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Nonlinear diffusion with a bounded stationary level surface ☆

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Abstract

We consider nonlinear diffusion of some substance in a container (not necessarily bounded) with bounded boundary of class C^2. Suppose that, initially, the container is empty and, at all times, the substance at its boundary is kept at density 1. We show that, if the container contains a proper C^2-subdomain on whose boundary the substance has constant density at each given time, then the boundary of the container must be a sphere. We also consider nonlinear diffusion in the whole $\mathbb{R}^N$ of some substance whose density is initially a characteristic function of the complement of a domain with bounded C^2 boundary, and obtain similar results. These results are also extended to the heat flow in the sphere $\mathbb{S}^N$ and the hyperbolic space $\mathbb{H}^N$.

Résumé

Nous considérons la diffusion non linéaire d'une substance dans un récipient (pas nécessairement borné) avec frontière bornée de classe C^2. Supposons qu'initialement, le récipient soit vide et, à sa frontière, la densité de la substance soit gardée à tout moment égale à 1. Nous montrons que, si le récipient contient un sous-domaine C^2 propre à la frontière duquel la substance est gardée à tout moment à densité constante, alors la frontière du récipient doit être une sphère. Nous considérons aussi la diffusion non linéaire dans tout $\mathbb{R}^N$ d'une substance dont la densité est initialement une fonction caractéristique du complémentaire d'un domaine ayant la frontière bornée et C^2, et nous obtenons des résultats semblables. Ces résultats sont aussi généralisés au cas du flux de chaleur dans la sphère $\mathbb{S}^N$ et l'espace hyperbolique $\mathbb{H}^N$.

MSC: primary 35K60; secondary 35B40, 35B25

Keywords: Nonlinear diffusion equation; Overdetermined problems; Stationary level surfaces

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1. Introduction

1.1. Background

In the paper [12], we considered the solution $u = u(x,t)$ of the following initial–boundary value problem for the heat equation:

$$u_t = \Delta u \quad \text{in } \Omega \times (0, +\infty), \tag{1.1}$$

$$u = 1 \quad \text{on } \partial\Omega \times (0, +\infty), \tag{1.2}$$

$$u = 0 \quad \text{on } \Omega \times \{0\}, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $N \geqslant 2$, and we obtained the following symmetry result.

Theorem A. *(See [12].) Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqslant 2$, satisfying the exterior sphere condition and suppose that D is a domain, with boundary ∂D, satisfying the interior cone condition, and such that $\overline{D} \subset \Omega$.*

Assume that the solution u of problem (1.1)–(1.3) *is such that*

$$u(x,t) = a(t), \quad (x,t) \in \partial D \times (0, +\infty), \tag{1.4}$$

for some function $a : (0, +\infty) \to (0, +\infty)$. Then Ω must be a ball.

We recall some terminology from [12]. A surface satisfying (1.4) is said to be a *stationary isothermic surface*; Ω satisfies the *exterior sphere condition* if for every $y \in \partial\Omega$ there exists a ball $B_r(z)$ such that $\overline{B_r(z)} \cap \overline{\Omega} = \{y\}$, where $B_r(z)$ denotes an open ball centered at $z \in \mathbb{R}^N$ and with radius $r > 0$; D satisfies the *interior cone condition* if for every $x \in \partial D$ there exists a finite right spherical cone K_x with vertex x such that $K_x \subset \overline{D}$ and $\overline{K_x} \cap \partial D = \{x\}$.

In order to better understand the background of the present paper, we outline the proof of Theorem A improved by a result in [13]. The proof is essentially based on three ingredients.

The first one is a result of Varadhan [21] which states that, as $t \to 0^+$, the function $-4t \log u(x,t)$ converges uniformly on $\overline{\Omega}$ to the function $d(x)^2$, where

$$d(x) = \operatorname{dist}(x, \partial\Omega), \quad x \in \Omega.$$

To apply this result one needs the boundary $\partial\Omega$ to be also the boundary of the exterior $\mathbb{R}^N \setminus \overline{\Omega}$. The assumption that Ω satisfies the exterior sphere condition is sufficient for that to happen. Hence, by (1.4) there exists $R > 0$ satisfying

$$d(x) = R \quad \text{for every } x \in \partial D. \tag{1.5}$$

The second ingredient consists of a *balance law* proved in [10] and [11] (see [12] for another proof). It states that, in any domain G in $\mathbb{R}^N$, a solution $v = v(x,t)$ of the heat equation is zero at some point $x_0 \in G$ *for every* $t > 0$ if and only if

$$\int\limits_{\partial B_r(x_0)} v(x,t)\, dS_x = 0, \quad \text{for every } r \in \big[0, \operatorname{dist}(x_0, \partial G)\big) \text{ and } t > 0. \tag{1.6}$$

We use (1.6) in two different ways. In the former one, we choose $G = \Omega$ and $v = u_{x_i}$, $i = 1, \dots, N$, and obtain, by some manipulations, that the gradient ∇u is zero at some point $x_0 \in \Omega$ for every $t > 0$ if and only if

$$\int\limits_{\partial B_r(x_0)} (x - x_0) u(x,t)\, dS_x = 0, \quad \text{for every } r \in \big[0, d(x_0)\big) \text{ and } t > 0. \tag{1.7}$$

This condition helps us show that both ∂D and $\partial\Omega$ must be analytic. Indeed, with the aid of the interior cone condition for D, by combining (1.7) and (1.5) with the short-time behavior of u described in Varadhan [21], we can see that for every point $x_0 \in \partial D$ there exists a time $t_0 > 0$ satisfying $\nabla u(x_0, t_0) \neq 0$; this implies that ∂D is analytic. Thus, by using the exterior sphere condition for Ω again, we can conclude that $\partial\Omega$ is analytic and parallel to ∂D.

In the latter way of using (1.6), we choose two distinct points $P, Q \in \partial\Omega$ and let $p, q \in \partial D$ be the points such that

$$\overline{B_R(p)} \cap \partial\Omega = \{P\} \quad \text{and} \quad \overline{B_R(q)} \cap \partial\Omega = \{Q\}.$$

Thence, we consider the function $v = v(x,t)$ defined by

$$v(x,t) = u(x+p,t) - u(x+q,t) \quad \text{for } (x,t) \in B_R(0) \times (0,+\infty).$$

Since v satisfies the heat equation and $v(0,t) = a(t) - a(t) = 0$ for every $t > 0$, it follows from (1.6) that

$$t^{-\frac{N+1}{4}} \int\limits_{B_R(p)} u(x,t)\,dx = t^{-\frac{N+1}{4}} \int\limits_{B_R(q)} u(x,t)\,dx \quad \text{for every } t > 0.$$

Therefore, by taking advantage of the boundary layer for u for short times, we let $t \to 0^+$ and by using a result in [13], we obtain that

$$C(N)\left\{\prod_{j=1}^{N-1}\left[\frac{1}{R} - \kappa_j(P)\right]\right\}^{-\frac{1}{2}} = C(N)\left\{\prod_{j=1}^{N-1}\left[\frac{1}{R} - \kappa_j(Q)\right]\right\}^{-\frac{1}{2}}, \tag{1.8}$$

where $\kappa_j(x)$, $j = 1,\dots,N-1$, denotes the j-th principal curvature of the surface $\partial\Omega$ at the point $x \in \partial\Omega$ with respect to the inward unit normal vector to $\partial\Omega$, and where $C(N)$ is a positive constant depending only on N (see [13, Theorem 4.2]).

With (1.8) in hand, we are ready to use our third ingredient: Aleksandrov's sphere theorem [1, p. 412]. (A special case of this theorem is the well-known *Soap-Bubble Theorem* (see also [17]).) Since (1.8) implies that $\prod_{j=1}^{N-1}[\frac{1}{R} - \kappa_j(x)]$ is constant for $x \in \partial\Omega$, by applying Aleksandrov's sphere theorem, we conclude that $\partial\Omega$ must be a sphere (see [12] and [13] for details).

1.2. Main results

In the present paper, we extend and improve the results described in Section 1.1 to the case of certain *nonlinear* diffusion equations. It is evident that the introduction of a nonlinearity immediately rules out the use of our second ingredient, e.g. the balance law.

Since this was crucial to prove the necessary regularity of $\partial\Omega$, we will have to change our assumptions on the domain Ω. Thus, we shall assume Ω to be a domain (not necessarily bounded) in $\mathbb{R}^N$, $N \geqslant 2$, having *bounded* boundary of class C^2, that is, $\partial\Omega$ consists of m ($m \geqslant 1$) connected components $S_1,\dots,S_m \subset \partial\Omega$ which are the boundaries of bounded C^2-domains $G^1,\dots,G^m$ in $\mathbb{R}^N$, respectively. Thus

$$\partial\Omega = \bigcup_{j=1}^{m} S_j \quad \text{and} \quad S_j = \partial G^j \text{ for each } j \in \{1,\dots,m\}. \tag{1.9}$$

It should also be noticed that the lack of a balance law precludes the proof of property (1.8) (unless we find an alternative proof) and hence Aleksandrov's sphere theorem cannot be put in action. We shall overcome this difficulty by a new and more direct proof of symmetry only based on our first ingredient (conveniently modified in Theorem 1.1) and Serrin's *method of moving planes* (see [18,16,19]). It is worth mentioning that our proof does not need Serrin's *corner lemma* but simply uses the strong maximum principle and Hopf boundary lemma (see Theorems 1.2 and 1.3).

We now set up our framework. We consider the unique bounded solution $u = u(x,t)$ of the nonlinear diffusion equation

$$u_t = \Delta\phi(u) \quad \text{in } \Omega \times (0,+\infty), \tag{1.10}$$

subject to conditions (1.2) and (1.3). Here $\phi : \mathbb{R} \to \mathbb{R}$ is such that

$$\phi \in C^2(\mathbb{R}), \quad \phi(0) = 0, \quad \text{and} \tag{1.11}$$

$$0 < \delta_1 \leqslant \phi'(s) \leqslant \delta_2 \quad \text{for } s \in \mathbb{R}, \tag{1.12}$$

where δ_1, δ_2 are positive constants. By the maximum principle we know that

$$0 < u < 1 \quad \text{in } \Omega \times (0,+\infty).$$

Moreover, by applying the comparison principle to $u(x,t+h)$ and $u(x,t)$ for $h > 0$, we get

$$u_t \geqslant 0 \quad \text{and hence} \quad \Delta\phi(u) \geqslant 0 \quad \text{in } \Omega \times (0,+\infty). \tag{1.13}$$

Let $\Phi = \Phi(s)$ be the function defined by

$$\Phi(s) = \int_1^s \frac{\phi'(\xi)}{\xi}\,d\xi \quad \text{for } s > 0. \tag{1.14}$$

Note that if $\phi(s) \equiv s$, then $\Phi(s) = \log s$.

We extend Varadhan's result to our setting by the following

Theorem 1.1. *Let u be the solution of problem* (1.10), (1.2)–(1.3).

Then,

$$\lim_{t\to 0^+} -4t\Phi\big(u(x,t)\big) = d(x)^2$$

uniformly on every compact set in Ω.

The proof of this theorem is constructed by adapting well-known results of the theory of viscosity solutions [2,7,3,4,9]. The techniques developed to prove Theorem 1.1 can be used to extend this result to the important case in which the homogeneous boundary condition (1.2) is replaced by the non-homogeneous one

$$u = f \quad \text{on } \partial\Omega \times (0, +\infty), \tag{1.15}$$

where $f = f(x)$ is a continuous function on $\partial\Omega$, bounded from above and away from zero by positive constants (see Theorem 3.7).

The following symmetry result corresponds to Theorem A and Theorem 3.1 in [14].

Theorem 1.2. *Let D be a C^2 domain in $\mathbb{R}^N$ satisfying $\overline{D} \subset \Omega$. Assume that the solution u of problem* (1.10), (1.2)–(1.3)*, satisfies* (1.4).

Then $m = 1$ and $\partial\Omega$ must be a sphere.

When Ω is limited to unbounded domains, we have

Theorem 1.3. *Let D be a C^2 unbounded domain in $\mathbb{R}^N$ satisfying $\overline{D} \subset \Omega$.*

Assume that, for any connected component Γ of ∂D, the solution u of problem (1.10), (1.2)–(1.3)*, satisfies the following condition*:

$$u(x,t) = a_\Gamma(t), \quad (x,t) \in \Gamma \times (0, +\infty), \tag{1.16}$$

for some function $a_\Gamma : (0, +\infty) \to (0, +\infty)$.

Then $m = 1$ and $\partial\Omega$ must be a sphere.

When $\phi(s) = s$ and Ω is bounded, Theorem A is clearly stronger than Theorem 1.2, since in the former we can use the balance law to infer better regularity. Furthermore, the same techniques used for the proof of Theorem A also yield a more general version of it (see Theorem 2.1).

The paper is then organized as follows. In Section 2, we prove all our symmetry results: Theorems 1.2, 1.3 and 2.1. In Section 3, with the help of the theory of viscosity solutions, we prove Theorem 1.1 and its extension, Theorem 3.7. Section 4 is devoted to show similar results for the unique bounded solution of the Cauchy problem for nonlinear diffusion equations. In Section 5, we mention that this kind of results also hold for the heat flow in the sphere $\mathbb{S}^N$ and the hyperbolic space $\mathbb{H}^N$ with $N \geqslant 2$.

2. Symmetry results

In this section, with the aid of Theorem 1.1, by applying the method of moving planes to problem (1.10), (1.2)–(1.3) directly, we prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. First of all, we consider the case where Ω is unbounded. In this case Ω is an exterior domain, that is, we have

$$\overline{G^i} \cap \overline{G^j} = \emptyset \quad \text{if } i \neq j,\ i, j = 1, \ldots, m, \quad \text{and} \quad \Omega = \mathbb{R}^N \setminus \left(\bigcup_{j=1}^m \overline{G^j} \right)$$

(see (1.9) for the definitions of S_j and G^j). Set

$$G = \bigcup_{j=1}^m G^j. \tag{2.1}$$

Then G is a bounded open set in $\mathbb{R}^N$ having m connected components $G^1, \ldots, G^m$. Theorem 1.1 and the assumption (1.4) yield (1.5). Furthermore, with the aid of our C^2-smoothness assumption on ∂D and $\partial\Omega$, we see that both $\partial\Omega$ and ∂D consist of m connected closed hypersurfaces and each component of $\partial\Omega$ is parallel, at distance R, to only one component of ∂D.

We apply the method of moving planes to the open set G. The proof runs similarly to those of Serrin's [18] — or Reichel's [16] and Sirakov's [19] for exterior domains — but with the major difference that, here, since the relevant overdetermination takes place *inside* Ω, Serrin's corner lemma — an extension of Hopf boundary lemma to domains with corners — is not needed.

Let ℓ be a unit vector in $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, and let π_λ be the hyperplane $x \cdot \ell = \lambda$. For large λ, π_λ will be disjoint from $\overline{G}$; as λ decreases, π_λ will intersect $\overline{G}$ and cut off from G an open cap G_λ (on the same side of $\lambda \to +\infty$).

Denote by G'_λ the reflection of G_λ in the plane π_λ. At the beginning, G'_λ will be and remain in G until one of the following occurs:

(i) G'_λ becomes internally tangent to ∂G at some point P not on π_λ;
(ii) π_λ reaches a position in which it is orthogonal to ∂G at some point Q.

Let λ_* denote the (minimal) value of λ at which the plane π_λ reaches one of these positions and suppose that G is not symmetric with respect to π_{λ_*}. Let Ω_ℓ be the connected component of $\Omega \cap \{x \in \mathbb{R}^N\colon\ x \cdot \ell < \lambda_*\}$ whose boundary contains the points P or Q in the respective cases (i) or (ii). Since, as already observed, $\partial\Omega$ and ∂D consist of connected closed pairwise parallel hypersurfaces, we can find points P^* and Q^* in ∂D such that $|P - P^*|$ or $|Q - Q^*|$ equal R, respectively, and we have that $P^* \in \Omega_\ell$ and $Q^* \in \partial\Omega_\ell \cap \pi_{\lambda_*}$.

Let $x^\lambda = x + 2[\lambda - (x \cdot \ell)]\ell$ denote the reflection of a point $x \in \mathbb{R}^N$ in the plane π_λ. For $(x, t) \in \Omega_\ell \times (0, \infty)$, consider the function $w = w(x, t)$ defined by

$$w(x, t) = u\big(x^{\lambda_*}, t\big).$$

Then it follows from (1.4) that

$$w(P^*, t) = u(P^*, t) \quad \text{or} \quad \frac{\partial u}{\partial \ell}(Q^*, t) = 0 \quad \text{for all } t > 0, \tag{2.2}$$

where in the second equality we have used the fact that the vector ℓ is tangential also to ∂D at $Q^* \in \partial D$.

Observe that w and u satisfy

$$\begin{aligned}
&w_t = \Delta\phi(w) \quad \text{and} \quad u_t = \Delta\phi(u) && \text{in } \Omega_\ell \times (0, +\infty),\\
&w = u && \text{on } (\partial\Omega_\ell \cap \pi_{\lambda_*}) \times (0, +\infty),\\
&w < 1 = u && \text{on } (\partial\Omega_\ell \setminus \pi_{\lambda_*}) \times (0, +\infty),\\
&w = u = 0 && \text{on } \Omega_\ell \times \{0\}.
\end{aligned}$$

Hence, by the strong comparison principle,

$$w < u \quad \text{in } \Omega_\ell \times (0, +\infty). \tag{2.3}$$

Indeed, (2.3) can be obtained by applying the strong comparison principle to the bounded solutions $W = \phi(w)$ and $U = \phi(u)$ of $W_t = \frac{1}{\psi'(W)} \Delta W$ and $U_t = \frac{1}{\psi'(U)} \Delta U$, respectively; here, ψ is the inverse function of ϕ.

If case (i) applies, (2.3) contradicts the first equality in (2.2), since $P^* \in \Omega_\ell$. If case (ii) applies, by using Hopf's boundary point lemma, we can infer that

$$\frac{\partial u}{\partial \ell}(Q^*, t) < 0 \quad \text{for all } t > 0,$$

which contradicts the second equality in (2.2).

In conclusion, G is symmetric for any direction $\ell \in \mathbb{R}^N$, and in view of the definition (2.1) of G, $m = 1$ and G must be a ball. Namely, Ω is the exterior of a ball and $\partial\Omega$ must be a sphere.

When Ω is bounded, it suffices to apply the method of moving planes directly to Ω. □

Proof of Theorem 1.3. With the aid of the C^2 smoothness assumption of both ∂D and $\partial\Omega$, Theorem 1.1 and the assumption (1.16), together with the fact that D is unbounded, yield that $\partial\Omega$ and ∂D consist of m pairs of connected closed hypersurfaces being parallel to each other respectively. (When D is bounded, ∂D may consist of two connected components being parallel to one component of $\partial\Omega$.) Hence, the proof runs similarly to that of Theorem 1.2, with the only difference that the components in each pair constituting $\partial\Omega \cup \partial D$ may be at different distance from one another. □

We conclude this section with a more general version of Theorem A.

Theorem 2.1. *Let Ω be a domain* (*not necessarily bounded*) *in $\mathbb{R}^N$, $N \geqslant 2$, satisfying the exterior sphere condition and suppose that $\partial\Omega$ is bounded. Let D be a domain with $\overline{D} \subset \Omega$, and let Γ be a connected component of ∂D satisfying*

$$\operatorname{dist}(\Gamma, \partial\Omega) = \operatorname{dist}(\partial D, \partial\Omega). \tag{2.4}$$

Suppose that D satisfies the interior cone condition on Γ. Assume that the solution u of problem (1.1)–(1.3) *satisfies* (1.16).

Then $\partial\Omega$ must be either a sphere or the union of two concentric spheres.

Proof. Because of the assumption (2.4), the proofs of Lemma 2.2 of [14] and Lemma 3.1 in [12] also work in this situation. Then, there exists a connected component S of $\partial\Omega$ such that both Γ and S are analytic and these are parallel with distance $R = \operatorname{dist}(\Gamma, \partial\Omega)$; also, $\prod_{j=1}^{N-1}[\frac{1}{R} - \kappa_j(x)]$ is constant for $x \in S$. Since S is bounded, by applying Aleksandrov's sphere theorem [1] to this equation, we see that S and Γ are concentric spheres.

Let E be the annulus with $\partial E = S \cup \Gamma$. With the help of the analyticity of u, by proceeding as in the proof of Theorem 3.1 in [14], we see that for any $i \neq j$

$$-(x_j - a_j)\frac{\partial u(x,t)}{\partial x_i} + (x_i - a_i)\frac{\partial u(x,t)}{\partial x_j} = 0 \quad \text{in } \Omega \times (0, +\infty),$$

where the point $a = (a_1, \dots, a_N) \in \mathbb{R}^N$ is the center of the sphere S. Hence u must be radially symmetric with respect to a. □

3. Short-time behavior of solutions of nonlinear diffusion equations

In this section, with the help of the theory of viscosity solutions, we prove our keystone result, Theorem 1.1. We begin with some preliminaries.

Lemma 3.1. *Let $w = \phi(u)$, where u is the solution of* (1.10), (1.2)–(1.3). *For $j = 1, 2$, let w_j solve the problem*:

$$(w_j)_t = \delta_j \Delta w_j \quad \text{in } \Omega \times (0, +\infty), \tag{3.1}$$

$$w_j = \phi(1) \quad \text{on } \partial\Omega \times (0, +\infty), \tag{3.2}$$

$$w_j = 0 \quad \text{on } \Omega \times \{0\}. \tag{3.3}$$

Then

$$w_1 \leqslant w \leqslant w_2 \quad \text{in } \Omega \times (0, +\infty).$$

Proof. Since $w_t = \phi'(u)\Delta w$, from (1.12) and (1.13) we have:

$$\delta_1 \Delta w \leqslant w_t \leqslant \delta_2 \Delta w \quad \text{in } \Omega \times (0, +\infty). \tag{3.4}$$

Hence, by the comparison principle we get our claim. □

Now, let $\Psi = \Psi(s)$ be the inverse function of Φ. Then

$$s = \Phi\big(\Psi(s)\big) = \int_1^{\Psi(s)} \frac{\phi'(\xi)}{\xi}\, d\xi$$

and

$$\Psi(s) = \phi'\big(\Psi(s)\big)\Psi'(s), \tag{3.5}$$

by differentiating in s.

As in Freidlin and Wentzell [5], for $0 < \varepsilon < 1$, define the function $u^\varepsilon = u^\varepsilon(x, t)$ by

$$u^\varepsilon(x, t) = u(x, \varepsilon t) \quad \text{for } (x, t) \in \Omega \times (0, +\infty).$$

Then u^ε satisfies

$$\begin{aligned} &u^\varepsilon_t = \varepsilon \Delta\phi\big(u^\varepsilon\big) && \text{in } \Omega \times (0, +\infty),\\ &u^\varepsilon = 1 && \text{on } \partial\Omega \times (0, +\infty),\\ &u^\varepsilon = 0 && \text{on } \Omega \times \{0\}. \end{aligned}$$

Moreover, the function $v^\varepsilon = v^\varepsilon(x, t)$ defined by

$$v^\varepsilon(x, t) = -\varepsilon\Phi\big(u^\varepsilon(x, t)\big) \quad \text{for } (x, t) \in \Omega \times (0, +\infty)$$

is such that $u^\varepsilon = \Psi(-\varepsilon^{-1}v^\varepsilon)$ and, by (3.5), we have that

$$v^\varepsilon_t = \varepsilon\phi'\Delta v^\varepsilon - \big|\nabla v^\varepsilon\big|^2 \quad \text{in } \Omega \times (0, +\infty), \tag{3.6}$$

$$v^\varepsilon = 0 \quad \text{on } \partial\Omega \times (0, +\infty), \tag{3.7}$$

$$v^\varepsilon = +\infty \quad \text{on } \Omega \times \{0\}, \tag{3.8}$$

where $\phi' = \phi'(\Psi(-\varepsilon^{-1}v^\varepsilon))$.

Lemma 3.2. *It holds that for $(x, t) \in \overline{\Omega} \times (0, +\infty)$*

$$\frac{\delta_1}{\delta_2} \cdot \frac{1}{4t} d(x)^2 \leqslant \liminf_{\varepsilon \to 0^+} v^\varepsilon(x, t) \leqslant \limsup_{\varepsilon \to 0^+} v^\varepsilon(x, t) \leqslant \frac{\delta_2}{\delta_1} \cdot \frac{1}{4t} d(x)^2,$$

where these limits as $\varepsilon \to 0^+$ are uniform in every compact set contained in $\overline{\Omega} \times (0, +\infty)$.

Proof. We observe that the following hold:

$$\delta_1 s \leqslant \phi(s) \leqslant \delta_2 s \quad \text{for } s \geqslant 0, \tag{3.9}$$

$$-\delta_1 \log s \leqslant -\Phi(s) \leqslant -\delta_2 \log s \quad \text{for } 0 < s \leqslant 1, \tag{3.10}$$

$$e^{s/\delta_1} \leqslant \Psi(s) \leqslant e^{s/\delta_2} \quad \text{for } -\infty < s \leqslant 0. \tag{3.11}$$

Let $w^\varepsilon_j = w^\varepsilon_j(x, t)$ $(j = 1, 2)$ be the functions defined by

$$w^\varepsilon_j(x, t) = w_j(x, \varepsilon t),$$

where the w_j's are defined in Lemma 3.1. With the aid of (3.9) and (3.10), it follows from Lemma 3.1 that

$$-\varepsilon\delta_1 \log\left(\frac{w^\varepsilon_2}{\delta_1}\right) \leqslant v^\varepsilon \leqslant -\varepsilon\delta_2 \log\left(\frac{w^\varepsilon_1}{\delta_2}\right) \quad \text{in } \Omega \times (0, +\infty).$$

By the result of Varadhan [21], we see that, as $\varepsilon \to 0^+$, the functions $-\varepsilon\delta_j \log w_j^\varepsilon$ converge to the function $\frac{1}{4t}d(x)^2$ uniformly on every compact set contained in $\overline{\Omega} \times (0, +\infty)$, since each scaled function $\frac{1}{\phi(1)} w_j(x, \delta_j^{-1}t)$ solves problem (1.1)–(1.3). Our claim then follows at once. $\square$

The next lemma easily follows from Lemma 3.2.

Lemma 3.3. *For any compact set K in $\Omega \times (0, +\infty)$, there exist three positive constants ε_0, c_1, and c_2 $(0 < c_1 \leqslant c_2)$ depending on K such that*

$$0 < c_1 \leqslant v^\varepsilon \leqslant c_2 \quad \text{in } K,$$

for $0 < \varepsilon \leqslant \varepsilon_0$.

The key point in the proof of Theorem 1.1 is to obtain the following gradient estimate which we shall prove at the end of this section.

Lemma 3.4. *For any compact set K in $\Omega \times (0, +\infty)$, there exist two positive constants ε_1 $(\varepsilon_1 \leqslant \varepsilon_0)$ and c_3 depending on K, such that*

$$\left|\nabla v^\varepsilon\right| \leqslant c_3 \quad \text{in } K,$$

for $0 < \varepsilon \leqslant \varepsilon_1$.

Then, by combining Lemmas 3.3 and 3.4 with Gilding's result [6], we obtain the following uniform Hölder estimate.

Lemma 3.5. *For any compact set K in $\Omega \times (0, +\infty)$, there exist two positive constants ε_2 $(\varepsilon_2 \leqslant \varepsilon_1)$ and c_4 depending on K, such that*

$$\left|v^\varepsilon(x,t) - v^\varepsilon(x,s)\right| \leqslant c_4 |t-s|^{\frac{1}{2}} \quad \text{for any pair } (x,t), (x,s) \in K$$

and for $0 < \varepsilon \leqslant \varepsilon_2$.

Theorem 3.6. *The following limit*

$$\lim_{\varepsilon \to 0^+} v^\varepsilon(x,t) = \frac{1}{4t} d(x)^2$$

holds uniformly on every compact set in $\Omega \times (0, +\infty)$.

Proof. Lemmas 3.3, 3.4, and 3.5 together with Ascoli–Arzelà's theorem and the Cantor diagonal process yield a positive vanishing sequence of numbers ε_n and a continuous function $v = v(x,t)$ in $\Omega \times (0, +\infty)$ such that, as $n \to \infty$, the v^{ε_n}'s converge to v uniformly on every compact set contained in $\Omega \times (0, +\infty)$. Hence, by Lemma 3.2,

$$\frac{\delta_1}{\delta_2} \cdot \frac{1}{4t} d(x)^2 \leqslant v(x,t) \leqslant \frac{\delta_2}{\delta_1} \cdot \frac{1}{4t} d(x)^2 \quad \text{for } (x,t) \in \Omega \times (0, +\infty). \tag{3.12}$$

Define a function $V = V(x,t)$ on $\mathbb{R}^N \times (0, +\infty)$ by

$$V(x,t) = \begin{cases} v(x,t) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

Since both d^2 and its gradient vanish on $\partial\Omega$, (3.12) yields that V is continuous on $\mathbb{R}^N \times (0, +\infty)$, differentiable at any point on $\partial\Omega \times (0, +\infty)$, and that both V and ∇V vanish on $\partial\Omega \times (0, +\infty)$. Also, (3.12) yields that $\lim_{t \to 0^+} v(x,t) = +\infty$ if $x \in \Omega$.

Therefore, by using the fact that v^ε solves problem (3.6)–(3.8), with the help of Crandall, Ishii, and Lions [2], we see that V is a viscosity solution of the following Cauchy problem:

$$\begin{aligned} &V_t = -|\nabla V|^2 && \text{in } \mathbb{R}^N \times (0, +\infty), \\ &V = 0 && \text{on } \left(\mathbb{R}^N \setminus \Omega\right) \times \{0\}, \\ &V = +\infty && \text{on } \Omega \times \{0\}. \end{aligned} \tag{3.13}$$

Moreover, since a uniqueness result of Strömberg [20] tells us that the Hopf–Lax formula provides the unique viscosity solution of the Cauchy problem (3.13), we must have that, for any $(x, t) \in \mathbb{R}^N \times (0, +\infty)$,

$$V(x, t) = \inf\left\{\varphi(\xi) + \frac{|x - \xi|^2}{4t} \colon \xi \in \mathbb{R}^N\right\} = \frac{(\operatorname{dist}(x, \mathbb{R}^N \setminus \Omega))^2}{4t},$$

where $\varphi = \varphi(\xi)$ is the lower semicontinuous initial data defined by

$$\varphi(\xi) = \begin{cases} +\infty & \text{if } \xi \in \Omega, \\ 0 & \text{if } \xi \notin \Omega. \end{cases}$$

By the uniqueness of V, the whole sequence $\{v^\varepsilon\}$ converges as $\varepsilon \to 0^+$, and we get our claim. $\square$

Proof of Theorem 1.1. The desired result follows by simply setting $t = 1$ and then $\varepsilon = t$ in Theorem 3.6. $\square$

By a simple argument, we can extend Theorem 1.1 to the important case of non-homogeneous boundary values.

Theorem 3.7. *Let $f = f(x)$ be a continuous function on $\partial\Omega$ such that*

$$0 < b_1 \leqslant f(x) \leqslant b_2 \quad \textit{for all } x \in \partial\Omega, \tag{3.14}$$

for some positive constants b_1 and b_2. Let u be the solution of problem (1.10), (1.15), (1.3).
Then,

$$\lim_{t \to 0^+} -4t\Phi\left(u(x, t)\right) = d(x)^2$$

uniformly on every compact set in Ω.

Proof. Consider the unique bounded solutions $u^j = u^j(x, t)$ $(j = 1, 2)$ of the following initial–boundary value problems:

$$\begin{aligned} &u^j_t = \Delta\phi\left(u^j\right) && \text{in } \Omega \times (0, +\infty), \\ &u^j = b_j && \text{on } \partial\Omega \times (0, +\infty), \\ &u^j = 0 && \text{on } \Omega \times \{0\}. \end{aligned}$$

Then it follows from (3.14) and the comparison principle that

$$u^1 \leqslant u \leqslant u^2 \quad \text{in } \Omega \times (0, +\infty). \tag{3.15}$$

With the help of Theorem 1.1, we see that, as $t \to 0^+$, the function $-4t\Phi(u^j(x, t))$ converges to the function $d(x)^2$ uniformly on every compact set in Ω for each $j = 1, 2$. Indeed, for each $j = 1, 2$, we set

$$U = \frac{u^j}{b_j}, \qquad \tilde\phi(s) = \frac{1}{b_j}\phi(b_j s) \quad \text{for } s \in \mathbb{R}, \qquad \text{and} \qquad \tilde\Phi(s) = \int_1^s \frac{\tilde\phi'(\xi)}{\xi}\, d\xi \quad \text{for } s > 0.$$

Then it follows that

$$\tilde\phi'(s) = \phi'(b_j s) \quad \text{for } s \in \mathbb{R}, \qquad \tilde\Phi(s) = \Phi(b_j s) - \Phi(b_j) \quad \text{for } s > 0, \tag{3.16}$$

and

$$U_t = \Delta\tilde\phi(U) \quad \text{in } \Omega \times (0, +\infty), \tag{3.17}$$

$$U = 1 \quad \text{on } \partial\Omega \times (0, +\infty), \tag{3.18}$$

$$U = 0 \quad \text{on } \Omega \times \{0\}. \tag{3.19}$$

Thus, applying Theorem 1.1 to U yields that, as $t \to 0^+$, the function $-4t\tilde{\Phi}(U(x,t))$ converges to the function $d(x)^2$ uniformly on every compact set in Ω. Hence, with the aid of the second equality of (3.16), this means that, as $t \to 0^+$, the function $-4t\Phi(u^j(x,t))$ converges to the function $d(x)^2$ uniformly on every compact set in Ω.

On the other hand, since Φ is increasing in $s > 0$, we have from (3.15) that

$$-4t\Phi\big(u^1\big) \geqslant -4t\Phi(u) \geqslant -4t\Phi\big(u^2\big) \quad \text{in } \Omega \times (0,+\infty),$$

which implies that, as $t \to 0^+$, the function $-4t\Phi(u(x,t))$ converges to the function $d(x)^2$ uniformly on every compact set in Ω. $\square$

Proof of Lemma 3.4. We use Bernstein's technique (see [3,7,4,9]). Let r, τ and T be positive numbers such that $\tau < 2\tau < T$ and $K \subset B_r(0) \times [2\tau, T]$. Take $\zeta \in C^\infty(B_{2r}(0) \times (\tau, T])$ satisfying

$$0 \leqslant \zeta \leqslant 1 \quad \text{and} \quad \zeta_t \geqslant 0 \quad \text{in } B_{2r}(0) \times (\tau, T],$$
$$\zeta = 1 \quad \text{on } B_r(0) \times [2\tau, T], \quad \text{and} \quad \operatorname{supp}\zeta \subset B_{2r}(0) \times (\tau, T].$$

In the sequel of this proof, we will use the constants ε_0, c_1 and c_2 of Lemma 3.3 relative to the compact set $\overline{B_{2r}(0)} \times [\tau, T]$.

Consider the function $z = z(x,t)$ defined by

$$z = \zeta^2\big|\nabla v^\varepsilon\big|^2 - \lambda v^\varepsilon, \tag{3.20}$$

where $\lambda > 0$ is a constant to be determined later, and $0 < \varepsilon \leqslant \varepsilon_0$. Suppose that (x_0, t_0) is a point in $B_{2r}(0) \times (\tau, T]$ satisfying

$$\zeta(x_0, t_0) > 0 \quad \text{and} \quad \max_{\overline{B_{2r}(0)} \times [\tau, T]} z = z(x_0, t_0).$$

At (x_0, t_0) we then have

$$z_t \geqslant 0, \qquad z_{x_i} = 0, \quad \text{and} \quad \Delta z \leqslant 0, \tag{3.21}$$

and hence

$$0 \leqslant z_t - \varepsilon\phi'\big(\Psi\big(-\varepsilon^{-1}v^\varepsilon\big)\big)\Delta z.$$

The following inequality holds at (x_0, t_0) for some positive constants A_1 and A_2 independent of (x_0, t_0) and ε:

$$\lambda\big|\nabla v^\varepsilon\big|^2 \leqslant A_1\big|\nabla v^\varepsilon\big|^2 + A_2\zeta\big|\nabla v^\varepsilon\big|^3 - 2\zeta^2\big|\nabla v^\varepsilon\big|^2\phi''\Psi'\Delta v^\varepsilon - \varepsilon\phi'\zeta^2\big|\nabla^2 v^\varepsilon\big|^2. \tag{3.22}$$

It is a consequence of (3.21) and some lengthy calculations that, for the reader's convenience, will be carried out in Appendix A.

Now, we want to bound the third and fourth summand on the right-hand side of (3.22). The bound for the latter summand,

$$-\varepsilon\phi'\zeta^2\big|\nabla^2 v^\varepsilon\big|^2 \leqslant -\varepsilon\delta_1\zeta^2\big|\nabla^2 v^\varepsilon\big|^2,$$

easily follows from (1.12). In order to bound the former one, we use the fact that $\phi \in C^2(\mathbb{R})$ and Lemma 3.3, the algebraic inequality $2ab \leqslant a^2 + b^2$, and the key inequality

$$0 < \Psi'\big(-\varepsilon^{-1}v^\varepsilon\big) = \frac{\Psi(-\varepsilon^{-1}v^\varepsilon)}{\phi'} \leqslant \frac{1}{\delta_1}e^{-\frac{v^\varepsilon}{\varepsilon\delta_2}} \leqslant \frac{1}{\delta_1}e^{-\frac{c_1}{\varepsilon\delta_2}}, \tag{3.23}$$

which follows from (3.5), (1.12) and (3.11). With these three ingredients, we show that

$$-2\zeta^2\big|\nabla v^\varepsilon\big|^2\phi''\Psi'\Delta v^\varepsilon \leqslant \frac{1}{\delta_1}e^{-\frac{c_1}{\varepsilon\delta_2}}\zeta^2\big(A_3\big|\nabla v^\varepsilon\big|^4 + \big|\nabla^2 v^\varepsilon\big|^2\big),$$

at (x_0, t_0), for some positive constant A_3 independent of (x_0, t_0) and ε.

Set

$$M = \max_{\overline{B_{2r}(0)} \times [\tau, T]} \zeta\big|\nabla v^\varepsilon\big|, \qquad \lambda = \frac{M^2 + 1}{2(c_2 + 1)},$$

and choose ε_* in $(0, \varepsilon_0]$ so small to obtain that

$$\frac{A_3}{\delta_1} e^{-\frac{c_1}{\varepsilon\delta_2}} \leqslant \frac{1}{4(c_2+1)} \quad \text{and} \quad \frac{1}{\delta_1} e^{-\frac{c_1}{\varepsilon\delta_2}} \leqslant \varepsilon\delta_1$$

for all $\varepsilon \in (0, \varepsilon_*]$. Then, with these choices of constants, from (3.22) and the aforementioned bounds on the second-order derivatives of v^ε, we have that

$$\frac{M^2+1}{4(c_2+1)} \left|\nabla v^\varepsilon\right|^2 \leqslant A_1 \left|\nabla v^\varepsilon\right|^2 + A_2 M \left|\nabla v^\varepsilon\right|^2 \tag{3.24}$$

at (x_0, t_0), for any $\varepsilon \in (0, \varepsilon_*]$.

Thus, if $\nabla v^\varepsilon(x_0, t_0) \neq 0$, from (3.24) we get

$$\frac{M^2+1}{4(c_2+1)} \leqslant A_1 + A_2 M,$$

which yields the desired gradient estimate at once. If $\nabla v^\varepsilon(x_0, t_0) = 0$, instead, we use the definition (3.20) of z to infer that

$$M^2 \leqslant \max z + \lambda \max v^\varepsilon \leqslant \lambda \max v^\varepsilon \leqslant \frac{M^2+1}{2(c_2+1)} c_2 \leqslant \frac{M^2}{2} + \frac{1}{2},$$

since $z(x_0, t_0) = -\lambda v^\varepsilon(x_0, t_0) < 0$. Therefore, $M \leqslant 1$ and this completes the proof. □

Remark. Lions, Souganidis, and Vázquez [9] consider the pressure equation for the porous medium equation:

$$(v_m)_t = (m-1) v_m \Delta v_m + |\nabla v_m|^2 \quad \text{for } m > 1,$$

and consider the asymptotic behavior as $m \to 1^+$. They get the interior gradient estimate for v_m independent of m by a technique similar to ours. We follow the outline of their proof but we use inequality (3.23) in order to overcome the difficulty caused by $\phi' = \phi'(\Psi(-\varepsilon^{-1} v^\varepsilon))$ in Eq. (3.6).

4. On the Cauchy problem

Let Ω be a domain given in (1.9) and consider the unique bounded solution $u = u(x,t)$ of the following Cauchy problem:

$$u_t = \Delta\phi(u) \quad \text{in } \mathbb{R}^N \times (0, +\infty), \quad \text{and} \quad u = \chi_{\mathbb{R}^N \setminus \Omega} \quad \text{on } \mathbb{R}^N \times \{0\}, \tag{4.1}$$

where $\chi_{\mathbb{R}^N \setminus \Omega}$ denotes the characteristic function of the set $\mathbb{R}^N \setminus \Omega$ and ϕ satisfies the assumptions (1.11)–(1.12). The purpose of this section is to prove the following result.

Theorem 4.1. *Theorems* 1.1*,* 1.2*, and* 1.3 *also hold for the unique bounded solution u of the Cauchy problem* (4.1).

Let us start with two lemmas.

Lemma 4.2. *There exist a small $\delta > 0$ and a C^2-function $f = f(\xi)$ on $\mathbb{R}$ satisfying*

$$\begin{aligned} &\left(\phi'(f) f'\right)' + \frac{1}{2}(\xi + 2\delta) f' = 0 \quad \textit{and} \quad f' < 0 \quad \textit{in } \mathbb{R}, \quad \textit{and} \\ &1 > f(-\infty) > f(0) > 0 > f(+\infty) > -\infty. \end{aligned} \tag{4.2}$$

Proof. It suffices to show that there exists a C^2-function $h = h(\xi)$ on $\mathbb{R}$ satisfying

$$\left(\phi'(h) h'\right)' + \frac{1}{2}\xi\, h' = 0 \quad \text{and} \quad h' < 0 \quad \text{in } \mathbb{R}, \quad \text{and} \tag{4.3}$$

$$1 > h(-\infty) > h(0) > 0 > h(+\infty) > -\infty. \tag{4.4}$$

Indeed, setting $f(\xi) = h(\xi + 2\delta)$ for sufficiently small $\delta > 0$ gives the desired solution f.

The assumptions (1.11)–(1.12) guarantee existence and uniqueness, *on the whole* $\mathbb{R}$, of the solution (h, H) of the Cauchy problem for the system of ordinary differential equations

$$h' = \frac{H}{\phi'(h)}, \qquad H' = -\frac{1}{2}\xi\frac{H}{\phi'(h)}, \quad \text{and} \quad \big(h(0), H(0)\big) = (h_0, H_0) \tag{4.5}$$

(obtained by letting $H = \phi'(h)h'$ in (4.3)); here $h_0 > 0$ and $H_0 < 0$ are given numbers. Also, by uniqueness we infer that $H < 0$ on $\mathbb{R}$ and hence $h' < 0$ on $\mathbb{R}$.

Thus, with the help of (1.12), by integrating the second equation in (4.5), we have that

$$H_0 \exp\left\{-\frac{\xi^2}{4\delta_2}\right\} \leqslant H(\xi) \leqslant H_0 \exp\left\{-\frac{\xi^2}{4\delta_1}\right\} < 0$$

and

$$\frac{H_0}{\delta_1}\exp\left\{-\frac{\xi^2}{4\delta_2}\right\} \leqslant h'(\xi) \leqslant \frac{H_0}{\delta_2}\exp\left\{-\frac{\xi^2}{4\delta_1}\right\} < 0$$

for $\xi \in \mathbb{R}$; hence

$$h_0 + \frac{H_0}{\delta_1}\int_0^\xi \exp\left\{-\frac{\eta^2}{4\delta_2}\right\} d\eta \leqslant h(\xi) \leqslant h_0 + \frac{H_0}{\delta_2}\int_0^\xi \exp\left\{-\frac{\eta^2}{4\delta_1}\right\} d\eta,$$

for $\xi > 0$, and

$$h_0 + \frac{H_0}{\delta_2}\int_0^\xi \exp\left\{-\frac{\eta^2}{4\delta_1}\right\} d\eta \leqslant h(\xi) \leqslant h_0 + \frac{H_0}{\delta_1}\int_0^\xi \exp\left\{-\frac{\eta^2}{4\delta_2}\right\} d\eta,$$

for $\xi < 0$. By letting $\xi \to +\infty$ and $\xi \to -\infty$, respectively, we get

$$h_0 + \frac{H_0}{\delta_1}\sqrt{\pi\delta_2} \leqslant h(+\infty) \leqslant h_0 + \frac{H_0}{\delta_2}\sqrt{\pi\delta_1},$$
$$h_0 - \frac{H_0}{\delta_2}\sqrt{\pi\delta_1} \leqslant h(-\infty) \leqslant h_0 - \frac{H_0}{\delta_1}\sqrt{\pi\delta_2}.$$

Therefore, (4.4) is obtained by setting

$$h_0 = \frac{\delta_1^{3/2}}{2(\delta_1^{3/2} + \delta_2^{3/2})} \quad \text{and} \quad H_0 = -\frac{\delta_1\delta_2}{\sqrt{\pi}(\delta_1^{3/2} + \delta_2^{3/2})}$$

in the last two formulas. □

Lemma 4.3. *There exists a constant $c_0 > 0$ satisfying*

$$c_0 \leqslant u < 1 \quad \text{on } \partial\Omega \times (0, 1].$$

Proof. First of all, by the strong maximum principle

$$0 < u < 1 \quad \text{in } \mathbb{R}^N \times (0, +\infty). \tag{4.6}$$

Consider the signed distance function $d^* = d^*(x)$ of $x \in \mathbb{R}^N$ to the boundary $\partial\Omega$ defined by

$$d^*(x) = \begin{cases} \operatorname{dist}(x, \partial\Omega) & \text{if } x \in \Omega, \\ -\operatorname{dist}(x, \partial\Omega) & \text{if } x \notin \Omega. \end{cases} \tag{4.7}$$

Since $\partial\Omega$ is C^2 and compact, there exists a number $\rho > 0$ such that $d^*(x)$ is C^2-smooth on a compact neighborhood $\mathcal{N}$ of the boundary $\partial\Omega$ given by

$$\mathcal{N} = \left\{x \in \mathbb{R}^N\colon -\rho \leqslant d^*(x) \leqslant \rho\right\}.$$

We set now

$$w(x,t) = f\big(t^{-\frac{1}{2}} d^*(x)\big) \quad \text{for } (x,t) \in \mathbb{R}^N \times (0,+\infty). \tag{4.8}$$

Then, it follows from a straightforward computation and the properties of f that

$$w_t - \Delta\phi(w) = -\frac{1}{t} f'\big(t^{-\frac{1}{2}} d^*\big)\big\{-\delta + \sqrt{t}\phi'(f)\Delta d^*\big\} \quad \text{in } \mathcal{N} \times (0,+\infty).$$

Notice that if $\sqrt{t} < \frac{\delta}{\delta_2 \max_{\mathcal{N}} |\Delta d^*|}$, then $w_t - \Delta\phi(w) < 0$. Hence, since $\partial\mathcal{N}$ is compact, in view of Lemma 4.2, (4.8), and (4.1), we observe that there exists a small $\tau > 0$ satisfying

$$\begin{aligned} &w_t - \Delta\phi(w) < 0 = u_t - \Delta\phi(u) && \text{in } \mathcal{N} \times (0,\tau],\\ &w \leqslant u && \text{on } \partial\mathcal{N} \times (0,\tau],\\ &w \leqslant u && \text{on } \mathcal{N} \times \{0\}. \end{aligned}$$

Here, we note that u and w are regarded as continuous mappings from $[0,\tau]$ to $L^1(\mathcal{N})$ and, by taking into account (4.2), the initial condition is satisfied by their limits as $t \to 0^+$ in $L^1(\mathcal{N})$.

Therefore it follows from the comparison principle that $w \leqslant u$ in $\mathcal{N} \times (0,\tau]$. In particular, we have

$$u \geqslant f(0)\ (>0) \quad \text{on } \partial\Omega \times (0,\tau].$$

Combining this with (4.6) completes the proof. □

Proof of Theorem 4.1. Consider the unique bounded solutions $u^\pm = u^\pm(x,t)$ of the following initial–boundary value problems:

$$\begin{aligned} &u^\pm_t = \Delta\phi\big(u^\pm\big) && \text{in } \Omega \times (0,+\infty),\\ &u^+ = 1 \quad \text{and} \quad u^- = c_0 && \text{on } \partial\Omega \times (0,+\infty),\\ &u^\pm = 0 && \text{on } \Omega \times \{0\}. \end{aligned}$$

Then it follows from Lemma 4.3 and the comparison principle that

$$u^- \leqslant u \leqslant u^+ \quad \text{in } \Omega \times (0,1]. \tag{4.9}$$

By applying Theorem 3.7 to $u^\pm$, we have that, as $t \to 0^+$, both functions $-4t\Phi(u^\pm(x,t))$ converge to the function $d(x)^2$ uniformly on every compact set in Ω.

On the other hand, since Φ is increasing in $s > 0$, we have from (4.9) that

$$-4t\Phi\big(u^-\big) \geqslant -4t\Phi(u) \geqslant -4t\Phi\big(u^+\big) \quad \text{in } \Omega \times (0,1],$$

which implies that, as $t \to 0^+$, the function $-4t\Phi(u(x,t))$ converges to the function $d(x)^2$ uniformly on every compact set in Ω. This means that Theorem 1.1 also holds for the Cauchy problem (4.1).

Finally, proceeding as in Section 2, with the aid of the strong comparison principle for the Cauchy problem, we can easily show that Theorems 1.2 and 1.3 also hold for the Cauchy problem (4.1). □

5. Sphere $\mathbb{S}^N$ and hyperbolic space $\mathbb{H}^N$

The purpose of this section is to show that similar results hold also for the heat flow in the sphere $\mathbb{S}^N$ and the hyperbolic space $\mathbb{H}^N$ with $N \geqslant 2$. In order to handle $\mathbb{S}^N$ and $\mathbb{H}^N$ together, let us put $\mathbb{M} = \mathbb{S}^N$ or $\mathbb{M} = \mathbb{H}^N$.

Let Ω be a domain in $\mathbb{M}$ with bounded C^2-smooth boundary $\partial\Omega$, and denote by L the Laplace–Beltrami operator on $\mathbb{M}$. Let $u = u(x,t)$ be the unique bounded solution either of the following initial–boundary value problem for the heat flow:

$$u_t = Lu \quad \text{in } \Omega \times (0,+\infty), \tag{5.1}$$

$$u = 1 \quad \text{on } \partial\Omega \times (0,+\infty), \tag{5.2}$$

$$u = 0 \quad \text{on } \Omega \times \{0\}, \tag{5.3}$$

or of the following Cauchy problem for the heat flow:

$$u_t = Lu \quad \text{in } \mathbb{M} \times (0, +\infty), \quad \text{and} \quad u = \chi_{\mathbb{M}\setminus\Omega} \quad \text{on } \mathbb{M} \times \{0\}, \tag{5.4}$$

where $\chi_{\mathbb{M}\setminus\Omega}$ denotes the characteristic function of the set $\mathbb{M} \setminus \Omega$.

Denote by $d(x) = \inf\{d(x, y)\colon\ y \in \partial\Omega\}$ the geodesic distance between x and $\partial\Omega$, where $d(x, y)$ is the geodesic distance between two points x and y in $\mathbb{M}$. Then, with the aid of a result of Norris [15, Theorem 1.1, p. 82] concerning the short-time asymptotics of the heat kernel of Riemannian manifolds, we later prove

Theorem 5.1. *Let u be the solution either of problem* (5.1)–(5.3) *or of problem* (5.4) *in $\mathbb{S}^N$ or $\mathbb{H}^N$. Then the function $-4t \log u(x, t)$ converges to the function $d(x)^2$ as $t \to 0^+$ uniformly on every compact set in $\overline{\Omega}$.*

Theorem 5.1 yields the following symmetry results.

Theorem 5.2. *Let u be the solution either of problem* (5.1)–(5.3) *or of problem* (5.4) *in $\mathbb{S}^N$ or $\mathbb{H}^N$. In the case of $\mathbb{S}^N$, assume that $\overline{\Omega}$ is contained in a hemisphere in $\mathbb{S}^N$.*

Then Theorems 1.2 *and* 1.3 *hold for $\mathbb{H}^N$ and Theorem* 1.2 *holds for $\mathbb{S}^N$ in the sense that $\partial\Omega$ must be a geodesic sphere in $\mathbb{H}^N$ or $\mathbb{S}^N$, respectively.*

Proof. By Theorem 5.1, each stationary isothermic surface of class C^2 in Ω is then parallel to a connected component of $\partial\Omega$ in the sense of the geodesic distance. The claims of our theorem can thus be proved by replacing the method of moving planes, used in Section 2 for $\mathbb{R}^N$, by a straightforward adaptation of the method of moving closed and totally geodesic hypersurfaces for $\mathbb{S}^N$ or $\mathbb{H}^N$ developed by Kumaresan and Prajapat in [8]. □

Proof of Theorem 5.1. Consider the signed distance function $d^* = d^*(x)$ of $x \in \mathbb{M}$ given by the same definition as (4.7). Since $\partial\Omega$ is C^2 and compact, there exists a number $\rho_0 > 0$ such that $d^*(x)$ is C^2-smooth on a compact neighborhood $\mathcal{N}$ of $\partial\Omega$ given by

$$\mathcal{N} = \left\{x \in \mathbb{M}\colon -\rho_0 \leqslant d^*(x) \leqslant \rho_0\right\}.$$

Set

$$\mathcal{N}^- = \left\{x \in \mathbb{M}\colon -\rho_0 \leqslant d^*(x) \leqslant 0\right\} \ (\subset \mathcal{N}).$$

Let $u^- = u^-(x, t)$ be the unique bounded solution of the following Cauchy problem:

$$u_t^- = Lu^- \quad \text{in } \mathbb{M} \times (0, +\infty), \quad \text{and} \quad u^- = \chi_{\mathcal{N}^-} \quad \text{on } \mathbb{M} \times \{0\}, \tag{5.5}$$

where $\chi_{\mathcal{N}^-}$ denotes the characteristic function of the set $\mathcal{N}^-$. Moreover, for each $0 < \rho < \rho_0$, we set

$$\mathcal{N}_\rho = \left\{x \in \mathbb{M}\colon -\rho \leqslant d^*(x) \leqslant \rho\right\} \ (\subset \mathcal{N}).$$

For each $0 < \rho < \rho_0$, let $u^{\rho+} = u^{\rho+}(x, t)$ be the unique bounded solution of the following Cauchy problem:

$$u_t^{\rho+} = Lu^{\rho+} \quad \text{in } \mathbb{M} \times (0, +\infty), \quad \text{and} \quad u^{\rho+} = 2\chi_{\mathcal{N}_\rho} \quad \text{on } \mathbb{M} \times \{0\}, \tag{5.6}$$

where $\chi_{\mathcal{N}_\rho}$ denotes the characteristic function of the set $\mathcal{N}_\rho$.

Let $\rho \in (0, \min\{1, \rho_0\})$ be arbitrarily small. Then, there exists a number $t_\rho > 0$ satisfying

$$u^{\rho+} > 1 \quad \text{in } \partial\Omega \times (0, t_\rho]. \tag{5.7}$$

Thus it follows from (5.7) and the comparison principle that for any $(x, t) \in \Omega \times (0, t_\rho]$

$$\int\limits_{\mathcal{N}^-} p(t, x, y)\, dy = u^-(x, t) \leqslant u(x, t) \leqslant u^{\rho+}(x, t) = 2 \int\limits_{\mathcal{N}_\rho} p(t, x, y)\, dy, \tag{5.8}$$

where $p = p(t, x, y)$ denotes the heat kernel or the fundamental solution of the heat equation on the whole $\mathbb{M}$.

On the other hand, it follows from a result of Norris [15, Theorem 1.1, p. 82] that the function $-4t\log p(t,x,y)$ converges to the function $d(x,y)^2$ as $t \to 0^+$ uniformly on every compact set in $\mathbb{M}\times\mathbb{M}$. If $\mathcal{K}$ is any compact set contained in $\overline{\Omega}$, there exists a number $t_{\rho,1}\in(0,t_\rho]$ satisfying

$$\left|-4t\log p(t,x,y) - d(x,y)^2\right| < \rho^2 \quad \text{for any } (t,x,y)\in(0,t_{\rho,1}]\times\mathcal{K}\times\mathcal{N}.$$

Then we have that for any $(t,x,y)\in(0,t_{\rho,1}]\times\mathcal{K}\times\mathcal{N}$

$$\exp\left(-\frac{d(x,y)^2+\rho^2}{4t}\right)\leqslant p(t,x,y)\leqslant \exp\left(-\frac{d(x,y)^2-\rho^2}{4t}\right). \tag{5.9}$$

Set $m=\max_{x\in\mathcal{K}} d(x)$. In view of (5.8) and (5.9), let us estimate u from above. Since $d(x,y)\geqslant\max\{0,d(x)-\rho\}$ for any $(x,y)\in\mathcal{K}\times\mathcal{N}_\rho$, we observe that

$$d(x,y)^2-\rho^2\geqslant d(x)^2-2m\rho \quad \text{for any } (x,y)\in\mathcal{K}\times\mathcal{N}_\rho.$$

Combining this inequality and (5.9) with (5.8) yields that for any $(x,t)\in\mathcal{K}\times(0,t_{\rho,1}]$

$$u(x,t)\leqslant 2\exp\left(-\frac{d(x)^2-2m\rho}{4t}\right)|\mathcal{N}_\rho|, \tag{5.10}$$

where $|\mathcal{N}_\rho|$ denotes the volume of $\mathcal{N}_\rho$.

Next, we proceed to estimating u from below. For each $x\in\mathcal{K}$, there exists a point $z\in\partial\Omega$ with $d(x)=d(x,z)$. Then there exists a geodesic ball B with radius $\frac{1}{2}\rho$ satisfying

$$B\subset\mathcal{N}^- \quad\text{and}\quad \overline{B}\cap\overline{\Omega}=\{z\}.$$

Since $d(x,y)\leqslant d(x,z)+d(z,y)\leqslant d(x)+\rho$ for any $y\in B$, we have

$$d(x,y)^2+\rho^2\leqslant d(x)^2+2(m+1)\rho \quad\text{for any } y\in B.$$

Combining this inequality and (5.9) with (5.8) yields that for any $(x,t)\in\mathcal{K}\times(0,t_{\rho,1}]$

$$u(x,t)\geqslant \exp\left(-\frac{d(x)^2+2(m+1)\rho}{4t}\right)|B|, \tag{5.11}$$

where $|B|$ denotes the volume of B and $|B|$ is independent of $x\in\mathcal{K}$.

Therefore, it follows from (5.10) and (5.11) that there exists a number $t_{\rho,2}\in(0,t_{\rho,1}]$ satisfying for any $(x,t)\in\mathcal{K}\times(0,t_{\rho,2}]$

$$d(x)^2-(2m+1)\rho\leqslant -4t\log u(x,t)\leqslant d(x)^2+(2m+3)\rho. \tag{5.12}$$

This completes the proof. $\square$

Appendix A. Proof of inequality (3.22)

Let us write $v=v^\varepsilon$ for simplicity. By (3.21) we have at (x_0,t_0)

$$z_t = 2\zeta\zeta_t|\nabla v|^2+2\zeta^2 v_{x_k}v_{x_k t}-\lambda v_t\geqslant 0, \tag{A.1}$$

$$z_{x_i} = 2\zeta\zeta_{x_i}|\nabla v|^2+2\zeta^2 v_{x_k}v_{x_k x_i}-\lambda v_{x_i}=0, \tag{A.2}$$

$$\begin{aligned}\Delta z ={}& 2|\nabla\zeta|^2|\nabla v|^2+2\zeta\Delta\zeta|\nabla v|^2+8\zeta\zeta_{x_i}v_{x_k}v_{x_k x_i}+2\zeta^2 v_{x_k x_i}v_{x_k x_i}\\ &+2\zeta^2 v_{x_k}(\Delta v)_{x_k}-\lambda\Delta v\leqslant 0,\end{aligned} \tag{A.3}$$

where the summation convention is understood. Hence, it follows from (A.1) and (A.3) that

$$\begin{aligned}0&\leqslant z_t-\varepsilon\phi'\Delta z\\ &=-\lambda(v_t-\varepsilon\phi'\Delta v)+2\zeta^2 v_{x_k}(v_t-\varepsilon\phi'\Delta v)_{x_k}+2\zeta^2 v_{x_k}\varepsilon\phi''\Psi'\cdot\left(-\varepsilon^{-1}v_{x_k}\right)\Delta v+2\zeta\zeta_t|\nabla v|^2\\ &\quad-\varepsilon\phi'\left\{2|\nabla\zeta|^2|\nabla v|^2+2\zeta\Delta\zeta|\nabla v|^2+8\zeta\zeta_{x_i}v_{x_k}v_{x_k x_i}+2\zeta^2 v_{x_k x_i}v_{x_k x_i}\right\}.\end{aligned}$$

Then with the aid of (3.6), we get

$$0 \leqslant \lambda|\nabla v|^2 + 2\zeta^2 v_{x_k}\left(-|\nabla v|^2\right)_{x_k} - 2\zeta^2|\nabla v|^2\phi''\Psi'\Delta v + 2\zeta\zeta_t|\nabla v|^2 - \varepsilon\phi'\left\{2|\nabla\zeta|^2|\nabla v|^2 + 2\zeta\Delta\zeta|\nabla v|^2 + 8\zeta\zeta_{x_i}v_{x_k}v_{x_kx_i} + 2\zeta^2 v_{x_kx_i}v_{x_kx_i}\right\}.$$

Here, by using (A.2) we obtain

$$\begin{aligned}2\zeta^2 v_{x_k}\left(-|\nabla v|^2\right)_{x_k} &= -4\zeta^2 v_{x_k}v_{x_i}v_{x_kx_i}\\ &= -4\left\{\frac{\lambda}{2}|\nabla v|^2 - \zeta\zeta_{x_i}v_{x_i}|\nabla v|^2\right\}\\ &= -2\lambda|\nabla v|^2 + 4\zeta\zeta_{x_i}v_{x_i}|\nabla v|^2.\end{aligned}$$

Then it follows that

$$\lambda|\nabla v|^2 \leqslant 4\zeta\zeta_{x_i}v_{x_i}|\nabla v|^2 - 2\zeta^2|\nabla v|^2\phi''\Psi'\Delta v + 2\zeta\zeta_t|\nabla v|^2 - \varepsilon\phi'\left\{2|\nabla\zeta|^2|\nabla v|^2 + 2\zeta\Delta\zeta|\nabla v|^2 + 8\zeta\zeta_{x_i}v_{x_k}v_{x_kx_i} + 2\zeta^2 v_{x_kx_i}v_{x_kx_i}\right\}.$$

Next, by using inequality $ab \leqslant \frac{1}{8}a^2 + 2b^2$, we get

$$\begin{aligned}-8\varepsilon\phi'\zeta\zeta_{x_i}v_{x_k}v_{x_kx_i} &\leqslant \varepsilon\phi'\left\{\zeta^2\left|\nabla^2 v\right|^2 + 16|\nabla\zeta|^2|\nabla v|^2\right\}\\ &\leqslant \varepsilon\phi'\zeta^2\left|\nabla^2 v\right|^2 + 16\varepsilon\delta_2|\nabla\zeta|^2|\nabla v|^2.\end{aligned}$$

Consequently, combining these inequalities with Lemma 3.3 yields inequality (3.22).

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