

# Multi-bump standing waves with critical frequency for nonlinear Schrödinger equations

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## Abstract

We glue together standing wave solutions concentrating around critical points of the potential  $V$  with different energy scales. We devise a hybrid method using simultaneously a Lyapunov–Schmidt reduction method and a variational method to glue together standing waves concentrating on local minimum points which possibly have no corresponding limiting equations and those concentrating on general critical points which converge to solutions of corresponding limiting problems satisfying a non-degeneracy condition.

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## Résumé

Nous recollons des ondes stationnaires d'ordres différents en énergie, se concentrant autour de points critiques d'un potentiel  $V$ . Nous introduisons une méthode hybride, utilisant à la fois une méthode de réduction de Lyapunov–Schmidt, et une méthode variationnelle pour recoller des ondes stationnaires, se concentrant en des minima locaux, éventuellement sans équation-limite correspondante, et d'autres se concentrant en des points critiques quelconques, convergeant vers des solutions de problèmes-limites correspondants, satisfaisant une condition de non-dégénérescence.

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## 1. Introduction and statement of main results

We consider a standing wave solution for the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2} \Delta \psi - V(x)\psi + f(\psi) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad (1.1)$$

where  $\hbar$  denotes the Plank constant,  $i$  is the imaginary unit and  $f(e^{i\theta} \psi) = e^{i\theta} f(\psi)$ . A solution of the form  $\psi(x, t) = \exp(-iEt/\hbar)v(x)$  is called a standing wave solution of the nonlinear Schrödinger equation (1.1). Then, a function  $\psi(x, t) \equiv \exp(-iEt/\hbar)v(x)$  is a standing wave solution of (1.1) if and only if the function  $v$  satisfies

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$$\frac{\hbar^2}{2} \Delta v - (V(x) - E) + f(v) = 0, \quad x \in \mathbb{R}^n. \quad (1.2)$$

For the physical background, refer to [8,29] and [30].

In this paper, we study standing waves of (1.1) for small  $\hbar > 0$ . For small  $\hbar > 0$ , these standing waves of the nonlinear Schrödinger equation (1.1) are referred to as semi-classical states. Thus we are concerned on the following equation

$$\begin{cases} \varepsilon^2 \Delta u - V(x)u + f(u) = 0, & u > 0 \quad \text{in } \mathbb{R}^n, \\ \lim_{|x| \rightarrow \infty} u(x) = 0. \end{cases} \quad (1.3)$$

In this paper, we are interested in the situation where  $E$  is a critical frequency in the sense that  $\min_{x \in \mathbb{R}^n} V(x) = 0$ . Since the pioneering work [21], there have been many further papers for the case  $\inf_{x \in \mathbb{R}^n} V(x) > E$  (refer to [1,2,4,5,14–21,24,25,27,28,31–34,37–39] and references therein). When  $\inf_{x \in \mathbb{R}^n} V(x) > 0$ , we see via a transformation  $v(x) \equiv u(\varepsilon x)$  that the following equations with constant  $c > 0$  serve as limiting equations of (1.3)

$$\begin{cases} \Delta u - cu + f(u) = 0, & u > 0 \quad \text{in } \mathbb{R}^n, \\ \lim_{|x| \rightarrow \infty} u(x) = 0. \end{cases} \quad (1.4)$$

Thus, if  $\inf_{x \in \mathbb{R}^n} V(x) > 0$ , for any solution  $u_\varepsilon$  of (1.3),  $\liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty} > 0$ . On the other hand, in the case of  $\min_{x \in \mathbb{R}^n} V(x) = 0$ , it was shown in [8] and [9] that there exists a locally minimal energy solution  $w_\varepsilon$  of (1.3) concentrating around an isolated component of global minimum points of  $V$  as  $\varepsilon \rightarrow 0$ . In contrast to the case of  $\inf_{x \in \mathbb{R}^n} V(x) > 0$ , the amplitude  $\|w_\varepsilon\|_{L^\infty}$  and energy of the localized solution  $w_\varepsilon$  concentrating around global minimum points of  $V$  decay to 0 as  $\varepsilon \rightarrow 0$ , and their decay rates depend subtly upon how the potential  $V$  decays to 0 around the concentration points. Moreover, if the decaying behavior of potential  $V$  to 0 is sufficiently *irregular*, there will be no corresponding limiting problem; in such a case, an exact estimation of the amplitude and energy of the corresponding solution may not be possible. This makes the gluing of the localized solutions very difficult. If the decaying behavior of the potential  $V$  to 0 around global minimum points is *regular* and the corresponding limiting problems have good properties, a gluing of solutions concentrating around global minimum points has been worked out in [7,10,12,11]. Recently, without requiring the existence of limiting problems, Sato [36] were able to glue localized solutions concentrating around local minimum points when  $f(u) = u^p$ ,  $p \in (1, 2^*)$ , where  $2^* = (n+2)/(n-2)$  for  $n \geq 3$  and  $2^* = \infty$  for  $n = 1, 2$ . He glues the solution via a minimization on a torus of finite codimension in a Sobolev space which depends strongly on the homogeneity of an  $f(u) = u^p$ . In this paper we devise a new approach to glue together the localized solution for a more general type of nonlinearity, where the solution cannot be obtained via a minimization argument. In fact, we glue together the localized solution concentrating around local minimum points for quite general nonlinearities  $f$  without a monotonicity assumption for  $f(t)/t$ . Furthermore, we use both a variational method and a Lyapunov–Schmidt reduction method to glue together the solutions concentrating around global minimum points (without requiring any existence of limiting problems) and the solutions concentrating around stable critical points of potential  $V$ . We have never seen this approach using simultaneously both variational method and Lyapunov–Schmidt reduction method in the literature.

To begin, we list some conditions for  $V$  and  $f$ :

- (V1)  $V \in C(\mathbb{R}^n)$ ;  $\liminf_{|x| \rightarrow \infty} V(x) > 0 = \inf_{x \in \mathbb{R}^n} V(x)$ ;
- (V2) there exist disjoint bounded open sets  $\Omega_i$  with smooth boundary  $\partial\Omega_i$ ,  $i = 1, \dots, k$ , satisfying  $0 \leq m_i = \inf_{x \in \Omega_i} V(x) < \min_{x \in \partial\Omega_i} V(x)$ ;
- (V3) there are  $x_1, \dots, x_m \in \mathbb{R}^n$  and disjoint bounded open sets  $\Omega_{k+1}, \dots, \Omega_{k+m}$  with smooth boundaries  $\partial\Omega_{k+j}$  such that  $x_j \in \Omega_{k+j}$ ,  $V \in C^2(\Omega_{k+j})$ ,  $\nabla V(x) \neq 0$  for  $x \in \Omega_{k+j} \setminus \{x_j\}$ ,  $\inf_{x \in \Omega_{k+j}} V(x) > 0$  and  $x_j$  is a non-degenerate critical point of  $V$ , for  $j \in \{1, \dots, m\}$ ;
- (f1)  $f \in C^1(\mathbb{R})$ ,  $f(t) = 0$  for  $t \leq 0$  and there exist some  $\mu_1 > 1$  and  $C > 0$  satisfying  $|f(t)| \leq Ct^{\mu_1}$  for  $t \in (0, 1)$ ;
- (f2-1) there exists some  $p \in (1, \frac{n+2}{n-2})$  for  $n \geq 3$  and  $p \in (1, \infty)$  for  $n = 1, 2$  such that  $\liminf_{t \rightarrow 0^+} \frac{1}{t^{p+1}} \int_0^t f(s) ds > 0$ ;
- (f2-2) there exists some  $p \in (1, \frac{n+2}{n-2})$  for  $n \geq 3$  and  $p \in (1, \infty)$  for  $n = 1, 2$  such that  $\limsup_{t \rightarrow \infty} \frac{|f(t)| + |f'(t)t|}{t^p} < \infty$ ;
- (f3-1) there exists  $\mu_2 > 1$  such that  $(\mu_2 + 1) \int_0^t f(s) ds \leq f(t)t$  for  $t > 0$ ;
- (f3-2) there exists  $\mu_3 > 1$  and  $t_1 > 0$  such that  $\mu_3 f(s) \leq f'(t)t$  for  $t \in [0, t_1]$ ;
- (f3-3) there exists  $\mu_3 > 1$  such that  $\mu_3 f(s) \leq f'(t)t$  for  $t > 0$ ;

(f4) for any  $a \in \{V(x_1), \dots, V(x_m)\}$ , the problem

$$\Delta u - au + f(u) = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n, \quad u \in H^{1,2}(\mathbb{R}^n) \tag{1.5}$$

has a radially symmetric solution  $U_a$  which is non-degenerate in  $H_r^{1,2}(\mathbb{R}^n) \equiv \{w \in H^{1,2}(\mathbb{R}^n); w(x) = w(|x|)\}$ , and  $f \in C_{loc}^{1,\gamma}(\mathbb{R})$  for some  $\gamma \in (0, 1)$ .

It is proved in [8] that if (f1), (f2-1) and (f3-1) hold, there exists a positive solution of (1.3) concentrating around an isolated component of zeros of  $V$ , and that if (f1), (f2-1), (f2-2) and (f3-1) hold, there exists a positive solution of (1.3) concentrating around an isolated component of local minimum points of  $V$ . In this paper, we will glue together the solutions found in [8] under the same conditions.

Throughout this paper we assume (f1), (f2-1) and (f3-1). Then we see that  $f(t) > 0$  for  $t > 0$ . Note that (f1), (f2-1) and (f3-1) hold for  $f(t) = t_+^p + t_+^q$  with  $p \in (1, 2^*), q > 1$ .

Some remarks about the above conditions on  $f$  are in order. If we take any  $\mu \in (1, \min\{\mu_1, \mu_2, \mu_3\})$ , then the conditions (f1), (f3-1), (f3-2) hold with  $\mu$  instead of  $\mu_1, \mu_2, \mu_3$ , respectively. Note that (f1) or (f3-2) implies  $\limsup_{t \rightarrow 0} \frac{f(t)}{t^\mu} < \infty$ , that (f3-1) implies  $\limsup_{t \rightarrow 0} \frac{1}{t^{\mu_2+1}} \int_0^t f(s) ds < \infty$ , and that  $\mu_1, \mu_2, \mu_3 \leq p$  if the conditions for  $\mu_1, \mu_2, \mu_3$  and  $p$  hold. For  $\mu < \mu_1$ , we can find  $t_0 \in (0, t_1)$  such that  $\mu f(t_0) \leq f'(t_0)t_0$ . If not, there exists a constant  $C > 0$  such that  $f(t) \geq Ct^\mu$  for  $t \in (0, 1)$ ; this contradicts (f1). Now, we define

$$\tilde{f}(t) = \begin{cases} f(t) & t \leq t_0, \\ f(t_0) + \frac{f'(t_0)}{\mu t_0^{\mu-1}}(t^\mu - t_0^\mu) & t > t_0. \end{cases}$$

Then we see that  $(\mu + 1) \int_0^t \tilde{f}(s) ds \leq \tilde{f}(t)t$  for all  $t > 0$  if (f3-1) holds, and that  $\mu \tilde{f}(t) \leq \tilde{f}'(t)t$  for all  $t > 0$  if (f3-2) holds. Refer to [26] for the result related to the non-degeneracy condition appearing in (f4).

To state our main results we give some definitions. We define  $A_i \equiv \{x \in \Omega_i \mid V(x) = m_i\}$  for  $i = 1, \dots, k$ ,  $\mathcal{Z} \equiv \{x \in \mathbb{R}^n \mid V(x) = 0\}$ ,  $F(t) \equiv \int_0^t f(s) ds$ ,  $\tilde{F}(t) \equiv \int_0^t \tilde{f}(s) ds$  and

$$L_a(u) \equiv \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 + au^2 dx - \int_{\mathbb{R}^n} F(u) dx, \quad u \in H^1(\mathbb{R}^n).$$

For any  $a > 0$ , let  $S_a$  be the set of all least energy solutions of the problem

$$\Delta u - au + f(u) = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n, \quad u(0) = \max_{x \in \mathbb{R}^n} u(x). \tag{1.6}$$

It is known in [37] that for each  $a > 0$ ,  $S_a$  is nonempty and if (f1), (f2-2) and (f3-1) are satisfied, and in [22] that any  $u \in S_a$  is radially symmetric. Moreover, any solution  $U \in S_a$  satisfies

$$U(x) \leq Ce^{-c|x|},$$

for some constants  $C, c > 0$ ,

$$\frac{n-2}{2} \int_{\mathbb{R}^n} |\nabla U|^2 dx + \frac{na}{2} \int_{\mathbb{R}^n} U^2 dx - n \int_{\mathbb{R}^n} F(U) dx = 0,$$

and

$$L_a(U) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla U|^2 dx + \frac{a}{2} \int_{\mathbb{R}^n} U^2 dx - \int_{\mathbb{R}^n} F(U) dx = \frac{1}{n} \int_{\mathbb{R}^n} |\nabla U|^2 dx.$$

For  $i = 1, \dots, k$ , we consider the following localized problem

$$\begin{cases} \varepsilon^2 \Delta u - V(x)u + h(u) = 0, & u > 0 \quad \text{in } \Omega_i, \\ u(x) = 0 & \text{on } \partial\Omega_i, \end{cases} \tag{1.7}$$

where  $h = f$  if  $m_i > 0$  and  $h = \tilde{f}$  if  $m_i = 0$ . We define

$$J_\varepsilon(u; \Omega_i) \equiv \frac{1}{2} \int_{\Omega_i} \varepsilon^2 |\nabla u|^2 + Vu^2 dx - \int_{\Omega_i} H(u) dx,$$

where

$$H(t) = \int_0^t h(s) ds = \begin{cases} F(t) & \text{if } m_i > 0, \\ \tilde{F}(t) & \text{if } m_i = 0. \end{cases} \quad (1.8)$$

Then a solution  $u_\varepsilon$  of (1.7) corresponds to a critical point of the energy functional  $J_\varepsilon(u; \Omega_i)$  on  $H_0^1(\Omega_i)$ .

If (f3-1) holds, for each nonnegative function  $h_i^\varepsilon \in H_0^1(\Omega_i) \setminus \{0\}$ , we can find  $t(h_i^\varepsilon) > 0$  such that for  $t \geq t(h_i^\varepsilon)$ ,  $J_\varepsilon(th_i^\varepsilon; \Omega_i) < 0$ . Then we define

$$C_\varepsilon^i \equiv \inf_{\gamma \in \Phi_\varepsilon^i} \max_{t \in [0,1]} J_\varepsilon(\gamma(t); \Omega_i),$$

where  $\Phi_\varepsilon^i \equiv \{\gamma \in C([0, 1], H_0^1(\Omega_i)) \mid \gamma(0) = 0, \gamma(1) = t(h_i^\varepsilon)h_i^\varepsilon\}$ . Then, it follows from the Mountain Pass Theorem (refer to [35]) that if (f1) and (f3-1) are satisfied when  $m_i = 0$ , and (f1), (f2-2) and (f3-1) when  $m_i > 0$ , there exists a mountain pass solution  $u_\varepsilon^i$  of (1.7) with  $J_\varepsilon(u_\varepsilon^i, \Omega_i) = C_\varepsilon^i$ . The main result in [9] implies that in case  $m_i = 0$ , if we further assume (f2-1),  $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-n} C_\varepsilon^i = 0$ ,

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon^i\|_{L^\infty(\Omega_i)} = 0, \quad \liminf_{\varepsilon \rightarrow 0} \varepsilon^{-\frac{2}{\mu-1}} \|u_\varepsilon^i\|_{L^\infty(\Omega_i)} > 0$$

and in case  $m_i > 0$ ,  $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-n} C_\varepsilon^i = L_{m_i}(U)$  for  $U \in S_{m_i}$  and there exists a maximum point  $x_\varepsilon^i$  of  $u_\varepsilon^i$  with  $\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon^i, A_i) = 0$  such that for some  $C, c > 0$ ,  $u(x) \leq C \exp(-\frac{c|x-x_\varepsilon^i|}{\varepsilon})$  and  $u_\varepsilon^i(\cdot + x_\varepsilon^i)$  converges (up to a subsequence) uniformly to a function  $U \in S_{m_i}$ .

For any set  $A \subset \mathbb{R}^n$  and  $d > 0$ , we define  $A^d \equiv \{x \in \mathbb{R}^n \mid \text{dist}(x, A) < d\}$ .

**Theorem 1.** *Suppose that (V1) and (V2) hold. Assume that  $m_1 = \dots = m_l = 0 < m_{l+1}, \dots, m_k$ . Suppose that (f1), (f2-1), (f3-1) hold if  $l = k$ , and that (f1), (f2-1), (f2-2), (f3-1) hold if  $l < k$ . Then, for sufficiently small  $\varepsilon > 0$ , there exists a positive solution  $u_\varepsilon$  of (1.3) such that*

(i) *for any sufficiently small  $d > 0$ , there exist  $C, c > 0$  satisfying*

$$u_\varepsilon(x) \leq C \exp(-c \text{dist}(x, (A_1 \cup \dots \cup A_k)^d) / \varepsilon);$$

(ii) *for  $i = 1, \dots, l$ ,*

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty(\Omega_i)} = 0, \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2/(\mu-1)} \|u_\varepsilon\|_{L^\infty(\Omega_i)} > 0;$$

*and for  $i = l+1, \dots, k$ , a least energy solution  $U \in S_{m_i}$  and some  $x_\varepsilon^i \in \mathbb{R}^n$  with  $\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon^i, A_i) = 0$ , a transformed solution  $u_\varepsilon(\varepsilon x + x_\varepsilon^i)$  converges (up to a subsequence) uniformly to  $U(x)$  on each bounded set in  $\mathbb{R}^n$ .*

*Moreover, if (f3-2) and (f3-3) are also satisfied when  $l = k$  and  $l < k$ , respectively, there exist some  $C, c > 0$  such that for  $i = 1, \dots, k$ ,*

$$|J_\varepsilon(u_\varepsilon; \Omega_i) - C_\varepsilon^i| \leq C e^{-c/\varepsilon}.$$

**Theorem 2.** *Assume that (V1), (V2) and (V3) hold. Suppose that (f1), (f2-1), (f3-1) and (f4) hold. Then for sufficiently small  $\varepsilon > 0$ , there exists a positive solution  $u_\varepsilon$  of (1.3) such that*

(i) *for any sufficiently small  $d > 0$ , there exist  $C, c > 0$  satisfying*

$$u_\varepsilon(x) \leq C \exp(-c \text{dist}(x, (A_1 \cup \dots \cup A_k \cup \{x_1\} \cup \dots \cup \{x_m\})^d) / \varepsilon);$$

(ii) *for  $m_i = 0$ ,*

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty(\Omega_i)} = 0, \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2/(\mu-1)} \|u_\varepsilon\|_{L^\infty(\Omega_i)} > 0,$$

*and for  $m_i > 0$ , there exists  $x_\varepsilon^i \in \mathbb{R}^n$  such that  $\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon^i, A_i) = 0$  and that  $u_\varepsilon(\varepsilon x + x_\varepsilon^i)$  converges uniformly (up to a subsequence) to some  $U_{m_i} \in S_{m_i}$  on each bounded set in  $\mathbb{R}^n$ ;*

(iii) for each  $i = 1, \dots, m$ , there exists  $y_\varepsilon^i \in \mathbb{R}^n$  such that  $\lim_{\varepsilon \rightarrow 0} y_\varepsilon^i = x_i$  and that  $u_\varepsilon(\varepsilon x + y_\varepsilon^i)$  converges uniformly (up to a subsequence) to  $U_{V(x_i)}$  on each bounded set in  $\mathbb{R}^n$ . Here  $U_{V(x_i)}$  is a function given in (f4).

Moreover, if (f3-2) and (f3-3) are also satisfied when  $l = k$  and  $l < k$ , respectively, there exist some  $C, c > 0$  such that for  $i = 1, \dots, k$ ,

$$|J_\varepsilon(u_\varepsilon; \Omega_i) - C_i^\varepsilon| \leq C e^{-c/\varepsilon}.$$

We conclude the introduction with an outline of our proof of the main results.

We look for a critical point of some energy functional  $\Gamma_\varepsilon$ . First we choose a bounded open set  $\Omega_0$  enclosing  $\mathcal{Z} \setminus \bigcup_{i=1}^k A_i$  (see Section 2 for a precise condition for  $\Omega_0$ ). Then we consider a modified problem outside  $\bigcup_{i=0}^{k+m} \Omega_i$  for each given function  $u_i$  on  $\Omega_i$ , and show the existence of a solution  $\varphi(u_0, \dots, u_{k+m})$  for the external problem solving by a minimization problem. We also show the existence of a solution  $P_i(u_i)$  of the modified problem by a minimization such that  $P_i(u_i)(x) = 0$  for  $\text{dist}(x, \Omega_i) \geq \delta > 0$  and  $P_i(u_i) = u_i$  on  $\Omega_i$ . Then, we will show that finding a solution is reduced to finding a critical point of a reduced functional  $I_\varepsilon(u_0, \dots, u_{k+m})$ . Finding a good estimate for

$$\tilde{\varphi}(u_0, \dots, u_{k+m}) \equiv \varphi(u_0, \dots, u_{k+m}) - \sum_{i=0}^{k+m} P_i(u_i)$$

is important in our proof. In fact, we show that  $\tilde{\varphi}$  is exponentially small with respect to small  $\varepsilon > 0$  (see Proposition 3). That enables us to regard the sum  $\sum_{i=0}^{k+m} \Gamma_\varepsilon(P_i(u_i))$  of localized functionals depending only on each  $u_i$  as an exponentially small perturbation of the reduced functional  $I_\varepsilon$ . This is a novelty of our argument.

To prove Theorem 1, we consider an energy gradient flow in a product of an appropriate small ball in  $H^1(\Omega_0)$  and appropriate annuli in  $H^1(\Omega_i), i = 1, \dots, k$ . To take appropriate radii of the ball and annuli is also important in our proof. If there exist no solutions in the product of the ball and the annuli, we show that via a gradient estimation near the boundary of the product of the ball and the annuli, we can deform a product of localized mountain paths into a surface where the maximum energy is less than a sum of independent local mountain pass levels by an algebraic order of  $\varepsilon > 0$ . Then, from the exponential smallness of  $\tilde{\varphi}(u_0, \dots, u_{k+m})$ , we will get a contradiction.

For the proof of Theorem 2, we use a Lyapunov–Schmidt reduction method in a region  $\Omega_{k+1} \cup \dots \cup \Omega_{k+m}$  before we use the variational argument in a region  $\Omega_0 \cup \dots \cup \Omega_k$  as in the proof of Theorem 1. For the typical case  $m = 1$  we will find a critical point of the functional  $u_{k+1} \mapsto I(u_0, \dots, u_{k+1})$  by the reduction method which depends smoothly on  $(u_0, \dots, u_k)$ , and then use the same argument as in the proof of Theorem 1. So we will skip the variational procedure in the proof of Theorem 2 since the required variational argument after the reduction is exactly the same as that of the proof of Theorem 1.

This paper is organized as follows. In Section 2, some preliminaries about the above reduction are given. Theorems 1 and 2 will be proved in Sections 3 and 4, respectively.

## 2. Preliminaries

We define  $A_0 := \mathcal{Z} \setminus \bigcup_{i=1}^k A_i$  and  $\Omega_0 \supset A_0$  a bounded open set with a smooth boundary such that  $\overline{\Omega_0} \cap \overline{\Omega_i} = \emptyset$  for  $i = 1, \dots, k + m$ . For  $\delta > 0$ , let  $\Omega_i^\delta = \{x \in \mathbb{R}^n \mid \text{dist}(x, \Omega_i) \leq \delta\}$ , and  $\Omega_i^{-\delta} = \{x \in \Omega_i \mid \text{dist}(x, \partial\Omega_i) \geq \delta\}$ . Taking sufficiently small  $\delta > 0$ , we may assume that  $A_i \subset \Omega_i^{-2\delta}$ ,  $\Omega_i^{2\delta} \cap \Omega_j^{2\delta} = \emptyset$  for each  $0 \leq i \neq j \leq k + m$ , and that  $\partial\Omega_i^{\delta'}$  is smooth for each  $0 \leq i \leq k + m$  and  $\delta' \in [-2\delta, 2\delta]$ . Reordering the index, we may assume that  $m_1 = \dots = m_l = 0$  and  $m_{l+1}, \dots, m_k > 0$  for some  $l \in \{1, \dots, k\}$ . In this section, we assume that (f1), (f2-1) and (f3-1) are satisfied and (f2-2) are also satisfied if  $l < k$ . As in the previous section, we can find  $t_0 > 0$  such that  $\mu f(t_0) \leq f'(t_0)t_0$ . Then, we define

$$\tilde{f}(t) = \begin{cases} f(t) & t \leq t_0, \\ f(t_0) + \frac{f'(t_0)}{\mu t_0^{\mu-1}}(t^\mu - t_0^\mu) & t > t_0. \end{cases}$$

Then we see that  $(\mu + 1) \int_0^t \tilde{f}(s) ds \leq \tilde{f}(t)t$  for all  $t > 0$  if (f3-1) holds, and that  $\mu \tilde{f}(t) \leq \tilde{f}'(t)t$  for all  $t > 0$  if (f3-2) holds. We define

$$b \equiv \inf\{V(x) \mid x \notin \Omega_0^{-\delta} \cup \dots \cup \Omega_k^{-\delta}\} > 0.$$

We define  $V_\varepsilon(x) := V(\varepsilon x)$ , and  $H_\varepsilon$  the completion of  $C_0^\infty(\mathbb{R}^n)$  with respect to a norm  $\|u\|_\varepsilon = (\int_{\mathbb{R}^n} |\nabla u|^2 + V_\varepsilon u^2 dx)^{1/2}$ . Note that  $H_\varepsilon \subset H^1(\mathbb{R}^n)$  by (V1). We also define  $\Omega_{i,\varepsilon} = \{x \mid \varepsilon x \in \Omega_i\}$ ,  $\Omega_{i,\varepsilon}^\delta = \{x \mid \varepsilon x \in \Omega_i^\delta\}$ . Then we see that

$$\inf\{V_\varepsilon(x) \mid x \notin \Omega_{0,\varepsilon}^{-\delta} \cup \dots \cup \Omega_{k,\varepsilon}^{-\delta}\} = b > 0.$$

From (f3-1), there exist some  $C_1, C_2 > 0$  such that  $f(t) \leq C_1 t^{\mu_2}$  for  $t \in [0, 1]$  and  $f(t) \geq C_2 t^{\mu_2}$  for  $t \in [1, \infty)$ . Thus, taking sufficiently small  $\lambda \in (0, \min\{b/10, b(1/2 - 1/(\mu + 1))\})$ , we can construct a function  $f_\lambda \in C^1(\mathbb{R})$  for small  $\lambda_0$  and large  $\lambda_1 > 0$  such that

$$f_\lambda(t) = \begin{cases} f(t) & t \leq \lambda_0, \\ \lambda t & t \geq \lambda_1 \end{cases}$$

with  $0 < \lambda_0 < \lambda_1$ , and  $f_\lambda(t) \leq \min\{f(t), \tilde{f}(t), \lambda t\}$ ,  $|f'_\lambda(t)| \leq 2\lambda$  for  $t \geq 0$ . Note that  $f_\lambda(t) = 0$  for  $t \leq 0$  and

$$\frac{|f_\lambda(t_1) - f_\lambda(t_2)|}{|t_1 - t_2|} \leq 2\lambda \quad \text{for } t_1 \neq t_2.$$

We find a function  $\chi \in C^1(\mathbb{R}^n)$  such that  $0 \leq \chi \leq 1$ ,  $\chi(x) = 1$  for  $x \in \bigcup_{i=0}^{k+m} \Omega_i^{-\delta}$  and  $\chi(x) = 0$  for  $x \notin \bigcup_{i=0}^{k+m} \Omega_i$ .

We define

$$g(x, t) = \begin{cases} \chi(\varepsilon x) \tilde{f}(t) + (1 - \chi(\varepsilon x)) f_\lambda(t) & x \in \bigcup_{i=0}^l \Omega_{i,\varepsilon}, \\ \chi(\varepsilon x) f(t) + (1 - \chi(\varepsilon x)) f_\lambda(t) & x \in \bigcup_{i=l+1}^{k+m} \Omega_{i,\varepsilon}, \\ f_\lambda(t) & x \notin \bigcup_{i=0}^{k+m} \Omega_{i,\varepsilon}, \end{cases}$$

$G(x, t) \equiv \int_0^t g(x, s) ds$ , and for  $u \in H_\varepsilon$ ,

$$\Gamma_\varepsilon(u) \equiv \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 + V_\varepsilon(x) u^2 dx - \int_{\mathbb{R}^n} G(x, u) dx.$$

Then  $\Gamma_\varepsilon \in C^2(H_\varepsilon)$ . For a measurable subset  $U \subset \mathbb{R}^n$ , we define

$$\Gamma_\varepsilon(u; U) = \frac{1}{2} \int_U |\nabla u|^2 + V_\varepsilon(x) u^2 dx - \int_U G(x, u) dx,$$

$$\|u\|_{\varepsilon,U}^2 = \int_U |\nabla u|^2 + V_\varepsilon(x) u^2 dx$$

for  $u \in H_\varepsilon$ . For  $\vec{u} = (u_0, \dots, u_{k+m}) \in H^1(\Omega_{0,\varepsilon}) \times \dots \times H^1(\Omega_{k+m,\varepsilon})$ , we define a norm

$$\|\vec{u}\|_\varepsilon^2 = \sum_{i=0}^{k+m} \|u_i\|_{\varepsilon,\Omega_{i,\varepsilon}}^2.$$

For  $u_i \in H^1(\Omega_{i,\varepsilon})$ , let

$$X_i^\varepsilon(u_i) = \{u \in H_0^1(\Omega_{i,\varepsilon}^\delta) \mid u = u_i \text{ on } \Omega_{i,\varepsilon}\}.$$

We regard  $u \in H_0^1(\Omega_{i,\varepsilon}^\delta)$  as an element in  $H^1(\mathbb{R}^n)$  by defining  $u(x) = 0$  for  $x \notin \Omega_{i,\varepsilon}^\delta$ .

**Proposition 1.** For each  $u_i \in H^1(\Omega_{i,\varepsilon})$ ,  $i \in \{0, \dots, k + m\}$ , there exists a unique minimizer  $P_i(u_i)$  of  $\Gamma_\varepsilon$  on  $X_i^\varepsilon(u_i)$ , which satisfies the following:

(i)  $w = P_i(u_i) \in H_0^1(\Omega_{i,\varepsilon}^\delta)$  solves

$$\begin{cases} \Delta w - V_\varepsilon w + f_\lambda(w) = 0 & \text{in } \Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}, \\ w = u_i & \text{on } \partial\Omega_{i,\varepsilon}, \\ w = 0 & \text{on } \partial\Omega_{i,\varepsilon}^\delta, \end{cases}$$

(ii)  $P_i : H^1(\Omega_{i,\varepsilon}) \rightarrow H_0^1(\Omega_{i,\varepsilon}^\delta)$  is of class  $C^1$ , and for all  $h \in H^1(\Omega_{i,\varepsilon})$ ,  $w = P_i'(u_i)h$  solves

$$\begin{cases} \Delta w - V_\varepsilon w + f'_\lambda(P_i(u_i))w = 0 & \text{in } \Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}, \\ w = h & \text{in } \Omega_{i,\varepsilon}, \\ w = 0 & \text{on } \partial\Omega_{i,\varepsilon}^\delta, \end{cases}$$

(iii) there exists a positive constant  $C$ , independent of small  $\varepsilon > 0$  such that

$$\|P_i(u_i)\|_\varepsilon \leq C \|u_i\|_{\varepsilon, \Omega_{i,\varepsilon}} \quad \text{for all } u_i \in H^1(\Omega_{i,\varepsilon}), i \in \{0, \dots, k+m\}.$$

This proposition can be proved in a similar way as in [36], so we omit the proof.

For  $\vec{u} = (u_0, \dots, u_{k+m})$ ,  $u_i \in H^1(\Omega_{i,\varepsilon})$ , let

$$X_*^\varepsilon(\vec{u}) = \{u \in H_\varepsilon \mid u = u_i \text{ on } \Omega_{i,\varepsilon}, i = 0, \dots, k+m\}.$$

By the same procedure as in the proof of Proposition 1, we can prove the following proposition.

**Proposition 2.** For each  $\vec{u} = (u_0, \dots, u_{k+m}) \in H^1(\Omega_{0,\varepsilon}) \times \dots \times H^1(\Omega_{k+m,\varepsilon})$ , there exists a unique minimizer  $\varphi(\vec{u})$  of  $\Gamma_\varepsilon$  on  $X_*^\varepsilon(\vec{u})$ , which satisfies the following:

(i)  $w = \varphi(\vec{u})$  solves

$$\begin{cases} \Delta w - V_\varepsilon w + f'_\lambda(w) = 0 & \text{in } (\Omega_{0,\varepsilon} \cup \dots \cup \Omega_{k,\varepsilon})^c, \\ w = u_i & \text{on } \partial\Omega_{i,\varepsilon} \ (i = 0, \dots, k), \end{cases}$$

(ii)  $\varphi : H^1(\Omega_{0,\varepsilon}) \times \dots \times H^1(\Omega_{k,\varepsilon}) \rightarrow H_\varepsilon$  is of class  $C^1$ , and for all  $h \in H^1(\Omega_{i,\varepsilon})$ ,  $v = \frac{\partial\varphi(\vec{u})}{\partial u_i} h$  solves

$$\begin{cases} \Delta v - V_\varepsilon v + f'_\lambda(\varphi(\vec{u}))v = 0 & \text{in } (\Omega_{0,\varepsilon} \cup \dots \cup \Omega_{k,\varepsilon})^c, \\ v = h & \text{in } \Omega_{i,\varepsilon}, \\ v = 0 & \text{in } \Omega_{j,\varepsilon}, j \neq i, \end{cases}$$

(iii) there exists a positive constant  $C$ , independent of small  $\varepsilon > 0$  such that

$$\|\varphi(\vec{u})\|_\varepsilon \leq C \|\vec{u}\|_\varepsilon.$$

Let  $\tilde{\varphi}(\vec{u}) = \varphi(\vec{u}) - \sum_{i=0}^{k+m} P_i(u_i)$ . Then it follows that  $\tilde{\varphi}(\vec{u}) \in X_*^\varepsilon(\vec{0})$ . Now we obtain the following estimates for  $\tilde{\varphi}(\vec{u})$ .

**Proposition 3.** For any  $R > 0$  and  $\varepsilon_0 > 0$ , there exist constants  $C, c > 0$  such that

$$\|\tilde{\varphi}(\vec{u})\|_\varepsilon \leq C e^{-c/\varepsilon}$$

for  $\varepsilon \in (0, \varepsilon_0)$  and  $\|\vec{u}\|_\varepsilon \leq R$ .

**Proof.** For  $u_i \in H^1(\Omega_{i,\varepsilon})$ ,  $i = 0, \dots, k+m$ ,  $w_i = P_i(u_i) \in H_0^1(\Omega_{i,\varepsilon}^\delta)$  solves

$$\begin{cases} \Delta w - V_\varepsilon w + f'_\lambda(w) = 0 & \text{in } \Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}, \\ w = u_i & \text{on } \partial\Omega_{i,\varepsilon}, \\ w = 0 & \text{on } \partial\Omega_{i,\varepsilon}^\delta. \end{cases}$$

By Proposition 1, there exists a constant  $C > 0$ , independent of  $\vec{u}$  with  $\|\vec{u}\|_\varepsilon \leq R$ , such that  $\|w_i\|_{H^1(\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon})} \leq C$ . Hence by elliptic estimates (refer to [23]), we deduce that for each  $s \in (0, 1)$ , there exists a constant  $c > 0$  such that  $\|w_i\|_{L^\infty(\Omega_{i,\varepsilon}^{s\delta} \setminus \Omega_{i,\varepsilon})} \leq C$ . By comparison principle, we conclude that for some  $C > 0$ , independent of small  $\varepsilon > 0$ ,

$$w_i \leq C e^{-c/\varepsilon} \quad \text{on } \Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}^{\delta/2}.$$

Then, it follows from boundary Schauder estimates [23, Corollary 8.36] that  $|w_i|_{C^{1,\theta}(\partial\Omega_{i,\varepsilon}^\delta)} \leq C e^{-c/\varepsilon}$ . Denoting  $\varphi = \varphi(\vec{u}) = \sum_{i=0}^{k+m} P_i(u_i) + \tilde{\varphi}(\vec{u})$ , we see that  $\varphi$  satisfies

$$\begin{cases} \Delta\varphi - V_\varepsilon\varphi + f_\lambda(\varphi) = 0 & \text{in } (\Omega_{0,\varepsilon} \cup \dots \cup \Omega_{k+m,\varepsilon})^c, \\ \varphi = u_i & \text{on } \partial\Omega_{i,\varepsilon}. \end{cases}$$

If  $\|\vec{u}\|_\varepsilon \leq R$ , it follows from (iii) of Proposition 2 that  $\|\varphi(\vec{u})\|_\varepsilon \leq C$ . Hence by elliptic estimates, we see that for each  $s \in (0, 1)$ ,

$$\|\varphi(\vec{u})\|_{L^\infty(\Omega_{0,\varepsilon}^\delta \cup \dots \cup \Omega_{k+m,\varepsilon}^\delta)^c} \leq C.$$

Then by comparison principle, we deduce that

$$\varphi \leq C e^{-c/\varepsilon} \quad \text{in } \bigcup_{i=0}^{k+m} \Omega_{i,\varepsilon}^{3\delta/2} \setminus \Omega_{i,\varepsilon}^{\delta/2}.$$

Then it follows from interior Schauder estimates [23, Theorem 8.32] that

$$|\varphi(\vec{u})|_{C^{1,\theta}(\bigcup_{i=0}^{k+m} \partial\Omega_{i,\varepsilon}^\delta)} \leq C e^{-c/\varepsilon}.$$

Now a function  $w_i = P_i(u_i)$  satisfies

$$\Delta(w_i - \varphi) - V_\varepsilon(w_i - \varphi) + \frac{f_\lambda(w_i) - f_\lambda(\varphi)}{w_i - \varphi}(w_i - \varphi) = 0 \quad \text{in } \Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}.$$

Multiplying by  $w_i - \varphi$  and integrating by parts, we have

$$\begin{aligned} & \int_{\partial\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} (w_i - \varphi) \frac{\partial(w_i - \varphi)}{\partial n} dS - \int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} |\nabla(w_i - \varphi)|^2 dx \\ & - \int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} V_\varepsilon(w_i - \varphi)^2 dx + \int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} \frac{f_\lambda(w_i) - f_\lambda(\varphi)}{w_i - \varphi}(w_i - \varphi)^2 dx = 0. \end{aligned}$$

Since  $w_i - \varphi = 0$  on  $\partial\Omega_{i,\varepsilon}$  and  $\|w_i - \varphi\|_{C^1(\partial\Omega_{i,\varepsilon}^\delta)} = O(e^{-c/\varepsilon})$ , we see that

$$\int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} |\nabla(w_i - \varphi)|^2 + \int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} (w_i - \varphi)^2 = O(e^{-c/\varepsilon});$$

hence

$$\|\tilde{\varphi}\|_{H^1(\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon})} = \|\varphi - w_i\|_{H^1(\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon})} = O(e^{-c/\varepsilon}). \tag{2.1}$$

We note that  $\tilde{\varphi}(\vec{u}) = \varphi(\vec{u})$  on  $\mathbb{R}^n \setminus (\Omega_{0,\varepsilon}^\delta \cup \dots \cup \Omega_{k+m,\varepsilon}^\delta)$ . From a decay property of  $\varphi$ , we see that  $\|\varphi\|_{H^1(\mathbb{R}^n \setminus \Omega_{0,\varepsilon}^\delta \cup \dots \cup \Omega_{k+m,\varepsilon}^\delta)} = O(e^{-c/\varepsilon})$ . Thus combining this with (2.1), we get the required estimate.  $\square$

Let  $I_\varepsilon(\vec{u}) = I_\varepsilon(\varphi(\vec{u}))$ . Then from Proposition 3, we conclude that for any  $R, \varepsilon_0 > 0$ , there exist constants  $C, c > 0$  such that

$$\left| I_\varepsilon(\vec{u}) - \sum_{i=0}^{k+m} \Gamma_\varepsilon(P_i(u_i)) \right| \leq C e^{-c/\varepsilon} \quad \text{for } \|\vec{u}\|_\varepsilon \leq R \text{ and } \varepsilon \in (0, \varepsilon_0). \tag{2.2}$$

Moreover, we have the following properties for  $I_\varepsilon(\vec{u})$ .

**Proposition 4.** *The following hold.*

- (i) A vector function  $\vec{u} = (u_0, \dots, u_{k+m}) \in H^1(\Omega_{0,\varepsilon}) \times \dots \times H^1(\Omega_{k+m,\varepsilon})$  is a critical point of  $I_\varepsilon$  if and only if  $\varphi(\vec{u})$  is a critical point of  $\Gamma_\varepsilon$ .



- (ii) The functional  $\vec{u} \mapsto I_\varepsilon(\vec{u})$  satisfies (PS) condition if  $\Gamma_\varepsilon$  does.
- (iii) For any  $R > 0, i = 0, \dots, k + m$  and  $\varepsilon_0 > 0$ , there exist constants  $C, c > 0$  such that

$$\left| \frac{\partial I_\varepsilon}{\partial u_i}(u_0, \dots, u_{k+m}) - \frac{d\Gamma_\varepsilon(P_i(u_i))}{du_i} \right| \leq C e^{-c/\varepsilon} \quad \text{for } \varepsilon \in (0, \varepsilon_0) \text{ and } \|\vec{u}\|_\varepsilon \leq R.$$

**Proof.** (i) For all  $\vec{\zeta} = (\zeta_0, \dots, \zeta_{k+m}) \in H^1(\Omega_{0,\varepsilon}) \times \dots \times H^1(\Omega_{k+m,\varepsilon})$ , we have  $\tilde{\varphi}'(\vec{u})\vec{\zeta} \in X_*^\varepsilon(\vec{0})$ . Then, it follows from the definition of  $\varphi$  that for all  $h \in X_*^\varepsilon(\vec{0})$ ,

$$\begin{aligned} I'_\varepsilon(\vec{u})\vec{\zeta} &= \Gamma'_\varepsilon(\varphi(\vec{u}))\varphi'(\vec{u})\vec{\zeta} \\ &= \Gamma'_\varepsilon(\varphi(\vec{u})) \left[ \sum_{i=0}^{k+m} P'_i(u_i)\zeta_i + \tilde{\varphi}'(\vec{u})\vec{\zeta} \right] \\ &= \Gamma'_\varepsilon(\varphi(\vec{u})) \left[ \sum_{i=0}^{k+m} P'_i(u_i)\zeta_i \right] \\ &= \Gamma'_\varepsilon(\varphi(\vec{u})) \left[ \sum_{i=0}^{k+m} P'_i(u_i)\zeta_i + h \right]. \end{aligned}$$

Then, from the fact  $H_\varepsilon = \{\sum_{i=0}^{k+m} P'_i(u_i)\zeta_i + h \mid \zeta_i \in H^1(\Omega_{i,\varepsilon}), h \in X_*^\varepsilon(\vec{0})\}$ , the equivalence of (i) follows.

- (ii) We see from Propositions 1 and 2 that for  $\vec{\zeta} = (\zeta_0, \dots, \zeta_{k+m}) \in H^1(\Omega_{0,\varepsilon}) \times \dots \times H^1(\Omega_{k+m,\varepsilon})$ ,

$$\begin{aligned} \left| \Gamma'_\varepsilon(\varphi(\vec{u})) \left( \sum_{i=0}^{k+m} P'_i(u_i)\zeta_i + h \right) \right| &\leq |I'_\varepsilon(\vec{u})\vec{\zeta}| \\ &\leq \|I'_\varepsilon(\vec{u})\| \|\vec{\zeta}\|_\varepsilon \\ &\leq \|I'_\varepsilon(\vec{u})\| \left\| \sum_{i=0}^{k+m} P'_i(u_i)\zeta_i + h \right\|_\varepsilon. \end{aligned}$$

Thus it follows that  $\|\Gamma'_\varepsilon(\varphi(\vec{u}))\| \leq \|I'_\varepsilon(\vec{u})\|$ . Note that  $\Gamma_\varepsilon(\varphi(\vec{u})) = I_\varepsilon(\vec{u})$  and for some  $C > 0, \|\vec{u}\|_\varepsilon \leq \|\varphi(u)\|_\varepsilon \leq C\|\vec{u}\|_\varepsilon$ . Thus, we conclude that  $I_\varepsilon$  satisfies (PS) condition if  $\Gamma_\varepsilon$  does.

- (iii) For  $\vec{h} = (h_0, \dots, h_{k+m}), \vec{u} = (u_0, \dots, u_{k+m})$  with  $h_i, u_i \in H^1(\Omega_{i,\varepsilon})$ , a function  $v = \sum_{i=0}^{k+m} \frac{\partial \varphi(\vec{u})}{\partial u_i} h_i$  solves

$$\begin{cases} \Delta v - V_\varepsilon v + f'_\lambda(\varphi(\vec{u}))v = 0 & \text{in } (\Omega_{0,\varepsilon} \cup \dots \cup \Omega_{k+m,\varepsilon})^c, \\ v = h_i & \text{in } \Omega_{i,\varepsilon}. \end{cases}$$

By the minimization characterization of  $v$ , we have  $\|v\|_\varepsilon \leq C\|\vec{h}\|_\varepsilon \leq C$  for a constant  $C$  independent of  $\vec{h}$  with  $\|\vec{h}\|_\varepsilon \leq 1$ .

Similarly, a function  $w_i = P'_i(u_i)h_i$  solves

$$\begin{cases} \Delta w - V_\varepsilon w + f'_\lambda(P_i(u_i))w = 0 & \text{in } \Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}, \\ w = h_i & \text{in } \Omega_{i,\varepsilon}, \\ w = 0 & \text{on } \partial\Omega_{i,\varepsilon}^\delta. \end{cases}$$

By the minimization characterization of  $w_i$ , we have  $\|w_i\|_\varepsilon \leq C\|h_i\|_\varepsilon \leq C$  for a constant  $C$  independent of  $\vec{h}$  with  $\|\vec{h}\|_\varepsilon \leq 1$ .

Since

$$\begin{aligned} &\left| \sum_{i=0}^{k+m} \Gamma'_\varepsilon(\varphi(\vec{u}))w_i - \sum_{i=0}^{k+m} \Gamma'_\varepsilon(P_i(u_i))w_i \right| \\ &\leq \left| \sum_{i=0}^{k+m} \int_{\Omega_{i,\varepsilon}^\delta} \nabla(\varphi(\vec{u}) - P_i(u_i)) \cdot \nabla w_i + V_\varepsilon(\varphi(\vec{u}) - P_i(u_i))w_i \, dx \right| \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=0}^{k+m} \int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} |f_\lambda(\varphi(\vec{u})) - f_\lambda(P_i(u_i))| |w_i| dx \\
 & \leq \sum_{i=0}^{k+m} \|\varphi(\vec{u}) - P_i(u_i)\|_{\varepsilon, \Omega_{i,\varepsilon}^\delta} \|w_i\|_{\varepsilon, \Omega_{i,\varepsilon}^\delta} + 2\lambda \sum_{i=0}^{k+m} \|\varphi(\vec{u}) - P_i(u_i)\|_{L^2(\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon})} \|w_i\|_{L^2(\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon})},
 \end{aligned}$$

we see from Proposition 3 that for any  $\varepsilon_0, R > 0$ , there exist  $C, c > 0$  satisfying

$$\left| \sum_{i=0}^{k+m} \Gamma'_\varepsilon(\varphi(\vec{u}))w_i - \sum_{i=0}^{k+m} \Gamma'_\varepsilon(P_i(u_i))w_i \right| \leq Ce^{-c/\varepsilon} \quad \text{for } \varepsilon \in (0, \varepsilon_0) \text{ and } \|\vec{u}\|_\varepsilon \leq R.$$

Then, it follows that

$$\begin{aligned}
 \left| I'_\varepsilon(\vec{u})\vec{h} - \sum_{i=0}^{k+m} \frac{d\Gamma_\varepsilon(P_i(u_i))}{du_i} h_i \right| & = \left| \Gamma'_\varepsilon(\varphi(\vec{u}))v - \sum_{i=0}^{k+m} \Gamma'_\varepsilon(P_i(u_i))w_i \right| \\
 & \leq \left| \Gamma'_\varepsilon(\varphi(\vec{u})) \left( v - \sum_{i=0}^{k+m} w_i \right) \right| + Ce^{-c/\varepsilon} \\
 & = Ce^{-c/\varepsilon}
 \end{aligned}$$

since  $v - \sum w_i \in X_*^\varepsilon(\vec{0})$ . This completes the proof.  $\square$

We define  $\Gamma_\varepsilon^j(u) = \Gamma_\varepsilon(P_j(u))$  for  $u \in H^1(\Omega_{j,\varepsilon})$  and  $j \in \{1, \dots, k+m\}$ .

**Proposition 5.** For each  $j \in \{1, \dots, k+m\}$ , the following hold.

- (i) A function  $u_j$  is a critical point of  $\Gamma_\varepsilon^j$  if and only if  $P_j(u_j) \in H_0^1(\Omega_{j,\varepsilon}^\delta)$  is a critical point of  $\Gamma_\varepsilon$  on  $H_0^1(\Omega_{j,\varepsilon}^\delta)$ .
- (ii) The functional  $\Gamma_\varepsilon^j$  on  $H^1(\Omega_{j,\varepsilon})$  satisfies (PS) condition if  $\Gamma_\varepsilon$  on  $H_0^1(\Omega_{j,\varepsilon}^\delta)$  does.

**Proof.** We can prove these following the same argument as in the proof of (i) and (ii) in Proposition 4.  $\square$

Let  $\alpha = 2\frac{\mu+1}{\mu-1} \geq 2 \max\{\frac{p+1}{p-1}, \frac{\mu+1}{\mu-1}\}$ . By Proposition 1(iii), we can choose a constant  $M > 1$ , independent of small  $\varepsilon > 0$ , such that

$$\|u_i\|_{\varepsilon, \Omega_{i,\varepsilon}} \leq \|P_i(u_i)\|_\varepsilon \leq M\|u_i\|_{\varepsilon, \Omega_{i,\varepsilon}} \quad \text{for all } u_i \in H^1(\Omega_{i,\varepsilon}), i \in \{0, \dots, k+m\}.$$

**Proposition 6.** Let  $i \in \{0, \dots, l\}$ . For sufficiently small  $\varepsilon > 0$ , it holds that  $\Gamma_\varepsilon(u) \geq 6M^2\varepsilon^{2\alpha}$  for all  $u \in H_0^1(\Omega_{i,\varepsilon}^\delta)$  with  $\|u\|_\varepsilon = 4M\varepsilon^\alpha$ .

**Proof.** Note that from conditions (f2-1) and (f3-1), we have  $\mu \leq p$ . Also it follows from (f3-1) and (f2-2) that  $G(x, t) \leq Ct^{p+1} + Ct^{\mu+1}$  for some  $C > 0$ . Then, for  $u \in H_0^1(\Omega_{i,\varepsilon}^\delta) \subset H_\varepsilon$ , denoting  $v(x) \equiv u(x/\varepsilon) \in H_0^1(\Omega_i^\delta)$ , we see that for sufficiently small  $\varepsilon > 0$ ,

$$\begin{aligned}
 \Gamma_\varepsilon(u) & \geq \int_{\Omega_{i,\varepsilon}^\delta} \frac{1}{2} (|\nabla u|^2 + V_\varepsilon u^2) - Cu^{p+1} - Cu^{\mu+1} dx \\
 & = \frac{1}{2} \|u\|_\varepsilon^2 - \varepsilon^{-n} \left\{ \int_{\Omega_i^\delta} Cv^{p+1} + Cv^{\mu+1} dx \right\} \\
 & \geq \frac{1}{2} \|u\|_\varepsilon^2 - C\varepsilon^{-n} \left\{ \left( \int_{\Omega_i^\delta} |\nabla v|^2 \right)^{(p+1)/2} + \left( \int_{\Omega_i^\delta} |\nabla v|^2 \right)^{(\mu+1)/2} \right\}
 \end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{2} \|u\|_\varepsilon^2 - C\varepsilon^{n(p-1)/2-(p+1)} \|u\|_\varepsilon^{p+1} - C\varepsilon^{n(\mu-1)/2-(\mu+1)} \|u\|_\varepsilon^{\mu+1} \\ &\geq \left( \frac{1}{2} - C\varepsilon^{n(p-1)/2-(p+1)} \|u\|_\varepsilon^{p-1} - C\varepsilon^{n(\mu-1)/2-(\mu+1)} \|u\|_\varepsilon^{\mu-1} \right) \|u\|_\varepsilon^2 \\ &\geq 6M^2\varepsilon^{2\alpha}. \end{aligned}$$

This completes the proof.  $\square$

**Proposition 7.** *Let  $i \in \{0, \dots, l\}$ . For sufficiently small  $\varepsilon > 0$ , it holds that  $|\Gamma_\varepsilon(u)| \leq 5M^2\varepsilon^{2\alpha}$  for all  $u \in H_0^1(\Omega_{i,\varepsilon}^\delta)$  with  $\|u\|_\varepsilon \leq 3M\varepsilon^\alpha$ , and  $\Gamma_\varepsilon(u) \geq 0$  for all  $u \in H_0^1(\Omega_{i,\varepsilon}^\delta)$  with  $\|u\|_\varepsilon \leq 5M\varepsilon^\alpha$ .*

**Proof.** As in the proof of Proposition 6, we note that  $\mu \leq p$  and  $G(x, t) \leq Ct^{p+1} + Ct^{\mu+1}$  for some  $C > 0$ . Then, for  $u \in H_0^1(\Omega_{i,\varepsilon}^\delta) \subset H_\varepsilon$ , we deduce as in Proposition 6 that for sufficiently small  $\varepsilon > 0$ ,

$$\begin{aligned} |\Gamma_\varepsilon(u)| &\leq \int_{\Omega_{i,\varepsilon}^\delta} \frac{1}{2} (|\nabla u|^2 + V_\varepsilon u^2) + Cu^{p+1} + Cu^{\mu+1} dx \\ &\leq \frac{1}{2} \|u\|_\varepsilon^2 + C\varepsilon^{n(p-1)/2-(p+1)} \|u\|_{H_0^1(\Omega_{i,\varepsilon}^\delta)}^{p+1} + C\varepsilon^{n(\mu-1)/2-(\mu+1)} \|u\|_{H_0^1(\Omega_{i,\varepsilon}^\delta)}^{\mu+1} \\ &= \left\{ \frac{1}{2} + C\varepsilon^{n(p-1)/2-(p+1)} \|u\|_\varepsilon^{p-1} + C\varepsilon^{n(\mu-1)/2-(\mu+1)} \|u\|_\varepsilon^{\mu-1} \right\} \|u\|_\varepsilon^2 \\ &\leq 5M^2\varepsilon^{2\alpha}. \end{aligned}$$

Similarly we see that for  $u \in H_0^1(\Omega_{i,\varepsilon}^\delta) \subset H_\varepsilon$  with  $\|u\|_\varepsilon \leq 5M\varepsilon^\alpha$ ,

$$\begin{aligned} \Gamma_\varepsilon(u) &\geq \int_{\Omega_{i,\varepsilon}^\delta} \frac{1}{2} (|\nabla u|^2 + V_\varepsilon u^2) - Cu^{p+1} - Cu^{\mu+1} dx \\ &\geq \left\{ \frac{1}{2} - C\varepsilon^{n(p-1)/2-(p+1)} \|u\|_\varepsilon^{p-1} - C\varepsilon^{n(\mu-1)/2-(\mu+1)} \|u\|_\varepsilon^{\mu-1} \right\} \|u\|_\varepsilon^2 \\ &\geq \left\{ \frac{1}{2} - C\varepsilon^{n(p-1)/2}(5M)^{p-1} - C\varepsilon^{n(\mu-1)/2}(5M)^{\mu-1} \right\} \|u\|_\varepsilon^2. \end{aligned}$$

Thus we see that if  $\varepsilon > 0$  is sufficiently small,  $\Gamma_\varepsilon(u) \geq 0$  for  $u \in H_0^1(\Omega_{i,\varepsilon}^\delta)$  with  $\|u\|_\varepsilon \leq 5M\varepsilon^\alpha$ . This completes the proof.  $\square$

**Proposition 8.** *For all  $u \in H^1(\Omega_{i,\varepsilon})$ ,  $i \in \{0, \dots, k+m\}$ , it holds that*

$$\begin{aligned} \Gamma'_\varepsilon(P_i(u))P'_i(u)u &= \int_{\Omega_{i,\varepsilon}} |\nabla u|^2 + V_\varepsilon u^2 dx - \int_{\Omega_{i,\varepsilon}} g(x, u)u dx + \int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} |\nabla P_i(u)|^2 + V_\varepsilon (P_i(u))^2 dx \\ &\quad - \int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} f_\lambda(P_i(u))P_i(u) dx. \end{aligned}$$

**Proof.** Note that  $w = P'_i(u)u$  solves

$$\begin{cases} \Delta w - V_\varepsilon w + f'_\lambda(P_i(u))w = 0 & \text{in } \Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}, \\ w = u & \text{in } \Omega_{i,\varepsilon}, \\ w = 0 & \text{on } \partial\Omega_{i,\varepsilon}^\delta. \end{cases}$$

Since  $w - P_i(u) \in H_0^1(\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon})$ , we deduce that  $\Gamma'_\varepsilon(P_i(u))P'_i(u)u = \Gamma'_\varepsilon(P_i(u))P_i(u)$ . Then, the claim follows.  $\square$

Note that we take  $\alpha = 2\frac{\mu+1}{\mu-1} \geq 2 \max\{\frac{p+1}{p-1}, \frac{\mu+1}{\mu-1}\}$ .

**Proposition 9.** For sufficiently small  $\varepsilon > 0$ ,

$$\|I'_\varepsilon(\vec{u})\| \geq \frac{1}{2}\varepsilon^\alpha \quad \text{if } 2\varepsilon^\alpha \leq \|P_i(u_i)\|_\varepsilon \leq 3M\varepsilon^\alpha \quad \text{for some } i \in \{0, \dots, k\}.$$

**Proof.** We denote  $w_i = P_i(u_i)$ . Note that  $\lambda < b/10$ . Then, from Proposition 8, conditions (f2-1) and (f3-2), we see that

$$\begin{aligned} \Gamma'_\varepsilon(P_i(u_i))P'_i(u_i)u_i &= \int_{\Omega_{i,\varepsilon}} |\nabla u_i|^2 + V_\varepsilon u_i^2 dx - \int_{\Omega_{i,\varepsilon}} g(x, u_i)u_i dx \\ &\quad + \int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} |\nabla w_i|^2 + V_\varepsilon(w_i)^2 dx - \int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} f_\lambda(w_i)w_i dx \\ &\geq \int_{\Omega_{i,\varepsilon}} |\nabla u_i|^2 + V_\varepsilon u_i^2 dx - C \int_{\Omega_{i,\varepsilon}} u_i^{\mu+1} + u_i^{p+1} dx \\ &\quad + \int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} |\nabla w_i|^2 + V_\varepsilon(w_i)^2 dx - \lambda \int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} (w_i)^2 dx \\ &\geq \int_{\Omega_{i,\varepsilon}} |\nabla u_i|^2 + V_\varepsilon u_i^2 dx - C \int_{\Omega_{i,\varepsilon}} u_i^{\mu+1} + u_i^{p+1} dx + \frac{1}{2} \int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} |\nabla w_i|^2 + V_\varepsilon(w_i)^2 dx \\ &\geq \frac{1}{2} \|w_i\|_\varepsilon^2 - C\varepsilon^{n(p-1)/2-(p+1)} \|w_i\|_{H_0^1}^{p+1} - C\varepsilon^{n(\mu-1)/2-(\mu+1)} \|w_i\|_{H_0^1}^{\mu+1} \\ &\geq \left\{ \frac{1}{2} - C\varepsilon^{n(p-1)/2-(p+1)} \|w_i\|_\varepsilon^{p-1} - C\varepsilon^{n(\mu-1)/2-(\mu+1)} \|w_i\|_\varepsilon^{\mu-1} \right\} \|w_i\|_\varepsilon^2. \end{aligned}$$

Hence for sufficiently small  $\varepsilon > 0$ ,

$$\begin{aligned} \left\| \frac{d\Gamma_\varepsilon(P_i(u_i))}{du_i} \right\| &\geq \frac{\Gamma'_\varepsilon(P_i(u_i))P'_i(u_i)u_i}{\|u_i\|_{\varepsilon, \Omega_{i,\varepsilon}}} \\ &\geq \left\{ \frac{1}{2} - C\varepsilon^{\frac{n(p-1)}{2}-(p+1)} \|P_i(u_i)\|_\varepsilon^{p-1} - C\varepsilon^{\frac{n(\mu-1)}{2}-(\mu+1)} \|P_i(u_i)\|_\varepsilon^{\mu-1} \right\} \frac{\|P_i(u_i)\|_\varepsilon^2}{\|u_i\|_{\varepsilon, \Omega_{i,\varepsilon}}} \\ &\geq \left\{ \frac{1}{2} - C\varepsilon^{\frac{n(p-1)}{2}-(p+1)} \|P_i(u_i)\|_\varepsilon^{p-1} - C\varepsilon^{\frac{n(\mu-1)}{2}-(\mu+1)} \|P_i(u_i)\|_\varepsilon^{\mu-1} \right\} \|P_i(u_i)\|_\varepsilon \\ &\geq \frac{1}{2}\varepsilon^\alpha. \end{aligned}$$

Then, the claim follows from Proposition 4.  $\square$

**Proposition 10.** Let  $E > 0$  be a given constant. Then for sufficiently large  $R_1 > 0$ , there exists a constant  $\varepsilon_0$  such that if  $I_\varepsilon(\vec{u}) \leq E$ ,  $\varepsilon \in (0, \varepsilon_0)$ ,  $\frac{R_1}{4M} \leq \|P_i(u_i)\|_\varepsilon \leq R_1$  for some  $i = 1, \dots, k$ , then  $\|I'_\varepsilon(\vec{u})\| \geq 1$ .

**Proof.** Denoting  $w_i = P_i(u_i)$  and  $F_\lambda(t) = \int_0^t f_\lambda(s) ds$ , we see from Proposition 8 and the condition (f3-1) that

$$\begin{aligned} &(\mu + 1)\Gamma_\varepsilon(P_i(u)) - \Gamma'_\varepsilon(P_i(u))P'_i(u)u \\ &= \frac{\mu - 1}{2} \int_{\Omega_{i,\varepsilon}} |\nabla u|^2 + V_\varepsilon u^2 dx + \int_{\Omega_{i,\varepsilon}} g(x, u)u - (\mu + 1)G(x, u) dx \end{aligned}$$

$$\begin{aligned}
 & + \int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} f_\lambda(w_i)w_i - (\mu + 1)F_\lambda(w_i) dx + \frac{\mu - 1}{2} \int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} |\nabla w_i|^2 + V_\varepsilon(w_i)^2 dx \\
 & \geq \frac{\mu - 1}{2} \int_{\Omega_{i,\varepsilon}} |\nabla u|^2 + V_\varepsilon u^2 dx + \frac{\mu - 1}{2} \int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} |\nabla w_i|^2 + V_\varepsilon(w_i)^2 dx - \int_{\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}} (\mu + 1) \frac{\lambda}{2} (w_i)^2 dx \\
 & \geq \frac{\mu - 1}{4} \|w_i\|_\varepsilon^2.
 \end{aligned}$$

By Proposition 4, we see that

$$I'_\varepsilon(\vec{u})(0, \dots, 0, u_i, 0, \dots, 0) = \frac{d\Gamma_\varepsilon(P_i(u_i))}{du_i} u_i + O(e^{-c/\varepsilon}).$$

Then it follows that

$$\begin{aligned}
 \|I'_\varepsilon(\vec{u})\| & \geq \frac{-\Gamma'_\varepsilon(P_i(u_i))P'_i(u_i)u_i + O(e^{-c/\varepsilon})}{\|(0, \dots, 0, u_i, 0, \dots, 0)\|_\varepsilon} \\
 & \geq \frac{\frac{\mu-1}{4} \|P_i u_i\|_\varepsilon^2 - (\mu + 1)\Gamma_\varepsilon(P_i(u_i)) + O(e^{-c/\varepsilon})}{\|(0, \dots, 0, u_i, 0, \dots, 0)\|_\varepsilon} \\
 & \geq \frac{\frac{\mu-1}{4^3 M^2} (R_1)^2 - (\mu + 1)I_\varepsilon(\vec{u}) + O(e^{-c/\varepsilon})}{R_1} \\
 & \geq \frac{\frac{\mu-1}{4^3 M^2} (R_1)^2 - (\mu + 1)E + O(e^{-c/\varepsilon})}{R_1}.
 \end{aligned}$$

Thus we conclude that for sufficiently large  $R_1 > 0$ , there exists  $\varepsilon_0 > 0$  satisfying  $\|I'_\varepsilon(\vec{u})\| \geq 1$ ,  $\varepsilon \in (0, \varepsilon_0)$ .  $\square$

**Proposition 11.** *Suppose that (f3-1) holds. Then, there exists  $h_i \in H_0^1(\Omega_{i,\varepsilon}^{-\delta}) \setminus \{0\}$  and  $t_i > 0$  for each  $i \in \{1, \dots, k\}$  such that  $\Gamma_\varepsilon(th_i) < 0$  for  $t > t_i$  and small  $\varepsilon > 0$ .*

**Proof.** Note that for  $u \in H_0^1(\Omega_{i,\varepsilon}^{-\delta})$ ,

$$\Gamma_\varepsilon(P_i(tu)) = \Gamma_\varepsilon(tu) \leq \frac{M^2 t^2}{2} \int_{\Omega_{i,\varepsilon}} |\nabla u|^2 + V_\varepsilon u^2 dx - \int_{\Omega_{i,\varepsilon}} H(tu) dx.$$

Here  $H$  is defined as in (1.8). By condition (f3-1) and the definition of  $\tilde{f}$ , there exist constants  $C > 0$ ,  $\mu > 1$  such that  $H(t) \geq Ct^{\mu+1}$  for all  $t \geq 1$ . This implies that for any nonnegative function  $u_i \in H_0^1(\Omega_{i,\varepsilon}) \setminus \{0\}$ , there exists  $t_i > 0$  satisfying

$$\Gamma_\varepsilon(tu_i) < 0, \quad t \geq t_i.$$

Then, the claim follows.  $\square$

**Proposition 12.** *For each  $\varepsilon > 0$ ,  $\Gamma_\varepsilon$  satisfies the (PS) condition.*

**Proof.** Note from the definition of  $g$  that  $(\mu + 1)G(x, t) \leq g(x, t)t$  for  $x \in \bigcup_{i=0}^{k+m} \Omega_{i,\varepsilon}^{-\delta}$ ,  $g(x, t)t - (\mu + 1)G(x, t) \geq -\frac{(\mu+1)\lambda}{2} t^2$  for  $x \notin \bigcup_{i=0}^{k+m} \Omega_{i,\varepsilon}^{-\delta}$ . Then, we can follow the same scheme as in the proof of Lemma 1.1 in [16].  $\square$

### 3. Proof of Theorem 1

We use the same notations as in Section 2 with  $m = 0$ . By the assumptions,  $m_1 = \dots = m_l = 0$  and  $m_{l+1}, \dots, m_k > 0$ . Recall that  $M > 1$  is a constant, independent of small  $\varepsilon > 0$ , such that

$$\|u_i\|_{\varepsilon, \Omega_{i,\varepsilon}} \leq \|P_i(u_i)\|_\varepsilon \leq M \|u_i\|_{\varepsilon, \Omega_{i,\varepsilon}} \quad \text{for all } u_i \in H^1(\Omega_{i,\varepsilon}), \quad i \in \{0, \dots, k\}.$$

Now we take  $\alpha \equiv 2\frac{\mu+1}{\mu-1} \geq 2 \max\{\frac{\mu+1}{\mu-1}, \frac{\mu+1}{\mu-1}\}$ . By Proposition 6, for each  $i \in \{0, \dots, k\}$ , it holds that for small  $\varepsilon > 0$ ,

$$\inf\{\Gamma_\varepsilon(u); u \in H_0^1(\Omega_{i,\varepsilon}^\delta), \|u\|_\varepsilon = 4M\varepsilon^\alpha\} \geq 6M^2\varepsilon^{2\alpha}.$$

By Proposition 11, we can take a nonnegative function  $h_i \in H_0^1(\Omega_{i,\varepsilon})$  such that for some  $t_i(h_i) > 0$ ,  $\Gamma_\varepsilon(P_i(th_i)) = \Gamma_\varepsilon(th_i) < 0$  for  $t \geq t_i(h_i)$  and each  $i \in \{1, \dots, k\}$ . Let  $G_\varepsilon^i = \{\gamma \in C([0, 1], H_0^1(\Omega_{i,\varepsilon}^\delta)) \mid \gamma(0) = 0, \gamma(1) = h_i\}$  and

$$c_\varepsilon^i := \inf_{\gamma \in G_\varepsilon^i} \max_{t \in [0,1]} \Gamma_\varepsilon(P_i(\gamma(t))) \geq 6M^2\varepsilon^{2\alpha}, \quad i \in \{1, \dots, k\}. \tag{3.1}$$

Some functions  $h_i^\varepsilon, i = 1, \dots, k$  were introduced in Section 1 when we define  $C_\varepsilon^i$ . Now we take  $h_i^\varepsilon(x) = h_i(x/\varepsilon)$ . Then, it is obvious that  $C_\varepsilon^i \geq \varepsilon^n c_\varepsilon^i$  for each  $i = 1, \dots, k$ . By Proposition 5, there exists a critical point  $u_i \in H^1(\Omega_{i,\varepsilon})$  of  $\Gamma_\varepsilon^i = \Gamma_\varepsilon \circ P_i$  which satisfying

$$\Delta P_i(u_i^\varepsilon) - V_\varepsilon P_i(u_i^\varepsilon) + g(x, P_i(u_i^\varepsilon)) = 0 \quad \text{in } \Omega_{i,\varepsilon}^\delta.$$

For each  $i \in \{1, \dots, k\}$ , let  $S_\varepsilon^i$  the set of critical points  $u \in H^1(\Omega_{i,\varepsilon})$  of  $\Gamma_\varepsilon^i \equiv \Gamma_\varepsilon \circ P_i$  with  $\Gamma_\varepsilon^i(u_i^\varepsilon) = c_\varepsilon^i$ . We define  $\Pi_\varepsilon^i$  the set of solution  $u \in H^1(\Omega_{i,\varepsilon})$  of  $\Gamma_\varepsilon^i$  with  $\Gamma_\varepsilon^i(u) \leq c_\varepsilon^i$ .

Then we see the following result.

**Proposition 13.** *For each  $i \in \{1, \dots, k\}$ ,  $\Pi_\varepsilon^i$  is compact, and for  $i \in \{1, \dots, l\}$ ,  $\limsup_{\varepsilon \rightarrow 0} \max\{\|u\|_\varepsilon \mid u \in \Pi_\varepsilon^i\} = 0$ , and for  $i \in \{l + 1, \dots, k\}$ ,  $\limsup_{\varepsilon \rightarrow 0} \max\{\|u\|_\varepsilon \mid u \in \Pi_\varepsilon^i\} < \infty$ . Furthermore, for each  $d > 0$ , there exist constants  $C, c > 0$  such that  $P_i(u_i^\varepsilon)(x) \leq C \exp(-\frac{c}{\varepsilon} \text{dist}(\varepsilon x, A_i^d))$  for  $i \in \{1, \dots, k\}$  and  $u_i^\varepsilon \in \Pi_\varepsilon^i$ .*

**Proof.** The compactness follows from Proposition 5.

It follows by the same argument as in the proof of Proposition 4 in [9] that for  $j = 1, \dots, l$ ,  $\limsup_{\varepsilon \rightarrow 0} \Gamma_\varepsilon^j(u_\varepsilon^j) = 0$  and for  $j = l + 1, \dots, k$ ,  $\limsup_{\varepsilon \rightarrow 0} \Gamma_\varepsilon^j(u_\varepsilon^j) \leq L_{m_j}(U)$  for  $U \in S_{m_j}$ . We see from (f3-1) and the definition of  $\tilde{f}$  that

$$\begin{aligned} \Gamma_\varepsilon^j(u_\varepsilon^j) &\geq \frac{1}{2} \int_{\Omega_{j,\varepsilon}^\delta} |\nabla P_j(u_\varepsilon^j)|^2 + V_\varepsilon(P_j(u_\varepsilon^j))^2 dx - \frac{1}{\mu+1} \int_{\Omega_{j,\varepsilon}} g(x, P_j(u_\varepsilon^j)) P_j(u_\varepsilon^j) dx \\ &\quad - \frac{\lambda}{2} \int_{\Omega_{j,\varepsilon}^\delta \setminus \Omega_{j,\varepsilon}^{-\delta}} (P_j(u_\varepsilon^j))^2 dx \\ &= \left(\frac{1}{2} - \frac{1}{\mu+1}\right) \int_{\Omega_{j,\varepsilon}^\delta} |\nabla P_j(u_\varepsilon^j)|^2 + V_\varepsilon(P_j(u_\varepsilon^j))^2 dx \\ &\quad + \frac{1}{\mu+1} \int_{\Omega_{j,\varepsilon}^\delta \setminus \Omega_{j,\varepsilon}} g(x, P_j(u_\varepsilon^j)) P_j(u_\varepsilon^j) dx - \frac{\lambda}{2} \int_{\Omega_{j,\varepsilon}^\delta \setminus \Omega_{j,\varepsilon}^{-\delta}} (P_j(u_\varepsilon^j))^2 dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu+1}\right) \int_{\Omega_{j,\varepsilon}^\delta} |\nabla P_j(u_\varepsilon^j)|^2 + V_\varepsilon(P_j(u_\varepsilon^j))^2 dx - \frac{\lambda}{2} \int_{\Omega_{j,\varepsilon}^\delta \setminus \Omega_{j,\varepsilon}^{-\delta}} (P_j(u_\varepsilon^j))^2 dx. \end{aligned}$$

Thus, we see from the condition  $\lambda < \min\{\frac{b}{10}, b(\frac{1}{2} - \frac{1}{\mu+1})\}$  that

$$\begin{aligned} \Gamma_\varepsilon^j(u_\varepsilon^j) &\geq \left(\frac{1}{2} - \frac{1}{\mu+1}\right) \int_{\Omega_{j,\varepsilon}^\delta} |\nabla P_j(u_\varepsilon^j)|^2 + V_\varepsilon(P_j(u_\varepsilon^j))^2 dx - \frac{\lambda}{2b} \int_{\Omega_{j,\varepsilon}^\delta \setminus \Omega_{j,\varepsilon}^{-\delta}} V_\varepsilon(P_j(u_\varepsilon^j))^2 dx \\ &\geq \frac{1}{2} \left(\frac{1}{2} - \frac{1}{\mu+1}\right) \int_{\Omega_{j,\varepsilon}^\delta} |\nabla P_j(u_\varepsilon^j)|^2 + V_\varepsilon(P_j(u_\varepsilon^j))^2 dx. \end{aligned}$$

Then the second claim follows.

From the elliptic estimates (see Proposition 3.5 in [3]), we see that for  $i \in \{1, \dots, l\}$ ,

$$\limsup_{\varepsilon \rightarrow 0} \max\{\|u_\varepsilon^i\|_{L^\infty} \mid u \in \Pi_\varepsilon^i\} = 0,$$

and for  $i \in \{l + 1, \dots, k\}$ ,  $\limsup_{\varepsilon \rightarrow 0} \max\{\|u\|_{L^\infty} \mid u \in \Pi_\varepsilon^i\} < \infty$ . Then, we can show as in [9] that for any small  $d > 0$ ,  $\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon^i\|_{L^\infty(\Omega_{i,\varepsilon}^\delta \setminus (A_i^d)_\varepsilon)} = 0$  uniformly with respect to  $u_\varepsilon^i \in S_\varepsilon^i$ . Then, we can show also as in [9] that for some  $c, C > 0$  that  $P_i(u_\varepsilon^i)(x) \leq C \exp(-\frac{c}{\varepsilon} \text{dist}(\varepsilon x, A_i^d))$  for  $i \in \{1, \dots, k\}$  and  $u_\varepsilon^i \in \Pi_\varepsilon^i$ . This completes the proof.  $\square$

**Proposition 14.** For each  $i \in \{1, \dots, k\}$  and small  $\varepsilon > 0$ , there exist  $\gamma_\varepsilon^i \in G_\varepsilon^i$ ,  $d_\varepsilon^i \in (c_\varepsilon^i, c_\varepsilon^i + \exp(-\frac{1}{\varepsilon}))$  and  $R_2 > 0$  such that  $\max_{t \in [0,1]} \Gamma_\varepsilon^i(\gamma_\varepsilon^i(t)) \leq d_\varepsilon^i$ , and  $\|\gamma_\varepsilon^i(t)\|_{\varepsilon, \Omega_{i,\varepsilon}} \leq R_2$  if  $\Gamma_\varepsilon^i(\gamma_\varepsilon^i(t)) \in [c_\varepsilon^i - \varepsilon^{2\alpha}, c_\varepsilon^i + \exp(-\frac{1}{\varepsilon})]$  for some  $t \in [0, 1]$ .

**Proof.** From Proposition 13, we see that  $R' \equiv \sup_{0 < \varepsilon < 1} \max_{u \in \Pi_\varepsilon^i} \|u\|_\varepsilon < \infty$ . Then, we can find  $d_\varepsilon^i \in (c_\varepsilon^i, c_\varepsilon^i + \exp(-\frac{1}{\varepsilon}))$  and  $a(\varepsilon) > 0$  such that when  $\Gamma_\varepsilon^i(u) \leq d_\varepsilon^i$  and  $\|u\|_\varepsilon \geq 2R'$ ,  $|(\Gamma_\varepsilon^i)'(u)| \geq a(\varepsilon)$ . From the definition of  $c_\varepsilon^i$ , we can find a path  $\zeta_\varepsilon^i$  such that  $\Gamma_\varepsilon^i(\zeta_\varepsilon^i(t)) \leq d_\varepsilon^i$  for  $t \in [0, 1]$ . Suppose that  $\Gamma_\varepsilon^i(\zeta_\varepsilon^i(t)) \in [c_\varepsilon^i - \varepsilon^\alpha, d_\varepsilon^i]$  and  $\|\zeta_\varepsilon^i(t)\|_\varepsilon > 2R'$  for some  $t \in (0, 1)$ . Then, by a deformation lemma, we can deform  $\zeta_\varepsilon^i$  to  $\gamma_\varepsilon^i \in G_\varepsilon^i$  such that if  $\|\gamma_\varepsilon^i(t)\|_\varepsilon \geq 2R'$  for  $t \in (0, 1)$ ,  $\Gamma_\varepsilon^i(\gamma_\varepsilon^i(t)) < c_\varepsilon^i - \varepsilon^\alpha$ . Thus, if we take  $R_2 = 2R'$ , our conclusion holds.  $\square$

For  $i \in \{l + 1, \dots, k\}$ , we denote

$$E_i = L_{m_i}(U), \quad U \in S_{m_i}.$$

Here we state a result proved in [6].

**Proposition 15.** For  $n = 1, 2$  and  $m_i > 0$ , there exist  $T > 0$  and a path  $\gamma_i : [0, T] \rightarrow H^1(\mathbb{R}^n)$ , such that

- (i)  $L_{m_i}(\gamma_i(T)) < -1$ ,  $\max_{s \in [0,T]} L_{m_i}(\gamma_i(s)) = E_i$ ;
- (ii) there exists  $T_0 \in (0, T)$  such that  $\gamma(T_0) \in S_{m_i}$  and  $L_{m_i}(\gamma_i(T_0)) = E_i$  and  $L_{m_i}(\gamma_i(t)) < E_i$  for  $\|\gamma_i(t) - \gamma_i(T_0)\| \geq \delta$ ;
- (iii) there exist  $C, c > 0$  such that for any  $t \in [0, T]$ ,

$$|\gamma_i(t)(x)| + |\nabla_x \gamma_i(t)(x)| \leq C e^{-c|x|}.$$

**Proposition 16.** For each  $i \in \{1, \dots, l\}$ ,  $\lim_{\varepsilon \rightarrow 0} c_\varepsilon^i = 0$ , and for each  $i \in \{l + 1, \dots, k\}$ ,  $\lim_{\varepsilon \rightarrow 0} c_\varepsilon^i \leq L_{m_i}(U)$  with  $U \in S_{m_i}$ .

**Proof.** The behavior  $\lim_{\varepsilon \rightarrow 0} c_\varepsilon^i = 0, i \in \{1, \dots, l\}$  can be proved by the same method as in the proof of Proposition 4 in [9].

For the proof of remained cases, we find a nonnegative function  $\phi \in C_0^\infty(\mathbb{R}^n)$  such that  $\phi(x) = 0$  for  $|x| \leq \delta/2$  and  $\phi(x) = 0$  for  $|x| \geq \delta$ . For  $x_i \in A_i$  and  $U \in S_{m_i}$ , we define

$$\gamma_\varepsilon(t)(x) = \begin{cases} \phi(\varepsilon(x - x_i))U(\frac{x-x_i}{t}) & n \geq 3, \\ \phi(\varepsilon(x - x_i))\gamma_i(t)(x) & n = 1, 2, \end{cases}$$

where  $\gamma_i$  is the curve satisfying the properties of Proposition 15. Then, we see that

$$\lim_{\varepsilon \rightarrow 0} \Gamma_\varepsilon^j(\gamma_\varepsilon(t)) = \begin{cases} \frac{t^{n-2}}{2} \int_{\mathbb{R}^n} |\nabla U|^2 dx + \frac{t^n}{2} \int_{\mathbb{R}^n} m_i U^2 dx - t^n \int_{\mathbb{R}^n} F(U) dx & \text{for } n \geq 3, \\ L_{m_i}(\gamma_i(t)) & \text{for } n = 1, 2. \end{cases}$$

Thus,

$$\lim_{\varepsilon \rightarrow 0} \max_{t \in (0, \infty)} \Gamma_\varepsilon^j(\gamma_\varepsilon(t)) \leq L_{m_i}(U).$$

This completes the proof.  $\square$

Now we define  $\vec{\gamma}(s) = (0, \gamma_\varepsilon^1(s_1), \dots, \gamma_\varepsilon^k(s_k))$  with  $s = (s_1, \dots, s_k) \in [0, 1]^k := \overbrace{[0, 1] \times \dots \times [0, 1]}^{k\text{-times}}$ . We define

$$d_\varepsilon := \max_{s \in [0, 1]^k} I_\varepsilon(\vec{\gamma}(s)).$$

It follows from  $\max_{s \in [0, 1]^k} I_\varepsilon(\vec{\gamma}(s)) = \sum_{i=1}^k \max_{s_i \in [0, 1]} \Gamma_\varepsilon^i(\gamma_\varepsilon^i(s_i)) + O(\exp(-\frac{c}{\varepsilon}))$  for some  $c > 0$ , the definition of  $c_\varepsilon^i$ , and Proposition 14 that for some  $c > 0$ ,

$$\begin{aligned} \sum_{i=1}^k c_\varepsilon^i + O\left(\exp\left(-\frac{c}{\varepsilon}\right)\right) &\leq \max_{s \in [0, 1]^k} I_\varepsilon(\vec{\gamma}(s)) \leq \sum_{i=1}^k d_\varepsilon^i + O\left(\exp\left(-\frac{c}{\varepsilon}\right)\right) \\ &\leq \sum_{i=1}^k c_\varepsilon^i + O\left(\exp\left(-\frac{c}{\varepsilon}\right)\right). \end{aligned}$$

Thus we see that  $d_\varepsilon = c_\varepsilon^1 + \dots + c_\varepsilon^k + O(e^{-c/\varepsilon})$  for some  $c > 0$ . From Proposition 16, we find that  $d_\varepsilon \leq E$  for some constant  $E$ , independent of small  $\varepsilon > 0$ .

Let  $R_1, R_2 > 0$  be constants given in Proposition 10 and Proposition 14, respectively. We define  $R = 4M \max\{R_1, R_2\}$ . In what follows, for the sake of simplicity, we write  $P_i(u)$  for  $P_i(u_i) = P_i(u|_{\Omega_{i,\varepsilon}})$ ,  $u \in H_\varepsilon$ . For each  $c, d \in \mathbb{R}$ , we define  $\Gamma_\varepsilon^c = \{u \in H_\varepsilon \mid \Gamma_\varepsilon(u) \leq c\}$  and

$$I_\varepsilon^d = \{\vec{u} \in H_\varepsilon(\Omega_0^\varepsilon) \times \dots \times H_\varepsilon(\Omega_k^\varepsilon) \mid I_\varepsilon(\vec{u}) \leq d\}.$$

Now we have the following existence result.

**Proposition 17.** *For sufficiently small  $\varepsilon > 0$ , there exists a critical point  $u_\varepsilon \in H_\varepsilon$  of  $\Gamma_\varepsilon$  such that  $u_\varepsilon \in (\Gamma_\varepsilon^{d_\varepsilon} \setminus \Gamma_\varepsilon^{d_\varepsilon - \varepsilon^{2\alpha}})$  and*

$$u_\varepsilon \in \{u \in H_\varepsilon \mid \|P_0(u)\|_\varepsilon \leq 5M\varepsilon^\alpha \text{ and } \varepsilon^\alpha \leq \|P_i(u)\|_\varepsilon \leq 2R, i = 1, \dots, k\}.$$

**Proof.** Recall from Proposition 4 that  $\vec{u}$  is a critical point of  $I_\varepsilon$  if and only if  $\varphi(\vec{u}) \in H_\varepsilon$  is a critical point of  $\Gamma_\varepsilon$ .

To the contrary, suppose that there are not such critical points of  $\Gamma_\varepsilon$ . Then, there are no critical points  $\vec{u}$  of  $I_\varepsilon$  on the set

$$Y_\varepsilon \equiv (I_\varepsilon^{d_\varepsilon} \setminus I_\varepsilon^{d_\varepsilon - \varepsilon^{2\alpha}}) \cap \{\vec{u} \mid \|P_0(u_0)\|_\varepsilon \leq 5M\varepsilon^\alpha, \varepsilon^\alpha \leq \|P_i(u)\|_\varepsilon \leq 2R \text{ for } i = 1, \dots, k\}.$$

We define

$$A_\varepsilon \equiv (I_\varepsilon^{d_\varepsilon} \setminus I_\varepsilon^{d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}}) \cap \{\vec{u} \mid \|P_0(u_0)\|_\varepsilon \leq 3M\varepsilon^\alpha, 2\varepsilon^\alpha \leq \|P_i(u)\|_\varepsilon \leq R \text{ for } i = 1, \dots, k\}.$$

Let  $\chi$  be a Lipschitz continuous function such that  $\chi(\vec{u}) = 1$  if  $\vec{u} \in A_\varepsilon$ , and  $\chi(\vec{u}) = 0$  if  $I_\varepsilon(\vec{u}) \leq d_\varepsilon - \varepsilon^{2\alpha}$  or

$$\vec{u} \notin \{\vec{u} \mid \|P_0(u_0)\|_\varepsilon \leq 5M\varepsilon^\alpha, \varepsilon^\alpha \leq \|P_i(u)\|_\varepsilon \leq 2R \text{ for } i = 1, \dots, k\}.$$

Let  $\vec{\gamma}(s, t)$ ,  $t \geq 0$  be the solution of

$$\frac{\partial}{\partial t} \vec{\gamma}(s, t) = -\chi(\vec{\gamma}(s, t)) \nabla I_\varepsilon(\vec{\gamma}(s, t)) / \|\nabla I_\varepsilon(\vec{\gamma}(s, t))\|, \quad \vec{\gamma}(s, 0) = \vec{\gamma}(s), \quad s \in [0, 1]^k.$$

Note that  $\vec{\gamma}$  is defined for all  $t \geq 0$ , and also that  $I_\varepsilon(\vec{\gamma}(s, t))$  is nonincreasing in  $t \geq 0$ . We shall prove that there exists  $t^* = t^*(\varepsilon) > 0$  such that

$$\max_{s \in [0, 1]^k} I_\varepsilon(\vec{\gamma}(s, t^*)) \leq d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}.$$

If  $I_\varepsilon(\vec{\gamma}(s)) < d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}$  for some  $s \in [0, 1]^k$ , then  $I_\varepsilon(\vec{\gamma}(s, t)) \leq I_\varepsilon(\vec{\gamma}(s)) \leq d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}$  for any  $t \geq 0$  by the monotonicity. Now let  $s$  be a point in  $[0, 1]^k$  such that  $I_\varepsilon(\vec{\gamma}(s)) \geq d_\varepsilon - \frac{1}{2}\varepsilon^{2\alpha}$ . Then we claim that  $\|P_i(\gamma_i(s))\|_\varepsilon > 3M\varepsilon^\alpha$  for all  $i \in \{1, \dots, k\}$ . Indeed, if it is not true, it follows from Proposition 7 that  $|\Gamma_\varepsilon(P_i(\gamma_i(s)))| \leq 5M^2\varepsilon^{2\alpha}$  for some  $i \in \{1, \dots, k\}$ . Then, we see from (3.1) that for sufficiently small  $\varepsilon > 0$ ,



$$\begin{aligned}
 I_\varepsilon(\vec{\gamma}(s)) &= \sum_{i=1}^k \Gamma_\varepsilon(P_i(\gamma_i(s_i))) + O(e^{-c/\varepsilon}) \\
 &\leq (c_\varepsilon^1 + \dots + \check{c}_\varepsilon^i + \dots + c_\varepsilon^k) + 5M^2\varepsilon^{2\alpha} + O(e^{-c/\varepsilon}) \\
 &\leq d_\varepsilon - c_\varepsilon^i + 5M^2\varepsilon^{2\alpha} + O(e^{-c/\varepsilon}) \\
 &< d_\varepsilon - 6M^2\varepsilon^{2\alpha} + 5M^2\varepsilon^{2\alpha} + O(e^{-c/\varepsilon}) \\
 &< d_\varepsilon - \varepsilon^{2\alpha}.
 \end{aligned}$$

This contradicts the above condition  $I_\varepsilon(\vec{\gamma}(s)) \geq d_\varepsilon - \frac{1}{2}\varepsilon^{2\alpha}$ , and the claim follows. Now from the definition of  $R$ , we see that  $\vec{\gamma}(s) \in A_\varepsilon$  if  $I_\varepsilon(\vec{\gamma}(s)) \geq d_\varepsilon - \frac{1}{2}\varepsilon^{2\alpha}$ .

By our assumption and (PS) condition proved in Proposition 4(ii), we see that for given  $\varepsilon > 0$ , there exists a positive constant  $c(\varepsilon) > 0$  such that  $\|I'_\varepsilon(\vec{u})\| \geq c(\varepsilon) > 0$  if  $\vec{u} \in Y_\varepsilon$ . Hence there exists a number  $t^* = t^*(\varepsilon) > 0$  such that if  $\gamma(s, t) \in A_\varepsilon$  for all  $0 \leq t \leq t^*$ , then

$$I_\varepsilon(\vec{\gamma}(s, t^*)) \leq d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}.$$

Hence we need only prove that if  $\vec{\gamma}(s, t_1) \notin A_\varepsilon$  for some  $t_1 \in (0, t^*]$ , then

$$I_\varepsilon(\vec{\gamma}(s, t_1)) \leq d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}. \tag{3.2}$$

Assume that  $\vec{\gamma}(s, t_1) \notin A_\varepsilon$  for some  $t_1 \in (0, t^*]$ . Then there holds one of the following

- (C0)  $\|P_0(\gamma_0(s, t_1))\|_\varepsilon \geq 3M\varepsilon^\alpha$ ;
- (C1)  $\|P_i(\gamma_i(s, t_1))\|_\varepsilon < 2\varepsilon^\alpha$  for some  $i \in \{1, \dots, k\}$ ;
- (C2)  $\|P_i(\gamma_i(s, t_1))\|_\varepsilon > R/2$  for some  $i \in \{1, \dots, k\}$ ;
- (C3)  $I_\varepsilon(\vec{\gamma}(s, t_1)) < d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}$ .

The case (C3) implies (3.2).

In case (C1), it follows from  $\|P_i(\gamma_i(s))\|_\varepsilon > 3M\varepsilon^\alpha$  and  $\|P_i(\gamma_i(s, t_1))\|_\varepsilon < 2\varepsilon^\alpha$  that  $\|\gamma_i(s)\|_{\varepsilon, \Omega_{i,\varepsilon}} > 3\varepsilon^\alpha$ ,  $\|\gamma_i(s, t_1)\|_{\varepsilon, \Omega_{i,\varepsilon}} < 2\varepsilon^\alpha$  for sufficiently small  $\varepsilon > 0$ . Thus there exists an interval  $[t_2, t_3] \subset [0, t_1]$  such that  $\|\gamma_i(s, t_2)\|_{\varepsilon, \Omega_{i,\varepsilon}} = 3\varepsilon^\alpha$ ,  $\|\gamma_i(s, t_3)\|_{\varepsilon, \Omega_{i,\varepsilon}} = 2\varepsilon^\alpha$  and  $2\varepsilon^\alpha \leq \|\gamma_i(s, t)\|_{\varepsilon, \Omega_{i,\varepsilon}} \leq 3\varepsilon^\alpha$  for all  $t \in [t_2, t_3]$ . Then we see that

$$\begin{aligned}
 \varepsilon^\alpha &\leq \|\vec{\gamma}(s, t_3) - \vec{\gamma}(s, t_2)\|_{\varepsilon, \Omega_{i,\varepsilon}} = \left\| \int_{t_2}^{t_3} \frac{\partial \vec{\gamma}}{\partial \tau}(s, \tau) d\tau \right\|_{\varepsilon, \Omega_{i,\varepsilon}} \\
 &= \left\| \int_{t_2}^{t_3} \chi(\vec{\gamma}(s, \tau)) \nabla I_\varepsilon(\vec{\gamma}(s, \tau)) / \|\nabla I_\varepsilon(\vec{\gamma}(s, \tau))\| d\tau \right\|_{\varepsilon, \Omega_{i,\varepsilon}} \leq \int_{t_2}^{t_3} \chi(\vec{\gamma}(s, \tau)) d\tau.
 \end{aligned}$$

Moreover since  $\|I'_\varepsilon(\vec{u})\|_\varepsilon \geq \frac{1}{2}\varepsilon^\alpha$  for  $2\varepsilon^\alpha \leq \|P_i(u_i)\|_\varepsilon \leq 3M\varepsilon^\alpha$  by (i) of Proposition 9, we see that for sufficiently small  $\varepsilon > 0$ ,

$$\begin{aligned}
 I_\varepsilon(\vec{\gamma}(s, t_1)) &= I_\varepsilon(\vec{\gamma}(s)) + \int_0^{t_1} I'_\varepsilon(\vec{\gamma}(s, \tau)) \frac{\partial \vec{\gamma}}{\partial \tau}(s, \tau) d\tau \\
 &= I_\varepsilon(\vec{\gamma}(s)) - \int_0^{t_1} \chi(\vec{\gamma}(s, \tau)) \|I'_\varepsilon(\vec{\gamma}(s, \tau))\| d\tau \\
 &\leq d_\varepsilon - \frac{1}{2}\varepsilon^\alpha \int_{t_2}^{t_3} \chi(\vec{\gamma}(s, \tau)) d\tau \leq d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}.
 \end{aligned}$$

We get (3.2) as desired.

In case (C0), it follows from  $\|P_0(\gamma_0(s))\|_\varepsilon = 0$  and  $\|P_0(\gamma_0(s, t_1))\|_\varepsilon \geq 3M\varepsilon^\alpha$  that there exists an interval  $[t_2, t_3] \subset [0, t_1]$  such that  $\|\gamma_0(s, t_2)\|_{\varepsilon, \Omega_0^\varepsilon} = 3\varepsilon^\alpha$ ,  $\|\gamma_0(s, t_3)\|_{\varepsilon, \Omega_0^\varepsilon} = 2\varepsilon^\alpha$  and  $2\varepsilon^\alpha \leq \|\gamma_0(s, t)\|_{\varepsilon, \Omega_0^\varepsilon} \leq 3\varepsilon^\alpha$  for all  $t \in [t_2, t_3]$ . Then, by the same procedure as in the proof of case (C1), we get (3.2).

In case (C2), we assume that  $\|P_i(\gamma_i(s, t_1))\|_\varepsilon > \frac{R}{2}$  for some  $i \in \{1, \dots, k\}$ . From the choice of  $R$ , it follows that for sufficiently small  $\varepsilon > 0$ ,  $\|\gamma_i(s)\|_{\varepsilon, \Omega_i^\varepsilon} \leq \frac{R}{4M}$  and  $\|\gamma_i(s, t_1)\|_{\varepsilon, \Omega_i^\varepsilon} > \frac{R}{2M}$ . Thus there exists an interval  $[t_2, t_3] \subset [0, t_1]$  such that  $\|\gamma_i(s, t_2)\|_{\varepsilon, \Omega_{i,\varepsilon}} = \frac{R}{4M}$ ,  $\|\gamma_i(s, t_3)\|_{\varepsilon, \Omega_{i,\varepsilon}} = \frac{R}{2M}$  and  $\frac{R}{4M} \leq \|\gamma_i(s, t)\|_{\varepsilon, \Omega_{i,\varepsilon}} \leq \frac{R}{2M}$  for all  $t \in [t_2, t_3]$ . We then have  $\frac{R}{4M} \leq \|P_i(\gamma_i(s, t))\|_\varepsilon \leq \frac{R}{2}$  for all  $t \in [t_2, t_3]$ . We also have  $\|P_i(\gamma_i(s, t))\|_\varepsilon \leq R$  for all  $i \in \{1, \dots, k\}$  and  $t \geq 0$  due to the cut-off function  $\chi$ . Note from Proposition 10 that  $\|I'_\varepsilon(\bar{u})\| \geq 1$  if  $\frac{R}{4M} \leq \|P_i(u_i)\|_\varepsilon \leq R$  for some  $i \in \{1, \dots, k\}$  and  $\|P_i(u_i)\|_\varepsilon \leq R$  for all  $i \in \{0, \dots, k\}$ . Then, by a similar procedure as in the case (C1), we get (3.2) for sufficiently small  $\varepsilon > 0$ . Therefore we conclude that

$$\max_{s \in [0, 1]^k} I_\varepsilon(\vec{\gamma}(s, t^*)) \leq d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}. \tag{3.3}$$

We note that

$$I_\varepsilon(\vec{\gamma}(s, t^*)) = \sum_{i=0}^k \Gamma_\varepsilon(P_i(\gamma_i(s, t^*))) + O(e^{-c/\varepsilon}),$$

and from Proposition 7 that

$$\Gamma_\varepsilon(P_0(\gamma_0(s, t^*))) \geq 0.$$

Thus, it follows that

$$I_\varepsilon(\vec{\gamma}(s, t^*)) \geq \sum_{i=1}^k \Gamma_\varepsilon(P_i(\gamma_i(s, t^*))) + O(e^{-c/\varepsilon}). \tag{3.4}$$

Furthermore, if  $s \in \partial[0, 1]^k$ , we have  $\vec{\gamma}(s, t) = \vec{\gamma}(s)$  for all  $t \geq 0$  since it follows from (3.1) that for sufficiently small  $\varepsilon > 0$ ,

$$\begin{aligned} I_\varepsilon(\vec{\gamma}(s)) &= \sum_{i=1}^k \Gamma_\varepsilon(P_i(\gamma_i(s_i))) + O(e^{-c/\varepsilon}) \\ &\leq \max_i (c_\varepsilon^1 + \dots + c_\varepsilon^i + \dots + c_\varepsilon^k) + O(e^{-c/\varepsilon}) \\ &\leq d_\varepsilon - \varepsilon^{2\alpha}. \end{aligned}$$

The result [13, Proposition 3.4] of Coti Zelati and Rabinowitz says that

$$\max_{s \in [0, 1]^k} \sum_{i=1}^k \Gamma_\varepsilon(P_i(\gamma_i(s, t^*))) \geq c_\varepsilon^1 + \dots + c_\varepsilon^k = d_\varepsilon + O(e^{-c/\varepsilon}).$$

This contradicts (3.3) and (3.4), and hence completes the proof.  $\square$

**Completion of Proof of Theorem 1.** From Proposition 17, there exists a solution  $u_\varepsilon$  satisfying  $\Delta u_\varepsilon - V_\varepsilon(x)u_\varepsilon + g(x, u_\varepsilon) = 0$  for  $x \in \mathbb{R}^n$ . It also follows from the strong maximum principle that  $u_\varepsilon > 0$ . By elliptic estimates (see Proposition 3.5 in [3]), we see that  $\{\|u_\varepsilon\|_{L^\infty}\}_{\varepsilon < 1}$  is bounded. Then by elliptic estimates, we see that for some  $\alpha \in (0, 1)$ ,  $\{\|u_\varepsilon\|_{C^{1,\alpha}}\}_{\varepsilon < 1}$  is bounded.

From the definition of  $\varphi$  on  $\mathbb{R}^n \setminus \bigcup_{i=0}^k \Omega_{i,\varepsilon}$ , we see that  $u_\varepsilon(x) \rightarrow 0$  uniformly with respect to small  $\varepsilon > 0$  as  $\text{dist}(x, \bigcup_{i=0}^k \Omega_{i,\varepsilon}) \rightarrow \infty$ . Then we can show by a comparison principle that for some  $C, c > 0$ ,  $u_\varepsilon(x) \leq C \exp(-c \text{dist}(x, \bigcup_{i=0}^k \Omega_{i,\varepsilon}))$ .

Note that  $\{\|u_\varepsilon\|_\varepsilon, \Gamma_\varepsilon(u_\varepsilon)\}_{\varepsilon < 1}$  is bounded. Then, it is standard to see (refer [24, Proposition 2.2]) that there exists  $\{y_1^\varepsilon, \dots, y_\ell^\varepsilon\} \subset \bigcup_{i=\ell+1}^k \bar{\Omega}_{i,\varepsilon}$  satisfying

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} u_\varepsilon(y_a^\varepsilon) &> 0 \quad (a = 1, \dots, e), \\ \liminf_{\varepsilon \rightarrow 0} \min\{|y_a^\varepsilon - y_b^\varepsilon|, 1 \leq a \neq b \leq e\} &= \infty, \end{aligned}$$

and

$$\lim_{\text{dist}(x, \{\Omega_{0,\varepsilon}, \dots, \Omega_{l,\varepsilon}, y_1^\varepsilon, \dots, y_e^\varepsilon\}) \rightarrow \infty} u_\varepsilon(x) = 0.$$

Then we deduce that for some  $C, c > 0$ ,

$$u_\varepsilon(x) \leq C \exp(-c \text{dist}(x, \{\Omega_{0,\varepsilon}, \dots, \Omega_{l,\varepsilon}, y_1^\varepsilon, \dots, y_e^\varepsilon\})). \tag{3.5}$$

Taking a subsequence if it is necessary, we can assume that  $\lim_{\varepsilon \rightarrow 0} \varepsilon y_a^\varepsilon = y_a \in \overline{\bigcup_{i=l+1}^k \Omega_i}$  for each  $a = 1, \dots, e$ . Suppose that for some  $i \in \{l+1, \dots, k\}$ ,  $\liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty(\Omega_{i,\varepsilon})} = 0$ . Then, from the definition of  $\varphi(\bar{u}_\varepsilon)$ , it is standard to see that for some  $C, c > 0$ ,  $\|u_\varepsilon\|_{L^\infty(\Omega_{i,\varepsilon})} \leq C \exp(-c/\varepsilon)$ . This implies that some  $C, c > 0$ ,  $\Gamma_\varepsilon(P_i(u_\varepsilon)) \leq C \exp(-c/\varepsilon)$ . This contradicts to the fact  $\|P_i(u_\varepsilon)\|_\varepsilon \geq \varepsilon^\alpha$ . Thus we see that for each  $i \in \{l+1, \dots, k\}$ , there exists  $a \in \{1, \dots, e\}$  with  $y_a \in \overline{\Omega_i}$ . Then, defining  $v_\varepsilon^i(x) \equiv u_\varepsilon(x + y_a^\varepsilon)$ , we see that  $v_\varepsilon$  converges locally uniformly to some  $v$  in  $C^1(\mathbb{R}^n)$  which satisfies

$$\Delta v - V(y_a)v + g(y_a, v) = 0, \quad v > 0 \quad \text{in } \mathbb{R}^n, \quad \lim_{|x| \rightarrow \infty} v(x) = 0.$$

We define

$$L(v; y_a) \equiv \frac{1}{2} \int_{\mathbb{R}^n} |\nabla v|^2 + V(y_a)v^2 dx - \int_{\mathbb{R}^n} G(y_a, v) dx,$$

where  $G(y_a, v) = \int_0^v g(y_a, s) ds \leq F(v)$ . Then, from Proposition 15 for  $n = 1, 2$  and by defining  $v_t(x) = v(x/t), t > 0$  for  $n \geq 3$ , we see that there exists a continuous path  $v_t, t \geq 0$  satisfying  $v_0 = 0, v_1 = v$  and  $\limsup_{t \rightarrow \infty} L(v_t; y_a) < 0, \max_{t \in (0, \infty)} L(v_t; y_a) = L(v; y_a)$ . Then, we see that

$$\begin{aligned} L(v; y_a) &= \max_{t \in (0, \infty)} L(v_t; y_a) \\ &\geq \frac{1}{2} \int_{\mathbb{R}^n} |\nabla v_t|^2 + V(y_a)(v_t)^2 dx - \int_{\mathbb{R}^n} G(y_a, v_t) dx \\ &\geq \frac{1}{2} \int_{\mathbb{R}^n} |\nabla v_t|^2 + V(y_a)(v_t)^2 dx - \int_{\mathbb{R}^n} F(v_t) dx. \end{aligned}$$

Then it follows from Pohozaev identity that for  $V \in S_{V(y_a)}$ ,

$$\begin{aligned} L_{V(y_a)}(V) &\leq \max_{t \in (0, \infty)} \left[ \frac{1}{2} \int_{\mathbb{R}^n} |\nabla v_t|^2 + V(y_a)(v_t)^2 dx - \int_{\mathbb{R}^n} F(v_t) dx \right] \\ &\leq L(v; y_a). \end{aligned}$$

We note that if  $c < d$ ,  $L_c(V) < L_d(W)$  for  $V \in S_c$  and  $W \in S_d$  (refer to [9, Proposition 5]). Furthermore, from condition (f3-1), the decaying property (3.5) and the fact  $f_\lambda(t) \leq \lambda t$  with  $\lambda < \min\{b/10, b(1/2 - 1/(\mu + 1))\}$ , we deduce that for some  $C, c > 0$ ,

$$\begin{aligned} &\frac{1}{2} \int_{\bigcup_{i=0}^l \Omega_{i,\varepsilon}^\delta} |\nabla u_\varepsilon|^2 + V_\varepsilon u_\varepsilon^2 dx - \int_{\bigcup_{i=0}^l \Omega_{i,\varepsilon}^\delta} G(x, u_\varepsilon) dx \\ &\geq \frac{1}{2} \int_{\bigcup_{i=0}^l \Omega_{i,\varepsilon}^\delta} |\nabla u_\varepsilon|^2 + V_\varepsilon u_\varepsilon^2 dx - \frac{1}{\mu + 1} \int_{\bigcup_{i=0}^l \Omega_{i,\varepsilon}^\delta} g(x, u_\varepsilon) u_\varepsilon dx \end{aligned}$$

$$\begin{aligned}
 & - \int_{\bigcup_{i=0}^l (\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}^{-\delta})} (1 - \chi(\varepsilon x)) \left( F_\lambda(u_\varepsilon) - \frac{1}{\mu + 1} f_\lambda(u_\varepsilon) u_\varepsilon \right) dx \\
 \geq & \frac{1}{2} \int_{\bigcup_{i=0}^l \Omega_{i,\varepsilon}^\delta} |\nabla u_\varepsilon|^2 + V_\varepsilon u_\varepsilon^2 dx - \frac{1}{\mu + 1} \int_{\bigcup_{i=0}^l \Omega_{i,\varepsilon}^\delta} g(x, u_\varepsilon) u_\varepsilon dx - \frac{\lambda}{2} \int_{\bigcup_{i=0}^l (\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}^{-\delta})} (u_\varepsilon)^2 dx \\
 \geq & \left( \frac{1}{2} - \frac{1}{\mu + 1} \right) \int_{\bigcup_{i=0}^l \Omega_{i,\varepsilon}^\delta} |\nabla u_\varepsilon|^2 + V_\varepsilon u_\varepsilon^2 dx - \frac{\lambda}{2} \int_{\bigcup_{i=0}^l (\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}^{-\delta})} (u_\varepsilon)^2 dx + O(e^{-c/\varepsilon}).
 \end{aligned}$$

Then we get

$$\frac{1}{2} \int_{\bigcup_{i=0}^l \Omega_{i,\varepsilon}^\delta} |\nabla u_\varepsilon|^2 + V_\varepsilon u_\varepsilon^2 dx - \int_{\bigcup_{i=0}^l \Omega_{i,\varepsilon}^\delta} G(x, u_\varepsilon) dx \geq \frac{1}{3} \left( \frac{1}{2} - \frac{1}{\mu + 1} \right) \int_{\bigcup_{i=0}^l \Omega_{i,\varepsilon}^\delta} |\nabla u_\varepsilon|^2 + V_\varepsilon u_\varepsilon^2 dx.$$

Thus, reordering the index  $\{a\}$  in  $\{y_a^\varepsilon\}$  if it is necessary, we conclude from Proposition 16 that  $e = k - l$ ,  $y_a \in A_{l+a}$  and

$$\lim_{\varepsilon \rightarrow 0} \int_{\bigcup_{i=0}^l \Omega_{i,\varepsilon}^\delta} |\nabla u_\varepsilon|^2 + V_\varepsilon u_\varepsilon^2 dx = 0.$$

This implies that for each  $d > 0$ ,

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty(\{x|\varepsilon x \notin (\bigcup_{i=l+1}^k A_i)^d\})} = 0. \tag{3.6}$$

Then we see that  $v_\varepsilon(x) \equiv u_\varepsilon(x/\varepsilon)$  satisfies (1.3). From comparison principle, we deduce that for each  $\sigma > 0$ , there exists constants  $C, c > 0$  satisfying

$$v_\varepsilon(x) \leq C \exp(-c \operatorname{dist}(x, (A_0 \cup \dots \cup A_k)^\sigma) / \varepsilon). \tag{3.7}$$

From (3.7), the fact  $\|P_0(u_\varepsilon)\|_\varepsilon \leq 5M\varepsilon^\alpha$  and an  $L^\infty$ -elliptic estimate [23, Theorem 8.25], we see that for some  $C > 0$ ,  $\|u_\varepsilon\|_{L^\infty(\Omega_0^\varepsilon)} \leq C\varepsilon^\alpha$ . Then we see that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-2/(\mu-1)} \|u_\varepsilon\|_{L^\infty(\Omega_0^\varepsilon)} \leq \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2/(\mu-1)} \varepsilon^{2\frac{\mu+1}{\mu-1}} = \lim_{\varepsilon \rightarrow 0} \varepsilon^{2\mu/(\mu-1)} = 0.$$

By the same method as in the proof of [9, Proposition 10], we see that since  $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-2/(\mu-1)} \|u_\varepsilon\|_{L^\infty(\Omega_{0,\varepsilon})} = 0$ , there exists  $C', c' > 0$  satisfying

$$\|u_\varepsilon\|_{L^\infty(\Omega_{0,\varepsilon})} \leq C' \exp(-c'/\varepsilon) \quad \text{for sufficiently small } \varepsilon > 0.$$

By the same method as in the proof of [9, Proposition 10], we see that if  $\liminf_{\varepsilon \rightarrow 0} \varepsilon^{-2/(\mu-1)} \|u_\varepsilon\|_{L^\infty(\Omega_{i,\varepsilon})} = 0$ ,  $i \in \{1, \dots, k\}$ , there exists  $C', c' > 0$  satisfying  $\|u_\varepsilon\|_{L^\infty(\Omega_{i,\varepsilon})} \leq C' \exp(-c'/\varepsilon)$  for some sufficiently small  $\varepsilon > 0$ . This means that for some  $C, c > 0$ ,  $\|P_i(u_\varepsilon)\|_\varepsilon \leq C \exp(-c/\varepsilon)$ ; this contradicts that  $\|P_i(u_\varepsilon)\|_\varepsilon \geq \varepsilon^\alpha$  for each  $i \in \{1, \dots, k\}$ . Thus we see that  $\liminf_{\varepsilon \rightarrow 0} \varepsilon^{-2/(1-\mu)} \|u_\varepsilon\|_{L^\infty(\Omega_{i,\varepsilon})} > 0$  for each  $i \in \{1, \dots, k\}$ . This proves (i) and (ii) of Theorem 1.

Now we assume that  $l = k$  and (f3-2) is also satisfied. We now estimate  $\Gamma_\varepsilon(u_\varepsilon; \Omega_{i,\varepsilon})$ . From Proposition 17 and (2.2), we have

$$\sum_{i=0}^k \Gamma_\varepsilon(P_i(u_\varepsilon)) \leq d_\varepsilon + O(e^{-\varepsilon/c}). \tag{3.8}$$

Since  $\Gamma_\varepsilon(P_0(u_\varepsilon)) \geq 0$ , we see that

$$\sum_{i=1}^k \Gamma_\varepsilon(P_i(u_\varepsilon)) \leq d_\varepsilon + O(e^{-\varepsilon/c}). \tag{3.9}$$

We define

$$\Gamma_\varepsilon(u) \equiv \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 + V_\varepsilon(x)u^2 dx - \int_{\mathbb{R}^n} F(u) dx.$$

Then we see from (3.6) and (3.7) that  $\Gamma_\varepsilon(u_\varepsilon) = \Gamma_\varepsilon(u_\varepsilon)$  and  $\Gamma_\varepsilon(P_i(u_\varepsilon)) = \Gamma_\varepsilon(P_i(u_\varepsilon))$ . By the definition of  $c_i^\varepsilon$ , we see that for each  $i \in \{1, \dots, k\}$ ,

$$c_i^\varepsilon \leq \max_{t \geq 0} \Gamma_\varepsilon(P_i(tu_\varepsilon)) \leq \max_{t \geq 0} \Gamma_\varepsilon(tP_i(u_\varepsilon)).$$

By the monotonicity of  $\tilde{f}(t)/t$ , we see that  $g_\varepsilon(t) := \Gamma_\varepsilon(tP_i(u_\varepsilon))$ ,  $t \geq 0$  has a unique maximum point. Let  $g_\varepsilon(t_\varepsilon) = \max_{t \geq 0} g_\varepsilon(t)$ .

Now we show  $t_\varepsilon = 1 + O(e^{-c/\varepsilon})$ . Indeed,

$$\begin{aligned} g'_\varepsilon(1) &= \Gamma'_\varepsilon(P_i(u_\varepsilon))P_i(u_\varepsilon) \\ &= \int_{\Omega_{i,\varepsilon}} |\nabla u_\varepsilon|^2 + V_\varepsilon(u_\varepsilon)^2 - f(u_\varepsilon)u_\varepsilon dx + O(e^{-c/\varepsilon}) \\ &= O(e^{-c/\varepsilon}). \end{aligned}$$

Here we used the fact that  $P_i(u_\varepsilon) = O(e^{-c/\varepsilon})$  in  $\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}$ .

Moreover we estimate for  $i = 1, \dots, k$ ,

$$\begin{aligned} g''_\varepsilon(1) &= \Gamma''_\varepsilon(P_i(u_\varepsilon))[P_i(u_\varepsilon), P_i(u_\varepsilon)] \\ &= \int_{\Omega_{i,\varepsilon}} |\nabla u_\varepsilon|^2 + V_\varepsilon(u_\varepsilon)^2 - f'(u_\varepsilon)(u_\varepsilon)^2 dx \\ &\leq \int_{\Omega_{i,\varepsilon}} |\nabla u_\varepsilon|^2 + V_\varepsilon(u_\varepsilon)^2 - \mu_3 f(u_\varepsilon)u_\varepsilon dx + O(e^{-c/\varepsilon}) \\ &= (1 - \mu_3) \int_{\Omega_{i,\varepsilon}^\delta} |\nabla u_\varepsilon|^2 + V_\varepsilon(u_\varepsilon)^2 + O(e^{-c/\varepsilon}) \\ &\leq -(\mu_3 - 1)\varepsilon^{2\alpha}, \end{aligned}$$

and so  $|g''_\varepsilon(1)| \geq (\mu_3 - 1)\varepsilon^{2\alpha}$ . Therefore we have  $t_\varepsilon = 1 + O(e^{-c/\varepsilon})$  as claimed.

Hence

$$c_i^\varepsilon \leq \max_{t \geq 0} \Gamma_\varepsilon(tP_i(u_\varepsilon)) = g_\varepsilon(t_\varepsilon) = g_\varepsilon(1) + O(e^{-c/\varepsilon}) = \Gamma_\varepsilon(P_i(u_\varepsilon)) + O(e^{-c/\varepsilon}). \tag{3.10}$$

Then we have

$$d_\varepsilon = \sum_{i=1}^k c_i^\varepsilon \geq \Gamma_\varepsilon(u_\varepsilon) \geq \sum_{i=1}^k \Gamma_\varepsilon(P_i(u_\varepsilon)) + O(e^{-c/\varepsilon}) \geq \sum_{i=1}^k c_i^\varepsilon + O(e^{-c/\varepsilon}).$$

This implies that for each  $i \in \{1, \dots, l\}$ ,

$$|c_i^\varepsilon - \Gamma_\varepsilon(P_i(u_\varepsilon))| \leq O(e^{-c/\varepsilon}).$$

From the decay property (3.7), we deduce that  $|c_i^\varepsilon - \Gamma_\varepsilon(u_\varepsilon; \Omega_{i,\varepsilon})| \leq O(e^{-c/\varepsilon})$  for  $i \in \{1, \dots, l\}$ . When (f3-3) holds for  $l < k$ , by the same argument, we conclude that  $|c_i^\varepsilon - \Gamma_\varepsilon(u_\varepsilon; \Omega_{i,\varepsilon})| \leq O(e^{-c/\varepsilon})$  for  $i \in \{1, \dots, k\}$ .

Finally, noting that  $C_\varepsilon^i = \varepsilon^n c_i^\varepsilon + O(e^{-c/\varepsilon})$ ,  $J_\varepsilon(v_\varepsilon; \Omega_i) = \varepsilon^n \Gamma_\varepsilon(u_\varepsilon; \Omega_{i,\varepsilon})$  for  $i \in \{1, \dots, k\}$ , we see that  $v_\varepsilon$  satisfies the desired properties.

The proof of Theorem 1 is completed.  $\square$

**4. Proof of Theorem 2**

For the notational simplicity, we only prove the case  $m = 1$ . We may assume that  $x_1 = 0$  and  $\Omega_{k+1} = B_\sigma(0)$ ,  $\sigma > 0$ . We use the same notations as in Section 2 with  $m = 1$ . In addition, we use the following notations:  $(\mathbf{u}, u_{k+1}) \in H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k+1,\varepsilon})$ ,  $\mathbf{u} = (u_0, \dots, u_k) \in H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k,\varepsilon})$ . We write  $\Omega_\varepsilon = \Omega_{k+1,\varepsilon}$ ,  $P = P_{k+1}$ . For a given  $\mathbf{u}$ , we define  $Q_\varepsilon(u_{k+1}) = I_\varepsilon(\mathbf{u}, u_{k+1})$ ,  $\psi(u_{k+1}) = \varphi(\mathbf{u}, u_{k+1})$ ,  $K_\varepsilon(u_{k+1}) = \Gamma_\varepsilon(P(u_{k+1}))$ .

In this section, for a given  $\mathbf{u}$ , we solve  $Q'_\varepsilon(u_{k+1}) = 0$  by the Lyapunov–Schmidt reduction method. In order to construct an approximate solution, we first find a solution  $\Psi_\varepsilon$  of the localized problem  $K'_\varepsilon(u) = 0$ . We then find a solution  $u_{k+1} = \Phi_\varepsilon(\mathbf{u})$  of  $Q'_\varepsilon(u_{k+1}) = 0$  in an exponentially small neighborhood of the solution  $\Psi_\varepsilon$  to the localized problem. The localized problem will be solved by the Lyapunov–Schmidt reduction method as well. To find a critical point of  $I_\varepsilon(\mathbf{u}, \Phi_\varepsilon(\mathbf{u}))$ , we consider the following functional

$$\bar{I}_\varepsilon(\mathbf{u}) \equiv I_\varepsilon(\mathbf{u}, \Phi_\varepsilon(\mathbf{u})) - K_\varepsilon(\Psi_\varepsilon).$$

Then we apply the same method as in the proof in Section 3 to find a critical point of  $\bar{I}_\varepsilon$  and finish the proof of Theorem 2. We need the non-degeneracy condition of  $D^2V(0)$  to ensure  $C^1$ -dependency of  $\Phi_\varepsilon(\mathbf{u})$ .

Let  $U$  be the radially symmetric solution of (1.5) with  $a = V(0)$  which is non-degenerate in  $H_r^{1,2}(\mathbb{R}^n)$ . Then we see that if  $\Delta\phi - V(0)\phi + f'(U)\phi \equiv 0$  for some  $\phi \in H^{1,2}(\mathbb{R}^n)$ , then  $\phi \in \text{span}\{\frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_n}\}$ . Moreover  $U$  satisfies the exponential decay estimate  $U(x) \leq Ce^{-c|x|}$  for some  $C, c > 0$ . We define  $U_{\varepsilon,y}(x) := U(x - y)$ ,  $x \in \Omega_\varepsilon = B_{\sigma/\varepsilon}(0)$ ,

$$E_{\varepsilon,y} := \left\{ w \in H^1(\Omega_\varepsilon); \left\langle w, \frac{\partial U_{\varepsilon,y}}{\partial x_i} \right\rangle_{\varepsilon,\Omega_\varepsilon} = 0, i \in \{1, \dots, n\} \right\}$$

for  $y \in B_1(0)$ . Here  $\langle u, v \rangle_{\varepsilon,\Omega_\varepsilon} \equiv \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v + V_\varepsilon(x)uv \, dx$ .

**Lemma 1.**  $Q_\varepsilon, K_\varepsilon$  is of class  $C^2$  and it holds that for  $u, h, k \in H^1(\Omega_\varepsilon)$ ,

- (i)  $Q''_\varepsilon(u)[h, k] = \Gamma''_\varepsilon(\psi(u))[\psi'(u)h, \psi'(u)k]$ ,
- (ii)  $K''_\varepsilon(u)[h, k] = \Gamma''_\varepsilon(P(u))[P'(u)h, P'(u)k]$ .

**Proof.** We prove only (i) since the proof of (ii) is similar. Since  $\psi'(u)h - \psi(h) \in X_x^\varepsilon(\vec{0})$ , there holds

$$\begin{aligned} Q'_\varepsilon(u)h &= \Gamma'_\varepsilon(\psi(u))\psi'(u)h \\ &= \Gamma'_\varepsilon(\psi(u))\psi(h). \end{aligned}$$

Hence we have

$$\begin{aligned} Q''_\varepsilon(u)[h, k] &= \Gamma''_\varepsilon(\psi(u))[\psi'(u)k, \psi(h)] \\ &= \Gamma''_\varepsilon(\psi(u))[\psi'(u)k, \psi'(u)h]. \end{aligned}$$

This completes the proof.  $\square$

We identify  $K'_\varepsilon(U_{\varepsilon,y} + \omega)$  with an element of  $H^1(\Omega_\varepsilon)$ , and  $K''_\varepsilon(U_{\varepsilon,y})$  with a linear operator on  $H^1(\Omega_\varepsilon)$  by Riesz representation theorem with respect to  $\langle \cdot, \cdot \rangle_{\varepsilon,\Omega_\varepsilon}$ . Let  $\Pi_y$  be the orthogonal projection from  $H^1(\Omega_\varepsilon)$  to  $E_{\varepsilon,y}$  with respect to  $\langle \cdot, \cdot \rangle_{\varepsilon,\Omega_\varepsilon}$ . Then we regard  $\Pi_y K''_\varepsilon(U_{\varepsilon,y})$  as a linear operator on  $E_{\varepsilon,y}$ .

**Lemma 2.** *There exists a constant  $c > 0$ , independent of small  $\varepsilon > 0$  and  $y \in B_1(0)$ , such that for all  $h \in E_{\varepsilon,y}$ , there holds*

$$\|\Pi_y K''_\varepsilon(U_{\varepsilon,y})[h]\|_{\varepsilon,\Omega_\varepsilon} \geq c\|h\|_{\varepsilon,\Omega_\varepsilon}.$$

**Proof.** Assume by contrary that there exist sequences  $\{\varepsilon_i\}$  with  $\varepsilon_i \rightarrow 0$ ,  $\{y_i\}$  and  $h_i \in E_{\varepsilon_i,y_i}$  such that  $\|h_i\|_{\varepsilon_i,\Omega_{\varepsilon_i}} = 1$  and

$$\lim_{i \rightarrow \infty} \|\Pi_{y_i} K''_{\varepsilon_i}(U_{\varepsilon_i,y_i})[h_i]\|_{\varepsilon_i,\Omega_{\varepsilon_i}} = 0.$$

For the sake of simplicity, we write  $\varepsilon = \varepsilon_i, y = y_i, h_\varepsilon = h_i$ . Set  $v_\varepsilon = P'(U_{\varepsilon,y})h_\varepsilon \in H_0^1(\Omega_\varepsilon^\delta) \subset H^1(\mathbb{R}^n)$ . By Lemma 1, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} |\nabla v_\varepsilon|^2 + V_\varepsilon(x)v_\varepsilon^2 - g'(x, P(U_{\varepsilon,y}))v_\varepsilon^2 dx = 0.$$

Since  $\|v_\varepsilon\|_\varepsilon \in [1, C]$  for some constant  $C > 1$ , we may assume that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} |\nabla v_\varepsilon|^2 + V_\varepsilon(x)v_\varepsilon^2 dx \in [1, \infty). \tag{4.1}$$

We deduce that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} g'(x, P(U_{\varepsilon,y}))v_\varepsilon^2 dx = \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} g'(x, P(U_{\varepsilon,y}))(h_\varepsilon)^2 dx.$$

We may assume that  $v_\varepsilon(\cdot + y)$  converges weakly in  $H^1(\mathbb{R}^n)$  to some function  $v \in H^1(\mathbb{R}^n)$ . Using  $\lim_{\varepsilon \rightarrow 0} \langle K_\varepsilon''(U_{\varepsilon,y})[h_\varepsilon], \Pi_y \eta(\cdot - y) \rangle_{\varepsilon, \Omega_\varepsilon} = 0$  for  $\eta \in C_0^\infty(\mathbb{R}^n)$ , and

$$\int_{\mathbb{R}^n} \nabla v \cdot \nabla \frac{\partial U}{\partial x_i} + V(0)v \frac{\partial U}{\partial x_i} - f'(U)v \frac{\partial U}{\partial x_i} dx = 0,$$

we then see that

$$\int_{\mathbb{R}^n} \nabla v \cdot \nabla \eta + V(0)v\eta - f'(U)v\eta dx = 0,$$

for all  $\eta \in C_0^\infty(\mathbb{R}^n)$ . However, since  $h_\varepsilon \in E_{\varepsilon,y}$ , we obtain that

$$\int_{\mathbb{R}^n} \nabla \frac{\partial U}{\partial x_i} \cdot \nabla v + V(0) \frac{\partial U}{\partial x_i} v dx = 0, \quad i \in \{1, \dots, n\}.$$

By the non-degeneracy condition (f4), we see that  $v = 0$ . It follows from  $v_\varepsilon(\cdot + y)$  converges weakly in  $H^1(\mathbb{R}^n)$  and strongly in  $L_{loc}^2(\mathbb{R}^n)$  to 0 that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} g'(x, P(U_{\varepsilon,y}))(h_\varepsilon)^2 dx = \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} f'(U_{\varepsilon,y})(h_\varepsilon)^2 dx = 0.$$

Then we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} |\nabla v_\varepsilon|^2 + V_\varepsilon(x)v_\varepsilon^2 dx = 0.$$

This contradicts (4.1) and completes the proof of lemma.  $\square$

**Proposition 18.** For any  $\theta \in (0, 1)$ , there exists a constant  $C, \varepsilon_0 > 0$  such that for any  $\varepsilon \in (0, \varepsilon_0)$  and  $y \in B_1(0)$ , there exist constants  $C_{i,\varepsilon,y}, i \in \{1, \dots, n\}$  and  $w_{\varepsilon,y} \in E_{\varepsilon,y} \cap H^2(\Omega_\varepsilon)$  satisfying  $K'_\varepsilon(U_{\varepsilon,y} + w_{\varepsilon,y}) = \sum_{i=1}^n C_{i,\varepsilon,y} \frac{\partial U_{\varepsilon,y}}{\partial x_i}$ ,  $\|w_{\varepsilon,y}\|_{\varepsilon, \Omega_\varepsilon} \leq C\varepsilon^{2-\theta}$ , and  $|C_{i,\varepsilon,y}| \leq C\varepsilon^{2-\theta}$ . In addition, a function  $y \mapsto (w_{\varepsilon,y}, C_{1,\varepsilon,y}, \dots, C_{n,\varepsilon,y}) \in H^1(\Omega_\varepsilon) \times \mathbb{R}^n$  is in  $C^1$ .

**Proof.** A linear operator  $\Pi_y K''_\varepsilon(U_{\varepsilon,y}) : E_{\varepsilon,y} \rightarrow E_{\varepsilon,y}$  has a bounded inverse  $[\Pi_y K''_\varepsilon(U_{\varepsilon,y})]^{-1}$  by Lemma 2. We define

$$G_\varepsilon(\omega) := -[\Pi_y K''_\varepsilon(U_{\varepsilon,y})]^{-1} [\Pi_y K'_\varepsilon(U_{\varepsilon,y} + \omega) - \Pi_y K'_\varepsilon(U_{\varepsilon,y})[\omega]]$$

for  $\omega \in E_{\varepsilon,y}$ . We shall show that there is a constant  $C' > 0$  such that  $G_\varepsilon$  has a fixed point  $\omega_{\varepsilon,y}$  on the set

$$S_\varepsilon := \{ \omega \in E_{\varepsilon,y}; \|\omega\|_{\varepsilon, \Omega_\varepsilon} \leq C'\varepsilon^{2-\theta}, \|\omega\|_{L^\infty(\Omega_\varepsilon)} \leq C'\varepsilon \}$$

for sufficiently small  $\varepsilon > 0$ . Note that  $\omega$  is a fixed point of  $G_\varepsilon$  if and only if  $\Pi_y K'_\varepsilon(U_{\varepsilon,y} + \omega) = 0$ . We claim that if we choose a sufficiently large  $C' > 0$  and small  $\varepsilon_0 > 0$ , then we have  $G_\varepsilon(\omega) \in S_\varepsilon$  for all  $\omega \in S_\varepsilon$ ,  $\varepsilon \in (0, \varepsilon_0)$  and  $\|G_\varepsilon(\omega) - G_\varepsilon(\tilde{\omega})\|_{\varepsilon, \Omega_\varepsilon} \leq \frac{1}{2} \|\omega - \tilde{\omega}\|_{\varepsilon, \Omega_\varepsilon}$  for all  $\omega, \tilde{\omega} \in S_\varepsilon$ ,  $\varepsilon \in (0, \varepsilon_0)$ .

We first estimate  $\|G_\varepsilon(\omega)\|_{\varepsilon, \Omega_\varepsilon}$  for  $\omega \in E_{\varepsilon,y}$ . We have

$$\|G_\varepsilon(\omega)\|_{\varepsilon, \Omega_\varepsilon} \leq c_0 \sup_{h \in E_{\varepsilon,y}, \|h\|_{\varepsilon, \Omega_\varepsilon} \leq 1} |K'_\varepsilon(U_{\varepsilon,y} + \omega)h - K''_\varepsilon(U_\varepsilon)[\omega, h]|$$

for some constant  $c_0 > 0$ . We denote  $g'(x, t) = \frac{\partial g}{\partial t}(x, t)$ . We note that

$$P'(U_{\varepsilon,y} + \omega)h - P'(U_{\varepsilon,y})h \in H_0^1(\Omega_\varepsilon^\delta \setminus \Omega_\varepsilon),$$

$$P(U_{\varepsilon,y} + \omega) - P(U_{\varepsilon,y})\omega - P(U_{\varepsilon,y}) \in H_0^1(\Omega_\varepsilon^\delta \setminus \Omega_\varepsilon).$$

Then, for  $h \in E_{\varepsilon,y}$ , it follows from Proposition 1 that

$$\begin{aligned} & K'_\varepsilon(U_{\varepsilon,y} + \omega)h - K''_\varepsilon(U_{\varepsilon,y})[\omega, h] \\ &= \Gamma'_\varepsilon(P(U_{\varepsilon,y} + \omega))P'(U_{\varepsilon,y} + \omega)h - \Gamma''_\varepsilon(P(U_{\varepsilon,y})) [P'(U_{\varepsilon,y})h, P'(U_{\varepsilon,y})\omega] \\ &= \Gamma'_\varepsilon(P(U_{\varepsilon,y} + \omega))P'(U_{\varepsilon,y})h - \Gamma''_\varepsilon(P(U_{\varepsilon,y})) [P'(U_{\varepsilon,y})h, P'(U_{\varepsilon,y})\omega] \\ &= \Gamma''_\varepsilon(P(U_{\varepsilon,y})) [P'(U_{\varepsilon,y})h, P(U_{\varepsilon,y} + \omega) - P'(U_{\varepsilon,y})\omega] \\ &\quad + \int_{\mathbb{R}^n} \{g'(x, P(U_{\varepsilon,y}))P(U_{\varepsilon,y} + \omega) - g(x, P(U_{\varepsilon,y} + \omega))\} P'(U_{\varepsilon,y})h \\ &= \Gamma''_\varepsilon(P(U_{\varepsilon,y})) [P'(U_{\varepsilon,y})h, P(U_{\varepsilon,y})] \\ &\quad + \int_{\mathbb{R}^n} \{g'(x, P(U_{\varepsilon,y}))P(U_{\varepsilon,y} + \omega) - g(x, P(U_{\varepsilon,y} + \omega))\} P'(U_{\varepsilon,y})h \\ &= \Gamma'_\varepsilon(P(U_{\varepsilon,y}))P'(U_{\varepsilon,y})h - \int_{\mathbb{R}^n} [g(x, P(U_{\varepsilon,y} + \omega)) - g(x, P(U_{\varepsilon,y})) \\ &\quad - g'(x, P(U_{\varepsilon,y}))\{P(U_{\varepsilon,y} + \omega) - P(U_{\varepsilon,y})\}] P'(U_{\varepsilon,y})h. \end{aligned} \tag{4.2}$$

The first term in (4.2) denotes an error of the approximate solution  $U_{\varepsilon,y}$ . For any  $\theta \in (0, 1)$ , there exist constants  $C, c > 0$  such that

$$\begin{aligned} \sup_{\substack{h \in E_{\varepsilon,y} \\ \|h\|_{\varepsilon, \Omega_\varepsilon} \leq 1}} |\Gamma'_\varepsilon(P(U_{\varepsilon,y}))P'(U_{\varepsilon,y})h| &\leq \sup_{\substack{h \in E_{\varepsilon,y} \\ \|h\|_{\varepsilon, \Omega_\varepsilon} \leq 1, \Omega_\varepsilon}} \int \nabla U_{\varepsilon,y} \cdot \nabla h + V_\varepsilon U_{\varepsilon,y} h - f(U_{\varepsilon,y})h \, dx + O(e^{-c/\varepsilon}) \\ &\leq \sup_{\substack{h \in E_{\varepsilon,y} \\ \|h\|_{\varepsilon, \Omega_\varepsilon} \leq 1, \Omega_\varepsilon}} \int (V_\varepsilon - V(0))U_{\varepsilon,y} h + O(e^{-c/\varepsilon}) \\ &\leq C \left( e^{-c\varepsilon^{-\theta/2}} + \sup_{|\varepsilon\xi - \varepsilon y| \leq \varepsilon^{1-\theta/2}} |V(\varepsilon\xi) - V(0)| \right) \\ &\leq C\varepsilon^{2-\theta}. \end{aligned}$$

Here we divided the integral into the parts in  $B_{\varepsilon^{-\theta/2}}(y)$  and  $\Omega_\varepsilon \setminus B_{\varepsilon^{-\theta/2}}(y)$ , and used the exponential decay of  $U_{\varepsilon,y}$ .

The remaining terms in (4.2) come from the nonlinearity of  $g$ . Using the uniform continuity of  $g'$  on compact sets, that is,

$$\limsup_{\delta \rightarrow 0} \{ |g'(x, s) - g'(x, t)| \mid s, t \in [0, d], |s - t| \leq \delta \} = 0 \tag{4.3}$$

for any constant  $d > 0$ , we deduce that there exists a positive constant  $c_1, C_1$ , independent of small  $\varepsilon > 0$  such that if  $\|\omega\|_{L^\infty} \leq c_1$  then

$$\|G_\varepsilon(\omega)\|_{\varepsilon, \Omega_\varepsilon} \leq C_1\varepsilon^{2-\theta} + \frac{1}{2} \|\omega\|_{\varepsilon, \Omega_\varepsilon}. \tag{4.4}$$

Then, taking  $C' \geq 2C_1$ , we see that  $\|G_\varepsilon(\omega)\|_{\varepsilon, \Omega_\varepsilon} \leq C'\varepsilon^{2-\theta}$ .



Similarly we deduce that if  $\|\omega\|_{L^\infty(\Omega_\varepsilon)} \leq c_1$  then

$$\|G_\varepsilon(\omega_1) - G_\varepsilon(\omega_2)\|_{\varepsilon, \Omega_\varepsilon} \leq \frac{1}{2} \|\omega_1 - \omega_2\|_{\varepsilon, \Omega_\varepsilon}.$$

Assume that  $\|\omega\|_{\varepsilon, \Omega_\varepsilon} \leq C'\varepsilon$ . We denote  $\tilde{\omega} \equiv G_\varepsilon(\omega)$  and we estimate  $\|\tilde{\omega}\|_{L^\infty(\Omega_\varepsilon)}$ . Then there exist some constants  $a_i$  for  $i \in \{1, \dots, n\}$  satisfying

$$-K''_\varepsilon(U_{\varepsilon, y})[\tilde{\omega}, h] = K'_\varepsilon(U_{\varepsilon, y} + \omega)h - K''_\varepsilon(U_{\varepsilon, y})[\omega, h] + \left\langle \sum_{i=1}^n a_i \frac{\partial U_{\varepsilon, y}}{\partial x_i}, h \right\rangle_{\varepsilon, \Omega_\varepsilon} \tag{4.5}$$

for all  $h \in H^1(\Omega_\varepsilon)$ . Substituting  $h = \frac{\partial U_{\varepsilon, y}}{\partial x_j}$ , and using that  $\|\tilde{\omega}\|_{\varepsilon, \Omega_\varepsilon} = o(\varepsilon)$  holds by (4.4), we see that  $a_i = o(\varepsilon)$ . Since there hold

$$\begin{aligned} & K'_\varepsilon(U_{\varepsilon, y} + \omega)h - K''_\varepsilon(U_{\varepsilon, y})[\omega, h] \\ &= \Gamma'_\varepsilon(P(U_{\varepsilon, y} + \omega))P'(U_{\varepsilon, y} + \omega)h - \Gamma''_\varepsilon(P(U_{\varepsilon, y})) [P'(U_{\varepsilon, y})h, P'(U_{\varepsilon, y})\omega] \\ &= \int_{\Omega_\varepsilon} \{-\Delta U_{\varepsilon, y} + V_\varepsilon U_{\varepsilon, y} - f(U_{\varepsilon, y} + \omega) + f'(U_{\varepsilon, y})\omega\} h \, dx \end{aligned}$$

and

$$-K''_\varepsilon(U_{\varepsilon, y})[\tilde{\omega}, h] = \int_{\Omega_\varepsilon} \{\Delta \tilde{\omega} - V_\varepsilon \tilde{\omega} + f'(U_{\varepsilon, y})\tilde{\omega}\} h \, dx$$

for any  $h \in H_0^1(\Omega_\varepsilon)$ , we see that  $\tilde{\omega}$  solves

$$\Delta \tilde{\omega} - V_\varepsilon \tilde{\omega} + f'(U_{\varepsilon, y})\tilde{\omega} = f_{\varepsilon, y}(x) \quad \text{in } \Omega_\varepsilon,$$

where

$$f_{\varepsilon, y}(x) \equiv -\Delta U_{\varepsilon, y} + V_\varepsilon U_{\varepsilon, y} - f(U_{\varepsilon, y} + \omega) + f'(U_{\varepsilon, y})\omega + \sum_{i=1}^n a_i (-\Delta + V_\varepsilon) \frac{\partial U_{\varepsilon, y}}{\partial x_i}.$$

Hence  $\xi = P'(U_{\varepsilon, y})\tilde{\omega}$  solves

$$\Delta \xi - V_\varepsilon \xi + g'(x, P(U_{\varepsilon, y}))\xi = \begin{cases} f_{\varepsilon, y}(x) & \text{in } \Omega_\varepsilon, \\ 0 & \text{in } \Omega_\varepsilon^\delta \setminus \Omega_\varepsilon. \end{cases}$$

Since  $a_i = o(\varepsilon)$ , we have  $|f_{\varepsilon, y}(x)| \leq o(\varepsilon) + o(\|\omega\|_{L^\infty(\Omega_\varepsilon)})$ . Note that  $\{\|V_\varepsilon - g'(x, P(U_{\varepsilon, y}))\|_{L^\infty(\Omega_\varepsilon^\delta)}\}_{\varepsilon < 1}$  is bounded and  $\|\xi\|_{L^2(\Omega_\varepsilon^\delta)} = o(\varepsilon)$ . Then, by elliptic estimates, we see that

$$\|\xi\|_{L^\infty(B_1(z))} \leq C(\|\xi\|_{L^2(B_2(z))} + \|f_{\varepsilon, y}\|_{L^\infty(B_2(z))}).$$

Hence  $\|\tilde{\omega}\|_{L^\infty(\Omega_\varepsilon)} \leq \|\xi\|_{L^\infty(\Omega_\varepsilon)} \leq C_2\varepsilon + \frac{1}{2}\|\omega\|_{L^\infty(\Omega_\varepsilon)}$  for some constant  $C_2 > 0$ , independent of  $C'$  for small  $\varepsilon > 0$ . If we take  $C' = \max\{2C_1, 2C_2\}$ , our claim follows.

By the contraction mapping theorem, there exists a unique fixed point  $w_{\varepsilon, y}$  of  $G_\varepsilon$  on  $S_\varepsilon$ . By the elliptic estimates, we see that  $w_{\varepsilon, y} \in H^2(\Omega_\varepsilon)$ . We note that  $(y, w) \mapsto \Pi_y K'_\varepsilon(U_{\varepsilon, y} + w)$  is of class  $C^1$ , and that

$$\lim_{\varepsilon \rightarrow 0} \|g'(x, P(U_{\varepsilon, y}))\|_{L^\infty(\Omega_\varepsilon^\delta \setminus \Omega_\varepsilon)} = \lim_{\varepsilon \rightarrow 0} \|g'(x, P(U_{\varepsilon, y} + w_{\varepsilon, y}))\|_{L^\infty(\Omega_\varepsilon^\delta \setminus \Omega_\varepsilon)} = 0.$$

Then, for  $h, k \in E_{\varepsilon, y}$ ,

$$\begin{aligned} \|K''_\varepsilon(U_{\varepsilon, y} + w)[h, k] - K''_\varepsilon(U_{\varepsilon, y})[h, k]\|_{\varepsilon, \Omega_\varepsilon} &\leq \int_{\Omega_\varepsilon} |(g'(x, P(U_{\varepsilon, y})) - g'(x, P(U_{\varepsilon, y})))hk| \\ &= o(1)\|h\|_{\varepsilon, \Omega_\varepsilon} \|k\|_{\varepsilon, \Omega_\varepsilon} \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

This and Lemma 2 imply  $\Pi_y K_\varepsilon''(U_{\varepsilon,y} + w_{\varepsilon,y}) \neq 0$ . Then, it follows from the implicit function theorem that  $y \mapsto w_{\varepsilon,y}$  is of  $C^1$ .

Substituting  $\omega = w_{\varepsilon,y}$ ,  $h = \frac{\partial U_{\varepsilon,y}}{\partial x_i}$ , and  $a_i = -C_{i,\varepsilon,y}$  in (4.5), we see that  $C_{i,\varepsilon,y} = O(\varepsilon^{2-\theta})$ . This completes the proof.  $\square$

We define  $\Psi_{\varepsilon,y} := U_{\varepsilon,y} + w_{\varepsilon,y}$ .

**Lemma 3.** *There exists a constant  $c > 0$ , independent of small  $\varepsilon > 0$  and  $y \in B_1(0)$ , such that for all  $h \in E_{\varepsilon,y}$ , there holds*

$$\|\Pi_y Q_\varepsilon''(\Psi_{\varepsilon,y})[h]\|_{\varepsilon,\Omega_\varepsilon} \geq c \|h\|_{\varepsilon,\Omega_\varepsilon}.$$

**Proof.** The proof is similar to one of Lemma 2, and so omitted.  $\square$

**Proposition 19.** *For any  $R > 0$ , there exist  $C, c, \varepsilon_0 > 0$  such that for any  $y \in B_1(0)$ ,  $\varepsilon \in (0, \varepsilon_0)$  and  $\mathbf{u} \in H^1(\Omega_{0,\varepsilon}) \times \dots \times H^1(\Omega_{k,\varepsilon})$  with  $\|\mathbf{u}\|_\varepsilon \leq R$ , there exist a unique  $z_{\varepsilon,y}(\mathbf{u}) \in E_{\varepsilon,y} \cap H^2(\Omega_\varepsilon)$  and constants  $\tilde{C}_{i,\varepsilon,y}(\mathbf{u})$ ,  $i \in \{1, \dots, n\}$  satisfying  $Q_\varepsilon'(\Psi_{\varepsilon,y} + z_{\varepsilon,y}(\mathbf{u})) = \sum_{i=1}^n \tilde{C}_{i,\varepsilon,y}(\mathbf{u}) \frac{\partial U_{\varepsilon,y}}{\partial x_i}$  and  $\|z_{\varepsilon,y}(\mathbf{u})\|_{\varepsilon,\Omega_\varepsilon} \leq Ce^{-c/\varepsilon}$ . In addition,  $(y, \mathbf{u}) \mapsto (z_{\varepsilon,y}(\mathbf{u}), \tilde{C}_{1,\varepsilon,y}(\mathbf{u}), \dots, \tilde{C}_{n,\varepsilon,y}(\mathbf{u})) \in H^1(\Omega_\varepsilon) \times \mathbb{R}^n$  is of  $C^1$ .*

**Proof.** The proof is almost similar to one of Proposition 18. We regard  $\Psi_{\varepsilon,y}$  as an approximate solution and use Lemma 3. The only difference is the error estimate of the approximate solutions. Indeed, by Propositions 4(iii) and 18, we have

$$\sup_{h \in E_{\varepsilon,y}, \|h\|_{\varepsilon,\Omega_\varepsilon} \leq 1} |\Gamma_\varepsilon'(\Psi_{\varepsilon,y})\psi'(\Psi_{\varepsilon,y})h| \leq Ce^{-c/\varepsilon}.$$

We omit the details.  $\square$

**Proposition 20.** *For sufficiently small  $\varepsilon > 0$ , there exists a  $y_\varepsilon \in B_1(0)$  such that  $K_\varepsilon'(U_{\varepsilon,y_\varepsilon} + w_{\varepsilon,y_\varepsilon}) = 0$ .*

**Proof.** We need only to solve  $C_{i,\varepsilon,y} = 0$  for  $i = 1, \dots, n$ . Note that  $K_\varepsilon'(U_{\varepsilon,y} + w_{\varepsilon,y}) \frac{\partial U_{\varepsilon,y}}{\partial x_i} = 0$ ,  $i = 1, \dots, n$  is equivalent to  $C_{i,\varepsilon,y} = 0$ ,  $i = 1, \dots, n$  since  $\{\frac{\partial U_{\varepsilon,y}}{\partial x_i}\}_{i=1,\dots,n}$  is linearly independent.

Then, using  $\frac{\partial U_{\varepsilon,y}}{\partial y_i} + \frac{\partial U_{\varepsilon,y}}{\partial x_i} = 0$  and the exponential decay property of  $U$ , we deduce that

$$\begin{aligned} & K_\varepsilon'(U_{\varepsilon,y} + w_{\varepsilon,y}) \frac{\partial U_{\varepsilon,y}}{\partial x_i} \\ &= \int_{\Omega_\varepsilon} \nabla U_{\varepsilon,y} \cdot \nabla \frac{\partial U_{\varepsilon,y}}{\partial x_i} + V_\varepsilon(0)U_{\varepsilon,y} \frac{\partial U_{\varepsilon,y}}{\partial x_i} - f(U_{\varepsilon,y}) \frac{\partial U_{\varepsilon,y}}{\partial x_i} + \int_{\Omega_\varepsilon} (V_\varepsilon(x) - V_\varepsilon(0))U_{\varepsilon,y} \frac{\partial U_{\varepsilon,y}}{\partial x_i} \\ & \quad + \int_{\Omega_\varepsilon} \nabla \frac{\partial U_{\varepsilon,y}}{\partial x_i} \cdot \nabla w_{\varepsilon,y} + V(0) \frac{\partial U_{\varepsilon,y}}{\partial x_i} w_{\varepsilon,y} - f'(U_{\varepsilon,y})w_{\varepsilon,y} \frac{\partial U_{\varepsilon,y}}{\partial x_i} + \int_{\Omega_\varepsilon} (V_\varepsilon(x) - V(0)) \frac{\partial U_{\varepsilon,y}}{\partial x_i} w_{\varepsilon,y} \\ & \quad - \int_{\Omega_\varepsilon} \{f(U_{\varepsilon,y} + w_{\varepsilon,y}) - f(U_{\varepsilon,y}) - f'(U_{\varepsilon,y})w_{\varepsilon,y}\} \frac{\partial U_{\varepsilon,y}}{\partial x_i} + o(\varepsilon^2) \\ &= \int_{\Omega_\varepsilon} (V_\varepsilon(x) - V_\varepsilon(0))U_{\varepsilon,y} \frac{\partial U_{\varepsilon,y}}{\partial x_i} + \int_{\Omega_\varepsilon} (V_\varepsilon(x) - V(0)) \frac{\partial U_{\varepsilon,y}}{\partial x_i} w_{\varepsilon,y} \\ & \quad - \int_{\Omega_\varepsilon} \{f(U_{\varepsilon,y} + w_{\varepsilon,y}) - f(U_{\varepsilon,y}) - f'(U_{\varepsilon,y})w_{\varepsilon,y}\} \frac{\partial U_{\varepsilon,y}}{\partial x_i} + o(\varepsilon^2) \\ &\equiv \text{I} + \text{II} + \text{III} + o(\varepsilon^2). \end{aligned}$$

We note that  $\|w_{\varepsilon,y}\|_{\varepsilon,\Omega_\varepsilon} \leq C\varepsilon^{2-\theta}$ . Now, we can take  $\theta \in (0, 1/2)$  so that  $(1 + \gamma)(2 - \theta) > 2$ . From the property  $f \in C^{1,\gamma}$  and an exponential decay of  $U$ , we deduce that  $\text{II} = o(\varepsilon^2)$  and  $\text{III} = o(\varepsilon^2)$ . As for I, we see that

$$\text{I} = -\frac{\varepsilon}{2} \int_{B_{\sigma/\varepsilon}(-y)} \frac{\partial V}{\partial x_i}(\varepsilon(x + y))(U(x))^2 dx + O(e^{-c/\varepsilon}).$$

We define

$$s_i^0(y) \equiv -\frac{1}{2} \sum_{j=1}^n \frac{\partial^2 V}{\partial x_i \partial x_j}(0) y_j \int_{\mathbb{R}^n} U^2.$$

Then since

$$s_i^0(y) = -\frac{1}{2} \int_{\mathbb{R}^n} \sum_{j=1}^n \frac{\partial^2 V}{\partial x_i \partial x_j}(0)(x_j + y_j)(U(x))^2 dx,$$

we have

$$\begin{aligned} |I - \varepsilon^2 s_i^0(y)| &= \frac{\varepsilon}{2} \int_{B_{\sigma/\varepsilon}(-y)} \left| \sum_{j=1}^n \frac{\partial^2 V}{\partial x_i \partial x_j}(0)\varepsilon(x_j + y_j) - \frac{\partial V}{\partial x_i}(\varepsilon(x + y)) \right| (U(x))^2 + O(e^{-c/\varepsilon}) \\ &\leq C\varepsilon \int_{B_{\sigma/\varepsilon}(-y)} |\varepsilon x + \varepsilon y|^2 (U(x))^2 + O(e^{-c/\varepsilon}) \\ &\leq C\varepsilon [(\varepsilon + \varepsilon^{1-\theta})^2 + e^{-c\varepsilon^{-\theta}}] + O(e^{-c/\varepsilon}) = o(\varepsilon^2). \end{aligned}$$

Setting  $s^\varepsilon(y) = (s_1^\varepsilon(y), \dots, s_n^\varepsilon(y))$ ,  $s_i^\varepsilon(y) := \varepsilon^{-2} K'_\varepsilon(U_{\varepsilon,y} + w_{\varepsilon,y}) \frac{\partial U_{\varepsilon,y}}{\partial x_i}$ ,  $y \in B_1(0)$ , we see that

$$s_i^\varepsilon(y) \rightarrow s_i^0(y)$$

uniformly in  $y \in B_1(0)$ . Since  $D^2V(0)$  is non-degenerate, using the degree theory, we see that there exists a point  $y_\varepsilon$  such that  $s^\varepsilon(y_\varepsilon) = 0$ . Then we have  $C_{i,\varepsilon,y_\varepsilon} = 0$ . The proof is completed.  $\square$

**Proposition 21.** For any  $R > 0$ , there exist  $C, c, \varepsilon_0 > 0$  such that for any  $\mathbf{u} \in H^1(\Omega_{0,\varepsilon}) \times \dots \times H^1(\Omega_{k,\varepsilon})$ ,  $\|\mathbf{u}\|_\varepsilon \leq R$  and  $\varepsilon \in (0, \varepsilon_0)$ , there exists a unique  $\tilde{y}_\varepsilon = \tilde{y}_\varepsilon(\mathbf{u}) \in B_1(0)$  satisfying  $Q'_\varepsilon(\Psi_{\varepsilon,\tilde{y}_\varepsilon} + z_{\varepsilon,\tilde{y}_\varepsilon}(\mathbf{u})) = 0$  and  $|\tilde{y}_\varepsilon(\mathbf{u}) - y_\varepsilon| \leq Ce^{-c/\varepsilon}$ . In addition,  $\mathbf{u} \mapsto \tilde{y}_\varepsilon(\mathbf{u})$  is of  $C^1$ .

**Proof.** By Proposition 20, we have  $Q'_\varepsilon(\Psi_{\varepsilon,y_\varepsilon}) = K'_\varepsilon(\Psi_{\varepsilon,y_\varepsilon}) + O(e^{-c/\varepsilon}) = O(e^{-c/\varepsilon})$ . Hence it follows that  $Q'_\varepsilon(\Psi_{\varepsilon,y_\varepsilon} + z_{\varepsilon,y_\varepsilon}) = O(e^{-c/\varepsilon})$ . We define

$$t(y) := (t_1(y), \dots, t_n(y)), \quad t_i(y) := \varepsilon^{-2} Q'_\varepsilon(\Psi_{\varepsilon,y} + z_{\varepsilon,y}) \frac{\partial U_{\varepsilon,y}}{\partial x_i}, \quad y \in B_1(0).$$

We have  $t_i(y_\varepsilon) = O(e^{-c/\varepsilon})$ , and we will find a solution of  $t(y) = 0$  in an exponentially small neighborhood of  $y_\varepsilon$ .

For a constant  $\gamma \in (0, 1)$  given in (f4), we choose a sufficiently small  $\theta \in (0, 1)$  in Proposition 18 to satisfy  $(2 - \theta)(1 + \gamma) > 2$ .

We write  $\tilde{w}_{\varepsilon,y} = w_{\varepsilon,y} + z_{\varepsilon,y}$  and then have

$$\begin{aligned} &\frac{\partial}{\partial y_j} Q'_\varepsilon(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \frac{\partial U_{\varepsilon,y}}{\partial x_i} \\ &= \left\langle \frac{\partial U_{\varepsilon,y}}{\partial y_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}, \frac{\partial U_{\varepsilon,y}}{\partial x_i} \right\rangle_{\varepsilon,\Omega_\varepsilon} - \int_{\Omega_\varepsilon} f'(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \left( \frac{\partial U_{\varepsilon,y}}{\partial y_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j} \right) \frac{\partial U_{\varepsilon,y}}{\partial x_i} \\ &\quad + \left\langle U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}, \frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i} \right\rangle_{\varepsilon,\Omega_\varepsilon} - \int_{\Omega_\varepsilon} f(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i} + O(e^{-c/\varepsilon}) \end{aligned}$$

$$\begin{aligned}
 &= \int_{\Omega_\varepsilon} (V_\varepsilon(x) - V_\varepsilon(0)) \left\{ \frac{\partial U_{\varepsilon,y}}{\partial x_i} \left( \frac{\partial U_{\varepsilon,y}}{\partial y_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j} \right) + (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i} \right\} \\
 &\quad + \int_{\Omega_\varepsilon} \nabla \frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i} \cdot \nabla \tilde{w}_{\varepsilon,y} + V_\varepsilon(0) \frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i} \tilde{w}_{\varepsilon,y} - f'(U_{\varepsilon,y}) \frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i} \tilde{w}_{\varepsilon,y} \\
 &\quad - \int_{\Omega_\varepsilon} \{f'(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) - f'(U_{\varepsilon,y})\} \left( \frac{\partial U_{\varepsilon,y}}{\partial y_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j} \right) \frac{\partial U_{\varepsilon,y}}{\partial x_i} + o(\varepsilon^2) \\
 &\equiv E_1 + E_2 + E_3 + o(\varepsilon^2).
 \end{aligned}$$

We estimate  $E_1$  as follows:

$$\begin{aligned}
 E_1 &= \int_{\Omega_\varepsilon} (V_\varepsilon(x) - V_\varepsilon(0)) \left\{ \frac{\partial U_{\varepsilon,y}}{\partial x_i} \left( \frac{\partial U_{\varepsilon,y}}{\partial y_j} \right) + (U_{\varepsilon,y}) \frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i} \right\} + o(\varepsilon^2) \\
 &= \frac{1}{2} \int_{\Omega_\varepsilon} \frac{\partial V_\varepsilon}{\partial x_j} \frac{\partial U_{\varepsilon,y}^2}{\partial x_i} + o(\varepsilon^2) = -\frac{1}{2} \int_{\Omega_\varepsilon} \frac{\partial^2 V_\varepsilon}{\partial x_i \partial x_j} U_{\varepsilon,y}^2 + o(\varepsilon^2) \\
 &= -\frac{\varepsilon^2}{2} \frac{\partial^2 V}{\partial x_i \partial x_j}(0) \int_{\mathbb{R}^n} U^2 + o(\varepsilon^2).
 \end{aligned}$$

Next we claim that  $\| \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} \|_{\varepsilon, \Omega_\varepsilon} = O(\varepsilon^{2-\theta})$ .

We write  $\frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} = w_1 + w_2$ ,  $w_1 \in \text{span}\{ \frac{\partial U_{\varepsilon,y}}{\partial x_1}, \dots, \frac{\partial U_{\varepsilon,y}}{\partial x_n} \}$ ,  $w_2 \in E_{\varepsilon,y}$ .

Through by integration by parts, we see that

$$\left| \left\langle \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j}, \frac{\partial U_{\varepsilon,y}}{\partial x_i} \right\rangle_{\varepsilon, \Omega_\varepsilon} \right| \leq \left| \left\langle \tilde{w}_{\varepsilon,y}, \frac{\partial U_{\varepsilon,y}}{\partial x_i \partial x_j} \right\rangle_{\varepsilon, \Omega_\varepsilon} \right| + \int_{\Omega_\varepsilon} \varepsilon \frac{\partial V}{\partial x_j}(\varepsilon x) \tilde{w}_{\varepsilon,y} \frac{\partial U_{\varepsilon,y}}{\partial x_i} + O(\varepsilon^{2-\theta}) \leq O(\varepsilon^{2-\theta}).$$

Since  $\tilde{w}_{\varepsilon,y} \in E_{\varepsilon,y}$ , we have  $|\langle \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}, \frac{\partial U_{\varepsilon,y}}{\partial x_i} \rangle_{\varepsilon, \Omega_\varepsilon}| = O(\varepsilon^{2-\theta})$ . Hence  $\|w_1\|_{\varepsilon, \Omega_\varepsilon} \leq O(\varepsilon^{2-\theta})$ . On the other hand, we have

$$\begin{aligned}
 \|w_2\|_{\varepsilon, \Omega_\varepsilon} &\leq C \| \Pi_y Q''_\varepsilon(\Psi_{\varepsilon,y}) w_2 \|_{\varepsilon, \Omega_\varepsilon} \\
 &\leq C \| \Pi_y Q''_\varepsilon(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) w_2 \|_{\varepsilon, \Omega_\varepsilon} + o(1) \|w_2\|_{\varepsilon, \Omega_\varepsilon}
 \end{aligned} \tag{4.6}$$

by Lemma 3. Differentiating the both sides of the following identity

$$Q'_\varepsilon(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})h = \sum_{i=1}^n \tilde{C}_i \left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_i}, h \right\rangle_{\varepsilon, \Omega_\varepsilon}, \quad h \in H^1(\Omega_\varepsilon), \quad \tilde{C}_i = \tilde{C}_{i,\varepsilon,y}(\mathbf{u})$$

with respect to  $y_j$ , we get

$$\begin{aligned}
 &\int_{\Omega_\varepsilon} \nabla \left( \frac{\partial U_{\varepsilon,y}}{\partial y_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j} \right) \cdot \nabla h + (V_\varepsilon - f'(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})) \left( \frac{\partial U_{\varepsilon,y}}{\partial y_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j} \right) h \, dx \\
 &= \sum_{i=1}^n \tilde{C}_i \int_{\Omega_\varepsilon} \nabla \frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i} \cdot \nabla h + V_\varepsilon \frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i} h \, dx + \sum_{i=1}^n \frac{\partial \tilde{C}_i}{\partial y_j} \left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_i}, h \right\rangle_{\varepsilon, \Omega_\varepsilon} + O(e^{-c/\varepsilon})
 \end{aligned} \tag{4.7}$$

for  $\|h\|_{\varepsilon, \Omega_\varepsilon} \leq 1$ . Note that  $\tilde{w}_{\varepsilon,y} \in H^2(\Omega_\varepsilon)$ . The first integration in (4.7) equals to

$$\int_{\Omega_\varepsilon} \nabla \left( \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j} - \frac{\partial(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})}{\partial x_j} \right) \cdot \nabla h$$

$$\begin{aligned}
 & + (V_\varepsilon - f'(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})) \left( \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j} - \frac{\partial (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})}{\partial x_j} \right) h \, dx \\
 = & \int_{\Omega_\varepsilon} \nabla (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \cdot \nabla \frac{\partial h}{\partial x_j} - f(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \frac{\partial h}{\partial x_j} + V_\varepsilon (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \frac{\partial h}{\partial x_j} \\
 & + \int_{\Omega_\varepsilon} \nabla \left( \frac{\partial w}{\partial x_j} + \frac{\partial w}{\partial y_j} \right) \cdot \nabla h + (V_\varepsilon - f'(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})) \left( \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j} \right) h \\
 & + \int_{\Omega_\varepsilon} \frac{\partial V_\varepsilon}{\partial x_j} (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) h \, dx \\
 = & Q'_\varepsilon (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \frac{\partial h}{\partial x_j} + Q''_\varepsilon (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \left[ \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}, h \right] \\
 & + \int_{\Omega_\varepsilon} \frac{\partial V_\varepsilon}{\partial x_j} (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) h \, dx + O(e^{-c/\varepsilon}).
 \end{aligned}$$

The next term of the identity (4.7) equals to

$$\begin{aligned}
 & - \sum_{i=1}^n \tilde{C}_i \int_{\Omega_\varepsilon} \nabla \frac{\partial^2 U_{\varepsilon,y}}{\partial x_j \partial x_i} \cdot \nabla h + V_\varepsilon \frac{\partial^2 U_{\varepsilon,y}}{\partial x_j \partial x_i} h \, dx + \sum_{i=1}^n \frac{\partial \tilde{C}_i}{\partial y_j} \left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_i}, h \right\rangle_{\varepsilon, \Omega_\varepsilon} + O(e^{-c/\varepsilon}) \\
 = & \sum_{i=1}^n \tilde{C}_i \int_{\Omega_\varepsilon} \nabla \frac{\partial U_{\varepsilon,y}}{\partial x_i} \cdot \nabla \frac{\partial h}{\partial x_j} + V_\varepsilon \frac{\partial U_{\varepsilon,y}}{\partial x_i} \frac{\partial h}{\partial x_j} + \frac{\partial V_\varepsilon}{\partial x_j} \frac{\partial U_{\varepsilon,y}}{\partial x_i} h \, dx \\
 & + \sum_{i=1}^n \frac{\partial \tilde{C}_i}{\partial y_j} \left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_i}, h \right\rangle_{\varepsilon, \Omega_\varepsilon} + O(e^{-c/\varepsilon}) \\
 = & Q'_\varepsilon (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \frac{\partial h}{\partial x_j} + \sum_{i=1}^n \tilde{C}_i \int_{\Omega_\varepsilon} \frac{\partial V_\varepsilon}{\partial x_j} \frac{\partial U_{\varepsilon,y}}{\partial x_i} h \, dx + \sum_{i=1}^n \frac{\partial \tilde{C}_i}{\partial y_j} \left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_i}, h \right\rangle_{\varepsilon, \Omega_\varepsilon} + O(e^{-c/\varepsilon}).
 \end{aligned}$$

Therefore we get

$$\begin{aligned}
 & Q''_\varepsilon (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \left[ \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}, h \right] \\
 = & \sum_{i=1}^n \tilde{C}_i \int_{\Omega_\varepsilon} \frac{\partial V_\varepsilon}{\partial x_j} \frac{\partial U_{\varepsilon,y}}{\partial x_i} h \, dx + \sum_{i=1}^n \frac{\partial \tilde{C}_i}{\partial y_j} \left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_i}, h \right\rangle_{\varepsilon, \Omega_\varepsilon} - \int_{\Omega_\varepsilon} \frac{\partial V_\varepsilon}{\partial x_j} (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) h \, dx + O(e^{-c/\varepsilon})
 \end{aligned}$$

for  $\|h\|_{\varepsilon, \Omega_\varepsilon} \leq 1$ . We then obtain

$$\begin{aligned}
 & \left\langle \Pi_y Q''_\varepsilon (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \left[ \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}, h \right], h \right\rangle_{\varepsilon, \Omega_\varepsilon} \\
 = & \left\langle Q''_\varepsilon (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \left[ \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}, \Pi_y h \right], \Pi_y h \right\rangle_{\varepsilon, \Omega_\varepsilon} \\
 = & \sum_{i=1}^n \tilde{C}_i \int_{\Omega_\varepsilon} \frac{\partial V_\varepsilon}{\partial x_j} \frac{\partial U_{\varepsilon,y}}{\partial x_i} \Pi_y h \, dx - \int_{\Omega_\varepsilon} \frac{\partial V_\varepsilon}{\partial x_j} (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \Pi_y h \, dx + O(e^{-c/\varepsilon})
 \end{aligned}$$

for  $\|h\|_{\varepsilon, \Omega_\varepsilon} \leq 1$ ,

$$\left\| \Pi_y Q''_\varepsilon (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \left[ \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}, h \right] \right\|_{\varepsilon, \Omega_\varepsilon}$$

$$\begin{aligned}
&= \sup_{\|h\|_{\varepsilon, \Omega_\varepsilon} \leq 1} \left| \left\langle \Pi_y Q''_\varepsilon(U_{\varepsilon, y} + \tilde{w}_{\varepsilon, y}) \left[ \frac{\partial \tilde{w}_{\varepsilon, y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon, y}}{\partial y_j} \right], h \right\rangle_{\varepsilon, \Omega_\varepsilon} \right| \\
&\leq \sup_{\|h\|_{\varepsilon, \Omega_\varepsilon} \leq 1} \left| \int_{\Omega_\varepsilon} \varepsilon \frac{\partial V}{\partial x_j}(\varepsilon x)(U_{\varepsilon, y} + w_{\varepsilon, y}) \Pi_y h \, dx \right| + O(\varepsilon^{2-\theta}) \leq O(\varepsilon^{2-\theta}),
\end{aligned}$$

and hence

$$\begin{aligned}
\| \Pi_y Q''_\varepsilon(U_{\varepsilon, y} + \tilde{w}_{\varepsilon, y}) w_2 \|_{\varepsilon, \Omega_\varepsilon} &\leq \left\| \Pi_y Q''_\varepsilon(U_{\varepsilon, y} + \tilde{w}_{\varepsilon, y}) \left[ \frac{\partial \tilde{w}_{\varepsilon, y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon, y}}{\partial y_j} \right] \right\|_{\varepsilon, \Omega_\varepsilon} + O(\varepsilon^{2-\theta}) \\
&\leq O(\varepsilon^{2-\theta}).
\end{aligned}$$

Thus from (4.6), we get  $\|w_2\|_{\varepsilon, \Omega_\varepsilon} = O(\varepsilon^{2-\theta})$ . The claim follows.

Using this claim, we see that

$$E_3 = - \int_{\Omega_\varepsilon} \{ f'(U_{\varepsilon, y} + \tilde{w}_{\varepsilon, y}) - f'(U_{\varepsilon, y}) \} \left( \frac{\partial U_{\varepsilon, y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon, y}}{\partial x_j} \right) \frac{\partial U_{\varepsilon, y}}{\partial x_i} + o(\varepsilon^2).$$

In addition, from the facts

$$\int_{\Omega_\varepsilon} f'(U_{\varepsilon, y} + \tilde{w}_{\varepsilon, y}) \left( \frac{\partial U_{\varepsilon, y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon, y}}{\partial x_j} \right) \frac{\partial U_{\varepsilon, y}}{\partial x_i} = - \int_{\Omega_\varepsilon} f(U_{\varepsilon, y} + \tilde{w}_{\varepsilon, y}) \frac{\partial^2 U_{\varepsilon, y}}{\partial x_j \partial x_i} + O(e^{-c/\varepsilon})$$

and

$$\int_{\Omega_\varepsilon} f'(U_{\varepsilon, y}) \frac{\partial U_{\varepsilon, y}}{\partial x_j} \frac{\partial U_{\varepsilon, y}}{\partial x_i} = - \int_{\Omega_\varepsilon} f(U_{\varepsilon, y}) \frac{\partial^2 U_{\varepsilon, y}}{\partial x_j \partial x_i} + O(e^{-c/\varepsilon}),$$

we get that

$$\begin{aligned}
&\int_{\Omega_\varepsilon} (f'(U_{\varepsilon, y} + \tilde{w}_{\varepsilon, y}) - f'(U_{\varepsilon, y})) \frac{\partial U_{\varepsilon, y}}{\partial x_j} \frac{\partial U_{\varepsilon, y}}{\partial x_i} \\
&= - \int_{\Omega_\varepsilon} (f(U_{\varepsilon, y} + \tilde{w}_{\varepsilon, y}) - f(U_{\varepsilon, y})) \frac{\partial^2 U_{\varepsilon, y}}{\partial x_j \partial x_i} - \int_{\Omega_\varepsilon} f'(U_{\varepsilon, y} + \tilde{w}_{\varepsilon, y}) \frac{\partial U_{\varepsilon, y}}{\partial x_i} \frac{\partial \tilde{w}_{\varepsilon, y}}{\partial x_j} + O(e^{-c/\varepsilon}) \\
&= - \int_{\Omega_\varepsilon} f'(U_{\varepsilon, y}) \tilde{w}_{\varepsilon, y} \frac{\partial^2 U_{\varepsilon, y}}{\partial x_j \partial x_i} - \int_{\Omega_\varepsilon} f'(U_{\varepsilon, y} + \tilde{w}_{\varepsilon, y}) \frac{\partial U_{\varepsilon, y}}{\partial x_i} \frac{\partial \tilde{w}_{\varepsilon, y}}{\partial x_j} + o(\varepsilon^2).
\end{aligned}$$

Since

$$E_2 = \left\{ \int_{\Omega_\varepsilon} \nabla \frac{\partial^2 U_{\varepsilon, y}}{\partial x_j \partial x_i} \cdot \nabla \tilde{w}_{\varepsilon, y} + V_\varepsilon(0) \frac{\partial^2 U_{\varepsilon, y}}{\partial x_j \partial x_i} \tilde{w}_{\varepsilon, y} - f'(U_{\varepsilon, y}) \frac{\partial^2 U_{\varepsilon, y}}{\partial x_j \partial x_i} \tilde{w}_{\varepsilon, y} \right\},$$

we see that

$$\begin{aligned}
E_2 + E_3 &= \int_{\Omega_\varepsilon} \nabla \frac{\partial^2 U_{\varepsilon, y}}{\partial x_j \partial x_i} \cdot \nabla \tilde{w}_{\varepsilon, y} + V_\varepsilon(0) \frac{\partial^2 U_{\varepsilon, y}}{\partial x_j \partial x_i} \tilde{w}_{\varepsilon, y} + f'(U_{\varepsilon, y}) \frac{\partial U_{\varepsilon, y}}{\partial x_i} \frac{\partial \tilde{w}_{\varepsilon, y}}{\partial x_j} + o(\varepsilon^2) \\
&= - \int_{\Omega_\varepsilon} \nabla \frac{\partial U_{\varepsilon, y}}{\partial x_i} \cdot \nabla \frac{\partial \tilde{w}_{\varepsilon, y}}{\partial x_j} + V_\varepsilon(0) \frac{\partial U_{\varepsilon, y}}{\partial x_i} \frac{\partial \tilde{w}_{\varepsilon, y}}{\partial x_j} - f'(U_{\varepsilon, y}) \frac{\partial U_{\varepsilon, y}}{\partial x_i} \frac{\partial \tilde{w}_{\varepsilon, y}}{\partial x_j} + o(\varepsilon^2) \\
&= o(\varepsilon^2).
\end{aligned}$$

We conclude that  $\frac{\partial}{\partial y_j} t_i(y) \rightarrow -\frac{1}{2} \frac{\partial^2 V}{\partial x_i \partial x_j}(0) \int_{\mathbb{R}^n} U^2 dx$ , and hence by the condition (f4), we have that  $\nabla t^\varepsilon(y)$ ,  $y \in B_1(0)$  is non-degenerate for small  $\varepsilon > 0$ . Therefore using a fixed point theorem, we see that there exists  $\tilde{y}_\varepsilon$  such that  $t(\tilde{y}_\varepsilon) = 0$  and  $|\tilde{y}_\varepsilon - y_\varepsilon| = O(e^{-c/\varepsilon})$ . Moreover by the implicit function theorem,  $\mathbf{u} \mapsto \tilde{y}_\varepsilon$  is of  $C^1$ .  $\square$

We define a  $C^1$  map  $\Phi_\varepsilon(\mathbf{u}) = \Psi_{\varepsilon, \tilde{y}_\varepsilon(\mathbf{u})} + z_{\varepsilon, \tilde{y}_\varepsilon(\mathbf{u})}(\mathbf{u})$  and a  $C^1$  functional

$$\tilde{I}_\varepsilon(\mathbf{u}) = I_\varepsilon(\mathbf{u}, \Phi_\varepsilon(\mathbf{u})) - K_\varepsilon(\Psi_{\varepsilon, y_\varepsilon}).$$

**Proposition 22.** *The following hold.*

(i) *For any  $R > 0$ , there exist constants  $C, c, \varepsilon_0 > 0$  such that*

$$\left| \tilde{I}_\varepsilon(\mathbf{u}) - \sum_{i=0}^k \Gamma_\varepsilon(P_i(u_i)) \right| \leq C e^{-c/\varepsilon}$$

*for all  $\mathbf{u} \in H^1(\Omega_{0,\varepsilon}) \times \dots \times H^1(\Omega_{k,\varepsilon})$  with  $\|\mathbf{u}\|_\varepsilon \leq R$  and  $\varepsilon \in (0, \varepsilon_0)$ .*

(ii) *The functional  $\tilde{I}_\varepsilon(\mathbf{u})$  satisfies (PS) condition.*

(iii)  *$\mathbf{u}$  is a critical point of  $\tilde{I}_\varepsilon$  if and only if  $(\mathbf{u}, \Phi_\varepsilon(\mathbf{u}))$  is a critical point of  $I_\varepsilon$ .*

**Proof.** (i) This follows from (2.2) and the fact

$$\|\Phi_\varepsilon(\mathbf{u}) - \Psi_{\varepsilon, y_\varepsilon}\|_{\varepsilon, \Omega_\varepsilon} \leq \|z_{\varepsilon, \tilde{y}_\varepsilon}(\mathbf{u})\|_{\varepsilon, \Omega_\varepsilon} = O(e^{-c/\varepsilon}).$$

We can prove (ii) and (iii) following the same argument as in the proof of (i) and (ii) in Proposition 4.  $\square$

Finally we can find a critical point  $\mathbf{u}_\varepsilon$  of  $\tilde{I}_\varepsilon$  similarly as in Section 3. Moreover it follows from the construction that  $\limsup_{\varepsilon \rightarrow 0} \|\mathbf{u}_\varepsilon\|_\varepsilon < \infty$ , and hence that  $\Phi_\varepsilon(\mathbf{u}_\varepsilon)(\cdot + \tilde{y}_\varepsilon)$  converges uniformly to  $U$  on each bounded set in  $\mathbb{R}^n$ . This proves (iv) of Theorem 2. Other properties can be proved similarly as in Section 3. The proof of Theorem 2 is completed.

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**References**

[1] A. Ambrosetti, M. Badiale, S. Cingolani, Semiclassical states of nonlinear Schrödinger equations, Arch. Ration. Mech. Anal. 140 (1997) 285–300.  
 [2] A. Ambrosetti, A. Malchiodi, S. Secchi, Multiplicity results for some nonlinear Schrödinger equations with potentials, Arch. Ration. Mech. Anal. 159 (2001) 253–271.  
 [3] J. Byeon, Existence of large positive solutions of some nonlinear elliptic equations on singularly perturbed domains, Comm. Partial Differential Equations 22 (1997) 1731–1769.  
 [4] J. Byeon, L. Jeanjean, Standing waves for nonlinear Schrödinger equations with a general nonlinearity, Arch. Ration. Mech. Anal. 185 (2007) 185–200.  
 [5] J. Byeon, L. Jeanjean, Multi-peak standing waves for nonlinear Schrödinger equations with a general nonlinearity, Discrete Contin. Dyn. Syst. 19 (2) (2007) 255–269.  
 [6] J. Byeon, L. Jeanjean, K. Tanaka, Standing waves for nonlinear Schrödinger equations with a general nonlinearity: one and two dimensional cases, Comm. Partial Differential Equations 33 (4–6) (2008) 1113–1136.  
 [7] J. Byeon, Y. Oshita, Existence of multi-bump standing waves with a critical frequency for nonlinear Schrödinger equations, Comm. Partial Differential Equations 29 (2004) 1877–1904.  
 [8] J. Byeon, Z.-Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations, Arch. Ration. Mech. Anal. 165 (2002) 295–316.  
 [9] J. Byeon, Z.-Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations, II, Calc. Var. Partial Differential Equations 18 (2003) 207–219.  
 [10] D. Cao, E. Noussair, Multi-bump standing waves with a critical frequency for nonlinear Schrödinger equations, J. Differential Equations 203 (2004) 292–312.

- [11] D. Cao, E. Noussair, S. Yan, Multiscale-bump standing waves with a critical frequency for nonlinear Schrödinger equations, *Trans. Amer. Math. Soc.* 360 (2008) 3813–3837.
- [12] D. Cao, S. Peng, Multi-bump bound states of Schrödinger equations with a critical frequency, *Math. Ann.* 336 (2006) 925–948.
- [13] V. Coti Zelati, P.H. Rabinowitz, Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials, *J. Amer. Math. Soc.* 4 (4) (1991) 693–727.
- [14] E.N. Dancer, K.Y. Lam, S. Yan, The effect of the graph topology on the existence of multipeak solutions for nonlinear Schrödinger equations, *Abstr. Appl. Anal.* 3 (1998) 293–318.
- [15] E.N. Dancer, S. Yan, On the existence of multipeak solutions for nonlinear field equations on  $\mathbb{R}^n$ , *Discrete Contin. Dyn. Syst.* 6 (2000) 39–50.
- [16] M. Del Pino, P.L. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, *Calc. Var. Partial Differential Equations* 4 (1996) 121–137.
- [17] M. Del Pino, P.L. Felmer, Semi-classical states for nonlinear Schrödinger equations, *J. Functional Analysis* 149 (1997) 245–265.
- [18] M. Del Pino, P.L. Felmer, Multi-peak bound states for nonlinear Schrödinger equations, *Ann. Inst. Henri Poincaré* 15 (1998) 127–149.
- [19] M. Del Pino, P.L. Felmer, Semi-classical states of nonlinear Schrödinger equations: a variational reduction method, *Math. Ann.* 324 (2002) 1–32.
- [20] M. Del Pino, P.L. Felmer, O.H. Miyagaki, Existence of positive bound states of nonlinear Schrödinger equations with saddle-like potential, *Nonlinear Analysis, TMA* 34 (1998) 979–989.
- [21] A. Floer, A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equations with a bounded potential, *J. Functional Analysis* 69 (1986) 397–408.
- [22] B. Gidas, W.N. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* 68 (1979) 209–243.
- [23] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, second ed., Grundlehren, vol. 224, Springer, Berlin, Heidelberg, New York, Tokyo, 1983.
- [24] C. Gui, Existence of multi-bump solutions for nonlinear Schrödinger equations via variational method, *Comm. Partial Differential Equations* 21 (1996) 787–820.
- [25] O. Kwon, Existence of multi-bump standing waves with a critical frequency for nonlinear Schrödinger equations with potentials vanishing at infinity, *Proc. Roy. Soc. Edinburgh Sect. A* 139 (4) (2009) 833–852.
- [26] K.M. Kwong, Uniqueness of positive solutions of  $\Delta u - u + u^p = 0$  in  $R^n$ , *Arch. Ration. Mech. Anal.* 105 (1989) 243–266.
- [27] X. Kang, J. Wei, On interacting bumps of semi-classical states of nonlinear Schrödinger equations, *Adv. Differential Equations* 5 (2000) 899–928.
- [28] Y.Y. Li, On a singularly perturbed elliptic equation, *Adv. Differential Equations* 2 (1997) 955–980.
- [29] P. Meystre, *Atom Optics*, Springer, 2001.
- [30] D.L. Mills, *Nonlinear Optics*, Springer, 1998.
- [31] Y.G. Oh, Existence of semiclassical bound states of nonlinear Schrödinger equations with potentials of the class  $(V)_a$ , *Comm. Partial Differential Equations* 13 (1988) 1499–1519.
- [32] Y.G. Oh, Correction to: Existence of semiclassical bound states of nonlinear Schrödinger equations with potentials of the class  $(V)_a$ , *Comm. Partial Differential Equations* 14 (1989) 833–834.
- [33] Y.G. Oh, On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential, *Comm. Math. Phys.* 131 (1990) 223–253.
- [34] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, *Z. Angew. Math. Phys.* 43 (1992) 270–291.
- [35] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Application to Differential Equations*, CBMS Regional Conference Series Math., vol. 65, AMS, 1986.
- [36] Y. Sato, Multi-peak positive solutions for nonlinear Schrödinger equations with critical frequency, *Calc. Var. Partial Differential Equations* 29 (2007) 365–395.
- [37] W. Strauss, Existence of solitary waves in higher dimensions, *Comm. Math. Phys.* 55 (1977) 149–162.
- [38] X. Wang, On concentration of positive bound states of nonlinear Schrödinger equations, *Comm. Math. Phys.* 153 (1993) 229–244.
- [39] Z.-Q. Wang, Existence and symmetry of multi-bump solutions for nonlinear Schrödinger equations, *J. Differential Equations* 159 (1999) 102–137.