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# Concentration of solutions for a singularly perturbed Neumann problem in non-smooth domains

Serena Dipierro

*SISSA, Sector of Mathematical Analysis, Via Bonomea 265, 34136 Trieste, Italy*

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**Abstract**

We consider the equation $-\epsilon^2 \Delta u + u = u^p$ in a bounded domain $\Omega \subset \mathbb{R}^3$ with edges. We impose Neumann boundary conditions, assuming $1 < p < 5$, and prove concentration of solutions at suitable points of $\partial\Omega$ on the edges.


## 1. Introduction

In this paper we study the following singular perturbation problem with Neumann boundary condition in a bounded domain $\Omega \subset \mathbb{R}^3$ whose boundary $\partial\Omega$ is non-smooth:

$$\begin{cases} -\epsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here $p \in (1, 5)$ is subcritical and $\nu$ denotes the outer unit normal at $\partial\Omega$.

Problem (1) or some of its variants arise in several physical and biological models. Consider, for example, the Nonlinear Schrödinger Equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta\psi + V\psi - \gamma|\psi|^{p-2}\psi, \tag{2}$$

where $\hbar$ is the Planck constant, $V$ is the potential, and $\gamma$ and $m$ are positive constants. Then standing waves of (2) can be found setting $\psi(x,t) = e^{-iEt/\hbar} v(x)$, where $E$ is a constant and the real function $v$ satisfies the elliptic equation

$$-\hbar^2 \Delta v + \tilde{V} v = |v|^{p-2} v$$

for some modified potential $\tilde{V}$. In particular, when one considers the semiclassical limit $\hbar \to 0$, the last equation becomes a singularly perturbed one; see for example [2,9], and references therein.

Concerning reaction–diffusion systems, this phenomenon is related to the so-called Turing's instability. More precisely, it is known that scalar reaction–diffusion equations in a convex domain admit only constant stable steady

*E-mail address:* dipierro@sissa.it.

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state solutions; see [4,27]. On the other hand, as noticed in [38], reaction–diffusion systems with different diffusivities might generate non-homogeneous stable steady states. A well-known example is the Gierer–Meinhardt system, introduced in [12] to describe some biological experiment. We refer to [30,34] for more details.

The study of the concentration phenomena at points for smooth domains is very rich and has been intensively developed in recent years. The search for such condensing solutions is essentially carried out by two methods. The first approach is variational and uses tools of the critical point theory or topological methods. A second way is to reduce the problem to a finite-dimensional one by means of Lyapunov–Schmidt reduction.

The typical concentration behavior of solution $U_{Q,\epsilon}$ to (1) is via a scaling of the variables in the form

$$U_{Q,\epsilon}(x) \sim U\left(\frac{x-Q}{\epsilon}\right), \tag{3}$$

where $Q$ is some point of $\bar{\Omega}$, and $U$ is a solution of the problem

$$-\Delta U + U = U^p \quad \text{in } \mathbb{R}^3 \ \left(\text{or in } \mathbb{R}^3_+ = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3\colon\ x_3 > 0\right\}\right), \tag{4}$$

the domain depending on whether $Q$ lies in the interior of $\Omega$ or at the boundary; in the latter case Neumann conditions are imposed. When $p < 5$ (and indeed only if this inequality is satisfied), problem (4) admits positive radial solutions which decay to zero at infinity; see [3,37]. Solutions of (1) with this profile are called *spike-layers*, since they are highly concentrated near some point of $\bar{\Omega}$.

Let us recall some known results. *Boundary-spike layers* are solutions of (1) with a concentration at one or more points of the boundary $\partial\Omega$ as $\epsilon \to 0$. They are peaked near critical point of the mean curvature. It was shown in [32,33] that mountain-pass solutions of (1) concentrate at $\partial\Omega$ near global maxima of the mean curvature. One can see this fact considering the variational structure of (1). In fact, its solutions can be found as critical points of the following Euler–Lagrange functional

$$\tilde{I}_\epsilon(u) = \frac{1}{2}\int_\Omega \left(\epsilon^2|\nabla u|^2 + u^2\right)dx - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx, \quad u \in W^{1,2}(\Omega).$$

Plugging into $\tilde{I}_\epsilon$ a function of the form (3) with $Q \in \partial\Omega$ one sees that

$$\tilde{I}_\epsilon(U_{Q,\epsilon}) = C_0\epsilon^3 - C_1\epsilon^4 H(Q) + o\left(\epsilon^4\right), \tag{5}$$

where $C_0, C_1$ are positive constants depending only on the dimension and $p$, and $H$ is the mean curvature; see for instance [2, Lemma 9.7]. To obtain this expansion one can use the radial symmetry of $U$ and parametrize $\partial\Omega$ as a normal graph near $Q$. From the above formula one can see that the bigger is the mean curvature the lower is the energy of this function: roughly speaking, boundary spike layers would tend to move along the gradient of $H$ in order to minimize their energy. Moreover one can say that the energy of spike-layers is of order $\epsilon^3$, which is proportional to the volume of their *support*, heuristically identified with a ball of radius $\epsilon$ centered at the peak. There is an extensive literature regarding the search of more general solutions of (1) concentrating at critical points of $H$; see [8,15–17,23, 25,31,40].

There are other types of solutions of (1) with interior and/or boundary peaks, possible multiple, which are constructed by using gluing techniques or topological methods; see [6,7,18–20,39]. For *interior spike* solutions the distance function $d$ from the boundary $\partial\Omega$ plays a role similar to that of the mean curvature $H$. In fact, solutions with interior peaks, as for the problem with the Dirichlet boundary condition, concentrate at critical points of $d$, in a generalized sense; see [24,35,41].

Concerning a singularly perturbed problem with mixed Dirichlet and Neumann boundary conditions, in [10,11] it was proved that, under suitable geometric conditions on the boundary of a smooth domain, there exist solutions which approach the intersection of the Neumann and the Dirichlet parts as the singular perturbation parameter tends to zero.

There is an extensive literature regarding this type of problems, but only the case $\Omega$ smooth was considered. Concerning the case $\Omega$ non-smooth, at our knowledge there is only a bifurcation result for the equation

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

obtained by Shi in [36] when $\Omega$ is a rectangle $(0,a) \times (0,b)$ in $\mathbb{R}^2$.

In this paper we consider the problem (1), where $\Omega$ is a bounded domain in $\mathbb{R}^3$ whose boundary $\partial\Omega$ has smooth edges. If we denote by $\Gamma$ an edge of $\partial\Omega$, we can consider the function $\alpha : \Gamma \to \mathbb{R}$ which associates to every $Q \in \Gamma$ the opening angle at $Q$, $\alpha(Q)$. As in the previous case, we can expect that the function $\alpha$ plays the same role as the mean curvature $H$ for a smooth domain. In fact, plugging into $\tilde{I}_\epsilon$ a function of the form (3) with $Q \in \Gamma$ one obtains an expression similar to (5), with $C_0\alpha(Q)$ instead of $C_0$; see Lemma 4.3. Roughly speaking, we can say that the energy of solutions is of order $\epsilon^3$, which is proportional to the volume of their *support*, heuristically identified with a ball of radius $\epsilon$ centered at the peak $Q \in \Gamma$; then, when we intersect this ball with the domain we obtain the dependence on the angle $\alpha(Q)$.

The main result of this paper is the following

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^3$ be a piecewise smooth bounded domain whose boundary $\partial\Omega$ has a finite number of smooth edges, and $1 < p < 5$. Fix an edge $\Gamma$, and suppose $Q \in \Gamma$ is a local strict maximum or minimum of the function $\alpha$, with $\alpha(Q) \neq \pi$. Then for $\epsilon > 0$ sufficiently small problem* (1) *admits a solution concentrating at $Q$.*

**Remark 1.2.** The condition that $Q$ is a local strict maximum or minimum of $\alpha$ can be replaced by the fact that there exists an open set $V$ of $\Gamma$ containing $Q$ such that $\alpha(Q) > \sup_{\partial V} \alpha$ or $\alpha(Q) < \inf_{\partial V} \alpha$.

**Remark 1.3.** The condition $\alpha(Q) \neq \pi$ is natural since it is needed to ensure that $\partial\Omega$ is not flat at $Q$.

**Remark 1.4.** We expect a similar result to hold in higher dimension, with substantially the same proof. For simplicity we only treat the 3-dimensional case.

The general strategy for proving Theorem 1.1 relies on a finite-dimensional reduction; see for example the book [2].

By the change of variables $x \mapsto \epsilon x$, problem (1) can be transformed into

$$\begin{cases} -\Delta u + u = u^p & \text{in } \Omega_\epsilon, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{6}$$

where $\Omega_\epsilon := \frac{1}{\epsilon}\Omega$. Solutions of (6) can be found as critical points of the Euler–Lagrange functional

$$I_\epsilon(u) = \frac{1}{2}\int_{\Omega_\epsilon} \left(|\nabla u|^2 + u^2\right) dx - \frac{1}{p+1}\int_{\Omega_\epsilon} |u|^{p+1}\, dx, \quad u \in W^{1,2}(\Omega_\epsilon). \tag{7}$$

Now, first of all, one finds a manifold $Z_\epsilon$ of approximate solutions to the given problem, which are of the form $U_{Q,\epsilon}(x) = \varphi_\mu(\epsilon x)U(x - Q)$, where $\varphi_\mu$ is a suitable cut-off function defined in a neighborhood of $Q \in \Gamma$; see the beginning of Section 4, Lemma 4.1.

To apply the method described in Section 2.1 one needs the condition that the critical manifold $Z_\epsilon$ is non-degenerate, in the sense that it satisfies property (ii) in Section 2.1. The result of non-degeneracy in $\Omega_\epsilon$, obtained in Lemma 4.2, follows from the non-degeneracy of a manifold $Z$ of critical points of the unperturbed problem in $K = \tilde{K} \times \mathbb{R} \subset \mathbb{R}^3$, where $\tilde{K} \subset \mathbb{R}^2$ is a cone of opening angle $\alpha(Q)$. In fact, one sees that $\Omega_\epsilon$ tends to $K$ as $\epsilon \to 0$. To show the non-degeneracy of the unperturbed manifold $Z$ we follow the line of Lemma 4.1 in the book [2] or Lemma 3.1 in [26]. We prove that $\lambda = 0$ is a simple eigenvalue of the linearized of the unperturbed problem at $U \in Z$; see Lemma 3.1. Moreover, if $\alpha(Q) < \pi$, it has only one negative simple eigenvalue; whereas, if $\alpha(Q) > \pi$, it has two negative simple eigenvalues; see Corollary 3.4. We note that in the case $\alpha(Q) = \pi$, that is when $\partial\Omega$ is flat at $Q$, $\lambda = 0$ is an eigenvalue of multiplicity 2. The proof relies on Fourier analysis, but in this case one needs spherical functions defined on a portion of the sphere instead of the whole $S^2$.

Then one solves the equation up to a vector parallel to the tangent plane of the manifold $Z_\epsilon$, and generates a new manifold $\tilde{Z}_\epsilon$ close to $Z_\epsilon$ which represents a natural constraint for the Euler functional (7); see the proof of Proposition 4.5. By *natural constraint* we mean a set for which constrained critical points of $I_\epsilon$ are true critical points.

We can finally apply the above mentioned perturbation method to reduce the problem to a finite-dimensional one, and study the functional constrained on $\tilde{Z}_\epsilon$. Lemma 4.3 provides an expansion of the energy of the approximate solution peaked at $Q$ and allows us to see that the dominant term in the expression of the reduced functional at $Q$ is $\alpha(Q)$. This implies Theorem 1.1.

The paper is organized in the following way. In Section 2 we collect preliminary material: we recall the abstract variational perturbative scheme and obtain some useful geometric results. In Section 3 we prove the non-degeneracy of the critical manifold for the unperturbed problem in the cone $K$. In Section 4 we construct the manifold of approximate solutions, showing that it is a non-degenerate pseudo-critical manifold, expand the functional on the natural constraint and deduce Theorem 1.1.

**Notation.** Generic fixed constant will be denoted by $C$, and will be allowed to vary within a single line or formula. The symbols $o_\epsilon(1)$, $o_R(1)$, $o_{\epsilon,R}(1)$ will denote respectively a function depending on $\epsilon$ that tends to 0 as $\epsilon \to 0$, a function depending on $R$ that tends to 0 as $R \to +\infty$ and a function depending on both $\epsilon$ and $R$ that tends to 0 as $\epsilon \to 0$ and $R \to +\infty$. We will work in the space $W^{1,2}(\Omega_\epsilon)$, endowed with the norm $\|u\|^2 = \int_{\Omega_\epsilon}(|\nabla u|^2 + u^2)\,dx$, which we denote simply by $\|u\|$, without any subscript.

## 2. Some preliminaries

In this section we introduce the abstract perturbation method which takes advantage of the variational structure of the problem, and allows us to reduce it to a finite-dimensional one. We refer the reader mainly to [2,26] and the bibliography therein.

In the second part we make some computations concerning the parametrization of $\partial\Omega$ and $\partial\Omega_\epsilon$, and in particular of the edge.

### 2.1. Perturbation in critical point theory

In this subsection we recall some results about the existence of critical points for a class of functionals which are perturbative in nature. Given a Hilbert space $H$, which might depend on the perturbation parameter $\epsilon$, let $I_\epsilon : H \to \mathbb{R}$ be a functional of class $C^2$ which satisfies the following properties

(i) there exists a smooth finite-dimensional manifold, compact or not, $Z_\epsilon \subseteq H$ such that $\|I'_\epsilon(z)\| \leqslant C\epsilon$ for every $z \in Z_\epsilon$ and for some fixed constant $C$, independent of $z$ and $\epsilon$; moreover $\|I''_\epsilon(z)[q]\| \leqslant C\epsilon\|q\|$ for every $z \in Z_\epsilon$ and every $q \in T_z Z_\epsilon$;
(ii) letting $P_z : H \to (T_z Z_\epsilon)^\perp$, for every $z \in Z_\epsilon$, be the projection onto the orthogonal complement of $T_z Z_\epsilon$, there exists $C > 0$, independent of $z$ and $\epsilon$, such that $P_z I''_\epsilon(z)$, restricted to $(T_z Z_\epsilon)^\perp$, is invertible from $(T_z Z_\epsilon)^\perp$ into itself, and the inverse operator satisfies $\|(P_z I''_\epsilon(z))^{-1}\| \leqslant C$.

We assume that $Z_\epsilon$ has a local $C^2$ parametric representation $z = z_\xi$, $\xi \in \mathbb{R}^d$. If we set $W = (T_z Z_\epsilon)^\perp$, we look for critical points of $I_\epsilon$ in the form $u = z + w$ with $z \in Z_\epsilon$ and $w \in W$. If $P_z : H \to W$ is as in (ii), the equation $I'_\epsilon(z + w) = 0$ is equivalent to the following system

$$\begin{cases} P_z I'_\epsilon(z+w) = 0 & \text{(the auxiliary equation)}, \\ (Id - P_z) I'_\epsilon(z+w) = 0 & \text{(the bifurcation equation)}. \end{cases} \tag{8}$$

**Proposition 2.1.** *(See Proposition 2.2 in [26].) Let* (i), (ii) *hold. Then there exists $\epsilon_0 > 0$ with the following property: for all $|\epsilon| < \epsilon_0$ and for all $z \in Z_\epsilon$, the auxiliary equation in* (8) *has a unique solution $w = w_\epsilon(z)$ such that:*

(j) *$w_\epsilon(z) \in W$ is of class $C^1$ with respect to $z \in Z_\epsilon$ and $w_\epsilon(z) \to 0$ as $|\epsilon| \to 0$, uniformly with respect to $z \in Z_\epsilon$, together with its derivative with respect to $z$, $w'_\epsilon$;*
(jj) *more precisely one has that $\|w_\epsilon(z)\| = O(\epsilon)$ as $\epsilon \to 0$, for all $z \in Z_\epsilon$.*

We shall now solve the bifurcation equation in (8). In order to do this, let us define the *reduced functional* $\Phi_\epsilon : Z_\epsilon \to \mathbb{R}$ by setting $\Phi_\epsilon(z) = I_\epsilon(z + w_\epsilon(z))$.

**Theorem 2.2.** *(See Theorem 2.3 in [26].) Suppose we are in the situation of Proposition* 2.1*, and let us assume that* $\Phi_\epsilon$ *has, for* $|\epsilon|$ *sufficiently small, a critical point* $z_\epsilon$*. Then* $u_\epsilon = z_\epsilon + w(z_\epsilon)$ *is a critical point of* $I_\epsilon$*.*

The next result is a useful criterion for applying Theorem 2.2, based on expanding $I_\epsilon$ on $Z_\epsilon$ in powers of $\epsilon$.

**Theorem 2.3.** *(See Theorem 2.4 in [26].) Suppose the assumptions of Proposition* 2.1 *hold, and that for* $\epsilon$ *small there is a local parametrization* $\xi \in \frac{1}{\epsilon}U \subseteq \mathbb{R}^d$ *of* $Z_\epsilon$ *such that, as* $\epsilon \to 0$*,* $I_\epsilon$ *admits the expansion* $I_\epsilon(z_\xi) = C_0 + \epsilon G(\epsilon\xi) + o(\epsilon)$*, for* $\xi \in \frac{1}{\epsilon}U$*, for some function* $G : U \to \mathbb{R}$*. Then we still have the expansion* $\Phi_\epsilon(z_\xi) = C_0 + \epsilon G(\epsilon\xi) + o(\epsilon)$*, as* $\epsilon \to 0$*. Moreover, if* $\bar{\xi} \in U$ *is a strict local maximum or minimum of G, then for* $|\epsilon|$ *small the functional* $I_\epsilon$ *has a critical point* $u_\epsilon$*. Furthermore, if* $\bar{\xi}$ *is isolated, we can take* $u_\epsilon - z_{\bar{\xi}/\epsilon} = o(1/\epsilon)$ *as* $\epsilon \to 0$*.*

**Remark 2.4.** The last statement asserts that, once we scale back in $\epsilon$, the solution concentrates near $\bar{\xi}$.

*2.2. Geometric preliminaries*

Let us describe $\partial\Omega_\epsilon$ near a generic point $Q$ on the edge $\Gamma$ of $\partial\Omega_\epsilon$. Without loss of generality, we can assume that $Q = 0 \in \mathbb{R}^3$, that $x_1$-axis is the tangent line at $Q$ to $\Gamma$ in $\partial\Omega_\epsilon$, or $\partial\Omega$. In a neighborhood of $Q$, let $\gamma : (-\mu_0, \mu_0) \to \mathbb{R}^2$ be a local parametrization of $\Gamma$, that is $(x_2, x_3) = \gamma(x_1) = (\gamma_1(x_1), \gamma_2(x_1))$. Then one has, for $|x_1| < \mu_0$,

$$\begin{aligned}(x_2, x_3) &= \gamma(x_1)\\ &= \gamma(0) + \gamma'(0)x_1 + \frac{1}{2}\gamma''(0)x_1^2 + O\big(|x_1|^3\big)\\ &= \frac{1}{2}\gamma''(0)x_1^2 + O\big(|x_1|^3\big).\end{aligned}$$

On the other hand, $\Gamma$ is parametrized by $(x_2, x_3) = \gamma_\epsilon(x_1) := \frac{1}{\epsilon}\gamma(\epsilon x_1)$, for which the following expansions hold

$$\begin{aligned}&\gamma_\epsilon(x_1) = \frac{\epsilon}{2}\gamma''(0)x_1^2 + O\big(\epsilon^2|x_1|^3\big),\\ &\frac{\partial\gamma_\epsilon}{\partial x_1} = \epsilon\gamma''(0)x_1 + O\big(\epsilon^2|x_1|^2\big).\end{aligned} \tag{9}$$

Now we introduce a new set of coordinates on $B_{\frac{\mu_0}{\epsilon}}(Q) \cap \Omega_\epsilon$:

$$y_1 = x_1, \qquad (y_2, y_3) = (x_2, x_3) - \gamma_\epsilon(x_1).$$

The advantage of these coordinates is that the edge identifies with $y_1$-axis, but the corresponding metric $g = (g_{ij})_{ij}$ will not be flat anymore. If $\gamma_\epsilon(x_1) = (\gamma_{\epsilon 1}(x_1), \gamma_{\epsilon 2}(x_1))$, the coefficients of $g$ are given by

$$(g_{ij}) = \left(\frac{\partial x}{\partial y_i} \cdot \frac{\partial x}{\partial y_j}\right) = \begin{pmatrix} 1 + \frac{\partial\gamma_{\epsilon 1}}{\partial y_1}\frac{\partial\gamma_{\epsilon 1}}{\partial y_1} + \frac{\partial\gamma_{\epsilon 2}}{\partial y_1}\frac{\partial\gamma_{\epsilon 2}}{\partial y_1} & \frac{\partial\gamma_{\epsilon 1}}{\partial y_1} & \frac{\partial\gamma_{\epsilon 2}}{\partial y_1} \\ \frac{\partial\gamma_{\epsilon 1}}{\partial y_1} & 1 & 0 \\ \frac{\partial\gamma_{\epsilon 2}}{\partial y_1} & 0 & 1 \end{pmatrix}.$$

From the estimates in (9) it follows that

$$g_{ij} = Id + \epsilon A + O\big(\epsilon^2|x_1|^2\big), \tag{10}$$

where

$$A = \begin{pmatrix} 0 & \gamma''(0)x_1 \\ \gamma''(0)^T x_1 & 0 \end{pmatrix}.$$

It is also easy to check that the inverse matrix $(g^{ij})$ is of the form $g^{ij} = Id - \epsilon A + O(\epsilon^2|x_1|^2)$. Furthermore one has $\det g = 1$. Therefore, by (10), for any smooth function $u$ there holds

$$\begin{aligned}\Delta_g u = \Delta u &- \epsilon\left[2\left(\gamma''(0)y_1 \cdot \nabla_{(y_2,y_3)}\frac{\partial u}{\partial y_1}\right) + \big(\gamma''(0) \cdot \nabla_{(y_2,y_3)}u\big)\right]\\ &+ O\big(\epsilon^2|x_1|^2\big)\big|\nabla^2 u\big| + O\big(\epsilon^2|x_1|^2\big)|\nabla u|.\end{aligned} \tag{11}$$

Now, let us consider a smooth domain $\tilde{\Omega} \subset \mathbb{R}^3$ and $\tilde{\Omega}_\epsilon = \frac{1}{\epsilon}\tilde{\Omega}$. In the same way we can describe $\partial\tilde{\Omega}_\epsilon$ near a generic point $Q \in \partial\tilde{\Omega}_\epsilon$. Without loss of generality, we can assume that $Q = 0 \in \mathbb{R}^3$, that $\{x_3 = 0\}$ is the tangent plane of $\partial\tilde{\Omega}_\epsilon$, or $\partial\tilde{\Omega}$, at $Q$, and that the outer normal $\nu(Q) = (0, 0, -1)$. In a neighborhood of $Q$, let $x_3 = \psi(x_1, x_2)$ be a local parametrization of $\partial\tilde{\Omega}$. Then one has, for $|(x_1, x_2)| < \mu_1$,

$$\begin{aligned} x_3 &= \psi(x_1, x_2) \\ &= \frac{1}{2}\big(A_Q(x_1, x_2)\cdot(x_1, x_2)\big) + C_Q(x_1, x_2) + O\big(\big|(x_1, x_2)\big|^4\big), \end{aligned}$$

where $A_Q$ is the Hessian of $\psi$ at $(0, 0)$ and $C_Q$ is a cubic polynomial, which is given precisely by

$$C_Q(x_1, x_2) = \frac{1}{6}\sum_{i,j,k=1}^{2}\frac{\partial^3\psi}{\partial x_i\partial x_j\partial x_k}(0, 0)x_i x_j x_k.$$

On the other hand, $\partial\tilde{\Omega}_\epsilon$ is parametrized by $x_3 = \psi_\epsilon(x_1, x_2) := \frac{1}{\epsilon}\psi(\epsilon x_1, \epsilon x_2)$, for which the following expansions hold

$$\begin{aligned} &\psi_\epsilon(x_1, x_2) = \frac{\epsilon}{2}\big(A_Q(x_1, x_2)\cdot(x_1, x_2)\big) + \epsilon^2 C_Q(x_1, x_2) + O\big(\epsilon^3\big|(x_1, x_2)\big|^4\big), \\ &\frac{\partial\psi_\epsilon}{\partial x_i}(x_1, x_2) = \epsilon\big(A_Q(x_1, x_2)\big)_i + \epsilon^2 D^i_Q(x_1, x_2) + O\big(\epsilon^3\big|(x_1, x_2)\big|^3\big), \end{aligned} \tag{12}$$

where $D^i_Q$ are quadratic forms in $(x_1, x_2)$ given by

$$D^i_Q(x_1, x_2) = \frac{1}{2}\sum_{j,k=1}^{2}\frac{\partial^3\psi}{\partial x_i\partial x_j\partial x_k}(0, 0)x_j x_k.$$

Concerning the outer normal $\nu$, we have also

$$\nu = \frac{(\frac{\partial\psi_\epsilon}{\partial x_1}, \frac{\partial\psi_\epsilon}{\partial x_2}, -1)}{\sqrt{1+|\nabla\psi_\epsilon|^2}} = \left(\epsilon\big(A_Q(x_1, x_2)\big) + \epsilon^2 D_Q(x_1, x_2), -1 + \frac{1}{2}\epsilon^2\big|A_Q(x_1, x_2)\big|^2\right) + O\big(\epsilon^3\big|(x_1, x_2)\big|^3\big). \tag{13}$$

Now we introduce a new set of coordinates on $B_{\frac{\mu_1}{\epsilon}}(Q) \cap \tilde{\Omega}_\epsilon$:

$$z_1 = x_1, \qquad z_2 = x_2, \qquad z_3 = x_3 - \psi_\epsilon(x_1, x_2).$$

The advantage of these coordinates is that $\partial\tilde{\Omega}_\epsilon$ identifies with $\{z_3 = 0\}$, but, as before, the corresponding metric $\tilde{g} = (\tilde{g}_{ij})_{ij}$ will not be flat anymore. Its coefficients are given by

$$(\tilde{g}_{ij}) = \left(\frac{\partial x}{\partial z_i}\cdot\frac{\partial x}{\partial z_j}\right) = \begin{pmatrix} 1 + \frac{\partial\psi_\epsilon}{\partial z_1}\frac{\partial\psi_\epsilon}{\partial z_1} & \frac{\partial\psi_\epsilon}{\partial z_1}\frac{\partial\psi_\epsilon}{\partial z_2} & \frac{\partial\psi_\epsilon}{\partial z_1} \\ \frac{\partial\psi_\epsilon}{\partial z_2}\frac{\partial\psi_\epsilon}{\partial z_1} & 1 + \frac{\partial\psi_\epsilon}{\partial z_2}\frac{\partial\psi_\epsilon}{\partial z_2} & \frac{\partial\psi_\epsilon}{\partial z_2} \\ \frac{\partial\psi_\epsilon}{\partial z_1} & \frac{\partial\psi_\epsilon}{\partial z_2} & 1 \end{pmatrix}.$$

From the estimates in (12) it follows that

$$\tilde{g}_{ij} = Id + \epsilon A + \epsilon^2 B + O\big(\epsilon^3\big|(z_1, z_2)\big|^3\big), \tag{14}$$

where

$$A = \begin{pmatrix} 0 & A_Q(z_1, z_2) \\ (A_Q(z_1, z_2))^T & 0 \end{pmatrix},$$

and

$$B = \begin{pmatrix} A_Q(z_1, z_2)\otimes A_Q(z_1, z_2) & D_Q(z_1, z_2) \\ (D_Q(z_1, z_2))^T & 0 \end{pmatrix}.^{1}$$

[1] If the vector $v$ has components $(v_i)_i$, the notation $v\otimes v$ denotes the square matrix with entries $(v_i v_j)_{ij}$.

It is also easy to check that the inverse matrix $(\tilde{g}^{ij})$ is of the form $\tilde{g}^{ij} = Id - \epsilon A + \epsilon^2 C + O(\epsilon^3|(z_1, z_2)|^3)$, where

$$C = \begin{pmatrix} 0 & -D_Q(z_1, z_2) \\ -(D_Q(z_1, z_2))^T & |A_Q(z_1, z_2)|^2 \end{pmatrix}.$$

Furthermore one has $\det \tilde{g} = 1$. Therefore, by (14), for any smooth function $u$ there holds

$$\begin{aligned}\Delta_{\tilde{g}} u = \Delta u &- \epsilon\left[2\left(A_Q(z_1, z_2) \cdot \nabla_{(z_1,z_2)} \frac{\partial u}{\partial z_3}\right) + tr\, A_Q \frac{\partial u}{\partial z_3}\right] \\ &+ \epsilon^2\left[-2\left(D_Q \cdot \nabla_{(z_1,z_2)} \frac{\partial u}{\partial z_3}\right) + \left|A_Q(z_1, z_2)\right|^2 \frac{\partial^2 u}{\partial z_3 \partial z_3} - div\, D_Q \frac{\partial u}{\partial z_3}\right] \\ &+ O\left(\epsilon^3\left|(z_1, z_2)\right|^3\right)\left|\nabla^2 u\right| + O\left(\epsilon^3\left|(z_1, z_2)\right|^3\right)|\nabla u|.\end{aligned}$$

Moreover, from (13), we obtain the expression of the unit outer normal to $\partial\tilde{\Omega}_\epsilon$, $\tilde{\nu}$, in the new coordinates $z$:

$$\tilde{\nu} = \left(\epsilon\left(A_Q(z_1, z_2)\right) + \epsilon^2 D_Q(z_1, z_2), -1 + \frac{3}{2}\epsilon^2\left|A_Q(z_1, z_2)\right|^2\right) + O\left(\epsilon^3\left|(z_1, z_2)\right|^3\right).$$

Finally the area-element of $\partial\tilde{\Omega}_\epsilon$ can be estimated as

$$d\sigma = \left(1 + O\left(\epsilon^2\left|(z_1, z_2)\right|^2\right)\right) dz_1\, dz_2.$$

Now, locally, in a suitable neighborhood of $Q \in \Gamma$, we can consider $\Omega$ as the intersection of two smooth domains $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ if the opening angle at $Q$ is less than $\pi$, or as the union of them if the opening angle is greater than $\pi$. In the first case one has $\partial\Omega = (\partial\tilde{\Omega}_1 \cap \tilde{\Omega}_2) \cup (\partial\tilde{\Omega}_2 \cap \tilde{\Omega}_1)$, whereas in the second case $\partial\Omega = (\partial\tilde{\Omega}_1 \cap \tilde{\Omega}_2^c) \cup (\partial\tilde{\Omega}_2 \cap \tilde{\Omega}_1^c)$. Then, locally, one can straighten $\Gamma$ and stretch the two parts of the boundary using the coordinates $z$ for the smooth domains $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$.

## 3. Study of the non-degeneracy for the unperturbed problem in the cone

Let us consider $K = \tilde{K} \times \mathbb{R} \subset \mathbb{R}^3$, where $\tilde{K} \subset \mathbb{R}^2$ is a cone of opening angle $\alpha$, and the problem

$$\begin{cases} -\Delta u + u = u^p & \text{in } K, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial K, \end{cases} \tag{15}$$

where $p > 1$.

If $p < 5$ and if $u \in W^{1,2}(K)$, solutions of (15) can be found as critical points of the functional $I_K : W^{1,2}(K) \to \mathbb{R}$ defined as

$$I_K(u) = \frac{1}{2}\int_K \left(|\nabla u|^2 + u^2\right) dx - \frac{1}{p+1}\int_K |u|^{p+1}\, dx. \tag{16}$$

Note that $I_k$ is well defined on $W^{1,2}(K)$; in fact, since $K$ is Lipschitz, the Sobolev embeddings hold for $p \leqslant 5$; see for instance [1,13].

Let us consider also the elliptic equation in $\mathbb{R}^3$

$$-\Delta u + u = u^p, \quad u \in W^{1,2}\left(\mathbb{R}^3\right),\ u > 0, \tag{17}$$

which has a positive radial solution $U$; see for instance [2,3,26,37]. It has been shown in [22] that such a solution is unique. Moreover $U$ and its radial derivatives decay to zero exponentially: more precisely satisfy the properties

$$\lim_{r\to+\infty} e^r r U(r) = c_{3,p}, \qquad \lim_{r\to+\infty} \frac{U'(r)}{U(r)} = -\lim_{r\to+\infty} \frac{U''(r)}{U(r)} = -1,$$

where $r = |x|$ and $c_{3,p}$ is a positive constant depending only on the dimension $n = 3$ and $p$; see [3].

Now, if $p$ is subcritical, the function $U$ is also a solution of problem (15). Moreover, if we consider a coordinate system with the $x_1$-axis coinciding with the edge of $K$, the problem (15) is invariant under a translation along the $x_1$-axis. This means that any

$$U_{x_1}(x) = U\big(x - (x_1, 0, 0)\big)$$

is also a solution of (15). Then the functional $I_k$ has a non-compact critical manifold given by

$$Z = \big\{U_{x_1}(x):\ x_1 \in \mathbb{R}\big\} \simeq \mathbb{R}.$$

Now, to apply the results of the previous section, we have to characterize the spectrum and some eigenfunctions of $I''_K(U_{x_1})$. More precisely we have to show the following

**Lemma 3.1.** *Suppose $\alpha \in (0, 2\pi) \setminus \{\pi\}$. Then the following properties are true*:

(a) $T_{U_{x_1}} Z = \mathrm{Ker}[I''_K(U_{x_1})]$, *for all* $x_1 \in \mathbb{R}$;
(b) $I''_K(U_{x_1})$ *is an index* 0 *Fredholm map,*[2] *for all* $x_1 \in \mathbb{R}$.

**Remark 3.2.** The properties (a) and (b) imply that $Z$ satisfies condition (ii) in Section 2.1 and then it is non-degenerate for $I_K$.

**Proof of Lemma 3.1.** We will prove the lemma by taking $x_1 = 0$, hence $U_0 = U$. The case of a general $x_1$ will follow immediately.

Let us show (a). It is known that there holds the inclusion $T_U Z \subset \mathrm{Ker}[I''_K(U)]$; see for instance [2, Section 2.2]. Then it is sufficient to prove that $\mathrm{Ker}[I''_K(U)] \subset T_U Z$. Now, $v \in W^{1,2}(K)$ belongs to $\mathrm{Ker}[I''_K(U)]$ if and only if

$$\begin{cases} -\Delta v + v = pU^{p-1}v & \text{in } K, \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial K. \end{cases} \tag{18}$$

We use the polar coordinates in $K$, $r$, $\theta$, $\varphi$, where $r \geqslant 0$, $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \varphi \leqslant \alpha$. Then we write $v \in W^{1,2}(K)$ in the form

$$v(x_1, x_2, x_3) = \sum_{k=0}^{\infty} v_k(r) Y_k(\theta, \varphi), \tag{19}$$

where the $Y_k(\theta, \varphi)$ are the spherical functions satisfying

$$\begin{cases} -\Delta_{S^2} Y_k = \lambda_k Y_k & \text{in } K, \\ \dfrac{\partial Y_k}{\partial \varphi} = 0 & \varphi = 0, \alpha. \end{cases} \tag{20}$$

Here $\Delta_{S^2}$ denotes the Laplace–Beltrami operator on $S^2$ (acting on the variables $\theta$, $\varphi$). To determine $\lambda_k$ and the expression of $Y_k$, let us split $Y_k$ as

$$Y_k(\theta, \varphi) = \sum_{m=0}^{\infty} \Theta_{k,m}(\theta) \Phi_{k,m}(\varphi)$$

so that

$$\begin{aligned} \Delta_{S^2} Y_k &= \sum_{m=0}^{\infty} \left[ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right] \Theta_{k,m} \Phi_{k,m} \\ &= \sum_{m=0}^{\infty} \left[ \frac{1}{\sin\theta} \frac{d}{d\theta} \big( \sin\theta\, \Theta'_{k,m} \big) \Phi_{k,m} + \frac{1}{\sin^2\theta} \Theta_{k,m} \Phi''_{k,m} \right]. \end{aligned}$$

[2] A linear map $T \in L(H, H)$ is Fredholm if the kernel is finite-dimensional and the image is closed and has finite codimension. The index of $T$ is $\dim(\mathrm{Ker}[T]) - \mathit{codim}(\mathit{Im}[T])$.

Then (20) becomes

$$\begin{cases} -\sum_{m=0}^{\infty}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\Theta'_{k,m}\right)\Phi_{k,m}+\frac{1}{\sin^2\theta}\Theta_{k,m}\Phi''_{k,m}\right]=\sum_{m=0}^{\infty}\lambda_{k,m}\Theta_{k,m}\Phi_{k,m} \quad \text{in } K, \\ \Phi'_{k,m}(0)=\Phi'_{k,m}(\alpha)=0. \end{cases} \tag{21}$$

If we require that for all $m$

$$\begin{cases} -\Phi''_{k,m}=\mu_m\Phi_{k,m} \quad \text{in } [0,\alpha], \\ \Phi'_{k,m}(0)=\Phi'_{k,m}(\alpha)=0, \end{cases} \tag{22}$$

we obtain that $\Phi_{k,m}(\varphi)=a_{k,m}\cos(\frac{\pi m}{\alpha}\varphi)$ satisfies (22) with $\mu_m=\frac{\pi^2m^2}{\alpha^2}$. Replacing this expression in (21) we have

$$\begin{cases} \sum_{m=0}^{\infty}\left[-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\Theta'_{k,m}\right)+\frac{1}{\sin^2\theta}\frac{\pi^2m^2}{\alpha^2}\Theta_{k,m}\right]\Phi_{k,m}=\sum_{m=0}^{\infty}\lambda_{k,m}\Theta_{k,m}\Phi_{k,m} \quad \text{in } K, \\ \Phi'_{k,m}(0)=\Phi'_{k,m}(\alpha)=0. \end{cases}$$

Since the $\Phi_{k,m}$ are independent, we have to solve, for every $m$, the Sturm–Liouville equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\Theta'_{k,m}\right)+\left[\lambda_{k,m}-\frac{1}{\sin^2\theta}\frac{\pi^2m^2}{\alpha^2}\right]\Theta_{k,m}=0. \tag{23}$$

Let us rewrite (23) in the following form

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\Theta'_{k,m}\right)+\frac{1}{\sin^2\theta}\frac{\pi^2m^2}{\alpha^2}\Theta_{k,m}=\lambda_k\Theta_{k,m}, \tag{24}$$

so that we have to determine the eigenvalues $\lambda_{k,m}$ and the eigenfunctions of the operator

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\Theta'(\theta)\right)+\frac{1}{\sin^2\theta}\frac{\pi^2m^2}{\alpha^2}\Theta(\theta).$$

In order to do this, let us consider the case $\alpha=\pi$, that is the following equation

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\Theta'_{k,m}\right)+\frac{1}{\sin^2\theta}m^2\Theta_{k,m}=\lambda_{k,m}\Theta_{k,m}. \tag{25}$$

Now, for every $m$, (25) has solution if $\lambda_{k,m}=k(k+1)$, with $k\geqslant|m|$, and the solutions are the Legendre polynomials $\Theta_{k,m}(\theta)=P_{k,m}(\cos\theta)$; see for instance [14,21,28,29]. Then, for a given value of $k$, there are $2k+1$ independent solutions of the form $\Theta_{k,m}(\theta)\Phi_{k,m}(\varphi)$, one for each integer $m$ with $-k\leqslant m\leqslant k$. Now, by the classical comparison principle, if we decrease $\alpha$ the corresponding eigenvalues $\lambda_{k,m}$, given by (24), should increase, whereas if we increase $\alpha$ they should decrease; see for instance [5]. More precisely, if $m=0$ Eqs. (24) and (25) are the same, therefore the eigenvalues do not change (and they are $0,2,6,\ldots$). If $m\geqslant 1$ we cannot give an explicit expression for the $\lambda_{k,m}$ for general $\alpha$, but we can use the comparison principle. In conclusion, we obtain that each $Y_k=\sum_{m=0}^{\infty}\Theta_{k,m}\Phi_{k,m}$ satisfies

$$-\Delta_{S^2}Y_k=\lambda_{k,m}Y_k. \tag{26}$$

Now, one has that

$$\Delta(v_kY_k)=\Delta_r(v_k)Y_k+\frac{1}{r^2}v_k\Delta_{S^2}Y_k, \tag{27}$$

where $\Delta_r$ denotes the Laplace operator in radial coordinates, that is $\Delta_r=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}$. Then, using (19), (26) and (27), the condition (18) becomes

$$\sum_{k=0}^{\infty}\left[-v''_k-\frac{2}{r}v'_k+v_k+\frac{\lambda_{k,m}}{r^2}v_k-pU^{p-1}v_k\right]Y_k=0.$$

Since the $Y_k$ are independent, we get the following equations for $v_k$:

$$A_{k,m}(v_k) := -v_k'' - \frac{2}{r}v_k' + v_k + \frac{\lambda_{k,m}}{r^2}v_k - pU^{p-1}v_k = 0, \quad m = 0, 1, 2, \dots, \ k \geqslant m.$$

Let us first consider the case $m = 0$. If $k = 0$, we have to find a $v_0$ such that

$$A_{0,0}(v_0) = -v_0'' - \frac{2}{r}v_0' + v_0 - pU^{p-1}v_0 = 0.$$

It has been shown in [22], Lemma 6, that all the solutions of $A_{0,0}(v) = 0$ are unbounded. Since we are looking for solutions $v_0 \in W^{1,2}(\mathbb{R})$, it follows that $v_0 = 0$.

For $k = 1$ we have to solve

$$A_{1,0}(v_1) = -v_1'' - \frac{2}{r}v_1' + v_1 + \frac{2}{r^2}v_1 - pU^{p-1}v_1 = 0.$$

Let $\hat{U}(r)$ denote the function such that $U(x) = \hat{U}(|x|)$, where $U(x)$ is the solution of (17). Reasoning as in the proof of Lemma 4.1 in [2], we obtain that the family of solutions of $A_{1,0}(v_1) = 0$, with $v_1 \in W^{1,2}(\mathbb{R})$, is given by $v_1(r) = c\hat{U}'(r)$, for some $c \in \mathbb{R}$.

Now, let us show that the equation $A_{k,0}(v_k) = 0$ has only the trivial solution in $W^{1,2}(\mathbb{R})$, provided that $k \geqslant 2$. First of all, note that the operator $A_{1,0}$ has the solution $\hat{U}'$ which does not change sign in $(0, \infty)$ and therefore is a non-negative operator. In fact, if $\sigma$ denotes its smallest eigenvalue, any corresponding eigenfunction $\psi_\sigma$ does not change sign. If $\sigma < 0$, then $\psi_\sigma$ should be orthogonal to $\hat{U}'$ and this is a contradiction. Thus $\sigma \geqslant 0$ and $A_{1,0}$ is non-negative. Now, we can write

$$A_{k,0} = A_{1,0} + \frac{\lambda_{k,0} - 2}{r^2}.$$

Since $\lambda_{k,0} - 2 > 0$ whenever $k \geqslant 2$, it follows that $A_{k,0}$ is a positive operator. Thus $A_{k,0}(v_k) = 0$ implies that $v_k = 0$.

If $m \geqslant 1$ and $\alpha < \pi$, using the comparison principle, we obtain that each $\lambda_{k,m}$ is greater than 2. Then, reasoning as above, we have that each $v_k = 0$.

Let us consider the case $\alpha > \pi$. If $m = 1$ and $k = 1$, using again the comparison principle, we have that $0 < \lambda_{1,1} < 2$; whereas for $m = 1$, $k \geqslant 2$, and for $m \geqslant 2$, $k \geqslant m$, we have that each $\lambda_{k,m} > 2$. Then in the last two cases we can use the non-negativity of the operator $A_{1,0}$ and conclude that $v_k = 0$. In the case $m = 1$ and $k = 1$ we note that the operator

$$A_{1,1}(v_1) := -v_1'' - \frac{2}{r}v_1' + v_1 + \frac{\lambda_{1,1}}{r^2}v_1 - pU^{p-1}v_1$$

has a negative eigenvalue, instead of the eigenvalue 0, since $\lambda_{1,1} < 2$. Then also $v_1 = 0$.

Putting together all the previous information, we deduce that any $v \in \mathrm{Ker}[I''(U)]$ has to be of the form

$$v(x_1, x_2, x_3) = c\hat{U}'(r)Y_1(\theta, \varphi).$$

Now, $Y_1$ is such that $-\Delta_{S^2}Y_1 = \lambda_{1,m}Y_1$, namely it belongs to the kernel of the operator $-\Delta_{S^2} - \lambda_{1,m}Id$, and such a kernel is 1-dimensional. In conclusion, we find that

$$v \in span\{\hat{U}'Y_1\} = span\left\{\frac{\partial U}{\partial x_1}\right\} = T_U Z.$$

This proves that (a) holds. It is also easy to check that the operator $I_K''(U)$ is a compact perturbation of the identity, showing that (b) holds true, too. This complete the proof of Lemma 3.1. □

**Remark 3.3.** Since $U$ is a Mountain–Pass solution of (17), the spectrum of $I_K''(U)$ has one negative simple eigenvalue, $1 - p$, with eigenspace spanned by $U$ itself. Moreover, we have shown in the preceding lemma that $\lambda = 0$ is an eigenvalue with multiplicity 1 and eigenspace spanned by $\frac{\partial U}{\partial x_1}$. If $\alpha < \pi$ the rest of the spectrum is positive. Whereas if $\alpha > \pi$ there is an other negative simple eigenvalue, corresponding to an eigenfunction $\tilde{U}$ given by

$$\tilde{U}(r,\theta,\varphi)=\tilde{u}(r)\cos\left(\frac{\pi}{\alpha}\varphi\right)\tilde{\Theta}(\theta),$$

where $\tilde{\Theta}$ satisfies (23) with $m=1$ and $k=1$, and $\tilde{u}$ satisfies the equation

$$-v''-\frac{2}{r}v'+v+\frac{\lambda_{1,1}}{r^2}v-pU^{p-1}v=0. \tag{28}$$

From (28) one has that there exists a positive constant $C$ such that, for $r$ sufficiently large, $\tilde{u}(r)\leqslant Ce^{-r/C}$. In conclusion, one has the following result:

**Corollary 3.4.** *Let $U$ and $\tilde{U}$ be as above and consider the functional $I_K$ given in* (16). *Then for every $x_1\in\mathbb{R}$, $U_{x_1}(x)=U(x-(x_1,0,0))$ is a critical point of $I_K$. Moreover, the kernel of $I_K''(U)$ is generated by $\frac{\partial U}{\partial x_1}$. If $\alpha<\pi$ the operator has only one negative eigenvalue, and therefore there exists $\delta>0$ such that*

$$I_K''(U)[v,v]\geqslant\delta\|v\|^2,\quad \textit{for every } v\in W^{1,2}(K),\ v\perp U,\frac{\partial U}{\partial x_1}.$$

*If $\alpha>\pi$ the operator has two negative eigenvalues, and therefore there exists $\delta>0$ such that*

$$I_K''(U)[v,v]\geqslant\delta\|v\|^2,\quad \textit{for every } v\in W^{1,2}(K),\ v\perp U,\tilde{U},\frac{\partial U}{\partial x_1}.$$

## 4. Proof of Theorem 1.1

For every $Q$ on the edge $\Gamma$ of $\partial\Omega_\epsilon$, let $\mu=\min\{\mu_i\}$, so that in $B_{\frac{\mu}{\epsilon}}(Q)\cap\Omega_\epsilon$ we can use the new set of coordinates $z$. Now we choose a cut-off function $\varphi_\mu$ with the following properties

$$\begin{cases}\varphi_\mu(x)=1 & \text{in } B_{\frac{\mu}{4}}(Q),\\ \varphi_\mu(x)=0 & \text{in } \mathbb{R}^3\setminus B_{\frac{\mu}{2}}(Q),\\ |\nabla\varphi_\mu|+\left|\nabla^2\varphi_\mu\right|\leqslant C & \text{in } B_{\frac{\mu}{2}}(Q)\setminus B_{\frac{\mu}{4}}(Q).\end{cases} \tag{29}$$

For any $Q\in\Gamma$, we define the following function, in the coordinates $(z_1,z_2,z_3)$,

$$U_{Q,\epsilon}(z):=\varphi_\mu(\epsilon z)U_Q(z), \tag{30}$$

where $U_Q(z)=U(z-Q)$. Then we consider the manifold

$$Z_\epsilon=\{U_{Q,\epsilon}\colon\ Q\in\Gamma\}.$$

Now, we estimate the gradient of $I_\epsilon$ at $U_{Q,\epsilon}$, showing that $Z_\epsilon$ constitute a manifold of pseudo-critical points of $I_\epsilon$.

**Lemma 4.1.** *There exists $C>0$ such that for $\epsilon$ small there holds*

$$\left\|I_\epsilon'(U_{Q,\epsilon})\right\|\leqslant C\epsilon,\quad \textit{for all } Q\in\Gamma.$$

**Proof.** Let $v\in W^{1,2}(\Omega_\epsilon)$. Since the function $U_{Q,\epsilon}$ is supported in $B:=B_{\frac{\mu}{2\epsilon}}(Q)$, see (30), we can use the coordinate $z$ in this set, and we obtain

$$\begin{aligned}I_\epsilon'(U_{Q,\epsilon})[v]&=\int\limits_{\partial\Omega_\epsilon}\frac{\partial U_{Q,\epsilon}}{\partial\tilde{\nu}}v\,d\tilde{\sigma}+\int\limits_{\Omega_\epsilon}\left(-\Delta_{\tilde{g}}U_{Q,\epsilon}+U_{Q,\epsilon}-|U_{Q,\epsilon}|^p\right)v\,dV_{\tilde{g}}(z)\\&\doteqdot I+II.\end{aligned}$$

Let us now estimate $I$:

$$I=\int\limits_{\partial\Omega_{\epsilon 1}}\frac{\partial U_{Q,\epsilon}}{\partial\tilde{\nu}_1}v\,d\tilde{\sigma}_1+\int\limits_{\partial\Omega_{\epsilon 2}}\frac{\partial U_{Q,\epsilon}}{\partial\tilde{\nu}_2}v\,d\tilde{\sigma}_2\doteqdot I_1+I_2.$$

If $K = K_{\alpha(Q)}$ denotes the cone of angle equal to the angle of the edge in $Q$, we have

$$\begin{aligned}
I_1 &= \int_{\partial K} \big(U_Q(z)\nabla\varphi_\mu(\epsilon z)\cdot\tilde{\nu}_1 + \varphi_\mu(\epsilon z)\nabla U_Q(z)\cdot\tilde{\nu}_1\big) v\, d\tilde{\sigma}_1 \\
&= \int_{\partial K} U_Q(z)\nabla\varphi_\mu(\epsilon z)\cdot\left(\epsilon\big(A_Q(z_1,z_2)\big)+\epsilon^2 D_Q(z_1,z_2), -1+\frac{3}{2}\epsilon^2\big|A_Q(z_1,z_2)\big|^2\right) \\
&\quad + \varphi_\mu(\epsilon z)\nabla U_Q(z)\cdot\left(\epsilon\big(A_Q(z_1,z_2)\big)+\epsilon^2 D_Q(z_1,z_2), -1+\frac{3}{2}\epsilon^2\big|A_Q(z_1,z_2)\big|^2\right) \\
&\quad \times v\big(1+O\big(\epsilon^2\big|(z_1,z_2)\big|^2\big)\big)\, dz_1\, dz_2 \\
&\doteqdot a+b.
\end{aligned}$$

Since $\nabla\varphi_\mu(\epsilon\cdot)$ is supported in $\mathbb{R}^3 \setminus B_{\frac{\mu}{4\epsilon}}(Q)$ and $U_Q$ has an exponential decay, we have that, for $\epsilon$ small,

$$|a| \leqslant C\epsilon e^{-\frac{\mu}{4\epsilon}} \int_{\partial K} |v|\, dz_1\, dz_2. \tag{31}$$

On the other hand

$$\begin{aligned}
b &= \int_{\frac{\mu}{4\epsilon}\leqslant|z-Q|\leqslant\frac{\mu}{2\epsilon}} \varphi_\mu(\epsilon z)\nabla U_Q(z)\cdot\left(\epsilon\big(A_Q(z_1,z_2)\big)+\epsilon^2 D_Q(z_1,z_2), -1+\frac{3}{2}\epsilon^2\big|A_Q(z_1,z_2)\big|^2\right) \\
&\quad \times v\big(1+O\big(\epsilon^2\big|(z_1,z_2)\big|^2\big)\big)\, dz_1\, dz_2 \\
&\quad + \int_{|z-Q|\leqslant\frac{\mu}{4\epsilon}} \varphi_\mu(\epsilon z)\nabla U_Q(z)\cdot\left(\epsilon\big(A_Q(z_1,z_2)\big)+\epsilon^2 D_Q(z_1,z_2), -1+\frac{3}{2}\epsilon^2\big|A_Q(z_1,z_2)\big|^2\right) \\
&\quad \times v\big(1+O\big(\epsilon^2\big|(y_1,y_2)\big|^2\big)\big)\, dy_1\, dy_2 \\
&\leqslant C\epsilon e^{-\frac{\mu}{4\epsilon}} \int_{\partial K} |v|\, dz_1\, dz_2 + C\epsilon \int_{\partial K} |\nabla U_Q|\cdot|v|\, dz_1\, dz_2.
\end{aligned} \tag{32}$$

The estimates (31) and (32), and the trace Sobolev inequalities imply $|I_1| \leqslant C\epsilon\|v\|$. In the same way we can estimate $I_2$, getting

$$|I| \leqslant C\epsilon\|v\|. \tag{33}$$

Now let's evaluate $II$. Using (11) one has

$$\begin{aligned}
II &= \int_K \big(-\Delta U_{Q,\epsilon} + U_{Q,\epsilon} - |U_{Q,\epsilon}|^p\big) v\, dV_{\tilde{g}}(z) \\
&\quad + \epsilon \int_K \left[2\left(\gamma''(0)z_1\cdot\nabla_{(z_2,z_3)}\frac{\partial U_{Q,\epsilon}}{\partial z_1}\right) + \big(\gamma''(0)\cdot\nabla_{(z_2,z_3)}U_{Q,\epsilon}\big)\right] v\, dV_{\tilde{g}}(z) \\
&\quad + O\big(\epsilon^2\big) \int_K \big(|z_1|^2\big|\nabla^2 U_{Q,\epsilon}\big| + |z_1|^2|\nabla U_{Q,\epsilon}|\big) v\, dV_{\tilde{g}}(z) \\
&\doteqdot II_1 + \epsilon II_2 + O\big(\epsilon^2\big) II_3.
\end{aligned}$$

Since $\Delta U_{Q,\epsilon} = U_Q\Delta\varphi_\mu(\epsilon z) + 2\nabla U_Q\cdot\nabla\varphi_\mu(\epsilon z) + \varphi_\mu(\epsilon z)\Delta U_Q$ and both $\Delta\varphi_\mu(\epsilon\cdot)$ and $\nabla\varphi_\mu(\epsilon\cdot)$ are supported in $\mathbb{R}^3 \setminus B_{\frac{\mu}{4\epsilon}}(Q)$, we get

$$II_1 = \int\limits_{\frac{\mu}{4\epsilon} \leqslant |z-Q| \leqslant \frac{\mu}{2\epsilon}} \big(-U_Q \Delta\varphi_\mu(\epsilon z) - 2\nabla U_Q \cdot \nabla\varphi_\mu(\epsilon z)\big) v\big(1 + O\big(\epsilon|z|\big)\big)\, dz$$
$$+ \int\limits_{\frac{\mu}{4\epsilon} \leqslant |z-Q| \leqslant \frac{\mu}{2\epsilon}} \big(-\varphi_\mu(\epsilon z)\Delta U_Q + U_{Q,\epsilon} - |U_{Q,\epsilon}|^p\big) v\big(1 + O\big(\epsilon|z|\big)\big)\, dz$$
$$+ \int\limits_{|z-Q| \leqslant \frac{\mu}{4\epsilon}} \big(-\Delta U_Q + U_Q - |U_Q|^p\big) v\big(1 + O\big(\epsilon|z|\big)\big)\, dz. \tag{34}$$

Since $U_Q$ is a solution in $\mathbb{R}^3$ the last term in (34) vanishes, and using the exponential decay of $U_Q$ at infinity and the properties of the cut-off function, see (29), one has

$$|II_1| \leqslant Ce^{-\frac{\mu}{4\epsilon}} \int\limits_K |v|\, dz.$$

By (30) we can compute also $\nabla_{(z_2,z_3)} \frac{\partial U_{Q,\epsilon}}{\partial z_1}$ and $\nabla_{(z_2,z_3)} U_{Q,\epsilon}$ and we have

$$II_2 = \int\limits_K 2\gamma''(0) z_1 \cdot \left[\nabla_{(z_2,z_3)} \frac{\partial\varphi_\mu(\epsilon z)}{\partial z_1} U_Q + \nabla_{(z_2,z_3)}\varphi_\mu(\epsilon z) \frac{\partial U_Q}{\partial z_1}\right]$$
$$+ 2\gamma''(0) z_1 \cdot \left[\frac{\partial\varphi_\mu(\epsilon z)}{\partial z_1} \nabla_{(z_2,z_3)} U_Q + \varphi_\mu(\epsilon z) \nabla_{(z_2,z_3)} \frac{\partial U_Q}{\partial z_1}\right]$$
$$+ \gamma''(0) \cdot \big[\nabla_{(z_2,z_3)}\varphi_\mu(\epsilon z) U_Q + \varphi_\mu(\epsilon z)\nabla_{(z_2,z_3)} U_Q\big] v\, dV_{\tilde{g}}(z)$$
$$= \int\limits_{\frac{\mu}{4\epsilon} \leqslant |z-Q| \leqslant \frac{\mu}{2\epsilon}} 2\gamma''(0) z_1 \cdot \left[\nabla_{(z_2,z_3)} \frac{\partial\varphi_\mu(\epsilon z)}{\partial z_1} U_Q + \nabla_{(z_2,z_3)}\varphi_\mu(\epsilon z) \frac{\partial U_Q}{\partial z_1} + \frac{\partial\varphi_\mu(\epsilon z)}{\partial z_1} \nabla_{(z_2,z_3)} U_Q\right]$$
$$+ \gamma''(0) \cdot \nabla_{(z_2,z_3)}\varphi_\mu(\epsilon z) U_Q v\, dV_{\tilde{g}}(z)$$
$$+ \int\limits_{|z-Q| \leqslant \frac{\mu}{2\epsilon}} \varphi_\mu(\epsilon z)\left[2\gamma''(0) z_1 \cdot \nabla_{(z_2,z_3)} \frac{\partial U_Q}{\partial z_1} + \gamma''(0) \cdot \nabla_{(z_2,z_3)} U_Q\right] v\, dV_{\tilde{g}}(z).$$

Hence

$$|II_2| \leqslant C \int\limits_{\frac{\mu}{4\epsilon} \leqslant |z-Q| \leqslant \frac{\mu}{2\epsilon}} \left[2\big|\gamma''(0)\big| \cdot |z_1| \left(|U_Q| + \left|\frac{\partial U_Q}{\partial z_1}\right| + |\nabla_{(z_2,z_3)} U_Q|\right) + \big|\gamma''(0)\big| \cdot |U_Q|\right] |v|\, dV_{\tilde{g}}(z)$$
$$+ \int\limits_{|z-Q| \leqslant \frac{\mu}{2\epsilon}} 2\big|\varphi_\mu(\epsilon z)\big| \cdot \sup_Q \big|\gamma''(0)\big| \left(|z_1| \cdot \left|\nabla_{(z_2,z_3)} \frac{\partial U_Q}{\partial z_1}\right| + |\nabla_{(z_2,z_3)} U_Q|\right) |v|\, dV_{\tilde{g}}(z).$$

Using again the exponential decay of $U_Q$ at infinity one can estimate the first term by $Ce^{-\frac{\mu}{4\epsilon}} \int_K |v|\, dz$ and conclude that the second term is bounded. In the same way we can estimate $II_3$, getting

$$|II| \leqslant C\epsilon \|v\|. \tag{35}$$

From (33) and (35) we obtain the conclusion. □

Now, we need a result of non-degeneracy, which allows us to say that the operator $I''_\epsilon(U_{Q,\epsilon})$ is invertible on the orthogonal complement of $T_{U_{Q,\epsilon}} Z_\epsilon$.

**Lemma 4.2.** *There exists $\bar{\delta} > 0$ such that for $\epsilon$ small, if $\alpha < \pi$, there holds*

$$I''_\epsilon(U_{Q,\epsilon})[v, v] \geqslant \bar{\delta}\|v\|^2, \quad \textit{for every } v \in W^{1,2}(\Omega_\epsilon),\ v \perp U_{Q,\epsilon}, \frac{\partial U_{Q,\epsilon}}{\partial Q},$$

*and, if $\alpha > \pi$, there holds*

$$I''_\epsilon(U_{Q,\epsilon})[v,v] \geqslant \bar{\delta}\|v\|^2, \quad \textit{for every } v \in W^{1,2}(\Omega_\epsilon),\ v \perp U_{Q,\epsilon}, \tilde{U}_{Q,\epsilon}\frac{\partial U_{Q,\epsilon}}{\partial Q},$$

*where $\tilde{U}_{Q,\epsilon}$ is defined as $U_{Q,\epsilon}$ in* (30).

**Proof.** Let us consider the case $\alpha < \pi$. Let $R \gg 1$; consider a radial smooth function $\chi_R : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\begin{cases} \chi_R(x) = 1 & \text{in } B_R(0), \\ \chi_R(x) = 0 & \text{in } \mathbb{R}^3 \setminus B_{2R}(0), \\ |\nabla \chi_R| \leqslant \dfrac{2}{R} & \text{in } B_{2R}(0) \setminus B_R(0), \end{cases} \tag{36}$$

and set

$$v_1(x) = \chi_R(x-Q)v(x), \qquad v_2(x) = \big(1-\chi_R(x-Q)\big)v(x).$$

A straight computation yields

$$\|v\|^2 = \|v_1\|^2 + \|v_2\|^2 + 2\int_{\Omega_\epsilon} (\nabla v_1 \cdot \nabla v_2 + v_1 v_2)\,dx.$$

We write $\int_{\Omega_\epsilon} (\nabla v_1 \cdot \nabla v_2 + v_1 v_2)\,dx = \gamma_1 + \gamma_2$, where

$$\gamma_1 = \int_{\Omega_\epsilon} \chi_R(1-\chi_R)\big(v^2 + |\nabla v|^2\big)\,dx,$$

$$\gamma_2 = \int_{\Omega_\epsilon} \big(v_2 \nabla v \cdot \nabla \chi_R - v_1 \nabla v \cdot \nabla \chi_R - v^2 |\nabla \chi_R|^2\big)\,dx.$$

Since the integrand in $\gamma_2$ is supported in $B_{2R}(Q) \setminus B_R(Q)$, using (36) and the Young's inequality we obtain that $|\gamma_2| = o_R(1)\|v\|^2$. As a consequence we have

$$\|v\|^2 = \|v_1\|^2 + \|v_2\|^2 + 2\gamma_1 + o_R(1)\|v\|^2.$$

Now let us evaluate $I''_\epsilon(U_{Q,\epsilon})[v,v] = \sigma_1 + \sigma_2 + \sigma_3$, where

$$\sigma_1 = I''_\epsilon(U_{Q,\epsilon})[v_1,v_1], \qquad \sigma_2 = I''_\epsilon(U_{Q,\epsilon})[v_2,v_2], \qquad \sigma_3 = 2I''_\epsilon(U_{Q,\epsilon})[v_1,v_2].$$

Similarly to the previous estimates, since $U_Q$ decays exponentially away from $Q$, we get

$$\begin{aligned} &\sigma_2 \geqslant C^{-1}\|v_2\|^2 + o_{\epsilon,R}(1)\|v\|^2, \\ &\sigma_3 \geqslant C^{-1}\gamma_1 + o_{\epsilon,R}(1)\|v\|^2. \end{aligned} \tag{37}$$

Hence it is sufficient to estimate the term $\sigma_1$. From the exponential decay of $U_Q$ and the fact that $v \perp U_{Q,\epsilon}, \frac{\partial U_{Q,\epsilon}}{\partial Q}$ it follows that

$$\begin{aligned} &(v_1, U_{Q,\epsilon})_{W^{1,2}(\Omega_\epsilon)} = -(v_2, U_{Q,\epsilon})_{W^{1,2}(\Omega_\epsilon)} = o_{\epsilon,R}(1)\|v\|^2, \\ &\left(v_1, \frac{\partial U_{Q,\epsilon}}{\partial Q}\right)_{W^{1,2}(\Omega_\epsilon)} = -\left(v_2, \frac{\partial U_{Q,\epsilon}}{\partial Q}\right)_{W^{1,2}(\Omega_\epsilon)} = o_{\epsilon,R}(1)\|v\|^2. \end{aligned} \tag{38}$$

Moreover, since $U_{Q,\epsilon}$ is supported in $B := B_{\frac{\mu}{2\epsilon}}(Q)$, see (30), we can use the coordinate $z$ in this set, and we obtain

$$\begin{aligned} (v_1, U_{Q,\epsilon})_{W^{1,2}(\Omega_\epsilon)} &= \int_{\partial\Omega_\epsilon} v_1 \frac{\partial U_{Q,\epsilon}}{\partial \tilde{\nu}} v\,d\tilde{\sigma} + \int_{\Omega_\epsilon} v_1(-\Delta_{\tilde{g}} U_{Q,\epsilon} + U_{Q,\epsilon})\,dV_{\tilde{g}}(z) \\ &= (v_1, U_Q)_{W^{1,2}(K)} + o_\epsilon(1)\|v_1\|, \end{aligned} \tag{39}$$

where $K = K_\alpha$ is the cone of opening angle equal to the angle of $\Gamma$ in $Q$. In the same way we can obtain that

$$\left(v_1, \frac{\partial U_{Q,\epsilon}}{\partial Q}\right)_{W^{1,2}(\Omega_\epsilon)} = \left(v_1, \frac{\partial U_Q}{\partial Q}\right)_{W^{1,2}(K)} + o_\epsilon(1)\|v_1\|. \tag{40}$$

From the estimates (38), (39) and (40), we deduce that for $R$ sufficiently large and $\epsilon$ sufficiently small

$$(v_1, U_Q)_{W^{1,2}(K)} = o_{\epsilon,R}(1)\|v_1\|,$$
$$\left(v_1, \frac{\partial U_Q}{\partial Q}\right)_{W^{1,2}(K)} = o_{\epsilon,R}(1)\|v_1\|.$$

Now we can apply Lemma 3.1, getting

$$I''(U_Q)[v_1, v_1] \geqslant \delta\|v_1\|_{W^{1,2}(K)} + o_{\epsilon,R}(1).$$

Then the following estimate holds

$$\begin{aligned}\sigma_1 &= I''(U_Q)[v_1, v_1] + o_\epsilon(1)\|v_1\| \geqslant \delta\|v_1\|_{W^{1,2}(K)} + o_{\epsilon,R}(1)\|v\| \\ &\geqslant \delta\|v_1\| + o_{\epsilon,R}(1)\|v\|.\end{aligned} \tag{41}$$

In conclusion, from (37) and (41) we deduce

$$I''_\epsilon(U_{Q,\epsilon})[v, v] \geqslant \delta\|v\| + o_{\epsilon,R}(1)\|v\| \geqslant \frac{\delta}{2}\|v\|,$$

provided $R$ is taken large and $\epsilon$ sufficiently small. This concludes the proof.

The case $\alpha > \pi$ has substantially the same proof, but we have to consider also the function $\tilde{U}$ and use the exponential decay of $\tilde{u}$ at infinity, see Remark 3.3. □

The following lemma provides an expansion of the functional $I_\epsilon(U_{Q,\epsilon})$ with respect to $Q$.

**Lemma 4.3.** *For $\epsilon$ small the following expansion holds*

$$I_\epsilon(U_{Q,\epsilon}) = C_0\alpha(Q) + O(\epsilon), \tag{42}$$

*where*

$$C_0 = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_0^\infty\int_0^\pi \left|U_Q(r)\right|^{p+1} r\sin^2\theta\, dr\, d\theta.$$

**Proof.** Since the function $U_{Q,\epsilon}$ is supported in $B := B_{\frac{\mu}{2\epsilon}}(Q)$, see (30), we can use the coordinate $z$ in this set, and we obtain

$$I_\epsilon(U_{Q,\epsilon}) = \frac{1}{2}\int_{B\cap\Omega_\epsilon} \left(|\nabla_{\tilde{g}} U_{Q,\epsilon}|^2 + U_{Q,\epsilon}^2\right) dV_{\tilde{g}}(z) - \frac{1}{p+1}\int_{B\cap\Omega_\epsilon} |U_{Q,\epsilon}|^{p+1}\, dV_{\tilde{g}}(z).$$

Integrating by parts, we get

$$\begin{aligned}I_\epsilon(U_{Q,\epsilon}) &= \frac{1}{2}\int_{B\cap\partial\Omega_\epsilon} U_{Q,\epsilon}\frac{\partial U_{Q,\epsilon}}{\partial\tilde{\nu}}\, d\tilde{\sigma} + \frac{1}{2}\int_{B\cap\Omega_\epsilon} U_{Q,\epsilon}(-\Delta_{\tilde{g}} U_{Q,\epsilon} + U_{Q,\epsilon})\, dV_{\tilde{g}}(z) \\ &\quad - \frac{1}{p+1}\int_{B\cap\Omega_\epsilon} |U_{Q,\epsilon}|^{p+1}\, dV_{\tilde{g}}(z) \\ &\doteqdot I + II,\end{aligned}$$

where $I$ is the surface integral over the boundary and $II$ refers to the last two terms. Now, $I$ can be split in two terms which correspond to the surface integrals on the "faces" of the edge $\Gamma$:

$$I = \frac{1}{2} \int\limits_{B \cap \partial\Omega_{\epsilon 1}} U_{Q,\epsilon} \frac{\partial U_{Q,\epsilon}}{\partial \tilde{\nu}_1} \, d\tilde{\sigma}_1 + \frac{1}{2} \int\limits_{B \cap \partial\Omega_{\epsilon 2}} U_{Q,\epsilon} \frac{\partial U_{Q,\epsilon}}{\partial \tilde{\nu}_2} \, d\tilde{\sigma}_2 \doteqdot I_1 + I_2.$$

It is sufficient to evaluate $I_1$, since the estimate of $I_2$ is similar. Using the expression of $U_{Q,\epsilon}$, see (30), we get

$$\begin{aligned} I_1 &= \frac{1}{2} \int\limits_{B \cap \partial\Omega_{\epsilon 1}} U_{Q,\epsilon} \big( U_Q \nabla \varphi_\mu(\epsilon z) + \varphi_\mu(\epsilon z) \nabla U_Q \big) \cdot \left( \epsilon \big( A_Q(z_1, z_2) \big) + \epsilon^2 D_Q(z_1, z_2), -1 + \frac{3}{2} \epsilon^2 \big| A_Q(z_1, z_2) \big|^2 \right) \\ &\quad \times \big( 1 + O\big( \epsilon^2 \big| (z_1, z_2) \big|^2 \big) \big) \, dz_1 \, dz_2 \\ &= \frac{1}{2} \int\limits_{\frac{\mu}{4\epsilon} \leqslant |z - Q| \leqslant \frac{\mu}{2\epsilon}} \varphi_\mu(\epsilon z) U_Q^2 \nabla \varphi_\mu(\epsilon z) \cdot \left( \epsilon \big( A_Q(z_1, z_2) \big) + \epsilon^2 D_Q(z_1, z_2), -1 + \frac{3}{2} \epsilon^2 \big| A_Q(z_1, z_2) \big|^2 \right) \\ &\quad \times \big( 1 + O\big( \epsilon^2 \big| (z_1, z_2) \big|^2 \big) \big) \, dz_1 \, dz_2 \\ &\quad + \frac{1}{2} \int\limits_{|z - Q| \leqslant \frac{\mu}{2\epsilon}} \varphi_\mu^2(\epsilon z) U_Q \nabla U_Q \cdot \left( \epsilon \big( A_Q(z_1, z_2) \big) + \epsilon^2 D_Q(z_1, z_2), -1 + \frac{3}{2} \epsilon^2 \big| A_Q(z_1, z_2) \big|^2 \right) \\ &\quad \times \big( 1 + O\big( \epsilon^2 \big| (z_1, z_2) \big|^2 \big) \big) \, dz_1 \, dz_2. \end{aligned}$$

Similarly to the previous estimates, we get $I_1 = O(e^{-\frac{\mu}{2\epsilon}}) + O(\epsilon)$. Then we obtain that

$$I = O(\epsilon). \tag{43}$$

Now, we have to evaluate $II$:

$$\begin{aligned} II &= \frac{1}{2} \int\limits_{B \cap \Omega_\epsilon} U_{Q,\epsilon} (-\Delta U_{Q,\epsilon} + U_{Q,\epsilon}) \big( 1 + O\big( \epsilon |z| \big) \big) \, dz \\ &\quad + \frac{\epsilon}{2} \int\limits_{B \cap \Omega_\epsilon} U_{Q,\epsilon} \left[ 2\gamma''(0) z_1 \cdot \nabla_{(z_2, z_3)} \frac{\partial U_{Q,\epsilon}}{\partial z_1} + \gamma''(0) \cdot \nabla_{(z_2, z_3)} U_{Q,\epsilon} \right] \big( 1 + O\big( \epsilon |z| \big) \big) \, dz \\ &\quad + O\big( \epsilon^2 |z_1|^2 \big) - \frac{1}{p+1} \int\limits_{B \cap \Omega_\epsilon} |U_{Q,\epsilon}|^{p+1} \big( 1 + O\big( \epsilon |z| \big) \big) \, dz. \end{aligned}$$

We have

$$II = \left( \frac{1}{2} - \frac{1}{p+1} \right) \alpha(Q) \int_0^\infty \int_0^\pi \big| U_Q(r) \big|^{p+1} r \sin^2\theta \, dr \, d\theta + O(\epsilon). \tag{44}$$

Putting together (43) and (44), we obtain (42) and this concludes the proof. □

Let $P_Q : W^{1,2}(\Omega_\epsilon) \to (T_{U_{Q,\epsilon}} Z_\epsilon)^\perp$ be the projection onto the orthogonal complement of $T_{U_{Q,\epsilon}} Z_\epsilon$, for all $Q$ on the edge $\Gamma$ of $\partial\Omega_\epsilon$. According to Lemma 4.2, we have that for $\epsilon$ sufficiently small the operator $L_Q = P_Q \circ I''_\epsilon(U_{Q,\epsilon}) \circ P_Q$ is invertible and there exists $C > 0$ such that

$$\big\| L_Q^{-1} \big\| \leqslant C.$$

Now, using the fact that $I''_\epsilon(U_{Q,\epsilon})$ is invertible on the orthogonal complement of $T_{U_{Q,\epsilon}} Z_\epsilon$, we will solve the auxiliary equation.

**Proposition 4.4.** *Let $I_\epsilon$ be the functional defined in* (7). *Then for $\epsilon > 0$ small there exists a unique $w = w(\epsilon, Q) \in (T_{U_{Q,\epsilon}} Z_\epsilon)^\perp$ such that $I'_\epsilon(U_{Q,\epsilon} + w(\epsilon, Q)) \in T_{U_{Q,\epsilon}} Z_\epsilon$. Moreover the function $w(\epsilon, Q)$ is of class $C^1$ with respect to $Q$*

*and there holds*

$$\|w(\epsilon, Q)\| \leqslant C\epsilon, \qquad \left\|\frac{\partial w(\epsilon, Q)}{\partial Q}\right\| \leqslant C\epsilon. \tag{45}$$

**Proof.** We want to find a solution $w \in (T_{U_{Q,\epsilon}} Z_\epsilon)^\perp$ of $P_Q I'_\epsilon(U_{Q,\epsilon} + w) = 0$. For every $w \in (T_{U_{Q,\epsilon}} Z_\epsilon)^\perp$ we can write

$$I'_\epsilon(U_{Q,\epsilon} + w) = I'_\epsilon(U_{Q,\epsilon}) + I''_\epsilon(U_{Q,\epsilon})[w] + R_{Q,\epsilon}(w),$$

where $R_{Q,\epsilon}(w)$ is given by

$$R_{Q,\epsilon}(w) = I'_\epsilon(U_{Q,\epsilon} + w) - I'_\epsilon(U_{Q,\epsilon}) - I''_\epsilon(U_{Q,\epsilon})[w].$$

Given $v \in W^{1,2}(\Omega_\epsilon)$ there holds

$$R_{Q,\epsilon}(w)[v] = -\int_{\Omega_\epsilon} \left(|U_{Q,\epsilon} + w|^p - |U_{Q,\epsilon}|^p - p|U_{Q,\epsilon}|^{p-1} w\right) v \, dx.$$

Using the following inequality

$$\left|(a+b)^p - a^p - pa^{p-1}b\right| \leqslant \begin{cases} C(p)|b|^p & \text{for } p \leqslant 2, \\ C(p)(|b|^2 + |b|^p) & \text{for } p > 2, \end{cases}$$

for $a, b \in \mathbb{R}$, $|a| \leqslant 1$, the Hölder's inequality and the Sobolev embeddings we obtain

$$\|R_{Q,\epsilon}(w)[v]\| \leqslant C \int_{\Omega_\epsilon} \left(|w|^2 + |w|^p\right)|v| \, dx \leqslant C\left(\|w\|^2 + \|w\|^p\right)\|v\|. \tag{46}$$

Similarly, from the inequality

$$\begin{aligned} &\left|(a+b_1)^p - (a+b_2)^p - pa^{p-1}(b_1 - b_2)\right| \\ &\quad \leqslant \begin{cases} C(p)(|b_1|^{p-1} + |b_2|^{p-1})|b_1 - b_2| & \text{for } p \leqslant 2, \\ C(p)(|b_1| + |b_2| + |b_1|^{p-1} + |b_2|^{p-1})|b_1 - b_2| & \text{for } p > 2, \end{cases} \end{aligned}$$

for $a, b_1, b_2 \in \mathbb{R}$, $|a| \leqslant 1$, we get

$$\begin{aligned} \|R_{Q,\epsilon}(w_1)[v] - R_{Q,\epsilon}(w_2)[v]\| &\leqslant C \int_{\Omega_\epsilon} \left(|w_1| + |w_2| + |w_1|^{p-1} + |w_2|^{p-1}\right)|w_1 - w_2| \cdot |v| \, dx \\ &\leqslant C\left(\|w_1\| + \|w_2\| + \|w_1\|^{p-1} + \|w_2\|^{p-1}\right)\|w_1 - w_2\| \cdot \|v\|. \end{aligned} \tag{47}$$

Now, by the invertibility of the operator $L_Q = P_Q \circ I''_\epsilon(U_{Q,\epsilon}) \circ P_Q$, we have that the function $w$ solves $P_Q I'_\epsilon(U_{Q,\epsilon} + w) = 0$ if and only if

$$w = -(L_Q)^{-1}\left[P_Q I'_\epsilon(U_{Q,\epsilon}) + P_Q R_{Q,\epsilon}(w)\right].$$

Setting

$$N_{Q,\epsilon}(w) = -(L_Q)^{-1}\left[P_Q I'_\epsilon(U_{Q,\epsilon}) + P_Q R_{Q,\epsilon}(w)\right],$$

we have to solve

$$w = N_{Q,\epsilon}(w).$$

The norm of $I'_\epsilon(U_{Q,\epsilon})$ has been estimated in Lemma 4.1. Then from (46) and (47) we obtain the two relations

$$\|N_{Q,\epsilon}(w)\| \leqslant C_1\epsilon + C_2\left(\|w\|^2 + \|w\|^p\right), \tag{48}$$

$$\|N_{Q,\epsilon}(w_1) - N_{Q,\epsilon}(w_2)\| \leqslant C\left(\|w_1\| + \|w_2\| + \|w_1\|^{p-1} + \|w_2\|^{p-1}\right)\|w_1 - w_2\|. \tag{49}$$

Now, for $\bar{C} > 0$, we define the set

$$W_{\bar{C}} = \{ w \in (T_{U_{Q,\epsilon}} Z_\epsilon)^\perp \colon \|w\| \leqslant \bar{C}\epsilon \}.$$

We show that $N_{Q,\epsilon}$ is a contraction in $W_{\bar{C}}$ for $\bar{C}$ sufficiently large and for $\epsilon$ small. Clearly, by (48), if $\bar{C} > 2C_1$ the set $W_{\bar{C}}$ is mapped into itself if $\epsilon$ is sufficiently small. Then, if $w_1, w_2 \in W_{\bar{C}}$, by (49) there holds

$$\| N_{Q,\epsilon}(w_1) - N_{Q,\epsilon}(w_2) \| \leqslant 2C\big(\bar{C}\epsilon + \bar{C}^{p-1}\epsilon^{p-1}\big)\|w_1 - w_2\|.$$

Therefore, again if $\epsilon$ is sufficiently small, the coefficient of $\|w_1 - w_2\|$ in the last formula is less than 1. Hence the Contraction Mapping Theorem applies, yielding the existence of a solution $w$ satisfying the condition

$$\|w\| \leqslant \bar{C}\epsilon. \tag{50}$$

This concludes the proof of the existence part.

Now the $C^1$-dependence of the function $w$ on $Q$ follows from the Implicit Function Theorem; see also [2], Proposition 8.7. In order to prove the second estimate in (45), let us consider the map $H : \mathbb{R}^3 \times W^{1,2}(\Omega_\epsilon) \times \mathbb{R} \times \mathbb{R} \to W^{1,2}(\Omega_\epsilon) \times \mathbb{R}$ defined by

$$H(Q, w, \alpha, \epsilon) = \begin{pmatrix} I'_\epsilon(U_{Q,\epsilon} + w) - \alpha \frac{\partial U_{Q,\epsilon}}{\partial Q} \\ (w, \frac{\partial U_{Q,\epsilon}}{\partial Q}) \end{pmatrix}.$$

Then $w \in (T_{U_{Q,\epsilon}} Z_\epsilon)^\perp$ is a solution of $P_Q I'_\epsilon(U_{Q,\epsilon} + w) = 0$ if and only if $H(Q, w, \alpha, \epsilon) = 0$. Moreover, for $v \in W^{1,2}(\Omega_\epsilon)$ and $\beta \in \mathbb{R}$, there holds

$$\begin{aligned} \frac{\partial H}{\partial(w,\alpha)}(Q, w, \alpha, \epsilon)[v, \beta] &= \begin{pmatrix} I''_\epsilon(U_{Q,\epsilon} + w)[v] - \beta \frac{\partial U_{Q,\epsilon}}{\partial Q} \\ (v, \frac{\partial U_{Q,\epsilon}}{\partial Q}) \end{pmatrix} \\ &= \begin{pmatrix} I''_\epsilon(U_{Q,\epsilon})[v] - \beta \frac{\partial U_{Q,\epsilon}}{\partial Q} \\ (v, \frac{\partial U_{Q,\epsilon}}{\partial Q}) \end{pmatrix} + O\big(\|w\| + \|w\|^{p-1}\big). \end{aligned} \tag{51}$$

To prove the last estimate it is sufficient to use the following inequality

$$\big|(a+b)^{p-1} - a^{p-1}\big| \leqslant \begin{cases} C(p)|b|^{p-1} & \text{for } p \leqslant 2, \\ C(p)(|b| + |b|^{p-1}) & \text{for } p > 2, \end{cases}$$

for $a, b \in \mathbb{R}$, $|a| \leqslant 1$, the Hölder's inequality and the Sobolev embedding. Using the invertibility of the operator $L_Q = P_Q \circ I''_\epsilon(U_{Q,\epsilon}) \circ P_Q$, it is easy to check that $\frac{\partial H}{\partial(w,\alpha)}(Q, 0, 0, \epsilon)$ is uniformly invertible in $Q$ for $\epsilon$ small. Hence, by (50) and (51), also $\frac{\partial H}{\partial(w,\alpha)}(Q, w, \alpha, \epsilon)$ is uniformly invertible in $Q$ for $\epsilon$ small. As a consequence, by the Implicit Function Theorem, the map $Q \mapsto (w_Q, \alpha_Q)$ is of class $C^1$. Now we are in position to provide the norm estimate of $\frac{\partial w(\epsilon, Q)}{\partial Q}$. Differentiating the equation

$$H(Q, w_Q, \alpha_Q, \epsilon) = 0$$

with respect to $Q$, we obtain

$$0 = \frac{\partial H}{\partial Q}(Q, w, \alpha, \epsilon) + \frac{\partial H}{\partial(w,\alpha)}(Q, w, \alpha, \epsilon)\frac{\partial(w_Q, \alpha_Q)}{\partial Q}.$$

Hence, by the uniform invertibility of $\frac{\partial H}{\partial(w,\alpha)}(Q, w, \alpha, \epsilon)$ it follows that

$$\begin{aligned} \left\| \frac{\partial(w_Q, \alpha_Q)}{\partial Q} \right\| &\leqslant C \left\| \begin{pmatrix} I''_\epsilon(U_{Q,\epsilon} + w)[\frac{\partial U_{Q,\epsilon}}{\partial Q}] - \alpha \frac{\partial^2 U_{Q,\epsilon}}{\partial Q^2} \\ (w, \frac{\partial^2 U_{Q,\epsilon}}{\partial Q^2}) \end{pmatrix} \right\| \\ &\leqslant C\left( \left\| I''_\epsilon(U_{Q,\epsilon} + w)\left[\frac{\partial U_{Q,\epsilon}}{\partial Q}\right] \right\| + |\alpha| \cdot \left\| \frac{\partial^2 U_{Q,\epsilon}}{\partial Q^2} \right\| + \|w\| \cdot \left\| \frac{\partial^2 U_{Q,\epsilon}}{\partial Q^2} \right\| \right) \\ &\leqslant C\left( \left\| I''_\epsilon(U_{Q,\epsilon} + w)\left[\frac{\partial U_{Q,\epsilon}}{\partial Q}\right] \right\| + |\alpha| + \|w\| + \epsilon \right). \end{aligned}$$

Note that $\alpha$, similarly to $w$, satisfies $|\alpha| \leqslant C\epsilon$. By the estimate in (51) we obtain

$$\left\| I''_\epsilon(U_{Q,\epsilon}+w)\left[\frac{\partial U_{Q,\epsilon}}{\partial Q}\right]\right\| \leqslant \left\| I''_\epsilon(U_{Q,\epsilon})\left[\frac{\partial U_{Q,\epsilon}}{\partial Q}\right]\right\| + C\big(\|w\| + \|w\|^{p-1}\big).$$

Using the fact that $I''(U_Q)[\frac{\partial U_Q}{\partial z_1}] = 0$ we obtain

$$\left\| I''_\epsilon(U_{Q,\epsilon}+w)\left[\frac{\partial U_{Q,\epsilon}}{\partial Q}\right]\right\| \leqslant \left\| I''_\epsilon(U_{Q,\epsilon})\left[\frac{\partial U_Q}{\partial z_1}\right] - I''(U_Q)\left[\frac{\partial U_Q}{\partial z_1}\right]\right\| + C\epsilon + C\big(\|w\| + \|w\|^{p-1}\big).$$

For any $v \in W^{1,2}(K)$, one finds

$$\left|\big(I''_\epsilon(U_{Q,\epsilon}) - I''(U_Q)\big)\left[\frac{\partial U_Q}{\partial z_1}, v\right]\right| \leqslant p \int\limits_{K\cap\Omega_\epsilon} |U_{Q,\epsilon} - U_Q| \frac{\partial U_Q}{\partial z_1} v + C\epsilon.$$

The last three formulas imply the estimate for $\frac{\partial w(\epsilon,Q)}{\partial Q}$. This concludes the proof. □

Now we can state the following result, which allows us to perform a finite-dimensional reduction of problem (6) on the manifold $Z_\epsilon$.

**Proposition 4.5.** *The functional $\Psi_\epsilon : Z_\epsilon \to \mathbb{R}$ defined by $\Psi_\epsilon(Q) = I_\epsilon(U_{Q,\epsilon} + w(\epsilon, Q))$ is of class $C^1$ in $Q$ and satisfies*

$$\Psi'_\epsilon(Q) = 0 \quad \Rightarrow \quad I'_\epsilon\big(U_{Q,\epsilon} + w(\epsilon, Q)\big) = 0.$$

**Proof.** This proposition can be proved using the arguments of Theorem 2.12 of [2]. From a geometric point of view, we consider the manifold

$$\tilde{Z}_\epsilon = \big\{U_{Q,\epsilon} + w(\epsilon, Q):\ Q \in \Gamma\big\}.$$

Since (45) holds, we have that for $\epsilon$ small

$$T_{U_{Q,\epsilon}} Z_\epsilon \sim T_{U_{Q,\epsilon}+w(\epsilon,Q)} \tilde{Z}_\epsilon. \tag{52}$$

If $U_{Q,\epsilon} + w(\epsilon, Q)$ is a critical point of $I_\epsilon$ constrained on $\tilde{Z}_\epsilon$, then $I'_\epsilon(U_{Q,\epsilon} + w(\epsilon, Q))$ is perpendicular to $T_{U_{Q,\epsilon}+w(\epsilon,Q)}\tilde{Z}_\epsilon$, and hence, from (52), is almost perpendicular to $T_{U_{Q,\epsilon}} Z_\epsilon$. Since, by construction of $\tilde{Z}_\epsilon$, it is $I'_\epsilon(U_{Q,\epsilon} + w(\epsilon, Q)) \in T_{U_{Q,\epsilon}} Z_\epsilon$, it must be $I'_\epsilon(U_{Q,\epsilon} + w(\epsilon, Q)) = 0$. This concludes the proof. □

### *4.1. Proof of Theorem* 1.1

First of all we have

$$\begin{aligned}\Psi_\epsilon(Q) &= I_\epsilon\big(U_{Q,\epsilon} + w(\epsilon, Q)\big)\\ &= I_\epsilon(U_{Q,\epsilon}) + I'_\epsilon(U_{Q,\epsilon})\big[w(\epsilon, Q)\big] + O\big(\big\|w(\epsilon, Q)\big\|^2\big).\end{aligned}$$

Now, using Lemma 4.1 and the estimate (45) we infer

$$\Psi_\epsilon(Q) = I_\epsilon(U_{Q,\epsilon}) + O\big(\epsilon^2\big).$$

Hence Lemma 4.3 yields

$$\Psi_\epsilon(Q) = C_0\alpha(Q) + O(\epsilon).$$

Therefore, if $Q \in \Gamma$ is a local strict maximum or minimum of the function $\alpha$, the thesis follows from Proposition 4.5.

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