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A refined Brunn–Minkowski inequality for convex sets

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Abstract

Starting from a mass transportation proof of the Brunn–Minkowski inequality on convex sets, we improve the inequality showing a sharp estimate about the stability property of optimal sets. This is based on a Poincaré-type trace inequality on convex sets that is also proved in sharp form.

Keywords: Brunn–Minkowski inequality; Sharp estimates; Stability results

1. Introduction

We deal with the *Brunn–Minkowski inequality*: given E and F non-empty subsets of $\mathbb{R}^n$, we have

$$|E+F|^{1/n} \geqslant |E|^{1/n} + |F|^{1/n}, \tag{1}$$

where $E+F = \{x+y\colon x \in E,\ y \in F\}$ is the *Minkowski sum of E and F*, and where $|\cdot|$ stands for the (outer) Lebesgue measure on $\mathbb{R}^n$. The central role of this inequality in many branches of Analysis and Geometry, and especially in the theory of convex bodies, is well explained in the excellent survey [11] by R. Gardner. Concerning the case E and F are *open bounded convex sets* (shortly: *convex bodies*), it may be proved (see [4,14]) that equality holds in (1) if and only if E and F are homothetic, i.e.

$$\exists \lambda > 0,\ x_0 \in \mathbb{R}^n\colon \quad E = x_0 + \lambda F. \tag{2}$$

Theorem 1 provides a refined Brunn–Minkowski inequality on convex bodies, in the spirit of [7,12,18,17]. We define the *relative asymmetry of E and F* as

$$A(E,F) := \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{|E \Delta (x_0 + \lambda F)|}{|E|}\colon \lambda = \left(\frac{|E|}{|F|}\right)^{1/n} \right\}, \tag{3}$$

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and the *relative size of* E *and* F as

$$\sigma(E,F) := \max\left\{\frac{|F|}{|E|}, \frac{|E|}{|F|}\right\}. \tag{4}$$

We note that $A(E,F) = A(F,E)$ and $\sigma(E,F) = \sigma(F,E)$.

Theorem 1. *If* E *and* F *are convex bodies, then*

$$|E+F|^{1/n} \geqslant \left(|E|^{1/n} + |F|^{1/n}\right)\left\{1 + \frac{A(E,F)^2}{C_0(n)\sigma(E,F)^{1/n}}\right\}. \tag{5}$$

In [10], inequality (5) was derived as a corollary of the sharp quantitative Wulff inequality, with a constant $C_0(n) \approx n^7$ and with explicit examples proving the sharpness of decay rate of $A(E,F)$ and $\sigma(E,F)$ in the regime $\beta(E,F) \to 0$. Here, we introduce the *Brunn–Minkowski deficit of the pair* (E,F) by setting

$$\beta(E,F) := \frac{|E+F|^{1/n}}{|E|^{1/n} + |F|^{1/n}} - 1,$$

so that (5) becomes equivalent to

$$C_0(n)\sqrt{\beta(E,F)\sigma(E,F)^{1/n}} \geqslant A(E,F). \tag{6}$$

As in [10], our approach to (5) is based on the theory of mass transportation. A one-dimensional mass transportation argument is at the basis of the beautiful proof of (1) by Hadwiger and Ohmann [13], see [9, 3.2.41] and [11, Proof of Theorem 4.1]. The impact of mass transportation theory in the field of sharp functional-geometric inequalities is now widely recognized, with many old and new inequalities treated from a unified and elegant viewpoint (see [19, Chapter 6] for an introduction). A proof of the Brunn–Minkowski inequality in this framework is already contained in the seminal paper by McCann [16], see also Step two in the proof of Theorem 1.

In Section 3 of this note we present a direct proof of (5), independent from the structure theory for sets of finite perimeter that was heavily used in [10]. As a technical drawback, this approach does not provide a polynomial bound on $C_0(n)$, but only an exponential behavior in n. However, we believe this proof is more broadly accessible and substantially simpler. A technical element of this proof that we believe of independent interest is the Poincaré-type trace inequality on convex sets proved in Section 2, with a constant having sharp dependence on the dimension n and on the ratio between the in-radius and the out-radius of the set (see Remark 3).

2. A Poincaré-type trace inequality on convex sets

In this section we aim to prove the following Poincaré-type trace inequality for a convex body:

Lemma 2. *Let* E *be a convex body such that* $B_r \subset E \subset B_R$, *for* $0 < r < R$. *Then*

$$\frac{n\sqrt{2}}{\log(2)}\frac{R}{r}\int_E |\nabla f| \geqslant \inf_{c\in\mathbb{R}} \int_{\partial E} |f - c|\, d\mathcal{H}^{n-1}, \tag{7}$$

for every $f \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

It is quite easy to prove (7) by a contradiction argument, if we allow to replace $n(R/r)$ by a constant generically depending on E. However, in order to prove Theorem 1, we need to express this dependence just in terms of n and R/r, and thus require a more careful approach. Let us also note that, by a standard density argument, (7) holds true for every $f \in BV(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ (see [1,8]), in the form

$$\frac{n\sqrt{2}}{\log(2)}\frac{R}{r}|Df|(E) \geqslant \inf_{c\in\mathbb{R}} \int_{\partial E} \left|\mathrm{tr}_E(f) - c\right| d\mathcal{H}^{n-1},$$

where $|Df|$ denotes the total variation measure of Df and where $\mathrm{tr}_E(f)$ is the trace of f on ∂E, defined as an element of $L^1(\mathcal{H}^{n-1}\llcorner\partial E)$ (see [1, Theorem 3.87]). However, we shall not need this stronger form of the inequality.

Given a convex body E containing the origin in its interior, we introduce a weight function on directions defined for $\nu \in S^{n-1}$ as

$$\|\nu\|_E := \sup\{x \cdot \nu: \ x \in E\}.$$

When F is a set with Lipschitz boundary and outer unit normal ν_F, we define the anisotropic perimeter of F with respect to E as

$$P_E(F) := \int_{\partial F} \|\nu_F(x)\|_E \, d\mathcal{H}^{n-1}(x),$$

and recall that $P_E(E) = n|E|$. Then, the anisotropic isoperimetric inequality, or Wulff inequality,

$$P_E(F) \geqslant n|E|^{1/n}|F|^{(n-1)/n}, \tag{8}$$

holds true, as it can be shown starting from (1) (see [11, Section 3]).

Proof of Lemma 2. Let us set

$$\tau(E) := \inf_F \frac{\mathcal{H}^{n-1}(E \cap \partial F)}{\mathcal{H}^{n-1}(F \cap \partial E)}$$

where F ranges over the class of open sets of $\mathbb{R}^n$ with smooth boundary such that $|E \cap F| \leqslant |E|/2$. Then, fixed $f \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we set $F_t = \{x \in \mathbb{R}^n: \ f(x) > t\}$ for every $t \in \mathbb{R}$. The proof of the lemma is then achieved on combining the following two statements.

Step one: We have that

$$\int_E |\nabla f| \geqslant \tau(E) \int_{\partial E} |f - m| \, d\mathcal{H}^{n-1},$$

where m is a median of f in E, i.e.

$$|F_t \cap E| \leqslant \frac{|E|}{2}, \quad \forall t \geqslant m,$$
$$|F_t \cap E| > \frac{|E|}{2}, \quad \forall t < m.$$

Indeed, let $g = \max\{f - m, 0\}$ and let $G_t = \{x \in \mathbb{R}^n: \ g(x) > t\}$. Then by the Coarea Formula, the choice of m and the definition of $\tau(E)$ (note that F_t is admissible in $\tau(E)$ for a.e. $t \geqslant m$ by Morse–Sard Lemma)

$$\begin{aligned}
\int_{E \cap F_m} |\nabla f| &= \int_E |\nabla g| = \int_0^\infty \mathcal{H}^{n-1}(E \cap \partial G_t) \, dt \\
&\geqslant \tau(E) \int_0^\infty \mathcal{H}^{n-1}(G_t \cap \partial E) \, dt = \tau(E) \int_{\partial E} g \, d\mathcal{H}^{n-1} \\
&= \tau(E) \int_{\partial E} \max\{f - m, 0\} \, d\mathcal{H}^{n-1}.
\end{aligned}$$

The choice of m allows to argue similarly with $\max\{m - f, 0\}$ in place of g and to eventually achieve the proof of Step one.

Step two: We have that

$$\tau(E) \geqslant \frac{r}{R}\left(1 - \frac{1}{2^{1/n}}\right).$$

To prove this, let us consider an admissible set F for $\tau(E)$ and set for simplicity

$$\lambda := \frac{\mathcal{H}^{n-1}(E \cap \partial F)}{\mathcal{H}^{n-1}(F \cap \partial E)}. \tag{9}$$

On denoting $F_1 = F \cap E$ and $F_2 = E \setminus \overline{F}$, we have that

$$E \cap \partial F_1 = E \cap \partial F_2 = E \cap \partial F, \quad \text{with } \nu_F = \nu_{F_1} = -\nu_{F_2} \text{ on } E \cap \partial F.$$

Therefore

$$\begin{aligned} P_E(E) &\geqslant P_E(F_1) + P_E(F_2) - \int_{E \cap \partial F_1} \|\nu_{F_1}\|_E \, d\mathcal{H}^{n-1} - \int_{E \cap \partial F_2} \|\nu_{F_2}\|_E \, d\mathcal{H}^{n-1} \\ &\geqslant P_E(F_1) + P_E(F_2) - 2R\mathcal{H}^{n-1}(E \cap \partial F) \\ &= P_E(F_1) + P_E(F_2) - 2R\lambda\mathcal{H}^{n-1}(F \cap \partial E) \\ &\geqslant P_E(F_1) + P_E(F_2) - 2R\lambda\mathcal{H}^{n-1}(\partial F_1) \\ &\geqslant \left(1 - 2\lambda\frac{R}{r}\right) P_E(F_1) + P_E(F_2), \end{aligned} \tag{10}$$

where we have used (9) and the elementary inequality

$$r \leqslant \|\nu\|_E \leqslant R,$$

for every $\nu \in S^{n-1}$. On combining (10), the anisotropic isoperimetric inequality (8) and the fact that $P_E(E) = n|E|$, we come to

$$n|E| \geqslant n|E|^{1/n}\left\{\left(1 - 2\lambda\frac{R}{r}\right)|F_1|^{1/n'} + |F_2|^{1/n'}\right\},$$

i.e. we have proved that

$$\lambda t^{1/n'} \geqslant \frac{r}{2R}\left(t^{1/n'} + (1-t)^{1/n'} - 1\right),$$

where $t = |F_1|/|E|$. As $t \in (0, 1/2]$ by construction and

$$s^{1/n'} + (1-s)^{1/n'} - 1 \geqslant \left(2 - 2^{1/n'}\right)s^{1/n'}, \quad \forall s \in (0, 1/2],$$

the proof of Step two is easily concluded. □

Remark 3. Let us point out that the dependence on n and R/r given in the above result, that is $n(R/r)$, is sharp. In $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, it suffices to consider the box E defined as

$$E = Q \times [-R_0, R_0], \quad Q = \left[-\frac{r}{2}, \frac{r}{2}\right]^{n-1}.$$

We clearly have that $B_r \subset E \subset B_R$, with $R = \sqrt{R_0^2 + (n-1)r^2}$. Now, let us consider as a test set for the trace constant the half-space $F = \mathbb{R}^{n-1} \times (0, \infty)$, so that

$$\partial F \cap E = Q \times \{0\}, \qquad \partial E \cap F = \left(\partial Q \times (0, R_0)\right) \cup \left(Q \times \{R_0\}\right).$$

The boundary ∂Q is the union of $2(n-1)$ cubes of dimension $(n-2)$ and size r. Thus,

$$\mathcal{H}^{n-1}(\partial F \cap E) = r^{n-1}, \qquad \mathcal{H}^{n-1}(\partial E \cap F) = 2(n-1)R_0 r^{n-2} + r^{n-1}.$$

For $R_0 \gg \sqrt{n-1}\, r$ we have $R \approx R_0$, and therefore

$$\frac{n\sqrt{2}}{\log(2)}\frac{R}{r} \leqslant \tau(E) \leqslant \frac{2(n-1)R_0 r^{n-2} + r^{n-1}}{r^{n-1}} \approx n\frac{R_0}{r} \approx n\frac{R}{r}.$$

This shows the sharpness of our trace constant, up to a numeric factor.

3. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. We consider two convex bodies E and F, and we aim to prove (6). Without loss of generality, we may assume that $|E| \geqslant |F|$. By approximation, we can also assume that E and F are smooth and uniformly convex. Eventually, we can directly consider the case

$$\beta(E,F)\sigma(E,F)^{1/n} \leqslant 1. \tag{11}$$

Indeed, as we always have $A(E,F) \leqslant 2$, if $\beta(E,F)\sigma(E,F)^{1/n} > 1$ then (6) holds trivially with $C_0(n) = 2$. Observe further that, since $\sigma(E,F) \geqslant 1$, (11) implies

$$\beta(E,F) \leqslant 1. \tag{12}$$

We divide the proof in several steps.

Step one: John's normalization. A classical result in the theory of convex bodies by F. John [15] ensures the existence of a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$B_1 \subset L(E) \subset B_n.$$

We note that

$$\beta(E,F) = \beta\big(L(E), L(F)\big), \qquad A(E,F) = A\big(L(E), L(F)\big), \qquad \big|L(E)\big| \geqslant \big|L(F)\big|.$$

Therefore in the proof of Theorem 1 we may also assume that

$$B_1 \subset E \subset B_n. \tag{13}$$

In particular, under this assumption one has $1 \leqslant r \leqslant R \leqslant n$, so that by Lemma 2 we can write

$$\frac{n^2\sqrt{2}}{\log(2)} \int_E |\nabla f| \geqslant \inf_{c\in\mathbb{R}} \int_{\partial E} |f - c| \, d\mathcal{H}^{n-1} \tag{14}$$

for every $f \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Step two: Mass transportation proof of Brunn–Minkowski. We prove the Brunn–Minkowski inequality by mass transportation. By the Brenier Theorem [2,3], there exists a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that its gradient $T = \nabla\varphi$ defines a map $T \in BV(\mathbb{R}^n, \overline{F})$ pushing forward $|E|^{-1}1_E(x)\,dx$ to $|F|^{-1}1_F(x)\,dx$, i.e.

$$\frac{1}{|F|}\int_F h(y)\,dy = \frac{1}{|E|}\int_E h\big(T(x)\big)\,dx, \tag{15}$$

for every Borel function $h : \mathbb{R}^n \to [0,\infty)$. As shown by Caffarelli [5,6], under our assumptions the Brenier map is smooth up to the boundary, i.e. $T \in C^\infty(\overline{E}, \overline{F})$. Moreover, the push-forward condition (15) takes the form

$$\det \nabla T(x) = \frac{|F|}{|E|}, \quad \forall x \in E. \tag{16}$$

We are going to consider the eigenvalues $\{\lambda_k(x)\}_{k=1,\dots,n}$ of $\nabla T(x) = \nabla^2\varphi(x)$, ordered so that $\lambda_k \leqslant \lambda_{k+1}$ for $1 \leqslant k \leqslant n-1$. We also define, for every $x \in E$,

$$\lambda_A(x) = \frac{\sum_{k=1}^n \lambda_k(x)}{n}, \qquad \lambda_G(x) = \left(\prod_{k=1}^n \lambda_k(x)\right)^{1/n}.$$

Thanks to (16) we have

$$\lambda_G(x) = \left(\frac{|F|}{|E|}\right)^{1/n}$$

for every $x \in E$. We are in the position to prove the Brunn–Minkowski inequality. Let $S(x) := x + T(x)$, then $S(E) \subset E + F$. As $\det \nabla S = \prod_{k=1}^n (1+\lambda_k) > 1$, we have $|\det \nabla S| = \det \nabla S$. Thus

$$|E+F|^{1/n} \geqslant \left|S(E)\right|^{1/n} = \left(\int_E \det \nabla S\right)^{1/n} = \left(\int_E \prod_{k=1}^n (1+\lambda_k)\right)^{1/n}. \tag{17}$$

We observe that

$$\prod_{k=1}^n (1+\lambda_k) = 1 + \sum_{m=1}^n \sum_{\{1\leqslant i_1<\cdots<i_m\leqslant n\}} \prod_{j=1}^m \lambda_{i_j}. \tag{18}$$

Note that the set of indexes $(i_1,\dots,i_m)$ with $1 \leqslant i_j < i_{j+1} \leqslant n$ counts $\binom{n}{m}$ elements. For each fixed $m \geqslant 1$, the arithmetic–geometric mean inequality implies that

$$\sum_{\{1\leqslant i_1<\cdots<i_m\leqslant n\}} \prod_{j=1}^m \lambda_{i_j} \geqslant \binom{n}{m} \prod_{\{1\leqslant i_1<\cdots<i_m\leqslant n\}} \left(\prod_{j=1}^m \lambda_{i_j}\right)^{1/\binom{n}{m}}. \tag{19}$$

This last term is equal to

$$\binom{n}{m} \prod_{k=1}^n \lambda_k^{\binom{n-1}{m-1}/\binom{n}{m}} = \binom{n}{m} \lambda_G^m. \tag{20}$$

On putting (18), (19) and (20) together, and applying the binomial formula to $(1+\lambda_G)^n$ we come to

$$\prod_{k=1}^n (1+\lambda_k) - (1+\lambda_G)^n = \sum_{m=1}^n \Gamma_m, \tag{21}$$

where Γ_m denotes the difference between the left- and the right-hand side of (19). We observe that $\Gamma_m \geqslant 0$ whenever $1 \leqslant m \leqslant n$, and in particular $\Gamma_1 = n(\lambda_A - \lambda_G)$. On combining this with (17), (16), and $\lambda_G = (\det \nabla T)^{1/n}$, we find that

$$|E+F|^{1/n} \geqslant \left(\int_E (1+\lambda_G)^n\right)^{1/n} = |E|^{1/n}\left(1 + \left(\frac{|F|}{|E|}\right)^{1/n}\right) = |E|^{1/n} + |F|^{1/n},$$

i.e. we prove the Brunn–Minkowski inequality for E and F.

Step three: Lower bounds on the deficit. In this step we aim to prove

$$\frac{1}{|E|}\int_E \left|\nabla T(x) - \lambda_G \operatorname{Id}\right| dx \leqslant C(n)\sqrt{\beta(E,F)}\sqrt{\beta(E,F) + \sigma(E,F)^{-1/n}}. \tag{22}$$

Let us set, for the sake of brevity,

$$s = \frac{1}{|E|}\int_E \det \nabla S, \qquad t = (1+\lambda_G)^n.$$

From Step two we deduce that

$$\frac{|E+F|^{1/n} - (|E|^{1/n} + |F|^{1/n})}{|E|^{1/n}} \geqslant s^{1/n} - t^{1/n} = \frac{s-t}{\sum_{h=1}^n s^{(n-h)/n} t^{(h-1)/n}}. \tag{23}$$

As $t \leqslant s$ and $|E|s = |S(E)| \leqslant |E+F|$,

$$\begin{aligned}\sum_{h=1}^n s^{(n-h)/n} t^{(h-1)/n} &\leqslant n s^{(n-1)/n} \leqslant n\left(\frac{|E+F|}{|E|}\right)^{(n-1)/n} \\ &= n\left(\bigl(1+\beta(E,F)\bigr)\frac{|E|^{1/n} + |F|^{1/n}}{|E|^{1/n}}\right)^{n-1} \leqslant C(n),\end{aligned} \tag{24}$$

where we have also made use of (12) and of the fact that $|F| \leqslant |E|$. A similar argument shows that the left-hand side of (23) is controlled by $2\beta(E, F)$, and therefore we conclude that

$$C(n)\beta(E, F) \geqslant s - t = \frac{1}{|E|} \int_E \left(\prod_{k=1}^n (1 + \lambda_k) - (1 + \lambda_G)^n \right) dx. \tag{25}$$

Then, by (25) and (21), as $\Gamma_m \geqslant 0$ whenever $1 \leqslant m \leqslant n$ and $\Gamma_1 = n(\lambda_A - \lambda_G)$, we get

$$C(n)\beta(E, F) \geqslant \frac{1}{|E|} \int_E \sum_{m=1}^n \Gamma_m(x)\, dx \geqslant \frac{1}{|E|} \int_E \Gamma_1(x)\, dx = \frac{n}{|E|} \int_E (\lambda_A - \lambda_G). \tag{26}$$

An elementary quantitative version of the arithmetic–geometric mean inequality proved in [10, Lemma 2.5], ensures that

$$7n^2(\lambda_A - \lambda_G) \geqslant \frac{1}{\lambda_n} \sum_{k=1}^n (\lambda_k - \lambda_G)^2.$$

In particular, as $(\lambda_n - \lambda_1)^2 \leqslant 2[(\lambda_n - \lambda_G)^2 + (\lambda_G - \lambda_1)^2]$ we obtain from (26)

$$C(n)\beta(E, F) \geqslant \frac{1}{|E|} \int_E \frac{(\lambda_n - \lambda_1)^2}{\lambda_n}\, dx. \tag{27}$$

By Hölder inequality

$$\frac{1}{|E|} \int_E (\lambda_n - \lambda_1)\, dx \leqslant C(n) \sqrt{\beta(E, F) \frac{1}{|E|} \int_E \lambda_n}. \tag{28}$$

As $\lambda_1 \leqslant (|F|/|E|)^{1/n} = \sigma(E, F)^{-1/n}$, from (28) we come to

$$\frac{1}{|E|} \int_E \lambda_n \leqslant C(n) \sqrt{\beta(E, F) \frac{1}{|E|} \int_E \lambda_n} + \sigma(E, F)^{-1/n},$$

which easily implies

$$\frac{1}{|E|} \int_E \lambda_n \leqslant C(n)\big(\beta(E, F) + \sigma(E, F)^{-1/n}\big) \tag{29}$$

by Young's inequality. We eventually combine (29) with (28), and prove that

$$\frac{1}{|E|} \int_E (\lambda_n - \lambda_1)\, dx \leqslant C(n)\sqrt{\beta(E, F)}\sqrt{\beta(E, F) + \sigma(E, F)^{-1/n}}. \tag{30}$$

Then (22) follows immediately.

Step four: Trace inequality. On combining (22) with (14), we conclude that, up to a translation of F,

$$C(n)\sqrt{\beta(E, F)}\sqrt{\beta(E, F) + \sigma(E, F)^{-1/n}}|E| \geqslant \int_{\partial E} \big|T(x) - \lambda_G x\big|\, d\mathcal{H}^{n-1}(x).$$

If $F' = \lambda_G^{-1} F$ and $P : \mathbb{R}^n \setminus F' \to \partial F'$ denotes the projection of $\mathbb{R}^n \setminus F'$ over F', then, since by construction T takes value in $\overline{F}$, we get

$$C(n)\sqrt{\beta(E, F)}\sqrt{\beta(E, F) + \sigma(E, F)^{-1/n}} \geqslant \frac{\lambda_G}{|E|} \int_{\partial E \setminus F'} \big|P(x) - x\big|\, d\mathcal{H}^{n-1}(x). \tag{31}$$

We now consider the map $\Phi : (\partial E \setminus F') \times (0,1) \to E \setminus F'$ defined by

$$\Phi(x,t) = tx + (1-t)P(x).$$

Let $\{\varepsilon_k(x)\}_{k=1}^{n-1}$ be a basis of the tangent space to ∂E at x. Since Φ is a bijection, we find

$$|E \setminus F'| = \int_0^1 dt \int_{(\partial E \setminus F')} \left| \big(x - P(x)\big) \wedge \left(\bigwedge_{k=1}^{n-1} \big(t\varepsilon_k(x) + (1-t)\,dP_x\big(\varepsilon_k(x)\big)\big) \right) \right| d\mathcal{H}^{n-1}(x), \tag{32}$$

where dP_x denotes the differential of the projection P at x. As P is the projection over a convex set, it decreases distances, i.e. $|dP_x(e)| \leqslant 1$ for every $e \in S^{n-1}$. Thus,

$$\big|t\varepsilon_k(x) + (1-t)dP_x\big(\varepsilon_k(x)\big)\big| \leqslant 1, \quad \forall k \in \{1, \ldots, n-1\}.$$

Recalling that $\lambda_G = \sigma(E,F)^{-1/n}$, we combine this last inequality with (31) and (32) to get

$$\begin{aligned}
\frac{|E \setminus F'|}{|E|} &\leqslant \frac{1}{|E|} \int_{\partial E \setminus F'} \big|x - P(x)\big|\, d\mathcal{H}^{n-1}(x) \\
&\leqslant C(n)\sigma(E,F)^{1/n} \sqrt{\beta(E,F)} \sqrt{\beta(E,F) + \sigma(E,F)^{-1/n}} \\
&\leqslant C(n)\sigma(E,F)^{1/n} \sqrt{\beta(E,F)} \big(\sqrt{\beta(E,F)} + \sigma(E,F)^{-1/2n}\big) \\
&= C(n)\big(\sqrt{\beta(E,F)\sigma(E,F)^{1/n}} + \beta(E,F)\sigma(E,F)^{1/n}\big) \\
&\leqslant C(n)\sqrt{\beta(E,F)\sigma(E,F)^{1/n}},
\end{aligned}$$

where in the last inequality we have used (11). As

$$A(E,F) \leqslant \frac{|E \Delta F'|}{|E|} = 2\frac{|E \setminus F'|}{|E|},$$

this proves (6) and we achieve the proof of the theorem.

We conclude noticing that the constant $C_0(n)$ in the above theorem can be taken to be

$$C_0(n) \approx p(n)c_0^n,$$

where $p(n)$ is a polynomial in n, and c_0 is any constant greater than $\sqrt{2}$. Indeed, a quick inspection of the proof shows that all the terms to be considered for $C(n)$ are polynomials, except for the estimate given in Step three – more precisely in (24) – which gives a term like nc^n, with $c > 2$ (recall that, up to loosing a numeric factor in $C_0(n)$, we can assume from the beginning that $\beta(E,F)$ is smaller than an arbitrarily small constant). Eventually, when applying Hölder inequality in (28) we take a square root of the constant $C(n)$ appearing in (27), thus coming to the choice $c_0 > \sqrt{2}$.

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