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Superharmonic functions are locally renormalized solutions

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Dedicated to Peter Lindqvist on the occasion of his 60th birthday

Abstract

We show that different notions of solutions to measure data problems involving p-Laplace type operators and nonnegative source measures are locally essentially equivalent. As an application we characterize singular solutions of multidimensional Riccati type partial differential equations.

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1. Introduction

Consider elliptic quasilinear type equations

$$-\operatorname{div}\big(\mathcal{A}(x, Du)\big) = \mu, \tag{1.1}$$

in an open set $\Omega \subset \mathbf{R}^n$, where μ is a nonnegative Radon measure and the operator $\operatorname{div}(\mathcal{A}(x, Du))$ is a measurable perturbation of the p-Laplacian operator

$$\Delta_p u = \operatorname{div}\big(|Du|^{p-2} Du\big), \quad 1 < p \leqslant n.$$

The natural domain of definition for the operator $\operatorname{div}(\mathcal{A}(x, Du))$ is $W^{1,p}_{\mathrm{loc}}(\Omega)$. Then, however, $u \mapsto \operatorname{div}(\mathcal{A}(x, Du))$ is locally in $W^{-1,p'}(\Omega)$. Consequently, Eq. (1.1) carries no solutions u in $W^{1,p}_{\mathrm{loc}}(\Omega)$ if the measure data μ is not in the dual. On the other hand, if $\mu \in W^{-1,p'}(\Omega)$, the existence of solutions is a straightforward consequence of duality methods in view of the weak continuity of the operator, see e.g. [23]. Moreover, the reader is asked to examine functions

$$u(x) = \int_{|x|}^{1} r^{\gamma-1}\, dr \tag{1.2}$$

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that for $\gamma = (p-n)/(p-1)$ yield a reasonable distributional solution to Eq. (1.1), where the operator is the p-Laplacian and μ is a multiple of the Dirac measure – a measure outside the dual. From this example we also infer that the maximal regularity for a general solution cannot reach n'-integrability of $|Du|^{p-1}$.

In conclusion, in order to solve Eq. (1.1) with a general Radon measure one is forced to look outside the natural domain of the operator (see Section 2 for a more accurate description). A relevant existence theory for equations with general signed measure data was developed by Boccardo and Gallouët [6] for $p > 2 - 1/n$ (this restriction, dictated by the fact that the fundamental solution in (1.2) does not have a distributional derivative at the origin, can be dispensed with by using a weaker derivative, see [21]). Their method is based on a suitable approximation of the measure μ. The main task pursued there was showing necessary a priori summability estimates for the gradients of solutions that allow for viable compactness arguments. The solutions produced in [6] are often called SOLA (Solutions Obtained as Limits of Approximations), emphasizing the fact that these are limit functions of solutions to equations with regularized source measures from the dual of $W^{1,p}$ converging weakly to the original measure. Regularity theory for SOLA is a widely studied field, see for example [29–31] and the references therein.

As known e.g. by the example given by Serrin [35] the distributional solutions to (1.1) do not solve the Dirichlet problem in a unique manner. Thus there arose attempts to arrive at the unique solvability by imposing new requirements for u to be a solution.

When μ belongs to L^1, alternative solutions were called entropy or renormalized solutions, introduced independently by Bénilan et al. [4], Dall'Aglio [7], and by Lions and Murat [25], and in these works also the uniqueness of renormalized solutions was settled, but only when $\mu \in L^1$. Later, Dal Maso et al. [11] generalized the concept for general measures. These *renormalized solutions* allow for testing the equation with Lipschitz functions of the solution itself provided that the derivative of the test function is compactly supported; see Section 2.3 for the precise definition. Again, renormalized solutions are SOLA in the above sense.

In the case of nonnegative measures, Kilpeläinen and Malý [21] established a clear connection between existence theory and nonlinear potential theory. In particular, it was shown that every nonnegative measure induces an $\mathcal{A}$-superharmonic solution for all $p > 1$ and that obtained solutions are SOLA as well. A class of $\mathcal{A}$-superharmonic functions consists of (pointwise defined) lower semicontinuous functions satisfying a comparison with respect to solutions to homogeneous equations. See Section 2.1 for definitions and [19] for the rich theory behind such functions. In the light of the fundamental convergence theorem, stating that under mild integrability conditions properly pointwise defined limits of $\mathcal{A}$-superharmonic functions remain $\mathcal{A}$-superharmonic, it is easy to see that SOLA have $\mathcal{A}$-superharmonic representatives whenever μ can be approximated with nonnegative smooth measures.

In this paper we study the connection between $\mathcal{A}$-superharmonic functions and renormalized solutions. Our main result is that *every $\mathcal{A}$-superharmonic function is locally a renormalized solution*. We also show the converse, i.e. that every renormalized solution has an $\mathcal{A}$-superharmonic representative. In this respect, *our result unifies the existence theory in the case of nonnegative measures* and allows for very sharp testing of superharmonic functions provided by the definition of renormalized solutions. More importantly, superharmonic functions form a class of pointwise defined solutions to (2.6) equivalent with SOLA and renormalized solutions whenever the source measure is nonnegative.

As an application for our main result we characterize all $W^{1,p}$ solutions to Riccati type equation

$$-\Delta_p u = |\nabla u|^p, \quad p > 1. \tag{1.3}$$

We show that the transformation

$$u \mapsto e^{\frac{u}{p-1}} \tag{1.4}$$

gives an one-to-one correspondence between the solutions to (1.3) and those p-superharmonic functions whose Riesz measures are singular with respect to the p-capacity. More precisely, for each nonnegative Radon measure μ, singular with respect to the p-capacity, any (SOLA) solution of $-\Delta_p v = \mu$ has a p-superharmonic representative and it can be transformed to a solution u to (1.3) by the inverse of the transformation (1.4). Conversely, if u is a solution to the Riccati equation (1.3), then $e^{\frac{u}{p-1}}$ is a p-superharmonic function whose Riesz measure is supported in a set of p-capacity zero.

A corresponding result was proved in the Laplacian case in [38] by using the linear potential theory. In the nonlinear case, results in the akin spirit were obtained independently by Abdel Hamid and Bidaut-Véron [1]; however our argument is fairly simple and our result completes the story.

The Riccati type equations, especially related existence and uniqueness questions, are widely studied, see for instance [1–3,8,12–17,20,26,27,33,34].

2. Tools from nonlinear potential theory

Throughout this paper we let Ω stand for an open set in $\mathbf{R}^n$, $n \geqslant 2$, and μ be a nonnegative Radon measure in Ω. Moreover, we let $1 < p < \infty$ be a fixed number. Throughout, c and C (and $c(a, b, d)$) will denote positive constants (depending on data a, b, d) whose value is not necessarily the same at each occurrence.

Let $\mathcal{A} : \Omega \times \mathbf{R}^n \mapsto \mathbf{R}^n$ be a Carathéodory function, that is, $(x, \xi) \mapsto \mathcal{A}(x, \xi)$ is measurable for every $\xi \in \mathbf{R}^n$ and $\xi \mapsto \mathcal{A}(x, \xi)$ is continuous for almost every $x \in \Omega$. We assume the growth conditions

$$\langle \mathcal{A}(x,\xi), \xi \rangle \geqslant \alpha_0 |\xi|^p, \quad \text{and} \quad \left|\mathcal{A}(x,\xi)\right| \leqslant \beta_0 |\xi|^{p-1}, \tag{2.1}$$

for all $\xi \in \mathbf{R}^n$ and for almost every $x \in \Omega$, and the monotonicity condition

$$\langle \mathcal{A}(x,\xi) - \mathcal{A}(x,\zeta), \xi - \zeta \rangle > 0 \tag{2.2}$$

for all $\xi \neq \zeta$ in $\mathbf{R}^n$ and for almost every $x \in \Omega$. Here α_0 and β_0 are positive constants.

2.1. $\mathcal{A}$-superharmonic functions

A continuous function $h \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is said to be $\mathcal{A}$-*harmonic* in Ω if it is a weak solution to

$$-\operatorname{div}\big(\mathcal{A}(x, \nabla h)\big) = 0,$$

that is,

$$\int_\Omega \langle \mathcal{A}(x, \nabla h), \nabla \varphi \rangle \, dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$.

A lower semicontinuous function $u : \Omega \to \mathbf{R} \cup \{\infty\}$ is called $\mathcal{A}$-*superharmonic* if $u \not\equiv \infty$ in each component of Ω, and for each open $U \Subset \Omega$ and each $h \in C(\overline{U})$ that is $\mathcal{A}$-harmonic in U, the inequality $u \geqslant h$ on ∂U implies $u \geqslant h$ in U.

The following characterization for $\mathcal{A}$-superharmonicity is our starting point. For the proof, see for example [19].

2.3. Proposition. *Suppose that u is an a.e. finite function in Ω. Then u has an $\mathcal{A}$-superharmonic representative if and only if the truncations $u_k = \min(u, k)$ are supersolutions to*

$$-\operatorname{div}\big(\mathcal{A}(x, \nabla u)\big) \geqslant 0$$

for each $k > 0$, i.e. $u_k \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega \langle \mathcal{A}(x, \nabla u_k), \nabla \varphi \rangle \, dx \geqslant 0$$

for all nonnegative $\varphi \in C_0^\infty(\Omega)$.

Recall that the pointwise values of an $\mathcal{A}$-superharmonic function are uniquely determined by its values a.e. since

$$u(x) = \operatorname*{ess\,lim\,inf}_{y \to x} u(y),$$

for each x; see [19, Theorem 7.22].

We denote by $T_k(t) = \min(k, \max(t, -k))$ the usual truncation operator. Following the tradition of the potential theory we use the very weak gradient

$$Du = \lim_{k \to \infty} \nabla T_k(u)$$

for such u whose truncations are Sobolev functions, see [19,21].

A frequently used property of $\mathcal{A}$-superharmonic functions is the local summability:

2.4. Theorem. *(See [19, Theorem 7.46].) If u is $\mathcal{A}$-superharmonic in Ω, then $u \in L^s_{\mathrm{loc}}(\Omega)$ and $|Du|^{p-1} \in L^q_{\mathrm{loc}}(\Omega)$ whenever*

$$0 < s < \frac{n(p-1)}{n-p} \quad \text{and} \quad 0 < q < \frac{n}{n-1};$$

for $p = n$ any finite s is allowed; for $p > n$, $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$.

A function u is a solution to

$$-\operatorname{div}\bigl(\mathcal{A}(x, \nabla u)\bigr) = \mu \tag{2.5}$$

if

$$\int_\Omega \bigl\langle \mathcal{A}(x, Du), \nabla\varphi \bigr\rangle\, dx = \int_\Omega \varphi\, d\mu \tag{2.6}$$

for all $\varphi \in C_0^\infty(\Omega)$. Here, of course, one must have that $\mathcal{A}(x, Du)$ is locally integrable. For an $\mathcal{A}$-superharmonic function u this assumption is satisfied by Theorem 2.4 and, indeed, for any nonnegative measure μ there is an $\mathcal{A}$-superharmonic function solving (2.5), see [21]. Conversely, for any $\mathcal{A}$-superharmonic function there exists a unique nonnegative Radon measure μ such that u solves Eq. (2.5). This measure μ is called the *Riesz measure* of u, and it is often denoted by $\mu[u]$.

We shall later employ the fact that the truncations $u_k = \min(u, k)$ are also $\mathcal{A}$-superharmonic and their Riesz measures $\mu[u_k]$ are locally in the dual of the Sobolev space $W^{1,p}$ (see Proposition 2.3); moreover $\mu[u_k] \to \mu[u]$ weakly in Ω.

Recall also the two-sided Wolff potential estimate [22,24,32,39,40]: if u is a nonnegative $\mathcal{A}$-superharmonic solution to (2.5) in $B(x, 2r) \subset \Omega$, then there is a constant $c = c(n, p, \alpha_0, \beta_0)$ such that

$$\frac{1}{c} W_{\mu,r}(x) \leqslant u(x) \leqslant c\Bigl(\operatorname*{ess\,inf}_{B(x,r)} u + W_{\mu,r}(x)\Bigr), \tag{2.7}$$

where

$$W_{\mu,r}(x) = \int_0^r \left(\frac{\mu(B(x,\varrho))}{\varrho^{n-p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho}.$$

Observe carefully that all $\mathcal{A}$-superharmonic functions with the Riesz measure μ satisfy the estimate. This fact suggests a definition of a class of functions, namely

$$S_{\mu,r,L}\bigl(\Omega'\bigr) = \left\{u\colon\ \frac{1}{c} W_{\mu,r}(x) \leqslant u(x) \leqslant L + cW_{\mu,r}(x)\ \forall x \in \Omega'\right\},$$

for some $r > 0$, $L \geqslant 0$, and $\Omega' \Subset \Omega$. We indeed have the following.

2.8. Proposition. *Let u be a nonnegative $\mathcal{A}$-superharmonic function with the Riesz measure μ in a bounded domain Ω. Let $\Omega' \Subset \Omega$. For every $0 < r < \operatorname{dist}(\Omega', \partial\Omega)/2$, there is a constant $L < \infty$ for which $u \in S_{\mu,r,L}(\Omega')$.*

Proof. The first inequality in the definition of $S_{\mu,r,L}(\Omega')$ readily follows from the Wolff potential estimate (2.7). To deduce the second inequality from the same estimate we need to have an upper bound for $\inf_{B(y,r)} u$ with an arbitrary $y \in \Omega'$. This easily follows from Theorem 2.4: there is $\gamma = \gamma(n, p) > 0$ such that

$$\inf_{B(y,r)} u \leqslant \biggl(\ \fint_{B(y,r)} u^\gamma\, dx\biggr)^{1/\gamma} \leqslant c\biggl(r^{-n} \int_{\Omega' + B_r} u^\gamma\, dx\biggr)^{1/\gamma} < \infty$$

for all $y \in \Omega'$, as desired. $\square$

2.2. Decomposition of measures

The *p-capacity* $\operatorname{cap}_p(B,\Omega)$ of any set $B\subset\Omega$ is defined in the standard way: the p-capacity of a compact set $K\subset\Omega$ is

$$\operatorname{cap}_p(K,\Omega)=\inf\left\{\int_\Omega |\nabla\varphi|^p\,dx\colon\ \varphi\in C_0^\infty(\Omega),\ \varphi\geqslant 1 \text{ on } K\right\}.$$

The p-capacity of an open set $U\subset\Omega$ is then

$$\operatorname{cap}_p(U,\Omega)=\sup\bigl\{\operatorname{cap}_p(K,\Omega)\colon\ K \text{ compact},\ K\subset U\bigr\};$$

and for an arbitrary set $E\subset\Omega$

$$\operatorname{cap}_p(E,\Omega)=\inf\bigl\{\operatorname{cap}_p(U,\Omega)\colon\ U \text{ open},\ E\subset U\bigr\}.$$

There is also a dual approach to the capacity. Indeed, define

$$\widetilde{\operatorname{cap}}_p(E,\Omega):=\sup\bigl\{\nu(E)\colon\ \nu\in\bigl(W_0^{1,p}(\Omega)\bigr)',\ \operatorname{supp}\nu\subset E,\ \nu\geqslant 0,\ -\Delta_p w=\nu \text{ such that } 0\leqslant w\leqslant 1\bigr\}$$

for $E\subset\Omega$. Then by [22, Theorem 3.5] we have

$$\widetilde{\operatorname{cap}}_p(E,\Omega)=\operatorname{cap}_p(E,\Omega)$$

whenever $E\subset\Omega$ is a Borel set.

A set E is called *polar* if there is an open neighborhood U of E and an $\mathcal{A}$-superharmonic function u in U such that $u=\infty$ on E. We will later employ the fact (see e.g. [19]) that a set E is polar if and only if it is of *p-capacity zero*, that is

$$\operatorname{cap}_p(E\cap U,U)=0$$

for all open sets $U\subset\mathbf{R}^n$.

For every Radon measure μ we denote with μ_0 the part which is *absolutely continuous with respect to the p-capacity* and with μ_s the *singular part with respect to the p-capacity*, i.e.

$$\mu=\mu_0+\mu_s,$$

where $\mu_0\ll\operatorname{cap}_p$ (meaning that $\mu_0(E)=0$ for each set E of p-capacity zero), and $\mu_s\perp\operatorname{cap}_p$ (meaning that there is a Borel set F of p-capacity zero for which $\mu_s(\mathbf{R}^n\setminus F)=0$). The support of the singular part is contained in the polar set of corresponding $\mathcal{A}$-superharmonic functions, as the next lemma shows.

2.9. Lemma. *Let u be $\mathcal{A}$-superharmonic with the Riesz measure μ. Then*

$$\mu_s\bigl(\{u<\infty\}\bigr)=0,$$

where μ_s is the singular part of μ (with respect to the p-capacity).

Proof. Our goal is to estimate the measure μ_s on the set $\{u<\infty\}$ by employing the dual definition of the capacity. To this end, we first recall a general fact that if

$$\int_\Omega W_{\nu,r}(x)\,d\nu<\infty$$

for a measure ν and for some $r>0$, then ν belongs to $(W_0^{1,p}(\Omega))'$, see [18] and also [28,41].

Let then $E\subset\Omega$ be a set such that $\operatorname{cap}_p(E)=0$ and $\mu_s(\Omega\setminus E)=0$. For every $k>0$ denote $E_k=E\cap\{u<k\}$. Fix $k>0$ and take a compact subset $K\subset E_k$. By the compactness, the distance of K and $\partial\Omega$, say r, is positive. Now the Wolff potential estimate (2.7) implies

$$W_{\mu\lfloor K,r/8}(x)\leqslant W_{\mu,r/8}(x)\leqslant cu(x)<ck$$

for all $x \in K$. Thus

$$\int_\Omega W_{\mu\lfloor_K, r/8}(x)\,d\mu\lfloor_K \leqslant ck\mu(K) < \infty$$

and hence $\mu\lfloor_K$ belongs to the dual of $W_0^{1,p}(\Omega)$.

Next, let v be a nonnegative $\mathcal{A}$-superharmonic function solving

$$-\Delta_p v = \mu\lfloor_K$$

in Ω with $v \in W_0^{1,p}(\Omega)$. By the Wolff potential estimate (2.7) we have that

$$v(x) \leqslant L + ck$$

for all $x \in K$ (see Proposition 2.8). Since v is $\mathcal{A}$-harmonic in $\Omega \setminus K$, the maximum principle yields

$$0 \leqslant v \leqslant L + ck$$

in Ω. Hence, for $M = L + ck$, $w = v/M$ solves

$$-\Delta_p w = M^{p-1}\mu\lfloor_K \in \big(W_0^{1,p}(\Omega)\big)', \quad 0 \leqslant w \leqslant 1,$$

and therefore it is an admissible function to test the dual capacity of K. It follows that

$$\mu(K) \leqslant M^{p-1}\widetilde{\mathrm{cap}}_p(K,\Omega) = M^{p-1}\mathrm{cap}_p(K,\Omega) \leqslant M^{p-1}\mathrm{cap}_p(E,\Omega) = 0,$$

where we used the equivalence of capacities. Thus $\mu(E_k) = 0$, and hence

$$\mu_s\big(\{u < \infty\}\big) \leqslant \mu_s\big(\Omega \setminus E\big) + \sum_{k=1}^{\infty} \mu_s(E_k) = 0. \qquad \square$$

2.3. Locally renormalized solutions

If μ is a nonnegative Radon measure in an open set Ω, we say that a function u is a *local renormalized solution* to (2.5) in Ω if

$$\begin{aligned} &T_k(u) \in W^{1,p}_{\mathrm{loc}}(\Omega) \quad \text{for all } k > 0,\\ &|u|^{p-1} \in L^s_{\mathrm{loc}}(\Omega) \quad \text{for all } 1 \leqslant s < \frac{n}{n-p},\\ &|Du|^{p-1} \in L^q_{\mathrm{loc}}(\Omega) \quad \text{for all } 1 \leqslant q < \frac{n}{n-1}, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &\int_\Omega \big\langle \mathcal{A}(x,Du), Du\big\rangle h'(u)\phi\,dx + \int_\Omega \big\langle \mathcal{A}(x,Du), \nabla\phi\big\rangle h(u)\,dx \\ &\quad = \int_\Omega h(u)\phi\,d\mu_0 + h(+\infty)\int_\Omega \phi\,d\mu_s \end{aligned} \tag{2.11}$$

is satisfied for all $\phi \in C_0^\infty(\Omega)$ and $h \in W^{1,\infty}(\mathbf{R})$ such that h' has a compact support; here

$$h(\infty) = \lim_{t\to\infty} h(t).$$

This definition is a local version for a nonnegative measure μ of a renormalized solution used by Dal Maso, Murat, Orsina, and Prignet in [11] for general signed measures. The localization was then made by Bidaut-Véron in [5]. The most important feature in the localization is that the test function ϕ is required to be compactly supported in (2.11).

We would like to write the condition (2.11) for short as

$$\int_\Omega \big\langle \mathcal{A}(x,Du), D\big(h(u)\phi\big)\big\rangle\,dx = \int_\Omega h(u)\phi\,d\mu, \tag{2.12}$$

where h and ϕ are as above. This, however, requires some care: the left-hand sides of both (2.11) and (2.12) clearly agree for all a.e. representatives of u. The same is not true for the right-hand sides. Indeed,

$$\int_\Omega h(u)\phi\,d\mu = \int_\Omega h(u)\phi\,d\mu_0 + \int_\Omega h(u)\phi\,d\mu_s.$$

The first integral on the right is easily settled: the integration against μ_0 is independent of the chosen p-quasicontinuous representative of u as these agree q.e. and hence μ_0-a.e. That

$$\int_\Omega h(u)\phi\,d\mu_s = h(\infty)\int_\Omega \phi\,d\mu_s$$

for all h and ϕ is more tricky: it requires u to be Borel measurable (or μ_s-measurable) and, more importantly, that $u=\infty$ μ_s-a.e. By Lemma 2.9 $\mathcal{A}$-superharmonic representatives (if exist) have these properties, since they are lower semicontinuous.

We will proceed in showing that locally renormalized supersolutions have $\mathcal{A}$-superharmonic representatives whenever μ is nonnegative. For such functions the condition (2.12) is a legitimate way to write Eq. (2.11). The first task is to show that renormalized solutions are locally bounded below. This will readily imply by the assumption $T_k(u)\in W^{1,p}_{\mathrm{loc}}(\Omega)$ that also $\min(u,k)\in W^{1,p}_{\mathrm{loc}}(\Omega)$ for all $k>0$.

2.13. Lemma. *Let μ be nonnegative and let u be a local renormalized solution to* (2.5) *in Ω. Then u is locally essentially bounded below.*

Proof. Choose first the test function

$$h_\varepsilon(u) = \frac{1}{\varepsilon}\min\{\varepsilon, u_+\} - 1, \quad \varepsilon > 0,$$

and let $h\in W^{1,\infty}(\mathbf{R})$ be nonnegative with h' having a compact support. Let $\phi\in C_0^\infty(\Omega)$ be nonnegative. On the one hand, we have

$$h_\varepsilon(+\infty)h(+\infty)\int_\Omega \phi\,d\mu_s = 0, \qquad \int_\Omega h_\varepsilon(u)h(u)\phi\,d\mu_0 \leqslant 0,$$

and

$$\int_\Omega \big\langle \mathcal{A}(x,Du), \nabla h_\varepsilon(u)\big\rangle h(u)\phi\,dx \geqslant 0.$$

On the other hand, the dominated convergence theorem gives

$$\int_\Omega \big\langle \mathcal{A}(x,Du), \nabla\big(h(u)\phi\big)\big\rangle h_\varepsilon(u)\,dx \to \int_\Omega \big\langle -\mathcal{A}(x,-Du_-), \nabla\big(h(u)\phi\big)\big\rangle\,dx$$

as $\varepsilon\to 0$. Thus $v := u_-$ satisfies

$$\min(v,k)\in W^{1,p}_{\mathrm{loc}}(\Omega), \quad k>0,$$

and

$$\int_\Omega \big\langle \widetilde{\mathcal{A}}(x,Dv), \nabla\big(h(v)\phi\big)\big\rangle\,dx \leqslant 0 \tag{2.14}$$

for all ϕ and h as above. Here $\widetilde{\mathcal{A}}(x,z) := -\mathcal{A}(x,-z)$. This means that v is a nonnegative distributional subsolution for which a priori integrability requirements are not necessarily fulfilled. We now proceed to show that v is actually locally bounded and thus a usual weak subsolution. We establish this using the method in [22].

Define

$$h_{k,d,\varepsilon}(v) = 1 - \left(1+\min\left(\frac{(v-k)_+}{d}, \frac{1}{\varepsilon}\right)\right)^{1-\tau}, \quad k,d,\varepsilon>0,\ \tau>1,$$

which we can substitute into (2.14). Note that $h'_{k,d,\varepsilon} \geqslant 0$. We then have by the monotone convergence together with the assumed summability of $|Dv|^{p-1}$ that

$$\begin{aligned} 0 &\geqslant \int_\Omega \langle \widetilde{\mathcal{A}}(x, Dv), \nabla\big(h_{k,d,\varepsilon}(v)\phi^p\big)\rangle\,dx \\ &\geqslant \frac{1}{C}\int_\Omega |Dv|^p h'_{k,d,\varepsilon}(v)\phi^p\,dx - C\int_\Omega |Dv|^{p-1}h_{k,d,\varepsilon}(v)\phi^{p-1}|D\phi|\,dx \\ &\to \frac{1}{C}\int_\Omega |Dv|^p h'_{k,d}(v)\phi^p\,dx - C\int_\Omega |Dv|^{p-1}h_{k,d}(v)\phi^{p-1}|D\phi|\,dx \end{aligned}$$

as $\varepsilon \to 0$, where

$$h_{k,d}(v) := 1 - \left(1 + \frac{(v-k)_+}{d}\right)^{1-\tau}, \quad k, d > 0,\ \tau > 1.$$

This energy estimate is enough for showing [22, Lemma 4.1]. Indeed, now one may mimic the proof starting from [22, (4.5)] with obvious changes $v \equiv u$ and $\mu \equiv 0$. We then continue as in the proof of [22, Theorem 4.8], with only slight differences: let $x_0 \in \Omega$ be a Lebesgue point of v^γ and let $r < \operatorname{dist}(x_0, \partial\Omega)/2$. Assume that

$$p - 1 < \gamma < \frac{n(p-1)}{n-p+1}.$$

Denote $B_j = B(x_0, r_j)$, where $r_j = 2^{1-j}r$. Let $a_0 = 0$ and for $j \geqslant 1$ let

$$a_{j+1} = a_j + \delta^{-1}\left(\fint_{B_{j+1}} (v - a_j)_+^\gamma\,dx\right)^{1/\gamma},$$

where $\delta > 0$ is a suitable small constant. Note that $a_j < \infty$ since $v = u_- \in L^\gamma_{\mathrm{loc}}(\Omega)$ by the assumptions. Now applying [22, Lemma 4.1] one can deduce as in the proof of [22, Theorem 4.8] that $a_{j+1} - a_j \leqslant (a_j - a_{j-1})/2$ implying by telescoping argument that

$$a := \lim_{j\to\infty} a_j \leqslant 2a_1 = C\left(\fint_{B_1} v^\gamma\,dx\right)^{1/\gamma}.$$

Hence the sequence (a_j) is bounded and increasing. Therefore, we have

$$\big(v(x_0) - a\big)_+^\gamma = \lim_{j\to\infty}\fint_{B_j}(v-a)_+^\gamma\,dx \leqslant \lim_{j\to\infty}\fint_{B_j}(v-a_j)_+^\gamma\,dx = \lim_{j\to\infty} C\delta(a_j - a_{j-1}) = 0.$$

Thus

$$u_-(x_0) = v(x_0) \leqslant a \leqslant C\left(\fint_{B_0} |u|^\gamma\,dx\right)^{1/\gamma}$$

and hence u is locally essentially bounded below by the assumed summability of u. □

We are ready to prove that for nonnegative measures μ each local renormalized solution has an $\mathcal{A}$-superharmonic representative.

2.15. Theorem. *Suppose that u is a local renormalized solution to* (2.5) *in Ω with a nonnegative μ. Then there is an $\mathcal{A}$-superharmonic function $\tilde u$ such that $\tilde u = u$ a.e. and, moreover, $\tilde u$ satisfies* (2.12)*, i.e.*

$$\int_\Omega \langle \mathcal{A}(x, D\tilde u), \nabla\big(h(\tilde u)\phi\big)\rangle\,dx = \int_\Omega h(\tilde u)\phi\,d\mu$$

for all $\phi \in C_0^\infty(\Omega)$ and $h \in W^{1,\infty}(\mathbf{R})$ such that h' has a compact support.

Proof. In the light of the discussion after (2.12) it suffices to find an $\mathcal{A}$-superharmonic representative for u. To this end, let $\phi \in C_0^\infty(\Omega)$ be nonnegative. For $\varepsilon > 0$ and $k > 0$ write

$$h_{k,\varepsilon}(t) = \frac{1}{\varepsilon}\min\bigl((k+\varepsilon-t)_+, \varepsilon\bigr).$$

Since $h'_{k,\varepsilon}(t) \leqslant 0$, we have

$$\int_\Omega \bigl\langle \mathcal{A}(x, Du), Du\bigr\rangle h'_{k,\varepsilon}(u)\phi\, dx \leqslant 0.$$

Moreover, the nonnegativity of μ and ϕ implies

$$\int_\Omega h_{k,\varepsilon}(u)\phi\, d\mu_0 + h_{k,\varepsilon}(+\infty)\int_\Omega \phi\, d\mu_s^+ \geqslant 0.$$

Thus, (2.11) yields

$$\int_\Omega \bigl\langle \mathcal{A}(x, \nabla u_k), \nabla\phi\bigr\rangle\, dx \geqslant 0$$

once we let $\varepsilon \to 0$ and refer to the dominated convergence theorem; here $u_k = \min(u, k)$.

Since u is locally bounded from below by Lemma 2.13, $u_k \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is an ordinary supersolution. Therefore each u_k has an $\mathcal{A}$-superharmonic representative $\tilde{u}_k$. We conclude the proof by observing that the desired representative of u is then given by

$$\tilde{u} = \lim_{k\to\infty} \tilde{u}_k$$

as it is $\mathcal{A}$-superharmonic, being an increasing limit of $\mathcal{A}$-superharmonic functions. □

3. Superharmonic functions are locally renormalized

Before proving our main theorem, we establish the existence of an auxiliary comparison function. The result relies on the existence of renormalized solutions.

3.1. Lemma. *Let μ be a nonnegative Radon measure supported in $B(0, R)$. Then there is an $\mathcal{A}$-superharmonic function w solving*

$$\begin{cases} -\operatorname{div}(\mathcal{A}(x, Dw)) = \mu & \text{in } B(0, 4R),\\ w = 0 & \text{on } \partial B(0, 4R),\end{cases}$$

such that for all $0 < r < R$ there is a positive constant $L < \infty$ such that

$$w \in \mathcal{S}_{\mu,r,L}\bigl(B(0, 4R)\bigr)$$

and

$$\int_{B(0,4R)} \bigl|\nabla\bigl(\min(w, 2\lambda) - \lambda\bigr)_+\bigr|^p\, dx \leqslant \frac{\lambda}{\alpha_0}\mu\bigl(\{W_{\mu,r} > \lambda/L\} \cap B(0, R)\bigr)$$

for all $\lambda > L$.

Proof. We first obtain by [11, Definition 2.25] the existence of a renormalized solution v to the equation in the formulation vanishing on $\partial B(0, 4R)$ in the $W_0^{1,p}$-sense. Theorem 2.15 implies that v has an $\mathcal{A}$-superharmonic representative w satisfying (2.12).

Observe next that since w is an $\mathcal{A}$-superharmonic function, it is nonnegative by the minimum principle. Proposition 2.8 gives a constant $L_0 < \infty$ such that

$$w \in \mathcal{S}_{\mu,r,L}\bigl(B(0, 2R)\bigr)$$

for $L \geqslant L_0$. The Wolff potential estimate (2.7) implies that w is locally bounded outside the support of μ, hence w is locally in $W^{1,p}$ there; in particular w is $\mathcal{A}$-harmonic in $B(0,4R) \setminus B(0,R)$ (cf. [28, Corollary 3.19]). Thus the maximum principle gives

$$\sup_{B(0,4R)\setminus B(0,2R)} w \leqslant L_0.$$

Consequently, we may take any $L \geqslant L_0$ to obtain

$$w \in \mathcal{S}_{\mu,r,L}\big(B(0,4R)\big).$$

The potential estimate also leads to the inclusion

$$\{w > L_0 + ck\} \subset \{W_{\mu,r} > k\}$$

for all $k \in \mathbf{R}$. Fix $k = \lambda/c - L_0$, $\lambda > 0$. We have for all $\lambda > 2cL_0$ that $k > \lambda/(2c)$. Thus,

$$\{w > \lambda\} \subset \big\{W_{\mu,r} > \lambda/(2c)\big\}$$

holds for all $\lambda > 2cL_0$.

Finally, we test the renormalized equation of w with

$$h(w) = \big(\min(w, 2\lambda) - \lambda\big)_+,$$

$\lambda > 2cL_0 > 0$, which is clearly admissible since h is Lipschitz continuous and h' has a compact support. Moreover, since w vanishes continuously on the boundary of $B(0,4R)$, $h(w)$ has a compact support in $B(0,4R)$ and, in particular, $h(w) \in W_0^{1,p}(B(0,4R))$. We have

$$\begin{aligned}
&\lambda\mu\big(\big\{W_{\mu,r} > \lambda/(2c)\big\} \cap B(0,R)\big) \\
&\quad \geqslant \int_{B(0,R)} h(w)\,d\mu \\
&\quad = \int_{B(0,4R)} \big\langle \mathcal{A}(x,\nabla w), \nabla h(w)\big\rangle\,dx \geqslant \alpha_0 \int_{\{\lambda<w<2\lambda\}\cap B(0,4R)} |\nabla w|^p\,dx \\
&\quad = \alpha_0 \int_{B(0,4R)} \big|\nabla\big(\min(w,2\lambda) - \lambda\big)_+\big|^p\,dx
\end{aligned}$$

and the result follows for $L := 2c\max\{L_0, 1\}$. □

The heart of this paper is the following.

3.2. Theorem. *Suppose that u is an $\mathcal{A}$-superharmonic solution to* (2.5). *Suppose further that v is $\mathcal{A}$-superharmonic and that for all $\Omega' \Subset \Omega$ and for all small $r > 0$ there is $L < \infty$ such that*

$$u, v \in \mathcal{S}_{\mu,r,L}\big(\Omega'\big).$$

Let $h : \mathbf{R} \times \mathbf{R} \mapsto \mathbf{R}$ be Lipschitz and let ∇h have a compact support. Then

$$\int_\Omega h(u,v)\phi\,d\mu = \int_\Omega \big\langle \mathcal{A}(x,Du), \nabla\big(h(u,v)\phi\big)\big\rangle\,dx$$

for all $\phi \in C_0^\infty(\Omega)$.

Proof. Denote $u_j = \min(u, j)$, $j > 0$. Let k be so large that $h(u,v) = h(u_k, v_k)$. Let $\phi \in C_0^\infty(\Omega)$ and let $\Omega' \Subset \Omega$ be a smooth domain such that the support of ϕ belongs to Ω'.

Let $\varepsilon > 0$ and

$$K_\varepsilon \subset \{W_{\mu,1} = +\infty\} \cap \Omega'$$

be a compact set such that

$$\mu_s\big(\Omega' \setminus K_\varepsilon\big) < \varepsilon.$$

Set

$$r = \frac{1}{2}\min\big\{\operatorname{dist}\big(\Omega', \partial\Omega\big), \operatorname{dist}\big(K_\varepsilon, \big\{\max(u,v) \leqslant k\big\}\big)\big\} > 0$$

and denote

$$S_\varepsilon := \big\{x \in \mathbf{R}^n\colon \operatorname{dist}(x, K_\varepsilon) \leqslant r\big\}.$$

Take

$$\theta_\varepsilon \in C_0^\infty\big(\big\{\min(u,v) > k\big\}\big), \quad 0 \leqslant \theta_\varepsilon \leqslant 1,$$

such that

$$\theta_\varepsilon = 1 \quad \text{on } S_\varepsilon.$$

In particular, in the support of $\theta_\varepsilon\phi$, $h(u,v) = h(k,k)$ is a constant. Define further

$$\mu_\varepsilon = \mu \llcorner_{\Omega' \setminus K_\varepsilon},$$

i.e., the restriction of μ to the set $\Omega' \setminus K_\varepsilon$. Note that $\mu_\varepsilon(E) \leqslant \mu_0(E) + \varepsilon$ whenever E is a Borel set. Observe that we have

$$W_{\mu,r} = W_{\mu_\varepsilon,r} \quad \text{in } \Omega' \setminus S_\varepsilon.$$

This yields the inclusion

$$\mathcal{S}_{\mu_\varepsilon,r,L}\big(\Omega'\big) \subset \mathcal{S}_{\mu,r,L}\big(\Omega' \setminus S_\varepsilon\big)$$

for all $L \geqslant 0$.

Next, let R be large enough so that $\Omega' \subset B(0,R)$ and let w_ε be an $\mathcal{A}$-superharmonic renormalized solution to

$$\begin{cases} -\operatorname{div}(\mathcal{A}(x, Dw_\varepsilon)) = \mu_\varepsilon & \text{in } B(0,4R), \\ w_\varepsilon = 0 & \text{on } \partial B(0,4R). \end{cases}$$

By Lemma 3.1, there is a constant $\widetilde{L} < \infty$ such that

$$w_\varepsilon \in \mathcal{S}_{\mu_\varepsilon,r,\widetilde{L}}\big(\Omega'\big) \subset \mathcal{S}_{\mu,r,\widetilde{L}}\big(\Omega' \setminus S_\varepsilon\big),$$

and for

$$\psi_\lambda = \frac{(\min\{w_\varepsilon, 2\lambda\} - \lambda)_+}{\lambda},$$

the estimate

$$\begin{aligned} \int_\Omega |\nabla\psi_\lambda|^p\,dx &\leqslant C\lambda^{1-p}\mu_\varepsilon\big(\{W_{\mu,r} > \lambda/C\} \cap \Omega'\big) \\ &\leqslant C\lambda^{1-p}\big(\mu_0\big(\{W_{\mu,r} > \lambda/C\} \cap \Omega'\big) + \varepsilon\big) \end{aligned} \tag{3.3}$$

holds for all $\lambda > \widetilde{L}$.

Furthermore, the assumption of the theorem provides us L such that

$$u, v \in \mathcal{S}_{\mu,r,L}\big(\Omega' \setminus S_\varepsilon\big)$$

and thus

$$w_\varepsilon, u, v \in \mathcal{S}_{\mu,r,\max\{L,\widetilde{L}\}}\big(\Omega' \setminus S_\varepsilon\big).$$

Consequently, w_ε, u, and v are comparable. In particular, there is a constant $C < \infty$ such that

$$\big\{\max(u,v) > C\lambda\big\} \cap \big(\Omega' \setminus S_\varepsilon\big) \subset \{w_\varepsilon > \lambda\} \cap \big(\Omega' \setminus S_\varepsilon\big) \subset \big\{\min(u,v) > C^{-1}\lambda\big\} \cap \big(\Omega' \setminus S_\varepsilon\big)$$

holds for all $\lambda > C$.

Next we observe that by the choice of θ_ε we have

$$\begin{aligned}\int_\Omega h(u,v)\phi\theta_\varepsilon\,d\mu &= h(k,k)\int_\Omega \phi\theta_\varepsilon\,d\mu \\ &= h(k,k)\int_\Omega \big\langle \mathcal{A}(x,Du), \nabla(\phi\theta_\varepsilon)\big\rangle\,dx \\ &= \int_\Omega \big\langle \mathcal{A}(x,Du), \nabla\big(h(u,v)\phi\theta_\varepsilon\big)\big\rangle\,dx. \end{aligned}\tag{3.4}$$

Indeed, θ_ε has been chosen so that its support does not intersect the support of ∇h. Our goal is hence to show that

$$\left|\int_\Omega h(u,v)\phi(1-\theta_\varepsilon)\,d\mu - \int_\Omega \big\langle \mathcal{A}(x,Du), \nabla\big(h(u,v)\phi(1-\theta_\varepsilon)\big)\big\rangle\,dx\right|$$

is small by means of ε, eventually leading to the result of the theorem. To prove this, we use the truncated equation of u, i.e.

$$-\operatorname{div}\big(\mathcal{A}(x,\nabla u_m)\big) = \mu[u_m],$$

$m \in \mathbf{N}$.

First, since both u_k and v_k are p-quasicontinuous and in $W^{1,p}(\Omega')$, there are sequences $u_{k,j}$ and $v_{k,j}$ of smooth functions converging in $W^{1,p}(\Omega')$ and p-quasieverywhere to u_k and v_k, respectively. In particular, $u_{k,j} \to u_k$ and $v_{k,j} \to v_k$ μ_0-almost everywhere. This readily implies that $h(u_{j,k}, v_{j,k})$ converges weakly to $h(u_k, v_k)$ in $W^{1,p}(\Omega')$. Recall that $h(u_k, v_k) = h(u,v)$ by the choice of k. We have by the weak convergence of $\mu[u_m]$ to μ that

$$\int_\Omega h(u_{k,j}, v_{k,j})(1-\theta_\varepsilon)\phi\,d\mu[u_m] \to \int_\Omega h(u_{k,j}, v_{k,j})(1-\theta_\varepsilon)\phi\,d\mu.$$

Furthermore, by the p-quasieverywhere convergence and the dominated convergence theorem, we obtain

$$\int_\Omega h(u_{k,j}, v_{k,j})(1-\theta_\varepsilon)\phi\,d\mu_0 \to \int_\Omega h(u,v)(1-\theta_\varepsilon)\phi\,d\mu_0$$

as $j \to \infty$ and the estimate

$$\int_\Omega h(u_{k,j}, v_{k,j})(1-\theta_\varepsilon)\phi\,d\mu_s \leqslant \varepsilon\|h\|_\infty\|\phi\|_\infty$$

holds by Lemma 2.9 and the choice of θ_ε. Hence we obtain

$$\limsup_{j\to\infty}\left|\int_\Omega h(u_{k,j}, v_{k,j})(1-\theta_\varepsilon)\phi\,d\mu - \int_\Omega h(u,v)(1-\theta_\varepsilon)\phi\,d\mu\right| \leqslant C\varepsilon \tag{3.5}$$

with C independent of ε.

Next, rewrite

$$\begin{aligned}&\int_\Omega h(u_{k,j}, v_{k,j})(1-\theta_\varepsilon)\phi\,d\mu[u_m] \\ &\quad= \int_\Omega \psi_\lambda h(u_{k,j}, v_{k,j})(1-\theta_\varepsilon)\phi\,d\mu[u_m] + \int_\Omega (1-\psi_\lambda)h(u_{k,j}, v_{k,j})(1-\theta_\varepsilon)\phi\,d\mu[u_m]. \end{aligned}\tag{3.6}$$

We estimate the first integral on the right as

$$\int_\Omega \psi_\lambda h(u_{k,j}, v_{k,j})(1-\theta_\varepsilon)\phi\,d\mu[u_m] \leqslant \|h\|_\infty \int_\Omega \psi_\lambda(1-\theta_\varepsilon)\phi\,d\mu[u_m] \tag{3.7}$$

and then use the structure of $\mathcal{A}$ to obtain

$$\begin{aligned}
&\int_\Omega \psi_\lambda(1-\theta_\varepsilon)\phi\,d\mu[u_m] \\
&\quad = \int_\Omega \bigl\langle \mathcal{A}(x,\nabla u_m), \nabla\bigl(\psi_\lambda(1-\theta_\varepsilon)\phi\bigr)\bigr\rangle\,dx \\
&\quad \leqslant \beta_0\|\phi\|_\infty \int_{\Omega'\setminus S_\varepsilon} |\nabla u_m|^{p-1}|\nabla\psi_\lambda|\,dx + \beta_0\bigl\|\nabla\bigl(\phi(1-\theta_\varepsilon)\bigr)\bigr\|_\infty \int_{\Omega'\cap\operatorname{supp}(\psi_\lambda)} |Du|^{p-1}\,dx. \qquad (3.8)
\end{aligned}$$

Since $w_\varepsilon, u \in \mathcal{S}_{\mu,r,C}(\Omega'\setminus S_\varepsilon)$,

$$u \leqslant C + cW_{\mu,r} \leqslant C + c^2 w_\varepsilon < C + 2c^2\lambda \quad \text{in } \{w_\varepsilon < 2\lambda\} \cap \bigl(\Omega'\setminus S_\varepsilon\bigr) \qquad (3.9)$$

and hence $u \leqslant C\lambda$ in the intersection of the support of $\nabla\psi_\lambda$ and $\Omega'\setminus S_\varepsilon$ for all sufficiently large λ. In this set $u_m = u_{C\lambda}$ for all $m > C\lambda$. It follows by Hölder's inequality and (3.3) that

$$\begin{aligned}
\int_{\Omega'\setminus S_\varepsilon} |\nabla u_m|^{p-1}|\nabla\psi_\lambda|\,dx &= \int_{\Omega'\setminus S_\varepsilon} |\nabla u_{C\lambda}|^{p-1}|\nabla\psi_\lambda|\,dx \\
&\leqslant \biggl(\int_{\Omega'} |\nabla u_{C\lambda}|^p\,dx\biggr)^{(p-1)/p} \biggl(\int_{B(0,4R)} |\nabla\psi_\lambda|^p\,dx\biggr)^{1/p} \\
&\leqslant C\lambda^{(p-1)/p} C\lambda^{-(p-1)/p}\bigl(\mu_0\bigl(\{W_{\mu,r} > \lambda/C\}\cap\Omega'\bigr)+\varepsilon\bigr)^{1/p} \\
&= C\bigl(\mu_0\bigl(\{W_{\mu,r} > \lambda/C\}\cap\Omega'\bigr)+\varepsilon\bigr)^{1/p} \\
&\to C\varepsilon^{1/p} \qquad (3.10)
\end{aligned}$$

as $\lambda\to\infty$ since $\operatorname{cap}_p(\{W_{\mu,r} > \lambda/C\}\cap\Omega') \to 0$; here we have also employed the estimate

$$\int_{\Omega'\cap\{u\leqslant\lambda\}} |\nabla u|^p\,dx \leqslant C\lambda, \qquad (3.11)$$

with some constant C independent of λ, from the proof of [21, Theorem 1.13]. Note that the upper bound in (3.10) is independent of j and m. Moreover, the local summability of $|Du|^{p-1}$, see Theorem 2.4, implies that

$$\int_{\Omega'\cap\operatorname{supp}(\psi_\lambda)} |Du|^{p-1}\,dx \to 0 \qquad (3.12)$$

as $\lambda\to\infty$ since $\operatorname{cap}_p(\{\psi_\lambda > 0\}\cap\Omega') \to 0$. Inserting estimates (3.10) and (3.12) into (3.8) and then using (3.7) leads to

$$\limsup_{\lambda,j,m\to\infty}\biggl|\int_\Omega h(u_{k,j},v_{k,j})\psi_\lambda(1-\theta_\varepsilon)\phi\,d\mu[u_m]\biggr| \leqslant C\varepsilon^{1/p}. \qquad (3.13)$$

Hence, by (3.5) and (3.6),

$$\limsup_{\lambda,j,m\to\infty}\biggl|\int_\Omega h(u_{k,j},v_{k,j})(1-\psi_\lambda)(1-\theta_\varepsilon)\phi\,d\mu[u_m] - \int_\Omega h(u,v)(1-\theta_\varepsilon)\phi\,d\mu\biggr| \leqslant C\bigl(\varepsilon+\varepsilon^{1/p}\bigr), \qquad (3.14)$$

with C independent of ε.

Next, we consider the first term on the left in (3.14). By (3.9) we have that $(1-\psi_\lambda)(1-\theta_\varepsilon)\phi$ vanishes outside $\{u \leqslant C\lambda\}\cap(\Omega'\setminus S_\varepsilon)$ for all sufficiently large λ. Hence, for all $m > C\lambda$,

$$
\begin{aligned}
\int\limits_{\Omega} & h(u_{k,j}, v_{k,j})(1-\psi_\lambda)(1-\theta_\varepsilon)\phi \, d\mu[u_m] \\
&= \int\limits_{\Omega'\setminus S_\varepsilon} \big\langle \mathcal{A}(x, \nabla u_{C\lambda}), \nabla \phi \big\rangle h(u_{k,j}, v_{k,j})(1-\theta_\varepsilon)(1-\psi_\lambda)\, dx \\
&\quad + \int\limits_{\Omega'\setminus S_\varepsilon} \big\langle \mathcal{A}(x, \nabla u_{C\lambda}), \nabla (1-\theta_\varepsilon) \big\rangle h(u_{k,j}, v_{k,j})(1-\psi_\lambda)\phi\, dx \\
&\quad + \int\limits_{\Omega'\setminus S_\varepsilon} \big\langle \mathcal{A}(x, \nabla u_{C\lambda}), \nabla h(u_{k,j}, v_{k,j}) \big\rangle (1-\theta_\varepsilon)(1-\psi_\lambda)\phi\, dx \\
&\quad - \int\limits_{\Omega'\setminus S_\varepsilon} \big\langle \mathcal{A}(x, \nabla u_{C\lambda}), \nabla \psi_\lambda \big\rangle h(u_{k,j}, v_{k,j})(1-\theta_\varepsilon)\phi\, dx.
\end{aligned}
$$

Now we fix the "marching order" for the limiting processes by sending first m, then j, and finally λ to infinity; observe that the estimates in previous limiting processes were independent of the particular order.

First, it follows by the dominated convergence theorem that

$$
\begin{aligned}
\lim_{\lambda\to\infty}\lim_{j\to\infty}\lim_{m\to\infty} & \int\limits_{\Omega'\setminus S_\varepsilon} \big\langle \mathcal{A}(x, \nabla u_{C\lambda}), \nabla \phi \big\rangle h(u_{k,j}, v_{k,j})(1-\theta_\varepsilon)(1-\psi_\lambda)\, dx \\
&= \int\limits_{\Omega} \big\langle \mathcal{A}(x, Du), \nabla \phi \big\rangle h(u, v)(1-\theta_\varepsilon)\, dx
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{\lambda\to\infty}\lim_{j\to\infty}\lim_{m\to\infty} & \int\limits_{\Omega'\setminus S_\varepsilon} \big\langle \mathcal{A}(x, \nabla u_{C\lambda}), \nabla (1-\theta_\varepsilon) \big\rangle h(u_{k,j}, v_{k,j})(1-\psi_\lambda)\phi\, dx \\
&= \int\limits_{\Omega} \big\langle \mathcal{A}(x, Du), \nabla (1-\theta_\varepsilon) \big\rangle h(u, v)\phi\, dx.
\end{aligned}
$$

Second, the weak convergence of $\nabla h(u_{k,j}, v_{k,j})$ to $\nabla h(u_k, v_k) = \nabla h(u, v)$ together with the dominated convergence gives

$$
\begin{aligned}
\lim_{\lambda\to\infty}\lim_{j\to\infty}\lim_{m\to\infty} & \int\limits_{\Omega'\setminus S_\varepsilon} \big\langle \mathcal{A}(x, \nabla u_{C\lambda}), \nabla h(u_{k,j}, v_{k,j}) \big\rangle (1-\theta_\varepsilon)(1-\psi_\lambda)\phi\, dx \\
&= \int\limits_{\Omega} \big\langle \mathcal{A}(x, Du), \nabla h(u, v) \big\rangle (1-\theta_\varepsilon)\phi\, dx.
\end{aligned}
$$

Third, estimating as in (3.8) and (3.10), we have

$$
\begin{aligned}
\Bigg| \int\limits_{\Omega'\setminus S_\varepsilon} \big\langle \mathcal{A}(x, \nabla u_{C\lambda}), \nabla \psi_\lambda \big\rangle h(u_{k,j}, v_{k,j})(1-\theta_\varepsilon)\phi\, dx \Bigg| &\leqslant C\|h\|_\infty \|\phi\|_\infty \int\limits_{\Omega'} |\nabla u_{C\lambda}|^{p-1} |\nabla \psi_\lambda|\, dx \\
&\leqslant C\big(\mu_0\big(\{W_{\mu,r} > \lambda/C\} \cap \Omega'\big) + \varepsilon\big)^{1/p},
\end{aligned}
$$

which readily implies

$$
\limsup_{\lambda, j, m\to\infty} \Bigg| \int\limits_{\Omega'\setminus S_\varepsilon} \big\langle \mathcal{A}(x, \nabla u_{C\lambda}), \nabla \psi_\lambda \big\rangle h(u_{k,j}, v_{k,j})(1-\theta_\varepsilon)\phi\, dx \Bigg| \leqslant C\varepsilon^{1/p}.
$$

Inserting above estimates into (3.14) we infer that

$$\left| \int_{\Omega} h(u,v)(1-\theta_\varepsilon)\phi\, d\mu - \int_{\Omega} \big\langle \mathcal{A}(x,Du), \nabla\big(h(u,v)(1-\theta_\varepsilon)\phi\big)\big\rangle\, dx \right| \leqslant C\big(\varepsilon + \varepsilon^{1/p}\big).$$

This together with (3.4) yields

$$\left| \int_{\Omega} h(u,v)\phi\, d\mu - \int_{\Omega} \big\langle \mathcal{A}(x,Du), \nabla\big(h(u,v)\phi\big)\big\rangle\, dx \right| \leqslant C\big(\varepsilon + \varepsilon^{1/p}\big),$$

concluding the proof after letting $\varepsilon \to 0$. □

Now we arrive at our main theorem by choosing $u = v$ in Theorem 3.2.

3.15. Theorem. *Let u be $\mathcal{A}$-superharmonic with Riesz measure*

$$\mu = -\operatorname{div} \mathcal{A}(x, \nabla u).$$

Then u is a local renormalized solution, i.e.,

$$\int_{\Omega} \big\langle \mathcal{A}(x,Du), \nabla\big(h(u)\phi\big)\big\rangle\, dx = \int_{\Omega} h(u)\phi\, d\mu$$

for all $\phi \in C_0^\infty(\Omega)$ and for all Lipschitz functions $h : \mathbf{R} \mapsto \mathbf{R}$ whose derivatives h' are compactly supported.

3.16. Remark. Dal Maso and Malusa [9] defined a concept of a *reachable solution*. They showed that such a solution satisfies the formula (for h and φ as in Theorem 3.15)

$$\int_{\Omega} \big\langle \mathcal{A}(x,Du), \nabla\big(h(u)\phi\big)\big\rangle\, dx = \int_{\Omega} h(u)\phi\, d\mu_1 + h(+\infty)\int_{\Omega} \phi\, d\mu_2 - h(-\infty)\int_{\Omega} \phi\, d\mu_3$$

for *some* decomposition $\mu = \mu_1 + \mu_2 - \mu_3$ for the measure μ. The novelty in our result is that we may now specify the decomposition by taking $\mu_1 \ll \mathrm{cap}_p$, $\mu_2 \perp \mathrm{cap}_p$ such that $\mathrm{spt}(\mu_2) \subset \bigcap_{k>0}\{u > k\}$, and $\mu_3 = 0$. Our theorem seems not to be easily deduced from results in [9], since the weak convergence of measures, used by Dal Maso and Malusa to obtain the measures μ_2 and μ_3, seems to be as such inadequate to conquer the concentration phenomenon.

4. Nonlinear Riccati type equations

Theorem 3.15 enables us to employ all the properties of the renormalized solution when studying equations of type (2.5), regardless of the nature of the solutions. As an example we consider the following two problems:

$$\begin{cases} -\Delta_p u = |\nabla u|^p \quad \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega) \end{cases} \tag{4.1}$$

and

$$\begin{cases} -\Delta_p v = \mu, \\ \mu \in \mathcal{M}_+(\Omega) \quad \text{and} \quad \mu \perp \mathrm{cap}_p, \\ 0 \leqslant \min(v,k) \in W_0^{1,p}(\Omega) \quad \text{for all } k > 0, \end{cases} \tag{4.2}$$

where Ω is bounded. Recall that, as emphasized in (2.5), the equations are understood in the sense of distributions.

In this section we show that these two problems are essentially equivalent:

4.3. Theorem. *There is a one-to-one correspondence between problems* (4.1) *and* (4.2) *via the transformation*

$$v = e^{\frac{u}{p-1}} - 1.$$

That is, if u solves (4.1), *then $v = e^{u/(p-1)} - 1$ solves* (4.2); *and conversely, if v is a solution to* (4.2), *then $u = (p-1)\log(v+1)$ is a solution to* (4.1).

Abdel Hamid and Bidaut-Véron have related results in their recent manuscript [1]. The novelty in our result is that we do not assume a priori that solutions are of special nature like renormalized or similar.

Before proving the correspondence we first analyze equations locally.

4.4. Lemma. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfy*

$$-\Delta_p u = |\nabla u|^p \tag{4.5}$$

in Ω. Then $v = e^{u/(p-1)}$ is p-superharmonic in Ω.

Proof. Observe first that u is a nonnegative weak supersolution of $-\Delta_p u \geqslant 0$. It follows that $\tilde{u}$ defined via

$$\tilde{u}(x) = \operatorname{ess}\liminf_{y \to x} u(y)$$

is a representative of u in the sense that $\tilde{u} = u$ almost everywhere. Thus we may assume that u is lower semicontinuous.

Write next $u_k = \min(u, k)$ and $v_k = e^{u_k/(p-1)}$. There is a nonnegative measure ν_k such that

$$\int_\Omega \langle |\nabla u_k|^{p-2} \nabla u_k, \nabla \eta \rangle \, dx = \int_\Omega |\nabla u_k|^p \eta \, dx + \int_\Omega \eta \, d\nu_k$$

for each $\eta \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$; indeed choosing

$$\frac{1}{\varepsilon} \min\bigl(\varepsilon, (k + \varepsilon - u)_+\bigr)\eta, \quad \varepsilon > 0,$$

as a test function in (4.5) and letting $\varepsilon \to 0$, we have by the dominated convergence that

$$\int_\Omega \eta \, d\nu_k = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\{k < u < k+\varepsilon\}} |\nabla u|^p \eta \, dx.$$

Substitute then the test function $\eta = e^{u_k}\varphi$, $\varphi \in C^\infty_0(\Omega)$, $\varphi \geqslant 0$, to obtain

$$\int_\Omega \langle |\nabla u_k|^{p-2} \nabla u_k, e^{u_k} \nabla \varphi \rangle \, dx + \int_\Omega \langle |\nabla u_k|^{p-2} \nabla u_k, \nabla u_k e^{u_k} \varphi \rangle \, dx$$
$$= \int_\Omega |\nabla u_k|^p e^{u_k} \varphi \, dx + \int_\Omega e^{u_k} \varphi \, d\nu_k.$$

Hence

$$(p-1)^{1-p} \int_\Omega \langle |\nabla v_k|^{p-2} \nabla v_k, \nabla \varphi \rangle \, dx = \int_\Omega e^{u_k} \varphi \, d\nu_k \geqslant 0$$

and therefore $v_k \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is p-superharmonic. Consequently also

$$v = \lim_{k \to \infty} v_k$$

is p-superharmonic. $\square$

Next we calculate how the equations are transformed.

4.6. Lemma. *Suppose that v is a nonnegative p-superharmonic function with the Riesz measure μ. Then $u = (p-1)\log(v)$ satisfies*

$$(p-1)^{p-1} \int_\Omega e^{-u} \varphi \, d\mu = \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle \, dx - \int_\Omega |\nabla u|^p \varphi \, dx$$

for all $\varphi \in C^\infty_0(\Omega)$.

Proof. Let $\varphi \in C_0^\infty(\Omega)$. Theorem 3.15 allows us to test the equation of v with the function $\eta_k = e^{-u_k}\varphi$, where $u_k = \min(u,k)$. We have

$$\begin{aligned}
&(p-1)^{p-1}\int_\Omega \eta_k\,d\mu\\
&= (p-1)^{p-1}\int_\Omega \bigl\langle |\nabla v|^{p-2}\nabla v, e^{-u_k}\nabla\varphi - \nabla u_k e^{-u_k}\varphi\bigr\rangle\,dx\\
&= \int_{\{u\leqslant k\}} \bigl\langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\bigr\rangle\,dx + (p-1)^{p-1}\int_{\{u>k\}} \bigl\langle |\nabla v|^{p-2}\nabla v, e^{-u_k}\nabla\varphi\bigr\rangle\,dx - \int_\Omega |\nabla u_k|^p\varphi\,dx\\
&\to \int_\Omega \bigl\langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\bigr\rangle\,dx - \int_\Omega |\nabla u|^p\varphi\,dx
\end{aligned}$$

as $k\to\infty$ since $|\nabla v|^{p-1}$ is locally integrable (Lemma 2.4) and $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, see [19, Theorem 7.48]. The dominated convergence guarantees that

$$\int_\Omega \eta_k\,d\mu \to \int_\Omega e^{-u}\varphi\,d\mu,$$

finishing the proof. $\square$

We are ready to prove the local version of Theorem 4.3:

4.7. Theorem. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution to*

$$-\Delta_p u = |\nabla u|^p$$

in Ω. Then $v = e^{u/(p-1)}$ is p-superharmonic and its Riesz measure $\mu = -\Delta_p v$ is singular with respect to the p-capacity, i.e.

$$\mu \perp \mathrm{cap}_p.$$

Conversely, if v is a nonnegative p-superharmonic function whose Riesz measure $\mu = -\Delta_p v$ is singular with respect to the p-capacity, then $u = (p-1)\log v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ solves weakly the equation

$$-\Delta_p u = |\nabla u|^p$$

in Ω.

Proof. Suppose first that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfies

$$-\Delta_p u = |\nabla u|^p \quad \text{in } \Omega.$$

Then, by Lemma 4.4, $v = e^{u/(p-1)}$ is p-superharmonic and Lemma 4.6 gives

$$\int_\Omega v^{1-p}\varphi\,d\mu = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Therefore $v=\infty$ μ-almost everywhere. Since the set $\{v=\infty\}$ is of p-capacity zero, we have $\mu \perp \mathrm{cap}_p$, as desired.

For the converse, let v be nonnegative and p-superharmonic with a singular Riesz measure. First observe that $u = (p-1)\log v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ by the standard logarithm estimate [19, Lemma 7.48]. The rest follows by Lemma 4.6, because $v=\infty$ μ-almost everywhere by Lemma 2.9 and thus

$$\int_\Omega e^{-u}\varphi\,d\mu = \int_\Omega v^{1-p}\varphi\,d\mu = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. $\square$

4.8. Corollary. *Suppose that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfies*

$$-\Delta_p u = |\nabla u|^p \quad \text{in } \Omega.$$

Then $e^{\lambda u} \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for all $0 < \lambda < 1/p$.

If $e^{u/p} \in W^{1,p}_{\mathrm{loc}}(\Omega)$, then $v = e^{u/(p-1)}$ is p-harmonic and hence u is $C^{1,\alpha}$ for some $\alpha > 0$.

Proof. Let $0 < \lambda < 1/p$. The integrability result easily follows from the estimate [19, Lemma 3.57]: if $\varepsilon > 0$,

$$\int_\Omega |\nabla v|^p v^{-1-\varepsilon} \eta^p \, dx \leqslant \left(\frac{p}{\varepsilon}\right)^p \int_\Omega v^{p-1-\varepsilon} |\nabla \eta|^p \, dx$$

for all cut-off functions $\eta \in C_0^\infty(\Omega)$, $\eta \geqslant 0$. Now, choosing $\varepsilon = (p-1)(1-p\lambda)$ we have that

$$\int_\Omega \left|\nabla e^{\lambda u}\right|^p \eta^p \, dx = c \int_\Omega |\nabla v|^p v^{-1-\varepsilon} \eta^p \, dx \leqslant c \int_\Omega v^{p-1-\varepsilon} |\nabla \eta|^p \, dx < \infty,$$

since v^{p-1} is locally integrable (Lemma 2.4).

To prove the latter claim, we use the test function $\eta = e^{u_k}\varphi$, where $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$, and $u_k = \min(u, k)$. Then

$$\int_\Omega \left\langle |\nabla u|^{p-2} \nabla u, e^{u_k} \nabla \varphi \right\rangle dx + \int_\Omega \left\langle |\nabla u|^{p-2} \nabla u, \nabla u_k e^{u_k} \varphi \right\rangle dx = \int_\Omega |\nabla u|^p e^{u_k} \varphi \, dx,$$

and hence

$$\int_\Omega \left\langle |\nabla u|^{p-2} \nabla u, e^{u_k} \nabla \varphi \right\rangle dx = \int_\Omega |\nabla u|^p e^{u_k} \varphi \, dx - \int_\Omega |\nabla u_k|^p e^{u_k} \varphi \, dx = \int_{\{u>k\}} e^k |\nabla u|^p \varphi \, dx \to 0$$

by the assumption $e^{u/p} \in W^{1,p}_{\mathrm{loc}}(\Omega)$. The right-hand side then converges to

$$(p-1)^{1-p} \int_\Omega \left\langle |\nabla v|^{p-2} \nabla v, \nabla \varphi \right\rangle dx,$$

which shows that the Riesz measure of v vanishes and therefore v is p-harmonic, see [28, Corollary 3.19]. Hence v and thereby also u is locally $C^{1,\alpha}$ for some $\alpha > 0$ (see for example [36,37,10]). □

Next we turn to the global problems (4.1) and (4.2).

Proof for Theorem 4.3. In the light of the local version 4.7, we only need to check that the transformations go into correct spaces. First, if $u \in W_0^{1,p}(\Omega)$, then it is clear that the truncations of $v = e^{u/(p-1)} - 1$ lie in $W_0^{1,p}(\Omega)$, since Ω is bounded.

Conversely, if v is a solution to the problem (4.2) and $u = \log(v+1)$, then

$$\int_{\{v \leqslant 1\}} |\nabla u|^p \, dx = \int_{\{v \leqslant 1\}} \frac{|\nabla v|^p}{(v+1)^p} \, dx \leqslant \int_{\{v \leqslant 1\}} |\nabla v|^p \, dx < \infty.$$

Also

$$\int_{\{v>1\}} |\nabla u|^p \, dx \leqslant \int_{\{v>1\}} |\nabla \log v|^p \min(v,1)^p \, dx \leqslant \int_\Omega \left|\nabla \min(v,1)\right|^p dx < \infty$$

by the standard log-estimate [19, Proof of Lemma 3.47]. The proof is complete. □

We finally record an estimate that might be of some interest:

4.9. Proposition. *Suppose that u is a solution to* (4.1) *in a bounded open set Ω. Then $e^{\lambda u} \in W^{1,p}(\Omega)$ for all $0 < \lambda < 1/p$. If $e^{u/p} \in W^{1,p}(\Omega)$, then $u = 0$. In particular, the only bounded solution is $u = 0$.*

Proof. Define first a decreasing function $f : \mathbf{R} \mapsto [0, \infty)$ as

$$f(k) = \int_{\Omega \cap \{u > k\}} |\nabla u|^p \, dx.$$

We then have

$$f(k + \varepsilon) - f(k) = -\int_{\Omega \cap \{k < u < k + \varepsilon\}} |\nabla u|^p \, dx.$$

As a monotone function, f is differentiable for almost every k.

Take then the test function

$$\eta_{k,\varepsilon} = \frac{1}{\varepsilon} \min\bigl\{(u - k)_+, \varepsilon\bigr\},$$

$\varepsilon > 0$. Now $\eta_{k,\varepsilon} \in W_0^{1,p}(\Omega)$ provided that $k \geqslant 0$. The monotone convergence theorem implies

$$\lim_{\varepsilon \to 0} \int_\Omega |\nabla u|^p \eta_{k,\varepsilon} \, dx = f(k)$$

for every $k \geqslant 0$. Inserting thus $\eta_{k,\varepsilon}$ into (4.1) and letting $\varepsilon \to 0$ implies

$$\lim_{\varepsilon \to 0} \int_\Omega \bigl\langle |\nabla u|^{p-2} \nabla u, \nabla \eta_{k,\varepsilon} \bigr\rangle \, dx = f(k).$$

The term on the left is

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \bigl(f(k) - f(k + \varepsilon)\bigr)$$

and the limit is obtained for every $k \geqslant 0$. Replacing k by $k - \varepsilon$, an analogous argument gives

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \bigl(f(k - \varepsilon) - f(k)\bigr) = f(k).$$

So $f'(k) = -f(k)$ for every $k > 0$ and $f'(0+) = -f(0)$. Solving the ordinary differential equation gives

$$\int_{\Omega \cap \{u > k\}} |\nabla u|^p \, dx = f(k) = e^{-k} f(0) = e^{-k} \int_\Omega |\nabla u|^p \, dx$$

for all $k \geqslant 0$.

We multiply this equation by $e^{\tilde{\lambda} k}$, $0 < \tilde{\lambda} < 1$, and integrate by the aid of the Fubini theorem to obtain the desired estimate:

$$\begin{aligned}
\int_\Omega e^{\tilde{\lambda} u} |\nabla u|^p \, dx &= \int_\Omega \bigl(e^{\tilde{\lambda} u} - 1\bigr) |\nabla u|^p \, dx + \int_\Omega |\nabla u|^p \, dx \\
&= \tilde{\lambda} \int_0^\infty e^{\tilde{\lambda} k} \int_{\{u > k\}} |\nabla u|^p \, dx \, dk + \int_\Omega |\nabla u|^p \, dx \\
&= \tilde{\lambda} \int_0^\infty e^{\tilde{\lambda} k - k} \int_\Omega |\nabla u|^p \, dx \, dk + \int_\Omega |\nabla u|^p \, dx
\end{aligned}$$

$$= \left(\tilde{\lambda} \int_0^\infty e^{\tilde{\lambda}k-k}\,dk + 1 \right) \int_\Omega |\nabla u|^p\,dx$$
$$= \frac{1}{1-\tilde{\lambda}} \int_\Omega |\nabla u|^p\,dx < \infty.$$

Hence $e^{\lambda u} \in W^{1,p}(\Omega)$ for all $0 < \lambda < 1/p$.

Should it happen that $e^{u/p} \in W^{1,p}(\Omega)$ we could let $\tilde{\lambda}$ increase to 1 in the calculation above. Since the term on the left remains bounded, this would force ∇u vanish throughout. Hence u is zero. □

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