional (see for instance [9, Section 4.4]): There exist $\tilde{\alpha}''$ depending only on α, and an energy density $Q^* \in \mathcal{Q}_{\tilde{\alpha}''}(D \times \mathbb{M}^d)$ such that $\mathcal{E}_\varepsilon \, \Gamma(L^2)$-converges (up to extraction) to the functional $\mathcal{E}^* : H^1(D) \to \mathbb{R}$ characterized by

$$\mathcal{E}^* : u \mapsto \int_D Q^*(x, \nabla u(x)) \, dx. \tag{6}$$

This shows that $\mathcal{E}^*$ is a quadratic integral functional.

We are now in position to define the map $Q_D : \mathbb{M}^d \to [0, +\infty)$ as

$$Q_D(G) := \inf_{v \in H_0^1(D)} \mathcal{E}^*(\varphi_G + v).$$

Because of the representation (6), the map Q_D is a quadratic form of class $\mathcal{Q}_{\tilde{\alpha}}$ for a positive constant $\tilde{\alpha}$ depending only on α.

We claim that $\frac{1}{|D|} W_D$ is of class $\mathcal{W}_{\alpha', \rho'}$ where ρ' is defined by (5). It is clear that $\frac{1}{|D|} W_D$ is frame indifferent. It also satisfies a property of type (W2) as an application of [13, Lemma 2] (the proof of which actually does not use periodicity, but only the asymptotic formula (3)). The expansion property (W3) is equivalent to (4). As in [13] we notice that it is sufficient to prove the following: For all families $(G_h) \in \mathbb{R}^d$ with $|G_h| = 1$, we have:

(lower bound) $\qquad \dfrac{1}{h^2} W_D(\mathrm{Id} + h G_h) \geqslant Q_D(G_h) - \dfrac{|D|}{2} \rho'(h),$

(upper bound) $\qquad \dfrac{1}{h^2} W_D(\mathrm{Id} + h G_h) \leqslant Q_D(G_h) + \dfrac{|D|}{2} \rho'(h).$

We prove both statements separately.

Step 2. Proof of the lower bound.

By definition of W_D (see (3)), for all $h > 0$,

$$0 \leqslant W_D(\mathrm{Id} + h G_h) \leqslant \limsup_{\varepsilon \to 0} \int_D W_\varepsilon(x, \mathrm{Id} + h G_h).$$

From (W3), the fact that $Q_\varepsilon(x, G) \leqslant \alpha |G|^2$ for a.e. $x \in D$, and the assumption $|G_h| = 1$, we infer that

$$0 \leqslant \frac{1}{h^2} W_D(\mathrm{Id} + h G_h) \leqslant |D|\big(\alpha + \rho(h)\big). \tag{7}$$

By definition of W_D, there exists a sequence $(u_{h,\varepsilon}) \in W^{1,p}(D)$ with the properties

$$u_{h,\varepsilon} - \varphi_{\mathrm{Id} + h G_h} \in W_0^{1,p}(D) \subset H_0^1(D), \tag{8}$$

$$\lim_{\varepsilon \to 0} \mathcal{I}_\varepsilon(u_{h,\varepsilon}) = W_D(\mathrm{Id} + h G_h). \tag{9}$$

We then define the following sequence of scaled displacements

$$g_{h,\varepsilon} := \frac{u_{h,\varepsilon} - \varphi_{\mathrm{Id} + h G_h}}{h}.$$

By construction $g_{h,\varepsilon} \in H_0^1(D)$ and the uniform non-degeneracy assumption (W2) on W_ε yields the estimate

$$\frac{1}{h^2} \int_D \mathrm{dist}^2\big(\mathrm{Id} + h\big(G_h + \nabla g_{h,\varepsilon}(x)\big), \mathrm{SO}(d)\big)\, \mathrm{d}x \leqslant \alpha \frac{1}{h^2} \mathcal{I}_\varepsilon(u_{h,\varepsilon}).$$

The quantitative geometric rigidity estimate (see [7, Theorem 3.1]) implies the existence of a rotation $R_{h,\varepsilon} \in \mathrm{SO}(d)$ such that

$$\| F_{h,\varepsilon} + \nabla g_{h,\varepsilon} \|_{L^2(D)}^2 \leqslant C \frac{1}{h^2} \mathcal{I}_\varepsilon(u_{h,\varepsilon}) \quad \text{with } F_{h,\varepsilon} := \frac{\mathrm{Id} - R_{h,\varepsilon}}{h} + G_h.$$

Except otherwise stated, C denotes a positive constant that may vary from line to line, but can be chosen only depending on α and on the geometry of D. Because $g_{h,\varepsilon}$ vanishes on ∂D, an integration by parts shows that $\nabla g_{h,\varepsilon}$ and the (constant) matrix $F_{h,\varepsilon}$ are orthogonal w.r.t. the inner product in $L^2(D; \mathbb{M}^d)$:

$$\| F_{h,\varepsilon} + \nabla g_{h,\varepsilon} \|_{L^2(D)}^2 = |D| |F_{h,\varepsilon}|^2 + \| \nabla g_{h,\varepsilon} \|_{L^2(D)}^2 \geqslant \| \nabla g_{h,\varepsilon} \|_{L^2(D)}^2.$$

Hence, the rigidity estimate, (7), and (9) yield

$$\limsup_{\varepsilon \to 0} \|\nabla g_{h,\varepsilon}\|_{L^2(D)}^2 \leqslant C|D|(\alpha + \rho(h)). \tag{10}$$

Next, in order to make use of the quadratic expansion in (W3), we focus on the set where $h(G_h + \nabla g_{h,\varepsilon})$ is bounded. To this end, we let $\chi_{h,\varepsilon}$ denote the indicator function of the set $X_{h,\varepsilon} := \{x \in D: |\nabla g_{h,\varepsilon}| \leqslant h^{-1/2}\}$, and note that

$$\frac{1}{h^2} \mathcal{I}_\varepsilon(u_{h,\varepsilon}) = \frac{1}{h^2} \int_D W_\varepsilon\big(x, \mathrm{Id} + h(G_h + \nabla g_{h,\varepsilon}(x))\big) \, dx$$

$$\geqslant \frac{1}{h^2} \int_D \chi_{h,\varepsilon}(x) W_\varepsilon\big(x, \mathrm{Id} + h(G_h + \nabla g_{h,\varepsilon}(x))\big) \, dx$$

$$= \frac{1}{h^2} \int_D W_\varepsilon\big(x, \mathrm{Id} + h\chi_{h,\varepsilon}(x)(G_h + \nabla g_{h,\varepsilon}(x))\big) \, dx$$

by the non-negativity of W_ε and the fact that W_ε vanishes at Id. We then write the r.h.s. in the form

$$\frac{1}{h^2} \int_D W_\varepsilon\big(x, \mathrm{Id} + h\chi_{h,\varepsilon}(x)(G_h + \nabla g_{h,\varepsilon}(x))\big) \, dx = \int_D \big(Q_\varepsilon\big(x, \chi_{h,\varepsilon}(x)(G_h + \nabla g_{h,\varepsilon}(x))\big) + \mathrm{r}_{h,\varepsilon}(x)\big) \, dx,$$

where, using assumption (W3), the remainder $\mathrm{r}_{h,\varepsilon}$ satisfies for all $x \in X_{h,\varepsilon}$

$$|\mathrm{r}_{h,\varepsilon}(x)| = |G_h + \nabla g_{h,\varepsilon}(x)|^2 \times \frac{|W_\varepsilon(x, \mathrm{Id} + h(G_h + \nabla g_{h,\varepsilon}(x))) - Q_\varepsilon(x, h(G_h + \nabla g_{h,\varepsilon}(x)))|}{h^2 |G_h + \nabla g_{h,\varepsilon}(x)|^2}$$

$$\leqslant \rho\big(h|G_h| + \sqrt{h}\big)|G_h + \nabla g_{h,\varepsilon}(x)|^2$$

$$= \rho(h + \sqrt{h})|G_h + \nabla g_{h,\varepsilon}(x)|^2,$$

and $r_{h,\varepsilon}(x) = 0$ for a.e. $x \in D \setminus X_{h,\varepsilon}$. Thus, we conclude that

$$\frac{1}{h^2} \mathcal{I}_\varepsilon(u_{h,\varepsilon}) \geqslant \int_D Q_\varepsilon\big(x, \chi_{h,\varepsilon}(G_h + \nabla g_{h,\varepsilon})\big) \, dx - \rho(h + \sqrt{h}) \|G_h + \nabla g_{h,\varepsilon}\|_{L^2(D)}^2. \tag{11}$$

Appealing to (10) and using $|G_h| = 1$, (11) turns into

$$\liminf_{\varepsilon \to 0} \frac{1}{h^2} \mathcal{I}_\varepsilon(u_{h,\varepsilon}) \geqslant \liminf_{\varepsilon \to 0} \int_D Q_\varepsilon\big(x, \chi_{h,\varepsilon}(G_h + \nabla g_{h,\varepsilon})\big) \, dx - C|D|\rho(\sqrt{h} + h)\big(1 + \alpha + \rho(h)\big). \tag{12}$$

Next, we wish to replace the integral term on the r.h.s. of (12) by the infimum of $\mathcal{E}_\varepsilon$ on the set $\varphi_{G_h} + H_0^1(D)$. By coercivity of $\mathcal{E}_\varepsilon$ on this set, this infimum problem is well-posed, and there exists $g_{h,\varepsilon}^* \in H_0^1(D)$ such that $v_{h,\varepsilon} := \varphi_{G_h} + g_{h,\varepsilon}^*$ satisfies

$$\mathcal{E}_\varepsilon(v_{h,\varepsilon}) = \int_D Q_\varepsilon\big(x, \nabla v_{h,\varepsilon}(x)\big) \, dx = \inf_{v \in H_0^1(D)} \int_D Q_\varepsilon\big(x, G_h + \nabla v(x)\big) \, dx. \tag{13}$$

Since $\mathcal{E}_\varepsilon$ is equi-coercive on $\varphi_{G_h} + H_0^1(D)$, and $\mathcal{E}_\varepsilon$ $\Gamma(L^2)$-converges to $\mathcal{E}^*$ on $H^1(D)$, the Γ-limit is coercive, and the sequence of minima converges to the minimum of $\mathcal{E}^*$ on $\varphi_{G_h} + H_0^1(D)$. This yields

$$\lim_{\varepsilon \to 0} \mathcal{E}_\varepsilon(v_{h,\varepsilon}) = \inf_{v \in \varphi_{G_h} + H_0^1(D)} \mathcal{E}^*(v) = Q_D(G_h). \tag{14}$$

We shall actually prove that there exists $\mu > 0$ depending only on α and on the geometry of D such that

$$\liminf_{\varepsilon \to 0} \int_D \left(Q_\varepsilon\big(x, \chi_{h,\varepsilon}(G + \nabla g_{h,\varepsilon})\big) - Q_\varepsilon\big(x, \nabla v_{h,\varepsilon}(x)\big) \right) \mathrm{d}x \geqslant -C|D| h^{\frac{\mu}{4(2+\mu)}} \big(1 + \alpha + \rho(h)\big)^{\frac{4+3\mu}{4(2+\mu)}}. \tag{15}$$

Combined with (9), (12) and (14), (15) yields the desired lower bound.

The proof of (15) is the heart of the matter. Let $\mathbb{L}_\varepsilon \in L^\infty(D, \mathbb{T}^d_{\mathrm{sym}})$ denote the unique symmetric 4th order tensor associated with Q_ε, i.e.

$$\langle \mathbb{L}_\varepsilon(x)A, B \rangle = \frac{Q_\varepsilon(x, A + B) - Q_\varepsilon(x, A) - Q_\varepsilon(x, B)}{2} \tag{16}$$

for all $A, B \in \mathbb{M}^d$ and a.e. $x \in D$. Note that $(\mathbb{L}_\varepsilon)$ is uniformly bounded in $L^\infty(D, \mathbb{T}^d_{\mathrm{sym}})$ because the operator norm of $Q_\varepsilon(x, \cdot)$ on $\mathbb{M}^d$ is bounded by α for all $\varepsilon > 0$ and a.e. $x \in D$. Since $Q_\varepsilon(x, \cdot)$ is a non-negative quadratic form, the inequality

$$Q_\varepsilon(x, A) - Q_\varepsilon(x, B) \geqslant 2\langle \mathbb{L}_\varepsilon(x)(A - B), B \rangle \tag{17}$$

holds for all $A, B \in \mathbb{M}^d$ and a.e. $x \in D$. We use this estimate with $A = \chi_{h,\varepsilon}(x)(G_h + \nabla g_{h,\varepsilon}(x))$ and $B = \nabla v_{h,\varepsilon}(x)$, which yields by integration over D:

$$\int_D \left(Q_\varepsilon\big(x, \chi_{h,\varepsilon}(G_h + \nabla g_{h,\varepsilon})(x)\big) - Q_\varepsilon\big(x, \nabla v_{h,\varepsilon}(x)\big) \right) \mathrm{d}x$$

$$\geqslant 2 \int_D \big\langle \mathbb{L}_\varepsilon\big(\chi_{h,\varepsilon}(G_h + \nabla g_{h,\varepsilon}) - \nabla v_{h,\varepsilon}\big), \nabla v_{h,\varepsilon} \big\rangle \mathrm{d}x. \tag{18}$$

Following [13, proof of Theorem 1], we rewrite the r.h.s. as

$$\int_D \big\langle \mathbb{L}_\varepsilon\big(\chi_{h,\varepsilon}(G_h + \nabla g_{h,\varepsilon}) - \nabla v_{h,\varepsilon}\big), \nabla v_{h,\varepsilon} \big\rangle \mathrm{d}x = I_{h,\varepsilon}^{(1)} - I_{h,\varepsilon}^{(2)},$$

with

$$I_{h,\varepsilon}^{(1)} := \int_D \big\langle \mathbb{L}_\varepsilon(G_h + \nabla g_{h,\varepsilon} - \nabla v_{h,\varepsilon}), \nabla v_{h,\varepsilon} \big\rangle \mathrm{d}x,$$

$$I_{h,\varepsilon}^{(2)} := \int_D \big\langle \mathbb{L}_\varepsilon(1 - \chi_{h,\varepsilon})(G_h + \nabla g_{h,\varepsilon}), \nabla v_{h,\varepsilon} \big\rangle \mathrm{d}x.$$

Because $I_{h,\varepsilon}^{(1)}$ is the weak form of the Euler–Lagrange equation of the minimization problem in (13) with admissible test-function $\varphi_{G_h} + g_{h,\varepsilon} - v_{h,\varepsilon} \in H_0^1(D)$, the first term $I_{h,\varepsilon}^{(1)}$ vanishes identically. We now deal with the second term, and claim that

$$\limsup_{\varepsilon \to 0} \big| I_{h,\varepsilon}^{(2)} \big| \leqslant C|D| h^{\frac{\mu}{4(2+\mu)}} \big(1 + \alpha + \rho(h)\big)^{\frac{4+3\mu}{4(2+\mu)}}. \tag{19}$$

Combined with (18) and $I_{h,\varepsilon}^{(1)} \equiv 0$, this implies the desired estimate (15). To prove (19), we apply the higher integrability result of Proposition 2.3 part (b) to $\nabla v_{h,\varepsilon}$. In particular, there exists a Meyers' exponent $\mu > 0$ and a positive constant C such that

$$\int_D |\nabla v_{h,\varepsilon}|^{2+\mu} \, \mathrm{d}x \leqslant C|D| |G_h|^{2+\mu} = C|D|. \tag{20}$$

By Cauchy–Schwarz' and Hölder's inequalities, we may estimate $I_{h,\varepsilon}^{(2)}$ by

$$\left|I_{h,\varepsilon}^{(2)}\right| \leqslant C\|G_h + \nabla g_{h,\varepsilon}\|_{L^2(D)}\left\|(1 - \chi_{h,\varepsilon})\nabla v_{h,\varepsilon}\right\|_{L^2(D)}$$

$$\leqslant C\|G_h + \nabla g_{h,\varepsilon}\|_{L^2(D)}\|1 - \chi_{h,\varepsilon}\|_{L^q(D)}\|\nabla v_{h,\varepsilon}\|_{L^{2+\mu}(D)} \tag{21}$$

with $q := \frac{2(2+\mu)}{\mu} \in (1,\infty)$. By definition of $\chi_{h,\varepsilon}$ there holds

$$\left|1 - \chi_{h,\varepsilon}(x)\right| \leqslant \sqrt{h}\left(1 - \chi_{h,\varepsilon}(x)\right)\left|\nabla g_{h,\varepsilon}(x)\right| \leqslant \sqrt{h}\left|\nabla g_{h,\varepsilon}(x)\right|$$

for a.e. $x \in D$, so that

$$\int_D |1 - \chi_{h,\varepsilon}|^q \, dx = \int_D |1 - \chi_{h,\varepsilon}| \, dx \leqslant \sqrt{h}\int_D |\nabla g_{h,\varepsilon}| \, dx.$$

Hence, by Cauchy–Schwarz' inequality,

$$\|1 - \chi_{h,\varepsilon}\|_{L^q(D)} \leqslant Ch^{\frac{1}{2q}}|D|^{\frac{1}{2q}}\|g_{h,\varepsilon}\|_{H^1(D)}^{\frac{1}{q}} = C|D|^{\frac{\mu}{4(2+\mu)}}h^{\frac{\mu}{4(2+\mu)}}\|g_{h,\varepsilon}\|_{H^1(D)}^{\frac{\mu}{2(2+\mu)}},$$

which, combined with (10), (20) and (21), proves (19). This concludes the proof of the lower bound.

Step 3. Proof of the upper bound.

As usual, the proof of the upper bound relies on an explicit construction. As a first step we apply the Lipschitz truncation argument of Proposition 2.3 part (c) with $\lambda = h^{-1/2}$: There exists a doubly indexed sequence $(g_{h,\varepsilon}) \subset H_0^1(D)$ and some $\mu > 0$ (only depending on α and the geometry of D) such that

$$\begin{cases} \|\nabla g_{h,\varepsilon}\|_{L^\infty(D)} \leqslant h^{-1/2}, \\ E_\varepsilon(\varphi_{G_h} + g_{h,\varepsilon}) - Q_D(G_h) \leqslant Ch^{\mu/2}|D|. \end{cases} \tag{22}$$

Here and below, C denotes a positive constant that may vary from line to line, but only depends on α and on the geometry of D.

Since for all $\varepsilon > 0$ the quadratic form $\mathcal{E}_\varepsilon$ is Korn-elliptic with some constant α' depending only on α, the second property in (22) and Poincaré's inequality imply that the sequence $(g_{h,\varepsilon})_\varepsilon$ is bounded in $H^1(D)$. Using in addition Step 1 in the form of $Q_D(G_h) \leqslant \tilde{\alpha}|D|$, this yields the estimate

$$\|G_h + \nabla g_{h,\varepsilon}\|_{L^2(D)}^2 \leqslant C(1 + h^{\mu/2})|D|. \tag{23}$$

We set

$$u_{h,\varepsilon} := \varphi_{\mathrm{Id}+hG_h} + hg_{h,\varepsilon}.$$

By definition we have

$$W_D(\mathrm{Id} + hG_h) = \lim_{\varepsilon \to 0}\left\{\inf_{v \in H_0^1(D)} \mathcal{I}_\varepsilon(\varphi_{\mathrm{Id}+hG_h} + v)\right\} \leqslant \liminf_{\varepsilon \to 0} \mathcal{I}_\varepsilon(u_{h,\varepsilon})$$

$$= \liminf_{\varepsilon \to 0} \int_D W_\varepsilon\big(x, \mathrm{Id} + h\big(G_h + \nabla g_{h,\varepsilon}(x)\big)\big) \, dx. \tag{24}$$

As in the proof of the lower bound, we expand the r.h.s. as

$$\int_D W_\varepsilon\big(x, \mathrm{Id} + h\big(G_h + \nabla g_{h,\varepsilon}(x)\big)\big) \, dx = h^2 \int_D \big(Q_\varepsilon\big(x, G_h + \nabla g_{h,\varepsilon}(x)\big) + \mathrm{r}_{h,\varepsilon}(x)\big) \, dx, \tag{25}$$

where, using assumption (W3) and property (22), the remainder is estimated by

$$\int_D \left|\mathrm{r}_{h,\varepsilon}(x)\right| dx \leqslant \rho(h + \sqrt{h})\|G_h + \nabla g_{h,\varepsilon}\|_{L^2(D)}^2.$$

The combination of (24), (25), (23), and the second property in (22) then yields

$$\frac{1}{h^2} W_D(\mathrm{Id} + hG_h) \leqslant \liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(\varphi_{G_h} + g_{h,\varepsilon}) + \limsup_{\varepsilon \to 0} \rho(h + \sqrt{h}) \|G_h + \nabla g_{h,\varepsilon}\|^2_{L^2(D)}$$

$$\leqslant Q_D(G_h) + C|D|\big(h^{\mu/2} + \rho(h + \sqrt{h})\big(1 + h^{\mu/2}\big)\big).$$

This proves the upper bound, and concludes the proof of the proposition. □

The proof of Theorem 2.1 relies on the quantitative version of Proposition 2.2 (see Remark 6) and on a localization argument allowed by the p-growth condition.

Proof of Theorem 2.1. We split the proof into four steps.

Step 1. Localization of the energy $\mathcal{I}_\varepsilon$.

Let $\mathcal{B}$ denote the collection of all open balls contained in D, and define for all $B \in \mathcal{B}$ and all $u \in W^{1,p}(B)$ the localized functionals

$$\mathcal{I}_\varepsilon(u; B) := \int_B W_\varepsilon\big(x, \nabla u(x)\big) \, dx \quad \text{and} \quad \mathcal{I}^*(u; B) := \int_B W^*\big(x, \nabla u(x)\big) \, dx.$$

Since W_ε satisfies the standard p-growth conditions, Γ-convergence is local (see [2, Theorem 12.5]), and for all $B \in \mathcal{B}$ the functionals $\mathcal{I}_\varepsilon(\cdot; B)$ $\Gamma(L^p)$-converge to $\mathcal{I}^*(\cdot; B)$.

Step 2. Localization of $\mathcal{E}_\varepsilon$.

We consider a subsequence (not relabeled) such that $\mathcal{E}_\varepsilon$ $\Gamma(L^2)$-converges to a functional $\mathcal{E}^*$ on $H^1(D)$. As in Step 1 of the proof of Proposition 2.2, $\mathcal{E}^*$ is of the form

$$\mathcal{E}^*(g) = \int_D Q^*\big(x, \nabla g(x)\big) \, dx$$

for some $Q^* \in \mathcal{Q}_{\alpha''}(D \times \mathbb{M}^d)$, where α'' only depends on α. Moreover, for all $B \in \mathcal{B}$ the localized functionals

$$\mathcal{E}_\varepsilon(g; B) := \int_B Q_\varepsilon\big(x, \nabla g(x)\big) \, dx$$

$\Gamma(L^2)$-converge on $H^1(B)$ to

$$\mathcal{E}^*(g; B) := \int_B Q^*\big(x, \nabla g(x)\big) \, dx.$$

Step 3. Characterization of Q^*.

For all $B \in \mathcal{B}$ and $G \in \mathbb{M}^d$ we define

$$Q_B(G) := \inf_{g \in H^1_0(B)} \int_B Q^*\big(x, G + \nabla g(x)\big) \, dx,$$

$$W_B(G) := \inf_{v \in W^{1,p}_0(B)} \int_B W^*\big(x, G + \nabla v(x)\big) \, dx.$$

Since $\mathcal{I}^*(\cdot; B)$ is the $\Gamma(L^p)$-limit of the sequence $\mathcal{I}_\varepsilon(\cdot; B)$, the functional $\mathcal{I}^*(\cdot; B)$ is lower semicontinuous and its energy density W^* satisfies a p-growth condition from below. Hence, infima converge, and we have

$$W_B(G) = \lim_{\varepsilon \to 0} \Big\{ \inf_{v \in W^{1,p}_0(B)} \mathcal{I}_\varepsilon(\varphi_G + v; B) \Big\}, \tag{26}$$

which proves that (3) is well defined. Since $B \in \mathcal{B}$ is of class C^1, we can apply Proposition 2.2 to each of the functionals $\mathcal{I}_\varepsilon(\cdot; B)$; and since each $B \in \mathcal{B}$ can be obtained by translation and dilation of the unit ball in $\mathbb{R}^d$, we deduce that there exist a constant α' and a modulus of approximation ρ' (both only depending on α, and on ρ), such that the following two properties are fulfilled: For all $B \in \mathcal{B}$ and $G \in \mathbb{M}^d$

$$\frac{1}{|B|} W_B \in \mathcal{W}_{\alpha', \rho'}, \tag{27}$$

$$\left| \frac{1}{|B|} W_B(\mathrm{Id} + G) - \frac{1}{|B|} Q_B(G) \right| \leqslant \rho'(|G|)|G|^2. \tag{28}$$

In particular, (28) holds for all balls $B(x, r)$ with center $x \in D$ and sufficiently small radius $r > 0$. Because the l.h.s. of (28) is independent of x and r, and since for almost every $x \in D$

$$\lim_{r \to 0} \frac{1}{|B(x,r)|} W_{B(x,r)}(\mathrm{Id} + G) = W^*(x, \mathrm{Id} + G),$$

$$\lim_{r \to 0} \frac{1}{|B(x,r)|} Q_{B(x,r)}(\mathrm{Id} + G) = Q^*(x, G)$$

(see e.g. [4]), the estimate

$$\left| W^*(x, \mathrm{Id} + G) - Q^*(x, G) \right| \leqslant \rho'(|G|)|G|^2 \tag{29}$$

holds for all $G \in \mathbb{M}^d$ and almost every $x \in D$. On the one hand, this implies that $W^* \in \mathcal{W}_{\alpha', \rho'}^p(D \times \mathbb{M}^d)$. On the other hand, this proves that Q^* can be characterized by

$$Q^*(x, G) := \lim_{h \to 0} \frac{1}{h^2} W^*(x, \mathrm{Id} + hG).$$

The limit on the r.h.s. does not depend on the extraction of Step 2, so that the entire sequence $\mathcal{E}_\varepsilon$ $\Gamma(L^2)$-converges to $\mathcal{E}^*$.

Step 4. Commutation diagram.

The proof of the diagram, which closely follows [13, Section 6], is left to the reader. $\quad\square$

We conclude this section with the Lipschitz truncation estimate of Proposition 2.3.

Proof of Proposition 2.3(c). The proof is divided into three steps. In the first two steps we prove the statement for a fixed domain D by combining Meyers' estimate and the Lipschitz truncation argument of [7, Proposition A.3]. We then prove in the third step that the constants can be chosen only depending on α and on the geometry of D.

Step 1. Control of energy differences.

Let $g^* \in H_0^1(D)$ denote the unique minimizer of the functional $\mathcal{E}_G$ on $H_0^1(D)$. We claim that for all $g \in H_0^1(D)$ there holds

$$\int_D Q(x, G + \nabla g) \, \mathrm{d}x - \int_D Q(x, G + \nabla g^*) \, \mathrm{d}x \leqslant \alpha \|\nabla g - \nabla g^*\|_{L^2(D)}^2. \tag{30}$$

To prove this we first expand the formula for $Q(x, \nabla g - \nabla g^*)$ (use (16) with $A = G + \nabla g^*$ and $B = \nabla g - \nabla g^*$):

$$Q(x, G + \nabla g) - Q(x, G + \nabla g^*) = Q(x, \nabla g - \nabla g^*) + 2\langle \mathbb{L}(x)(G + \nabla g^*), \nabla g - \nabla g^* \rangle.$$

We integrate this identity over D and note that the second term on the r.h.s. vanishes as the first variation of the minimization problem characterizing g^*. Thus, (30) follows from the fact that $Q(x, \cdot) \in \mathcal{Q}_\alpha$ for a.e. $x \in D$.

Step 2. Proof of the claim for a fixed domain.

We assert that there exist $C > 0$ and $\mu > 0$ such that for all $\lambda > 0$ there exists a map $g \in W_0^{1,\infty}(D)$ that satisfies

$$\|g\|_{W^{1,\infty}(D)} \leqslant \lambda, \tag{31}$$

$$\|\nabla g - \nabla g^*\|_{L^2(D)}^2 \leqslant C\lambda^{-\mu}\|\nabla g^*\|_{L^{2+\mu}(D)}^{2+\mu}. \tag{32}$$

We construct the map as follows. By Meyers' estimate (see Proposition 2.3 part (b)), there exist $C > 0$ and $\mu > 0$ depending only on α and D such that for all $G \in \mathbb{M}^d$,

$$\|\nabla g^*\|_{L^{2+\mu}(D)}^{2+\mu} \leqslant C|D||G|^{2+\mu}.$$

Hence, [7, Proposition A.2] yields a map $g \in W_0^{1,\infty}(D)$ that satisfies (31), and the estimate

$$|D_\lambda| \leqslant \frac{C}{\lambda^{2+\mu}}\|\nabla g^*\|_{L^{2+\mu}(D)}^{2+\mu}, \qquad D_\lambda := \{x \in D \colon g(x) \neq g^*(x)\} \tag{33}$$

for some C independent of λ and g^*. Let us prove that g also satisfies (32). From Hölder's inequality with exponents $(\frac{2+\mu}{2}, \frac{2+\mu}{\mu})$, we have

$$\|\nabla g - \nabla g^*\|_{L^2(D)}^2 = \int_{D_\lambda} |\nabla g - \nabla g^*|^2 \, \mathrm{d}x \leqslant |D_\lambda|^{\frac{\mu}{2+\mu}} \left(\int_D |\nabla g - \nabla g^*|^{2+\mu} \, \mathrm{d}x \right)^{\frac{2}{2+\mu}}.$$

On the one hand, the combination of (31) and (33) yields

$$\int_D |\nabla g - \nabla g^*|^{2+\mu} \, \mathrm{d}x = \int_{D_\lambda} |\nabla g - \nabla g^*|^{2+\mu} \, \mathrm{d}x$$

$$\leqslant C \left(\int_{D_\lambda} |\nabla g|^{2+\mu} \, \mathrm{d}x + \int_D |\nabla g^*|^{2+\mu} \, \mathrm{d}x \right)$$

$$\leqslant C \left(|D_\lambda|\lambda^{2+\mu} + \int_D |\nabla g^*|^{2+\mu} \, \mathrm{d}x \right)$$

$$\leqslant C\|\nabla g^*\|_{L^{2+\mu}(D)}^{2+\mu}.$$

On the other hand, (33) also implies

$$|D_\lambda|^{\frac{\mu}{2+\mu}} \leqslant C\lambda^{-(2+\mu)\frac{\mu}{2+\mu}}\|\nabla g^*\|_{L^{2+\mu}(D)}^{(2+\mu)\frac{\mu}{2+\mu}} = C\lambda^{-\mu}\|\nabla g^*\|_{L^{2+\mu}(D)}^{\mu}.$$

Estimate (31) follows from these last three inequalities.

Step 3. Dependence of the constants on D.

Step 2 provides a function g satisfying the desired properties with some constants μ and C which only depend on α and the domain D. Let us quickly show that both constants can be chosen invariant under dilations, translations, and rotations of D. Assume that D is a translated, rotated, and dilated version of some reference domain D_0, i.e. $D := \tau + rRD_0$ with $\tau \in \mathbb{R}^d$, $R \in \mathrm{SO}(d)$, and $r > 0$. We shall prove that C and μ only depend on α and D_0. To this end we set

$$g_0^*(x) := \frac{1}{r}g^*(\tau + rRx) \quad \text{and} \quad Q_0(x, G) := Q(\tau + rRx, G).$$

Then g_0^* is the unique minimizer of

$$H_0^1(D_0) \ni g_0 \mapsto \int_{D_0} Q_0(x, G + \nabla g_0) \, \mathrm{d}x.$$

For all $\lambda > 0$, Step 2 yields a map $g_0 \in W_0^{1,\infty}(D_0)$ with

$$\left|\nabla g_0(x)\right| \leqslant \lambda \quad \text{for a.e. } x \in D_0,$$

$$\int_{D_0} Q_0\big(x, G + \nabla g_0(x)\big) \, dx - \int_{D_0} Q_0\big(x, G + \nabla g_0^*(x)\big) \, dx \leqslant C_0 \lambda^{-\mu_0} |D_0| |G|^{2+\mu_0},$$

where μ_0 and C_0 only depend on α and D_0. A simple change of variables shows that the map

$$g(x) := r g_0\left(R^{-1} \frac{x - \tau}{r}\right)$$

satisfies (31) and (32) with $\mu = \mu_0$ and $C = C_0$. $\square$

2.3. Extension to general energy densities

In finite elasticity it is desirable to consider energy densities with the physical behavior

$$W(x, F) = +\infty \quad \text{for all } F \in \mathbb{M}^d \text{ with } \det F \leqslant 0.$$

In order to allow such energy densities we have to drop the p-growth condition from above. As we shall show in Theorem 2.4 below, provided homogenization holds, linearization and homogenization commute at identity in the homogeneous case (that is, when the limiting energy density does not depend on the space variable). Yet homogenization of such energy functionals is currently an open problem, so that the assumption of Theorem 2.4 is quite strong. On the other hand, the commutativity result is a *local* statement close to $SO(d)$, so that the behavior of $W(x, F)$ as $\det F$ is close to 0 should not rule out our arguments—in the spirit of the linearization result by Dal Maso, Negri and Percivale in [6]. To emphasize this point we shall prove the following statement: Taking the Γ-limsup (or liminf) and linearization commute. In particular, whether the Γ-limit in ε (homogenization) exists or not does not matter, since whatever the chosen (converging) subsequence in ε, it commutes with the limit in h (linearization). In particular, this is a mild statement of the fact that nonlinear elastic energy densities without growth condition can be homogenized in a neighborhood of identity. This is made precise in Theorem 2.5.

Theorem 2.4. *Let the domain D be C^1, and consider a family of energy densities $(W_\varepsilon) \subset \mathcal{W}_{\alpha,\rho}(D \times \mathbb{M}^d)$. Suppose that there exist $p \geqslant 2$, and homogeneous-in-space energy densities $W_{\text{hom}} : \mathbb{M}^d \to [0, +\infty]$ and $Q_{\text{hom}} : \mathbb{M}^d \to \mathbb{R}$, such that as $\varepsilon \to 0$:*

(i) *the energy functionals $\mathcal{I}_\varepsilon|_{W_0^{1,p}(D)}$ defined in (1) $\Gamma(L^p)$-converge to*

$$\mathcal{I}_{\text{hom}} : W_0^{1,p}(D) \to [0, +\infty], \quad u \mapsto \int_D W_{\text{hom}}\big(\nabla u(x)\big) \, dx;$$

(ii) *the quadratic energy functionals $\mathcal{E}_\varepsilon|_{H_0^1(D)}$ defined in (2) $\Gamma(L^2)$-converge to $\mathcal{E}_{\text{hom}}$ given by*

$$\mathcal{E}_{\text{hom}} : H_0^1(D) \to \mathbb{R}, \quad u \mapsto \int_D Q_{\text{hom}}\big(\nabla u(x)\big) \, dx.$$

If the homogenized density W_{hom} satisfies for all $F \in \mathbb{M}^d$ the asymptotic formula

$$W_{\text{hom}}(F) = \lim_{\varepsilon \to 0} \frac{1}{|D|} \inf\big\{\mathcal{I}_\varepsilon(\varphi_F + v) : v \in W_0^{1,p}(D)\big\}, \tag{34}$$

then W_{hom} admits a quadratic expansion at Id given by Q_{hom}, i.e. for all $G \in \mathbb{M}^d$, there holds

$$W_{\text{hom}}(\text{Id} + G) = Q_{\text{hom}}(G) + o\big(|G|^2\big). \tag{35}$$

In addition the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{G}_{h,\varepsilon} & \xrightarrow{\ (1)\ } & \mathcal{E}_{\varepsilon} \\
{\scriptstyle(2)}\downarrow & & \downarrow{\scriptstyle(3)} \\
\mathcal{G}_{h,\mathrm{hom}} & \xrightarrow[\ (4)\ ]{} & \mathcal{E}_{\mathrm{hom}}
\end{array}
$$

where $\mathcal{G}_{h,\varepsilon}$ and $\mathcal{G}_{h,\mathrm{hom}}$ denote the functionals from $H_0^1(D)$ to $[0,+\infty]$ defined as

$$
\mathcal{G}_{h,\varepsilon}(g) := \frac{1}{h^2}\mathcal{I}_\varepsilon(\varphi_{\mathrm{Id}} + hg), \quad \mathcal{G}_{h,\mathrm{hom}}(g) := \frac{1}{h^2}\mathcal{I}_{\mathrm{hom}}(\varphi_{\mathrm{Id}} + hg);
$$

and (1), (4), and (2), (3) mean Γ-convergence in $H_0^1(D)$ with respect to the strong topology of $L^2(D)$ as $h \to 0$ and $\varepsilon \to 0$, respectively. Moreover, the families $(\mathcal{G}_{h,\varepsilon})$, $(\mathcal{G}_{h,\mathrm{hom}})$ and $(\mathcal{E}_\varepsilon)$ are equi-coercive w.r.t. weak convergence in $H_0^1(D)$.

Remark 8. The quadratic expansion (35) of W_{hom} does not depend on the exponent for which the $\Gamma(L^p)$-convergence holds.

Theorem 2.4 is an immediate consequence Proposition 2.2:

Proof of Theorem 2.4. By assumption the Γ-limits $\mathcal{I}_{\mathrm{hom}}$ and $\mathcal{E}_{\mathrm{hom}}$ are integral functionals with homogeneous integrands W_{hom} and Q_{hom}, respectively. Thus, the expansion (4) in Proposition 2.2 simplifies to

$$
\inf_{v\in W_0^{1,p}(D)}\int_D W_{\mathrm{hom}}\big(\mathrm{Id}+G+\nabla v(x)\big)\,\mathrm{d}x = \inf_{v\in H_0^1(D)}\int_D Q_{\mathrm{hom}}\big(G+\nabla v(x)\big)\,\mathrm{d}x + o\big(|G^2|\big).
$$

The functional $\mathcal{I}_{\mathrm{hom}}$ is (as a $\Gamma(L^p)$-limit) lower semicontinuous w.r.t. strong convergence in $L^p(D)$. Hence, W_{hom} is $W^{1,p}$-quasiconvex, and

$$
\inf_{v\in W_0^{1,p}(D)}\int_D W_{\mathrm{hom}}\big(\mathrm{Id}+G+\nabla v(x)\big)\,\mathrm{d}x = |D|W_{\mathrm{hom}}(\mathrm{Id}+G).
$$

By convexity of Q_{hom}, we also have

$$
\inf_{v\in H_0^1(D)}\int_D Q_{\mathrm{hom}}\big(G+\nabla v(x)\big)\,\mathrm{d}x = |D|Q_{\mathrm{hom}}(G).
$$

This proves (35). The proof of the diagram, which closely follows [13, Section 6], is left to the reader. $\quad\square$

If we do not assume the homogenization property to hold, we still have a commutation diagram, where the Γ-limit is replaced by the Γ-liminf or the Γ-limsup (which are well-defined quantities):

Theorem 2.5. Let the domain D be C^1, and consider a family of energy densities $(W_\varepsilon) \subset \mathcal{W}_{\alpha,\rho}(D \times \mathbb{M}^d)$. Suppose that there exists $Q^* : D \times \mathbb{M}^d \to \mathbb{R}$, such that as $\varepsilon \to 0$ the quadratic energy functionals $\mathcal{E}_\varepsilon|_{H_0^1(D)}$ defined in (2) $\Gamma(L^2)$-converge to $\mathcal{E}^*$ given by

$$
\mathcal{E}^* : H_0^1(D) \to \mathbb{R}, \quad g \mapsto \int_D Q^*\big(x, \nabla g(x)\big)\,\mathrm{d}x.
$$

Then setting for all $h > 0$

$$
\mathcal{G}_{h,\mathrm{hom}}^+ := \Gamma\big(L^2\big)\text{-}\limsup_{\varepsilon\to 0}\mathcal{G}_{h,\varepsilon},
$$

$$
\mathcal{G}_{h,\mathrm{hom}}^- := \Gamma\big(L^2\big)\text{-}\liminf_{\varepsilon\to 0}\mathcal{G}_{h,\varepsilon},
$$

where $\mathcal{G}_{h,\varepsilon}$ denotes the functional from $H_0^1(D)$ to $[0, +\infty]$ defined as

$$\mathcal{G}_{h,\varepsilon}(g) := \frac{1}{h^2} \int_D W_\varepsilon\big(x, \mathrm{Id} + h\nabla g(x)\big)\,\mathrm{d}x,$$

the following diagrams commute

$$
\begin{array}{ccc}
\mathcal{G}_{h,\varepsilon} & \xrightarrow{\ (1)\ } & \mathcal{E}_\varepsilon \\
{\scriptstyle (2)^+}\downarrow & & \downarrow{\scriptstyle (3)} \\
\mathcal{G}_{h,\mathrm{hom}}^+ & \xrightarrow[(4)]{} & \mathcal{E}^*
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{G}_{h,\varepsilon} & \xrightarrow{\ (1)\ } & \mathcal{E}_\varepsilon \\
{\scriptstyle (2)^-}\downarrow & & \downarrow{\scriptstyle (3)} \\
\mathcal{G}_{h,\mathrm{hom}}^- & \xrightarrow[(4)]{} & \mathcal{E}^*
\end{array}
$$

where (1), (4), and $(2)^+$ (resp. $(2)^-$), (3) stand for Γ-convergence in $H_0^1(D)$ with respect to the strong topology of $L^2(D)$ as $h \to 0$ and $\Gamma(L^2)$-$\limsup$ (resp. $\Gamma(L^2)$-$\liminf$) as $\varepsilon \to 0$, respectively. In addition the families $(\mathcal{G}_{h,\varepsilon})$, $(\mathcal{G}_{h,\mathrm{hom}}^\pm)$ and $(\mathcal{E}_\varepsilon)$ are equi-coercive w.r.t. weak convergence in $H_0^1(D)$.

Proof. In the diagram $(2)^+$, $(2)^-$ and (3) hold by assumption. For the convergence in arrow (1), which corresponds to the passage from nonlinear to linear elasticity, we refer to [13, Theorem 5.1] (see also [6]). Hence, only (4) has to be proved. Since by definition $\mathcal{G}_{h,\mathrm{hom}}^- \leqslant \mathcal{G}_{h,\mathrm{hom}}^+$, it suffices to check the following two statements.

(lower bound) For every sequence $(g_h) \subset H_0^1(D)$ with $g_h \to g$ in $L^2(D)$ it is

$$\liminf_{h \to 0} \mathcal{G}_{h,\mathrm{hom}}^-(g_h) \geqslant \begin{cases} \mathcal{E}^*(g) & \text{for } g \in H_0^1(D), \\ +\infty & \text{else.} \end{cases}$$

(upper bound) For all $g \in H_0^1(D)$ there exists a sequence $(g_h) \subset H_0^1(D)$ such that $g_h \to g$ in $L^2(D)$ and

$$\lim_{h \to 0} \mathcal{G}_{h,\mathrm{hom}}^+(g_h) = \mathcal{E}^*(g).$$

Both statements follow by adaptation of the proofs of Proposition 2.2 and Proposition 2.3(c). For the sake of completeness, we briefly present the argument.

Step 1. Construction of a recovery sequence with higher integrability.

We need the following modification of Proposition 2.3(c): Let $g \in W_0^{1,\infty}(D)$ and let g_ε be the unique minimizer on $H_0^1(D)$ of the functional

$$\varphi \mapsto \mathcal{E}_\varepsilon(\varphi) + L_g(\varphi), \quad \text{where } L_g(\varphi) := -2 \int_D \langle \mathbb{L}^*(x)\nabla g, \nabla\varphi \rangle\,\mathrm{d}x,$$

and $\mathbb{L}^*$ denotes the symmetric fourth order tensor associated with Q^*. Then there exist an exponent $\mu > 0$, a constant C and a family $(g_{h,\varepsilon}) \subset W_0^{1,\infty}(D)$ such that for all positive ε and h we have

$$g_\varepsilon \rightharpoonup g \quad \text{weakly in } H^1(D) \quad \text{and} \quad \mathcal{E}_\varepsilon(g_\varepsilon) \to \mathcal{E}^*(g), \tag{36}$$

$$\nabla g_\varepsilon \rightharpoonup \nabla g \quad \text{weakly in } L^{2+\mu}(D), \tag{37}$$

$$\|g_{h,\varepsilon}\|_{W^{1,\infty}(D)} \leqslant Ch^{-1/2}, \tag{38}$$

$$\|\nabla g_{h,\varepsilon} - \nabla g_\varepsilon\|_{L^2(D)} \leqslant Ch^{\mu/2}. \tag{39}$$

Since the function g is the unique minimizer on $H_0^1(D)$ of the functional $\mathcal{E}^* + L_g$, the assertion (36) follows from the property that $\mathcal{E}_\varepsilon + L_g$ Γ-converges to $\mathcal{E}^* + L_g$. For (37) it suffices to prove the higher integrability statement

$$\|\nabla g_\varepsilon\|_{L^{2+\mu}(D)} \leqslant C, \tag{40}$$

where μ and C are independent of ε. Indeed, as a minimizer the function g_ε satisfies the Euler–Lagrange equation

$$\int_D \langle \mathbb{L}_\varepsilon \nabla g_\varepsilon, \nabla \varphi \rangle \, dx = \int_D \langle \mathbb{L} \nabla g, \nabla \varphi \rangle \, dx \quad \text{for all } \varphi \in H_0^1(D). \tag{41}$$

Since $\mathbb{L}_\varepsilon$ is uniformly elliptic, and since $\mathbb{L} \nabla g \in L^\infty(D, \mathbb{M}^d)$, we can apply Meyers' estimate [17, Theorem 1.1] and (40) follows. As in the proof Proposition 2.3(c), Step 2, the doubly indexed family $(g_{h,\varepsilon})$ satisfying estimates (38) and (39) can be constructed by a Lipschitz truncation argument.

Step 2. Proof of (upper bound).

Since $W_0^{1,\infty}(D)$ is dense in $H_0^1(D)$, it suffices to prove the statement for $g \in W_0^{1,\infty}(D)$. Let (g_ε) and $(g_{h,\varepsilon})$ be as in Step 1. By (38) for each h the family $(g_{h,\varepsilon})_\varepsilon$ is bounded in $H_0^1(D)$. Therefore, for each h there exist a function $g_h \in H_0^1(D)$ and a vanishing sequence of positive numbers $(\varepsilon_i^h)_{i \in \mathbb{N}}$ such that $g_{h,\varepsilon_i^h} \to g_h$ strongly in $L^2(D)$ and weakly in $H_0^1(D)$ as $i \to \infty$. Hence, by appealing to the definition of the Γ-limsup we get

$$\mathcal{G}_{h,\mathrm{hom}}^+(g_h) = \inf \left\{ \limsup_{i \to \infty} \mathcal{G}_{h,\varepsilon_i}(\varphi_i) : \varphi_i \to g_h \text{ strongly in } L^2(D) \text{ and } \varepsilon_i \downarrow 0 \right\}$$

$$\leqslant \limsup_{i \to \infty} \mathcal{G}_{h,\varepsilon_i^h}(g_{h,\varepsilon_i^h}) = \limsup_{i \to \infty} \frac{1}{h^2} \int_D W_{\varepsilon_i^h}\left(x, \mathrm{Id} + h \nabla g_{h,\varepsilon_i^h}(x)\right) dx.$$

For $h \ll 1$ we can expand the integral of the r.h.s. as follows

$$\frac{1}{h^2} \int_D W_{\varepsilon_i^h}(x, \mathrm{Id} + h \nabla g_{h,\varepsilon_i^h}) \, dx$$

$$\overset{(W3),\ (38)\ \text{and}\ (39)}{\leqslant} \int_D Q_{\varepsilon_i^h}(x, \nabla g_{h,\varepsilon_i^h}) \, dx + \|\nabla g_{h,\varepsilon_i^h}\|_{L^2(D)}^2 \rho(C\sqrt{h})$$

$$\leqslant \int_D Q_{\varepsilon_i^h}(x, \nabla g_{\varepsilon_i^h}) \, dx + \int_D Q_{\varepsilon_i^h}(x, \nabla g_{h,\varepsilon_i^h} - \nabla g_{\varepsilon_i^h}) + 2\langle \mathbb{L}_{\varepsilon_i^h} \nabla g_{\varepsilon_i^h}, \nabla g_{h,\varepsilon_i^h} - \nabla g_{\varepsilon_i^h}\rangle \, dx$$

$$+ 2\left(\|\nabla g_{\varepsilon_i^h}\|_{L^2(D)}^2 + \|\nabla g_{h,\varepsilon_i^h} - \nabla g_{\varepsilon_i^h}\|_{L^2(D)}^2\right) \rho(C\sqrt{h}).$$

The last line follows from an expansion of $Q_{\varepsilon_i^h}$ and the triangle inequality. Because $Q_{\varepsilon_i^h} \in \mathcal{Q}_{\alpha'}(D \times \mathbb{M}^d)$ for some α' independent of i and h, and by (36), (37) and (39) we deduce that

$$\limsup_{i \to \infty} \frac{1}{h^2} \int_D W_{\varepsilon_i^h}(x, \mathrm{Id} + h \nabla g_{h,\varepsilon_i^h}) \, dx \leqslant \mathcal{E}^*(g) + O\left(h^{\mu/2} + \rho(C\sqrt{h})\right).$$

Now, the statement follows by passing to the limit $h \to 0$ on both sides.

Step 3. Proof of (lower bound).

It suffices to consider a sequence $g_h \to g$ in $L^2(D)$ with $(g_h) \subset H_0^1(D)$ and

$$\liminf_{h \to 0} \mathcal{G}_{h,\mathrm{hom}}^-(g_h) = \limsup_{h \to 0} \mathcal{G}_{h,\mathrm{hom}}^-(g_h) < \infty.$$

By appealing to the definition of the Γ-liminf there exists a doubly indexed family $(g_{h,\varepsilon}) \subset H_0^1(D)$ such that for each h we have $\|g_{h,\varepsilon} - g_h\|_{L^2(D)} \to 0$ as $\varepsilon \to 0$ and $\liminf_{\varepsilon \to 0} \mathcal{G}_{h,\varepsilon}(g_{h,\varepsilon}) \leqslant \mathcal{G}_{h,\mathrm{hom}}^-(g_h) + h$. By appealing to (W2) and geometric rigidity, we deduce that $g \in H_0^1(D)$ and that for each h there exists a vanishing sequence $(\varepsilon_i^h)_{i \in \mathbb{N}}$ such that

$$g_{h,\varepsilon_i^h} \rightharpoonup g_h \quad \text{and} \quad g_h \rightharpoonup g \quad \text{weakly in } H^1(D) \text{ as } i \to \infty \text{ and } h \to 0, \text{ respectively,} \tag{42}$$

$$\lim_{i \to \infty} \mathcal{G}_{h,\varepsilon_i^h}(g_{\varepsilon_h^i,h}) \leqslant \mathcal{G}_{h,\mathrm{hom}}^-(g_h) + h. \tag{43}$$

As in the proof of Proposition 2.2, Step 2, we get for $h \ll 1$ by expansion of $W_{\varepsilon_i^h}$ the estimate

$$\mathcal{G}_{h,\varepsilon_i^h}(g_{h,\varepsilon_i^h}) \geqslant \int_D Q_{\varepsilon_i^h}(x, \nabla g_{h,\varepsilon_i^h}) \chi_{h,\varepsilon_i^h}\, dx - \|\nabla g_{h,\varepsilon_i^h}\|_{L^2(D)}^2 \rho(\sqrt{h})$$

where $\chi_{h,\varepsilon}$ is the indicator function of the set $\{x \in D: |\nabla g_{h,\varepsilon}(x)| \leqslant h^{-1/2}\}$. To estimate the quadratic term, fix a function $\tilde{g} \in W_0^{1,\infty}(D)$, which we think of as an approximation of g, and let $\tilde{g}_\varepsilon$ be defined as in Step 1. Application of the expansion formula (17) with $A = \chi_{h,\varepsilon_i^h}\nabla g_{h,\varepsilon_i^h}$ and $B = \nabla \tilde{g}_{\varepsilon_i^h}$ yields

$$\int_D Q_{\varepsilon_i^h}(x, \nabla g_{h,\varepsilon_i^h})\chi_{h,\varepsilon_i^h}\, dx \geqslant \int_D Q_{\varepsilon_i^h}(x, \nabla \tilde{g}_{\varepsilon_i^h})\, dx + 2\underbrace{\int_D \langle \mathbb{L}_{\varepsilon_i^h}(x)(\nabla g_{h,\varepsilon_i^h} - \nabla \tilde{g}_{\varepsilon_i^h}), \nabla \tilde{g}_{\varepsilon_i^h}\rangle\, dx}_{=:I_{h,\varepsilon_i^h}^{(1)}}$$

$$- 2\underbrace{\int_D \langle \mathbb{L}_{\varepsilon_i^h}(x)(1 - \chi_{h,\varepsilon_i^h})\nabla g_{h,\varepsilon_i^h}, \nabla \tilde{g}_{\varepsilon_i^h}\rangle\, dx}_{I_{h,\varepsilon_i^h}^{(2)}}. \tag{44}$$

By (36) we have

$$\lim_{i \to \infty} \int_D Q_{\varepsilon_i^h}(x, \nabla \tilde{g}_{\varepsilon_i^h})\, dx = \int_D Q^*(x, \nabla \tilde{g})\, dx. \tag{45}$$

Next, we treat $I_{h,\varepsilon_i^h}^{(1)}$:

$$\lim_{h \to 0}\lim_{i \to \infty} I_{h,\varepsilon_i^h}^{(1)} \overset{(41)}{=} \lim_{h \to 0}\lim_{i \to \infty} L_{\tilde{g}}(g_{h,\varepsilon_i^h} - \tilde{g}_{\varepsilon_i^h}) \overset{(42)}{=} L_{\tilde{g}}(g - \tilde{g}). \tag{46}$$

As in the proof of Proposition 2.2, Step 2, the contribution of $I_{h,\varepsilon_i^h}^{(2)}$ vanishes in the limit $i \to \infty$, $h \to 0$. Thus, the combination of (43), (44), (45), and (46) yields

$$\liminf_{h \to 0} \mathcal{G}_{h,\mathrm{hom}}^-(g_h) \geqslant \int_D Q^*(x, \nabla \tilde{g})\, dx + L_{\tilde{g}}(g - \tilde{g}).$$

Since $\tilde{g}$ can be chosen arbitrarily close to g, the statement follows. $\quad \square$

3. Application to stochastic homogenization

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We shall say that the family of mappings $(\tau_z)_{z \in \mathbb{R}^d}$ from Ω to Ω is a strongly continuous measure-preserving ergodic translation group if:

– $(\tau_z)_{z \in \mathbb{R}^d}$ has the group property: $\tau_0 = \mathrm{Id}$ (the identity mapping), and for all $x, y \in \mathbb{R}^d$, $\tau_{x+y} = \tau_x \circ \tau_y$;
– $(\tau_z)_{z \in \mathbb{R}^d}$ preserves the measure: For all $x \in \mathbb{R}^d$, and every measurable set $F \in \mathcal{F}$, $\tau_x F$ is measurable and $\mathbb{P}(\tau_x F) = \mathbb{P}(F)$;
– $(\tau_z)_{z \in \mathbb{R}^d}$ is strongly continuous: For any measurable function f on Ω, the function $(\omega, x) \mapsto f(\tau_x \omega)$ defined on $\Omega \times \mathbb{R}^d$ is measurable (with the Lebesgue measure on $\mathbb{R}^d$);
– $(\tau_z)_{z \in \mathbb{R}^d}$ is ergodic: For all $F \in \mathcal{F}$, if for all $x \in \mathbb{R}^d$, $\tau_x F \subset F$, then $\mathbb{P}(F) \in \{0, 1\}$.

We are in position to recall the following standard stochastic homogenization result (see the original contribution [5] in the convex case, and its generalization [12] to the quasiconvex case).

Theorem 3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\tau_z)_{z \in \mathbb{R}^d}$ be a strongly continuous measure-preserving ergodic translation group, and let $W : \mathbb{R}^d \times \mathbb{M}^d \times \Omega \to \mathbb{R}^+$ be a map such that*

(i) *W is Lebesgue-measurable in its first variable,*
(ii) *W is $\mathcal{F}$-measurable in its third variable,*
(iii) *$W(x, \cdot, \omega) \in \mathcal{W}_\alpha^p$ for $\mathbb{P}$-almost every $\omega \in \Omega$, almost every $x \in \mathbb{R}^d$ and some $p \in (1, \infty)$,*
(iv) *W is stationary in the sense that for $\mathbb{P}$-almost every $\omega \in \Omega$, almost every $x \in \mathbb{R}^d$, every $F \in \mathbb{M}^d$ and every $z \in \mathbb{R}^d$*

$$W(x + z, F, \omega) = W(x, F, \tau_z \omega).$$

Then for $\mathbb{P}$-almost every $\omega \in \Omega$, the integral functional $\mathcal{I}_\varepsilon(\omega) : W^{1,p}(D) \to \mathbb{R}^+$ given for all $\varepsilon > 0$ by

$$\mathcal{I}_\varepsilon(\omega)(u) = \int_D W(x/\varepsilon, \nabla u(x), \omega) \, dx$$

$\Gamma(L^p)$-converges, as ε vanishes, to the integral functional $\mathcal{I}_{\mathrm{hom}} : W^{1,p}(D) \to \mathbb{R}^+$ given by

$$\mathcal{I}_{\mathrm{hom}}(u) = \int_D W_{\mathrm{hom}}(\nabla u(x)) \, dx,$$

where the deterministic homogeneous-in-space energy density W_{hom} is quasiconvex, satisfies (W4) *and the asymptotic formula*

$$W_{\mathrm{hom}}(F) = \lim_{R \to \infty} \frac{1}{R^d} \inf \left\{ \int_{(0,R)^d} W(x, F + \nabla \phi(x), \omega) \, dx, \ \phi \in W_0^{1,p}((0, R)^d) \right\} \tag{47}$$

for all $F \in \mathbb{M}^d$ and $\mathbb{P}$-almost every $\omega \in \Omega$.

The combination of Theorems 2.4 and 3.1 yields

Theorem 3.2. *Let W and W_{hom} be as in Theorem 3.1 and assume in addition that for some $p \geqslant 2$ and a modulus of approximation ρ*

$$W(x, \cdot, \omega) \in \mathcal{W}_{\alpha, \rho}^p$$

for almost every $x \in \mathbb{R}^d$ and $\mathbb{P}$-almost every $\omega \in \Omega$. Let Q denote the quadratic term of the Taylor expansion of W at identity. Then:

(a) *the density W_{hom} is of class $\mathcal{W}_{\alpha', \rho'}^p$, with α' and ρ' as in Theorem 2.1;*
(b) *the energy functionals*

$$\mathcal{E}_\varepsilon(\omega) : H^1(D) \to \mathbb{R}^+, \quad u \mapsto \int_D Q_\varepsilon(x/\varepsilon, \nabla u(x), \omega) \, dx$$

$\Gamma(L^2)$-converge for $\mathbb{P}$-almost every $\omega \in \Omega$ to

$$\mathcal{E}_{\mathrm{hom}} : H^1(D) \to \mathbb{R}, \quad u \mapsto \int_D Q_{\mathrm{hom}}(\nabla u(x)) \, dx$$

where Q_{hom} is the deterministic homogeneous-in-space quadratic energy density that is determined by the expansion

$$\forall G \in \mathbb{M}^d: \quad W_{\mathrm{hom}}(\mathrm{Id} + G) = Q_{\mathrm{hom}}(G) + o(|G|^2);$$

(c) *for $\mathbb{P}$-almost every $\omega \in \Omega$ the following diagram commutes*

$$
\begin{array}{ccc}
\mathcal{G}_{h,\varepsilon}(\omega) & \xrightarrow{\ (1)\ } & \mathcal{E}_\varepsilon(\omega) \\[4pt]
{\scriptstyle (2)}\Big\downarrow & & \Big\downarrow{\scriptstyle (3)} \\[4pt]
\mathcal{G}_{\mathrm{hom},h} & \xrightarrow[\ (4)\ ]{} & \mathcal{E}_{\mathrm{hom}}
\end{array}
$$

where $\mathcal{G}_{h,\varepsilon}(\omega)$ and $\mathcal{G}_{\mathrm{hom},h}$ denote the functionals from $H_0^1(D)$ to $[0, +\infty]$ defined as

$$
\mathcal{G}_{h,\varepsilon}(\omega)(g) := \frac{1}{h^2} \int_D W\big(x/\varepsilon, \mathrm{Id} + h\nabla g(x), \omega\big)\, dx,
$$

$$
\mathcal{G}_{\mathrm{hom},h}(g) := \frac{1}{h^2} \int_D W_{\mathrm{hom}}\big(\mathrm{Id} + h\nabla g(x)\big)\, dx
$$

and (1), (4), and (2), (3) mean Γ-convergence in $H_0^1(D)$ with respect to the strong topology of $L^2(D)$ as $h \to 0$ and $\varepsilon \to 0$, respectively. Moreover, the families $(\mathcal{I}_\varepsilon(\omega))$ and $(\mathcal{E}_\varepsilon(\omega))$ are equi-coercive w.r.t. weak convergence in $H_0^1(D)$ (for $\mathbb{P}$-almost every $\omega \in \Omega$).

Proof. Let $\mathcal{I}_\varepsilon(\omega)$ and $\mathcal{I}_{\mathrm{hom}}(\omega)$ be defined as in Theorem 3.1. Then $\mathcal{I}_\varepsilon(\omega)$ $\Gamma(L^p)$-converges to $\mathcal{I}_{\mathrm{hom}}(\omega)$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Now, the statement is a direct consequence of Theorem 2.1 which applies for $\mathbb{P}$-almost every $\omega \in \Omega$. $\quad\square$

Corollary 3.3. *Within the notation and assumptions of Theorem 3.2, we also have*

$$
Q_{\mathrm{hom}}(G) = \lim_{R \to \infty} \frac{1}{R^d} \inf\left\{ \int_{(0,R)^d} Q\big(x, G + \nabla\phi(x), \omega\big)\, dx,\ \phi \in H_0^1\big((0,R)^d\big) \right\} \tag{48}
$$

for all $G \in \mathbb{M}^d$, and for $\mathbb{P}$-almost every $\omega \in \Omega$, where

$$
Q(y, G, \omega) := \liminf_{h \to 0} \frac{W(y, \mathrm{Id} + hG, \omega)}{h^2}.
$$

Proof. Once we know that $\mathcal{E}_\varepsilon$ $\Gamma(L^2)$-converges to $\mathcal{E}_{\mathrm{hom}}$, the uniform coercivity of $\mathcal{E}_\varepsilon$ and $\mathcal{E}_{\mathrm{hom}}$ implies the convergence of the infima, which yields the desired formula (48). The Γ-convergence result is either a consequence of Theorem 3.2 part (b), or of the G-convergence of the associated elliptic operator proved for instance in [10, Section 12.3] (by definition, $Q(y, G, \omega)$ is stationary for the ergodic translation group). $\quad\square$

Remark 9. As can be easily seen, Theorem 3.2 holds as well in the almost-periodic case (see for instance [3] or [2, Section 17.2]) and in variants of the stochastic case (see for instance [8]).

4. Locality of the Γ-closure at identity

This section is devoted to the locality of the Γ-closure in $\mathcal{W}_{\alpha,\rho}^p$ at identity. Given k homogeneous energy densities $\{W_i\}_{i \in \{1,\dots,k\}} \in \mathcal{W}_{\alpha,\rho}^p$, we are interested in characterizing the set of maps $W^*(x, \cdot)$ that can be reached as energy densities of $\Gamma(L^p)$-limits $\mathcal{I}^* : W^{1,p}(D) \to \mathbb{R}$

$$
u \mapsto \mathcal{I}^*(u) = \int_D W^*\big(x, \nabla u(x)\big)\, dx, \tag{49}
$$

of energy functionals $\mathcal{I}_{\chi^n} : W^{1,p}(D) \to \mathbb{R}$ of the form

$$
u \mapsto \mathcal{I}_{\chi^n}(u) = \int_D \sum_{i=1}^k W_i\big(\nabla u(x)\big)\chi_i^n(x)\, dx \tag{50}
$$

as n goes to infinity. Above, $\chi^n \in L^\infty(D, \{0, 1\}^k)$ denotes a vector field with $\sum_{i=1}^k \chi_i^n \equiv 1$ and satisfies $\chi^n \rightharpoonup^* \theta$ weakly-* in $L^\infty(D, [0, 1]^k)$. The components of χ^n can be seen as the characteristic functions of the k phases. Note that also in the limit we have $\sum_{i=1}^k \theta_i \equiv 1$.

The Γ-closure of $\{W_i\}_{i \in \{1,...,k\}}$ is said to be local if the set of such integrands $W^*(x, \cdot)$ for almost every $x \in D$ coincides with the closure for the pointwise convergence of the set of energy densities W_{hom} obtained by periodic homogenization of mixtures of $\{W_i\}_{i \in \{1,...,k\}}$ in the proportions $\{\theta_i(x)\}_{i \in \{1,...,k\}}$. To turn this into a rigorous statement, let us recall some definitions related to periodic homogenization.

Definition 5. (See [14,3].) Let $1 < p < \infty$, and $W \in \mathcal{W}_\alpha^p(\mathbb{R}^d \times \mathbb{M}^d)$ be $U = (0, 1)^d$-periodic in its first variable. The homogenized energy density associated with W is denoted by $W_{\text{hom}} : \mathbb{M}^d \to \mathbb{R}$ and characterized by

$$W_{\text{hom}}(F) := \lim_{R \to \infty} \frac{1}{R^d} \inf \left\{ \int_{(0,R)^d} W(y, F + \nabla u(y)) \, dy, \ u \in W_0^{1,p}((0, R)^d) \right\}. \tag{51}$$

We are now in position to define the set of periodic homogenized energy densities.

Definition 6. Let $1 < p < \infty$, $\{W_i\}_{i \in \{1,...,k\}} \in \mathcal{W}_\alpha^p$, and $\theta \in [0, 1]^k$ be such that $\sum_{i=1}^k \theta_i = 1$. We define the set of periodic homogenized energy densities associated with $\{W_i, \theta_i\}_{i \in \{1,...,k\}}$ as

$$\mathcal{P}_\theta = \left\{ (W_\chi)_{\text{hom}} : \mathbb{M}^d \to \mathbb{R} : \exists \chi \in L^\infty(\mathbb{R}^d, \{0, 1\}^k) \text{ such that } \chi \text{ is } U\text{-periodic with } \int_U \chi_i \, dy = \theta_i \right.$$
$$\left. \text{and } (W_\chi)_{\text{hom}} \text{ is associated with } W_\chi : (y, F) \mapsto \sum_{i=1}^k W_i(F) \chi_i(y) \text{ through (51)} \right\},$$

and its closure for the pointwise convergence by

$$\mathcal{G}_\theta = \left\{ W^* : \mathbb{M}^d \to \mathbb{R} : \exists (W_{\chi^n})_{\text{hom}} \in \mathcal{P}_\theta, (W_{\chi^n})_{\text{hom}} \to W^* \text{ pointwise} \right\}.$$

The definition of locality of the Γ-closure now reads:

Definition 7. Let $1 < p < \infty$, $\{W_i\}_{i \in \{1,...,k\}} \in \mathcal{W}_\alpha^p$. We say that the Γ-closure of $\{W_i\}_{i \in \{1,...,k\}}$ is local if and only if for every sequence $\chi^n \in L^\infty(D, \{0, 1\}^k)$ with $\sum_{i=1}^k \chi_i^n \equiv 1$ and such that

- $\chi^n \rightharpoonup^* \theta$ weakly-* in $L^\infty(D, [0, 1]^k)$,
- the functional $\mathcal{I}_{\chi^n} : W^{1,p}(D) \to \mathbb{R}$ defined in (50) $\Gamma(L^p)$-converges to the functional $\mathcal{I}^* : W^{1,p}(D) \to \mathbb{R}$ defined in (49),

one has

$$W^*(x, \cdot) \in \mathcal{G}_{\theta(x)}$$

for almost every $x \in D$.

If the k energy densities $\{W_i\}_{i \in \{1,...,k\}}$ are convex functions, then the associated Γ-closure is local (see for instance [1, Theorem 5.1]). In the case of quasiconvex non-convex functions, the locality (or non-locality) of the Γ-closure is an open problem. In the specific case when $W_i \in \mathcal{W}_{\alpha,\rho}^p$ for all $i \in \{1, \ldots, k\}$, Theorem 2.4 allows us to prove that the Γ-closure is "local at identity". This notion is made precise by the following two definitions.

Definition 8. Let $2 \leqslant p < \infty$, $\{W_i\}_{i \in \{1,...,k\}} \in \mathcal{W}_{\alpha,\rho}^p$, and $\theta \in [0, 1]^k$ such that $\sum_{i=1}^k \theta_i = 1$. We define the set of periodic homogenized energy densities associated with $\{W_i, \theta_i\}_{i \in \{1,...,k\}}$ at identity as

$$\mathcal{P}_\theta^{\text{Id}} = \left\{ W^* : \mathbb{M}^d \to \mathbb{R} : \exists (W_\chi)_{\text{hom}} \in \mathcal{P}_\theta \text{ such that } \left| W^*(\text{Id} + G) - (W_\chi)_{\text{hom}}(\text{Id} + G) \right| = o(|G|^2) \right\},$$

and its closure:

$$\mathcal{G}_\theta^{\mathrm{Id}} = \Big\{ W^* : \mathbb{M}^d \to \mathbb{R} : \text{ there exists a sequence } (W_{\chi^n})_{\mathrm{hom}} \in \mathcal{P}_\theta$$

$$\text{such that } \Big| W^*(\mathrm{Id} + G) - \lim_{n \to 0} (W_{\chi^n})_{\mathrm{hom}}(\mathrm{Id} + G) \Big| = o\big(|G|^2\big) \Big\}.$$

Definition 9. Let $2 \leqslant p < \infty$, $\{W_i\}_{i \in \{1,\dots,k\}} \in \mathcal{W}_{\alpha,\rho}^p$. We say that the Γ-closure of $\{W_i\}_{i \in \{1,\dots,k\}}$ is local at identity if and only if for every sequence $\chi^n \in L^\infty(D, \{0, 1\}^k)$ with $\sum_{i=1}^k \chi_i^n \equiv 1$ and such that

- $\chi^n \rightharpoonup^* \theta$ weakly-* in $L^\infty(D, [0, 1]^k)$,
- the functional $\mathcal{I}_{\chi^n} : W^{1,p}(D) \to \mathbb{R}$ defined in (50) $\Gamma(L^p)$-converges to the functional $\mathcal{I}^* : W^{1,p}(D) \to \mathbb{R}$ defined in (49),

one has

$$W^*(x, \cdot) \in \mathcal{G}_{\theta(x)}^{\mathrm{Id}}$$

for almost every $x \in D$.

The above definition is a weakened version of the locality of the Γ-closure of Definition 7 obtained by restricting the property of approximation by periodic homogenized energy densities to a neighborhood of identity via a Taylor expansion. We have:

Theorem 4.1. *Let $2 \leqslant p < \infty$ and $\{W_i\}_{i \in \{1,\dots,k\}} \in \mathcal{W}_{\alpha,\rho}^p$, then the Γ-closure of $\{W_i\}_{i \in \{1,\dots,k\}}$ is local at identity.*

Proof. By [1, Theorem 3.5], it is enough to prove the locality property in the so-called homogeneous case, that is with a repartition function $\chi^n \in L^\infty(D, \{0, 1\}^k)$ such that

- χ^n weakly-* converges to a constant function θ in $L^\infty(D, [0, 1]^k)$,
- the functional $\mathcal{I}_{\chi^n} : W^{1,p}(D) \to \mathbb{R}$ defined in (50) $\Gamma(L^p)$-converges to the functional $\mathcal{I}^* : W^{1,p}(D) \to \mathbb{R}$ defined in (49), where W^* does not depend on the space variable.

Let $\mathcal{E}_{\chi^n} : H^1(D) \to \mathbb{R}^+$ denote the quadratic energy functional associated with $\mathcal{I}_{\chi^n}$, that is

$$\mathcal{E}_{\chi^n}(u) := \int_D \sum_{i=1}^k Q_i\big(\nabla u(x)\big) \chi_i^n(x) \, \mathrm{d}x$$

where $Q_i \in \mathcal{Q}_{\alpha'}$ denotes the quadratic form associated with W_i through (W3). We then apply Theorem 2.1 and deduce that $\mathcal{E}_\varepsilon \, \Gamma(L^2)$-converges to

$$\mathcal{E}^* : H^1(D) \to [0, +\infty), \quad \mathcal{E}^*(u) := \int_D Q^*\big(\nabla u(x)\big) \, \mathrm{d}x,$$

where $Q^* \in \mathcal{Q}_{\hat\alpha}$ for some $\hat\alpha > 0$, and is characterized by the expansion

$$W^*(\mathrm{Id} + G) = Q^*(G) + o\big(|G|^2\big). \tag{52}$$

Next, we appeal to the locality of the Γ-closure for convex linear problems. In particular, there exists a U-periodic sequence $\tilde\chi^n \in L^\infty(\mathbb{R}^d, \{0, 1\}^k)$ satisfying $\int_Q \tilde\chi_i^n(y) \, \mathrm{d}y = \theta_i$ for all $n \in \mathbb{N}$ and all $i \in \{1, \dots, k\}$, and such that the homogenized quadratic functions $\tilde{Q}_{\mathrm{hom}}^n$ associated with the periodic quadratic energy densities $\tilde{Q}^n : \mathbb{R}^d \times \mathbb{M}^d \to \mathbb{R}$

$$\tilde{Q}^n : (y, G) \mapsto \sum_{i=1}^k Q_i(G) \tilde\chi_i^n(y)$$

approximate Q^* in the sense that for all $G \in \mathbb{M}^d$,

$$\lim_{n \to \infty} Q_{\mathrm{hom}}^n(G) = Q^*(G). \tag{53}$$

We are now in position to prove the claim. To this aim, we define a sequence of periodic integrands $\widetilde{W}^n : \mathbb{R}^d \times \mathbb{M}^d \to \mathbb{R}$ as

$$\widetilde{W}^n : (y, G) \mapsto \sum_{i=1}^{k} W_i(G) \tilde{\chi}_i^n(y).$$

With this sequence of periodic integrands we associate a sequence of homogenized integrands W_{hom}^n through (51) with $\widetilde{W}^n$ in place of W. Combined with standard periodic homogenization results (see for instance [14,3] or [2, Section 14.2]), Theorem 2.1 then shows that

$$\frac{|W_{\mathrm{hom}}^n(\mathrm{Id} + G) - Q_{\mathrm{hom}}^n(G)|}{|G|^2} \leqslant \rho'(|G|),$$

and the theorem follows from (52), (53), and the uniformity of the validity of the Taylor expansion since for all n, W_{hom}^n and W^* are of class $\mathcal{W}_{\alpha', \rho'}^p$ for the same function ρ'. $\quad\square$

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