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Ann. I. H. Poincaré – AN 27 (2010) 997–1016

ANNALES DE L'INSTITUT HENRI POINCARÉ ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Non-uniqueness of weak solutions for the fractal Burgers equation

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Received 26 August 2009; received in revised form 13 January 2010; accepted 13 January 2010

Available online 20 January 2010

Abstract

The notion of Kruzhkov entropy solution was extended by the first author in 2007 to conservation laws with a fractional Laplacian diffusion term; this notion led to well-posedness for the Cauchy problem in the L^∞-framework. In the present paper, we further motivate the introduction of entropy solutions, showing that in the case of fractional diffusion of order strictly less than one, uniqueness of a weak solution may fail.

Résumé

La notion de solution entropique de Kruzhkov a été étendue par Alibaud en 2007 aux lois de conservation avec un terme diffusif fractionnaire ; ceci a permis de démontrer que le prolème de Cauchy est bien posé dans le cadre L^∞. Dans cet article, on montre que si l'ordre de l'opérateur de diffusion est strictement plus petit que un, alors il peut exister plusieurs solutions faibles ; on apporte ainsi une motivation supplémentaire à l'utilisation des solutions entropiques.

MSC: 35L65; 35L67; 35L82; 35S10; 35S30

Keywords: Fractional Laplacian; Non-local diffusion; Conservation law; Lévy–Khintchine's formula; Entropy solution; Admissibility of solutions; Oleĭnik's condition; Non-uniqueness of weak solutions

1. Introduction

This paper contributes to the study of the so-called fractal/fractional Burgers equation

$$\partial_t u(t,x) + \partial_x\left(\frac{u^2}{2}\right)(t,x) + \mathcal{L}_\lambda[u](t,x) = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $\mathcal{L}_\lambda$ is the non-local operator defined for all Schwartz function $\varphi \in \mathcal{S}(\mathbb{R})$ through its Fourier transform by

$$\mathcal{F}\big(\mathcal{L}_\lambda[\varphi]\big)(\xi) := |\xi|^\lambda \mathcal{F}(\varphi)(\xi) \quad \text{with } \lambda \in (0,1); \tag{1.3}$$

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2010.01.008

i.e. $\mathcal{L}_\lambda$ denotes the fractional power of order $\lambda/2$ of the Laplacian operator $-\Delta$ with respect to (w.r.t. for short) the space variable.

This equation is involved in many different physical problems, such as overdriven detonation in gas [13] or anomalous diffusion in semiconductor growth [29]; it appeared in a number of papers, such as [5–8,17,20,21,18,1,2,22,24,15,26,27,11,4,16,23,10,12]. Recently, the notion of entropy solution has been introduced by Alibaud in [1] to show the well-posedness in the L^∞-framework, globally in time.

For $\lambda > 1$, the notion of weak solution (*i.e.* a solution in the sense of distributions; *cf.* Definition 2.4 below) is sufficient to ensure the uniqueness and stability result; see the work [17] of Droniou, Gallouët and Vovelle. This result has been generalized to the critical case $\lambda = 1$ by Kiselev, Nazarov and Shterenberg in [24], Dong, Du and Li in [15], Miao and Wu in [27] and Chan and Czubak in [11] for a large class of initial data (periodic data, or L^2 data, or data in the critical Besov space).

In this paper, we focus on the range of exponent $\lambda \in (0, 1)$. By analogy with the purely hyperbolic equation $\lambda = 0$ (*cf.* Oleĭnik [28] and Kruzhkov [25]), a natural conjecture was that a weak solution to the Cauchy problem (1.1)–(1.2) need not be unique in this case. Indeed, it has been shown by Alibaud, Droniou and Vovelle in [2] that the assumption $\lambda < 1$ makes the diffusion term too weak to prevent the appearance of discontinuities in solutions of (1.1); see also Kiselev, Nazarov and Shterenberg [24] and Dong, Du and Li [15]. To the best of our knowledge, yet it was unclear whether such discontinuities in a weak solution can violate the entropy conditions of [1].

Here we construct a stationary weak solution of (1.1)–(1.2), $\lambda < 1$, which does violate the entropy constraint (constraint that can be expressed under the form of Oleĭnik's inequality, *cf.* [28]). Thus the main result of this paper is the following.

Theorem 1.1. *Let $\lambda \in (0, 1)$. There exists initial data $u_0 \in L^\infty(\mathbb{R})$ such that uniqueness of a weak solution to the Cauchy problem* (1.1)–(1.2) *fails.*

The paper is organized as follows. The next section presents the notations, definitions and basic results on fractal conservation laws. The Oleĭnik inequality for the fractal Burgers equation is stated and proved in Section 3. In Section 4, we study an equation regularized by means of the viscous term $\varepsilon\partial^2_{xx}u$. This equation is solved in the domains $\{\pm x > 0\}$ with boundary conditions at $\{x = 0^\pm\}$ chosen in such a way that the Burgers flux is continuous but the solution is discontinuous, moreover, the jump across the line $\{x = 0\}$ is increasing (contrarily to what is needed for the Oleĭnik condition to hold). In Section 5, we pass to the limit as $\varepsilon \to 0$ and show that the increasing jump across $\{x = 0\}$ is persistent; at the limit, we obtain a non-entropy stationary weak solution to (1.1). Section 6 is devoted to the proof of the main properties of the fractional Laplacian (see Lemma 4.1) that are used in both preceding sections. Finally, technical proofs and results are gathered in Appendices A–B.

2. Preliminaries

In this section, we fix some notation, recall the Lévy–Khintchine formula for the fractional Laplacian and the associated notions of generalized solutions to fractal conservation laws.

2.1. Notations

Sets. Throughout this paper, $\mathbb{R}^\pm$ denote the sets $(-\infty, 0)$ and $(0, +\infty)$, respectively; the set $\mathbb{R}_*$ denotes $\mathbb{R} \setminus \{0\}$ and $\overline{\mathbb{R}}$ denotes $\{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$.

Right-differentiability. A function $m : \mathbb{R}^+ \to \mathbb{R}$ is said to be right-differentiable at $t_0 > 0$ if there exists the limit $\lim_{t\downarrow t_0} \frac{m(t)-m(t_0)}{t-t_0}$ in $\mathbb{R}$; the limit itself is denoted by $m'_r(t_0)$.

Function spaces. Further, $C_c^\infty = \mathcal{D}$ denotes the space of infinitely differentiable compactly supported test functions, $\mathcal{S}$ is the Schwartz space, $\mathcal{D}'$ is the space of distributions and $\mathcal{S}'$ is the space of tempered distributions. The space of k times continuously differentiable functions is denoted by C^k and C_b^k denotes the subspace of functions with bounded derivatives up to order k (if $k = 0$, the superscripts are omitted); C_c denotes the subspace of C of functions with compact support; C_0 denotes the closure of C_c for the norm of the uniform convergence; L^p, L^p_{loc}

and $W^{k,p}$, $W^{k,p}_{\mathrm{loc}}$ (if $p=2$ the latter two spaces are denoted by H^k, H^k_{loc}, respectively) stand for the classical Lebesgue and Sobolev spaces, respectively; BV and BV_{loc} denote the spaces of functions of bounded variation (with globally or locally bounded variation, respectively).

Unless the topology of a space is indicated explicitly, we mean that $\mathcal{D}'$ and $\mathcal{S}'$ are endowed with their usual weak-$\star$ topologies and the other spaces are endowed with their usual strong topologies (of Banach spaces, of Fréchet spaces, *etc.*).

Weak-$\star$ topology in BV. Let $\partial_x : \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ denote the gradient (w.r.t x) operator in the sense of distributions. We let $L^1(\mathbb{R}) \cap (BV(\mathbb{R}))_{w\text{-}\star}$ denote the linear space $L^1(\mathbb{R}) \cap BV(\mathbb{R})$ endowed with the most coarse topology such that the inclusion $L^1(\mathbb{R}) \cap BV(\mathbb{R}) \subset L^1(\mathbb{R})$ and the mapping $\partial_x : L^1(\mathbb{R}) \cap BV(\mathbb{R}) \to (C_0(\mathbb{R}))'$ are continuous, where $L^1(\mathbb{R})$ is endowed with its strong topology and $(C_0(\mathbb{R}))'$ with its weak-$\star$ topology. Hence, one has:

$$\big[v_k \to v \text{ in } L^1(\mathbb{R}) \cap \big(BV(\mathbb{R})\big)_{w\text{-}\star}\big] \quad \Longleftrightarrow \quad \begin{cases} v_k \to v & \text{in } L^1(\mathbb{R}), \\ \partial_x v_k \overset{w\text{-}\star}{\rightharpoonup} \partial_x v & \text{in } \big(C_0(\mathbb{R})\big)'. \end{cases}$$

We define in the same way the space $(BV_{\mathrm{loc}}(\mathbb{R}))_{w\text{-}\star} \cap H^1_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\})$, in which notion of convergent sequences is the following one:

$$\big[v_k \to v \text{ in } \big(BV_{\mathrm{loc}}(\mathbb{R})\big)_{w\text{-}\star} \cap H^1_{\mathrm{loc}}\big(\overline{\mathbb{R}} \setminus \{0\}\big)\big] \quad \Longleftrightarrow \quad \begin{cases} v_k \to v & \text{in } H^1\big(\mathbb{R} \setminus [-R,R]\big), \forall R>0, \\ \partial_x v_k \overset{w\text{-}\star}{\rightharpoonup} \partial_x v & \text{in } \big(C_c(\mathbb{R})\big)'. \end{cases}$$

From the Banach–Steinhaus theorem, one sees that $(v_k)_k$ is (strongly) bounded in $BV_{\mathrm{loc}}(\mathbb{R}) \cap H^1_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\})$, *i.e.*:

$$\forall R>0, \quad \sup_{k \in \mathbb{N}_*} \big(\|v_k\|_{H^1(\mathbb{R}\setminus[-R,R])} + |v_k|_{BV((-R,R))}\big) < +\infty,$$

where $|\cdot|_{BV}$ denotes the BV semi-norm.

Spaces of odd functions. In our construction, a key role is played by the spaces of odd functions v which are in the Sobolev space H^1:

$$H^1_{\mathrm{odd}} := \big\{v \in H^1 \ \big|\ v \text{ is odd}\big\};$$

notice that $v \in H^1_{\mathrm{odd}}(\mathbb{R}_*)$ can be discontinuous at zero so that $v(0^-) = -v(0^+)$ in the sense of traces, whereas $v(0^-) = v(0^+) = 0$ if $v \in H^1_{\mathrm{odd}}(\mathbb{R})$.

The space $H^1_{\mathrm{odd}}(\mathbb{R}_*)$ and, more generally, the space $H^1(\mathbb{R}_*)$ can be considered as a subspace of $L^2(\mathbb{R})$; to avoid confusion, $\partial_x v$ always denotes the gradient of v in $\mathcal{D}'(\mathbb{R})$, so that $(\partial_x v)_{|\mathbb{R}_*} \in L^2(\mathbb{R})$ is the gradient in $\mathcal{D}'(\mathbb{R}_*)$. One has $(\partial_x v)_{|\mathbb{R}_*} = \partial_x v$ almost everywhere (a.e. for short) on $\mathbb{R}$ if and only if (iff for short) v is continuous at zero; in the other case, one has $\partial_x v \notin L^1_{\mathrm{loc}}(\mathbb{R})$. When the context is clear, the products $\int_{\mathbb{R}} \varphi(\partial_x v)_{|\mathbb{R}_*}$ and $\int_{\mathbb{R}} (\partial_x v)_{|\mathbb{R}_*}(\partial_x \psi)_{|\mathbb{R}_*}$ with $\varphi \in L^2(\mathbb{R})$ and $\psi \in H^1(\mathbb{R}_*)$ are simply denoted by $\int_{\mathbb{R}_*} \varphi \partial_x v$ and $\int_{\mathbb{R}_*} \partial_x v \partial_x \psi$, respectively.

Identity and Fourier operators. By Id we denote the identity function. The Fourier transform $\mathcal{F}$ on $\mathcal{S}'(\mathbb{R})$ is denoted by $\mathcal{F}$; for explicit computations, we use the following definition on $L^1(\mathbb{R})$:

$$\mathcal{F}(v)(\xi) := \int_{\mathbb{R}} e^{-2i\pi x\xi} v(x)\,dx.$$

Entropy-flux pairs. By η, we denote a convex function on $\mathbb{R}$; following Kruzhkov [25], we call it an *entropy* and $q : u \mapsto \int_0^u s\,d\eta(s)$ is the associated *entropy flux*.

Truncation functions. The sign function is defined by:

$$u \mapsto \operatorname{sign} u := \begin{cases} \pm 1 & \text{if } \pm u > 0, \\ 0 & \text{if } u = 0. \end{cases}$$

In the proofs, we will need to regularize the truncation function $u \mapsto \min\{|u|, n\} \operatorname{sign} u$, where $n \in \mathbb{N}_*$ will be fixed; thus T_n denotes a function on $\mathbb{R}$ satisfying

$$\begin{cases} T_n \in C_b^\infty(\mathbb{R}) & \text{is odd,} \\ T_n = \mathrm{Id} & \text{on } [-n+1, n-1], \\ |T_n| \leqslant n. \end{cases} \tag{2.1}$$

2.2. Lévy–Khintchine's formula

Let $\lambda \in (0,1)$. For all $\varphi \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$, we have

$$\mathcal{L}_\lambda[\varphi](x) = -G_\lambda \int_{\mathbb{R}} \frac{\varphi(x+z) - \varphi(x)}{|z|^{1+\lambda}}\, dz, \tag{2.2}$$

where $G_\lambda = \frac{\lambda \Gamma(\frac{1+\lambda}{2})}{2\pi^{\frac{1}{2}+\lambda}\Gamma(1-\frac{\lambda}{2})} > 0$ and Γ is Euler's function, see *e.g.* [9,19] or [18, Theorem 2.1].

2.3. Entropy and weak solutions

Formula (2.2) motivates the following notion of entropy solution introduced in [1].

Definition 2.1 *(Entropy solutions).* Let $\lambda \in (0,1)$ and $u_0 \in L^\infty(\mathbb{R})$. A function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is said to be an entropy solution to (1.1)–(1.2) if for all non-negative test function $\varphi \in C_c^\infty([0,+\infty) \times \mathbb{R})$, all entropy $\eta \in C^1(\mathbb{R})$ and all $r > 0$,

$$\int_{\mathbb{R}} \eta(u_0)\varphi(0) + \int_{\mathbb{R}^+}\int_{\mathbb{R}} \big(\eta(u)\partial_t\varphi + q(u)\partial_x\varphi\big) + G_\lambda \int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{|z|>r} \eta'\big(u(t,x)\big)\frac{u(t,x+z)-u(t,x)}{|z|^{1+\lambda}}\varphi(t,x)\,dt\,dx\,dz$$
$$+ G_\lambda \int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{|z|\leqslant r} \eta\big(u(t,x)\big)\frac{\varphi(t,x+z)-\varphi(t,x)}{|z|^{1+\lambda}}\,dt\,dx\,dz \geqslant 0. \tag{2.3}$$

Remark 2.2. In the above definition, r plays the role of a cut-off parameter; taking $r > 0$ in (2.3), one avoids the technical difficulty while treating the singularity in the Lévy–Khintchine formula (by doing this, one looses some information; the information is recovered at the limit $r \to 0$). Let us refer to the recent paper of Karlsen and Ulusoy [23] for a different definition of the entropy solution, equivalent to the above one; note that the framework of [23] encompasses Lévy mixed hyperbolic/parabolic equations.

The notion of entropy solutions provides a well-posedness theory for the Cauchy problem for the fractional conservation law (1.1); the results are very similar to the ones for the classical Burgers equation (*cf. e.g.* [28,25]).

Theorem 2.3. *(See [1].) For all $u_0 \in L^\infty(\mathbb{R})$, there exists one and only one entropy solution $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ to* (1.1)–(1.2). *Moreover, $u \in C([0,+\infty); L^1_{\mathrm{loc}}(\mathbb{R}))$ (so that $u(0) = u_0$), and the solution depends continuously in $C([0,+\infty); L^1(\mathbb{R}))$ on the initial data in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.*

As explained in the introduction, the purpose of this paper is to prove that the weaker solution notion below would not ensure uniqueness.

Definition 2.4 *(Weak solutions).* Let $u_0 \in L^\infty(\mathbb{R})$. A function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is said to be a weak solution to (1.1)–(1.2) if for all $\varphi \in C_c^\infty([0,+\infty) \times \mathbb{R})$,

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(u\partial_t\varphi + \frac{u^2}{2}\partial_x\varphi - u\mathcal{L}_\lambda[\varphi]\right) + \int_{\mathbb{R}} u_0\varphi(0) = 0. \tag{2.4}$$

3. The Oleĭnik inequality

Notice that it can be easily shown that an entropy solution is also a weak one. The converse statement is false, which we will prove by constructing a weak non-entropy solution. A key fact here is the well-known Oleĭnik inequality (see [28]); in this section, we generalize it to entropy solutions of the fractal Burgers equation.

Proposition 3.1 *(Oleĭnik's inequality). Let $u_0 \in L^\infty(\mathbb{R})$. Let $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ be the entropy solution to* (1.1)–(1.2)*. Then, we have for all $t > 0$,*

$$\partial_x u(t) \leqslant \frac{1}{t} \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{3.1}$$

Remark 3.2. This result can be adapted to general uniformly convex fluxes. Moreover, we think that the Oleĭnik inequality gives a necessary and sufficient condition for a weak solution to be an entropy solution (as for pure scalar conservation laws, *cf.* [28,25]). Nevertheless, for the sake of simplicity, we only prove the above result, which is sufficient for our purpose.

In order to prove this proposition, we need the following technical result:

Lemma 3.3. *Let $v \in C^1(\mathbb{R}^+ \times \mathbb{R})$ be such that for all $b > a > 0$,*

$$\lim_{|x| \to +\infty} \sup_{t \in (a,b)} v(t,x) = -\infty. \tag{3.2}$$

Define $m(t) := \max_{x\in\mathbb{R}} v(t,x)$ and $K(t) := \arg\max_{x\in\mathbb{R}} v(t,x)$. Then m is continuous and right-differentiable on $\mathbb{R}^+$ with $m_r'(t) = \max_{x\in K(t)} \partial_t v(t,x)$.

For a proof of this result, see *e.g.* the survey book of Danskyn [14] on the min max theory; for the reader's convenience, a short proof is also given in Appendix A. We can now prove the Oleĭnik inequality.

Proof of Proposition 3.1. For $\varepsilon > 0$ consider the regularized problem

$$\partial_t u_\varepsilon + \partial_x\left(\frac{u_\varepsilon^2}{2}\right) + \mathcal{L}_\lambda[u_\varepsilon] - \varepsilon \partial_{xx}^2 u_\varepsilon = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}, \tag{3.3}$$

$$u_\varepsilon(0) = u_0 \quad \text{on } \mathbb{R}. \tag{3.4}$$

It was shown in [17] that there exists a unique solution $u_\varepsilon \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ to (3.3)–(3.4) in the sense of the Duhamel formula, and that $u_\varepsilon \in C_b^\infty((a,+\infty) \times \mathbb{R})$ for all $a > 0$. Furthermore, it has been proved in [4] that for all $T > 0$, u_ε converges to u in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$ as $\varepsilon \to 0$. Inequality (3.1) being stable by this convergence, it suffices to prove that u_ε satisfies (3.1).

To do so, let us differentiate (3.3) w.r.t. x. We get

$$\partial_t v_\varepsilon + v_\varepsilon^2 + u_\varepsilon \partial_x v_\varepsilon + \mathcal{L}_\lambda[v_\varepsilon] - \varepsilon \partial_{xx}^2 v_\varepsilon = 0, \tag{3.5}$$

with $v_\varepsilon := \partial_x u_\varepsilon$. Fix $0 < \lambda' < \lambda$ and introduce the "barrier function" $\Phi(x) := (1 + |x|^2)^{\frac{\lambda'}{2}}$. Then Φ is positive with

$$\lim_{|x| \to +\infty} \Phi(x) = +\infty; \tag{3.6}$$

moreover Φ is smooth with

$$C_\Phi := \|\partial_x \Phi\|_\infty + \left\|\partial_{xx}^2 \Phi\right\|_\infty + \left\|\mathcal{L}_\lambda[\Phi]\right\|_\infty < +\infty$$

(notice that Lemma B.2 in Appendix B ensures that $\mathcal{L}_\lambda[\Phi] \in C_b(\mathbb{R})$ is well defined by (2.2)). For $\delta > 0$ and $t > 0$, define

$$m_\delta(t) := \max_{x\in\mathbb{R}} \left\{ v_\varepsilon(t,x) - \delta \Phi(x) \right\}.$$

Define $K_\delta(t) := \arg\max_{x\in\mathbb{R}}\{v_\varepsilon(t,x) - \delta\Phi(x)\}$. This set is non-empty and compact, thanks to the regularity of v_ε and to (3.6); moreover, by Lemma 3.3, m_δ is right-differentiable w.r.t. t with

$$(m_\delta)'_r(t) = \max_{x\in K_\delta(t)} \partial_t v_\varepsilon(t,x) = \partial_t v_\varepsilon\big(t, x_\delta(t)\big)$$

for some $x_\delta(t) \in K_\delta(t)$. This point is also a global maximum point of $v_\varepsilon(t) - \delta\Phi$, so that

$$\partial_x v_\varepsilon\big(t, x_\delta(t)\big) = \delta\partial_x\Phi(x_\delta), \qquad \partial^2_{xx} v_\varepsilon\big(t, x_\delta(t)\big) \leqslant \delta\partial^2_{xx}\Phi(x_\delta) \quad \text{and} \quad \mathcal{L}_\lambda[v_\varepsilon] \geqslant \delta\mathcal{L}_\lambda[\Phi]\big(t, x_\delta(t)\big)$$

(the last inequality is easily derived from (2.2)). We deduce that

$$\big|\partial_x v_\varepsilon\big(t, x_\delta(t)\big)\big| \leqslant \delta C_\Phi, \qquad \partial^2_{xx} v_\varepsilon\big(t, x_\delta(t)\big) \leqslant \delta C_\Phi \quad \text{and} \quad \mathcal{L}_\lambda[v_\varepsilon]\big(t, x_\delta(t)\big) \geqslant -\delta C_\Phi.$$

By (3.5), we get $(m_\delta)'_r(t) + v_\varepsilon^2(t, x_\delta(t)) \leqslant C\delta$, for some constant C that only depends on ε, $\|u_\varepsilon\|_\infty$ and C_Φ. By construction $m_\delta(t) = v_\varepsilon(t, x_\delta(t)) - \delta\Phi(x_\delta(t))$, furthermore, Φ is non-negative, so that

$$(m_\delta)'_r(t) + \big(m_\delta(t) + \delta\Phi\big(x_\delta(t)\big)\big)^2 \leqslant C\delta \quad \text{and} \quad (m_\delta)'_r(t) - C\delta + \big(\max\big\{m_\delta(t), 0\big\}\big)^2 \leqslant 0.$$

Now we set $\widetilde{m}_\delta(t) := m_\delta(t) - C\delta t$. Because the function $r \in \mathbb{R} \mapsto (\max\{r, 0\})^2 \in \mathbb{R}$ is non-decreasing, we infer that $\widetilde{m}_\delta \in C(\mathbb{R}^+)$ is right-differentiable with

$$(\widetilde{m}_\delta)'_r(t) + \big(\max\big\{\widetilde{m}_\delta(t), 0\big\}\big)^2 \leqslant 0$$

for all $t > 0$. By Lemma B.1 in Appendix B, we can integrate this equation and conclude that $\widetilde{m}_\delta(t) \leqslant \frac{1}{t}$ for all $t > 0$.

Finally, it is easy to prove that $\widetilde{m}_\delta(t) = m_\delta(t) - C\delta t \to \sup_{x\in\mathbb{R}} v_\varepsilon(t,x)$ as $\delta \to 0$, so that $\sup_{x\in\mathbb{R}} \partial_x u_\varepsilon(t,x) \leqslant \frac{1}{t}$ (pointwise, for all $t > 0$). This proves (3.1) for u_ε in the place of u, and thus completes the proof of the proposition. □

4. A stationary regularized problem

The plan to show Theorem 1.1 consists in proving the existence of an odd weak stationary solution to (1.1) with a discontinuity at $x = 0$ not satisfying the Oleĭnik inequality. This non-entropy solution is constructed as limit of solutions to regularized problems, see Eqs. (4.2)–(4.3) below. This section focuses on the solvability of these problems. This is done in the second subsection; the first one lists some properties of $\mathcal{L}_\lambda$ that will be needed.

4.1. Main properties of the non-local operator

In the sequel, $\mathcal{L}_\lambda$ is always defined by the Lévy–Khintchine formula (2.2).

Lemma 4.1. *Let* $\lambda \in (0, 1)$. *The operator* $\mathcal{L}_\lambda$ *defined by the Lévy–Khintchine formula* (2.2) *enjoys the following properties*:

(i) *The operators* $\mathcal{L}_\lambda$ *and* $\mathcal{L}_{\lambda/2}$ *are continuous as operators*:
  (a) $\mathcal{L}_\lambda : C_b(\mathbb{R}_*) \cap C^1(\mathbb{R}_*) \to C(\mathbb{R}_*)$;
  (b) $\mathcal{L}_\lambda : H^1(\mathbb{R}_*) \to L^1_{\mathrm{loc}}(\mathbb{R}) \cap L^2_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\})$;
  (c) $\mathcal{L}_{\lambda/2} : H^1(\mathbb{R}_*) \to L^2(\mathbb{R})$.
  Moreover, $\mathcal{L}_\lambda$ *is sequentially continuous as an operator*:
  (d) $\mathcal{L}_\lambda : L^1(\mathbb{R}) \cap (BV(\mathbb{R}))_{w\text{-}\star} \to L^1(\mathbb{R})$.

(ii) *If* $v \in H^1(\mathbb{R}_*)$, *then the definition of* $\mathcal{L}_\lambda$ *by Fourier transform* (*see* (1.3)) *makes sense*; *more precisely,*

$$\mathcal{L}_\lambda[v] = \mathcal{F}^{-1}\big(\xi \to |\xi|^\lambda \mathcal{F}(v)(\xi)\big) \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

(iii) *For all* $v, w \in H^1(\mathbb{R}_*)$,

$$\int_{\mathbb{R}} \mathcal{L}_\lambda[v]w = \int_{\mathbb{R}} v\mathcal{L}_\lambda[w] = \int_{\mathbb{R}} \mathcal{L}_{\lambda/2}[v]\mathcal{L}_{\lambda/2}[w].$$

(iv) *If* $v \in H^1(\mathbb{R}_*)$ *is odd* (*resp. even*)*, then* $\mathcal{L}_\lambda[v]$ *is odd* (*resp. even*).

(v) *Let* $0 \not\equiv v \in C_b(\mathbb{R}_*) \cap C^1(\mathbb{R}_*)$ *be odd. Assume that* $x_* > 0$ *is an extreme point of* v *such that*

$$v(x_*) = \max_{\mathbb{R}^+} v \quad \text{and} \quad v(x_*) \geqslant 0 \quad \Big(\text{resp. } v(x_*) = \min_{\mathbb{R}^+} v \text{ and } v(x_*) \leqslant 0\Big).$$

Then, we have $\mathcal{L}_\lambda[v](x_*) > 0$ (*resp.* $\mathcal{L}_\lambda[v](x_*) < 0$).

Remark 4.2. Item (v) can be interpreted as a positive reverse maximum principle for the fractional Laplacian acting on the space of odd functions.

The proofs of these results are gathered in Section 6.

4.2. The regularized problem

Throughout this section, $\varepsilon > 0$ is a fixed parameter. Consider the space $H^1_{\text{odd}}(\mathbb{R}_*)$ with the scalar product

$$\langle v, w\rangle := \int_{\mathbb{R}_*} \big\{\varepsilon(vw + \partial_x v \partial_x w) + \mathcal{L}_{\lambda/2}[v]\mathcal{L}_{\lambda/2}[w]\big\}. \tag{4.1}$$

By the item (i)(c) of Lemma 4.1, $\langle\cdot,\cdot\rangle$ is well defined and its associated norm $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$ is equivalent to the usual $H^1(\mathbb{R}_*)$-norm; in particular, $H^1_{\text{odd}}(\mathbb{R}_*)$ is a Hilbert space.

Let us construct a solution $v \in H^1_{\text{odd}}(\mathbb{R}_*)$ to the problem

$$\varepsilon\big(v_\varepsilon - \partial^2_{xx} v_\varepsilon\big) + \partial_x\left(\frac{v_\varepsilon^2}{2}\right) + \mathcal{L}_\lambda[v_\varepsilon] = 0 \quad \text{in } \mathbb{R}_*, \tag{4.2}$$

$$v_\varepsilon\big(0^\pm\big) = \pm 1, \tag{4.3}$$

where Eq. (4.2) is understood in the weak sense (*e.g.* in $\mathcal{D}'(\mathbb{R}_*)$) and the constraint (4.3) is understood in the sense of traces. Setting

$$\theta(x) := \big(1 - |x|\big)^+ \operatorname{sign} x, \tag{4.4}$$

we equivalently look for a weak solution of (4.2) living in the affine subspace of $H^1_{\text{odd}}(\mathbb{R}_*)$ given by

$$E := \theta + H^1_{\text{odd}}(\mathbb{R}) = \big\{v \in H^1_{\text{odd}}(\mathbb{R}_*) \;\big|\; v\big(0^\pm\big) = \pm 1 \text{ in the sense of traces}\big\}.$$

Here is the main result of this section.

Proposition 4.3. *Let* $\lambda \in (0, 1)$ *and* $\varepsilon > 0$. *Eq.* (4.2) *admits a weak solution* $v_\varepsilon \in E$ *satisfying*

$$0 \leqslant v_\varepsilon(x) \operatorname{sign} x \leqslant 1 \quad \text{for all } x \in \mathbb{R}_*, \tag{4.5}$$

$$\sup_{\varepsilon \in (0,1)} \int_{\mathbb{R}} \big\{\varepsilon(\partial_x v_\varepsilon)^2_{|\mathbb{R}_*} + \big(\mathcal{L}_{\lambda/2}[v_\varepsilon]\big)^2\big\} < +\infty. \tag{4.6}$$

Proof. The proof is divided into several steps.

Step one. We first fix $\bar{v} \in E$ and introduce the auxiliary equation with modified convection term:

$$\varepsilon\big(v - \partial^2_{xx} v\big) + \rho_n \partial_x\left(\frac{(\rho_n T_n(\bar{v}))^2}{2}\right) + \mathcal{L}_\lambda[v] = 0, \tag{4.7}$$

where for $n \in \mathbb{N}_*$, the truncation functions T_n and ρ_n are given, respectively, by (2.1) and by the formula $\rho_n(x) := \rho(\frac{x}{C(n,\varepsilon)})$ with

$$\begin{cases} \rho \in C_c^\infty(\mathbb{R}) \quad \text{even}, \\ 0 \leqslant \rho \leqslant \rho(0) = 1, \\ -1 \leqslant \rho' \leqslant 0 \quad \text{on } \mathbb{R}^+ \end{cases}$$

and with

$$C(n,\varepsilon) := \frac{n^2}{\varepsilon} \tag{4.8}$$

(this choice of the constant is explained in Step three). Note the property

$$\rho_n \underset{n\to+\infty}{\longrightarrow} 1 \quad \text{uniformly on compact subsets of } \mathbb{R}. \tag{4.9}$$

It is straightforward to see that solving (4.7), (4.3) in the variational sense below,

$$\left| \begin{array}{l} \text{find } v \in E \text{ such that for all } \varphi \in H^1_{\mathrm{odd}}(\mathbb{R}), \\ \displaystyle\int_{\mathbb{R}_*} \left\{ \varepsilon(v\varphi + \partial_x v \partial_x \varphi) + \mathcal{L}_{\lambda/2}[v]\mathcal{L}_{\lambda/2}[\varphi] \right\} = \int_{\mathbb{R}} \frac{(\rho_n T_n(\bar{v}))^2}{2} \partial_x(\rho_n \varphi), \end{array} \right. \tag{4.10}$$

is equivalent to finding a minimizer $v \in E$ for the functional

$$\mathcal{J}_{\bar{v},n} : \begin{array}{rcl} E & \to & \mathbb{R} \\ u & \mapsto & \displaystyle\frac{1}{2}\int_{\mathbb{R}_*} \left\{ \varepsilon\left(u^2 + (\partial_x u)^2\right) + \left(\mathcal{L}_{\lambda/2}[u]\right)^2 - \left(\rho_n T_n(\bar{v})\right)^2 \partial_x(\rho_n u) \right\}. \end{array}$$

Notice that $\rho_n T_n(\bar{v}) \in L^\infty(\mathbb{R})$ and $\rho_n \in H^1(\mathbb{R})$, so that

$$\left(\rho_n T_n(\bar{v})\right)^2 \left(\partial_x(\rho_n u)\right)_{|\mathbb{R}_*} \in L^1(\mathbb{R}) \quad \text{with} \int_{\mathbb{R}_*} \left| \left(\rho_n T_n(\bar{v})\right)^2 \partial_x(\rho_n u) \right| \leqslant C_n \|u\|;$$

let us precise that here and until the end of this proof, C_n denotes a generic constant that depends only on n and eventually on the fixed parameter ε (and which can change from one expression to another). Then the functional $\mathcal{J}_{\bar{v},n}$ is well defined on E and coercive, because

$$\mathcal{J}_{\bar{v},n}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}_*} \left(\rho_n T_n(\bar{v})\right)^2 \partial_x(\rho_n u) \geqslant \frac{1}{2}\|u\|^2 - C_n\|u\| \tag{4.11}$$

tends to infinity as $\|u\| \to +\infty$.

Finally, it is clear that $\mathcal{J}_{\bar{v},n}$ is strictly convex and strongly continuous. Thus we conclude that there exists a unique minimizer of $\mathcal{J}_{\bar{v},n}$, which is the unique solution of (4.10). We denote this solution by $F_n(\bar{v})$, which defines a map $F_n : E \to E$.

Step two: apply the Schauder fixed-point theorem to the map F_n. Note that $F_n(E)$ is contained in the closed ball $\overline{B}_{R_n} := \overline{B}(0_{H^1_{\mathrm{odd}}(\mathbb{R}_*)}, R_n)$ of $H^1_{\mathrm{odd}}(\mathbb{R}_*)$ for some radius $R_n > 0$ (only depending on n and ε). Indeed, let $v := F_n(\bar{v})$; then by using (4.11), replacing the minimizer v with the function $\theta \in E$ in (4.4), and applying the Young inequality we get

$$\|v\|^2 \leqslant 2\mathcal{J}_{\bar{v},n}(v) + C_n\|v\| \leqslant 2\mathcal{J}_{\bar{v},n}(\theta) + \frac{1}{2}\|v\|^2 + C_n.$$

We can restrict F_n to the closed convex set $\mathcal{C} := E \cap \overline{B}_{R_n}$ of the Banach space $H^1_{\mathrm{odd}}(\mathbb{R}_*)$. It remains to show that $F_n : \mathcal{C} \to \mathcal{C}$ is continuous and compact.

In order to justify the compactness of $F_n(\mathcal{C})$, take a sequence $(v_k)_k \subset F_n(\mathcal{C})$ and an associated sequence $(\bar{v}_k)_k \subset \mathcal{C}$ with $v_k = F_n(\bar{v}_k)$. Because $\mathcal{C}$ is bounded, by standard embedding theorems there exists a (not relabelled) subsequence of $(\bar{v}_k)_k$ that converges weakly in $H^1(\mathbb{R}_*)$ and strongly in $L^2_{\mathrm{loc}}(\mathbb{R})$; let $\bar{v}_\infty$ be its limit. One has $\bar{v}_\infty \in \mathcal{C}$ because $\mathcal{C}$ is a strongly closed convex subset in $H^1(\mathbb{R}_*)$, thus it is weakly closed. We can assume without loss of generality that the corresponding subsequence of $(v_k)_k$ converges weakly to some $v_\infty \in \mathcal{C}$ in $H^1_{\text{odd}}(\mathbb{R}_*)$. Let us prove that v_k converges strongly to v_∞ in $H^1_{\text{odd}}(\mathbb{R}_*)$ and that $v_\infty = F_n(\bar{v}_\infty)$.

By the above convergences, using the facts that $T_n \in C_b^\infty(\mathbb{R})$ and $\rho_n \in C_c^\infty(\mathbb{R})$, one sees that

$$\int_{\mathbb{R}_*} \big(\rho_n T_n(\bar{v}_k)\big)^2 \partial_x(\rho_n v_k) \to \int_{\mathbb{R}_*} \big(\rho_n T_n(\bar{v}_\infty)\big)^2 \partial_x(\rho_n v_\infty) \quad \text{as } k \to +\infty.$$

Moreover, because v_k is the minimizer of $\mathcal{J}_{\bar{v}_k,n}$, we have

$$\|v_k\|^2 - \int_{\mathbb{R}_*} \big(\rho_n T_n(\bar{v}_k)\big)^2 \partial_x(\rho_n v_k) = 2\mathcal{J}_{\bar{v}_k,n}(v_k) \leqslant 2\mathcal{J}_{\bar{v}_k,n}(v_\infty) = \|v_\infty\|^2 - \int_{\mathbb{R}_*} \big(\rho_n T_n(\bar{v}_k)\big)^2 \partial_x(\rho_n v_\infty).$$

Thus passing to the limit as $k \to +\infty$ in this inequality we get $\|v_\infty\| \geqslant \limsup_{k\to+\infty} \|v_k\|$. It follows that the convergence of v_k to v_∞ is actually strong in $H^1_{\text{odd}}(\mathbb{R}_*)$. Passing to the limit as $k \to +\infty$ in the variational formulation (4.10) written for v_k and $\bar{v}_k$, we deduce by the uniqueness of a solution to (4.10) that $v_\infty = F_n(\bar{v}_\infty)$; this completes the proof of the compactness of $F_n(\mathcal{C})$.

To prove the continuity of F_n, one simply assumes that $\bar{v}_k \to \bar{v}_\infty$ strongly in $H^1_{\text{odd}}(\mathbb{R}_*)$ and repeats the above reasoning for each subsequence of $(v_k)_k$. One finds that from all subsequence of $(v_k)_k$ one can extract a subsequence strongly converging to $v_\infty = F_n(\bar{v}_\infty)$; hence, the proof of the continuity of F_n is complete.

We conclude that there exists a fixed point u_n of F_n in $\mathcal{C}$. Then $v := \bar{v} = u_n$ satisfies the formulation (4.10). In addition, (4.10) is trivially satisfied with a test function $\varphi \in H^1(\mathbb{R})$ which is even. Indeed, using the definitions of T_n and ρ_n and Lemma 4.1(iv), we see that

$$\varepsilon(v\varphi + \partial_x v \partial_x \varphi) + \mathcal{L}_{\lambda/2}[v]\mathcal{L}_{\lambda/2}[\varphi] - \frac{(\rho_n T_n(\bar{v}))^2}{2} \partial_x(\rho_n \varphi)$$

is an odd function, so that its integral on $\mathbb{R}_*$ is null. Since all function in $H^1(\mathbb{R})$ can be split into the sum of an odd function in $H^1_{\text{odd}}(\mathbb{R})$ and an even function in $H^1(\mathbb{R})$, we have proved that the fixed point $u_n \in E$ of F_n satisfies for all $\varphi \in H^1(\mathbb{R})$,

$$\int_{\mathbb{R}} \big\{\varepsilon\big(u_n\varphi + (\partial_x u_n)_{|\mathbb{R}_*} \partial_x \varphi\big) + \mathcal{L}_{\lambda/2}[u_n]\mathcal{L}_{\lambda/2}[\varphi]\big\} = \int_{\mathbb{R}} \frac{(\rho_n T_n(u_n))^2}{2} \partial_x(\rho_n \varphi) \tag{4.12}$$

(notice that the Rankine–Hugoniot condition is satisfied automatically because of the fact that $\frac{u_n^2}{2}$ is even). In particular, using Lemma 4.1 item (iii), one has

$$\varepsilon\big(\partial^2_{xx} u_n - u_n\big) = \rho_n \partial_x \left(\frac{(\rho_n T_n(u_n))^2}{2} \right) + \mathcal{L}_\lambda[u_n] \quad \text{in } \mathcal{D}'(\mathbb{R}_*). \tag{4.13}$$

Step three: uniform estimates on the sequence $(u_n)_n$. First, in order to prove a maximum principle for u_n let us point out that u_n is regular. Indeed, thanks to Lemma 4.1(i)(b) and the facts that $T_n \in C_b^\infty(\mathbb{R})$ and $\rho_n \in C_c^\infty(\mathbb{R})$, the right-hand side of (4.13) belongs to $L^1(I)$ for all compact interval $I \subset \mathbb{R}_*$. Eq. (4.13) then implies that $u_n \in W^{2,1}_{\text{loc}}(\mathbb{R}_*) \subset C^1(\mathbb{R}_*)$. Recall that $u_n \in H^1(\mathbb{R}_*) \subset C_b(\mathbb{R}_*)$; thus using Lemma 4.1(i)(a), we see that the right-hand side of (4.13) belongs to $C(I)$. Exploiting once more Eq. (4.13), we infer that $u_n \in C^2(\mathbb{R}_*)$ and (4.13) holds pointwise on $\mathbb{R}_*$.

Now, we are in a position to prove that for all $x > 0$ and $n \in \mathbb{N}_*$, $0 \leqslant u_n(x) \leqslant 1$. Indeed, because $u_n \in H^1(\mathbb{R}^+)$, we have $\lim_{x\to+\infty} u_n(x) = 0$; in addition, $u_n(0^+) = 1$. Thus if $u_n(x) \notin [0,1]$ for some $x \in \mathbb{R}^+$, there exists $x_* \in \mathbb{R}^+$ such that

$$\text{either} \quad u_n(x_*) = \max_{\mathbb{R}^+} u > 1 \quad \text{or} \quad u_n(x_*) = \min_{\mathbb{R}^+} u < 0.$$

Consider the first case. Since $u_n \in C^2(\mathbb{R}_*)$, we have $\partial_x u_n(x_*) = 0$ and $\partial^2_{xx} u_n(x_*) \leqslant 0$. In addition, by Lemma 4.1(v) we have $\mathcal{L}[u_n](x_*) > 0$. Therefore using (4.13) at the point x_*, by the choice of ρ_n and $C(n,\varepsilon)$ in (4.8) we infer

$$\varepsilon u_n(x_*) = \varepsilon \partial^2_{xx} u_n(x_*) - \mathcal{L}_\lambda[u_n](x_*) - \rho_n(x_*)\partial_x\left(\frac{(\rho_n(x_*)T_n(u_n(x_*)))^2}{2}\right)$$
$$\leqslant -\big(\rho_n(x_*)T_n\big(u(x_*)\big)\big)^2\partial_x\rho_n(x_*) \leqslant n^2\frac{1}{C(n,\varepsilon)}\sup_{\mathbb{R}^+}(-\partial_x\rho) \leqslant \varepsilon.$$

Thus $u_n(x_*) \leqslant 1$, which contradicts the definition of x_*. The case $u_n(x_*) = \min_{\mathbb{R}^+} u < 0$ is similar; we use in addition the fact that $\partial_x\rho_n \leqslant 0$ on $\mathbb{R}^+$.

The function u_n being even, from the maximum principle of Step three we have $|u_n| \leqslant 1$ on $\mathbb{R}_*$. Since $T_n = \mathrm{Id}$ on $[-n+1, n-1]$, we have $T_n(u_n) = u_n$ in (4.13) for all $n \geqslant 2$.

Let us finally derive the uniform $H^1_{\mathrm{odd}}(\mathbb{R}_*)$-bound on $(u_n)_n$. To do so, replace the minimum u_n of the functional $\mathcal{J}_{u_n,n}$ by the fixed function $\theta \in E$ in (4.4); we find

$$\|u_n\|^2 = 2\mathcal{J}_{u_n,n}(u_n) + \int_{\mathbb{R}_*} (\rho_n u_n)^2\partial_x(\rho_n u_n) \leqslant 2\mathcal{J}_{u_n,n}(\theta) + \int_{\mathbb{R}_*} \partial_x\left(\frac{(\rho_n u_n)^3}{3}\right).$$

Since $\rho_n(0) = 1 = \pm u_n(0^\pm)$, we get

$$\|u_n\|^2 \leqslant 2\mathcal{J}_{u_n,n}(\theta) - \frac{2}{3} = \|\theta\|^2 - \int_{\mathbb{R}_*} (\rho_n u_n)^2\partial_x(\rho_n\theta) - \frac{2}{3}.$$

To estimate the integral term, we use that θ is supported by $[-1, 1]$ with $|\partial_x(\rho_n\theta)| \leqslant 1 + \frac{\varepsilon}{n^2}$ (once more, this is due to the choice of ρ_n in (4.8)). Finally, using the bound $|u_n| \leqslant 1$ derived above, we get

$$-\int_{\mathbb{R}_*} (\rho_n u_n)^2\partial_x(\rho_n\theta) \leqslant 2 + \frac{2\varepsilon}{n^2};$$

hence, we obtain the following uniform estimate:

$$\|u_n\|^2 \leqslant \|\theta\|^2 + \frac{4}{3} + \frac{2\varepsilon}{n^2}. \tag{4.14}$$

Step four: passage to the limit as $n \to +\infty$. The $H^1_{\mathrm{odd}}(\mathbb{R}_*)$-estimate of Step three permits to extract a (not relabelled) subsequence $(u_n)_n$ which converges (weakly in $H^1(\mathbb{R}_*)$ and strongly in $L^2_{\mathrm{loc}}(\mathbb{R})$) to a limit that we denote v_ε. We have $(u_n)_n \subset E$ which is a closed affine subspace of $H^1(\mathbb{R}_*)$, so that $v_\varepsilon \in E$. The above convergences and the convergence of ρ_n in (4.9) are enough to pass to the limit in (4.12); at the limit, we conclude that v_ε is a weak solution of (4.2). Notice that v_ε inherits the bounds on u_n, namely the bound (4.14) and the maximum principle $0 \leqslant u_n(x)\,\mathrm{sign}\,x \leqslant 1$. This yields (4.5) and (4.6), thanks to the definition of $\|\cdot\|$ via the scalar product (4.1). □

Remark 4.4. When passing to the limit as $n \to +\infty$ in (4.12) in the last step, one gets:

$$\int_{\mathbb{R}} \big\{\varepsilon\big(v_\varepsilon\varphi + (\partial_x v_\varepsilon)_{|\mathbb{R}_*}\partial_x\varphi\big) + v_\varepsilon\mathcal{L}_\lambda[\varphi]\big\} = \int_{\mathbb{R}} \frac{{v_\varepsilon}^2}{2}\partial_x\varphi \quad \text{for all } \varphi \in H^1(\mathbb{R}). \tag{4.15}$$

5. A non-entropy stationary solution

We are now able to construct a stationary non-entropy solution to (1.1) by passing to the limit in v_ε as $\varepsilon \to 0$. Let us explain our strategy. First, we have to use the uniform estimates of Proposition 4.3 to get compactness; this is done *via* the following lemma which is proved in Appendix A.

Lemma 5.1. *Assume that for all $\varepsilon \in (0, 1)$, $v_\varepsilon \in H^1(\mathbb{R}_*)$ satisfies* (4.5)–(4.6). *Then the family $\{v_\varepsilon | \varepsilon \in (0, 1)\}$ is relatively compact in $L^2_{\mathrm{loc}}(\mathbb{R})$.*

With Lemma 5.1 in hand, we can prove the convergence of a subsequence of v_ε, as $\varepsilon \to 0$, to some stationary weak solution v of (1.1). Next, we need to control the traces of v at $x = 0^\pm$. This is done by reformulating Definition 2.4 and by exploiting the Green–Gauss formula.

Let us begin with giving a characterization of odd weak stationary solutions of the fractional Burgers equation.

Proposition 5.2. *An odd function $v \in L^\infty(\mathbb{R})$ satisfies*

$$\partial_x\left(\frac{v^2}{2}\right) + \mathcal{L}_\lambda[v] = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}), \tag{5.1}$$

iff (i) *and* (ii) *below hold true*:

(i) *there exists the trace $\gamma v^2 := \lim_{h\to 0^+} \frac{1}{h}\int_0^h v^2(x)\,dx$;*
(ii) *for all odd compactly supported in $\mathbb{R}$ test function $\varphi \in C_b^\infty(\mathbb{R}_*)$,*

$$\int_{\mathbb{R}_*} \left(v\mathcal{L}_\lambda[\varphi] - \frac{v^2}{2}\partial_x\varphi\right) = \varphi(0^+)\gamma v^2.$$

Proof. Assume (5.1). For all $h > 0$, let us set $\psi_h(x) := \frac{1}{h}(h-|x|)^+ \operatorname{sign} x$. Let us recall that $\theta(x) = (1-|x|)^+ \operatorname{sign} x$. First consider

$$\theta_h(x) := \begin{cases} \theta(x), & x<0, \\ -\psi_h(x), & x \geqslant 0 \end{cases} \quad \text{and} \quad \theta_0(x) = \begin{cases} \theta(x), & x<0, \\ 0, & x \geqslant 0. \end{cases}$$

By construction, $\theta_h \in H^1(\mathbb{R})$; therefore θ_h can be approximated in $H^1(\mathbb{R})$ by functions in $\mathcal{D}(\mathbb{R})$ and thus taken as a test function in (5.1). This gives

$$-\int_{\mathbb{R}^+} \frac{v^2}{2}\partial_x\psi_h = -\int_{\mathbb{R}^-}\frac{v^2}{2}\partial_x\theta + \int_{\mathbb{R}} v\mathcal{L}_\lambda[\theta_h].$$

But, it is obvious that $\theta_h \to \theta_0$ in $L^1(\mathbb{R}) \cap (BV(\mathbb{R}))_{w\text{-}*}$ as $h \to 0^+$; thus using Lemma 4.1(i)(d), we conclude that the limit in item (i) of Proposition 5.2 does exist, and

$$\gamma v^2 := \lim_{h\to 0^+}\frac{1}{h}\int_0^h v^2 = -\lim_{h\to 0^+}\int_{\mathbb{R}^+} v^2\partial_x\psi_h = -\int_{\mathbb{R}^-} v^2\partial_x\theta + 2\int_{\mathbb{R}} v\mathcal{L}_\lambda[\theta_0]. \tag{5.2}$$

Further, take a function φ as in item (ii) of Proposition 5.2 and set $\varphi_h(x) := \varphi(x) - \varphi(0^+)\psi_h(x)$. One can take $\varphi_h \in H^1(\mathbb{R})$ as a test function in (5.1). Taking into account the fact that $\frac{v^2}{2}\partial_x\varphi_h$ and $v\mathcal{L}_\lambda[\varphi_h]$ are even, thanks to Lemma 4.1(iv), we get

$$2\int_{\mathbb{R}^+}\left(v\mathcal{L}_\lambda[\varphi] - \frac{v^2}{2}\partial_x\varphi\right) = 2\varphi(0^+)\int_{\mathbb{R}^+}\left(v\mathcal{L}[\psi_h] - \frac{v^2}{2}\partial_x\psi_h\right).$$

Now we pass to the limit as $h \to 0^+$. As previously, because $\psi_h \to 0$ in $L^1(\mathbb{R}) \cap (BV(\mathbb{R}))_{w\text{-}*}$, the term $\mathcal{L}_\lambda[\psi_h]$ vanishes in $L^1(\mathbb{R})$. Using (5.2), we get item (ii) of Proposition 5.2.

Conversely, assume that an odd function v satisfies the properties of items (i) and (ii) of Proposition 5.2. Take a test function $\xi \in \mathcal{D}(\mathbb{R})$ and write $\xi = \varphi + \psi$ with $\varphi \in \mathcal{D}(\mathbb{R})$ odd (so that $\varphi(0^+) = 0$) and $\psi \in \mathcal{D}(\mathbb{R})$ even. Then (ii) and the symmetry considerations, including Lemma 4.1(iv), show that

$$\int_{\mathbb{R}}\left(v\mathcal{L}_\lambda[\varphi] - \frac{v^2}{2}\partial_x\varphi\right) = \varphi(0^+)\gamma v^2 = 0, \qquad \int_{\mathbb{R}}\left(v\mathcal{L}_\lambda[\psi] - \frac{v^2}{2}\partial_x\psi\right) = 0.$$

Hence we deduce that v satisfies (5.1). □

Here is the result of existence of a non-entropy stationary solution.

Proposition 5.3. *Let $\lambda \in (0,1)$. There exists $v \in L^\infty(\mathbb{R})$ that satisfies* (5.1) *and such that for all $c > 0$, v does not satisfy $\partial_x v \leqslant \frac{1}{c}$ in $\mathcal{D}'(\mathbb{R})$.*

Proof. First, by Proposition 4.3 and Lemma 5.1 there exist $v \in L^\infty(\mathbb{R})$ and a sequence $(\varepsilon_k)_k$, $\varepsilon_k \downarrow 0$ as $k \to +\infty$, such that the solution v_{ε_k} of (4.2) with $\varepsilon = \varepsilon_k$ tends to v in $L^2_{\mathrm{loc}}(\mathbb{R})$; v is bounded by 1 in the L^∞-norm. Using in particular (4.6) to make disappear the term $\sqrt{\varepsilon}(\sqrt{\varepsilon}\partial_x v_\varepsilon)_{|\mathbb{R}_*}$, we can pass to the limit in (4.15) and infer (5.1).

In order to conclude the proof, we only need to show that there exist the limits

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h v = 1, \qquad \lim_{h\to 0^+} \frac{1}{h}\int_{-h}^0 v = -1. \tag{5.3}$$

Indeed, (5.3) readily implies that for all $c > 0$, the function $(v - \frac{1}{c}\mathrm{Id})$ does not admit a non-increasing representative. Since $\partial_x(v - \frac{1}{c}\mathrm{Id}) = \partial_x v - \frac{1}{c}$, the inequality $\partial_x v - \frac{1}{c} \leqslant 0$ in the distribution sense fails to be true.

Thus it remains to show (5.3). To do so, we exploit the formulation (i)–(ii) of Proposition 5.2, the analogous formulation of the regularized problem (4.2), the fact that $v_{\varepsilon_k}(0^\pm) = \pm 1$, and (4.5).

Let us fix some odd compactly supported in $\mathbb{R}$ function $\varphi \in C_b^\infty(\mathbb{R}_*)$ such that $\varphi(0^+) = 1$. Let us take the test function $\varphi_h(x) := \varphi(x) - \psi_h(x) \in H^1(\mathbb{R})$ in (4.15). We infer

$$\int_{\mathbb{R}_*} \left\{ \varepsilon(v_\varepsilon\varphi_h + \partial_x v_\varepsilon \partial_x \varphi_h) + v_\varepsilon \mathcal{L}_\lambda[\varphi_h] - \frac{v_\varepsilon^2}{2}\partial_x\varphi_h \right\} = 0.$$

Each term in the above integrand is even; moreover, letting $h \to 0^+$ and using Lemma 4.1(i)(d) on $\mathcal{L}_\lambda[\psi_h]$, we infer

$$\begin{aligned}\int_{\mathbb{R}_*} \left\{ \varepsilon(v_\varepsilon\varphi + \partial_x v_\varepsilon \partial_x \varphi) + v_\varepsilon \mathcal{L}_\lambda[\varphi] - \frac{v_\varepsilon^2}{2}\partial_x\varphi \right\} &= 2\lim_{h\to 0^+}\frac{1}{h}\int_0^h \left(\frac{v_\varepsilon^2}{2} - \varepsilon\partial_x v_\varepsilon\right) = 1 - \frac{2\varepsilon}{h}[v_\varepsilon]_0^h \\ &= 1 - \frac{2\varepsilon}{h}\big(v_\varepsilon(h) - 1\big) \geqslant 1;\end{aligned} \tag{5.4}$$

here in the last inequality, we have used the fact that $0 \leqslant v_\varepsilon(x) \leqslant 1 = v_\varepsilon(0^+)$ for $x > 0$.

Letting $\varepsilon_k \to 0$ in (5.4), using again (4.6) to make disappear the term $\int_{\mathbb{R}_*} \varepsilon\partial_x v_\varepsilon \partial_x\varphi$, we infer

$$\int_{\mathbb{R}_*} \left\{ v\mathcal{L}_\lambda[\varphi] - \frac{v^2}{2}\partial_x\varphi \right\} \geqslant 1. \tag{5.5}$$

Recall that v is odd and solves (5.1); thus it satisfies items (i) and (ii) of Proposition 5.2. From item (ii), we infer that $\lim_{h\to 0^+}\frac{1}{h}\int_0^h v^2 = \gamma v^2 \geqslant 1$. But we also have $0 \leqslant v \leqslant 1$ on $[0, h]$. Therefore

$$\lim_{h\to 0^+}\frac{1}{h}\int_0^h |1 - v| = \lim_{h\to 0^+}\frac{1}{h}\int_0^h \frac{1 - v^2}{1 + v} \leqslant \lim_{h\to 0^+}\frac{1}{h}\int_0^h \big(1 - v^2\big) = 1 - \gamma v^2 \leqslant 0.$$

Whence the first equality in (5.3) follows. The second one is clear because v is an odd function. This concludes the proof. □

From Propositions 3.1 and 5.3, Theorem 1.1 readily follows.

Proof of Theorem 1.1. Take $u_0 := v$. From (5.1) we derive that the function defined by $u(t) := v$ for all $t \geqslant 0$ is a weak solution to (1.1)–(1.2). But it is not the entropy solution, because it fails to satisfy (3.1). □

Remark 5.4. Let us stress that the solution v_ε to the regularized problem (4.2)–(4.3) does not satisfy Eq. (4.2) on all the domain $\mathbb{R}$; indeed, one has

$$\varepsilon\big(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\big) + \partial_x\left(\frac{v_\varepsilon^2}{2}\right) + \mathcal{L}_\lambda[v_\varepsilon] = -2\varepsilon\partial_x(\delta_0) \quad \text{in } \mathbb{R}, \tag{5.6}$$

where δ_0 is the Dirac distribution at zero.

It is natural to compare (5.6) and (3.3); within the class of stationary solutions, the two regularizations are very similar. Yet the presence of a vanishing (as $\varepsilon \to 0$) but singular source term in (5.6) is responsible for the failure of the Oleĭnik inequality: this source term creates the increasing jump in v_ε across the line $\{x = 0\}$. This explains why v_ε converges as $\varepsilon \to 0$ to a weak non-entropy solution of Eq. (1.1), whereas we have seen in the proof of Proposition 3.1 that the solution u_ε of the regularized problem (3.3)–(3.4) converges to an entropy solution.

6. Proof of Lemma 4.1

We end this paper by proving the main properties of the fractional Laplacian acting on spaces of odd functions. First, we have to state and prove some technical lemmata.

Here are the embedding and density results that will be needed; for the reader's convenience, short proofs are given in Appendix A.

Lemma 6.1. *The inclusions*

$$H^1(\mathbb{R}_*) \subset BV_{\mathrm{loc}}(\mathbb{R}) \cap H^1_{\mathrm{loc}}\big(\overline{\mathbb{R}} \setminus \{0\}\big) \subset L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}) \tag{6.1}$$

and

$$\big(BV_{\mathrm{loc}}(\mathbb{R})\big)_{w\text{-}\star} \cap H^1_{\mathrm{loc}}\big(\overline{\mathbb{R}} \setminus \{0\}\big) \subset L^2(\mathbb{R}) \tag{6.2}$$

are continuous and sequentially continuous, respectively.

Lemma 6.2. *The space $\mathcal{D}(\mathbb{R})$ is dense in $H^1(\mathbb{R}_*)$ for the $(BV_{\mathrm{loc}}(\mathbb{R}))_{w\text{-}\star} \cap H^1_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\})$-topology.*

The next lemma states weak continuity results for the fractional Laplacian. Until the end of this section, $\mathcal{L}_\lambda$ denotes the operator defined by (2.2) and $\mathcal{L}^{\mathcal{F}}_\lambda$ denotes the one defined by (1.3).

Lemma 6.3. *Let* $\lambda \in (0, 1)$. *Then the following operators are sequentially continuous*:

$$\mathcal{L}_\lambda : \big(BV_{\mathrm{loc}}(\mathbb{R})\big)_{w\text{-}\star} \cap H^1_{\mathrm{loc}}\big(\overline{\mathbb{R}} \setminus \{0\}\big) \to L^1_{\mathrm{loc}}(\mathbb{R}) \cap L^2_{\mathrm{loc}}\big(\overline{\mathbb{R}} \setminus \{0\}\big),$$
$$\mathcal{L}^{\mathcal{F}}_{\lambda/2} : \big(BV_{\mathrm{loc}}(\mathbb{R})\big)_{w\text{-}\star} \cap H^1_{\mathrm{loc}}\big(\overline{\mathbb{R}} \setminus \{0\}\big) \to L^2(\mathbb{R}).$$

Proof. The proof is divided in several steps. *Step one: strong continuity of $\mathcal{L}_\lambda$.* Let $v \in BV_{\mathrm{loc}}(\mathbb{R}) \cap H^1_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\})$ and let us derive some estimates on $\mathcal{L}_\lambda[v]$. For all $r, R > 0$, using the Fubini theorem one has

$$\begin{aligned}
&\int_{-R}^{R}\int_{\mathbb{R}} \frac{|v(x+z)-v(x)|}{|z|^{1+\lambda}}\,dx\,dz \\
&\quad = \int_{-R}^{R}\int_{|z|\leqslant r} \frac{|v(x+z)-v(x)|}{|z|^{1+\lambda}}\,dx\,dz + \int_{-R}^{R}\int_{|z|>r} \frac{|v(x+z)-v(x)|}{|z|^{1+\lambda}}\,dx\,dz \\
&\quad \leqslant |v|_{BV((-R-r,R+r))}\int_{|z|\leqslant r} |z|^{-\lambda}\,dz + \Big(\sup_{|z|>r}\|v\|_{L^1((-R+z,R+z))} + \|v\|_{L^1((-R,R))}\Big)\int_{|z|>r} |z|^{-1-\lambda}\,dz \\
&\quad = \frac{2r^{1-\lambda}}{1-\lambda}|v|_{BV((-R-r,R+r))} + \frac{2}{\lambda r^\lambda}\Big(\sup_{|z|>r}\|v\|_{L^1((-R+z,R+z))} + \|v\|_{L^1((-R,R))}\Big).
\end{aligned} \tag{6.3}$$

By (6.1) of Lemma 6.1, using the Cauchy–Schwarz inequality to control the L^1-norms by the L^2-norms, one sees that integral term in (2.2) makes sense a.e. with

$$\big\|\mathcal{L}_\lambda[v]\big\|_{L^1((-R,R))} \leqslant \frac{2G_\lambda r^{1-\lambda}}{1-\lambda}|v|_{BV((-R-r,R+r))} + \frac{4G_\lambda}{\lambda r^\lambda}\sqrt{2R}\|v\|_{L^2(\mathbb{R})}, \tag{6.4}$$

for all r, $R > 0$. In the same way, by Minkowski's integral inequality one has for $R > r > 0$,

$$\begin{aligned}
&\left(\int_{\mathbb{R}\setminus[-R,R]}\left(\int_{\mathbb{R}}\frac{|v(x+z)-v(x)|}{|z|^{1+\lambda}}\,dz\right)^2dx\right)^{\frac{1}{2}}\\
&\quad\leqslant\int_{\mathbb{R}}\left(\int_{\mathbb{R}\setminus[-R,R]}\frac{|v(x+z)-v(x)|^2}{|z|^{2+2\lambda}}\,dx\right)^{\frac{1}{2}}dz\\
&\quad=\int_{|z|\leqslant r}|z|^{-1-\lambda}\left(\int_{\mathbb{R}\setminus[-R,R]}\left|v(x+z)-v(x)\right|^2dx\right)^{\frac{1}{2}}dz+\int_{|z|>r}|z|^{-1-\lambda}\left(\int_{\mathbb{R}\setminus[-R,R]}\left|v(x+z)-v(x)\right|^2dx\right)^{\frac{1}{2}}dz\\
&\quad\leqslant\frac{2r^{1-\lambda}}{1-\lambda}\|\partial_x v\|_{L^2(\mathbb{R}\setminus[-R+r,R-r])}+\frac{4}{\lambda r^\lambda}\|v\|_{L^2(\mathbb{R})};
\end{aligned}$$

therefore, one gets for all $R > r > 0$,

$$\left\|\mathcal{L}_\lambda[v]\right\|_{L^2(\mathbb{R}\setminus[-R,R])}\leqslant\frac{2G_\lambda r^{1-\lambda}}{1-\lambda}\|\partial_x v\|_{L^2(\mathbb{R}\setminus[-R+r,R-r])}+\frac{4G_\lambda}{\lambda r^\lambda}\|v\|_{L^2(\mathbb{R})}. \tag{6.5}$$

Now (6.4)–(6.5) imply that $\mathcal{L}_\lambda : BV_{\mathrm{loc}}(\mathbb{R})\cap H^1(\overline{\mathbb{R}}\setminus\{0\})\to L^1_{\mathrm{loc}}(\mathbb{R})\cap L^2_{\mathrm{loc}}(\overline{\mathbb{R}}\setminus\{0\})$ is well defined and continuous.

Step two: weak-$\star$ sequential continuity of $\mathcal{L}_\lambda$. Consider a sequence $(v_k)_k$ converging to zero in $(BV_{\mathrm{loc}}(\mathbb{R}))_{w\text{-}\star}\cap H^1_{\mathrm{loc}}(\overline{\mathbb{R}}\setminus\{0\})$. For all $R > 0$, $(v_k)_k$ is bounded in the norm of $H^1(\mathbb{R}\setminus[-R,R])$ and the semi-norm of $BV((-R,R))$ by some constant C_R. By (6.4), one deduces that

$$\limsup_{k\to+\infty}\left\|\mathcal{L}_\lambda[v_k]\right\|_{L^1((-R,R))}\leqslant\frac{2G_\lambda r^{1-\lambda}}{1-\lambda}C_{R+r}.$$

Letting $r\to 0$, one concludes that $\mathcal{L}_\lambda[v_k]$ converges to zero in $L^1((-R,R))$. In the same way, one can prove that $\mathcal{L}_\lambda[v_k]$ converges to zero in $L^2(\mathbb{R}\setminus[-R,R])$ by using (6.5). Since R is arbitrary, the proof of Lemma 6.3 is complete.

Step three: strong continuity of $\mathcal{L}^{\mathcal{F}}_{\lambda/2}$. Let us derive an L^2-estimate on $\mathcal{L}^{\mathcal{F}}_{\lambda/2}[v]$. Recall that by property (6.1) of Lemma 6.1, one has $v\in L^2(\mathbb{R})$ so that $|\cdot|\mathcal{F}(v)(\cdot)\in L^1_{\mathrm{loc}}(\mathbb{R})$ and $\mathcal{L}^{\mathcal{F}}_{\lambda/2}[v]$ is well defined in $\mathcal{S}'(\mathbb{R})$.

Further, consider some fixed $\rho\in C^\infty_c(\mathbb{R})$ such that $\rho = 1$ on some neighborhood of the origin, say on $[-1/2, 1/2]$, and $\operatorname{supp}\rho\subseteq[-1,1]$. Then one has $v=\rho v+(1-\rho)v$ with $\operatorname{supp}(\rho v)\subseteq[-1,1]$, $\rho v\in L^1(\mathbb{R})\cap BV(\mathbb{R})$ (since $v\in L^2(\mathbb{R})$) and $(1-\rho)v\in H^1(\mathbb{R})$; moreover, one readily see that

$$\|\rho v\|_{L^1(\mathbb{R})}\leqslant C_\rho\|v\|_{L^2(\mathbb{R})}, \tag{6.6}$$

$$|\rho v|_{BV(\mathbb{R})}\leqslant C_\rho\big(|v|_{BV((-1/2,1/2))}+\|v\|_{H^1(\mathbb{R}\setminus[-1/2,1/2])}\big), \tag{6.7}$$

$$\left\|(1-\rho)v\right\|_{L^2(\mathbb{R})}\leqslant C_\rho\|v\|_{L^2(\mathbb{R})}, \tag{6.8}$$

$$\left\|\partial_x\big((1-\rho)v\big)\right\|_{L^2(\mathbb{R})}\leqslant C_\rho\|v\|_{H^1(\mathbb{R}\setminus[-1/2,1/2])}; \tag{6.9}$$

here and until the end of the proof, C_ρ denotes a generic constant only depending on ρ.

By Plancherel's equality, we have

$$\begin{aligned}
\left\|\mathcal{L}^{\mathcal{F}}_{\lambda/2}[v]\right\|_{L^2(\mathbb{R})}&=\int_{\mathbb{R}}|\xi|^\lambda\left|\mathcal{F}(v)(\xi)\right|^2d\xi\\
&=\int_{\mathbb{R}}|\xi|^\lambda\left|\mathcal{F}(\rho v)(\xi)\right|^2d\xi+\int_{\mathbb{R}}|\xi|^\lambda\left|\mathcal{F}\big((1-\rho)v\big)(\xi)\right|^2d\xi=:I+J.
\end{aligned} \tag{6.10}$$

Let us first bound J from above. For all $r > 0$, one has

$$\begin{aligned} J &= \int_{|\xi|>r} |\xi|^\lambda \big|\mathcal{F}\big((1-\rho)v\big)(\xi)\big|^2\,d\xi + \int_{|\xi|\leqslant r} |\xi|^\lambda \big|\mathcal{F}\big((1-\rho)v\big)(\xi)\big|^2\,d\xi \\ &\leqslant \int_{|\xi|>r} |\xi|^{\lambda-2}|\xi|^2 \big|\mathcal{F}\big((1-\rho)v\big)(\xi)\big|^2\,d\xi + r^\lambda \big\|\mathcal{F}\big((1-\rho)v\big)\big\|^2_{L^2(\mathbb{R})} \\ &\leqslant \frac{1}{r^{2-\lambda}} \int_{|\xi|>r} |\xi|^2 \big|\mathcal{F}\big((1-\rho)v\big)(\xi)\big|^2\,d\xi + r^\lambda \big\|(1-\rho)v\big\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Using the formula

$$\mathcal{F}(\partial_x w) = 2i\pi\xi\mathcal{F}(w), \tag{6.11}$$

using once more Plancherel's equality, one gets $J \leqslant \frac{1}{4\pi^2 r^{2-\lambda}}\|\partial_x((1-\rho)v)\|^2_{L^2(\mathbb{R})} + r^\lambda\|(1-\rho)v\|^2_{L^2(\mathbb{R})}$; hence by (6.8)–(6.9), one has

$$J \leqslant \frac{C_\rho}{r^{2-\lambda}}\|v\|^2_{H^1(\mathbb{R}\setminus[-1/2,1/2])} + C_\rho r^\lambda \|v\|^2_{L^2(\mathbb{R})}. \tag{6.12}$$

To bound I from above, one uses the boundeness of $\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ together with the pointwise estimate $|\xi||\mathcal{F}(w)(\xi)| \leqslant \frac{1}{2\pi}|w|_{BV(\mathbb{R})}$ that comes from (6.11). We get

$$\begin{aligned} I &= \int_{|\xi|>r} |\xi|^\lambda \big|\mathcal{F}(\rho v)(\xi)\big|^2\,d\xi + \int_{|\xi|\leqslant r} |\xi|^\lambda \big|\mathcal{F}(\rho v)(\xi)\big|^2\,d\xi \\ &\leqslant \frac{1}{4\pi^2}|\rho v|^2_{BV(\mathbb{R})} \int_{|\xi|>r} |\xi|^{\lambda-2}\,d\xi + \|\rho v\|^2_{L^1(\mathbb{R})} \int_{|\xi|\leqslant r} |\xi|^\lambda\,d\xi \\ &= \frac{1}{2\pi^2(1-\lambda)r^{1-\lambda}}|\rho v|^2_{BV(\mathbb{R})} + \frac{2r^{1+\lambda}}{1+\lambda}\|\rho v\|^2_{L^1(\mathbb{R})}; \end{aligned}$$

therefore by (6.6)–(6.7), one has

$$I \leqslant \frac{C_\rho}{(1-\lambda)r^{1-\lambda}}\big(|v|_{BV((-1/2,1/2))} + \|v\|_{H^1(\mathbb{R}\setminus[-1/2,1/2])}\big)^2 + \frac{C_\rho r^{1+\lambda}}{1+\lambda}\|v\|^2_{L^2(\mathbb{R})}. \tag{6.13}$$

From (6.10), (6.12) and (6.13), we deduce the final estimate:

$$\begin{aligned} \big\|\mathcal{L}^{\mathcal{F}}_{\lambda/2}[v]\big\|_{L^2(\mathbb{R})} &\leqslant C_\rho\left(r^\lambda + \frac{r^{1+\lambda}}{1+\lambda}\right)\|v\|^2_{L^2(\mathbb{R})} \\ &\quad + C_\rho\left(\frac{1}{r^{2-\lambda}} + \frac{1}{(1-\lambda)r^{1-\lambda}}\right)\big(|v|_{BV((-1/2,1/2))} + \|v\|_{H^1(\mathbb{R}\setminus[-1/2,1/2])}\big)^2, \end{aligned} \tag{6.14}$$

for all $r > 0$.

One infers that $\mathcal{L}^{\mathcal{F}}_{\lambda/2} : BV_{\mathrm{loc}}(\mathbb{R}) \cap H^1_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\}) \to L^2(\mathbb{R})$ is continuous.

Step four: weak-$\star$ sequential continuity of $\mathcal{L}^{\mathcal{F}}_{\lambda/2}$. By (6.2) of Lemma 6.1, one sees that if $v_k \to 0$ in the topological space $(BV_{\mathrm{loc}}(\mathbb{R}))_{w\text{-}\star} \cap H^1_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\})$, then $v_k \to 0$ in $L^2(\mathbb{R})$. One then argues exactly as in Step two by using (6.14) instead of (6.4)–(6.5); one deduces that $\mathcal{L}^{\mathcal{F}}_{\lambda/2}[v_k] \to 0$ in $L^2(\mathbb{R})$ and this completes the proof of the lemma. □

We can now prove the main properties of $\mathcal{L}_\lambda$ stated in Subsection 4.1.

Proof of Lemma 4.1. Let us prove the different items step by step.

Step one: item (i) (a) and (b). Item (i)(a) is an immediate consequence of the theorem of continuity under the integral sign; the details are left to the reader. Item (i)(b) is clear from Lemmata 6.1 and 6.3.

Step two: item (i)(d). Passing to the limit $R \to +\infty$ in (6.3), one gets

$$\left\|\mathcal{L}_\lambda[v]\right\|_{L^1(\mathbb{R})} \leqslant \frac{2G_\lambda r^{1-\lambda}}{1-\lambda}|v|_{BV(\mathbb{R})} + \frac{4G_\lambda}{\lambda r^\lambda}\|v\|_{L^1(\mathbb{R})}, \tag{6.15}$$

for all $v \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$ and $r > 0$. With this estimate in hands, we can argue as in the second step of the proof of Lemma 6.3 to show item (i)(d).

Step three: items (ii) and (i)(c). Let us prove item (ii) first. By Lemma 6.2, $v \in H^1(\mathbb{R}_*)$ can be approximated by $v_k \in \mathcal{S}(\mathbb{R})$ in $(BV_{\mathrm{loc}}(\mathbb{R}))_{w\text{-}\star} \cap H^1_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\})$. One has $\mathcal{L}_\lambda[v_k] = \mathcal{L}^{\mathcal{F}}_\lambda[v_k]$ thanks to the classical Lévy–Khintchine formula. By Lemma 6.3, we infer that $\mathcal{L}_\lambda[v_k]$ converges toward $\mathcal{L}_\lambda[v]$ in $\mathcal{S}'(\mathbb{R})$ as $k \to +\infty$. But the embedding (6.2) of Lemma 6.1 implies that $v_k \to v$ in $L^2(\mathbb{R})$, so that $\mathcal{F}(v_k) \to \mathcal{F}(v)$ in $L^2(\mathbb{R})$. It follows that $|\cdot|^\lambda \mathcal{F}(v_k)(\cdot) \to |\cdot|^\lambda \mathcal{F}(v)(\cdot)$ in $\mathcal{S}'(\mathbb{R})$; hence, taking the inverse Fourier transform, one sees that $\mathcal{L}^{\mathcal{F}}_\lambda[v_k] \to \mathcal{L}^{\mathcal{F}}_\lambda[v]$ in $\mathcal{S}'(\mathbb{R})$. By uniqueness of the limit, one has $\mathcal{L}_\lambda[v] = \mathcal{L}^{\mathcal{F}}_\lambda[v]$ and the proof of item (ii) is complete.

As an immediate consequence, one deduces item (i)(c) by using in particular Lemmata 6.1 and 6.3.

Step four: item (iii). Take $v_k, w_k \in \mathcal{S}(\mathbb{R})$ converging in $(BV_{\mathrm{loc}}(\mathbb{R}))_{w\text{-}\star} \cap H^1_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\})$ to $v, w \in H^1(\mathbb{R}_*)$. For such functions, it is immediate from the definition by Fourier transform (1.3) that

$$\int_{\mathbb{R}} \mathcal{L}_\lambda[v_k]w_k = \int_{\mathbb{R}} v_k \mathcal{L}_\lambda[w_k] = \int_{\mathbb{R}} \mathcal{L}_{\lambda/2}[w_k]\mathcal{L}_{\lambda/2}[v_k].$$

By Lemma 6.3, one has $\mathcal{L}_\lambda[u_k] \to \mathcal{L}_\lambda[u]$ in $L^1_{\mathrm{loc}}(\mathbb{R}) \cap L^2_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\})$ for $u = v, w$. By Lemma 6.1 and Banach–Alaoglu–Bourbaki's theorem, one has the following convergence (up to a subsequence):

$$u_k \to u \quad \text{in } L^2(\mathbb{R}) \text{ and in } L^\infty(\mathbb{R}) \text{ weak-}\star$$

for $u = v, w$. Indeed, (6.2) implies the strong convergence in L^2 and (6.1) implies that $(u_k)_k$ is bounded in L^∞, because it is (strongly) bounded in $BV_{\mathrm{loc}}(\mathbb{R}) \cap H^1_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\})$ as a convergent sequence in $(BV_{\mathrm{loc}}(\mathbb{R}))_{w\text{-}\star} \cap H^1_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\})$. Hence, one clearly can pass to the limit:

$$\int_{\mathbb{R}} \mathcal{L}_\lambda[v]w = \lim_{k\to+\infty} \int_{\mathbb{R}} \mathcal{L}_\lambda[v_k]w_k = \lim_{k\to+\infty} \int_{\mathbb{R}} v_k \mathcal{L}_\lambda[w_k] = \int_{\mathbb{R}} v \mathcal{L}_\lambda[w].$$

To pass to the limit in $\int_{\mathbb{R}} \mathcal{L}_{\lambda/2}[w_k]\mathcal{L}_{\lambda/2}[v_k]$, one uses Lemma 6.3 and item (ii). The proof of item (iii) is complete.

Step five: item (iv). It suffices to change the variable by $z \to -z$ in (2.2).

Step six: item (v). We consider only the case where $v(x_*) = \max_{\mathbb{R}^+} v \geqslant 0$, since the case $v(x_*) = \min_{\mathbb{R}^+} v \leqslant 0$ is symmetric. Simple computations show that

$$\begin{aligned}
\mathcal{L}_\lambda[v](x_*) &= -G_\lambda \int_{\mathbb{R}} \frac{v(x_* + z) - v(x_*)}{|z|^{1+\lambda}}\,dz \\
&= -G_\lambda \int_{-x_*}^{+\infty} \frac{v(x_* + z) - v(x_*)}{|z|^{1+\lambda}}\,dz - G_\lambda \int_{-\infty}^{-x_*} \frac{v(x_* + z) - v(x_*)}{|z|^{1+\lambda}}\,dz \\
&= -G_\lambda \int_{-x_*}^{+\infty} \frac{v(x_* + z) - v(x_*)}{|z|^{1+\lambda}}\,dz - G_\lambda \int_{-x_*}^{+\infty} \frac{v(-x_* - z') - v(x_*)}{|z' + 2x_*|^{1+\lambda}}\,dz',
\end{aligned}$$

after having changed the variable by $z' = -z - 2x_*$. By the oddity of v, we get

$$\mathcal{L}_\lambda[v](x_*) = -G_\lambda \int_{-x_*}^{+\infty} \left\{ \frac{v(x_* + z) - v(x_*)}{|z|^{1+\lambda}} - \frac{v(x_* + z) + v(x_*)}{|z + 2x_*|^{1+\lambda}} \right\} dz.$$

Let $f(z)$ denote the integrand above. Let us prove that for $0 \neq z > -x_*$, this integrand is non-positive. It is readily seen that for such z, one always has $\{\frac{1}{|z|^{1+\lambda}} - \frac{1}{|z+2x_*|^{1+\lambda}}\} > 0$. Then, one has

$$f(z) = v(x_* + z)\left\{\frac{1}{|z|^{1+\lambda}} - \frac{1}{|z+2x_*|^{1+\lambda}}\right\} - v(x_*)\left\{\frac{1}{|z|^{1+\lambda}} + \frac{1}{|z+2x_*|^{1+\lambda}}\right\}$$
$$\leqslant v(x_*)\left\{\frac{1}{|z|^{1+\lambda}} - \frac{1}{|z+2x_*|^{1+\lambda}}\right\} - v(x_*)\left\{\frac{1}{|z|^{1+\lambda}} + \frac{1}{|z+2x_*|^{1+\lambda}}\right\};$$

indeed, $x_* + z \in \mathbb{R}^+$, so that $v(x_* + z) \leqslant v(x_*)$. We infer that $f(z) \leqslant -v(x_*)\frac{2}{|z+2x_*|^{1+\lambda}} \leqslant 0$ and conclude that $\mathcal{L}_\lambda[v](x_*) \geqslant 0$. To finish, observe that f cannot be identically equal to zero, whenever v is non-trivial. This proves that $\mathcal{L}_\lambda[v](x_*) > 0$ and completes the proof of the lemma. □

Acknowledgements

The first author would like to thank the Department of Mathematics of Prince of Songkla University (Hat Yai campus) in Thailand, for having ensured an excellent working environment for him.

Appendix A. Proofs of Lemmata 3.3, 5.1, 6.1 and 6.2

Proof of Lemma 3.3. The supremum $m(t)$ is achieved because of (3.2), so that $K(t) \neq \emptyset$; moreover, one has for all $b > a > 0$,

$$\sup_{t\in(a,b),\, x\in K(t)} |x| < +\infty. \tag{A.1}$$

It is quite easy to show that m is continuous and we only detail the proof of the derivability from the right.

Let $t_0 > 0$ be fixed and $(t_k)_k$, $(x_k)_k$ be such that $\lim_{k\to+\infty} t_k = t_0$, $t_k > t_0$ and $x_k \in K(t_k)$, $m(t_k) = v(t_k, x_k)$ for all $k \geqslant 1$. By (A.1), $(x_k)_k$ is bounded; hence, taking a subsequence if necessary, one can assume that x_k converges toward some limit x_0. One has

$$\limsup_{k\to+\infty} \frac{m(t_k) - m(t_0)}{t_k - t_0} = \limsup_{k\to+\infty} \frac{v(t_k, x_k) - m(t_0)}{t_k - t_0} \leqslant \limsup_{k\to+\infty} \frac{v(t_k, x_k) - v(t_0, x_k)}{t_k - t_0} = \partial_t v(t_0, x_0),$$

thanks to the C^1-regularity of v. But, one has $x_0 \in K(t_0)$; indeed, for all $x \in \mathbb{R}$, one has $v(t_k, x_k) \geqslant v(t_k, x)$ so that the limit as $k \to +\infty$ gives $v(t_0, x_0) \geqslant v(t_0, x)$. Hence, one has proved that $\limsup_{k\to+\infty} \frac{m(t_k)-m(t_0)}{t_k-t_0} \leqslant \sup_{x\in K(t_0)} \partial_t v(t_0, x)$. In the same way, for all $x \in K(t_0)$ one has

$$\liminf_{k\to+\infty} \frac{m(t_k) - m(t_0)}{t_n - t_0} \geqslant \liminf_{k\to+\infty} \frac{v(t_k, x) - v(t_0, x)}{t_k - t_0} = \partial_t v(t_0, x).$$

This shows that

$$\liminf_{k\to+\infty} \frac{m(t_k) - m(t_0)}{t_k - t_0} \geqslant \max_{x\in K(t_0)} \partial_t v(t_0, x) \geqslant \limsup_{k\to+\infty} \frac{m(t_k) - m(t_0)}{t_k - t_0},$$

for all $t_0 > 0$ and $(t_k)_k$ such that $t_k \to t_0$, $t_k > t_0$. This means that m is right-differentiable with $m_r'(t_0) = \max_{x\in K(t_0)} \partial_t v(t_0, x)$ on $\mathbb{R}^+$. □

Proof of Lemma 5.1. Let us estimate the translations of v_ε. Fix $h \in \mathbb{R}$ and define $\mathcal{T}_h v_\varepsilon(x) := v_\varepsilon(x - h)$. Classical formula gives $\mathcal{F}(\mathcal{T}_h v_\varepsilon)(\xi) = e^{-2i\pi\xi h}\mathcal{F}(v_\varepsilon)(\xi)$. By Plancherel's equality, we deduce that

$$\int_{\mathbb{R}} |\mathcal{T}_h v_\varepsilon - v_\varepsilon|^2 = \int_{\mathbb{R}} \left|e^{-2i\pi\xi h} - 1\right|^2 \left|\mathcal{F}(v_\varepsilon)(\xi)\right|^2 d\xi$$
$$= \int_{\mathbb{R}} \frac{|e^{-2i\pi\xi h} - 1|^2}{|\xi|^\lambda} |\xi|^\lambda \left|\mathcal{F}(v_\varepsilon)(\xi)\right|^2 d\xi$$
$$\leqslant M_h \int_{\mathbb{R}} |\xi|^\lambda \left|\mathcal{F}(v_\varepsilon)(\xi)\right|^2 d\xi,$$

where $M_h := \max_{\xi\in\mathbb{R}} \frac{|e^{-2i\pi\xi h}-1|^2}{|\xi|^\lambda}$. Lemma 4.1 item (ii) and Plancherel's equality imply that

$$\int_{\mathbb{R}} |\mathcal{T}_h v_\varepsilon - v_\varepsilon|^2 \leqslant M_h \int_{\mathbb{R}} \left|\mathcal{L}_{\lambda/2}[v_\varepsilon]\right|^2.$$

By the assumptions of the lemma, we deduce that $\int_{\mathbb{R}} |\mathcal{T}_h v_\varepsilon - v_\varepsilon|^2 \leqslant C_0 M_h$ for some constant C_0 (the constant comes from (4.6)). Using that $e^z - 1 = O(|z|)$ in a neighborhood, it is easy to see that $\lim_{h\to 0} M_h = 0$, because $\lambda \in (0,2]$. The family $\{v_\varepsilon | \varepsilon \in (0,1)\}$ is bounded in $L^\infty(\mathbb{R})$, and thus also in $L^2_{\mathrm{loc}}(\mathbb{R})$. By the Fréchet–Kolmogorov theorem, it is relatively compact in $L^2_{\mathrm{loc}}(\mathbb{R})$. □

Proof of Lemma 6.1. For all $v \in H^1(\mathbb{R}_*)$, there exist the traces $v(0^\pm) \in \mathbb{R}$; it is not difficult to show that $|v(0^\pm)| \leqslant \|v\|_{H^1(\mathbb{R}_*)}$. Further, for all $\pm x > 0$,

$$v(x) = v(0^\pm) + \int_0^x (\partial_x v)_{|\mathbb{R}_*}(y)\,dy. \tag{A.2}$$

It follows that for all $R > 0$, one has $v \in BV((-R,R))$ with

$$|v|_{BV((-R,R))} \leqslant \left|v(0^+) - v(0^-)\right| + \left\|(\partial_x v)_{|\mathbb{R}_*}\right\|_{L^1((-R,R))} \leqslant (2+\sqrt{2R})\|v\|_{H^1(\mathbb{R}_*)}.$$

This shows that the inclusion $H^1(\mathbb{R}_*) \subset BV_{\mathrm{loc}}(\mathbb{R}) \cap H^1_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\})$ is continuous.

Now take $v \in BV_{\mathrm{loc}}(\mathbb{R}) \cap H^1_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\})$. Then v is continuous on $\mathbb{R}_*$ and $v(x) = v(1) + \int_1^x \partial_x v(y)\,dy$, where $\partial_x v$ can be a Radon measure with singular part supported by $\{0\}$. By the continuity of the inclusion $H^1(\mathbb{R} \setminus [-1,1]) \subset C_b(\mathbb{R} \setminus (-1,1))$, one deduces that v is bounded outside $(-1,1)$; since v is bounded by $|v(1)| + |v|_{BV((-1,1))}$ on $[-1,1]$, the inclusion $BV_{\mathrm{loc}}(\mathbb{R}) \cap H^1_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\}) \subset L^\infty(\mathbb{R})$ is continuous. From this result, it is easy to show (6.1).

The sequential embedding (6.2) is clear from (6.1). Indeed, Helly's theorem and L^q, L^p interpolation inequalities imply that the inclusion $L^\infty(\mathbb{R}) \cap BV_{\mathrm{loc}}(\mathbb{R}) \subset L^p_{\mathrm{loc}}(\mathbb{R})$ is continuous and compact for all $p \in [1,+\infty)$; since each converging sequence in $(BV_{\mathrm{loc}}(\mathbb{R}))_{w\text{-}\star} \cap H^1_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\})$ is (strongly) bounded in $BV_{\mathrm{loc}}(\mathbb{R}) \cap H^1_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\})$, the inclusions

$$\left(BV_{\mathrm{loc}}(\mathbb{R})\right)_{w\text{-}\star} \cap H^1_{\mathrm{loc}}\left(\overline{\mathbb{R}} \setminus \{0\}\right) \subset L^p_{\mathrm{loc}}(\mathbb{R}) \cap L^2_{\mathrm{loc}}\left(\overline{\mathbb{R}} \setminus \{0\}\right) \subset L^2(\mathbb{R})$$

are sequentially continuous. □

Proof of Lemma 6.2. From (A.2), one deduces that if $v \in H^1(\mathbb{R}_*)$ then $\partial_x v = (\partial_x v)_{|\mathbb{R}_*} + (v(0^+) - v(0^-))\delta_0$, where one has $(\partial_x v)_{|\mathbb{R}_*} \in L^2(\mathbb{R})$. Let $(\rho_k)_k \subset \mathcal{D}(\mathbb{R})$ be an approximate unit and define $v_k := \rho_k * v$. Then it is easy to check that $v_k \to v$ in $L^2(\mathbb{R})$ and that $\partial_x v_k = (\partial_x v)_{|\mathbb{R}_*} * \rho_k + (v(0^+) - v(0^-))\rho_k$ converges to $\partial_x v$ in $L^2_{\mathrm{loc}}(\overline{\mathbb{R}} \setminus \{0\})$ and in $(C_c(\mathbb{R}))'$ weak-$\star$. □

Appendix B. Technical results

Lemma B.1. *Let $m \in C(\mathbb{R}^+)$ be right-differentiable with*

$$m'_r(t) + \left(\max\{m,0\}\right)^2 \leqslant 0 \quad \text{on } \mathbb{R}^+. \tag{B.1}$$

Then one has $m(t) \leqslant \frac{1}{t}$ for all $t > 0$.

Proof. Let $t_0 > 0$ be such that $m(t_0)$ is positive. The function m has to be positive in some neighborhood of t_0; since (B.1) implies that m is non-increasing, this neighborhood has to contain the interval $(0,t_0]$. Dividing (B.1) by $m^2 = (\max\{m,0\})^2$ on this interval, we get: $(-\frac{1}{m})'_r \leqslant -1$ in $(0,t_0)$. Integrating this inequality, one deduces that for all $t < t_0$, $\frac{1}{m(t)} - \frac{1}{m(t_0)} \leqslant t - t_0$, which implies that $m(t_0) \leqslant (\frac{1}{m(t)} + t_0 - t)^{-1} \leqslant (t_0 - t)^{-1}$. Letting $t \to 0$, we conclude that $m(t_0) \leqslant \frac{1}{t_0}$ whenever $m(t_0)$ is positive. The proof is complete. □

Lemma B.2. *Let $\lambda \in (0,1)$ and $\Phi : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz-continuous and such that there exist $0 < \lambda' < \lambda$, M_Φ and L_Φ with*

$$\left|\Phi(x)\right| \leqslant M_\Phi\left(1+|x|^{\lambda'}\right) \quad \text{and} \quad \left|\partial_x \Phi(x)\right| \leqslant \frac{L_\Phi}{1+|x|^{1-\lambda'}}$$

for a.e. $x \in \mathbb{R}$. Then $\mathcal{L}_\lambda[\Phi]$ is well defined by (2.2) *and belongs to $C_b(\mathbb{R})$.*

The idea of the proof of this technical result comes from [3]; we give here a short proof for the reader's convenience.

Proof. In the sequel, C denotes a constant only depending on λ', λ, M_Φ and L_Φ. For all $x \in \mathbb{R}$ and $r > 0$, one has

$$\begin{aligned}\int_{\mathbb{R}} \frac{|\Phi(x+z)-\Phi(x)|}{|z|^{1+\lambda}}\,dz &\leqslant \|\partial_x\Phi\|_{L^\infty((x-r,x+r))} \int_{|z|\leqslant r} |z|^{-\lambda}\,dz + \int_{|z|>r} \frac{|\Phi(x+z)-\Phi(x)|}{|z|^{1+\lambda}}\,dz,\\ &\leqslant Cr^{1-\lambda}\|\partial_x\Phi\|_{L^\infty((x-r,x+r))} + \int_{|z|>r} \frac{|\Phi(x+z)-\Phi(x)|}{|z|^{1+\lambda}}\,dz.\end{aligned}$$

Since $|x+z|^{\lambda'} \leqslant |x|^{\lambda'} + |z|^{\lambda'}$ for all $x, z \in \mathbb{R}$, the last integral term is bounded above by

$$C\int_{|z|>r} \frac{2+2|x|^{\lambda'}+|z|^{\lambda'}}{|z|^{1+\lambda}}\,dz \leqslant Cr^{-\lambda}\left(1+|x|^{\lambda'}+r^{\lambda'}\right).$$

Finally, we get

$$\int_{\mathbb{R}} \frac{|\Phi(x+z)-\Phi(x)|}{|z|^{1+\lambda}}\,dz \leqslant Cr^{-\lambda}\left(1+|x|^{\lambda'}+r^{\lambda'}+r\|\partial_x\Phi\|_{L^\infty((x-r,x+r))}\right) \tag{B.2}$$

(for some constant C not depending on $x \in \mathbb{R}$ and $r > 0$).

This proves that $\mathcal{L}_\lambda[\Phi](x)$ is well defined by (2.2) for all $x \in \mathbb{R}$; moreover, we let the reader check that the continuity of $\mathcal{L}_\lambda[\Phi]$ can be easily deduced from the dominated convergence theorem. What is left to study is the behavior of $\mathcal{L}_\lambda[\Phi]$ at infinity. To do so, one takes $r = \frac{|x|}{2}$; from (B.2), one gets the following estimate: $|\mathcal{L}_\lambda[\Phi](x)| \leqslant C(|x|^{-\lambda} + |x|^{\lambda'-\lambda})$ for large x. The proof is complete. □

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