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Controllability on the group of diffeomorphisms

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Abstract

Given a compact manifold M, we prove that any bracket generating family of vector fields on M, which is invariant under multiplication by smooth functions, generates the connected component of the identity of the group of diffeomorphisms of M.

Résumé

Soit M une variété compacte, nous montrons que toute famille de champs de vecteurs satisfaisant la condition du rang et étant invariante par multiplication par fonctions lisses engendre la composante connexe de l'identité du groupe $\mathrm{Diff}\, M$.

MSC: 93B05; 58D05

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1. Introduction

In this paper we give a simple sufficient condition for a family of flows on a smooth compact manifold M to generate the group $\mathrm{Diff}_0(M)$ of all diffeomorphisms of M that are isotopic to the identity.

If all flows are available then the result follows from the simplicity of the group $\mathrm{Diff}_0(M)$ (see [9]). Indeed, flows are just one-parametric subgroups of $\mathrm{Diff}_0(M)$ and all one-parametric subgroups generate a normal subgroup. In other words, any isotopic to the identity diffeomorphism of M can be presented as composition of exponentials of smooth vector fields.

In this paper we prove that a stronger result holds for a proper subset of the space of smooth vector fields on M. Our main result is as follows.

Theorem 1.1. *Let $\mathcal{F} \subset \mathrm{Vec}\, M$ be a family of smooth vector fields and let $\mathrm{Gr}\, \mathcal{F} = \{e^{t_1 f_1} \circ \cdots \circ e^{t_k f_k}\colon\ t_i \in \mathbb{R},\ f_i \in \mathcal{F},\ k \in \mathbb{N}\}$.*

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If $\operatorname{Gr}\mathcal{F}$ *acts transitively on* M *then there exist a neighborhood* $\mathcal{O}$ *of the identity in* $\operatorname{Diff}_0(M)$ *and a positive integer* m *such that every* $P \in \mathcal{O}$ *can be presented in the form*

$$P = e^{a_1 f_1} \circ \cdots \circ e^{a_m f_m},$$

for some $f_1, \dots, f_m \in \mathcal{F}$ *and* $a_1, \dots, a_m \in C^\infty(M)$.

In particular, if $\mathcal{F}$ is a bracket generating family of vector fields then any diffeomorphism in $\operatorname{Diff}_0(M)$ can be presented as composition of exponentials of vector fields in $\mathcal{F}$ rescaled by smooth functions. In fact, a stronger result is valid. The theorem states that every diffeomorphism sufficiently close to the identity can be presented as the composition of m exponentials, where the number m depends only on $\mathcal{F}$.

The structure of the paper is the following. In Section 2 we fix the notation used throughout the paper and we make some remarks about tools used in the sequel. In Section 3 we state some simple corollaries of the main result and explain its meaning for geometric control theory. Then we start the proof of Theorem 1.1 showing, in Section 4, an auxiliary result concerning local diffeomorphisms in $\mathbb{R}^n$. Namely, given n vector fields over $\mathbb{R}^n$, $X_1, \dots, X_n$, linearly independent at the origin, we find a closed neighborhood V of the origin in $\mathbb{R}^n$ such that the image of the map

$$\phi : (a_1, \dots, a_n) \mapsto e^{a_1 X_1} \circ \cdots \circ e^{a_n X_n}|_V$$

from $C_0^\infty(\mathbb{R}^n)^n$ to $C_0^\infty(V)^n$ has nonempty interior. In Section 5 we show how to reduce the proof of Theorem 1.1 to the mentioned auxiliary fact using a geometric idea that goes back to the Orbit Theorem of Sussmann [8].

2. Preliminaries

Let M be a smooth n-dimensional compact connected manifold. Throughout the paper smooth means C^∞.

We denote by $\operatorname{Vec} M$ the Lie algebra of smooth vector fields on M and by $\operatorname{Diff}_0 M$ the connected component of the identity of the group of diffeomorphisms of M.

If V is a neighborhood of the origin in $\mathbb{R}^n$ we set $C_0^\infty(V) = \{a \in C^\infty(V)\colon a(0) = 0\}$. Similarly, if U is an open subset of M then $C_q^\infty(U, M)$ is the Fréchet manifold of smooth maps $F : U \to M$ such that $F(q) = q$. All the spaces above are endowed with the standard C^∞ topology.

Given an autonomous vector field $f \in \operatorname{Vec} M$ we denote by $t \mapsto e^{tf}$, with $t \in \mathbb{R}$, the flow on M generated by f, which is a one-parametric subgroup of $\operatorname{Diff}_0(M)$.

If f_τ is a nonautonomous vector field, using "chronological" notation (see [1]), we denote by $\overrightarrow{\exp} \int_0^t f_\tau \, d\tau$ the "nonautonomous flow" at time t of the time-varying vector field f_τ.

Given a family of vector fields $\mathcal{F} \subset \operatorname{Vec} M$ we associate to $\mathcal{F}$ the subgroup of $\operatorname{Diff}_0(M)$

$$\operatorname{Gr}\mathcal{F} = \left\{ e^{t_1 f_1} \circ \cdots \circ e^{t_k f_k}\colon\ t_i \in \mathbb{R},\ f_i \in \mathcal{F},\ k \in \mathbb{N} \right\}.$$

$\operatorname{Lie}\mathcal{F}$ is the Lie subalgebra of $\operatorname{Vec} M$ generated by $\mathcal{F}$ and the algebra of vector fields in $\operatorname{Lie}\mathcal{F}$ evaluated at $q \in M$ is $\operatorname{Lie}_q \mathcal{F} = \{V(q)\colon\ V \in \operatorname{Lie}\mathcal{F}\}$.

Definition 1. A family $\mathcal{F} \in \operatorname{Vec} M$ is called *bracket generating*, or *completely nonholonomic*, if

$$\operatorname{Lie}_q \mathcal{F} = T_q M, \quad \text{for every } q \in M.$$

A classical result in Control Theory is the Rashevsky–Chow Theorem (see [7,2]) that gives a sufficient condition for controllability.

Theorem 2.1 *(Rashevsky–Chow). Let* M *be a compact connected manifold. If* $\mathcal{F}$ *is bracket generating then* $\operatorname{Gr}\mathcal{F}$ *acts transitively on* M.

Another classical result, due to Lobry [5], claims that $\operatorname{Gr}\{f_1, f_2\}$ acts transitively on M for a generic pair of smooth vector fields (f_1, f_2). Namely, the set of pairs of vector fields (f_1, f_2) such that $\operatorname{Gr}\{f_1, f_2\}$ acts transitively on M is an open dense (in the C^∞ topology) subset of the product space $\operatorname{Vec} M \times \operatorname{Vec} M$.

Moreover, if M has the structure of a semisimple Lie group then the set of pairs (f_1, f_2) of left-invariant vector fields such that $\mathrm{Gr}\{f_1, f_2\}$ acts transitively on M is an open dense subset in the Cartesian square of the Lie algebra (see [4]).

3. Corollaries of the main result

A direct consequence of Theorem 1.1 is the following.

Corollary 3.1. *Let $\mathcal{F} \subset \operatorname{Vec} M$, if $\operatorname{Gr}\mathcal{F}$ acts transitively on M, then*

$$\operatorname{Gr}\bigl\{af\colon a \in C^\infty(M),\ f \in \mathcal{F}\bigr\} = \operatorname{Diff}_0(M).$$

Last corollary shows the relation between the Rashevsky–Chow Theorem and Theorem 1.1. Indeed, if $\mathcal{F}$ is a bracket generating family of vector fields, then, by Rashevsky–Chow Theorem, for every pair of points $q_0, q_1 \in M$ there exist $t_1, \dots, t_k \in \mathbb{R}$ and $f_1, \dots, f_k \in \mathcal{F}$ such that

$$q_0 = q_1 \circ e^{t_1 f_1} \circ \cdots \circ e^{t_k f_k},$$

and, by Corollary 3.1, for every diffeomorphism $P \in \operatorname{Diff}_0(M)$ there exist $a_1, \dots, a_\ell \in C^\infty(M)$ and $g_1, \dots, g_\ell \in \mathcal{F}$ such that

$$P = e^{a_1 g_1} \circ \cdots \circ e^{a_\ell g_\ell}.$$

In other words, we have that controllability of a system of vector fields on the manifold implies a certain "controllability" on the group of diffeomorphisms. Namely, if it is possible to join every two points of the manifold M by exponentials of vector field in $\mathcal{F}$ then we can realize every diffeomorphism as composition of exponentials of vector fields in $\mathcal{F}$ rescaled by suitable smooth functions.

Let us reformulate Corollary 3.1 in terms of control systems. Consider the control system on M

$$\dot{q} = \sum_{i=1}^{k} u_i(t, q) f_i(q), \quad q \in M, \tag{1}$$

where $\{f_1, \dots, f_k\}$ is a bracket generating family of vector fields and $u_1, \dots, u_k$ are *time varying feedback* controls, that is

$$u_i : [0, 1] \times M \to \mathbb{R},$$

such that $u_i(t, q)$ is piecewise constant in t for every q and smooth with respect to q for every t. Then Corollary 3.1 states that for every $P \in \operatorname{Diff}_0(M)$ there exist time-varying feedback controls $u_1, \dots, u_k$, such that $q(1) = P(q(0))$ for any solution $q(\cdot)$ of system (1); in other words,

$$P = \overrightarrow{\exp} \int_0^1 \sum_{i=1}^{k} u_i(t, \cdot) f_i \, dt.$$

The next corollary is stated from a geometric viewpoint, in terms of completely nonholonomic vector distributions.

Corollary 3.2. *Let $\Delta \subset TM$ be a completely nonholonomic vector distribution. Then every diffeomorphism of M that is isotopic to the identity can be written as $e^{f_1} \circ \cdots \circ e^{f_k}$, where $f_1, \dots, f_k$ are sections of Δ.*

4. An auxiliary result

Proposition 4.1. *Let $X_1, \dots, X_n \in \operatorname{Vec} \mathbb{R}^n$ be such that*

$$\operatorname{span}\bigl\{X_1(0), \dots, X_n(0)\bigr\} = \mathbb{R}^n.$$

Then there exist a compact neighborhood V of the origin in $\mathbb{R}^n$ and an open subset $\mathcal{V}$ of $C_0^\infty(V)^n$ such that every $F \in \mathcal{V}$ can be written as

$$F = e^{a_1 X_1} \circ \cdots \circ e^{a_n X_n}|_V,$$

for some $a_1, \dots, a_n \in C_0^\infty(\mathbb{R}^n)$.

In order to prove this result we need the following lemma.

Lemma 4.2. *Let $X_1, \dots, X_n \in \operatorname{Vec} \mathbb{R}^n$ be such that*

$$\operatorname{span}\{X_1(0), \dots, X_n(0)\} = \mathbb{R}^n,$$

and let $\mathcal{U}_0$ be a neighborhood of the identity in $C_0^\infty(\mathbb{R}^n)^n$. Then there exist a neighborhood V of the origin in $\mathbb{R}^n$ and a neighborhood $\mathcal{U}$ of the identity in $C_0^\infty(V)^n$ such that for every $F \in \mathcal{U}$ there exist $\varphi_1, \dots, \varphi_n \in \mathcal{U}_0$ such that

$$F = \varphi_1 \circ \cdots \circ \varphi_n|_V,$$

where φ_k preserves the 1-foliation generated by the trajectories of the equation $\dot q = X_k(q)$ for $k = 1, \dots, n$.

Proof. Since $X_1, \dots, X_n$ are linearly independent at 0 then there exists a neighborhood of the origin $V \subset \mathbb{R}^n$ such that

$$\operatorname{span}\{X_1(q), \dots, X_n(q)\} = \mathbb{R}^n, \quad \text{for every } q \in \bar V.$$

Now, there exists a ball $B \subset \mathbb{R}^n$ containing $0 \in \mathbb{R}^n$ such that, for every $q \in V$, the map

$$(t_1, \dots, t_n) \mapsto q \circ e^{t_1 X_1} \circ \cdots \circ e^{t_n X_n} \tag{2}$$

is a local diffeomorphism from B to a neighborhood of q. Let

$$\mathcal{U}_\varepsilon = \{F \in C_0^\infty(V)^n \colon \|F - I\|_{C^1} < \varepsilon\},$$

where I denotes the identical map and ε is to be chosen later. If ε is sufficiently small, then, for every $F \in \mathcal{U}_\varepsilon$ and $q \in V$, $F(q)$ belongs to the image of map (2). Therefore, given every $F \in \mathcal{U}_\varepsilon$, it is possible to associate with every $q \in V$ an n-uple of real numbers $(t_1(q), \dots, t_n(q)) \in B$ such that

$$F(q) = q \circ e^{t_1(q) X_1} \circ \cdots \circ e^{t_n(q) X_n}.$$

We claim that there exists $\eta(\varepsilon)$ such that $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\|t_i\|_{C^1} < \eta(\varepsilon)$ for every $i = 1, \dots, n$ and for $F \in \mathcal{U}_\varepsilon$. Indeed $\|F - I\|_{C^0} < \varepsilon$ implies that $\|t_i\|_{C^0} < c\varepsilon$, for $i = 1, \dots, n$ and for some constant c. Moreover, if $q \in V$, for every $\xi \in \mathbb{R}^n$ we have

$$D_q F \xi = \left(e^{t_1(q) X_1} \circ \cdots \circ e^{t_n(q) X_n}\right)_* \xi + \sum_{i=1}^n e^{t_1(q) X_1} \circ \cdots \circ \frac{dt_i}{dq} \cdot \xi X_i \circ \cdots \circ e^{t_n(q) X_n}.$$

Therefore $\|D_q F \xi - \xi\|_{C^0} < \varepsilon$ implies $\|t_i\|_{C^1} < \eta(\varepsilon)$, where $\eta \to 0$ as $\varepsilon \to 0$.

Now consider, for every $k = 1, \dots, n$, the map

$$\Phi_k(q) = q \circ e^{t_1(q) X_1} \circ \cdots \circ e^{t_k(q) X_k}.$$

Note that $\Phi_0 = I$ and $\Phi_n = F$. For every k, Φ_k is a smooth diffeomorphism being smooth and invertible by the Implicit Function Theorem. Indeed, for every $q \in V$ the differential of Φ_k at q is

$$D_q \Phi_k \xi = \left(e^{t_1(q) X_1} \circ \cdots \circ e^{t_k(q) X_k}\right)_* \xi + \sum_{i=1}^k e^{t_1(q) X_1} \circ \cdots \circ \frac{dt_i}{dq} \xi X_i \circ \cdots \circ e^{t_k(q) X_k}.$$

Denote $T(\xi) = D_q \Phi_k \xi - \xi$. If ε is sufficiently small we have $\|T\|_0 < 1$. Therefore $D_q \Phi_k$ is of the form $I + T$, with T contraction, and thus invertible.

Finally call $\mathcal{U} = \mathcal{U}_\varepsilon$ and define for every $k = 1, \dots, n$, the smooth maps

$$\varphi_k(q) = q \circ e^{t_k(\Phi_{k-1}^{-1}(q)) X_k},$$

then the statement follows. $\square$

Thanks to the last lemma our problem is to find an appropriate exponential representation of every of the functions φ_k.

The main idea of the proof of Proposition 4.1 lies in the fact that a linear diffeomorphism is the exponential of a linear vector field. So, our goal is to find a change of coordinates that linearizes φ_k along trajectories of the equation $\dot q = X_k(q)$.

Proof of Proposition 4.1. Let V, $\mathcal{U}$, and $\mathcal{U}_0$ be as in Lemma 4.2. We denote by $\mathcal{X}_k$ the set of all $\varphi \in \mathcal{U}_0$ such that φ preserves 1-foliation generated by the equation $\dot q = X_k(q)$. Every $F \in \mathcal{U}$ can be written as $F = \varphi_1 \circ \cdots \circ \varphi_n|_V$. Now consider the open subset of $C_0^\infty(V)^n$

$$\mathcal{V} \subseteq \bigl\{F \in \mathcal{U}\colon\ F = \varphi_1 \circ \cdots \circ \varphi_n|_V,\ \varphi_k \in \mathcal{X}_k,\ (D_q\varphi_k)X_k(q) \neq X_k(q),\ q \in \varphi_k^{-1}(0),\ k = 1, \ldots, n\bigr\}.$$

Since every $F \in \mathcal{U}$ is close to the identity, then so is φ_k for every k. Moreover, $\varphi_k(0) = 0$ and X_k transversal to the hypersurface $\varphi_k^{-1}(0)$ at any point. Therefore we may rectify the field X_k in a neighborhood of the origin in such a way that, in new coordinates, $\varphi_k(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n) = 0$ and $X_k = \frac{\partial}{\partial x_k}$. Set $x := x_k$ and $y := (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$.

Since the following argument does not depend on $k = 1, \ldots, n$ the subscript k is omitted.

Let $\alpha(y) = \log(\frac{\partial}{\partial x}\varphi(0, y))$. Note that by the definition of $\mathcal{V}$ we have $\alpha(y) \neq 0$ for every y. In what follows we treat y as an $(n-1)$-dimensional parameter and, for the sake of readability, we omit it. We will show, step by step, that the argument holds for every value of the parameter y and all maps and vector fields under consideration depend smoothly on y. Consider the homotopy from φ to the identity

$$\varphi_t(x) = e^{\alpha(t-1)}\varphi(tx)/t, \quad t \in [0, 1].$$

There exists a nonautonomous vector field $a(t, x)\frac{\partial}{\partial x}$ such that

$$\varphi_t = \overrightarrow{\exp} \int_0^t a(\tau, \cdot)\frac{\partial}{\partial x}\, d\tau.$$

It is easy to see that $\frac{\partial a}{\partial x}(t, 0) = \alpha$. Let $a(t, x) = \alpha x + b(t, x)x$ with $b(t, 0) = 0$. We want to find a time-dependent change of coordinates $\psi(t, x)$ that linearizes the flow generated by $a(t, x)$. Namely if $x(t)$ is a solution of $\dot x = a(t, x)$ and $z(t) = \psi(t, x(t))$ then we want $\dot z(t) = \alpha z(t)$. We can suppose $\psi(t, 0) = 0$ and write $\psi(t, x) = xu(t, x)$, where $u(0, x) = 1$. On one hand we have

$$\frac{d}{dt}z = \frac{d}{dt}\bigl(xu(t, x)\bigr) = \dot x u(t, x) + x\dot x\frac{\partial u}{\partial x}(t, x) + x\frac{\partial u}{\partial t}(t, x) = a(t, x)u(t, x) + xa(t, x)\frac{\partial u}{\partial x}(t, x) + x\frac{\partial u}{\partial t}(t, x),$$

and, on the other hand,

$$\frac{d}{dt}z = \alpha z = \alpha xu(t, x).$$

Therefore, we can find u by solving

$$x\left(a(t, x)\frac{\partial u}{\partial x}(t, x) + \frac{\partial u}{\partial t}(t, x) + b(t, x)u(t, x)\right) = 0.$$

The first-order linear PDE

$$a(t, x)\frac{\partial u}{\partial x}(t, x) + \frac{\partial u}{\partial t}(t, x) + b(t, x)u(t, x) = 0 \tag{3}$$

can be solved by the method of characteristics. The characteristic lines of (3) are of the form $\xi_t = (t, \varphi_t(x_0))$ with initial data $(0, x_0)$. Note that these characteristic lines depend smoothly on y and are well defined for every y. Along ξ_t, Eq. (3) becomes the linear (parametric with parameter y) ODE

$$\dot u = -\tilde b(t)u,$$

where $\tilde{b}(t) = b(\xi_t)$. Now we can define $u(\xi_t) = e^{-\int_0^t \tilde{b}(\tau)\,d\tau}$. This formula, being applied to all characteristics, defines a smooth solution to Eq. (3). In particular $u(t,0) = 1$ since $b(t,0) = 0$.

We have constructed a time-dependent change of coordinates $\psi(t,x)$ such that

$$\psi(t,\cdot) \circ \overrightarrow{\exp} \int_0^t a(\tau,\cdot)\frac{\partial}{\partial x}\,d\tau \circ \psi(t,\cdot)^{-1} = e^{t\alpha x \frac{\partial}{\partial x}}, \quad \text{for every } t \in [0,1].$$

Recall that $\overrightarrow{\exp} \int_0^1 a(\tau,\cdot)\frac{\partial}{\partial x}\,d\tau = \varphi$. Therefore

$$\varphi = \psi(1,\cdot)^{-1} \circ e^{\alpha x \frac{\partial}{\partial x}} \circ \psi(1,\cdot) = e^{\psi(1,\cdot)_* \alpha x \frac{\partial}{\partial x}}.$$

Hence, we provide the desired exponential representation for every of the functions $\varphi_1, \dots, \varphi_n$ from Lemma 4.2 and the proposition follows. □

5. Proof of Theorem 1.1

Set

$$\mathcal{P} = \mathrm{Gr}\big\{af\colon\ a \in C^\infty(M),\ f \in \mathcal{F}\big\}$$

and

$$\mathcal{P}_q = \big\{P \in \mathcal{P}\colon\ P(q) = q\big\}, \quad q \in M.$$

Lemma 5.1. *Any $q \in M$ possesses a neighborhood $U_q \subset M$ such that the set*

$$\{P|_{U_q}\colon\ P \in \mathcal{P}_q\} \tag{4}$$

has nonempty interior in $C^\infty_q(U_q, M)$.

Proof. According to the Orbit Theorem of Sussmann [8] (see also the textbook [1]), the transitivity of the action of $\mathrm{Gr}\,\mathcal{F}$ on M implies that

$$T_qM = \mathrm{span}\big\{P_* f(q)\colon\ P \in \mathrm{Gr}\,\mathcal{F},\ f \in \mathcal{F}\big\}.$$

Take $X_i = P_{i*} f_i$ for $i = 1, \dots, n$ with $P_i \in \mathrm{Gr}\,\mathcal{F}$ and $f_i \in \mathcal{F}$ in such a way that $X_1(q), \dots, X_n(q)$ form a basis of T_qM. Then, for all smooth functions $a_1, \dots, a_n$, vanishing at q, the diffeomorphism

$$e^{a_1X_1} \circ \dots \circ e^{a_nX_n} = P_1 \circ e^{(a_1 \circ P_1)f_1} \circ P_1^{-1} \circ \dots \circ P_n \circ e^{(a_n \circ P_n)f_n} \circ P_n^{-1}$$

belongs to the group $\mathcal{P}_q$. The desired result now follows from Proposition 4.1. □

Corollary 5.2. *The interior of set* (4) *contains the identical map.*

Proof. Let $\mathcal{A}$ be an open subset of $C^\infty_q(U_q, M)$ that is contained in (4) and take $P_0|_{U_q} \in \mathcal{A}$. Then $P_0^{-1} \circ \mathcal{A}$ is a neighborhood of the identity contained in (4). □

Definition 2. Given $P \in \mathrm{Diff}(M)$, we set $\mathrm{supp}\,P = \overline{\{x \in M\colon\ P(x) \neq x\}}$.

Lemma 5.3. *Let $\mathcal{O}$ be a neighborhood of the identity in* $\mathrm{Diff}(M)$. *Then for any $q \in M$ and any neighborhood $U_q \subset M$ of q, we have*

$$q \in \mathrm{int}\big\{P(q)\colon\ P \in \mathcal{O} \cap \mathcal{P},\ \mathrm{supp}\,P \subset U_q\big\}.$$

Proof. Consider n vector fields $X_1, \dots, X_n$ as in the proof of Lemma 5.1 and let $b \in C^\infty(M)$ be a cut-off function such that $\operatorname{supp} b \subset U_q$ and $q \in \operatorname{int} b^{-1}(1)$. Then the diffeomorphism

$$Q(s_1, \dots, s_n) = e^{s_1 b X_1} \circ \cdots \circ e^{s_n b X_n}$$

belongs to $\mathcal{O} \cap \mathcal{P}$ for any n-uple of real numbers $(s_1, \dots, s_n)$ sufficiently close to 0. Moreover $\operatorname{supp} Q(s_1, \dots, s_n) \subset U_q$. On the other hand, the map

$$(s_1, \dots, s_n) \mapsto Q(s_1, \dots, s_n)(q)$$

is a local diffeomorphism in a neighborhood of 0. □

The next lemma is due to Palis and Smale (see [6, Lemma 3.1]).

Lemma 5.4. *Let $\bigcup_j U_j = M$ be a covering of M by open subsets and let $\mathcal{O}$ be a neighborhood of identity in* $\mathrm{Diff}(M)$. *Then the group* $\mathrm{Diff}_0(M)$ *is generated by the subset* $\{P \in \mathcal{O}\colon \exists j$ *such that* $\operatorname{supp} P \subset U_j\}$.

Proof. The group $\mathrm{Diff}_0(M)$ is a path-connected topological group. Therefore it is generated by any neighborhood of the identity $\mathcal{O}$.

Since M is compact we can assume that the covering $\{U_j\}$ is finite, namely $U_1 \cup \cdots \cup U_k = M$. Now let $P \in \mathcal{O}$ and consider the isotopy $H : M \times [0,1] \to M$ such that $H(0,\cdot) = I$ and $H(1,\cdot) = P$. Consider a partition of unity

$$\{\lambda_j : M \to \mathbb{R} \mid \operatorname{supp} \lambda_j \subset U_j\}$$

subordinated to the covering $\{U_j\}_{j=1}^k$. Let $\operatorname{supp} \lambda_j = \overline{V_j}$ and let $\mu_j : M \to M \times [0,1]$ the map $\mu_j = (I, \lambda_1 + \cdots + \lambda_j)$. Consider $Q_j = H \circ \mu_j$, then $Q_k = P$ and $Q_j = Q_{j-1}$ on $M \setminus V_j$. Finally, setting $P_j = Q_j \circ Q_{j-1}^{-1}$, we have $P = P_k \circ \cdots \circ P_1$ and $\operatorname{supp} P_j \subset U_j$. The lemma is proved. □

Proof of the theorem. According to Lemma 5.4, it is sufficient to prove that, for every $q \in M$, there exist a neighborhood $U_q \subset M$ and a neighborhood of the identity $\mathcal{O} \subset \mathrm{Diff}(M)$ such that any diffeomorphism $P \in \mathcal{O}$, whose support is contained in U_q, belongs to $\mathcal{P}$. Moreover, Lemma 5.3 allows to assume that $P(q) = q$. Finally, Corollary 5.2 completes the proof. □

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References

[1] A.A. Agrachev, Yu.L. Sachkov, Control Theory from the Geometric Viewpoint, Springer-Verlag, Berlin, 2004, xiv+412 pp.
[2] W.-L. Chow, Über Systeme von linearen partiellen Differentialgleichungen erster Ordinung, Math. Ann. 117 (1939) 98–105.
[3] B. Khesin, P. Lee, A nonholonomic Moser theorem and optimal mass transport, J. Symplectic Geom. 7 (4) (2009) 1–34.
[4] M. Kuranishi, On everywhere dense imbedding of free groups in Lie groups, Nagoya Math. J. 2 (1951) 63–71.
[5] C. Lobry, Une propriété générique des couples de champs de vecteurs, Czechoslovak Math. J. 22 (97) (1972) 230–237.
[6] J. Palis, S. Smale, Structural stability theorems, in: Global Analysis, Berkeley, CA, 1968, in: Proc. Sympos. Pure Math., vol. XIV, Amer. Math. Soc., 1970, pp. 223–231.
[7] P.K. Rashevsky, About connecting two points of complete nonholonomic space by admissible curve, Uch. Zap. Ped. Inst. Libknehta 2 (1938) 83–94.
[8] H.J. Sussmann, Orbits of families of vector fields and integrability of distributions, Trans. Amer. Math. Soc. 180 (1973) 171–188.
[9] W. Thurston, Foliations and groups of diffeomorphisms, Bull. Amer. Math. Soc. 80 (1974) 304–307.