

Vanishing viscosity limit for an expanding domain in space

James P. Kelliher^a, Milton C. Lopes Filho^b, Helena J. Nussenzveig Lopes^{b,*}

^a *Department of Mathematics, Brown University, Box 1917, Providence, RI 02912, United States*

^b *IMECC, Caixa Postal 6065, University of Campinas – UNICAMP, 13083-970, Campinas, SP, Brazil*

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Abstract

We study the limiting behavior of viscous incompressible flows when the fluid domain is allowed to expand as the viscosity vanishes. We describe precise conditions under which the limiting flow satisfies the full space Euler equations. The argument is based on truncation and on energy estimates, following the structure of the proof of Kato's criterion for the vanishing viscosity limit. This work complements previous work by the authors, see Iftimie et al. (2009) [5], Kelliher (2008) [8].

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Résumé

Nous étudions le comportement à la limite des écoulements incompressibles visqueux en admettant que l'évanouissement de la viscosité est accompagné d'une expansion du domaine fluide. Nous décrivons des conditions précises sous lesquelles l'écoulement limite satisfait les équations d'Euler spatiales complètes. L'argument est fondé sur la troncature et sur des estimations d'énergie, suivant une stratégie pareille à la preuve du critère de Kato pour la limite de viscosité tendant à zéro. Ce résultat complète les travaux précédents des auteurs (Iftimie et al., 2009 [5]; Kelliher, 2008 [8]).

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* Corresponding author.

E-mail addresses: kelliher@math.brown.edu (J.P. Kelliher), mlopes@ime.unicamp.br (M.C.L. Filho), hlopes@ime.unicamp.br (H.J.N. Lopes).

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1. Introduction

In [5], the second and third authors, in collaboration with Dragoş Iftimie, showed that, if an obstacle is scaled by a factor ϵ , then in the limit as viscosity vanishes the solutions to the Navier–Stokes equations external to the obstacle converge strongly in $L^\infty([0, T]; L^2)$ to a solution to the Euler equations in the whole space, as long as $\epsilon < a\nu$ for a specific constant a . They also give the rate of convergence in terms of ν and ϵ .

In [8], the first author considered the complementary problem of large domain asymptotics, studying convergence to full plane flow of solutions of Euler or Navier–Stokes in a large domain. The present article is a natural continuation of both [5] and [8].

For a domain with boundary, it is a classical open problem whether solutions of the Navier–Stokes equations converge to solutions of the Euler equations when viscosity vanishes. In [5] the authors are considering two limits simultaneously: the vanishing viscosity limit and the limit as the obstacle shrinks to a point, solving the external problem for the Navier–Stokes equations. This means studying the way in which a small boundary obstructs the vanishing viscosity convergence. Here, we consider what happens as a bounded domain expands by a factor R to fill the whole space, giving the convergence rate in the vanishing viscosity limit for the internal problem in terms of ν and R . In the same spirit as [5], the present work regards the effect of distant boundaries in the vanishing viscosity limit.

More precisely, let Ω be a simply connected bounded domain in \mathbb{R}^d , $d = 2$ or 3 , with C^2 -boundary Γ and let $\Omega_R = R\Omega$ and $\Gamma_R = R\Gamma = \partial\Omega_R$, where we assume that the origin lies inside Ω .

A classical solution (u, p) to the Euler equations without forcing in all of \mathbb{R}^d satisfies

$$(E) \quad \begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0 & \text{in } (0, T) \times \mathbb{R}^d, \\ \operatorname{div} u = 0 & \text{in } [0, T] \times \mathbb{R}^d, \\ u = u_0 & \text{on } \{0\} \times \mathbb{R}^d, \end{cases}$$

where $\operatorname{div} u_0 = 0$. A classical solution $(u^{v,R}, p^{v,R})$ to the Navier–Stokes equations without forcing on Ω_R satisfies

$$(NS) \quad \begin{cases} \partial_t u^{v,R} + u^{v,R} \cdot \nabla u^{v,R} + \nabla p^{v,R} = \nu \Delta u^{v,R} & \text{in } (0, T) \times \Omega_R, \\ \operatorname{div} u^{v,R} = 0 & \text{in } [0, T] \times \Omega_R, \\ u^{v,R} = 0 & \text{on } [0, T] \times \Gamma_R, \\ u^{v,R} = u_0^{v,R} & \text{on } \{0\} \times \Omega_R, \end{cases}$$

where $u_0^{v,R} = 0$ on Γ_R .

We will work, however, with weak solutions to the Navier–Stokes equations (to avoid having to deal with the dependence of the solutions’ existence times on the viscosity).

We consider the classical functions spaces,

$$V(\Omega_R) = \{u \in H^1(\Omega_R): \operatorname{div} u = 0 \text{ in } \Omega_R \text{ and } u = 0 \text{ on } \Gamma_R\},$$

$$H(\Omega_R) = \{u \in L^2(\Omega_R): \operatorname{div} u = 0 \text{ in } \Omega_R \text{ and } u \cdot \mathbf{n} = 0 \text{ on } \Gamma_R\},$$

where \mathbf{n} is the outward directed unit normal vector field to Γ_R .

The spaces $V(\mathbb{R}^d)$ and $H(\mathbb{R}^d)$ are analogously defined.

We define the space

$$V_C(\mathbb{R}^d) = \{u \in V(\mathbb{R}^d): \operatorname{supp}(\operatorname{curl} u) \text{ is compact}\}.$$

We will use the notation $\omega(u) \equiv \operatorname{curl} u$ for the vorticity associated to a given velocity u .

In dimension two the condition $u \in V_C(\mathbb{R}^2)$ requires that the total mass of the vorticity be zero, see Section 3.1.3 of [13] for a discussion. Hence, if we want to allow vorticities with distinguished sign we must allow for infinite energy. To this end we recall the affine spaces E_m , introduced by J.-Y. Chemin in [1], following a construction by R.J. DiPerna and A. Majda, see [2]. We say that $u \in E_m$ if $u = v + \sigma$ for some $v \in H(\mathbb{R}^2)$ and for some stationary solution σ of the Euler equations whose vorticity is smooth, compactly supported and has integral m . More precisely, for σ given by

$$\sigma = \sigma(x) = \frac{x^\perp}{|x|^2} \int_0^{|x|} s\varphi(s) ds, \tag{1.1}$$

for some $\varphi \in C_c^\infty(\mathbb{R}_+)$ and $2\pi \int \varphi(s)s ds = m$. Given the arbitrariness in the choice of φ we will assume, without loss of generality, that φ is of distinguished sign. Above we used the notation $x^\perp = (-x_2, x_1)$ if $x = (x_1, x_2)$. Notice that $\omega(\sigma)(x) = \varphi(|x|)$.

The classical well-posedness results for weak solutions to (E) for u_0 in subspaces of $E_0 = H(\mathbb{R}^2)$ remain true when E_0 is replaced by E_m ; see, for instance, Theorem 5.1.1, p. 85 of [1] (Yudovich’s theorem). In particular, fixing a value of $T > 0$, if u_0 is in $E_m \cap \dot{H}^1(\mathbb{R}^2)$ with compactly supported initial vorticity then the solution u to (E) will lie in $C([0, T]; E_m) \cap L^\infty([0, T]; \dot{H}^1(\mathbb{R}^2))$.

Throughout this paper we will assume that the initial velocity u_0 for solutions to (E) lies in $C^s(\mathbb{R}^d)$ for $s > 1$ so that a unique solution u to the Euler equations (E) with initial velocity u_0 exists in the space $C^s([0, T] \times \mathbb{R}^d)$ for all $T < T^*$; see, for instance, Theorem 4.2.1, p. 77 of [1] (or see Theorem 7.1, below). The time T^* can be assumed to be arbitrary in two dimensions, where we also assume that u_0 lies in E_m (see Theorem 4.2.4, p. 82 of [1]), but only finite time existence is known in three dimensions. We assume that the initial vorticity is compactly supported with its support contained in a ball of radius R_0 and define

$$R(T) = \inf_{r \geq 0} \{r : \text{supp } \omega(u) \subseteq [0, T] \times B_r(0)\}.$$

That $R(T)$ is finite in two dimensions follows from the transport of vorticity by the flow associated to u , u being bounded uniformly over finite time. But $R(T)$ is also finite in three dimension, as we show in Theorem 7.1.

Definition 1.1 (Classes of initial velocities). Let $s > 1$. We treat the following three classes of initial velocities:

- I. u_0 is in $C^s(\mathbb{R}^2) \cap V_C(\mathbb{R}^2)$,
- II. u_0 is in $C^s(\mathbb{R}^2) \cap E_m \cap \dot{H}^1(\mathbb{R}^2)$, the support of $\omega(u_0)$ is compact, and Ω_1 is a disk,
- III. u_0 is in $C^s(\mathbb{R}^3) \cap V_C(\mathbb{R}^3)$.

We assume that the initial velocity $u_0^{v,R}$ is in $H(\Omega_R)$. For such initial velocities it is a classical result of Leray, see Chapter III of [14], that there exists a weak solution $u^{v,R}$ to the Navier–Stokes equations (NS); in two dimensions this solution is unique, a result due to Ladyzhenskaya. In three dimensions, global-in-time existence is known, but not uniqueness, so we arbitrarily choose one such solution for each value of v .

Our main result is the following:

Theorem 1.2. Let u_0 be in one of the three classes of initial velocities in Definition 1.1 and set $F(v, R) \equiv \|u_0^{v,R} - u_0\|_{L^2(\Omega_R)}$. For all $T < T^*$ there exists a constant $C = C(s, T, \Omega, u_0) > 0$ such that

(1) if $s > 1$,

$$\|u^{v,R} - u\|_{L^\infty([0,T]; L^2(\Omega_R))} \leq (C(v^{1/2} + R^{-\alpha}) + F(v, R))e^{CT};$$

(2) if $s \geq 2$,

$$\|u^{v,R} - u\|_{L^\infty([0,T]; L^2(\Omega_R))} \leq (C(v + R^{-\alpha}) + F(v, R))e^{CT},$$

for all sufficiently large R .

The exponent α is defined for each of the three cases as follows:

- I. $\alpha = 1$,
- II. $\alpha = 1/3$,
- III. $\alpha = 1/2$.

Of particular interest is when we define $u_0^{v,R}$, independently of v , to be that unique divergence-free vector field tangent to the boundary of Ω_R whose vorticity on Ω_R is the same as that of u_0 . In Section 10 such a vector field is denoted $u_0^{v,R} = \mathcal{W}_R u_0$. We will see in Corollary 10.2 that

$$\|u_0^{v,R} - u_0\|_{L^2(\Omega_R)} = F(v, R) \equiv F(R) \leq CR^{-\alpha} \tag{1.2}$$

for all $R \geq 2R_0$, with α defined as in Theorem 1.2. In this case, the term $F(R)$ in the bounds in Theorem 1.2 is dominated by the other term and so, in effect, it disappears.

It follows immediately from Theorem 1.2 that, as long as $R = R(v) \rightarrow \infty$ as $v \rightarrow 0$ and $F(v, R) \rightarrow 0$ as $R \rightarrow \infty$, $\|u^{v,R(v)} - u\|_{L^\infty([0,T];L^2(\Omega_{R(v)}))} \rightarrow 0$ as $v \rightarrow 0$.

It was shown in [8] for Case I that if u^R is the solution to the Euler equations on Ω_R with initial velocity $\mathcal{T}_R u_0$ then $\|u^R - u\|_{L^\infty([0,T];L^2(\Omega_{R(v)}))} \rightarrow 0$ as $R \rightarrow \infty$. Here, \mathcal{T}_R is a truncation operator, which will be defined precisely in Section 3, see (3.2). This result is extended in [9] to cover Case II and to use the projector $P_{V(\Omega_R)}$ —restriction to Ω_R followed by projection into $V(\Omega_R)$ —in place of \mathcal{T}_R . This gives the following corollary:

Corollary 1.3. *Let $T < T^*$ and set $u_0^{v,R} = P_{V(\Omega_R)} u_0$. Then, for u_0 as in Cases I or II,*

$$\|u^{v,R(v)} - u^{R(v)}\|_{L^\infty([0,T];L^2(\Omega_{R(v)}))} \rightarrow 0 \quad \text{as } v \rightarrow 0$$

as long as $R = R(v) \rightarrow \infty$ as $v \rightarrow 0$.

The energy argument in our proof of Theorem 1.2 follows fairly closely the argument in [5], which itself is closely connected to Kato’s argument in [7]. We can describe in a unified way the approach of all three papers—[7,5], and this one—as follows. Let u^{NS} be the solution to (NS) in a domain Ω and let u^E be the solution to (E) either in the whole space or, as in [7], in Ω itself. In [7], Ω is a fixed bounded domain; in [5], Ω is an external domain which is scaled to a point by a parameter ϵ ; for us, Ω is a bounded domain which is scaled by a parameter R to fill the whole space.

Define a correction velocity u^C to u^E such that $u^C = u^E$ on $\partial\Omega$ and is equal to zero outside a boundary layer Γ_δ of width δ . In [7], $\delta = Cv$; in [5], $\delta = \epsilon$; in this paper, $\delta = CR^\alpha$. Let $u^A = u^E - u^C$ be an “approximate solution” to (E), and observe that $u^A = 0$ on $\partial\Omega$.

The goal is to bound the norm of $u^{NS} - u^E = u^{NS} - u^A - u^C$ in the space $X = L^\infty([0, T]; L^2(\Omega))$. To do so, one first shows that $\|u^C\|_X \rightarrow 0$ as $\delta \rightarrow 0$ or ∞ as the case may be. Then one bounds $W = u^{NS} - u^A$ in X by making an energy argument, the nature of the argument differing in each case. Because $u^A = 0$ on $\partial\Omega$, no troublesome boundary terms appear, though certain other terms appear because u^A is only an approximate solution to (E).

Kato’s energy argument in [7] is designed to estimate all of the uncontrollable terms by the quantity

$$v \int_0^T \|\nabla u^{NS}\|_{L^2(\Gamma_{Cv})}^2, \tag{1.3}$$

which, by the most basic energy argument for solutions to (NS), is bounded uniformly for all T and must vanish if the vanishing viscosity limit is to hold. Kato’s innovation is to show that the vanishing of this term is sufficient for the vanishing viscosity limit to hold.

The results achieved in the three papers differ most fundamentally because for Kato ∇u^C scales like $1/v$, which is detrimental (but unavoidable), introducing terms into the energy argument that cannot quite be controlled. For us, ∇u^C scales like $R^{-\alpha}$ which allows us to control all of these terms. In [5], ∇u^C scales like $1/\epsilon$, but the domain shrinks in area like ϵ^2 , which largely counteracts the detrimental effects of ∇u^C .

The research presented here is part of a series of papers aimed at studying asymptotic behavior of incompressible flows under singular domain perturbations. The first result in this line of research concerned ideal 2D flow in the exterior of a small obstacle, see [3], followed by a study of viscous 2D flow in the same limit, see [4]. Beyond these, this research has included ideal 2D flows in bounded domains with multiple holes one of which vanishes, see [12], ideal or viscous 2D flow in a large domain, see [8], 3D viscous flow in the exterior of a small obstacle, [6] and, most recently, 2D flow exterior to a smooth obstacle approaching a segment of a curve, see [10] for the ideal flow case and [11] for the viscous case. The classical open problem of vanishing viscosity in the presence of boundaries motivated the coupling of singularly perturbed domain problems with vanishing viscosity, specifically when the boundary disappears as viscosity vanishes. The first result in this direction was obtained in [5] for the small obstacle limit and the current work can be regarded as a natural continuation of [8] in the same spirit.

This paper is organized as follows: Section 2 contains certain notation we use and conventions we follow. In Section 3 we describe an approximate solution u^R to the Euler equations on Ω_R which we use in Section 4 to prove Theorem 1.2. The proof of Theorem 1.2 relies, however, on a long series of estimates involving u^R , which require us to understand how to take a divergence-free vector field defined in the whole plane or space and “truncate” it in such a way that it is unchanged in the central part of the domain Ω_R , vanishes on the boundary of Ω_R , and yet differs in the pertinent norms on Ω_R as little as possible from the original vector field. We describe the two-dimensional version of such a truncation operator in Section 5 and use it in Section 6 to define and obtain the necessary estimates on u^R .

The definition and analysis of the truncation operator in three dimensions are markedly different from those in two dimensions. In Section 7 we derive uniform-in-time bounds on the decay of the velocity and its gradient for a solution to (E). We then define the truncation operator in three dimensions in Section 8 and obtain the estimates on u^R in three dimensions in Section 9. In Section 10 we prove (1.2). In Section 11 we make some comments and state a couple of open problems.

2. Preliminaries

The symbol C stands for a positive constant that can hold different values on either side of an inequality, though always has the same value on each side of an equality.

For a scalar function f in two dimensions we write $\nabla^\perp f := (-\partial_2 f, \partial_1 f)$. In two dimensions we define the vorticity of a vector field u to be the scalar curl, $\omega = \omega(u) := \partial_1 u^2 - \partial_2 u^1 \equiv \nabla^\perp \cdot u$. In three dimensions, we define the vorticity to be $\omega = \omega(u) := \text{curl } u$; that is, ω is the three-vector,

$$\omega = (\partial_2 u^3 - \partial_3 u^2, \partial_3 u^1 - \partial_1 u^3, \partial_1 u^2 - \partial_2 u^1).$$

It is sometimes convenient in three dimensions to view the vorticity as the anti-symmetric 3×3 matrix $\mathbb{A} = \mathbb{A}(u)$ whose entry in the i -th row, k -th column is $\omega_k^i = \omega_k^i(u) := (\partial_k u^i - \partial_i u^k)/2$. Thus,

$$\mathbb{A} = \frac{1}{2} \begin{pmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{pmatrix}.$$

Observe that the L^p -norms of \mathbb{A} and ω are equivalent, differing only by a multiplicative constant.

Given a divergence-free C^1 vector field u on \mathbb{R}^2 let $\omega = \omega(u)$ be its vorticity, which we assume to have compact support. We define the associated two-dimensional stream function ψ as

$$\psi = \psi(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x - y| \omega(y) dy, \tag{2.1}$$

so that $\Delta\psi = \omega$ and $u = \nabla^\perp \psi$.

Given a divergence-free C^1 vector field u on \mathbb{R}^3 with compactly supported vorticity $\omega = \omega(u)$ we define the associated three-dimensional (vector-valued) stream function Ψ as

$$\Psi = \Psi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|} \omega(y) dy. \tag{2.2}$$

Hence, $-\Delta\Psi = \omega$ and $u = \text{curl } \Psi$, the latter statement following since $\text{div } \Psi = 0$, which in turn can be seen from the equation $\Delta \text{div } \Psi = 0, \text{div } \Psi \rightarrow 0$ at ∞ .

We note in passing that an alternative to this vector-valued stream function is to define the matrix-valued stream function

$$\psi_{ik} := \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|} \omega_k^i(y) dy,$$

which has the property that $u^i = \sum_k \partial_k \psi_{ik}$. The advantage of defining the stream function in this way is that it can be generalized to higher dimensions.

3. Approximate solution to the Euler equations

Define a cutoff function φ^R in two dimensions as follows. Fix θ in $[0, 1]$. (We will ultimately choose a value of θ that optimizes the convergence rate in Theorem 1.2.) Let $\delta_1 = 1/2\bar{\kappa}$, where $\bar{\kappa}$ is the maximum curvature of $\Gamma = \partial\Omega$. Let Σ_R be a tubular neighborhood of Γ_R in Ω_R of uniform width $\delta_1 R^\theta$ for all R in $[1, \infty)$. (Decrease the value of δ_1 if necessary to insure that the origin is not contained in Σ_R .) Put coordinates (s, r) on Σ_R , where s is arc length along Γ , which locates a point on Γ , and r is the distance along the inward normal at that point.

Let g in $C^\infty([0, \delta_1])$ taking values in $[0, 1]$ be defined so that $g(0) = g'(0) = 0$ and $g = 1$ on $[\frac{\delta_1}{2}, \delta_1]$. Then define φ^R in $C^\infty(\Omega_R)$ by $\varphi^R(s, r) = g(R^{-\theta}r)$ for points (s, r) in Σ_R , and $\varphi^R = 1$ on $\Omega_R \setminus \Sigma_R$. Observe that

$$\|\nabla\varphi^R\|_{L^\infty(\Omega_R)} \leq CR^{-\theta}, \quad \|\nabla\nabla\varphi^R\|_{L^\infty(\Omega_R)} \leq CR^{-2\theta}, \tag{3.1}$$

and similarly for higher derivatives of φ^R , where C is independent of R in $[1, \infty)$, and $\varphi^R = 0$ and $\nabla\varphi^R = 0$ on Γ .

We define φ^R in three dimensions more simply. Let

$$\Sigma = \{x \in \Omega: \text{dist}(x, \Gamma) < 1/2\bar{\kappa}\},$$

where $\bar{\kappa}$ is the maximum of all sectional curvatures over all points of Γ . Let φ in $C^\infty(\Omega)$ taking values in $[0, 1]$ be defined so that $\varphi = 1$ on $\Omega \setminus \Sigma$ and $\varphi = 0, \nabla\varphi = 0$ on Γ , and let $\varphi^R(\cdot) = \varphi(\cdot/R)$ and $\Sigma_R = R\Sigma$. Then (3.1) holds with $\theta = 1$.

Let ψ be the two-dimensional stream function associated to the full-plane Euler velocity u , as in (2.1). We define the vector field u^R on Ω_R by

$$u^R = \mathcal{T}_R u := \nabla^\perp(\varphi^R \psi). \tag{3.2}$$

Notice that this defines an operator \mathcal{T}_R whose properties we will explore later.

If Ψ is the three-dimensional stream function associated to the full-space Euler velocity u , as in (2.2), then we define the approximation u^R on Ω_R by

$$u^R = \mathcal{T}_R u := \nabla \times (\varphi^R \Psi). \tag{3.3}$$

The operator \mathcal{T}_R in both cases has the property that $u^R = \mathcal{T}_R u$ lies not just in $H(\Omega_R)$ but in $V(\Omega_R)$, and so vanishes on the boundary. It also satisfies (E) in $\Omega_R \setminus \Sigma_R$. In this sense, it is an approximate solution to (E).

Clearly, u^R satisfies the identity

$$\partial_t u^R = -\varphi^R u \cdot \nabla u - \varphi^R \nabla p + \partial_t \psi \nabla^\perp \varphi^R \tag{3.4}$$

in two dimensions and

$$\partial_t u^R = -\varphi^R u \cdot \nabla u - \varphi^R \nabla p + \nabla \varphi^R \times \partial_t \Psi, \tag{3.5}$$

in three dimensions.

Next we state a proposition which contains the key estimates on u^R that we will use in Section 4 to prove Theorem 1.2. We prove the two-dimensional case of this proposition in Section 6 and the three-dimensional case in Section 9.

Proposition 3.1. *For all $T < T^*$, for all sufficiently large R , we have*

- (1) $\|\nabla u^R\|_{L^\infty([0, T]; L^2(\Omega_R))} \leq C,$
- (2) $\|u^R\|_{L^\infty([0, T] \times \Omega_R)} \leq C,$

- (3) $\|\nabla u^R\|_{L^\infty([0,T] \times \Omega_R)} \leq C,$
- (4a) $\|p \nabla \varphi^R\|_{L^\infty([0,T]; L^2(\Omega_R))} + \|\partial_t \psi \nabla \varphi^R\|_{L^\infty([0,T]; L^2(\Omega_R))} \leq CR^{-\theta}$ in 2D,
- (4b) $\|p \nabla \varphi^R\|_{L^\infty([0,T]; L^2(\Omega_R))} + \|\nabla \varphi^R \times \partial_t \Psi\|_{L^\infty([0,T]; L^2(\Omega_R))} \leq CR^{-1}$ in 3D,
- (5) $\|\Delta u^R\|_{L^\infty([0,T]; L^2(\Omega_R))} \leq C$ when $s \geq 2,$
- (6) $\|u^R - u\|_{L^\infty([0,T]; L^2(\Omega_R))} + \|u^R - \varphi^R u\|_{L^\infty([0,T]; L^2(\Omega_R))} \leq CR^{-\alpha},$
- (7) $\|\nabla(u - u^R)\|_{L^\infty([0,T]; L^2(\Omega))} \leq CR^{-\beta}.$

Above, α and β are given by:

$$\alpha = \begin{cases} 1/2 + \theta/2 & \text{if } m = 0, \\ 1/2 - \theta/2 & \text{if } m \neq 0 \end{cases} \tag{3.6}$$

and

$$\beta = \begin{cases} 1/2 + 3\theta/2 & \text{if } m = 0, \\ 1/2 + \theta/2 & \text{if } m \neq 0, \end{cases} \tag{3.7}$$

in two dimensions, while $\alpha = 1/2$ and $\beta = 3/2$ in three dimensions.

For Case I of Definition 1.1 the constants above depend only on Ω ; for Cases II and III some of the constants also depend on T .

4. Energy argument

Proof of Theorem 1.2. The proof proceeds much as in Section 2 of [5]: Using our approximate solution u^R to (E) we make an energy argument to bound the difference

$$W = u^{v,R} - u^R$$

in the L^2 norm. Then using inequality (6) of Proposition 3.1 we apply the triangle inequality to complete the proof. We give the argument in 2 dimensions only; it is valid with minor adaptations in 3 dimensions. The only delicate point in adapting to 3 dimensions is that we deal with weak Leray solutions for which we cannot perform energy estimates. However the energy inequality is equivalent to the necessary estimates; see [5] for a more detailed discussion of this issue.

Subtracting the identity in (3.4) from (NS) we obtain

$$\partial_t W - v \Delta W = -u^{v,R} \cdot \nabla u^{v,R} - \nabla p^{v,R} + v \Delta u^R + \varphi^R u \cdot \nabla u + \varphi^R \nabla p - \partial_t \psi \nabla^\perp \varphi^R.$$

Multiplying both sides by W and integrating over Ω_R give

$$\frac{1}{2} \frac{d}{dt} \|W\|_{L^2}^2 + v \|\nabla W\|_{L^2}^2 = I_1 + I_2 + I_3 + I_4 + I_5,$$

where when $s > 1,$

$$\begin{aligned} I_1 &= -v \int_{\Omega_R} \nabla W \cdot \nabla u^R, & I_2 &= - \int_{\Omega_R} (u^{v,R} \cdot \nabla u^{v,R}) \cdot W, \\ I_3 &= \int_{\Omega_R} (\varphi^R u \cdot \nabla u) \cdot W, & I_4 &= \int_{\Omega_R} \varphi^R \nabla p \cdot W, \\ I_5 &= - \int_{\Omega_R} \partial_t \psi \nabla^\perp \varphi^R \cdot W. \end{aligned}$$

In I_1 we integrated by parts to remove Δu^R , but when $s > 2$ it is more advantageous to retain it, using

$$I_1 = v \int_{\Omega_R} W \cdot \Delta u^R.$$

When $s > 1$ we apply the Cauchy–Schwarz and Young’s inequalities to the first form of I_1 to get

$$|I_1| \leq \frac{\nu}{2} (\|\nabla W\|_{L^2}^2 + \|\nabla u^R\|_{L^2}^2) \leq \frac{\nu}{2} \|\nabla W\|_{L^2}^2 + C\nu,$$

and when $s > 2$ we apply the Cauchy–Schwarz inequality to the second form of I_1 to get

$$|I_1| \leq C\nu \|\Delta u^R\|_{L^2(\Omega_R)} \|W\|_{L^2(\Omega_R)} \leq C\nu \|W\|_{L^2(\Omega_R)}.$$

Summing I_2 and I_3 and using

$$\int_{\Omega_R} (u^{v,R} \cdot \nabla W) \cdot W = 0$$

we have

$$\begin{aligned} |I_2 + I_3| &= \left| \int_{\Omega_R} (u^R \cdot \nabla(u - u^R)) \cdot W - (W \cdot \nabla u^R) \cdot W + [(\varphi^R u - u^R) \cdot \nabla u] \cdot W \right| \\ &\leq \|u^R\|_{L^\infty} \|\nabla(u - u^R)\|_{L^2} \|W\|_{L^2} + \|\nabla u^R\|_{L^\infty} \|W\|_{L^2}^2 + \|\varphi^R u - u^R\|_{L^2} \|\nabla u\|_{L^\infty} \|W\|_{L^2} \\ &\leq CR^{-\beta} \|W\|_{L^2} + C\|W\|_{L^2}^2 + CR^{-\alpha} \|W\|_{L^2} \\ &\leq C(R^{-\alpha} + \|W\|_{L^2}) \|W\|_{L^2}, \end{aligned}$$

where we used inequalities (6) and (7) from Proposition 3.1 and also that $\alpha \leq \beta$.

Summing I_4 and I_5 and integrating the first term by parts give

$$\begin{aligned} |I_4 + I_5| &= \left| \int_{\Omega_R} p \nabla \varphi^R \cdot W + \partial_t \psi \nabla^\perp \varphi^R \cdot W \right| \\ &\leq (\|p \nabla \varphi^R\|_{L^2} + \|\partial_t \psi \nabla^\perp \varphi^R\|_{L^2}) \|W\|_{L^2} \leq CR^{-\theta} \|W\|_{L^2} \\ &\leq CR^{-2\theta} + C\|W\|_{L^2}^2. \end{aligned}$$

When $s > 1$, we conclude that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|W\|_{L^2}^2 + \nu \|\nabla W\|_{L^2}^2 &\leq \frac{\nu}{2} \|\nabla W\|_{L^2}^2 + C\nu + CR^{-\alpha} \|W\|_{L^2} + CR^{-2\theta} + C\|W\|_{L^2}^2 \\ &\leq \frac{\nu}{2} \|\nabla W\|_{L^2}^2 + C\nu + C(R^{-2\alpha} + R^{-2\theta}) + C\|W\|_{L^2}^2. \end{aligned}$$

Integrating in time gives

$$\|W(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla W\|_{L^2}^2 \leq \|W(0)\|_{L^2}^2 + CT\nu + CT(R^{-2\alpha} + R^{-2\theta}) + C \int_0^t \|W\|_{L^2}^2.$$

It follows from Gronwall’s lemma that

$$\begin{aligned} \|W\|_{L^\infty([0,T];L^2(\Omega_R))} &\leq (F(R)^2 + C(\nu + R^{-2\alpha} + R^{-2\theta}))^{1/2} e^{CT} \\ &\leq (F(R) + C(\nu^{1/2} + R^{-\alpha} + R^{-\theta})) e^{CT}. \end{aligned}$$

Then from the triangle inequality and inequality (6) of Proposition 3.1,

$$\begin{aligned} \|u^{v,R} - u\|_{L^\infty([0,T];L^2(\Omega_R))} &\leq \|u^R - u\|_{L^\infty([0,T];L^2(\Omega_R))} + \|W\|_{L^\infty([0,T];L^2(\Omega_R))} \\ &\leq CR^{-\alpha} + (F(R) + C(\nu^{1/2} + R^{-\alpha} + R^{-\theta})) e^{CT} \\ &\leq (F(R) + C(\nu^{1/2} + R^{-\alpha} + R^{-\theta})) e^{CT}. \end{aligned}$$

When $s > 2$, we have instead that

$$\begin{aligned} \|W\|_{L^2} \frac{d}{dt} \|W\|_{L^2} &= \frac{1}{2} \frac{d}{dt} \|W\|_{L^2}^2 \leq \frac{1}{2} \frac{d}{dt} \|W\|_{L^2}^2 + \nu \|\nabla W\|_{L^2}^2 \\ &\leq C\nu \|W\|_{L^2} + C(R^{-\alpha} + R^{-\theta}) \|W\|_{L^2} + C \|W\|_{L^2}^2. \end{aligned}$$

Dividing both sides by $\|W\|_{L^2}$ (it is easy to see that division by zero will not invalidate the following inequality after integrating in time) gives

$$\frac{d}{dt} \|W\|_{L^2} \leq C\nu + CR^{-\alpha} + CR^{-\theta} + C \|W\|_{L^2}.$$

Integrating in time and applying Gronwall’s lemma, we have

$$\|W\|_{L^2} \leq (F(R) + C(\nu + CR^{-\alpha} + R^{-\theta}))e^{CT}.$$

The bound on $\|u^{v,R} - u\|_{L^\infty([0,T];L^2(\Omega_R))}$ follows from the triangle inequality as for $s > 1$.

The value of α in the statement of Theorem 1.2 is chosen so that $\alpha = \alpha(\theta)$ gives the optimal rate of convergence in each case; this corresponds to $\theta = 1$ for Case I; $\theta = 1/3$ for Case II so that $\theta = \alpha$; and $\theta = 1$ was fixed for Case III. \square

5. Truncation operator in 2D

Let u be in $E_m \cap C^1$ for some m in \mathbb{R} with vorticity $\omega(u)$ having compact support in a ball of radius R_0 . Let ψ be the stream function, as defined by the expression in (2.1).

Let φ^R and Σ_R be defined as in Section 3 and recall the definition of u^R and \mathcal{T}_R given in (3.2), $u^R = \mathcal{T}_R u = \nabla^\perp(\varphi^R \psi)$. To explore the properties of \mathcal{T}_R we must first establish some bounds on the L^2 norms of u , ∇u , and ψ in Σ_R . To this end we write

$$u = v + \sigma,$$

where v is in $V_C(\mathbb{R}^2)$ and σ is a stationary solution with radially symmetric, smooth, compactly supported vorticity of integral m ; we assume that $\omega(\sigma)$ is of distinguished sign. We can assume, without loss of generality, that the support of $\omega(\sigma)$ is also contained in the ball of radius R_0 , from which it follows that the support of $\omega(v)$ is contained in this same ball. Now, v and σ are also C^1 divergence-free vector fields and hence we can define their associated stream functions ψ_v and ψ_σ using the expression in (2.1). But then $v = \nabla^\perp \psi_v$ and $\sigma = \nabla^\perp \psi_\sigma$. It follows in particular that v can be written in terms of $\omega(v)$ through the Biot–Savart law $v = K * \omega(v)$, an integral operator with kernel

$$K = K(z) = \frac{1}{2\pi} \frac{z^\perp}{|z|^2}. \tag{5.1}$$

From this explicit expression and using the fact that the integral of $\omega(v)$ vanishes, together with the easily obtained estimate

$$\|\omega(v)\|_{L^1(\mathbb{R}^2)} \leq 2\|\omega(u)\|_{L^1(\mathbb{R}^2)},$$

it follows that there exists $C = C(R_0) > 0$ such that

$$|v(x)| \leq CC_0/|x|^2, \quad |\nabla v(x)| \leq CC_0/|x|^3 \tag{5.2}$$

for all $|x| \geq 2R_0$, with $C_0 = 2\|\omega(u)\|_{L^1(\mathbb{R}^2)}$. Similarly, it follows from the explicit expression for ψ_v , (2.1), that

$$|\psi_v(x)| \leq CC_0/|x| \quad \text{for all } |x| \geq 2R_0. \tag{5.3}$$

Put coordinates on Σ_R as in the definition of φ_R in Section 3. Letting a be the length of Γ_1 it follows that the length of Γ_R is aR . Then

$$\|\psi_v\|_{L^2(\Sigma_R)}^2 = \int_0^{aR} \int_0^{R^\theta} |J(s, r)| |\psi_v(s, r)|^2 dr ds,$$

where J is the Jacobian of the transformation from rectangular coordinates to (s, r) -coordinates. Because of the way we constructed Σ_R and because $\theta \leq 1$, $|J| \leq C$ and Σ_R lies outside a ball of radius $C(\Omega)R$. Thus by (5.3), $|\psi_v(s, r)| \leq CC_0/R$ in the integral above as long as $C(\Omega)R \geq 2R_0$; that is, as long as

$$R \geq \mu(\Omega)R_0, \tag{5.4}$$

where $\mu(\Omega) = 2/C(\Omega)$ depends only on the geometry of Ω . Then

$$\|\psi_v\|_{L^2(\Sigma_R)} \leq (CC_0^2 R^{-2} a R \delta_1 R^\theta)^{1/2} \leq CC_0 R^{\theta/2-1/2}.$$

Since each of ∇^\perp and ∇ introduces an extra factor of $1/|x|$, it follows that

$$\|v\|_{L^2(\Sigma_R)} \leq CC_0 R^{\theta/2-3/2}, \quad \|\nabla v\|_{L^2(\Sigma_R)} \leq CC_0 R^{\theta/2-5/2}.$$

As previously pointed out, see (1.1), σ is given by

$$\sigma(x_1, x_2) = \left(-\frac{x_2}{|x|^2} \int_0^{|x|} r \omega(\sigma)(r) dr, \frac{x_1}{|x|^2} \int_0^{|x|} r \omega(\sigma)(r) dr \right)$$

so that $|\sigma(x)| = |m|(2\pi)^{-1}/|x|$ for $|x| \geq R_0$. Thus, σ decays like ψ_v so we can see that

$$\|\sigma\|_{L^2(\Sigma_R)} \leq C|m|R^{\theta/2-1/2}, \quad \|\nabla\sigma\|_{L^2(\Sigma_R)} \leq C|m|R^{\theta/2-3/2}.$$

The expression for ψ_σ can be calculated directly using (2.1) together with the radial symmetry of $\omega(\sigma)$. Of course, we can add an arbitrary constant to ψ_σ and still satisfy the equations $\sigma = \nabla^\perp \psi_\sigma$ and $\Delta \psi_\sigma = \omega(\sigma)$. For $|x| \geq R_0$ we obtain:

$$\psi_\sigma(x) = \frac{m}{2\pi} \log|x| + C.$$

Since when $m \neq 0$ we assume that Ω is a disk centered at the origin, we can choose the constant C_R so that $\psi_\sigma = 0$ on Γ_R . The value of $\partial_t \psi$ is unaffected by the choice of C_R , however, and $\partial_t \psi$ in inequality (4a) of Proposition 3.1 is the only direct use of ψ that we make, so the choice of C_R , though it depends on R , will not affect any of our estimates.

Applying Poincaré’s inequality (or integrating ψ_σ directly) gives

$$\|\psi_\sigma\|_{L^2(\Sigma_R)} \leq CR^\theta \|\sigma\|_{L^2(\Sigma_R)} \leq C|m|R^{3\theta/2-1/2}.$$

The factor of R^θ here comes from the thickness of Σ_R .

Adding the corresponding bounds for $u = v + \sigma$ and $\psi = \psi_v + \psi_\sigma$,

$$\|\psi\|_{L^2(\Sigma_R)} \leq CC_0 R^{\theta/2-1/2} + C|m|R^{3\theta/2-1/2}, \tag{5.5}$$

and for the velocity

$$\|u\|_{L^2(\Sigma_R)} \leq CC_0 R^{\theta/2-3/2} + C|m|R^{\theta/2-1/2} \tag{5.6}$$

and

$$\|\nabla u\|_{L^2(\Sigma_R)} \leq CC_0 R^{\theta/2-5/2} + C|m|R^{\theta/2-3/2}. \tag{5.7}$$

These inequalities each hold as long as (5.4) holds.

Let X be the subspace of all vector fields in $E_m \cap \dot{H}_1(\mathbb{R}^2)$ whose vorticity has compact support. We can now describe the relevant properties of the two-dimensional truncation operator, adapting Lemma 4.2 of [8].

Proposition 5.1. *Let Ω be a disk centered at the origin and let the truncation operator \mathcal{T}_R be defined as in (3.2). Then $\mathcal{T}_R: X \rightarrow V(\Omega_R)$ with the property that for all u in X*

$$\|u - \mathcal{T}_R u\|_{L^2(\Omega_R)} \leq C \|\omega(u)\|_{L^1(\mathbb{R}^2)} R^{-\alpha} \tag{5.8}$$

and

$$\|\nabla(u - \mathcal{T}_R u)\|_{L^2(\Omega_R)} \leq C \|\omega(u)\|_{L^1(\mathbb{R}^2)} R^{-\beta} \tag{5.9}$$

for all R satisfying (5.4), where $C = C(\Omega)$, α is defined in (3.6), and β is defined in (3.7). Also,

$$\|\nabla \mathcal{T}_R u\|_{L^2(\Omega_R)} \leq C \|\omega(u)\|_{L^2(\mathbb{R}^2)}, \tag{5.10}$$

where $C = C(R_0)$.

Proof. Using the inequalities in (3.1), (5.5), (5.6), and (5.7), we have,

$$\begin{aligned} \|u - \mathcal{T}_R u\|_{L^2(\Omega_R)} &= \|\nabla^\perp \psi - \varphi^R \nabla^\perp \psi - \psi \nabla^\perp \varphi^R\|_{L^2(\Omega_R)} \\ &\leq \|(1 - \varphi^R) \nabla^\perp \psi\|_{L^2(\Omega_R)} + \|\psi \nabla^\perp \varphi^R\|_{L^2(\Omega_R)} \\ &\leq \|u\|_{L^2(\Sigma_R)} + \|\nabla \varphi^R\|_{L^\infty(\Sigma_R)} \|\psi\|_{L^2(\Sigma_R)} \\ &\leq CC_0 R^{\theta/2-3/2} + C|m|R^{\theta/2-1/2} + CR^{-\theta} C_0 R^{\theta/2-1/2} + CR^{-\theta} |m| R^{3\theta/2-1/2} \\ &\leq CC_0 R^{\theta/2-3/2} + C|m|R^{\theta/2-1/2} + CC_0 R^{-\theta/2-1/2} + C|m|R^{\theta/2-1/2} \\ &\leq CC_0 R^{-\theta/2-1/2} + C|m|R^{\theta/2-1/2}. \end{aligned}$$

In the final inequality we needed to only include the two terms that dominate (depending on whether $m = 0$) for all θ in $[0, 1]$. Clearly, $|m| \leq \|\omega(u)\|_{L^1(\mathbb{R}^2)}$, so this gives (5.8).

Similarly,

$$\begin{aligned} \|\nabla u - \nabla \mathcal{T}_R u\|_{L^2(\Omega_R)} &= \|\nabla u - \nabla \nabla^\perp(\varphi^R \psi)\|_{L^2(\Omega_R)} \\ &= \|\nabla u - \nabla(\varphi^R \nabla^\perp \psi) - \nabla(\psi \nabla^\perp \varphi^R)\|_{L^2(\Omega_R)} \\ &= \|\nabla u - \varphi^R \nabla \nabla^\perp \psi - \nabla \varphi^R \otimes \nabla^\perp \psi - \nabla \psi \otimes \nabla^\perp \varphi^R - \psi \nabla \nabla^\perp \varphi^R\|_{L^2(\Omega_R)} \\ &= \|(1 - \varphi^R) \nabla u - \nabla \varphi^R \otimes \nabla^\perp \psi - \nabla \psi \otimes \nabla^\perp \varphi^R - \psi \nabla \nabla^\perp \varphi^R\|_{L^2(\Omega_R)} \\ &\leq \|\nabla u\|_{L^2(\Sigma_R)} + 2\|\nabla \varphi^R\|_{L^\infty(\Sigma_R)} \|u\|_{L^2(\Sigma_R)} + \|\nabla \nabla \varphi^R\|_{L^\infty(\Sigma_R)} \|\psi\|_{L^2(\Sigma_R)} \\ &\leq CC_0 R^{\theta/2-5/2} + C|m|R^{\theta/2-3/2} + CR^{-\theta} (C_0 R^{\theta/2-3/2} + |m| R^{\theta/2-1/2}) \\ &\quad + CR^{-2\theta} (C_0 R^{\theta/2-1/2} + |m| R^{3\theta/2-1/2}) \\ &= CC_0 R^{\theta/2-5/2} + C|m|R^{\theta/2-3/2} + CC_0 R^{-\theta/2-3/2} + C|m|R^{-\theta/2-1/2} \\ &\quad + CC_0 R^{-3\theta/2-1/2} + C|m|R^{-\theta/2-1/2} \\ &\leq CC_0 R^{-3\theta/2-1/2} + C|m|R^{-\theta/2-1/2}. \end{aligned}$$

Since, as we noted, $|m| \leq \|\omega(u)\|_{L^1(\mathbb{R}^2)}$, the estimate above gives (5.9). The bound on $\|\nabla \mathcal{T}_R u\|_{L^2(\Omega_R)}$ is obtained in the same way except that $\|\nabla u\|_{L^2(\Sigma_R)}$ is replaced by $\|\nabla u\|_{L^2(\Omega_R \setminus \Sigma_R)} \leq \|\nabla u\|_{L^2(\mathbb{R}^2)} = \|\omega(u)\|_{L^2(\mathbb{R}^2)}$. But $\|\omega\|_{L^1(\mathbb{R}^2)} \leq C\|\omega\|_{L^2(\mathbb{R}^2)}$ because the support of ω is contained in B_{R_0} , giving (5.10). \square

Remark 5.2. Had we not added the restriction that Ω is a disk, then since ψ_σ is defined only up to an additive constant, we could choose this constant so that $\int_{\Sigma_R} \psi_\sigma = 0$. By normalizing ψ_σ this way, though, the factor of R^θ in Poincaré’s inequality becomes the diameter CR of Ω_R and we have

$$\|\psi_\sigma\|_{L^2(\Sigma_R)} \leq CR \|\sigma\|_{L^2(\Sigma_R)} \leq CmR^{\theta/2+1/2}.$$

This gives

$$\|\psi\|_{L^2(\Sigma_R)} \leq CC_0 R^{\theta/2-1/2} + CmR^{\theta/2+1/2} \tag{5.11}$$

and we would have had the following bound in the proof of Proposition 5.1:

$$\begin{aligned} \|u - \mathcal{T}_R u\|_{L^2(\Omega_R)} &\leq CC_0 R^{\theta/2-3/2} + CmR^{\theta/2-1/2} + CC_0 R^{-\theta/2-1/2} + CmR^{-\theta/2+1/2} \\ &\leq CC_0 R^{\theta/2-1/2} + CmR^{-\theta/2+1/2}. \end{aligned}$$

No value of θ in $[0, 1]$ will allow this to vanish as $R \rightarrow \infty$ when $m \neq 0$.

6. Estimates in 2D

Proof of Proposition 3.1 in 2D. Define u^R, φ^R , and Σ_R as in Section 3. Then

$$u^R = \nabla^\perp(\varphi^R \psi) = \varphi^R u + \psi \nabla^\perp \varphi^R, \tag{6.1}$$

where $\psi = \psi_v + \psi_\sigma$ as in Section 5.

The L^1 and L^2 norms (indeed, all L^p norms) of the vorticity for solutions to (E) are conserved over time, while the bounds on the L^2 (for $m = 0$) and L^∞ norms of the velocity are bounded over any finite time interval (the former is conserved, and a bound on the latter is uniform). Thus, the estimates in Proposition 5.1 when applied to our solution u to (E) are uniform in time. Hence, in the estimates that follow we will generally not explicitly refer to the bounds in time.

Inequality (1). Follows directly from Proposition 5.1.

Inequality (2). We have

$$\begin{aligned} \|u^R\|_{L^\infty(\Omega_R)} &\leq \|\varphi^R u\|_{L^\infty(\Omega_R)} + \|\psi \nabla^\perp \varphi^R\|_{L^\infty(\Omega_R)} \\ &\leq \|u\|_{L^\infty(\mathbb{R}^2)} + CR^{-\theta} \|\psi\|_{L^\infty(\Sigma_R)}. \end{aligned}$$

From (5.3), $\|\psi_v\|_{L^\infty(\Sigma_R)} \leq CC_0/R$ and applying Poincaré’s inequality in the L^∞ norm,

$$\|\psi_\sigma\|_{L^\infty(\Sigma_R)} \leq CR^\theta \|\sigma\|_{L^\infty(\Sigma_R)} \leq C|m|R^\theta R^{-1} \leq CC_0R^{\theta-1}.$$

We conclude that inequality (2) holds for sufficiently large R .

Inequality (3). We have,

$$\begin{aligned} \|\nabla u^R\|_{L^\infty} &\leq \|\nabla u\|_{L^\infty(\Omega_R)} + 2\|\nabla \varphi^R\|_{L^\infty(\Sigma_R)} \|u\|_{L^\infty(\Sigma_R)} + \|\nabla \nabla^\perp \varphi^R\|_{L^\infty(\Sigma_R)} \|\psi\|_{L^\infty(\Sigma_R)} \\ &\leq C + CR^{-\theta} + CR^{-2\theta} C_0 R^{\theta-1}. \end{aligned}$$

We conclude that inequality (3) holds for sufficiently large R .

Inequality (4a). We begin with the observation that

$$\Delta \psi_t = -\omega(\operatorname{div}(u \otimes u)) \tag{6.2}$$

and

$$-\Delta p = \operatorname{div} \operatorname{div}(u \otimes u). \tag{6.3}$$

Now, u is bounded, uniformly over a finite time interval, in $L^4(\mathbb{R}^2)$. To see this write $u = v + \sigma$; clearly $\sigma \in L^4(\mathbb{R}^2)$ and $v \in L^\infty \cap L^2(\mathbb{R}^2)$. It follows that the right-hand sides of both (6.2) and (6.3) are second derivatives of an L^2 vector field. We can express ψ_t and p as linear combinations of Riesz transforms of terms which are uniformly bounded in $L^2(\mathbb{R}^2)$ and hence, by the Calderon–Zygmund inequality, it follows that ψ_t and p are both bounded, uniformly over a finite time interval, in $L^2(\mathbb{R}^2)$. Thus,

$$\begin{aligned} \|p \nabla \varphi^R\|_{L^2(\Omega_R)} + \|\partial_t \psi \nabla \varphi^R\|_{L^2(\Omega_R)} &\leq \|p\|_{L^2(\Sigma_R)} \|\nabla \varphi^R\|_{L^\infty(\Sigma_R)} + \|\partial_t \psi\|_{L^2(\Sigma_R)} \|\nabla \varphi^R\|_{L^\infty(\Sigma_R)} \\ &\leq CR^{-\theta} + CR^{-\theta} = CR^{-\theta}. \end{aligned}$$

Inequality (5). For f a scalar and v a vector field we have that

$$\Delta(fv) = \Delta f v + 2\nabla f \cdot (\nabla v)^T + f \Delta v,$$

so

$$\begin{aligned} \Delta u^R &= \Delta(\varphi^R u) + \Delta(\psi \nabla^\perp \varphi^R) \\ &= \Delta \varphi^R u + 2\nabla \varphi^R \cdot (\nabla u)^T + \varphi^R \Delta u + \Delta \psi \nabla^\perp \varphi^R + 2\nabla \psi \cdot (\nabla \nabla^\perp \varphi^R)^T + \psi \Delta \nabla^\perp \varphi^R. \end{aligned}$$

Then because $\Delta\psi = \omega$ and $u = \nabla^\perp\psi$,

$$\begin{aligned} \|\Delta u^R\|_{L^2(\Omega_R)} &\leq CR^{-2\theta}\|u\|_{L^2(\Sigma_R)} + CR^{-\theta}\|\nabla u\|_{L^2(\Sigma_R)} + C\|\Delta u\|_{L^2(\Omega_R)} + CR^{-\theta}\|\omega\|_{L^2(\Sigma_R)} \\ &\quad + CR^{-2\theta}\|u\|_{L^2(\Sigma_R)} + CR^{-3\theta}\|\psi\|_{L^2(\Sigma_R)} \\ &\leq C\|\Delta u\|_{L^2(\mathbb{R}^2)} + C, \end{aligned}$$

for sufficiently large R by the bounds in (5.5), (5.6), and (5.7).

Inequality (6). We have, $\|u^R - u\|_{L^2(\Omega_R)} \leq CC_0R^{-\alpha}$ by (5.8) and

$$\begin{aligned} \|u^R - \varphi^R u\|_{L^2(\Omega_R)} &\leq \|u^R - u\|_{L^2(\Omega_R)} + \|(1 - \varphi_R)u\|_{L^2(\Omega_R)} \\ &\leq \|u^R - u\|_{L^2(\Omega_R)} + \|u\|_{L^2(\Sigma_R)} \\ &\leq CC_0R^{-\alpha} + CC_0R^{\theta/2-3/2} + C|m|R^{\theta/2-1/2} \\ &\leq CC_0R^{-\alpha} \end{aligned}$$

where we used (5.6) and (5.8).

Inequality (7). The two-dimensional case follows directly from Proposition 5.1. \square

7. Decay of velocity in 3D

The basic existence result for a solution to (E) is given in Theorem 7.1.

Theorem 7.1. *Assume that u_0 is in $C^s(\mathbb{R}^3) \cap V_C(\mathbb{R}^3)$ for $s > 1$. Then there exists T^* in $(0, \infty]$ such that for all T in $(0, T^*)$ the solution u to (E) in the whole space lies in $L^\infty([0, T]; C^s(\mathbb{R}^3))$ with ∇p in $L^\infty([0, T]; L^a(\mathbb{R}^3))$ for all a in $(1, \infty]$. Also, the vorticity $\omega = \omega(u)$ lies in $L^\infty([0, T]; L^a(\mathbb{R}^3))$ for all a in $[1, \infty]$ with a bound on its norm that is independent of a . Furthermore, ω remains compactly supported for all time, with the support contained in a ball of radius $R(t) \leq R_0 + \|u\|_{L^\infty([0, T] \times \mathbb{R}^3)}T$ for all t in $[0, T]$, where the support of ω_0 is contained in $B_{R_0}(0)$.*

Proof. The initial vorticity $\omega = \omega(u_0)$ is compactly supported and continuous by assumption so ∇v_0 is in $L^a(\mathbb{R}^3)$ for all a in $[1, \infty]$. Thus by Theorem 4.2.3, p. 79 of [1], there exists T^* in $(0, \infty]$ such that for all T in $(0, T^*)$ the solution u to (E) lies in $L^\infty([0, T]; C^s(\mathbb{R}^3))$ with ∇p in $L^\infty([0, T]; L^a(\mathbb{R}^3))$ for all a in $(1, \infty)$.

Taking the vorticity of (E) gives

$$\frac{D\omega}{Dt} = \partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u,$$

where $D\omega/Dt$ is the rate of change of the vorticity along the flow lines. Assume that the support of ω^0 is contained in a ball $B_{R(0)}$ of radius $R(0)$. Then until some flow line starting from within $B_{R(0)}$ reaches a point outside of $B_{R(0)}$ there can be no change in vorticity at that point. This gives the compact support of the vorticity for all time along with the bound on its support. \square

Remark 7.2. Theorem 7.1 is stated for three dimensions but holds in two dimensions as well, with $T^* = \infty$, with minor adaptations to allow for $u_0 \in C^s(\mathbb{R}^2) \cap E_m$, $m \neq 0$.

Corollary 7.3. *Let u be a solution to (E) as in Theorem 7.1. Then u is in $L^\infty([0, T]; L^p(\mathbb{R}^3))$ for all p in $(3/2, \infty]$ and ∇u is in $L^\infty([0, T]; L^p(\mathbb{R}^3))$ for all p in $(1, \infty]$. Also,*

$$|u(x)| \leq C/|x|^2, \quad |\nabla u(x)| \leq C/|x|^3$$

for all t in $[0, T]$ and $|x| \geq 2R(T)$, and

$$\|u(t)\|_{L^2(\Sigma_R)} \leq CR^{-1/2}, \quad \|\nabla u(t)\|_{L^2(\Sigma_R)} \leq CR^{-3/2},$$

for all t in $[0, T]$ and all R sufficiently large that Σ_R is contained in $B_{2R(T)}^C$.

Proof. Let Ψ be the associated stream function in three dimensions and recall the expression for Ψ given in (2.2). In particular, $u = \text{curl } \Psi$ and $-\Delta \Psi = \omega$, where $\omega = \omega(u)$ is the associated vorticity. This gives rise to the three-dimensional Biot–Savart law, allowing us to write the velocity in terms of vorticity as:

$$u = u(x) = \frac{-1}{4\pi} \int \frac{x - y}{|x - y|^3} \times \omega(y) dy. \tag{7.1}$$

The decay of u and ∇u then follows from the compact support of ω and the Biot–Savart law. Because u and ∇u are in $L^\infty([0, T] \times \mathbb{R}^3)$, the membership of u and ∇u in the stated spaces and the decay of their L^2 norms then follow from their decay at infinity. \square

8. Truncation operator in 3D

Let φ^R and Σ_R be as in Section 3 and recall the definition of the truncation operator in three dimensions: If u in $H^1(\mathbb{R}^3)$ then $\mathcal{T}_R u$ is given by (3.3); that is, $\mathcal{T}_R u = \nabla \times (\varphi^R \Psi)$. Set $E = E(x) = (4\pi|x|)^{-1}$, the fundamental solution of $-\Delta$ in \mathbb{R}^3 .

Lemma 8.1. *If f in $L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is compactly supported in the ball B_L of radius L centered at the origin then for all $R \geq R^*$,*

$$\begin{aligned} \|E * f\|_{L^2(\Sigma_R)} &\leq CL^{3/2} \|f\|_{L^2(\mathbb{R}^3)} R^{1/2}, \\ \|\partial_k E * f\|_{L^2(\Sigma_R)} &\leq CL^{3/2} \|f\|_{L^2(\mathbb{R}^3)} R^{-1/2}, \\ \|E * f\|_{L^\infty(\Sigma_R)} &\leq CL^{3/2} \|f\|_{L^2(\mathbb{R}^3)} R^{-1}, \\ \|\partial_k E * f\|_{L^\infty(\Sigma_R)} &\leq CL^{3/2} \|f\|_{L^2(\mathbb{R}^3)} R^{-2}, \end{aligned}$$

where C is an absolute constant and R^* depends only upon Ω_1 and L .

Proof. On B_{2L}^C the compact support of f gives $|E * f| \leq C \|f\|_{L^1(\mathbb{R}^3)} / |x| \leq CL^{3/2} \|f\|_{L^2(\mathbb{R}^3)} / |x|$. Therefore, as long as R is large enough so that Σ_R is wholly contained in B_{2L}^C ,

$$\|E * f\|_{L^2(B_R)} \leq CL^{3/2} \|f\|_{L^2(\mathbb{R}^3)} R^{1/2},$$

and similarly for the L^∞ norm and for the inequalities for $\partial_k E * f$. \square

Proposition 8.2. *Let u be a solution to (E) as in Theorem 7.1. Then there exists $R^* > 0$ such that for all $R \geq R^*$ and all t in $[0, T]$, $\mathcal{T}_R u$ is in $V(\Omega_R)$, and*

$$\|u - \mathcal{T}_R u\|_{L^2(\Omega_R)} \leq C_1 R^{-1/2}, \tag{8.1}$$

$$\|\nabla(u - \mathcal{T}_R u)\|_{L^2(\Omega_R)} \leq C_2 R^{-3/2}. \tag{8.2}$$

The constants C_1 and C_2 depend on u and T , and R^* depends on $R(T)$ and Ω_1 .

Proof. On Γ_R , $\mathcal{T}_R u$ vanishes identically, while clearly $\text{div } \mathcal{T}_R u = 0$ on Ω_R .

Now, $\mathcal{T}_R u = \varphi_R u + \nabla \varphi_R \times \Psi$, so

$$\begin{aligned} \|u - \mathcal{T}_R u\|_{L^2(\Omega_R)} &\leq \|u\|_{L^2(\Sigma_R)} + C \|\nabla \varphi^R\|_{L^\infty(\Sigma_R)} \|\Psi\|_{L^2(\Sigma_R)} \\ &\leq \|u\|_{L^2(\Sigma_R)} + CR^{-1} \sum_{i,k} \|E * \omega(u)\|_{L^2(\Sigma_R)} \leq CR^{-1/2}. \end{aligned}$$

We used Corollary 7.3 and Lemma 8.1 in the final inequality.

Also,

$$\partial_k \mathcal{T}_R u = \varphi_R \partial_k u + \nabla \partial_k \varphi_R \times \Psi + (\partial_k \varphi_R) u + \nabla \varphi_R \times \partial_k \Psi. \tag{8.3}$$

Hence,

$$\begin{aligned} \|\nabla \mathcal{T}_R u - \nabla u\|_{L^2(\Omega_R)} &\leq \|\nabla u\|_{L^2(\Sigma_R)} + \|\nabla \nabla \varphi^R\|_{L^\infty(\Sigma_R)} \|\Psi\|_{L^2(\Sigma_R)} + \|\nabla \varphi^R\|_{L^\infty(\Sigma_R)} \|u\|_{L^2(\Sigma_R)} \\ &\quad + \|\nabla \varphi^R\|_{L^\infty(\Sigma_R)} \sum_k \|\partial_k E * \omega(u)\|_{L^2(\Sigma_R)} \\ &\leq \|\nabla u\|_{L^2(\Sigma_R)} + CR^{-2} \|\Psi\|_{L^2(\Sigma_R)} + CR^{-1} \|u\|_{L^2(\Sigma_R)} + CR^{-1} \sum_k \|\partial_k E * \omega(u)\|_{L^2(\Sigma_R)} \\ &\leq CR^{-3/2}, \end{aligned}$$

where we used Corollary 7.3 and Lemma 8.1 in the final inequality. Thus, $\mathcal{T}_R u \in V(\Omega_R)$, as desired. \square

9. Estimates in 3D

Proof of Proposition 3.1 in 3D. Define φ^R and Σ_R as in Section 3. Then

$$u^R = \mathcal{T}_R u = \varphi_R u + \nabla \varphi_R \times \Psi. \tag{9.1}$$

Inequality (1). Follows from (8.2).

Inequality (2). From (9.1),

$$\begin{aligned} \|\nabla u^R\|_{L^\infty(\Omega_R)} &\leq \|\varphi^R u\|_{L^\infty(\Omega_R)} + \|\nabla \varphi^R\|_{L^\infty(\Omega_R)} \|\Psi\|_{L^\infty(\Sigma_R)} \\ &\leq \|u\|_{L^\infty(\mathbb{R}^2)} + CR^{-1} \|E * \omega\|_{L^\infty(\Sigma_R)}, \end{aligned}$$

which, using Lemma 8.1, is enough to establish inequality (2).

Inequality (3). Recall the expression for ∇u^R in (8.3). Then, using Lemma 8.1, we have

$$\begin{aligned} \|\nabla u^R\|_{L^\infty} &\leq \|\nabla u\|_{L^\infty(\Omega^R)} + \|\nabla \nabla \varphi^R\|_{L^\infty(\Sigma^R)} \|\Psi\|_{L^\infty(\Sigma^R)} + \|\nabla \varphi^R\|_{L^\infty(\Sigma^R)} \|u\|_{L^\infty(\Sigma^R)} \\ &\quad + \|\nabla \varphi^R\|_{L^\infty(\Sigma^R)} \sum_k \|\partial_k E * \omega\|_{L^\infty(\Sigma^R)} \\ &\leq C + CR^{-3} + CR^{-1} + CR^{-3} \leq C. \end{aligned}$$

Inequality (4b). The proof of this inequality is similar to that of inequality (4a). We begin by observing that Ψ_t satisfies an equation analogous to (6.2), namely

$$\Delta \Psi_t = \text{curl div } u \otimes u.$$

Now, $u \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, uniformly in time up to T^* . Therefore, using again the Calderon–Zygmund inequality, it follows that $\Psi_t \in L^2(\mathbb{R}^3)$, uniformly in time up to T^* . The estimate for the pressure p in three dimensions is exactly the same as for two dimensions, using (6.3). Thus, using the scaling of φ_R , we get

$$\begin{aligned} \|p \nabla \varphi^R\|_{L^2(\Omega_R)} + \|\nabla \varphi^R \cdot \partial_t \Psi\|_{L^2(\Omega_R)} &\leq \|p\|_{L^2(\Sigma_R)} \|\nabla \varphi^R\|_{L^\infty(\Sigma_R)} + \|\partial_t \Psi\|_{L^2(\Sigma_R)} \|\nabla \varphi^R\|_{L^\infty(\Sigma_R)} \\ &\leq CR^{-1} + CR^{-1} = CR^{-1}. \end{aligned}$$

Inequality (5). We must compute Δu^R ; for the sake of simplicity we keep track mostly of the order of derivatives. We have then that

$$\Delta u^R = (\Delta \varphi^R)u + 2\nabla \varphi^R \cdot Du + \varphi^R \Delta u + \nabla \varphi^R \times \omega + 2D^2 \varphi^R \cdot D\Psi + \nabla \Delta \varphi^R \times \Psi.$$

Using Corollary 7.3 and Lemma 8.1, we find

$$\begin{aligned} \|\Delta u^R\|_{L^2(\Omega_R)} &\leq CR^{-2} \|u\|_{L^2(\Sigma_R)} + CR^{-1} \|\nabla u\|_{L^2(\Sigma_R)} + C \|\Delta u\|_{L^2(\Omega_R)} + CR^{-1} \|\omega\|_{L^2(\Sigma_R)} \\ &\quad + CR^{-2} \|D\Psi\|_{L^2(\Sigma_R)} + CR^{-3} \|\Psi\|_{L^2(\Sigma_R)} \\ &\leq C + CR^{-5/2} \leq C. \end{aligned}$$

Inequality (6). We have, $\|u^R - u\|_{L^2(\Omega_R)} \leq CR^{-1/2}$ by (8.1) and

$$\begin{aligned} \|u^R - \varphi^R u\|_{L^2(\Omega_R)} &\leq \|u^R - u\|_{L^2(\Omega_R)} + \|(1 - \varphi_R)u\|_{L^2(\Omega_R)} \\ &\leq \|u^R - u\|_{L^2(\Omega_R)} + \|u\|_{L^2(\Sigma_R)} \leq CR^{-1/2}, \end{aligned}$$

where we used Corollary 7.3.

Inequality (7). The three-dimensional case follows directly from Proposition 8.2. \square

10. Truncation of the initial velocity

The following observations hold in any dimension.

Define $\mathcal{W}_R : V(\mathbb{R}^d) \rightarrow H(\Omega_R)$ as follows: For any u in $V(\mathbb{R}^d)$ let $\mathcal{W}_R u$ be that unique vector in $H(\Omega_R)$ having the same vorticity as u (Ω is simply connected so the vector is unique), see the paragraph following the statement of Theorem 1.2. Let $P_{H(\Omega_R)} : V(\mathbb{R}^2) \rightarrow H(\Omega_R)$ be the projection operator into $H(\Omega_R)$.

Lemma 10.1. *The operators \mathcal{W}_R and $P_{H(\Omega_R)}$ are the same.*

Proof. Let u be in $V(\mathbb{R}^d)$. Then by the Leray–Helmholtz decomposition there exist v in $H(\Omega_R) \cap H^1(\Omega_R)$ and p in $H^2(\Omega_R)$ such that $u = v + \nabla p$ on Ω_R , with $\Delta p = 0$ and with v and ∇p unique. Then v is orthogonal to ∇p in $L^2(\Omega_R)$ (that is, in the $H(\Omega_R)$ -norm) so $v = P_{H(\Omega_R)} u$. But the curl of a gradient is zero, so $\omega(v) = \omega(u)$ which shows that $v = \mathcal{W}_R u$ as well. \square

Corollary 10.2 follows immediately from Lemma 10.1, given the well-known properties of the projection operator.

Corollary 10.2. *\mathcal{W}_R has the same convergence properties in the $L^2(\Omega_R)$ -norm of the velocity as does \mathcal{T}_R . In particular, we obtain (1.2).*

Remark 10.3. The operator \mathcal{W}_R is a more natural “truncation” operator than \mathcal{T}_R since it involves literally truncating the vorticity. It is perfectly adequate for truncating the initial velocity for the Navier–Stokes equations, but because we do not know how to control the remainder following truncation in the H^1 -norm, we cannot use it to construct the approximate solution u^R to the Euler equations.

11. Comments and conclusions

As a first comment, we note that the estimates in Sections 7 and 8 for three dimensions are considerably simplified by the assumption that the initial vorticity is compactly supported, whereas for two dimensions this assumption is merely a minor convenience that allows us in Theorem 1.2 to give an explicit rate of convergence in R . If one drops the assumption of compact support, it does not seem possible to obtain a uniform-in-time bound on the decay of the gradient of the velocity. One can obtain a bound on the decay of vorticity, however, if one assumes that the L^2 norm of the initial vorticity on a ball of radius R decays for large R at least as fast as $CR^{-1/2}$. One then modifies the energy argument in Section 4, integrating by parts differently to use the decay of the vorticity in place of the decay of the velocity. The value of α in Theorem 1.2, however, must be strictly less than $1/2$.

The case of initial vorticity with nonzero integral in two dimensions corresponds to the situation where the limiting full plane flow has infinite energy. Since our argument is based on energy estimates, it is natural that this situation would be complicated. In this situation we have only studied the case of the expanding disk, but, as explained in Section 5, this is not a matter of convenience. Our argument makes essential use of the fact that we are working in a disk. However, the restriction to the expanding disk is by no means natural, and removing this assumption becomes an interesting open problem.

Finally, we note that our result was proved for smooth flows but that, in two dimensions, it would be very reasonable to consider initial vorticities in L^p , $p > 2$, compactly supported, given that, in our proof, the high regularity was needed in estimates near the boundary, where vorticity acts as far field. If $p > 2$ then the support of vorticity can be controlled since the Euler velocity is *a priori* bounded.

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