Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 27 (2010) 37–56

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Concentration of solutions for some singularly perturbed mixed problems: Asymptotics of minimal energy solutions

Jesus Garcia Azorero [a], Andrea Malchiodi [b,*], Luigi Montoro [c], Ireneo Peral [a]

[a] *Departamento de Matemáticas, UAM, 28049 Madrid, Spain*
[b] *SISSA, Via Beirut 2-4, 34014 Trieste, Italy*
[c] *Dipartimento di Matematica, UNICAL, Ponte Pietro Bucci 31 B, 87036 Arcavacata di Rende, Cosenza, Italy*

Received 6 March 2009; accepted 24 June 2009

Available online 10 July 2009

Abstract

In this paper we carry on the study of asymptotic behavior of some solutions to a singularly perturbed problem with mixed Dirichlet and Neumann boundary conditions, started in the first paper [J. Garcia Azorero, A. Malchiodi, L. Montoro, I. Peral, Concentration of solutions for some singularly perturbed mixed problems: Existence results, Arch. Ration. Mech. Anal., in press]. Here we are mainly interested in the analysis of the location and shape of *least energy solutions* when the singular perturbation parameter tends to zero. We show that in many cases they coincide with the new solutions produced in [J. Garcia Azorero, A. Malchiodi, L. Montoro, I. Peral, Concentration of solutions for some singularly perturbed mixed problems: Existence results, Arch. Ration. Mech. Anal., in press].

MSC: 35B25; 35B34; 35J20; 35J60

Keywords: Singularly perturbed elliptic problems; Finite-dimensional reductions; Local inversion

1. Introduction

We study positive solutions to the problem

$$\begin{cases} -\varepsilon^2 \Delta u + u = f(u) \quad \text{in } \Omega; \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial_{\mathcal{N}}\Omega; \qquad u = 0 \quad \text{on } \partial_{\mathcal{D}}\Omega; \\ u > 0 \quad \text{in } \Omega, \end{cases} \tag{$\tilde{M}_\varepsilon$}$$

where Ω is a smooth bounded subset of $\mathbb{R}^n$, $\varepsilon > 0$ a small parameter, and $\partial_{\mathcal{N}}\Omega$, $\partial_{\mathcal{D}}\Omega$ two disjoint subsets of the boundary of Ω such that the union of their closures coincides with the whole $\partial\Omega$. We are interested in the case $f(u) = u^p$, with $p \in (1, \frac{n+2}{n-2})$. These stationary mixed boundary problems appear in different situations and generally

* Corresponding author.
E-mail addresses: jesus.azorero@uam.es (J. Garcia Azorero), malchiod@sissa.it (A. Malchiodi), montoro@mat.unical.it (L. Montoro), ireneo.peral@uam.es (I. Peral).

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2009.06.005

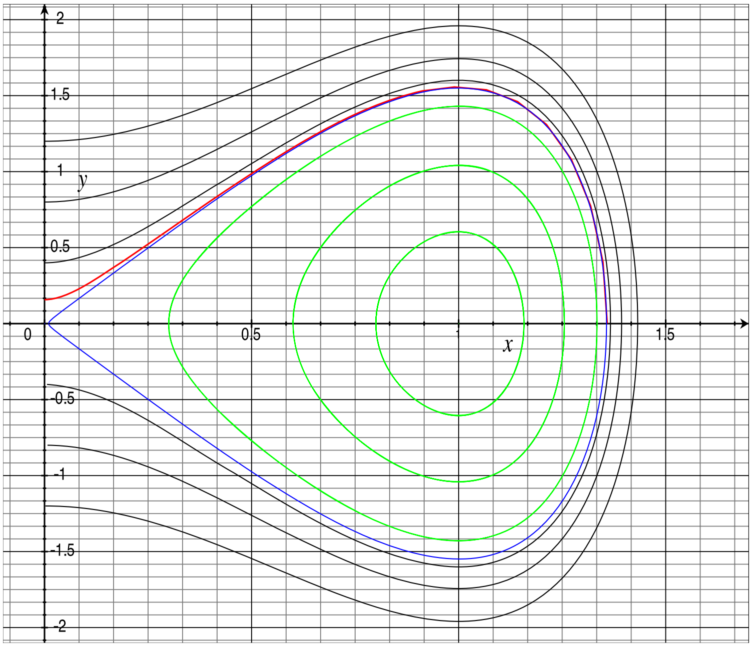

Fig. 1. Phase plane diagram for the equation $-u'' + u = u^p$.

the Dirichlet condition is equivalent to impose some *state* on the physical parameter represented by u while instead the Neumann conditions give a meaning at the flux parameter crossing $\partial_{\mathcal{N}}\Omega$, see the introduction of [9] for more specific comments. Singularly perturbed equations with Neumann or Dirichlet boundary conditions have been studied in detail. There is a wide literature regarding the solutions of this type of problems (see for example [13–16]). Many times they have a sharply peaked profile, so they are called *spike layers*, since they are highly concentrated near some points of $\overline{\Omega}$.

W.M. Ni and I. Takagi, in [15], studied the homogeneous Neumann boundary problem

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega; \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial_{\mathcal{N}}\Omega; \\ u > 0 & \text{in } \Omega. \end{cases} \tag{N_ε}$$

They showed that *least energy solutions* of (N_ε), that is mountain pass solutions, have only one local maximum over $\overline{\Omega}$ achieved at a point that must lie on the boundary, when the perturbation parameter ε tends to 0. The same authors in a subsequent paper [16] clarified the location of the point where the maximum of *least energy solutions* is attained, proving that it occurs near maxima of the mean curvature of $\partial\Omega$.

W.M. Ni and J. Wei in [14] studied the corresponding Dirichlet problem

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega; \\ u = 0 & \text{on } \partial_{\mathcal{D}}\Omega; \\ u > 0 & \text{in } \Omega. \end{cases} \tag{D_ε}$$

They proved that least energy solutions of (D_ε) have only one local maximum over $\overline{\Omega}$ achieved at the *most centered* point of Ω when the perturbation parameter goes to zero. Later P.L. Felmer and M. Del Pino [8] generalized the works of Ni–Takagi and Ni–Wei by enlarging considerably the class of treatable nonlinearities in both problems (N_ε) and (D_ε).

We would like here to establish the corresponding result when (generic) mixed boundary conditions are imposed. First of all we notice that also problem $(\tilde{M}_\varepsilon)$ has variational and mountain-pass structure, see Section 2 for more details, and hence we have still least energy solutions. In one dimension, as one can easily see using a phase-portrait analysis (see Fig. 1) there is only one positive solution, up to reflection, which vanishes on one end of an interval and has zero normal derivative on the other one. In higher dimension, when each connected component of $\partial\Omega$ is either contained in $\partial_{\mathcal{N}}\Omega$ or in $\partial_{\mathcal{D}}\Omega$, E.N. Dancer and S. Yan (see [7]) showed that as ε tends to zero there can be indeed a large number of solutions.

We are interested here in cases in which some component of $\partial\Omega$ contains both subsets (or all) of $\partial_{\mathcal{N}}\Omega$ or of $\partial_{\mathcal{D}}\Omega$, namely when there exists an *interface* $\mathcal{I}_\Omega := \overline{\partial_{\mathcal{D}}\Omega} \cap \overline{\partial_{\mathcal{N}}\Omega} \neq \emptyset$ which separates the Dirichlet and the Neumann parts. Our goal is to study the following issues.

Question 1. Determine the geometric conditions that a point P has to satisfy, to have some sequence of solutions $\{u_\varepsilon\}$ whose maximum points converge to P.

Question 2. Given a precise sequence $\{u_\varepsilon\}$, the least energy solutions, determine (up to subsequences) the limit of the maximum points.

Question 1 is analyzed in [9], and Question 2 is the subject of the present paper. In [9] we proved that when the gradient of the mean curvature of $\partial\Omega$ at $\mathcal{I}_\Omega$ points toward the Dirichlet part, then there are solutions of new type consisting of spike-layers which approach $\mathcal{I}_\Omega$ when ε tends to zero. More precisely we proved the following result, where H stands for the mean curvature of $\partial\Omega$.

Theorem 1.1. *(See [9].) Suppose $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a smooth bounded domain, and that $1 < p < \frac{n+2}{n-2}$ ($1 < p < +\infty$ if $n = 2$). Suppose $\partial_{\mathcal{D}}\Omega$, $\partial_{\mathcal{N}}\Omega$ are disjoint open sets of $\partial\Omega$ such that the union of the closures is the whole boundary of Ω and such that their intersection $\mathcal{I}_\Omega$ is an embedded hypersurface. Suppose $\overline{Q} \in \mathcal{I}_\Omega$ is such that $H|_{\mathcal{I}_\Omega}$ is critical and non-degenerate at $\overline{Q}$, and that $\nabla H \neq 0$ points toward $\partial_{\mathcal{D}}\Omega$. Then for $\varepsilon > 0$ sufficiently small problem $(\tilde{M}_\varepsilon)$ admits a solution u_ε concentrating at $\overline{Q}$, with a unique maximum point in $\partial_{\mathcal{N}}\Omega$, at distance of order $\varepsilon|\log\varepsilon|$ from $\mathcal{I}_\Omega$.*

Remark 1.2. The non-degeneracy condition on $\overline{Q}$ can be relaxed requiring that either $\overline{Q}$ is a strict local maximum or minimum for $H|_{\mathcal{I}_\Omega}$ or that the local degree of $\nabla H|_{\mathcal{I}_\Omega}$ is non-zero for any connected (small) neighborhood of $\overline{Q}$ in $\mathcal{I}_\Omega$.

In the present paper we show that, under generic assumptions on Ω and $\mathcal{I}_\Omega$, in many cases the least energy solutions of $(\tilde{M}_\varepsilon)$ are of the type found in Theorem 1.1. Our main goal is to complement the result in [9] with the following one.

Theorem 1.3. *Suppose $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is a smooth bounded domain, and that $1 < p < \frac{n+2}{n-2}$ ($1 < p < +\infty$ if $n = 2$). Suppose $\partial_{\mathcal{D}}\Omega$, $\partial_{\mathcal{N}}\Omega$ are disjoint open sets of $\partial\Omega$ such that the union of the closures is the whole boundary of Ω and such that their intersection $\mathcal{I}_\Omega$ is an embedded hypersurface. Then, as $\varepsilon \to 0$, the least energy solution of $(\tilde{M}_\varepsilon)$ have a unique maximum point which converges to $\overline{Q} \in \overline{\partial_{\mathcal{N}}\Omega}$ such that $H(\overline{Q}) = \max_{\overline{\partial_{\mathcal{N}}\Omega}} H$.*

Under some more (mild) assumptions, we can better characterize the case in which the maximum of H, restricted to $\overline{\partial_{\mathcal{N}}\Omega}$, is attained on the interface or at both the interface and in the interior of $\partial_{\mathcal{N}}\Omega$.

Theorem 1.4. *Suppose we are in the situation of Theorem 1.3. Assume also that $\max_{\overline{\partial_{\mathcal{N}}\Omega}} H$ is attained only at $\overline{Q} \in \mathcal{I}_\Omega$, where $\overline{Q}$ is an isolated maximum point of $H|_{\mathcal{I}_\Omega}$ for which $\nabla H(\overline{Q}) \neq 0$ points toward $\partial_{\mathcal{D}}\Omega$. Then the least energy solution of $(\tilde{M}_\varepsilon)$ behaves as in Theorem 1.1. If instead $\max_{\overline{\partial_{\mathcal{N}}\Omega}} H$ is attained both at $\mathcal{I}_\Omega$ and at an isolated interior point $\check{Q}$ of $\partial_{\mathcal{N}}\Omega$, and if $\overline{Q}$ is as in the previous case, the maxima of least energy solutions to $(\tilde{M}_\varepsilon)$ converge to $\check{Q}$.*

For proving Theorems 1.3 and 1.4 we have to combine mainly the asymptotic analysis in [15], new ingredients and some of the arguments in [9].

The first step consists in showing that the Dirichlet energy of mountain pass solutions cannot concentrate at more than one point and that, scaling the variables by a factor $\frac{1}{\varepsilon}$ around their global maximum, they converge locally to a solution U of the *limit equation* given below in (6). The limit profile U is defined in $\mathbb{R}^n$ or in the half space, depending on how fast the maxima approach the boundary of Ω compared to ε. Since in the first case these entire solutions would carry an *energy* double compared to the second case, it is not difficult to see that the limit profile of ground states must be defined in the half space only. As a consequence the maxima, which we denote by P_ε, have to approach $\partial\Omega$ as $\varepsilon \to 0$, and at a rate faster or comparable to ε.

We have then to distinguish three main cases, depending on whether the P_ε's converge to points in $\partial_{\mathcal{D}}\Omega$, $\partial_{\mathcal{N}}\Omega$ or $\mathcal{I}_\Omega$. The first case is excluded since the limit profile of a rescaling of the solutions, U, would vanish on the boundary of the half space, which is prevented by a result by Berestycki, Caffarelli and Nirenberg in [4]. The other two cases instead can both occur, depending on the geometry of Ω and of the interface.

When the maxima of the ground state approach points in the interior of $\partial_{\mathcal{N}}\Omega$, by the exponential decay of the rescaled solutions the situation is rather similar to the case of pure Neumann boundary conditions, and the analysis in [15] shows that concentration has to occur when the mean curvature is maximal. The new situation we have to analyze is when the maxima converge to the interface $\mathcal{I}_\Omega$.

We prove in this case that the maximum is still attained in $\partial_{\mathcal{N}}\Omega$, and that the rate of convergence to the interface is slower than ε. If this were not the case, we would obtain in the limit (after rescaling) a solution to (6) in the half space $\{x_n > 0\}$ which satisfies Dirichlet data on $\{x_n = 0\} \cap \{x_1 \leqslant 0\}$ and Neumann data on $\{x_n = 0\} \cap \{x_1 \geqslant 0\}$. Adapting an argument by Damascelli and Gladiali in [6], we show using moving plane techniques that the latter problem only admits trivial solutions, and we obtain the desired conclusion.

At this stage we can use arguments in [1,9] to prove that this asymptotic regime allows to study the problem via a finite-dimensional one (through a Lyapunov–Schmidt reduction), see Section 4 for more details. If concentration occurs at the interior of $\partial_{\mathcal{N}}\Omega$ we can apply some of the analysis in [15] (and [1]) to show that this happens at maxima of $H|_{\partial_{\mathcal{N}}\Omega}$. Concerning instead concentration at $\mathcal{I}_\Omega$, it was shown in [9] that the energy of an approximate solution to $(\tilde{M}_\varepsilon)$ centered at a point $Q \in \partial_{\mathcal{N}}\Omega$ has an expansion whose main order terms are given by

$$\varepsilon^n\bigl(\tilde{C}_0 - \varepsilon\tilde{C}_1 H(Q) + e^{-2\frac{d}{\varepsilon}}\bigr),$$

where $\tilde{C}_0$, $\tilde{C}_1$ are dimensional constants, and where d stands for the distance between Q and $\mathcal{I}_\Omega$. It is convenient to denote by $\check{Q}$ the closest point to Q on $\mathcal{I}_\Omega$, so that $\check{Q}$ and d determine Q univocally (for d small). With this notation, the previous formula becomes

$$\varepsilon^n\bigl(\tilde{C}_0 - \varepsilon\tilde{C}_1\bigl(H(\check{Q}) + d\nabla_d H(\check{Q}) + O\bigl(d^2\bigr)\bigr) + e^{-2\frac{d}{\varepsilon}}\bigr),$$

where $\nabla_d H(\check{Q})$ stands for the component of ∇H at $\check{Q}$ normal to $\mathcal{I}_\Omega$. From the latter expansion it is easy to see that the above quantity is critical, naively, if and only if $H|_{\mathcal{I}_\Omega}$ is also critical and if $d \simeq \varepsilon|\log\varepsilon|$. These arguments allow us then to prove Theorem 1.4.

The plan of the paper is the following. In Section 2 we collect some preliminary material for the asymptotic analysis of mountain pass solutions: a concentration compactness argument which rules out *spreading* of the Dirichlet energy and the derivation of the limit profile. In Section 3 we prove that the global maxima must lie on the Neumann boundary part and that they cannot approach the interface at a rate faster or of order ε. Finally in Section 4 we localize the position of global maxima depending on Ω and on the geometry of the interface, proving Theorems 1.3 and 1.4.

Notation. Generic fixed constants will be denoted by C, and will be allowed to vary within a single line or formula. The symbols $O(t)$ (respectively $o(t)$) will denote quantities for which $\frac{O(t)}{|t|}$ stays bounded (respectively $\frac{o(t)}{|t|}$ tends to zero) as the argument t goes to zero or to infinity. We will sometimes use the notation $d(1 + o(1))$, where $o(1)$ stands for a quantity which tends to zero as $d \to +\infty$.

2. Preliminaries

Problem $(\tilde{M}_\varepsilon)$ has variational structure and the associated Euler functional $\tilde{I}_\varepsilon : X \to \mathbb{R}$ is defined by

$$\tilde{I}_\varepsilon(u) = \frac{1}{2}\int_\Omega \bigl(\varepsilon^2|\nabla u|^2 + u^2\bigr)\,dx - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx; \quad u \in X, \tag{1}$$

where

$$X = \bigl\{u \in H^1_{\mathcal{D}}(\Omega):\ u \not\equiv 0 \text{ and } u \geqslant 0 \text{ in } \Omega\bigr\} \tag{2}$$

and where $H^1_{\mathcal{D}}(\Omega)$ stands for the space of functions in $H^1(\Omega)$ which have zero trace on $\partial_{\mathcal{D}}\Omega$:

$$H^1_{\mathcal{D}}(\Omega) = \{u \in H^1(\Omega):\ u|_{\partial_{\mathcal{D}}\Omega} = 0\}. \tag{3}$$

Definition 2.1. $c_\varepsilon \in \mathbb{R}$ is called a critical level of $\tilde{I}_\varepsilon$ if there exists a critical point u_ε such that $\tilde{I}_\varepsilon = c_\varepsilon$.

In general, critical levels can be found by min–max procedures. The least (nontrivial) energy level comes from the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [3].

Theorem 2.2 *(Mountain Pass). Let $\mathcal{B}$ a Banach space and $I \in C^1(\mathcal{B}, \mathbb{R})$. Suppose there exist $u_0, u_1 \in \mathcal{B}$ and $\alpha, r > 0$ such that*

(MP1) $\inf_{\|u-u_0\|=r} I(u) \geqslant \alpha > I(u_0)$;
(MP2) $\|u_1\| > r$ *and* $I(u_1) \leqslant I(u_0)$.

Then there exists a sequence $\{u_n\} \subset \mathcal{B}$ such that

(i) $I(u_n) \to c$, *where*

$$c := \inf_{\gamma \in \tilde{\Gamma}} \max_{t \in [0,1]} I(\gamma(t)),$$
$$\tilde{\Gamma} = \{\gamma \in C([0,1], \mathcal{B}):\ \gamma(0) = u_0,\ \gamma(1) = u_1\}.$$

(ii) $I'(u_n) \to 0$ *strongly in* $\mathcal{B}^*$.

Remark 2.1. If the functional I satisfies the so-called *Palais–Smale condition* (see [2] for example), then the number c, defined above, is a critical level of I. In our case, since the exponent p is subcritical, the *Palais–Smale condition* is clearly fulfilled by the functional $\tilde{I}_\varepsilon$ in (1).

Since $\tilde{I}_\varepsilon$ is doubly homogeneous, mountain pass solutions can be characterized in the following useful way.

Proposition 2.3. *Let $\tilde{I}_\varepsilon$ and X defined in* (1) *and* (2). *Then the mountain pass critical level c_ε can be characterized as*

$$c_\varepsilon = \min_{v \in X} \max_{t>0} \tilde{I}_\varepsilon(tv). \tag{4}$$

It is interesting to point out an important and simple fact that comes from the characterization of the critical value c_ε.

Corollary 2.4. *If we denote with $c_\varepsilon^{\mathcal{N}}$ the critical value of the functional $\tilde{I}_{\varepsilon,\mathcal{N}}$ associated to the semilinear Neumann problem (N_ε), $c_\varepsilon^{\mathcal{D}}$ the critical value of the functional $\tilde{I}_{\varepsilon,\mathcal{D}}$ associated to the semilinear Dirichlet problem (D_ε) and with $c_\varepsilon^{\mathcal{M}}$ the critical value of the functional $\tilde{I}_\varepsilon$ associated to the mixed semilinear problem $(\tilde{M}_\varepsilon)$, then one has that*

$$c_\varepsilon^{\mathcal{N}} \leqslant c_\varepsilon^{\mathcal{M}} \leqslant c_\varepsilon^{\mathcal{D}}. \tag{5}$$

Proof. It is sufficient to observe the definition (4) and that

$$H^1(\Omega) \supset H^1_{\mathcal{D}}(\Omega) \supset H^1_0(\Omega),$$

from which we immediately deduce the conclusion. □

To study concentration of the solutions u_ε to problems (N_ε) or (D_ε), one usually employs a blow-up argument consisting in a suitable scaling of the variables, which allows to prove that $u_\varepsilon(x) \sim U(\frac{x-Q}{\varepsilon})$, where U denotes a solution of the following problem:

$$-\Delta U + U = U^p \quad \text{in } \mathbb{R}^n \text{ (or in a half space)}, \tag{6}$$

the domain depending on whether Q lies in the interior of Ω or at the boundary; in the latter case Neumann conditions are imposed. When $p < \frac{n+2}{n-2}$ (and indeed only if this inequality is satisfied), problem (6) admits positive radial solutions which decay exponentially fast to zero at infinity (see [1]) and which satisfy

$$\lim_{r\to+\infty} e^r r^{\frac{n-1}{2}} U(r) = \alpha_{n,p}, \tag{7}$$

for some positive constant $\alpha_{n,p}$ depending only on n and p, with

$$\lim_{r\to+\infty} \frac{U'(r)}{U(r)} = -1; \qquad \lim_{r\to+\infty} \frac{U''(r)}{U(r)} = 1. \tag{8}$$

We denote the energy associated to the problem (6) as

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla u|^2 + u^2\right) dx - \frac{1}{p+1}\int_{\mathbb{R}^n} u^{p+1}\, dx, \tag{9}$$

where, clearly, the integral has to be taken in $\mathbb{R}^n_+$ in the second case.

2.1. Concentration-compactness arguments and limit profile

Here we collect some preliminary material that will be useful later in the asymptotic analysis of ground state solutions. Points in $\partial\Omega \subset \mathbb{R}^n$, will be denoted as couples (x', x_n) with $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. To simplify the notation, without loss of generality, we can assume that some fixed $P_0 \in \partial\Omega$ coincides with the origin of $\mathbb{R}^n$. Since Ω is regular, we describe $\partial\Omega$ near P_0 as a graph: there exists a constant $\delta > 0$ and a smooth function ψ_{P_0} defined for $|x'| < \delta$ such that

- $\psi_{P_0}(0) = 0$, $\frac{\partial\psi_{P_0}}{\partial x_j}(0) = 0$, $1 \leqslant j \leqslant n-1$;
- given a neighborhood of P_0, denoted by $\mathcal{U}$, then

$$\Omega \cap \mathcal{U} = \left\{(x', x_n) \,\middle|\, x_n > \psi_{P_0}(x')\right\} \quad \text{and} \quad \partial\Omega \cap \mathcal{U} = \left\{(x', x_n) \,\middle|\, x_n = \psi_{P_0}(x')\right\}.$$

The first condition implies that $\{x_n = 0\}$ is the tangent plane of $\partial\Omega$ at P_0.

We frequently need to change coordinates near a point of $\partial\Omega$ to flatten the boundary, in fact, there exists a constant $\delta' > 0$ such that if $|y| < \delta'$ then we can define $x = \Phi(y) = (\Phi_1(y), \ldots, \Phi_n(y))$ in the following way:

$$\Phi_j(y) = y_j - y_n \frac{\partial\psi_{P_0}}{\partial x_j}(y'), \quad 1 \leqslant j \leqslant n-1;$$
$$\Phi_n(y) = y_n + \psi_{P_0}(y').$$

In particular, $D\Phi(0) = Id$ and Φ has an inverse that we denote by $\Psi \equiv \Phi^{-1}$, defined in a neighborhood of the origin. Then if we change variables, and set

$$v_\varepsilon(y) = u_\varepsilon(x) = u_\varepsilon\big(\Phi(y)\big),$$

v_ε satisfies

$$\varepsilon^2\left(\sum_{i,j}^n a_{ij}(y)\frac{\partial^2 v_\varepsilon}{\partial y_i y_j} + \sum_j^n b_j(y)\frac{\partial v_\varepsilon}{\partial y_j}\right) - v_\varepsilon(y) + v_\varepsilon^p(y) = 0 \quad \text{in } \{y_n > 0\} \cap B_\rho, \tag{10}$$

where B_ρ stands for the open ball centered at zero with radius ρ (small enough). The boundary conditions depend on where P_0 lies in the boundary:

$$\begin{cases} v_\varepsilon = 0 \quad \text{on } \{y_n = 0\} \cap B_\rho, \text{ if } P_0 \in \partial_{\mathcal{D}}\Omega, \\ \dfrac{\partial v_\varepsilon}{\partial y_n} = 0 \quad \text{on } \{y_n = 0\} \cap B_\rho, \text{ if } P_0 \in \partial_{\mathcal{N}}\Omega, \\ v_\varepsilon = 0 \quad \text{on } \{y_1 > 0,\ y_n = 0\} \cap B_\rho \quad \text{and} \quad \dfrac{\partial v_\varepsilon}{\partial y_n} = 0 \quad \text{on } \{y_1 < 0,\ y_n = 0\} \cap B_\rho, \text{ if } P_0 \in \mathcal{I}_\Omega. \end{cases} \tag{11}$$

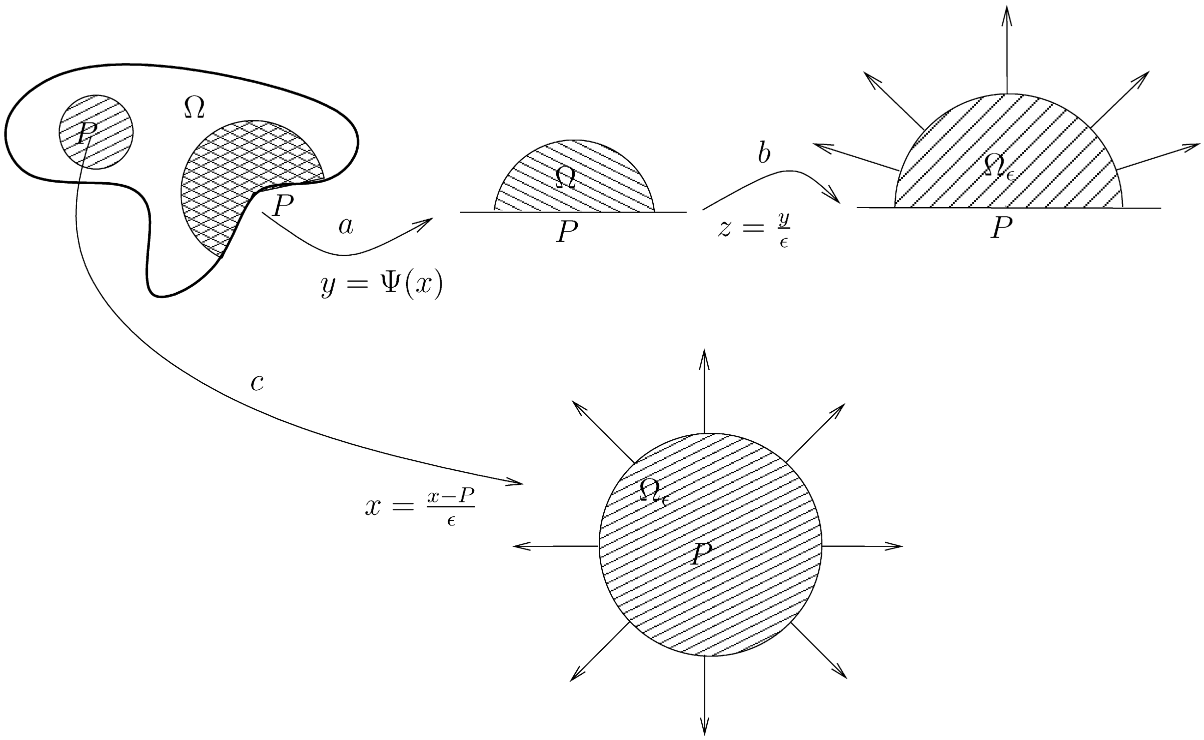

Fig. 2. Flattening and scaling.

The coefficients in Eq. (10) are given by

$$\begin{cases} a_{ij}(y)=\displaystyle\sum_k \frac{\partial \Psi_i}{\partial x_k}\big(\Phi(y)\big)\cdot\frac{\partial \Psi_j}{\partial x_k}\big(\Phi(y)\big), & 1\leqslant i,j\leqslant n;\\ b_j(y)=(\Delta\Psi_j)\big(\Phi(y)\big), \quad 1\leqslant j\leqslant n. \end{cases} \tag{12}$$

In particular, since $D\Psi(0)=Id$, then

$$a_{ij}(0)=\delta_{ij}. \tag{13}$$

Remark 2.5. To study the asymptotic profile of (mountain pass) solutions it will be useful to scale the variables in ε around some point P_ε, converging to some P_0 on the boundary of domain or in the interior. Depending on this possibility, we will both flatten and scale the domain or scale only, see (a–b) and (c) in Fig. 2.

From now on, we assume that $\{u_\varepsilon\}$ is a sequence of least energy solutions of $(\tilde{M}_\varepsilon)$, and suppose that P_ε is a global maximum point of u_ε in $\overline{\Omega}$. Up to a subsequence, we have that $P_\varepsilon \to P_0 \in \overline{\Omega}$.

We start noticing a useful property of the solution u_ε to the problem $(\tilde{M}_\varepsilon)$ in a local maximum point. Denoting $u_\varepsilon(P_\varepsilon)=\max_{x\in\overline{\Omega}} u_\varepsilon(P)$, one has that

$$u_\varepsilon(P_\varepsilon)\ \geqslant\ 1. \tag{14}$$

This is evident, using the equation, when P_ε lies in the interior of Ω. On the other hand if we suppose that $P_\varepsilon\in\partial_{\mathcal{N}}\Omega$ and by contradiction that $u_\varepsilon(P_\varepsilon)<1$, then we get

$$-\varepsilon^2\Delta u=u^p-u\leqslant 0$$

in a neighborhood of P_ε. From the boundary Hopf lemma follows that $\frac{\partial u_\varepsilon}{\partial\nu}>0$ for some point in $\partial_{\mathcal{N}}\Omega$ that gives us the desired contradiction. We do not have to consider the case $P_\varepsilon\in\overline{\partial_{\mathcal{D}}\Omega}$ since we are looking for nontrivial positive solutions to the problem $(\tilde{M}_\varepsilon)$.

The first result asserts that P_0 has to be a point in the boundary.

Proposition 2.6. *Let $\{u_\varepsilon\}$ be the family of least energy solutions to problem $(\tilde{M}_\varepsilon)$ and $\{P_\varepsilon\}$ a sequence of local maximum points. Then, up to a subsequence, $P_\varepsilon \to P_0 \in \partial\Omega$ and more precisely*

$$dist(P_\varepsilon, \partial\Omega) \leqslant C\varepsilon \tag{15}$$

for some fixed positive constant C.

Proof. The proof can be derived as in [15]: to make the paper self-contained, and for later purposes, we give a sketch of the arguments. In a first step one can get an upper bound for the critical level c_ε while in the second, using a contradiction argument, one shows that if (15) fails then c_ε should be nearly double than this upper bound.

Let c_ε be as in (4) and define

$$M[u] = \sup_{t>0} \tilde{I}_\varepsilon(tu), \quad u \in X.$$

First one can prove that, if $\partial_{\mathcal{N}}\Omega \neq \emptyset$, then

$$c_\varepsilon \leqslant \varepsilon^n \left\{ \frac{I(U)}{2} + o(\varepsilon) \right\}, \tag{16}$$

where U and I are given in (6) and (9). In fact, choose $P \in \partial_{\mathcal{N}}\Omega$ and consider a smooth non-increasing and non-negative radial cutoff functions $\chi_R : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{cases} \chi_R(y) = 1 & \text{in } B_R; \\ \chi_R(y) = 0 & \text{in } \mathbb{R}^n \setminus B_{2R}; \\ |\nabla \chi_R(y)| < C & \text{in } B_{2R} \setminus B_R. \end{cases} \tag{17}$$

Define then

$$\varphi_\varepsilon(x) := \chi_R(x-P)U(x-P):$$

from the definition of the mountain pass level, see Proposition 2.3, one has that

$$c_\varepsilon \leqslant M[\varphi_\varepsilon].$$

It was shown in [15, Proposition 2.3] that indeed

$$M[\varphi_\varepsilon] = \varepsilon^n \left\{ \tilde{C}_0 - \varepsilon \tilde{C}_1 H(P) + o(\varepsilon) \right\} \quad \text{as } \varepsilon \to 0, \tag{18}$$

where

$$\tilde{C}_0 = \frac{1}{2} I(U); \qquad \tilde{C}_1 = \frac{1}{n+1} \int\limits_{\mathbb{R}^n_+} U'(|x|)^2 x_n \, dx,$$

see also Proposition 2.7 in [9]. For this step to apply in our case, since we have mixed boundary conditions, it is sufficient to choose R so small that $B_{2R}(P) \cap \partial_{\mathcal{D}}\Omega = \emptyset$. The latter formulas clearly yield (16).

The last claim of the proposition can be proved following Step 1 in the proof of Theorem 2.1 of [15] (see page 830 in that reference). To show this, one can argue by contradiction assuming that $\frac{dist(P_{\varepsilon_j}, \partial\Omega)}{\varepsilon_j} \to +\infty$ along a sequence ε_j. As starting point, we use the following estimate:

$$\int\limits_\Omega u_\varepsilon^r \, dx \leqslant C_r \varepsilon^n, \tag{19}$$

where r is any positive number and where C_r is a constant independent from ε. This estimate has been proved in [13] (Lemma 2.3) for the Neumann problem (N_ε), but in fact it is also valid under mixed boundary conditions since, for the functional space X defined in (2), we have also classical results about Sobolev embedding theorems.

We define then the scaled functions w_{ε_j} as

$$w_{\varepsilon_j}(x) = u_{\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j x). \tag{20}$$

From a change of variables one immediately finds that

$$\int\limits_{\frac{1}{\varepsilon_j}(\Omega - P_{\varepsilon_j})} |w_{\varepsilon_j}|^r \, dx \leqslant C_r,$$

independently of j. Since then the functions w_{ε_j} satisfy the equation

$$-\Delta w_{\varepsilon_j} + w_{\varepsilon_j} = w_{\varepsilon_j}^p \quad \text{in } \frac{1}{\varepsilon_j}(\Omega - P_{\varepsilon_j}),$$

we obtain uniform Hölder estimates on w_{ε_j} on a sequence of expanding domains

$$\|w_{\varepsilon_j}\|_{C^{2,\alpha}(\overline{B_{\rho_j}})} < C, \tag{21}$$

where $\alpha \in (0,1)$ and $C > 0$ are fixed constants, and where $\rho_j \to +\infty$ (slowly) as $j \to +\infty$. This fact and a diagonal argument, recalling also that w_{ε_j} has a local maximum at 0, implies that w_{ε_j} converge in $C^{2,\alpha}_{loc}(\mathbb{R}^n)$ to a (nontrivial) solution $\overline{U}$ of the limit problem (6) in $\mathbb{R}^n$ with bounded energy. Since also $\overline{U}$ has to possess a local maximum at the origin (and by (14)), the result in [10] implies that $\overline{U}$ is radial, and hence by the uniqueness result in [12] it has to coincide with the function U introduced above.

Using this and the fact that

$$\tilde{I}_\varepsilon(u_\varepsilon) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int\limits_\Omega \left(\varepsilon^2 |\nabla u_\varepsilon|^2 + u_\varepsilon^2\right) dx \tag{22}$$

on solutions of (N_ε) (which gives positivity of the energy), Ni and Takagi were able to prove that, under the contradiction hypothesis one has $c_\varepsilon^{\mathcal{N}} \geqslant \varepsilon^n\{I(U) + o(1)\}$ as $\varepsilon \to 0$. The same reasoning applies here and gives

$$c_\varepsilon = \varepsilon^n\left\{I(U) + o(1)\right\}, \tag{23}$$

which contradicts (18) and gives the desired conclusion. □

Remark 2.7. We can also obtain a Hölder estimate on the w_ε's in a different way, following [17] or [5]. We can write the mixed problem $(\tilde{M}_\varepsilon)$ as

$$-\Delta u_\varepsilon = \varepsilon^{-2} f_\varepsilon,$$

where $f_\varepsilon \equiv u_\varepsilon^p - u_\varepsilon$. We know uniform a priori bounds for $\|u_\varepsilon\|_{L^r}$, for any r, see Eq. (19); in particular it follows

$$\|f_\varepsilon\|_{L^r} \leqslant C_r \varepsilon^{n/r}.$$

We will look for a uniform C^α estimate in the rescaled problem, that is when we take the sequence $\{w_\varepsilon\}$, defined by Eq. (20). We follow the proof of the C^α regularity result by G. Stampacchia in [17], to get the dependence on ε of the final estimate. Roughly, the proof consists in a careful estimate of the measure of the sets

$$A(k,R) \equiv \left\{x \in B_R(x_0) \cap \Omega \;\middle|\; u(x) > k\right\}, \quad x_0 \in \Omega,\ K \in \mathbb{R},$$

which leads to a Caccioppoli type inequality. This inequality, combined with an iterative argument, allows us to control the oscillation of u, proving the Hölder estimate. However, in our case we need an explicit control of the dependence with respect to ε of the constants which appear in all the process, since we need to extend the Hölder estimates to the rescaled problem. Then we get the following result:

$$\left|u_\varepsilon(x) - u_\varepsilon(y)\right| \leqslant C_\varepsilon |x-y|^\alpha,$$

where

- $C_\varepsilon \approx \|f_\varepsilon/\varepsilon^2\|_{m/2}$;
- $\alpha = \min(\alpha_0, -2(\frac{n}{m} - 1))$,

where α_0 is a constant which depends only on Ω.

In particular, if we take $m > n$, but $m \approx n$, then we have that

$$\alpha = -2\left(\frac{n}{m} - 1\right).$$

Let us fix such an m from now on. Therefore, in our case we have

$$C_\varepsilon \approx C_n \varepsilon^{2(\frac{n}{m}-1)}.$$

We note that the exponent is negative, and therefore

$$\left|u_\varepsilon(x) - u_\varepsilon(y)\right| \leqslant C_n \varepsilon^{2(\frac{n}{m}-1)} |x-y|^{-2(\frac{n}{m}-1)}. \tag{24}$$

In particular, taking the rescaled functions v_ε we get

$$\left|w_\varepsilon(r) - w_\varepsilon(s)\right| = \left|u_\varepsilon(\varepsilon r + P_\varepsilon) - u_\varepsilon(\varepsilon s + P_\varepsilon)\right| \leqslant \cdots \leqslant C_n |r-s|^{-2(\frac{n}{m}-1)}, \tag{25}$$

that is a uniform Hölder estimate for the sequence $\{w_\varepsilon\}$. In particular, taking a subsequence we can assume that w_ε converges uniformly on compact sets to a continuous function w.

Let us make some further comments about convergence in the general case, when points P_ε can converge to the boundary. Since we are dealing with mixed boundary conditions, at the interface we can expect just C^α regularity; see for instance [5]. However outside the interface we have $C^{2,\alpha}_{loc}$ convergence, on compact sets which do not intersect the interface (see [11]).

The previous argument rules out one of the alternatives in Remark 2.5, namely the concentration point cannot be in the interior of Ω. We then analyze the following three possible alternatives: when the limit point lies in the Neumann part, in the Dirichlet part or on the interface $\mathcal{I}_\Omega$.

Case 1. Up to a subsequence $\frac{dist(P_\varepsilon, \partial_{\mathcal{D}}\Omega)}{\varepsilon} \to +\infty$.

We know that there exists a neighborhood $\mathcal{U}$ of this point and a regular map, see Section 2.1, $\Psi = \Phi^{-1}$ such that $\Psi(\mathcal{U} \cap \Omega) \subset \mathbb{R}^n_+$. We can define as before the functions ($\rho_j \to +\infty$ slowly)

$$v_\varepsilon(y) \equiv u_\varepsilon\big(\Phi(y)\big), \quad y \in \overline{B^+_{\rho_j}} := \{y \in \overline{B_{\rho_j}},\ y_n \geqslant 0\},$$

and then the scaling

$$w_\varepsilon(z) = v_\varepsilon(Q_\varepsilon + \varepsilon z), \quad Q_\varepsilon = \psi(P_\varepsilon) \tag{26}$$

which solves an elliptic problem in some (dilated) domain with zero normal derivative on a flat boundary. If P_ε is a local maximum of u_ε (ground state solution of $(\tilde{M}_\varepsilon)$), and if $Q_\varepsilon = \Psi(P_\varepsilon)$, after some computations, from Eq. (10) we obtain

$$\varepsilon^2\left(\sum_{i,j}^N \frac{1}{\varepsilon^2} a^\varepsilon_{ij}(z) \frac{\partial^2 w_\varepsilon}{\partial z_i z_j} + \varepsilon \sum_j^N \frac{1}{\varepsilon^2} b^\varepsilon_j(z) \frac{\partial w_\varepsilon}{\partial z_j}\right) - w_\varepsilon(z) + w_\varepsilon(z)^p = 0, \tag{27}$$

where the coefficients a^ε_{ij} and b^ε_j depend on the subsequences $\{\varepsilon\}_r$ and $\{Q_\varepsilon\}_r$ and on the coefficients a_{ij} and b_j defined before by Eq. (12). It turns out that these coefficients are Lipschitz continuous with constant uniformly bounded with respect to the index of subsequence. In the limit $\varepsilon \to 0$, by the last statement of Proposition 2.6 we get convergence of the limit domain to a half space of the form $\{z_n \geqslant c\}$, for some $c \leqslant 0$. Also, by arguments similar to those in the previous proof (see also Remark 2.7), since the limit point is in the Neumann boundary far away of the interface, we get a convergent subsequence (in $C^2_{loc}(\{z_n \geqslant c\})$),

$$w_{\varepsilon_r} \to w \in C^2\big(\{z_n \geqslant c\}\big) \cap W^{1,2}\big(\{z_n \geqslant c\}\big) \tag{28}$$

with w solving (6) in $\{z_n \geqslant c\}$. Since the convergence is in C^2 sense and since every w_{ε_r} has zero normal derivative at the boundary, we find that $\frac{\partial w}{\partial x_n} = 0$ on $\{z_n = c\}$. By Proposition 2.6, the function w has to possess a local maximum at zero and since $w_{\varepsilon_r}(0) \geqslant 1$ (by (14)), also $w(0) \geqslant 1$.

Case 2. Up to a subsequence, $\frac{dist(P_\varepsilon, \partial_{\mathcal{N}}\Omega)}{\varepsilon} \to +\infty$.

Straightening again a portion of boundary, as usual we set $v_{\varepsilon_r}(y) = u_\varepsilon(\Phi(y))$, with y defined in a half ball $B_s(\tilde{P}_\varepsilon)$ of $\mathbb{R}^n$, centered around some point $\tilde{P}_\varepsilon$. Denote

$$w_\varepsilon(z) = v_\varepsilon(Q_\varepsilon + \varepsilon z), \quad z \in B_{\frac{s}{\varepsilon}}, \tag{29}$$

with $Q_\varepsilon = \Phi^{-1}(P_\varepsilon)$. Then, by Proposition 2.6 there exist $c_\varepsilon \to c \leqslant 0$ and $\rho_\varepsilon \to +\infty$ such that w_ε satisfies

$$\begin{cases} \varepsilon^2\left(\displaystyle\sum_{i,j}^N \frac{1}{\varepsilon^2} a_{ij}^\varepsilon(z)\frac{\partial^2 w_\varepsilon}{\partial z_i z_j} + \varepsilon \sum_j^N \frac{1}{\varepsilon^2} b_j(z)^\varepsilon \frac{\partial w_\varepsilon}{\partial z_j}\right) - w_\varepsilon(z) + w_\varepsilon(z)^p = 0 & \text{in } B_{\rho_\varepsilon} \cap \{z_n > c_\varepsilon\}; \\ w_\varepsilon(z) = 0 & \text{in } B_{\rho_\varepsilon} \cap \{z_n = c_\varepsilon\}, \end{cases}$$

where the coefficients a_{ij}^ε and b_j^ε, depend on the subsequences $\{\varepsilon\}_r$ and $\{Q_\varepsilon\}_r$ and on the coefficients a_{ij} and b_j defined before by Eq. (12). From Eq. (19), following the previous arguments we get

$$w_\varepsilon \to w \in C^2(\mathbb{R}^n_+) \cap W^{1,2}(\mathbb{R}^n_+),$$

where w solves (6) in $\{z_n \geqslant c\}$.

Case 3. Up to a subsequence, $\frac{dist(P_\varepsilon, \mathcal{I}_\Omega)}{\varepsilon}$ stays bounded.

We can repeat the arguments of Case 2, and define w_ε as in (29). This time we get local convergence of the w_ε's to a function $w \in W^{1,2} \cap C^\alpha$ that moreover outside the interface is in the class $C^{2,\alpha}_{loc}$, which satisfies

$$\begin{cases} -\Delta w + w = w^p \quad \text{in } \{z_n \geqslant c\}; \\ w = 0 \quad \text{on } \{z_1 > \tilde{c}\} \cap \{z_n = c\}; \qquad \dfrac{\partial w}{\partial \nu} = 0 \quad \text{on } \{z_1 < \tilde{c}\} \cap \{z_n = c\}, \end{cases} \tag{30}$$

where $c \leqslant 0$ and $\tilde{c}$ is some fixed real number.

3. Analysis of the limit points P_ε

In this section we specify the limit behavior of the local maximum points P_ε when ε tends to zero. We will establish where this can be attained and what is the rate of convergence, in case the limit point lies on the interface, see Corollary 3.7. This is going to be very relevant for us, since it will allow to complement the present results with those in [9].

We first rule out the second case in the previous section.

Proposition 3.1. *Let u_ε be a least-energy solution to the problem $(\tilde{M}_\varepsilon)$ and $\{P_\varepsilon\}$ a sequence of local maximum points of u_ε. Then Case 2 at the end of Section 2 cannot occur.*

Proof. Assuming by contradiction that such a possibility may happen, we showed that the functions w_ε converge in C^2_{loc} to a solution of the problem

$$\begin{cases} -\Delta w + w = w^p & \text{in } \{z_n \geqslant c\}; \\ w = 0 & \text{on } \{z_n = c\}, \end{cases} \tag{31}$$

where $c \leqslant 0$ and where $w \geqslant 0$, $w \in W^{1,2}$ and w has a local maximum at 0, with $w(0) \geqslant 1$.

If $c = 0$ we immediately get the contradiction, since otherwise w would vanish in an open set. If $c < 0$, we can invoke the following theorem by H. Berestycki, L. Caffarelli, L. Nirenberg [4, Corollary 1.3].

Theorem 3.2. *(See [4].) Let u be a solution of*

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \mathbb{R}^n_+; \\ u > 0 & \text{in } \mathbb{R}^n_+, \end{cases} \tag{32}$$

satisfying

$$u(x, 0) = 0, \quad \forall x \in \mathbb{R}^{n-1}. \tag{33}$$

Assume that f is Lipschitz and that $f(0) \geqslant 0$. Then $\frac{\partial u(x)}{\partial x_n} > 0$ for every $x \in \mathbb{R}^n_+$.

Applying this result (with an obvious shift in the z_n variable) we obtain strict monotonicity of w, which contradicts the fact that w is globally of class $W^{1,2}$, and hence L^2 integrable. □

Next, we also rule out the third case among the ones discussed above.

Proposition 3.3. *Let u_ε be a least-energy solution to the problem $(\tilde{M}_\varepsilon)$ and $\{P_\varepsilon\}$ a sequence of local maximum points of u_ε. Then Case 3 at the end of Section 2 cannot occur.*

Proof. The proof is an immediate consequence of (30) and Proposition 3.4 below. □

Proposition 3.4. *Let us consider the problem*

$$\begin{cases} -\Delta u + u = u^p & \text{in } \mathbb{R}^n_+; \\ u \geqslant 0 & \text{in } \mathbb{R}^n_+; \\ u = 0 & \text{on } \Gamma_0; \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1, \end{cases} \tag{34}$$

where

$$\Gamma_0 = \{x = (x^1, \dots, x^n)\colon x^n = 0,\ x^1 > 0\}, \qquad \Gamma_1 = \{x = (x^1, \dots, x^n)\colon x^n = 0,\ x^1 < 0\}$$

and where $1 < p < \frac{n+2}{n-2}$. Define

$$\tilde{X} = \left\{\phi \in W^{1,2}(\mathbb{R}^n_+)\colon \phi = 0 \text{ on } \Gamma_0, \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \Gamma_1\right\}.$$

Then if $u \in \tilde{X}$ is a solution, $u \equiv 0$.

Proof. Here we want to obtain a Liouville type result for the problem (34). Previously Damascelli and Gladiali in [6] obtained a similar classification result for a mixed problem that unfortunately does not apply to our case. They considered the differential problem $-\Delta u = f(u)$, getting a nonexistence result when the nonlinear term $f(u)$ satisfies different conditions. Some of them, like

(i) $g(t) := \frac{f(t)}{t^{\frac{n+2}{n-2}}}$ is non-increasing in $(0, +\infty)$;
(ii) $f(t) > 0$ for every $t > 0$,

do not apply in our case, being $f(t) = t^p - t$.

However we can still use some of the ideas in [6], getting a nonexistence result. In the proof we utilize the moving plane method, therefore we introduce some notation. For $\lambda > 0$ we let

$$T_\lambda = \{x \in \mathbb{R}^n_+\colon x_1 = \lambda\}, \qquad \Sigma_\lambda = \{x \in \mathbb{R}^n_+\colon x_1 > \lambda\},$$

and also we define

$$R_\lambda(x) = x_\lambda = (2\lambda - x_1, x_2, \dots, x_n), \quad x \in \mathbb{R}^n_+, \qquad u_\lambda(x) = u(x_\lambda).$$

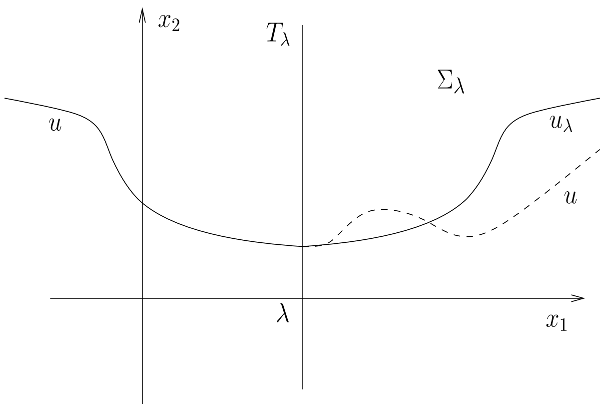

Fig. 3. Moving plane.

Note the x_λ is the reflection trough the plane T_λ and $u_\lambda(x)$ the reflected function (see Fig. 3).

We note that, for any $\lambda \in \mathbb{R}$, the function $(u - u_\lambda)^+$ vanishes in the set $\Sigma_\lambda \cap \Gamma_0$. In particular, if $\lambda \geqslant 0$ then $\Sigma_\lambda \cap \{x_n = 0\} \subset \Gamma_0$. If $\lambda < 0$, then $(\Sigma_\lambda \cap \{x_n = 0\}) \cap \Gamma_1 \neq \emptyset$, but in this part of the boundary the normal derivatives of u and u_λ vanish. Then if we consider the problem (34) in Σ_λ and the similar one satisfied by the reflected function u_λ (see [6]), we multiply both problems by the function $(u - u_\lambda)^+$, integrate by parts and subtract the equations we obtain

$$\int_{\Sigma_\lambda} \left|\nabla(u-u_\lambda)^+\right|^2 dx + \int_{\Sigma_\lambda} \left[(u-u_\lambda)^+\right]^2 dx = \int_{\Sigma_\lambda} \left[u^p - u_\lambda^p\right](u-u_\lambda)^+ dx$$
$$\leqslant p \int_{\Sigma_\lambda} \left(u^{p-1} + u_\lambda^{p-1}\right)\left[(u-u_\lambda)^+\right]^2 dx.$$

Then we have to consider separately two cases:

Case a. $\frac{4}{n} + 1 \leqslant p < \frac{n+2}{n-2}$.

Case b. $1 < p \leqslant \frac{4}{n} + 1$.

In Case a, via Hölder and Sobolev inequalities, we get

$$\int_{\Sigma_\lambda} \left|\nabla(u-u_\lambda)^+\right|^2 + \left[(u-u_\lambda)^+\right]^2 dx \leqslant p\left(\int_{\Sigma_\lambda \cap \{u \geqslant u_\lambda\}} \left(u_\lambda^{p-1} + u^{p-1}\right)^{\frac{n}{2}}\right)^{\frac{2}{n}} \left(\int_{\Sigma_\lambda} \left[(u-u_\lambda)^+\right]^{2^*}\right)^{\frac{2}{2^*}};$$

$$\int_{\Sigma_\lambda} \left|\nabla(u-u_\lambda)^+\right|^2 + \left[(u-u_\lambda)^+\right]^2 dx \leqslant pC\left(\int_{\Sigma_\lambda \cap \{u \geqslant u_\lambda\}} \left(u_\lambda^{p-1} + u^{p-1}\right)^{\frac{n}{2}}\right)^{\frac{2}{n}} \left(\int_{\Sigma_\lambda} \left|\nabla(u-u_\lambda)^+\right|^2\right).$$

Then

$$\int_{\Sigma_\lambda} \left|\nabla(u-u_\lambda)^+\right|^2 + \left[(u-u_\lambda)^+\right]^2 dx$$
$$\leqslant pC\left(\int_{\Sigma_\lambda \cap \{u \geqslant u_\lambda\}} \left(u_\lambda^{p-1} + u^{p-1}\right)^{\frac{n}{2}}\right)^{\frac{2}{n}} \left(\int_{\Sigma_\lambda} \left|\nabla(u-u_\lambda)^+\right|^2 + \left[(u-u_\lambda)^+\right]^2\right), \tag{35}$$

where C is the Sobolev constant.

On the other hand, in Case b, using again Hölder and Sobolev inequalities, we are led to consider the following

$$\int_{\Sigma_\lambda} \left|\nabla(u-u_\lambda)^+\right|^2 + \left[(u-u_\lambda)^+\right]^2 dx \leqslant p\left(\int_{\Sigma_\lambda\cap\{u\geqslant u_\lambda\}} \left(u_\lambda^{p-1}+u^{p-1}\right)^{\frac{2}{p-1}}\right)^{\frac{p-1}{2}} \left(\int_{\Sigma_\lambda} \left[(u-u_\lambda)^+\right]^{\frac{4}{3-p}}\right)^{\frac{3-p}{2}}.$$

Then

$$\int_{\Sigma_\lambda} \left|\nabla(u-u_\lambda)^+\right|^2 + \left[(u-u_\lambda)^+\right]^2 dx$$
$$\leqslant pC\left(\int_{\Sigma_\lambda\cap\{u\geqslant u_\lambda\}} \left(u_\lambda^{p-1}+u^{p-1}\right)^{\frac{2}{p-1}}\right)^{\frac{p-1}{2}} \left(\int_{\Sigma_\lambda} \left|\nabla(u-u_\lambda)^+\right|^2 + \left[(u-u_\lambda)^+\right]^2\right). \tag{36}$$

In both cases we get

$$\left\|(u-u_\lambda)^+\right\|_{W^{1,2}(\Sigma_\lambda)} \leqslant pC\left(\int_{\Sigma_\lambda\cap\{u\geqslant u_\lambda\}} \left(u_\lambda^{p-1}+u^{p-1}\right)^r\right)^{\frac{1}{r}} \left\|(u-u_\lambda)^+\right\|_{W^{1,2}(\Sigma_\lambda)}, \tag{37}$$

where $r>0$ is some number such that $(p-1)r\in[2,2^*]$.

We want to show that $u\leqslant u_\lambda$ in Σ_λ, $\forall\lambda\in\mathbb{R}$, i.e. the solution is decreasing in the x_1 direction, which implies that the solution cannot have finite energy. Now, it is easy to see that there is at least one $\widetilde{\lambda}$ such that $u\leqslant u_\lambda$ in Σ_λ, $\forall\lambda>\widetilde{\lambda}$. In fact if λ is big enough, then it readily follows that

$$\int_{\Sigma_\lambda\cap\{u\geqslant u_\lambda\}} \left(u_\lambda^{p-1}+u^{p-1}\right)^r < 1,$$

and then

$$\left\|(u-u_\lambda)^+\right\|_{W^{1,2}(\mathbb{R}^n_+)} \leqslant 0, \quad \forall\lambda>\widetilde{\lambda}.$$

Then we can infer that $\forall\lambda>\widetilde{\lambda}$, $u\leqslant u_\lambda$ in Σ_λ. From the Hopf lemma and the strong maximum principle, we can claim that $u<u_\lambda$ in Σ_λ, $\forall\lambda>\widetilde{\lambda}$. In fact if $u\equiv u_\lambda$ in Σ_λ, then u would be a nontrivial solution of problem (34) with $\frac{\partial u}{\partial\nu}<0$ on $R_\lambda(\Gamma_1)$. This is a contradiction since $\frac{\partial u_\lambda}{\partial\nu}=0$ on $R_\lambda(\Gamma_1)$. Then by maximum principle we have the conclusion.

Now, we will show that

$$\inf\{\widetilde{\lambda}:\ u\leqslant u_\lambda \text{ in } \Sigma_\lambda \text{ if } \lambda>\widetilde{\lambda}\}=-\infty.$$

We suppose by contradiction that such infimum $\hat{\lambda}$ is finite. We write a generic point $x\in\mathbb{R}^n_+$ in the form $x=(x_1,y)$ where $x_1\in\mathbb{R}$ and $y\in\mathbb{R}^{n-2}\times\mathbb{R}_+$. We define a change of variable $z=2\hat{\lambda}-x_1$ in such a way that

$$u_{\hat{\lambda}}(x_1,y)=u(2\hat{\lambda}-x_1,y):=u(z,y).$$

Then if we consider $\mu<\hat{\lambda}$ we have

$$u_\mu(x_1,y)=u(2\mu-x_1,y)=u\big(z-2(\hat{\lambda}-\mu),y\big)=u(z-\delta,y),$$

where $\delta:=2(\hat{\lambda}-\mu)$ is bigger than zero. It is easy to show the following claim.

Claim 3.5. *If* $2\leqslant g\leqslant 2^*$

$$\int_{\mathbb{R}^n_+\cap\{z>\hat{\lambda}\}} u^g(z-\delta,y)\,dz\,dy \to \int_{\mathbb{R}^n_+\cap\{z>\hat{\lambda}\}} u^g(z,y)\,dz\,dy \quad \textit{as } \delta\to 0.$$

Now we fix a compact set K_0, such that $K_0 \subset \mathbb{R}^n_+ \cap \Sigma_{\hat{\lambda}}$ and $\int_{\Sigma_{\hat{\lambda}} \setminus K_0} u^g_{\hat{\lambda}}(x)\, dx < \rho$. Over the compact K_0, we have

$$u_{\hat{\lambda}} - u > c_{K_0} > 0,$$

where c_{K_0} is some constant depending on K_0. The continuity of the reflection implies

$$u_\mu - u > \frac{c_{K_0}}{2} \tag{38}$$

in fact if $\mu < \hat{\lambda}$ we have

$$|u_{\hat{\lambda}} - u_\mu| < \varepsilon \quad \Leftrightarrow \quad |\hat{\lambda} - \mu| < \widetilde{\delta};$$

$$u_{\hat{\lambda}} - u = |u_{\hat{\lambda}} - u| = |u_\mu + u_{\hat{\lambda}} - u_\mu - u| \geqslant c_{K_0}; \tag{39}$$

$$|u_\mu - u| = u_\mu - u \geqslant c_{K_0} - \varepsilon > \frac{c_{K_0}}{2} \tag{40}$$

for some $\widetilde{\delta} > 0$. The inequality (40) is satisfied only if $u_\mu - u \geqslant c_{K_0} - \varepsilon$. In fact, by contradiction, if we suppose that $u_\mu - u < \varepsilon - c_{K_0}$ then

$$u_\mu - u_{\hat{\lambda}} + c_{K_0} < u_{\hat{\lambda}} - u + u_\mu - u_{\hat{\lambda}} + < \varepsilon - c_{K_0}. \tag{41}$$

From (41) and by the continuity of the reflection we get the contradiction:

$$u_{\hat{\lambda}} - u_\mu > 2c_{K_0} - \varepsilon > \varepsilon.$$

By continuity, we can choose $\widetilde{\delta}$ in (39) such that, on the compact K_0, $u_\mu - u > \frac{c_{K_0}}{2}$. Then Claim 3.5 (note that the compact K_0 does not depend on μ) implies

$$\int\limits_{\Sigma_\mu \setminus K_0} u^g_\mu \to \int\limits_{\Sigma_{\hat{\lambda}} \setminus K_0} u^g_\mu < \rho. \tag{42}$$

Then from Eqs. (42) and (38) we obtain

$$\int\limits_{\Sigma_\mu \cap \{u \geqslant u_\mu\}} u^g_\mu \leqslant \int\limits_{\Sigma_\mu \setminus K_0} u^g_\mu \leqslant 2\rho. \tag{43}$$

Finally if we choose $\rho \ll \frac{1}{2}$, via Eqs. (37) and (43) we get the contradiction concluding the proof of the lemma. □

Remark 3.6. We want to point out that if $\frac{4}{n} + 1 \leqslant p < \frac{n+2}{n-2}$ then $g := (p-1)\frac{n}{2}$ belongs to $[2, 2^*]$ and similarly if $1 < p \leqslant \frac{4}{n} + 1$ then $g := \frac{4}{3-p}$ belongs to $[2, 2^*]$.

As a corollary of Propositions 2.6, 3.1 and 3.3 we get immediately the following result.

Corollary 3.7. *Let u_ε be a least-energy solution to the problem $(\tilde{M}_\varepsilon)$ and $\{P_\varepsilon\}$ a sequence of local maximum points of u_ε. Then one has*

$$\frac{dist(P_\varepsilon, \partial\Omega)}{\varepsilon} \leqslant C \quad \text{and} \quad \frac{dist(P_\varepsilon, \partial_{\mathcal{D}}\Omega)}{\varepsilon} \to \infty \quad \text{as } \varepsilon \to 0,$$

where C is a fixed positive constant.

In [15] the authors proved that minimal energy solutions of the Neumann problem have a single local maximum, which is located precisely at the boundary. We see that the result extends rather easily to the mixed boundary conditions case.

Proposition 3.8. *Let u_ε be a least-energy solution to the problem $(\tilde{M}_\varepsilon)$ and $\{P_\varepsilon\}$ a sequence of local maximum points of u_ε. Then for ε small enough one has*

$$P_\varepsilon \in \partial_{\mathcal{N}}\Omega,$$

and moreover P_ε is unique.

Proof. This fact can be obtained using the previous analysis and some easy adaptations of the result in [15]. By Corollary 3.7 and by the arguments obtained above concerning Case 1, we know that the scaled functions w_ε converge to a solution of the problem

$$\begin{cases} -\Delta w + w = w^p & \text{in } \{z_n \geqslant c\}; \\ \dfrac{\partial w}{\partial z_n} = 0 & \text{on } \{z_n = c\}, \end{cases} \tag{44}$$

where $c \leqslant 0$, and where $w \in W^{1,2}(\{z_n \geqslant c\})$ has a local maximum at the origin.

First of all we claim that it must be $c = 0$: in fact, since w satisfies Neumann boundary conditions, by even reflection in z_n we can extend it to an entire $W^{1,2}$ solution of the first equation in (44). By the result in [10] then this extension must be radially symmetric around some point of $\mathbb{R}^n$ (and radially decreasing): the even symmetry in z_n then implies that this point must be on $\{z_n = 0\}$. Since this radial function has a unique critical point, it must be $c = 0$.

Next, by the uniqueness result of Kwong [12], w must coincide with U. Using the fact that w_ε satisfy (27) and that they converge to U in C^2_{loc}, Ni and Takagi were able to prove that the w_ε has a unique local maximum in any set of the form $B_R \cap \mathbb{R}^n_+$, provided ε is sufficiently small. This is evident when we remove a fixed small neighborhood of the origin: near 0 instead this is shown in Lemma 4.2 in [15]. Since we have vanishing of the normal derivative of w_ε in the Neumann part of the boundary (recall that we are in Case 1), there can be no interior critical points.

A similar argument allows to prove uniqueness of the maxima: assume by contradiction that there are at least two of them: $\widetilde{P}_\varepsilon$ and P_ε. Calling $p_\varepsilon = |\frac{\widetilde{P}_\varepsilon - P_\varepsilon}{\varepsilon}|$, up to a subsequence, we have three cases:

1. $p_\varepsilon \to 0$ as $\varepsilon \to 0$,
2. $p_\varepsilon \to c$ with $0 < c < +\infty$ as $\varepsilon \to 0$,
3. $p_\varepsilon \to +\infty$ as $\varepsilon \to 0$.

By the above mentioned uniqueness result and by the C^2 convergence of w_ε we can immediately rule out the first two possibilities. To exclude also the third, one can use energy estimates in the spirit of the (last part of the) proof of Proposition 2.6. Scaling u_ε near both P_ε and $\widetilde{P}_\varepsilon$, one finds two disjoint subsets of Ω where u_ε has the profile U (see (6)), centered at the boundary. In each of these regions the contribution to $\tilde{I}_\varepsilon$ is at least $\frac{1}{2}\varepsilon^n I(U) + o(\varepsilon^n)$, which implies that $\tilde{I}_\varepsilon(u_\varepsilon) \geqslant \varepsilon^n I(U) + o(\varepsilon^n)$, contradicting (16). This concludes the proof of the proposition. □

4. Proof of the main theorems

We first recall some of the notation and results in [9]. For this purpose, it is convenient to scale the variables in ε, so that problem $(\tilde{M}_\varepsilon)$ becomes

$$\begin{cases} -\Delta v + v = v^p & \text{in } \Omega_\varepsilon; \\ \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial_{\mathcal{N}}\Omega_\varepsilon; & v = 0 \quad \text{on } \partial_{\mathcal{D}}\Omega_\varepsilon; \\ v > 0 & \text{in } \Omega_\varepsilon. \end{cases} \tag{M_ε}$$

Here Ω_ε stands for $\frac{1}{\varepsilon}\Omega$, and $\partial_{\mathcal{N}}\Omega_\varepsilon$, $\partial_{\mathcal{D}}\Omega_\varepsilon$ for the dilations of $\partial_{\mathcal{N}}\Omega$ and $\partial_{\mathcal{D}}\Omega$ respectively.

The Euler functional corresponding to (M_ε) is the following

$$I_\varepsilon(v) = \frac{1}{2}\int_{\Omega_\varepsilon} \left(|\nabla v|^2 + v^2\right) dx - \frac{1}{p+1}\int_{\Omega_\varepsilon} |v|^{p+1}\, dx; \quad v \in H^1_{\mathcal{D}}(\Omega_\varepsilon), \tag{45}$$

where $H^1_{\mathcal{D}}(\Omega_\varepsilon)$ denotes the family of functions in $H^1(\Omega_\varepsilon)$ with zero trace on $\partial_{\mathcal{D}}\Omega_\varepsilon$.

In Sections 2 and 3 in [9], some approximate solutions to (M_ε) were introduced, depending on whether their maximum Q lies far or close to the scaled interface $\mathcal{I}_{\Omega_\varepsilon}$ (but always in the Neumann part $\partial_{\mathcal{N}}\Omega_\varepsilon$). To characterize precisely these two possibilities, we fix a small constant μ_0 which depends only on Ω and $\mathcal{I}_\Omega$ (which will be allows to assume smaller and smaller values), and distinguish two cases

$$\begin{cases} \text{Case a:} \quad dist(Q, \mathcal{I}_{\Omega_\varepsilon}) \geqslant \frac{\mu_0}{\varepsilon}; \ \text{that is, the maximum is far away from the interface.} \\ \text{Case b:} \quad dist(Q, \mathcal{I}_{\Omega_\varepsilon}) \leqslant \frac{\mu_0}{\varepsilon}; \ \text{that is, the maximum is close to the interface.} \end{cases} \tag{46}$$

In Case a, we fix a point $Q \in \partial\Omega_\varepsilon$ and use the corresponding coordinates y introduced in Section 2.1 (with a scaling in ε), assuming that $\mu_0 < \delta'$. Recall that, in particular, the change of coordinates implies that Q becomes the origin. Then we define a radial cutoff function χ_{μ_0} satisfying

$$\begin{cases} \chi_{\mu_0}(y) = 1 & \text{in } B_{\frac{\mu_0}{4}}; \\ \chi_{\mu_0}(y) = 0 & \text{in } \mathbb{R}^n \setminus B_{\frac{\mu_0}{2}}; \\ |\nabla \chi_{\mu_0}| + |\nabla^2 \chi_{\mu_0}| \leqslant C & \text{in } B_{\frac{\mu_0}{2}}(Q) \setminus B_{\frac{\mu_0}{4}}, \end{cases} \tag{47}$$

and we set $A_Q = Hess(\psi_Q)$. In [9] it was shown that the following problem admits a unique solution $\overline{w}_Q$, which decays exponentially to zero at infinity

$$\begin{cases} L_U w = -2\langle A_Q y', \nabla_{y'} \partial_{y_n} U\rangle - (tr\, A_Q)\, \partial_{y_n} U & \text{in } \mathbb{R}^n_+; \\ \dfrac{\partial}{\partial y_n} w = \langle A_Q y', \nabla_{y'} U\rangle & \text{on } \partial\mathbb{R}^n_+, \end{cases} \quad y = (y', y_n).$$

In the last formula L_U stands for the operator $L_U u = -\Delta u + u - pU^{p-1}u$. We then set

$$z_{\varepsilon,Q}(y) = \chi_{\mu_0}(\varepsilon y)\big(U(y) + \varepsilon \overline{w}_Q(y)\big). \tag{48}$$

Notice that, fixed ε, this is a $(n-1)$-dimensional set, parameterized by $Q \in \partial\Omega_\varepsilon$. Moreover, the functions $z_{\varepsilon,Q}(y)$ concentrate around the origin for ε small, which corresponds to concentration around Q in the original variables.

The function $\chi_{\mu_0}(\varepsilon y)U(y)$ solves (M_ε) up to order ε, in the sense that $\|I'_\varepsilon(\chi_{\mu_0}U)\| = O(\varepsilon)$: the term $\varepsilon\overline{w}_Q$ is a correction which improves the accuracy, and in fact one has that

$$\big\| I'_\varepsilon(z_{\varepsilon,Q}) \big\| \leqslant C\varepsilon^2 \quad \text{for some fixed } C > 0, \tag{49}$$

which follows from Proposition 2.5 in [9] (see also Section 9 in [1]). In the same proposition it was also shown that

$$I_\varepsilon(z_{\varepsilon,Q}) = \tilde{C}_0 - \tilde{C}_1 \varepsilon H(\varepsilon Q) + O\big(\varepsilon^2\big).$$

In Case b, one can define a similar system of coordinates which stretches not only $\partial\Omega_\varepsilon$, but also the scaled interface $\mathcal{I}_{\Omega_\varepsilon}$. We still call y these coordinates, and refer to the end of Section 2 in [9] for a detailed construction. As a result of if, there exist a small $\tilde{\delta} > 0$ and a number $0 < d < \frac{\mu_0}{\varepsilon}$ such that in the new coordinates one has

$$\partial\Omega_\varepsilon \cap B_{\tilde{\delta}(Q)} \subseteq \{y_n = 0\}; \qquad \mathcal{I}_{\Omega_\varepsilon} \cap B_{\tilde{\delta}(Q)} \subseteq \{y_n = 0\} \cap \{y_1 = d\}.$$

In [9, Section 3], we defined another $(n-1)$-dimensional family of approximate solutions $\hat{z}_{\varepsilon,Q}$ by the formula

$$\hat{z}_{\varepsilon,Q}(y) = \chi_{\mu_0}(\varepsilon y)\big[\big(U(y) - \varXi_d(y)\big)\chi_D(y) + \varepsilon\tilde{w}_Q(y)\chi_0(y_1 - d)\big], \tag{50}$$

where χ_D, χ_0 are other cutoff functions satisfying

$$\begin{cases} \chi_D(y) = 1 & \text{for } |y| \leqslant \dfrac{dD}{16}; \\ \chi_D(y) = 0 & \text{for } |y| \geqslant \dfrac{dD}{8}; \\ |\nabla\chi_D| \leqslant \dfrac{32}{dD} & \text{on } \mathbb{R}^n, \end{cases} \qquad \begin{cases} \chi_0(y) = 0 & \text{for } y \leqslant -1; \\ \chi_0(y) = 1 & \text{for } y \geqslant 0; \\ \chi_0 \text{ is non-decreasing} & \text{on } \mathbb{R}. \end{cases}$$

In (50), the function $\tilde{w}_Q$ corresponds to the above correction $\overline{w}_Q$, but it does not have the same expression since we are using a different system of coordinates. The constant D is required to be large but is kept fixed.

We also refer to Section 3 in [9] for the definition of Ξ_d, whose construction is one of the most delicate parts of the analysis in the paper. Its role is to find a correction to $z_{\varepsilon,Q}$ which takes into account the mixed boundary conditions of problem (M_ε). In Propositions 3.12, 4.1 and 4.2 of [9] it was shown that

$$\begin{aligned}
&\left\| I'_\varepsilon(\hat{z}_{\varepsilon,Q})\right\| \leqslant C\left(\varepsilon^2 + \varepsilon e^{-d(1+o(1))} + e^{-\frac{(p+1)d}{2}(1+o(1))} + e^{-\frac{3d}{2}(1+o(1))}\right);\\
&I_\varepsilon(\hat{z}_{\varepsilon,Q}) = \tilde{C}_0 - \tilde{C}_1\varepsilon H(\varepsilon Q) + e^{-2d(1+o(1))} + O\left(\varepsilon^2\right);\\
&\frac{\partial}{\partial Q_T} I_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1\varepsilon^2\nabla_T H(\varepsilon Q) + o\left(\varepsilon^2\right);\\
&\frac{\partial}{\partial Q_d} I_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1\varepsilon^2\nabla_d H(\varepsilon Q) - e^{-2d(1+o(1))} + o\left(\varepsilon^2\right),
\end{aligned} \tag{51}$$

as $\varepsilon \to 0$ and $d \to +\infty$. Q_T and Q_d denote tangent and normal variations of Q with respect to the interface $\mathcal{I}_\Omega$. The role of the correction Ξ_d is to improve the estimate in (51): using only cutoff functions on U and $\tilde{w}_Q$ (to get zero data on the Dirichlet part), one would obtain an error of order $\varepsilon^2 + O(e^{-d(1+o(1))})$, which is too large for our purposes.

The following proposition can be obtained from the results in Section 2 (and Lemma 4.4) in [9], see also Chapter 9 in [1].

Proposition 4.1. *For $Q \in \partial\Omega_\varepsilon$ as in* (46) *Case* a (*resp. Case* b)*, define*

$$Z_\varepsilon = \left\{ z_{\varepsilon,Q}\colon\ dist(Q, \mathcal{I}_{\Omega_\varepsilon}) \geqslant \frac{\mu_0}{\varepsilon}\right\} \quad \left(resp.\ Z_\varepsilon = \left\{\hat{z}_{\varepsilon,Q}\colon\ \frac{1}{\mu_0} dist(Q, \mathcal{I}_{\Omega_\varepsilon}) \geqslant \frac{\mu_0}{\varepsilon}\right\}\right).$$

Then for any $z \in Z_\varepsilon$ there exists a correction $\omega_\varepsilon(z)$ perpendicular to $T_z Z_\varepsilon$ and a small positive number c for which the following property holds. In a c neighborhood of Z_ε (with respect to the metric in $H^1_D(\Omega_\varepsilon)$) v is a critical point of I_ε if and only if $v = \bar{z} + \omega_\varepsilon(\bar{z})$, with $\bar{z}$ critical for the function $\mathbf{I}_\varepsilon : Z_\varepsilon \to \mathbb{R}$ defined as $\mathbf{I}_\varepsilon(z) = I_\varepsilon(z + \omega_\varepsilon(z))$.

In Chapter 9 of [1] it was shown that, in (46) Case a one has

$$\mathbf{I}_\varepsilon(z_{\varepsilon,Q}) = \tilde{C}_0 - \tilde{C}_1\varepsilon H(\varepsilon Q) + O\left(\varepsilon^2\right), \tag{52}$$

$$\frac{\partial}{\partial Q}\mathbf{I}_\varepsilon(z_{\varepsilon,Q}) = -\tilde{C}_1\varepsilon^2\nabla H(\varepsilon Q) + o\left(\varepsilon^2\right), \tag{53}$$

while in Proposition 4.5 of [9] it was proven that, in (46) Case b

$$\mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = \tilde{C}_0 - \tilde{C}_1\varepsilon H(\varepsilon Q) + O\left(\varepsilon^2\right) + e^{-2d(1+o(1))}; \tag{54}$$

$$\frac{\partial}{\partial Q_T}\mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1\varepsilon^2\nabla_T H(\varepsilon Q) + o\left(\varepsilon^2\right) + e^{-(p+1)d(1+o(1))} + e^{-3d(1+o(1))}; \tag{55}$$

$$\frac{\partial}{\partial Q_d}\mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1\varepsilon^2\nabla_d H(\varepsilon Q) - e^{-2d(1+o(1))} + o\left(\varepsilon^2\right) + e^{-(p+1)d(1+o(1))} + e^{-3d(1+o(1))}, \tag{56}$$

where we are using the same notation as above. We are now in position to prove our main theorems.

Proof of Theorem 1.3. By Propositions 3.1 and 3.3 we proved that only Case 1, among the three described in Section 3 can occur; that is, the maximum points converge to the Neumann part of the boundary. Also, by the proof of Proposition 3.8, we know that the functions w_ε (the scaling of u_ε at P_ε) converge locally in $\mathbb{R}^n_+$ to the soliton U.

The counterpart of (22) for the scaled solutions v_ε of (M_ε) is the following

$$I_\varepsilon(v_\varepsilon) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\Omega_\varepsilon} \left(|\nabla v_\varepsilon|^2 + v_\varepsilon^2\right) dx.$$

Fixed any compact set $K \subset \mathbb{R}^n_+$, for ε large, by the local convergence of v_ε to U we have

$$\int_{\Omega_\varepsilon} \left(|\nabla v_\varepsilon|^2 + v_\varepsilon^2\right) dx \geqslant \int_K \left(|\nabla v_\varepsilon|^2 + v_\varepsilon^2\right) dx = \int_K \left(|\nabla U|^2 + U^2\right) dx + o(1)$$

and therefore

$$\int_{\Omega_\varepsilon} \left(|\nabla v_\varepsilon|^2 + v_\varepsilon^2\right) dx \geqslant \int_{\mathbb{R}^n_+} \left(|\nabla U|^2 + U^2\right) dx + o(1) \quad \text{as } \varepsilon \to 0.$$

On the other hand by the fundamental estimate (16) once we scale in ε we get that $I_\varepsilon(v_\varepsilon) \leqslant \frac{1}{2} I(U)$, which implies that the H^1 norm of v_ε is small away from the scaling point.

As a consequence we deduce that, given any fixed (and small) $c > 0$, v_ε lies in a c neighborhood of Z_ε, for some of the choices of Z_ε in Proposition 4.1. In any case this proposition applies, and we have that v_ε can be critical only if $v_\varepsilon = \bar{z} + \omega_\varepsilon(\bar{z})$ for some $\bar{z}$ which is stationary for $\mathbf{I}_\varepsilon$.

If Case a in (46) occurs, then by (18) (where P can be taken arbitrarily in $\partial_{\mathcal{N}}\Omega$), (52) and (53) εQ must converge to a maximum point of H which realizes $\sup_{\partial_{\mathcal{N}}\Omega} H$.

In Case b in (46) then by (56) we have that, fixing any small $\delta > 0$ we cannot have $e^{-d} > \varepsilon^{1-\delta}$ for ε small, otherwise $\mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q})$ would not be critical in d. Therefore, for δ sufficiently small, we have that $e^{-2d} = o(\varepsilon^{\frac{3}{2}})$: hence from (18), (54) and (55) we have that εQ must converge to a maximum point of $H_{\mathcal{I}_\Omega}$ which realizes $\sup_{\mathcal{I}_\Omega} H$.

In either case, εQ converges to a maximum point $\overline{Q}$ of $H|_{\overline{\partial_{\mathcal{N}}\Omega}}$, and this concludes the proof. □

Proof of Theorem 1.4. Let $\overline{Q}$ be a global maximum of $H|_{\mathcal{I}_\Omega}$ as in the statement of the theorem. Then by the previous analysis we are in (46) Case b, namely the second alternative of Proposition 4.1, and Q is a minimum for $Q \mapsto \mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q})$.

We only need to prove that d is asymptotic to $|\log\varepsilon|$: in fact, if $\hat{c}$ is a fixed small constant and if $d \leqslant (1-\hat{c})|\log\varepsilon|$ along a sequence $\varepsilon_n \to 0$, by (56) we would find

$$\frac{\partial}{\partial Q_d} \mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\varepsilon^{2(1-\hat{c})(1+o(1))} + O\left(\varepsilon^2\right) < 0,$$

which is a contradiction to the criticality. Similarly, if $d \geqslant (1+\hat{c})|\log\varepsilon|$ along a sequence $\varepsilon_n \to 0$, still by (56) we would get

$$\frac{\partial}{\partial Q_d} \mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1 \varepsilon^2 \nabla_d H(\varepsilon Q) - \varepsilon^{2(1+\hat{c})(1+o(1))} + o\left(\varepsilon^2\right) > 0,$$

which is still a contradiction, and implies that $d \simeq |\log\varepsilon|$ as $\varepsilon \to 0$.

If (46) Case a holds, by (52) we have that

$$\mathbf{I}_\varepsilon(z_{\varepsilon,Q}) = \tilde{C}_0 - \tilde{C}_1 \varepsilon H(\check{Q}) + O\left(\varepsilon^2\right).$$

In we have (46) Case b instead, since $d \simeq |\log\varepsilon|$ as $\varepsilon \to 0$, by (54) and by the fact that $\nabla_d H$ at $\overline{Q}$ is negative we deduce that

$$\mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) \geqslant \tilde{C}_0 - \tilde{C}_1 \varepsilon H(\overline{Q}) + \tilde{c}\varepsilon|\log\varepsilon| + O\left(\varepsilon^{\frac{3}{2}}\right), \quad \varepsilon \to 0.$$

Since $H(\check{Q})$ and $H(\overline{Q})$ coincide, the energy in the first case is smaller, so the mountain pass solution has to be as in Case a. □

Acknowledgements

A.M. has been supported by MURST within the PRIN 2006 *Variational methods and nonlinear differential equations*, and by the *Giorgio and Elena Petronio Fellowship* while visiting IAS in Princeton in the Fall Semester 2008–2009. J.G. and I.P. are supported by the Project MTM2007-65018, MEC, Spain. This work was carried out while L.M. was visiting Universidad Autónoma de Madrid and the International School for Advanced Studies in Trieste. He would like to express his gratitude to Departamento de Matemáticas de Universidad Autónoma and to ISAS for their warm hospitality.

References

[1] A. Ambrosetti, A. Malchiodi, Perturbation Methods and Semilinear Elliptic Problems on $\mathbb{R}^n$, Progr. Math., vol. 240, Birkhäuser, 2005.

[2] A. Ambrosetti, A. Malchiodi, Nonlinear Analysis and Semilinear Elliptic Problems, Stud. Adv. Math., vol. 104, Cambridge Univ. Press, Cambridge, 2007.

[3] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.

[4] H. Berestycki, L. Caffarelli, L. Nirenberg, Further qualitative properties for elliptic equations in unbounded domains, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 25 (1–2) (1997) 69–94.

[5] E. Colorado, I. Peral, Eigenvalues and bifurcation for elliptic equations with mixed Dirichlet–Neumann boundary conditions related to Caffarelli–Kohn–Nirenberg inequalities, Topol. Methods Nonlinear Anal. 23 (2) (2004) 239–273.

[6] L. Damascelli, F. Gladiali, Some nonexistence results for positive solutions of elliptic equations in unbounded domains, Rev. Mat. Iberoamericana 20 (1) (2004) 67–86.

[7] E.N. Dancer, S. Yan, Interior and boundary peak solutions for a mixed boundary value problem, Indiana Univ. Math. J. 48 (4) (1999) 1177–1212.

[8] M. Del Pino, P. Felmer, Spike-layered solutions of singularly perturbed elliptic problems in a degenerate setting, Indiana Univ. Math. J. 48 (3) (1999) 883–898.

[9] J. Garcia Azorero, A. Malchiodi, L. Montoro, I. Peral, Concentration of solutions for some singularly perturbed mixed problems: Existence results, Arch. Ration. Mech. Anal., in press.

[10] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, Adv. Math. (Suppl. Stud. A) 7 (1981) 369–402.

[11] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, second ed., Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1983.

[12] M.K. Kwong, Uniqueness of positive solutions of $-\Delta u + u - u^p = 0$ in $\mathbb{R}^n$, Arch. Ration. Mech. Anal. 105 (1989) 243–266.

[13] C.S. Lin, W.M. Ni, I. Takagi, Large amplitude stationary solutions to a chemotaxis systems, J. Differential Equations 72 (1988) 1–27.

[14] W.M. Ni, J. Wei, On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems, Comm. Pure Appl. Math. 48 (1995) 731–768.

[15] W.M. Ni, I. Takagi, On the shape of least-energy solution to a semilinear Neumann problem, Comm. Pure Appl. Math. 41 (1991) 819–851.

[16] W.M. Ni, I. Takagi, Locating the peaks of least-energy solutions to a semilinear Neumann problem, Duke Math. J. 70 (1993) 247–281.

[17] G. Stampacchia, Problemi al contorno ellitici, con dati discontinui, dotati di soluzionie hölderiane, Ann. Mat. Pura Appl. (4) 51 (1960) 1–37.