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Generalised twists, $\mathbf{SO}(n)$, and the p-energy over a space of measure preserving maps

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Abstract

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and consider the energy functional

$$\mathbb{F}_p[\mathbf{u}, \Omega] := p^{-1} \int_{\Omega} \left|\nabla \mathbf{u}(\mathbf{x})\right|^p d\mathbf{x},$$

with $p \in]1, \infty[$ over the space of measure preserving maps

$$\mathcal{A}_p(\Omega) = \left\{\mathbf{u} \in W^{1,p}\left(\Omega, \mathbb{R}^n\right)\colon \mathbf{u}|_{\partial\Omega} = \mathbf{x},\ \det \nabla \mathbf{u} = 1 \text{ a.e. in } \Omega\right\}.$$

In this paper we introduce a class of maps referred to as *generalised* twists and *examine* them in connection with the Euler–Lagrange equations associated with $\mathbb{F}_p$ over $\mathcal{A}_p(\Omega)$. The main result is a surprising discrepancy between *even* and *odd* dimensions. Here we show that in even dimensions the latter system of equations admit *infinitely* many smooth solutions, modulo isometries, amongst such maps. In odd dimensions this number reduces to *one*. The result relies on a careful analysis of the *full* versus the *restricted* Euler–Lagrange equations where a key ingredient is a *necessary* and *sufficient* condition for an associated vector field to be a *gradient*.

Résumé

Soit $\Omega \subset \mathbb{R}^n$ un domaine de Lipschitz borné, on considère la fonctionnelle d'énergie

$$\mathbb{F}_p[\mathbf{u}, \Omega] := p^{-1} \int_{\Omega} \left|\nabla \mathbf{u}(\mathbf{x})\right|^p d\mathbf{x},$$

où $p \in]1, \infty[$ sur l'espace de fonctions conservant la mesure

$$\mathcal{A}_p(\Omega) = \left\{\mathbf{u} \in W^{1,p}\left(\Omega, \mathbb{R}^n\right)\colon \mathbf{u}|_{\partial\Omega} = \mathbf{x},\ \det \nabla \mathbf{u} = 1 \text{ a.a. dans } \Omega\right\}.$$

On introduit une classe de fonctions appellée des torsions généralisée qui est examinée dans le cadre des équations d'Euler–Lagrange associée à $\mathbb{F}_p$ sur $\mathcal{A}_p(\Omega)$. Le résultat principal est une surprenante différence de proprieté selon le parité de le dimension n. On démontre que pour n pair, ces équations admettent une infinité de solutions régulières qui sont des isometries, alors qu'en dimension impaire la solution est unique. Le résultat repose sur une analyse minutieuse de la version complète des équations d'Euler–Lagrange où l'ingrédient clé est une condition nécessaire et suffisante pour qu'un champ vectoriel soit un gradient.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and consider the energy functional

$$\mathbb{F}_p[\mathbf{u}, \Omega] := \int_\Omega \mathbf{F}_p\big(\nabla \mathbf{u}(\mathbf{x})\big)\, d\mathbf{x}, \tag{1.1}$$

with $\mathbf{F}_p(\xi) = p^{-1}|\xi|^p$ and $p \in]1, \infty[$ over the space of *admissible* maps

$$\mathcal{A}_p(\Omega) := \big\{\mathbf{u} \in W^{1,p}_\varphi\big(\Omega, \mathbb{R}^n\big)\colon \det \nabla \mathbf{u} = 1 \text{ a.e. in } \Omega\big\}, \tag{1.2}$$

where

$$W^{1,p}_\varphi\big(\Omega, \mathbb{R}^n\big) = \big\{\mathbf{u} \in W^{1,p}\big(\Omega, \mathbb{R}^n\big)\colon \mathbf{u}|_{\partial\Omega} = \varphi\big\},$$

and φ is the *identity* map.

In this paper we are concerned with the problem of *extremising* the energy functional (1.1) over the space (1.2) and *examining* a class of maps of *topological* significance as *solutions* to the associated system of Euler–Lagrange equations

$$\begin{cases} div\, \mathfrak{S}_p[\mathbf{x}, \nabla\mathbf{u}(\mathbf{x})] = 0, & \mathbf{x} \in \Omega, \\ \det \nabla \mathbf{u}(\mathbf{x}) = 1, & \mathbf{x} \in \Omega, \\ \mathbf{u}(\mathbf{x}) = \varphi(\mathbf{x}), & \mathbf{x} \in \partial\Omega. \end{cases}$$

Here, we have that

$$\begin{aligned} \mathfrak{S}_p[\mathbf{x}, \xi] &= \mathbf{F}_p'(\xi) - \mathfrak{p}(\mathbf{x})\xi^{-t} \\ &=: \mathfrak{T}_p[\mathbf{x}, \xi]\xi^{-t}, \end{aligned} \tag{1.3}$$

for $\mathbf{x} \in \Omega$, $\xi \in \mathbb{R}^{n\times n}$ satisfying $\det \xi = 1$ and $\mathfrak{p}$ a suitable *Lagrange* multiplier while

$$\mathfrak{T}_p[\mathbf{x}, \xi] = \mathbf{F}_p'(\xi)\xi^t - \mathfrak{p}(\mathbf{x})\mathbf{I}. \tag{1.4}$$

A motivating source for this type of problem is nonlinear elasticity where (1.1) and (1.2) represent a simple *model* of a homogeneous *incompressible* hyperelastic material and solutions to the above system of equations serve as the corresponding *equilibrium* states (cf., e.g., Ball [1]).[1]

While the *linear* map $\mathbf{u} = \varphi$ serves as the *unique* minimiser of $\mathbb{F}_p$ over $\mathcal{A}_p(\Omega)$ little is known about the *structure* and *features* of the solution *set* to this system of Euler–Lagrange equations [e.g., multiplicity *versus* uniqueness, existence of *strong* local minimisers, *partial* regularity, the nature and form of *singularities*, symmetries, *etc.* (*see*, e.g., [2,3,6,8,9,12,14])].

In this article we contribute towards understanding aspects of these questions by way of presenting *multiple* solutions to the above system of equations. Indeed we focus attention on the case where the domain $\Omega \subset \mathbb{R}^n$ is an n-dimensional *annulus*, i.e., $\Omega = \{\mathbf{x} \in \mathbb{R}^n\colon a < |\mathbf{x}| < b\}$ with $0 < a < b < \infty$.[2] We proceed by introducing a class of maps, referred to as *generalised* twists, characterised and defined by

$$\mathbf{u} = \mathbf{Q}(r)\mathbf{x},$$

where $\mathbf{Q} \in C([a, b], \mathbf{SO}(n))$ and $r = |\mathbf{x}|$. To ensure *admissibility*, i.e., $\mathbf{u} \in \mathcal{A}_p(\Omega)$ it suffices to impose a *further* p-summability on $\dot{\mathbf{Q}} := d\mathbf{Q}/dr$ along with $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$. Restricting the p-energy to the space of such *twists* we can write

$$\begin{aligned} \mathbb{E}_p[\mathbf{Q}] &:= p\mathbb{F}_p\big[\mathbf{Q}(r)\mathbf{x}, \Omega\big] \\ &= \int_a^b \mathbf{E}(r, \dot{\mathbf{Q}}) r^{n-1}\, dr \end{aligned}$$

[1] In the language of elasticity, the *tensor* fields (1.3) and (1.4) are referred to as the *Piola–Kirchhoff* and the *Cauchy* stress tensors respectively and the *Lagrange* multiplier $\mathfrak{p}$ is better known as the *hydrostatic* pressure.

[2] Recall that for *star-shaped* domains and subject to *linear* boundary conditions there is a *uniqueness* result associated with [*sufficiently* regular] *equilibrium* states in both *compressible* and *incompressible* hyperelasticity. (*See* [8].)

where the *integrand* itself is given through an integral over the *unit* sphere, i.e.,

$$\mathbf{E}(r,\xi) := \int_{\mathbb{S}^{n-1}} \big(n + r^2|\xi\theta|^2\big)^{\frac{p}{2}}\, d\mathcal{H}^{n-1}(\theta).$$

Here, the Euler–Lagrange equation can be shown to be the *second* order ordinary *differential* equation

$$\frac{d}{dr}\big\{r^{n-1}\big[\mathbf{E}'(r,\dot{\mathbf{Q}})\mathbf{Q}^t - \mathbf{Q}\mathbf{E}'^t(r,\dot{\mathbf{Q}})\big]\big\} = 0.$$

Now in order to characterise among solutions to the above equation, *all* those which grant a solution to the Euler–Lagrange equations associated with $\mathbb{F}_p$ over $\mathcal{A}_p(\Omega)$ we are *confronted* with the of task of obtaining *necessary* and *sufficient* conditions on the *vector* field

$$[\nabla\mathbf{u}]^t\Delta_p\mathbf{u} = \nabla\mathbf{s} + \bigg\{r\mathbf{s}\mathbf{A}^2 - r^2\mathbf{s}\langle\mathbf{A}\theta,\dot{\mathbf{A}}\theta\rangle\mathbf{I}_n + \frac{1}{r^n}\frac{d}{dr}\big(r^{n+1}\mathbf{s}\mathbf{A}\big) + \frac{1}{r^{n-1}}\frac{d}{dr}\big(r^{n+1}\mathbf{s}|\mathbf{A}\theta|^2\big)\mathbf{I}_n\bigg\}\theta$$

with $\mathbf{A} = \mathbf{Q}^t\dot{\mathbf{Q}}$ and $\mathbf{s} = (n + r^2|\dot{\mathbf{Q}}\theta|^2)^{\frac{p-2}{2}}$ for it to be a *gradient*, specifically, to coincide with $\nabla\mathfrak{p}$. This *analysis* occupies a major part of the paper and is *fully* settled in Theorems 5.1 and 5.2.

The conclusion that the above analysis bares on to the *original* Euler–Lagrange equations turns to be a surprising discrepancy between *even* and *odd* dimensions. Indeed it follows that in *even* dimensions the latter system of equations admit *infinitely* many *smooth* solutions, modulo isometries, in the form of *generalised* twists whilst in *odd* dimensions this number *severely* reduces to *one*.[3]

2. Generalised twists

Definition 2.1 *(Generalised twist).* Let $\Omega = \{\mathbf{x}\in\mathbb{R}^n\colon\ a < |\mathbf{x}| < b\}$. A map $\mathbf{u}\in C(\bar{\Omega},\bar{\Omega})$ is a *generalised* twist if and only if

$$\mathbf{u}(\mathbf{x}) = \mathbf{Q}(r)\mathbf{x} \tag{2.1}$$

for *some* $\mathbf{Q}\in C([a,b],\mathbf{SO}(n))$ and all $\mathbf{x}\in\bar{\Omega}$ with $r = |\mathbf{x}|$.[4]

[3] Note that for the choice of $\Omega\subset\mathbb{R}^n$ an n-dimensional annulus the space of its continuous self-maps, that is,

$$\mathfrak{A}(\Omega) = \big\{\phi\in C(\bar{\Omega},\bar{\Omega})\colon\ \phi(\mathbf{x}) = \mathbf{x} \text{ for } \mathbf{x}\in\partial\Omega\big\}$$

equipped with the topology of *uniform* convergence consist of *infinitely* many components for $n = 2$ and precisely *two* for $n \geqslant 3$. (*See* [13,15].) Thus with regards to $\mathcal{A}_p(\Omega)$ we distinguish the following *two* cases.

(**1**) When $p \geqslant n$ taking advantage of the embedding $\mathcal{A}_p(\Omega)\subset\mathfrak{A}(\Omega)$ enables one to *partition* $\mathcal{A}_p(\Omega)$ into a corresponding collection of pairwise disjoint *sequentially* weakly closed subsets on each of which minimising $\mathbb{F}_p$ gives rise to a *strong* local minimiser (*see* [14]).

(**2**) When $1 \leqslant p < n$ the above argument encounters *two* serious obstacles, *firstly*, there is *no* embedding of $\mathcal{A}_p(\Omega)$ into $\mathfrak{A}(\Omega)$, and *secondly*, the *determinant* function *fails* to be sequentially *weakly* continuous.

Thus in case (**2**) the question of existence and multiplicity of *strong* local minimisers as well as solutions to the system of Euler–Lagrange equations seem at large *open*. Luckily the approach developed in this paper overcomes this obstacle and leads to explicit constructions of *infinitely* many smooth solutions to the later system of Euler–Lagrange equations for *any* $p\in\,]1,\infty[$ when n is *even*. An interesting question is *if* the *strong* local minimisers in case (**1**) (n being *even*) lie *amongst* this class of *twist* solutions. Equally interesting is a full *characterisation* of these minimisers when n is *odd*. (*See* [11].)

Recall that a map $\bar{\mathbf{u}}\in\mathcal{A}_p(\Omega)$ is a *strong* local minimiser of $\mathbb{F}_p$ if and only if there exists $\delta = \delta(\bar{\mathbf{u}}) > 0$ such that $\mathbb{F}_p[\bar{\mathbf{u}},\Omega] \leqslant \mathbb{F}_p[\mathbf{v},\Omega]$ for all $\mathbf{v}\in\mathcal{A}_p(\Omega)$ satisfying $\|\bar{\mathbf{u}} - \mathbf{v}\|_{L^1(\Omega)} < \delta$.

[4] When $n = 2$ a *generalised* twist can be shown to take, in *polar* coordinates, the alternative form

$$(r,\theta)\mapsto\big(r,\theta + g(r)\big) \tag{2.2}$$

for a suitable $g\in C[a,b]$. Maps of the type (2.2) frequently arise in the study of *mapping* class groups of *surfaces* and are better known as *Dehn-twists*. In *higher* dimensions, by contrast, no such simple representation of (2.1) is feasible in generalised *spherical* coordinates, however, the terminology here is suggested by analogy with (2.2) when $n = 2$.

The continuous function $\mathbf{Q}$ in the above definition will be referred to as the twist *path*. When additionally $\mathbf{Q}(a) = \mathbf{Q}(b)$ we refer to $\mathbf{Q}$ as the twist *loop*.

Proposition 2.1. *Let $\Omega = \{\mathbf{x} \in \mathbb{R}^n\colon\ a < |\mathbf{x}| < b\}$. A generalised twist $\mathbf{u}$ lies in $\mathcal{A}_p = \mathcal{A}_p(\Omega)$ with $p \in [1, \infty[$ provided that the following hold.*

(**1**) $\mathbf{Q} \in W^{1,p}([a,b], \mathbf{SO}(n))$,
(**2**) $\mathbf{Q}(a) = \mathbf{I}_n$,
(**3**) $\mathbf{Q}(b) = \mathbf{I}_n$.

Thus, in particular, when a *generalised* twist $\mathbf{u}$ lies in $\mathcal{A}_p$ its corresponding twist path forms a *loop* in the pointed space $(\mathbf{SO}(n), \mathbf{I}_n)$.

Proof. Assume that $\mathbf{u}$ is a *generalised* twist. Then $\mathbf{u} \in \mathcal{A}_p(\Omega)$ if and only if the following hold.

(i) $\mathbf{u} = \mathbf{x}$ on $\partial\Omega$,
(ii) $\det \nabla\mathbf{u} = 1$ in Ω, and,
(iii) $\|\mathbf{u}\|_{W^{1,p}(\Omega)} < \infty$.

Evidently (**2**) and (**3**) give (i). Moreover, a straight-forward calculation gives

$$\begin{aligned}\nabla\mathbf{u} &= \mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta \\ &= \mathbf{Q}\big(\mathbf{I}_n + r\mathbf{Q}^t\dot{\mathbf{Q}}\theta \otimes \theta\big)\end{aligned} \tag{2.3}$$

where $r = |\mathbf{x}|$, $\theta = \mathbf{x}/|\mathbf{x}|$ and $\dot{\mathbf{Q}} := d\mathbf{Q}/dr$. Hence in view of $\det \mathbf{Q} = 1$ we can write

$$\begin{aligned}\det \nabla\mathbf{u} &= \det(\mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta) \\ &= \det\big(\mathbf{I}_n + r\mathbf{Q}^t\dot{\mathbf{Q}}\theta \otimes \theta\big) \\ &= 1 + r\big\langle \mathbf{Q}^t\dot{\mathbf{Q}}\theta, \theta\big\rangle \\ &= 1 + r\langle \dot{\mathbf{Q}}\theta, \mathbf{Q}\theta\rangle = 1,\end{aligned}$$

where in the last identity we have used the fact that $\langle \mathbf{Q}\theta, \mathbf{Q}\theta\rangle = |\theta|^2 = 1$ for all $\theta \in \mathbb{S}^{n-1}$ and so as a result

$$\frac{d}{dr}\langle \mathbf{Q}\theta, \mathbf{Q}\theta\rangle = \langle \mathbf{Q}\theta, \dot{\mathbf{Q}}\theta\rangle + \langle \dot{\mathbf{Q}}\theta, \mathbf{Q}\theta\rangle = 0.$$

This therefore gives (ii). Finally, to justify (iii) we *first* note that

$$\begin{aligned}|\nabla\mathbf{u}|^2 &= \operatorname{tr}\big\{[\nabla\mathbf{u}][\nabla\mathbf{u}]^t\big\} \\ &= \operatorname{tr}\big\{(\mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta)\big(\mathbf{Q}^t + r\theta \otimes \dot{\mathbf{Q}}\theta\big)\big\} \\ &= \operatorname{tr}\big\{\mathbf{I}_n + r\mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta + r\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta + r^2\dot{\mathbf{Q}}\theta \otimes \dot{\mathbf{Q}}\theta\big\} \\ &= n + 2r\langle \mathbf{Q}\theta, \dot{\mathbf{Q}}\theta\rangle + r^2\langle \dot{\mathbf{Q}}\theta, \dot{\mathbf{Q}}\theta\rangle.\end{aligned}$$

Therefore as a result of $\langle \mathbf{Q}\theta, \dot{\mathbf{Q}}\theta\rangle = 0$ for any $p \in [1, \infty[$ we have that

$$|\nabla\mathbf{u}|^p = \big(n + r^2|\dot{\mathbf{Q}}\theta|^2\big)^{\frac{p}{2}}. \tag{2.4}$$

Hence in view of $|\mathbf{u}| = r\sqrt{\langle \mathbf{Q}\theta, \mathbf{Q}\theta\rangle} = r$ we can write

$$\int_\Omega |\mathbf{u}|^p + |\nabla\mathbf{u}|^p = \int_a^b \int_{\mathbb{S}^{n-1}} \big\{r^p + \big(n + r^2|\dot{\mathbf{Q}}\theta|^2\big)^{\frac{p}{2}}\big\} r^{n-1}\, d\mathcal{H}^{n-1}(\theta)\, dr,$$

and so referring to (**1**) the conclusion follows. □

Proposition 2.2. *Suppose that* $\mathbf{u}$ *is a generalised twist with the associated twist path* $\mathbf{Q} \in C^2(]a,b[,\mathbf{SO}(n))$. *Then for* $p \in [1,\infty[$ *we have that*

$$\begin{aligned}\Delta_p \mathbf{u} &:= div\big(|\nabla \mathbf{u}|^{p-2}\big)\nabla \mathbf{u}\\ &= \mathbf{Q}\left[\nabla \mathbf{s} \otimes \theta + \frac{1}{r^n}\frac{d}{dr}\big(r^{n+1}\mathbf{s}\mathbf{A}\big) + r\mathbf{s}\mathbf{A}^2\right]\theta,\end{aligned}$$

where $\mathbf{A} = \mathbf{Q}^t\dot{\mathbf{Q}}$ *and* $\mathbf{s} = \mathbf{s}(r,\theta) := (n + r^2|\mathbf{A}\theta|^2)^{\frac{p-2}{2}}$.

Proof. (**1**) ($p = 2$) Referring to Definition 2.1 and using the notation $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ we can write with the aid of (2.3) in Proposition 2.1 that

$$\begin{aligned}\Delta u_i &= \sum_{j=1}^n \frac{\partial}{\partial x_j}\left\{\mathbf{Q}_{ij} + r\sum_{k=1}^n \dot{\mathbf{Q}}_{ik}\theta_k\theta_j\right\}\\ &= \sum_{j=1}^n \left\{\dot{\mathbf{Q}}_{ij}\theta_j + \theta_j\sum_{k=1}^n \dot{\mathbf{Q}}_{ik}\theta_k\theta_j + r\sum_{k=1}^n \ddot{\mathbf{Q}}_{ik}\theta_j\theta_k\theta_j\right.\\ &\quad \left.+ \sum_{k=1}^n \dot{\mathbf{Q}}_{ik}(\delta_{kj} - \theta_j\theta_k)\theta_j + \sum_{k=1}^n \dot{\mathbf{Q}}_{ik}\theta_k(1-\theta_j\theta_j)\right\}\\ &= 2\sum_{j=1}^n \dot{\mathbf{Q}}_{ij}\theta_j + r\sum_{j=1}^n \ddot{\mathbf{Q}}_{ij}\theta_j + (n-1)\sum_{j=1}^n \dot{\mathbf{Q}}_{ij}\theta_j\\ &= (n+1)\sum_{k=1}^n \dot{\mathbf{Q}}_{ik}\theta_k + r\sum_{j=1}^n \ddot{\mathbf{Q}}_{ij}\theta_j.\end{aligned}$$

As this is true for $1 \leqslant i \leqslant n$ going back to the original *vector* notation and using the substitutions $\dot{\mathbf{Q}} = \mathbf{QA}$ and $\ddot{\mathbf{Q}} = \mathbf{Q}[\dot{\mathbf{A}} + \mathbf{A}^2]$ we have that

$$\begin{aligned}\Delta \mathbf{u} &= \big[(n+1)\dot{\mathbf{Q}} + r\ddot{\mathbf{Q}}\big]\theta\\ &= \mathbf{Q}\big[(n+1)\mathbf{A} + r\dot{\mathbf{A}} + r\mathbf{A}^2\big]\theta\\ &= \mathbf{Q}\left[\frac{1}{r^n}\frac{d}{dr}\big(r^{n+1}\mathbf{A}\big) + r\mathbf{A}^2\right]\theta,\end{aligned}$$

which is the required result for $p = 2$. [Note that in this case $\mathbf{s} = \mathbf{s}(r,\theta) \equiv 1$.]

(**2**) ($p \in [1,\infty[$) According to definition we have that

$$\begin{aligned}\Delta_p \mathbf{u} &= div\big(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\big)\\ &= div(\mathbf{s}\nabla \mathbf{u}) = \nabla \mathbf{u}\nabla \mathbf{s} + \mathbf{s}\Delta \mathbf{u}.\end{aligned}$$

Now a straight-forward *differentiation* gives

$$\begin{aligned}\nabla \mathbf{s} &= \nabla\big(n + r^2|\dot{\mathbf{Q}}\theta|^2\big)^{\frac{p-2}{2}}\\ &= \nabla\big(n + r^2|\mathbf{A}\theta|^2\big)^{\frac{p-2}{2}}\\ &= \beta\big[r\mathbf{A}^t\mathbf{A} + r^2\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle \mathbf{I}_n\big]\theta,\end{aligned} \tag{2.5}$$

where $\beta = \beta(r,\theta,p) := (p-2)(n + r^2|\mathbf{A}\theta|^2)^{\frac{p-4}{2}}$. Thus we can write

$$\begin{aligned}\Delta_p \mathbf{u} &= \nabla \mathbf{u}\nabla \mathbf{s} + \mathbf{s}\Delta \mathbf{u}\\ &= \mathbf{Q}[\mathbf{I}_n + r\mathbf{A}\theta \otimes \theta]\nabla \mathbf{s}\\ &\quad + \mathbf{s}\mathbf{Q}\big[(n+1)\mathbf{A} + r\dot{\mathbf{A}} + r\mathbf{A}^2\big]\theta\end{aligned}$$

$$= \mathbf{Q}\nabla\mathbf{s} + r\beta\mathbf{Q}[\mathbf{A}\theta\otimes\theta]\big[r\mathbf{A}^t\mathbf{A} + r^2\langle\mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle\mathbf{I}_n\big]\theta$$
$$+ \mathbf{s}\mathbf{Q}\big[(n+1)\mathbf{A} + r\dot{\mathbf{A}} + r\mathbf{A}^2\big]\theta.$$

In order to further simplify the *second* term on the *right* in the last identity we *first* notice that

$$\mathbf{s}_r := \frac{\partial\mathbf{s}}{\partial r} = \frac{\partial}{\partial r}\big(n + r^2|\mathbf{A}\theta|^2\big)^{\frac{p-2}{2}}$$
$$= \beta\big[r|\mathbf{A}\theta|^2 + r^2\langle\mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle\big]$$

and consequently

$$r\mathbf{s}_r\mathbf{Q}\mathbf{A}\theta = r\beta\mathbf{Q}\big[r|\mathbf{A}\theta|^2 + r^2\langle\mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle\big]\mathbf{A}\theta$$
$$= r\beta\mathbf{Q}[\mathbf{A}\theta\otimes\theta]\big[r\mathbf{A}^t\mathbf{A} + r^2\langle\mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle\mathbf{I}_n\big]\theta.$$

Therefore substituting back gives

$$\Delta_p\mathbf{u} = \mathbf{Q}\big[\nabla\mathbf{s}\otimes\theta + r\mathbf{s}_r\mathbf{A} + (n+1)\mathbf{s}\mathbf{A} + r\mathbf{s}\dot{\mathbf{A}} + r\mathbf{s}\mathbf{A}^2\big]\theta$$
$$= \mathbf{Q}\left[\nabla\mathbf{s}\otimes\theta + \frac{1}{r^n}\frac{d}{dr}\big(r^{n+1}\mathbf{s}\mathbf{A}\big) + r\mathbf{s}\mathbf{A}^2\right]\theta$$

which is the required conclusion. □

Proposition 2.3. *Suppose that* $\mathbf{u}$ *is a generalised twist with the associated twist path* $\mathbf{Q} \in C^2(]a,b[, \mathbf{SO}(n))$. *Then for* $p \in [1,\infty[$ *we have that*

$$[\nabla\mathbf{u}]^t\Delta_p\mathbf{u} = \nabla\mathbf{s} + \bigg\{r\mathbf{s}\mathbf{A}^2 - r^2\mathbf{s}\langle\mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle\mathbf{I}_n$$
$$+ \frac{1}{r^n}\frac{d}{dr}\big(r^{n+1}\mathbf{s}\mathbf{A}\big) + \frac{1}{r^{n-1}}\frac{d}{dr}\big(r^{n+1}\mathbf{s}|\mathbf{A}\theta|^2\big)\mathbf{I}_n\bigg\}\theta \tag{2.6}$$

where $\mathbf{A} = \mathbf{Q}^t\dot{\mathbf{Q}}$ *and* $\mathbf{s} = \mathbf{s}(r,\theta) = (n + r^2|\mathbf{A}\theta|^2)^{\frac{p-2}{2}}$.

Proof. In view of (2.3) we have that

$$[\nabla\mathbf{u}]^t = [\mathbf{Q} + r\dot{\mathbf{Q}}\theta\otimes\theta]^t = \big[\mathbf{Q}^t + r\theta\otimes\dot{\mathbf{Q}}\theta\big] = [\mathbf{I}_n + r\theta\otimes\mathbf{A}\theta]\mathbf{Q}^t.$$

Therefore by substituting for $[\nabla\mathbf{u}]^t$ and $\Delta_p\mathbf{u}$ (from the previous proposition) we arrive at

$$[\nabla\mathbf{u}]^t\Delta_p\mathbf{u} = [\mathbf{I}_n + r\theta\otimes\mathbf{A}\theta]$$
$$\times\left[\nabla\mathbf{s}\otimes\theta + \frac{1}{r^n}\frac{d}{dr}\big(r^{n+1}\mathbf{s}\mathbf{A}\big) + r\mathbf{s}\mathbf{A}^2\right]\theta$$
$$= \left[\nabla\mathbf{s}\otimes\theta + \frac{1}{r^n}\frac{d}{dr}\big(r^{n+1}\mathbf{s}\mathbf{A}\big) + r\mathbf{s}\mathbf{A}^2\right]\theta$$
$$+ \left[r\langle\nabla\mathbf{s}, \mathbf{A}\theta\rangle + \frac{1}{r^{n-1}}\left\langle\frac{d}{dr}\big(r^{n+1}\mathbf{s}\mathbf{A}\big)\theta, \mathbf{A}\theta\right\rangle + r^2\mathbf{s}\big\langle\mathbf{A}^2\theta, \mathbf{A}\theta\big\rangle\right]\theta.$$

However, in view of $\mathbf{A}$ being *skew*-symmetric it can be easily verified that $\langle\mathbf{A}^2\theta, \mathbf{A}\theta\rangle = 0$ and in a similar way referring to (2.5)

$$\langle\nabla\mathbf{s}, \mathbf{A}\theta\rangle = \big\langle\beta\big[r\mathbf{A}^t\mathbf{A} + r^2\langle\mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle\mathbf{I}_n\big]\theta, \mathbf{A}\theta\big\rangle$$
$$= \beta r\big\{\big\langle\mathbf{A}^3\theta, \theta\big\rangle + r\langle\mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle\langle\mathbf{A}\theta, \theta\rangle\big\} = 0.$$

Thus *summarising*, we have that

$$\begin{aligned}
[\nabla \mathbf{u}]^t \Delta_p \mathbf{u} &= \nabla \mathbf{s} + \biggl\{ r\mathbf{s}\mathbf{A}^2 + \frac{1}{r^n}\frac{d}{dr}\bigl(r^{n+1}\mathbf{s}\mathbf{A}\bigr) \\
&\quad + \frac{1}{r^{n-1}}\biggl\langle \frac{d}{dr}\bigl(r^{n+1}\mathbf{s}\mathbf{A}\bigr)\theta, \mathbf{A}\theta\biggr\rangle \mathbf{I}_n \biggr\}\theta \\
&= \nabla \mathbf{s} + \biggl\{ r\mathbf{s}\mathbf{A}^2 - r^2\mathbf{s}\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle \mathbf{I}_n + \frac{1}{r^n}\frac{d}{dr}\bigl(r^{n+1}\mathbf{s}\mathbf{A}\bigr) \\
&\quad + \frac{1}{r^{n-1}}\frac{d}{dr}\bigl(r^{n+1}\mathbf{s}|\mathbf{A}\theta|^2\bigr)\mathbf{I}_n \biggr\}\theta.
\end{aligned}$$

The proof is thus complete. □

3. The p-energy restricted to the loop space

For a *generalised* twist $\mathbf{u}$ referring to (2.4) we have for any $p \in [1, \infty[$ that

$$\int_\Omega |\nabla \mathbf{u}|^p = \int_a^b \int_{\mathbb{S}^{n-1}} \bigl(n + r^2|\dot{\mathbf{Q}}\theta|^2\bigr)^{\frac{p}{2}} r^{n-1}\, d\mathcal{H}^{n-1}(\theta)\, dr.$$

Motivated by the above representation in this section we introduce the energy functional

$$\mathbb{E}_p[\mathbf{Q}] := \int_a^b \mathbf{E}(r, \dot{\mathbf{Q}}) r^{n-1}\, dr$$

where the *integrand* itself is given through the integral

$$\mathbf{E}(r, \xi) = \int_{\mathbb{S}^{n-1}} \bigl(n + r^2|\xi\theta|^2\bigr)^{\frac{p}{2}}\, d\mathcal{H}^{n-1}(\theta).$$

Associated with the energy functional $\mathbb{E}_p$ and in line with Proposition 2.1 we introduce the space of *admissible* loops

$$\mathcal{E}_p = \bigl\{\mathbf{Q} = \mathbf{Q}(r):\ \mathbf{Q} \in W^{1,p}\bigl([a,b], \mathbf{SO}(n)\bigr),\ \mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n\bigr\}.$$

Our primary objective here is to obtain the Euler–Lagrange equation associated with the energy functional $\mathbb{E}_p$ over the space of loops $\mathcal{E}_p$. In doing so the following observation will prove useful.

Proposition 3.1. *Let* $\mathbf{Q} \in \mathbf{SO}(n)$ *and* $\mathbf{R} \in \mathbb{M}_{n\times n}$. *Then the followings are equivalent*:

(**1**) $\mathbf{R}\mathbf{Q}^t + \mathbf{Q}\mathbf{R}^t = \mathbf{0}$,
(**2**) $\mathbf{R} = (\mathbf{F} - \mathbf{F}^t)\mathbf{Q}$ *for some* $\mathbf{F} \in \mathbb{M}_{n\times n}$.

Moreover, $\mathbf{F}$ *in* (**2**) *is unique if it is assumed skew-symmetric, i.e.,* $\mathbf{F}^t = -\mathbf{F}$.

Proof. The implication (**2**) ⇒ (**1**) follows from a direct verification. For the reverse implication it suffices to assume $\mathbf{F}^t + \mathbf{F} = \mathbf{0}$ and then take $2\mathbf{F} = \mathbf{R}\mathbf{Q}^t$. □

Proposition 3.2. *Let* $p \in [1, \infty[$. *Then the Euler–Lagrange equation associated with* $\mathbb{E}_p$ *over* $\mathcal{E}_p$ *takes the form*

$$\frac{d}{dr}\bigl\{r^{n-1}\bigl[\mathbf{E}_\xi(r, \dot{\mathbf{Q}})\mathbf{Q}^t - \mathbf{Q}\mathbf{E}^t_\xi(r, \dot{\mathbf{Q}})\bigr]\bigr\} = 0. \tag{3.1}$$

Proof. Fix $\mathbf{Q} \in W^{1,p}([a,b], \mathbf{SO}(n))$ and pick a *variation* $\mathbf{H} \in C_0^\infty([a,b], \mathbb{M}_{n\times n})$. For $\varepsilon \in \mathbb{R}$ put $\mathbf{Q}_\varepsilon = \mathbf{Q} + \varepsilon\mathbf{H}$. Then,

$$\begin{aligned}
\mathbf{Q}_\varepsilon\mathbf{Q}_\varepsilon^t &= [\mathbf{Q} + \varepsilon\mathbf{H}][\mathbf{Q} + \varepsilon\mathbf{H}]^t \\
&= \mathbf{I}_n + \varepsilon\bigl[\mathbf{H}\mathbf{Q}^t + \mathbf{Q}\mathbf{H}^t\bigr] + \varepsilon^2\mathbf{H}\mathbf{H}^t.
\end{aligned}$$

Hence for $\mathbf{Q}_\varepsilon$ to take values on $\mathbf{SO}(n)$ to the *first* order it suffices to have

$$\mathbf{H}\mathbf{Q}^t + \mathbf{Q}\mathbf{H}^t = \mathbf{0},$$

on $[a, b]$. In view of Proposition 3.1 this is equivalent to assuming that for some $\mathbf{F} \in C_0^\infty([a, b], \mathbb{M}_{n\times n})$ the *variation* $\mathbf{H}$ has the form

$$\mathbf{H} = \big(\mathbf{F} - \mathbf{F}^t\big)\mathbf{Q}.$$

With this assumption in place we examine the vanishing of the *first* derivative of the *energy*, i.e., that indeed

$$\begin{aligned} 0 &= \frac{d}{d\epsilon}\mathbb{E}_p[\mathbf{Q}_\varepsilon]\bigg|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}\int_a^b \mathbf{E}(r, \dot{\mathbf{Q}}_\varepsilon) r^{n-1}\,dr\bigg|_{\varepsilon=0} \\ &= \int_a^b \bigg\{\frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}}_\varepsilon) : \frac{d}{d\varepsilon}\dot{\mathbf{Q}}_\varepsilon\bigg\} r^{n-1}\,dr\bigg|_{\varepsilon=0} \\ &= \int_a^b \bigg\{\frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}}) : \big[\big(\dot{\mathbf{F}} - \dot{\mathbf{F}}^t\big)\mathbf{Q} + \big(\mathbf{F} - \mathbf{F}^t\big)\dot{\mathbf{Q}}\big]\bigg\} r^{n-1}\,dr \\ &=: \mathbf{I} + \mathbf{II}. \end{aligned}$$

We now proceed by evaluating each term separately. Indeed, with regards to the *first* term we have that

$$\begin{aligned} \mathbf{I} &= \int_a^b \bigg\{\frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}}) : \big(\dot{\mathbf{F}} - \dot{\mathbf{F}}^t\big)\mathbf{Q}\bigg\} r^{n-1}\,dr \\ &= \int_a^b \bigg\{\frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}})\mathbf{Q}^t : \big(\dot{\mathbf{F}} - \dot{\mathbf{F}}^t\big)\bigg\} r^{n-1}\,dr \\ &= \int_a^b \bigg\{-\frac{d}{dr}\bigg[r^{n-1}\frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}})\mathbf{Q}^t\bigg] : \big(\mathbf{F} - \mathbf{F}^t\big)\bigg\}\,dr. \end{aligned}$$

Note that in the *third* line we have used *integration* by *parts* which together with the *boundary* conditions $\mathbf{F}(a) = \mathbf{F}(b) = \mathbf{0}$ gives

$$\begin{aligned} 0 &= r^{n-1}\frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}})\mathbf{Q}^t : \big(\mathbf{F} - \mathbf{F}^t\big)\bigg|_a^b \\ &= \int_a^b r^{n-1}\frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}})\mathbf{Q}^t : \big(\dot{\mathbf{F}} - \dot{\mathbf{F}}^t\big)\,dr \\ &\quad + \int_a^b \frac{d}{dr}\bigg[r^{n-1}\frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}})\mathbf{Q}^t\bigg] : \big(\mathbf{F} - \mathbf{F}^t\big)\,dr. \end{aligned}$$

On the other hand for the *second* term a direct verification reveals that

$$\mathbf{II} = \int_a^b \bigg\{\frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}}) : \big(\mathbf{F} - \mathbf{F}^t\big)\dot{\mathbf{Q}}\bigg\} r^{n-1}\,dr$$

$$= \int_a^b \int_{\mathbb{S}^{n-1}} p\big(n + r^2|\dot{\mathbf{Q}}\theta|^2\big)^{\frac{p-2}{2}} \big\langle \dot{\mathbf{Q}}\theta, \big(\mathbf{F} - \mathbf{F}^t\big)\dot{\mathbf{Q}}\theta \big\rangle r^{n+1}\, dr = 0$$

as a result of the *pointwise* identity $\langle \dot{\mathbf{Q}}\theta, (\mathbf{F} - \mathbf{F}^t)\dot{\mathbf{Q}}\theta\rangle = 0$. Thus, *summarising*, we have that

$$\frac{d}{d\varepsilon}\mathbb{E}_p[\mathbf{Q}_\varepsilon]\bigg|_{\varepsilon=0} = \int_a^b \left\{ -\frac{d}{dr}\left[r^{n-1} \frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}})\mathbf{Q}^t \right] : \big(\mathbf{F} - \mathbf{F}^t\big) \right\} dr = 0.$$

As this is true for every $\mathbf{F} \in C_0^\infty([a,b], \mathbb{M}_{n\times n})$ it follows that the *skew*-symmetric part of the *tensor* field in the brackets in the equation above is zero. This gives the required conclusion. □

Proposition 3.3. *The Euler–Lagrange equation associated with* $\mathbb{E}_p$ *over* $\mathcal{E}_p$ *can be alternatively expressed as*

$$\int_a^b \int_{\mathbb{S}^{n-1}} \left\langle \left\{ \frac{d}{dr}\big(r^{n+1}\mathbf{sA}\big) \right\}\theta, \big(\mathbf{F} - \mathbf{F}^t\big)\theta \right\rangle d\mathcal{H}^{n-1}(\theta)\, dr = 0$$

for all $\mathbf{F} \in C_0^\infty(]a,b[, \mathbb{M}_{n\times n})$ *where* $\mathbf{A} = \mathbf{Q}^t\dot{\mathbf{Q}}$ *and* $\mathbf{s} = (n + r^2|\mathbf{A}\theta|^2)^{\frac{p-2}{2}}$.

Proof. Referring to the proof of Proposition 3.2 and making the substitutions described above for **A** and **s** we can write

$$\begin{aligned}
0 = \frac{d}{d\varepsilon}\mathbb{E}_p[\mathbf{Q}_\varepsilon]\bigg|_{\varepsilon=0} &= \mathbf{I} \\
&= \int_a^b \left\{ \frac{\partial \mathbf{E}}{\partial \xi}(r, \dot{\mathbf{Q}}) : \big(\dot{\mathbf{F}} - \dot{\mathbf{F}}^t\big)\mathbf{Q} \right\} r^{n-1}\, dr \\
&= \int_a^b \int_{\mathbb{S}^{n-1}} p\big\langle r^{n+1}\mathbf{sA}\theta, \big(\dot{\mathbf{F}} - \dot{\mathbf{F}}^t\big)\theta \big\rangle\, d\mathcal{H}^{n-1}(\theta)\, dr \\
&= \int_a^b \int_{\mathbb{S}^{n-1}} -p\left\langle \left\{ \frac{d}{dr}\big(r^{n+1}\mathbf{sA}\big) \right\}\theta, \big(\mathbf{F} - \mathbf{F}^t\big)\theta \right\rangle d\mathcal{H}^{n-1}(\theta)\, dr
\end{aligned}$$

which is the required conclusion. □

Any twist *loop* forming a solution to the Euler–Lagrange equation associated with $\mathbb{E}_p$ over $\mathcal{E}_p$ (as described in the above proposition) will be referred to as a p-*stationary* loop.

Remark 3.1. In view of Proposition 3.3 a *sufficient* condition for an admissible loop $\mathbf{Q} \in \mathcal{E}_p$ to be p-*stationary* is the *stronger* condition

$$\frac{d}{dr}\big(r^{n+1}\mathbf{sA}\big) = 0. \tag{3.2}$$

Interestingly for $p = 2$ the latter is *equivalent* to the Euler–Lagrange equation described in Proposition 3.3 (*see* [10]). However, in general, i.e., for $p \neq 2$, this need not be the case as in the *original* Euler–Lagrange equation the function **s** depends on both r and θ.[5]

[5] In fact, *if*, **s** were to be *independent* of θ then the Euler–Lagrange equation described in Proposition 3.3 could be easily shown to be *equivalent* to (3.2).

4. Minimising p-stationary loops

Consider as in the previous section for $p \in [1, \infty[$ the energy functional

$$\mathbb{E}_p[\mathbf{Q}] = \int_a^b \mathbf{E}(r, \dot{\mathbf{Q}}) r^{n-1}\, dr,$$

with the *integrand*

$$\mathbf{E}(r, \xi) = \int_{\mathbb{S}^{n-1}} \left(n + r^2|\xi\theta|^2\right)^{\frac{p}{2}}\, d\mathcal{H}^{n-1}(\theta),$$

over the space of *admissible* loops

$$\mathcal{E}_p = \left\{\mathbf{Q} = \mathbf{Q}(r):\ \mathbf{Q} \in W^{1,p}\left([a,b], \mathbf{SO}(n)\right),\ \mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n\right\}.$$

According to an elementary version of *Sobolev* embedding theorem any $\mathbf{Q} \in \mathcal{E}_p$ has a *continuous* representative (again denoted $\mathbf{Q}$). Thus each such $\mathbf{Q}$ represents an element of the fundamental group $\pi_1[\mathbf{SO}(n)]$ which is denoted by $]\mathbf{Q}[$. As is well known (*see*, e.g., [4])

$$\pi_1\left[\mathbf{SO}(n)\right] \cong \begin{cases} \mathbb{Z} & \text{when } n = 2, \\ \mathbb{Z}_2 & \text{when } n \geqslant 3, \end{cases}$$

and so these facts combined enable one to introduce the following *partitioning* of the loop space $\mathcal{E}_p$.

(**1**) ($n = 2$) for each $m \in \mathbb{Z}$ put

$$\mathfrak{c}_m[\mathcal{E}_p] := \left\{\mathbf{Q} \in \mathcal{E}_p:\ ]\mathbf{Q}[= m\right\}. \tag{4.1}$$

As a result the latter are *pairwise* disjoint and that

$$\mathcal{E}_p = \bigcup_{m \in \mathbb{Z}} \mathfrak{c}_m[\mathcal{E}_p].$$

(**2**) ($n \geqslant 3$) for each $\alpha \in \mathbb{Z}_2 = \{0, 1\}$ put

$$\mathfrak{c}_\alpha[\mathcal{E}_p] := \left\{\mathbf{Q} \in \mathcal{E}_p:\ ]\mathbf{Q}[= \alpha\right\}. \tag{4.2}$$

As a result, again, the latter are *pairwise* disjoint and that

$$\mathcal{E}_p = \bigcup_{\alpha \in \mathbb{Z}_2} \mathfrak{c}_\alpha[\mathcal{E}_p].$$

When $p > 1$ an application of the direct methods of the *calculus of variations* to the energy functional $\mathbb{E}_p$ together with the observation that the homotopy classes $\mathfrak{c}_\star[\mathcal{E}_p] \subset \mathcal{E}_p$ are *sequentially* weakly closed gives the existence of [multiple] *minimising* p-stationary loops.[6]

The only missing ingredient in this regard is the following statement implying the *coercivity* of $\mathbb{E}_p$ over $\mathcal{E}_p$.

Proposition 4.1. *Let* $p \in [1, \infty[$. *Then there exists* $c = c(n, p) > 0$ *such that*

$$\int_{\mathbb{S}^{n-1}} |\mathbf{F}\theta|^p\, d\mathcal{H}^{n-1}(\theta) \geqslant c|\mathbf{F}|^p,$$

for every $\mathbf{F} \in \mathbb{M}_{n\times n}$.

[6] The *sequential* weak closedness of the homotopy classes $\mathfrak{c}_\star[\mathcal{E}_p]$ is a result of $\mathbf{SO}(n)$ having a *tubular* neighbourhood that projects back onto itself and this in turn follows from $\mathbf{SO}(n)$ being a smooth *compact* manifold.

Proof. Fix $\mathbf{F} \in \mathbb{M}_{n\times n}$. Then the non-negative *symmetric* matrix $\mathbf{F}^t\mathbf{F}$ is orthogonally *diagonalisable*, that is, $\mathbf{F}^t\mathbf{F} = \mathbf{P}^t\mathbf{D}\mathbf{P}$ where $\mathbf{D} = \operatorname{diag}(\lambda_1[\mathbf{F}^t\mathbf{F}], \dots, \lambda_n[\mathbf{F}^t\mathbf{F}])$ and $\mathbf{P} \in \mathbf{O}(n)$. As a result for $\theta \in \mathbb{S}^{n-1}$ we can write

$$|\mathbf{F}\theta| = \left|\langle \mathbf{F}\theta, \mathbf{F}\theta\rangle\right|^{\frac{1}{2}} = \left|\langle \mathbf{F}^t\mathbf{F}\theta, \theta\rangle\right|^{\frac{1}{2}} = \left|\langle \mathbf{P}^t\mathbf{D}\mathbf{P}\theta, \theta\rangle\right|^{\frac{1}{2}} = \left|\langle \mathbf{D}\mathbf{P}\theta, \mathbf{P}\theta\rangle\right|^{\frac{1}{2}}.$$

Setting $\omega := \mathbf{P}\theta$ and noting that $\mathbf{O}(n)$ acts as the group of *isometries* on $\mathbb{S}^{n-1}$, an application of Jensen's inequality followed by Hölder inequality [on *finite* sequences] gives

$$\begin{aligned}
\left\{ \fint_{\mathbb{S}^{n-1}} |\mathbf{F}\theta|^p \, d\mathcal{H}^{n-1}(\theta) \right\}^{\frac{1}{p}} &\geqslant \fint_{\mathbb{S}^{n-1}} |\mathbf{F}\theta| \, d\mathcal{H}^{n-1}(\theta) \\
&\geqslant \fint_{\mathbb{S}^{n-1}} \left\{ \sum_{j=1}^n \lambda_j\left[\mathbf{F}^t\mathbf{F}\right]\omega_j^2(\theta) \right\}^{\frac{1}{2}} d\mathcal{H}^{n-1}(\theta) \\
&\geqslant \frac{1}{\sqrt{n}} \sum_{j=1}^n \lambda_j^{\frac{1}{2}}\left[\mathbf{F}^t\mathbf{F}\right] \fint_{\mathbb{S}^{n-1}} \left|\omega_j(\theta)\right| d\mathcal{H}^{n-1}(\theta) \\
&\geqslant \frac{\alpha_n}{\sqrt{n}} \left\{ \sum_{j=1}^n \lambda_j\left[\mathbf{F}^t\mathbf{F}\right] \right\}^{\frac{1}{2}} = \frac{\alpha_n}{\sqrt{n}} |\mathbf{F}|.
\end{aligned}$$

Hence the conclusion follows with the choice of

$$c = \alpha_n^p n^{1-\frac{p}{2}} \omega_n = \min_{1\leqslant j\leqslant n} \left\{ \fint_{\mathbb{S}^{n-1}} |\theta_j| \, d\mathcal{H}^{n-1}(\theta) \right\}^p n^{1-\frac{p}{2}} \omega_n > 0. \qquad \square$$

Proposition 4.2. *Let $p \in [1, \infty[$. Then there exists $d = d(n, p, \Omega) > 0$ such that*

$$\mathbb{E}_p[\mathbf{Q}] \geqslant d\|\mathbf{Q}\|_{1,p}^p$$

for all $\mathbf{Q} \in \mathcal{E}_p$.

Proof. In view of Proposition 4.1 it is enough to note that for $\mathbf{Q} \in \mathcal{E}_p$ we can write

$$\begin{aligned}
\mathbb{E}_p[\mathbf{Q}] &= \int_a^b \int_{\mathbb{S}^{n-1}} \left(n + r^2|\dot{\mathbf{Q}}\theta|^2\right)^{\frac{p}{2}} r^{n-1} \, d\mathcal{H}^{n-1}(\theta) \, dr \\
&\geqslant \int_a^b \int_{\mathbb{S}^{n-1}} r^{p+n-1} |\dot{\mathbf{Q}}\theta|^p \, d\mathcal{H}^{n-1}(\theta) \, dr \\
&\geqslant c \int_a^b r^{p+n-1} |\dot{\mathbf{Q}}|^p \, dr,
\end{aligned}$$

and so the conclusion follows by an application of Poincaré inequality. $\square$

Theorem 4.1. *Let $p \in]1, \infty[$. Then the following hold.*

(1) $(n = 2)$ *for each $m \in \mathbb{Z}$ there exists $\mathbf{Q}_m \in \mathfrak{c}_m[\mathcal{E}_p]$ such that*

$$\mathbb{E}_p[\mathbf{Q}_m] = \inf_{\mathfrak{c}_m[\mathcal{E}_p]} \mathbb{E}_p,$$

(2) ($n \geqslant 3$) *for each* $\alpha \in \mathbb{Z}_2$ *there exists* $\mathbf{Q}_\alpha \in \mathfrak{c}_\alpha[\mathcal{E}_p]$ *such that*

$$\mathbb{E}_p[\mathbf{Q}_\alpha] = \inf_{\mathfrak{c}_\alpha[\mathcal{E}_p]} \mathbb{E}_p.$$

In either case the resulting minimisers satisfy the corresponding Euler–Lagrange equations (3.1).

We return to the question of existence of *multiple* p-stationary loops having *specific* relevance to the *original* energy functional $\mathbb{F}_p$ over the space $\mathcal{A}_p$ towards the end of the paper. Before this, however, we pause to discuss in detail the implications that the *original* Euler–Lagrange equations [*see* Definition 5.1 below] will exert upon the *twist* loop associated with a *generalised* twist.

5. Generalised twists as classical solutions

The aim of this section is to give a complete *characterisation* of *all* those p-stationary loops $\mathbf{Q} \in \mathcal{E}_p$ whose resulting *generalised* twist

$$\mathbf{u} = \mathbf{Q}(r)\mathbf{x}$$

furnishes a solution to the Euler–Lagrange equations associated with the energy functional $\mathbb{F}_p$ over the space $\mathcal{A}_p$. To this end we begin by *clarifying* the notion of a [*classical*] solution.

Definition 5.1 *(Classical solution).* A pair $(\mathbf{u}, \mathfrak{p})$ is said to be a *classical* solution to the Euler–Lagrange equations associated with the energy functional (1.1) and subject to the constraint (1.2) if and only if

(1) $\mathbf{u} \in C^2(\Omega, \mathbb{R}^n) \cap C(\bar{\Omega}, \mathbb{R}^n)$,
(2) $\mathfrak{p} \in C^1(\Omega) \cap C(\bar{\Omega})$, and
(3) $(\mathbf{u}, \mathfrak{p})$ satisfy the *system* of equations[7]

$$\begin{cases} [cof\, \nabla\mathbf{u}(\mathbf{x})]^{-1}\Delta_p \mathbf{u}(\mathbf{x}) = \nabla\mathfrak{p}(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \det \nabla\mathbf{u}(\mathbf{x}) = 1, & \mathbf{x} \in \Omega, \\ \mathbf{u}(\mathbf{x}) = \mathbf{x}, & \mathbf{x} \in \partial\Omega. \end{cases}$$

In view of Proposition 2.3 the task outlined at the start of this section amounts to verifying that under what *additional* conditions would the *vector* field described by the expression on the *right* in (2.6) be a gradient. The answer to this question is given by the following *two* theorems.

Theorem 5.1. *Let* $\Omega = \{\mathbf{x} \in \mathbb{R}^n\colon\ a < |\mathbf{x}| < b\}$ *and consider the vector field* $\mathbf{v} \in C^1(\Omega, \mathbb{R}^n)$ *defined in spherical coordinates through*

$$\mathbf{v} = \left\{ r\mathbf{s}\mathbf{A}^2 - r^2\mathbf{s}\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle \mathbf{I}_n + \frac{1}{r^n}\frac{d}{dr}\left(r^{n+1}\mathbf{s}\mathbf{A}\right) + \frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n+1}\mathbf{s}|\mathbf{A}\theta|^2\right)\mathbf{I}_n \right\}\theta$$

where $r \in]a, b[$, $\theta \in \mathbb{S}^{n-1}$, $\mathbf{A} = \mathbf{A}(r) \in C^1(]a, b[, \mathbb{M}_{n\times n})$ *is skew-symmetric and*

$$\begin{aligned} \mathbf{s} &= \mathbf{s}(r, \theta) \\ &=: \left(n + r^2|\mathbf{A}\theta|^2\right)^{\frac{p-2}{2}} \end{aligned} \tag{5.1}$$

with $p \in [1, \infty[$. *Then the following are equivalent.*

(1) $\mathbf{v}$ *is a gradient,*
(2) $\mathbf{A}^2 = -\sigma \mathbf{I}_n$ *for some* $\sigma \in C^1]a, b[$ *with* $\sigma \geqslant 0$ *and*

$$\frac{d}{dr}\left(r^{n+1}\mathbf{s}\mathbf{A}\right) = 0. \tag{5.2}$$

[7] Note that $\Delta_p \mathbf{u} := div(|\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u})$.

Proof. (**2**) ⇒ (**1**) Assuming $\mathbf{A}$ to be *skew*-symmetric and $\mathbf{A}^2 = -\sigma \mathbf{I}_n$ it follows that

$$\begin{aligned}\mathbf{s} &= \left(n + r^2|\mathbf{A}\theta|^2\right)^{\frac{p-2}{2}} \\ &= \left(n - r^2\langle \mathbf{A}^2\theta, \theta\rangle\right)^{\frac{p-2}{2}} \\ &= \left(n + \sigma r^2\right)^{\frac{p-2}{2}}\end{aligned}$$

and so in particular $\mathbf{s} = \mathbf{s}(r)$. Now referring to (5.2) we can write

$$\begin{aligned}0 &= \frac{1}{r^n}\left\langle \frac{d}{dr}\left(r^{n+1}\mathbf{s}\mathbf{A}\right)\theta, \mathbf{A}\theta\right\rangle \\ &= (n+1)\mathbf{s}|\mathbf{A}\theta|^2 + r\mathbf{s}_r|\mathbf{A}\theta|^2 + r\mathbf{s}\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle \\ &= \frac{1}{r^n}\frac{d}{dr}\left(r^{n+1}\mathbf{s}|\mathbf{A}\theta|^2\right) - r\mathbf{s}\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle. \end{aligned} \tag{5.3}$$

As a result the vector field $\mathbf{v}$ can be simplified and hence *re-written* in the form

$$\mathbf{v} = r\mathbf{s}\mathbf{A}^2\theta = -\left(n + \sigma r^2\right)^{\frac{p-2}{2}}\sigma\theta.$$

Denoting now by F a suitable *primitive* of $f(r) := -(n + \sigma r^2)^{\frac{p-2}{2}}\sigma$ it is evident that

$$\mathbf{v} = \nabla F$$

and so $\mathbf{v}$ is a gradient. This gives (**1**).

(**1**) ⇒ (**2**) For the sake of clarity and convenience we break this part into *two* steps. In the *first* step we establish (5.2) and in the *second* one the particular *diagonal* form of $\mathbf{A}^2$.[8]

Step 1. [Justification of (5.2)] We begin by *extracting* a gradient out of $\mathbf{v}$ and hence *re-writting* it in the form

$$\mathbf{v} = \nabla\mathbf{t} + \left\{\frac{1}{r^n}\frac{d}{dr}\left(r^{n+1}\mathbf{s}\mathbf{A}\right) + \frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n+1}\mathbf{s}|\mathbf{A}\theta|^2\right)\mathbf{I}_n\right\}\theta \tag{5.4}$$

where $\mathbf{t} = -p^{-1}(n + r^2|\mathbf{A}\theta|^2)^{\frac{p}{2}}$.

To the vector field $\mathbf{v} = (v_1, \ldots, v_n)$ we now assign the differential 1-*form* $\omega = v_1 dx_1 + \cdots + v_n dx_n$. Then in view of $\mathbf{v}$ being a *gradient*, for any *closed* path $\gamma \in C^1([0, 2\pi], \mathbb{S}^{n-1})$ it *must* be that

$$\begin{aligned}0 &= \int_{r\gamma} \omega \\ &= \int_0^{2\pi} \left\langle \mathbf{v}\left(r\gamma(t)\right), r\gamma'(t)\right\rangle dt \\ &= \frac{1}{r^n}\int_0^{2\pi} \left\langle \frac{d}{dr}\left[r^{n+1}\mathbf{s}\left(r, \gamma(t)\right)\mathbf{A}\right]\gamma(t), r\gamma'(t)\right\rangle dt \\ &\quad + \frac{1}{r^{n-1}}\int_0^{2\pi} \left\langle \frac{d}{dr}\left[r^{n+1}\mathbf{s}\left(r, \gamma(t)\right)\left|\mathbf{A}\gamma(t)\right|^2\right]\gamma(t), r\gamma'(t)\right\rangle dt \\ &= \frac{1}{r^n}\int_0^{2\pi} \left\langle \frac{d}{dr}\left[r^{n+1}\mathbf{s}\left(r, \gamma(t)\right)\mathbf{A}\right]\gamma(t), r\gamma'(t)\right\rangle dt, \end{aligned} \tag{5.5}$$

[8] Thus it is important to note that in the first *two* steps the function $\mathbf{s}$ depends on both r and θ!

where in concluding the *last* line we have used the *pointwise* identity $\langle\gamma,\gamma'\rangle = 0$ which holds as a result of γ taking values on $\mathbb{S}^{n-1}$ and consequently implying that

$$\begin{aligned}0 &= \int_0^{2\pi} \left\langle \frac{d}{dr}\left[r^{n+1}\mathbf{s}\big(r,\gamma(t)\big)\big|\mathbf{A}\gamma(t)\big|^2\right]\gamma(t), r\gamma'(t)\right\rangle dt \\ &= \int_0^{2\pi} \frac{d}{dr}\left[r^{n+1}\mathbf{s}\big(r,\gamma(t)\big)\big|\mathbf{A}\gamma(t)\big|^2\right] r\big\langle\gamma(t),\gamma'(t)\big\rangle\, dt.\end{aligned}$$

Anticipating on (5.2) we *first* note that in view of **A** being *skew*-symmetric it can be *orthogonally* diagonalised, i.e.,[9]

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^t, \tag{5.6}$$

where $\mathbf{P} = \mathbf{P}(r) \in \mathbf{SO}(n)$ and $\mathbf{D} = \mathbf{D}(r) \in \mathbb{M}_{n\times n}$ is in *special* block diagonal form, i.e.,

(**1**) $(n = 2k)$

$$\mathbf{D} = \mathrm{diag}(d_1\mathbf{J}, d_2\mathbf{J}, \ldots, d_k\mathbf{J}),$$

(**2**) $(n = 2k+1)$

$$\mathbf{D} = \mathrm{diag}(d_1\mathbf{J}, d_2\mathbf{J}, \ldots, d_k\mathbf{J}, 0),$$

with $\{\pm d_1 i, \pm d_2 i, \ldots, \pm d_k i\}$ or $\{\pm d_1 i, \pm d_2 i, \ldots, \pm d_k i, 0\}$ denoting the *eigen*-values of the *skew-symmetric* matrix **A** [as well as **D**] respectively.[10]

With the aid of (5.6) and for the sake of convenience we now introduce the *skew-symmetric* matrix

$$\mathbf{F} = \mathbf{F}(r,\theta) := \mathbf{P}^t \frac{d}{dr}\big(r^{n+1}\mathbf{s}\mathbf{A}\big)\mathbf{P}. \tag{5.7}$$

Then a straight-forward *differentiation* shows that

$$\begin{aligned}\mathbf{F} &= \mathbf{P}^t \frac{d}{dr}\big(r^{n+1}\mathbf{s}\mathbf{A}\big)\mathbf{P} \\ &= \mathbf{P}^t\big\{r^n\big[(n+1)\mathbf{s} + r\mathbf{s}_r\big]\mathbf{A} + r^{n+1}\mathbf{s}\dot{\mathbf{A}}\big\}\mathbf{P} \\ &= \mathbf{P}^t\big\{r^n\big[(n+1)\mathbf{s} + r\mathbf{s}_r\big]\mathbf{P}\mathbf{D}\mathbf{P}^t + r^{n+1}\mathbf{s}\dot{\mathbf{A}}\big\}\mathbf{P} \\ &= r^n\big[(n+1)\mathbf{s} + r\mathbf{s}_r\big]\mathbf{D} + r^{n+1}\mathbf{s}\mathbf{P}^t\dot{\mathbf{A}}\mathbf{P}.\end{aligned} \tag{5.8}$$

Evidently establishing (5.2) is *equivalent* to showing that

$$\mathbf{F}(r,\theta) = 0 \tag{5.9}$$

for *all* $r \in\,]a,b[$ and *all* $\theta \in \mathbb{S}^{n-1}$.

On the other hand for each *fixed* $r \in\,]a,b[$ setting $\omega := \mathbf{P}^t\gamma$ [also a *closed* path in $C^1([0,2\pi],\mathbb{S}^{n-1})$] in (5.5) we have that expressed as

$$0 = \int_0^{2\pi} \left\langle \frac{d}{dr}\big(r^{n+1}\mathbf{s}\mathbf{A}\big)\gamma, \gamma'\right\rangle dt$$

[9] At this stage the reader is encouraged to consult Appendix A at the end of the paper where some *notation* as well as basic properties related to the matrix *exponential* as a mapping between the space of *skew-symmetric* matrices and the *special* orthogonal group is discussed.

[10] We emphasise that *nowhere* in this proof have we assumed *continuity* or *differentiability* on $\mathbf{P} = \mathbf{P}(r)$ or $\mathbf{D} = \mathbf{D}(r)$ with respect to r. These in general need not even be true! [*See*, e.g., [7].]

$$= \int_0^{2\pi} \left\langle \frac{d}{dr}\left(r^{n+1}\mathbf{sA}\right)\mathbf{P}\omega, \mathbf{P}\omega' \right\rangle dt$$

$$= \int_0^{2\pi} \left\langle \mathbf{P}^t \frac{d}{dr}\left(r^{n+1}\mathbf{sA}\right)\mathbf{P}\omega, \omega' \right\rangle dt$$

$$= \int_0^{2\pi} \langle \mathbf{F}\omega, \omega' \rangle \, dt$$

where in the above $\mathbf{s} = \mathbf{s}(r, \mathbf{P}\omega)$ and $\mathbf{F} = \mathbf{F}(r, \mathbf{P}\omega)$. Thus the *necessary* condition (5.5) can be equivalently expressed as

$$\int_0^{2\pi} \left\langle \mathbf{F}(r, \mathbf{P}\omega)\omega, \omega' \right\rangle dt = 0 \tag{5.10}$$

for every *closed* path $\omega \in C^1([0, 2\pi], \mathbb{S}^{n-1})$.

With this introduction the conclusion in Step 1 now amounts to proving the implication

$$(5.10) \implies (5.9).$$

This will be established below in a *componentwise* fashion. Note that in view of the *skew-symmetry* of $\mathbf{F}$ it suffices to justify the latter in the form $\mathbf{F}_{pq}(r, \theta) = 0$ *only* when $1 \leqslant p < q \leqslant n$.

Indeed consider a *parameterised* family of *closed* paths $\rho \in C^\infty([0, 2\pi], \mathbb{S}^{n-1})$ given by

$$\rho : [0, 2\pi] \ni t \mapsto \rho(t) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n \tag{5.11}$$

with

$$\begin{cases} \rho_1 = \sin t \sin\phi_2 \sin\phi_3 \cdots \sin\phi_{n-1}, \\ \rho_2 = \cos t \sin\phi_2 \sin\phi_3 \cdots \sin\phi_{n-1}, \\ \rho_3 = \cos\phi_2 \sin\phi_3 \cdots \sin\phi_{n-1}, \\ \vdots \\ \rho_{n-1} = \cos\phi_{n-2} \sin\phi_{n-1}, \\ \rho_n = \cos\phi_{n-1}, \end{cases}$$

where $\phi_j \in [0, \pi]$ for all $2 \leqslant j \leqslant n-1$. For *fixed* $1 \leqslant p < q \leqslant n$ we introduce the matrix Γ^{pq} as that obtained by *simultaneously* interchanging the *first* and pth and the *second* and qth rows of $\mathbf{I}_n$, i.e.,

$$\Gamma^{pq} e_j = \begin{cases} e_p & \text{if } j = 1, \\ e_1 & \text{if } j = p, \\ e_q & \text{if } j = 2, \\ e_2 & \text{if } j = q, \\ e_j & \text{otherwise,} \end{cases}$$

where $\{e_1, e_2, \ldots, e_n\}$ denotes the *standard* basis of $\mathbb{R}^n$. In view of $\Gamma^{pq} \in \mathbf{O}(n)$ setting $\omega = \Gamma^{pq}\rho$ it is clear that ω is a *closed* path in $C^\infty([0, 2\pi], \mathbb{S}^{n-1})$.

Claim 1. *For any skew-symmetric matrix $\mathbf{F} \in \mathbb{M}_{n\times n}$ and $\omega = \Gamma^{pq}\rho$ as above we have that*

$$\int_0^{2\pi} \left\langle \mathbf{F}\omega(t), \omega'(t) \right\rangle dt = 2\pi\left(\rho_1^2 + \rho_2^2\right)\mathbf{F}_{pq}.$$

The proof of this claim follows by direct *verification* noting that here $\omega'(t) = \Gamma^{pq}\rho'(t) = \Gamma^{pq}(\rho_2, -\rho_1, 0, \ldots, 0)$.

We now proceed by substituting ω as described above into (5.10) and then considering the following two *distinct* cases.

(**1**) ($p = 2j-1$, $q = 2j$ for some $1 \leqslant j \leqslant k = [n/2]$) In this case by utilising the *special* block diagonal form of $\mathbf{D}$ a straight-forward calculation shows that

$$\begin{aligned}\mathbf{s} &= \mathbf{s}\big(r, \mathbf{P}\omega(t)\big)\\ &= \big(n - r^2\big\langle \mathbf{D}^2\omega(t), \omega(t)\big\rangle\big)^{\frac{p-2}{2}}\\ &= \big(n - r^2\big\langle \mathbf{D}^2\Gamma^{pq}\rho(t), \Gamma^{pq}\rho(t)\big\rangle\big)^{\frac{p-2}{2}}\\ &= \big(n + r^2\big[d_1^2\rho_p^2 + d_1^2\rho_q^2 + \cdots + d_j^2\big(\rho_1^2+\rho_2^2\big) + \cdots\big]\big)^{\frac{p-2}{2}}\end{aligned}$$

is indeed *independent* of the t *variable* [as $\rho_1^2 + \rho_2^2$ does *not* depend on t]. Hence the same is true of $\mathbf{F}(r, \mathbf{P}\omega)$ and so referring to (5.10) and utilising Claim 1 we can write

$$\begin{aligned}0 &= \int_0^{2\pi} \big\langle \mathbf{F}(r, \mathbf{P}\omega)\omega, \omega'\big\rangle\, dt\\ &= \int_0^{2\pi} \big\langle \mathbf{F}\big(r, \mathbf{P}\Gamma^{pq}\rho(t)\big)\Gamma^{pq}\rho(t), \Gamma^{pq}\rho'(t)\big\rangle\, dt\\ &= 2\pi\big(\rho_1^2+\rho_2^2\big)\mathbf{F}_{pq}(r, \mathbf{P}\omega)\end{aligned}$$

which in turn for $\rho_1^2 + \rho_2^2 \neq 0$ gives[11]

$$\mathbf{F}_{pq}(r, \mathbf{P}\omega) = 0. \tag{5.12}$$

Now to get (5.9) for the latter choice of p, q pick $\theta \in \mathbb{S}^{n-1}$ and set $\alpha = [\Gamma^{pq}]^t\mathbf{P}^t\theta$. Then $\alpha \in \mathbb{S}^{n-1}$ and thus can be written in *generalised* spherical coordinates as

$$\begin{cases}\alpha_1 = \sin\phi_1 \sin\phi_2 \sin\phi_3 \cdots \sin\phi_{n-1},\\ \alpha_2 = \cos\phi_1 \sin\phi_2 \sin\phi_3 \cdots \sin\phi_{n-1},\\ \alpha_3 = \cos\phi_2 \sin\phi_3 \cdots \sin\phi_{n-1},\\ \vdots\\ \alpha_{n-1} = \cos\phi_{n-2}\sin\phi_{n-1},\\ \alpha_n = \cos\phi_{n-1},\end{cases}$$

where $\phi_1 \in [0, 2\pi]$ and $\phi_j \in [0, \pi]$ for all $2 \leqslant j \leqslant n-1$. Considering now the closed *path* ρ in (5.11) for the latter choice of *parameters* $\phi_2, \ldots, \phi_{n-1}$ a straight-forward calculation gives

$$\begin{aligned}\mathbf{s}(r,\theta) &= \big(n + r^2|\mathbf{A}\theta|^2\big)^{\frac{p-2}{2}}\\ &= \big(n + r^2\big|\mathbf{D}\Gamma^{pq}\alpha\big|^2\big)^{\frac{p-2}{2}}\\ &= \big(n + r^2\big|\mathbf{D}\Gamma^{pq}\rho\big|^2\big)^{\frac{p-2}{2}}\\ &= \big(n + r^2|\mathbf{A}\mathbf{P}\omega|^2\big)^{\frac{p-2}{2}}\\ &= \mathbf{s}(r, \mathbf{P}\omega)\end{aligned}$$

and so referring to (5.12) for $\rho_1^2 + \rho_2^2 \neq 0$ we obtain

$$\mathbf{F}_{pq}(r,\theta) = \mathbf{F}_{pq}(r, \mathbf{P}\omega) = 0$$

as required.

[11] Note that $(\rho_1^2 + \rho_2^2) = \prod_{2\leqslant j\leqslant n-1} \sin^2\phi_j$ and so $\rho_1^2 + \rho_2^2 = 0 \iff \sum_{3\leqslant j\leqslant n}\rho_j^2 = 1 \iff \phi_j \in \{0, \pi\}$ for some $2 \leqslant j \leqslant n-1$. This set is a copy of $\mathbb{S}^{n-3}$ lying in $\mathbb{S}^{n-1}$.

(**2**) (p, q not as in (**1**)) Unlike the case with (**1**) here $\mathbf{s}$ depends *explicitly* on the t *variable* [yet in a *specific* manner (*see* below)] whilst $\mathbf{D}_{pq} = 0$ as can be verified by inspecting its block diagonal representation.

Now referring, again, to (5.10) and noting that the pth and qth components of ω' are given by $\omega'_p = \rho'_1 = \rho_2$ and $\omega'_q = \rho'_2 = -\rho_1$ [with *all* the remaining derivatives vanishing] we can write using $\mathbf{F} = \mathbf{F}(r, \mathbf{P}\omega)$

$$
\begin{aligned}
0 &= \int_0^{2\pi} \langle \mathbf{F}\omega, \omega' \rangle \, dt \\
&= \int_0^{2\pi} \left\{ \sum_{j=1}^{n} \mathbf{F}_{pj}\omega_j\omega'_p + \sum_{j=1}^{n} \mathbf{F}_{qj}\omega_j\omega'_q \right\} dt \\
&= \int_0^{2\pi} \left\{ \left(\mathbf{F}_{pq}\rho_2^2 - \mathbf{F}_{qp}\rho_1^2\right) + \rho_2 \sum_{\substack{j=1 \\ j\neq q}}^{n} \mathbf{F}_{pj}\omega_j - \rho_1 \sum_{\substack{j=1 \\ j\neq p}}^{n} \mathbf{F}_{qj}\omega_j \right\} dt \\
&= \mathbf{I} + \mathbf{II} - \mathbf{III}.
\end{aligned} \tag{5.13}
$$

In order to evaluate the above terms we *first* observe that here $\mathbf{s}$ takes the form

$$
\begin{aligned}
\mathbf{s} &= \mathbf{s}\big(r, \mathbf{P}\omega(t)\big) \\
&= \left(n - r^2\big\langle \mathbf{D}^2\omega(t), \omega(t)\big\rangle\right)^{\frac{p-2}{2}} \\
&= \left(n - r^2\big\langle \mathbf{D}^2\Gamma^{pq}\rho(t), \Gamma^{pq}\rho(t)\big\rangle\right)^{\frac{p-2}{2}} \\
&= \left(n + r^2\big[d_1^2\rho_p^2 + d_2^2\rho_q^2 + \cdots + d_\xi^2\rho_1^2 + \cdots + d_\zeta^2\rho_2^2 + \cdots\big]\right)^{\frac{p-2}{2}} \\
&=: \mathfrak{s}\big(\sin^2 t, \cos^2 t\big).
\end{aligned} \tag{5.14}
$$

Returning to (5.13) we have that

$$
\begin{aligned}
\mathbf{II} &= \int_0^{2\pi} \rho_2 \sum_{\substack{j=1 \\ j\neq q}}^{n} \mathbf{F}_{pj}\omega_j \, dt \\
&= \int_0^{2\pi} \rho_2 \sum_{\substack{j=1 \\ j\neq q}}^{n} \left[\mathbf{P}^t \frac{d}{dr}\big(r^{n+1}\mathbf{s}\mathbf{A}\big)\mathbf{P}\right]_{pj} \omega_j \, dt \\
&= \sum_{\substack{j=1 \\ j\neq q}}^{n} \left[\mathbf{P}^t \frac{d}{dr}\left(r^{n+1}\left\{\int_0^{2\pi} \rho_2 \mathbf{s} \, dt\right\}\mathbf{A}\right)\mathbf{P}\right]_{pj} \omega_j,
\end{aligned}
$$

and in a similar way

$$
\begin{aligned}
\mathbf{III} &= \int_0^{2\pi} \rho_1 \sum_{\substack{j=1 \\ j\neq p}}^{n} \mathbf{F}_{qj}\omega_j \, dt \\
&= \int_0^{2\pi} \rho_1 \sum_{\substack{j=1 \\ j\neq p}}^{n} \left[\mathbf{P}^t \frac{d}{dr}\big(r^{n+1}\mathbf{s}\mathbf{A}\big)\mathbf{P}\right]_{qj} \omega_j \, dt
\end{aligned}
$$

$$= \sum_{\substack{j=1\\ j\neq p}}^{n} \left[\mathbf{P}^t \frac{d}{dr}\left(r^{n+1} \left\{ \int_0^{2\pi} \rho_1 \mathbf{s}\, dt \right\} \mathbf{A} \right) \mathbf{P} \right]_{qj} \omega_j,$$

where in concluding the *last* line in both equalities we have used the fact that the only components of ω depending explicitly on the t *variable* are $\omega_p = \rho_1$ and $\omega_q = \rho_2$ where in each case *one* is excluded from the *summation* sign and the other has a *zero* coefficient in view of the *skew-symmetry* of the matrix preceding it.

However in view of the specific manner in which $\mathbf{s}$ depends on t [*see* (5.14)] it follows that both integrals *vanish* and so as a result $\mathbf{II} = \mathbf{III} = 0$.[12] Hence returning to (5.13) and utilising the *skew*-symmetry on $\mathbf{F}$ and (5.8) we can write

$$\begin{aligned}
\mathbf{I} &= \int_0^{2\pi} \left(\mathbf{F}_{pq}\rho_2^2 - \mathbf{F}_{qp}\rho_1^2 \right) dt \\
&= \int_0^{2\pi} \left(\rho_1^2 + \rho_2^2 \right) \mathbf{F}_{pq}\, dt \\
&= \int_0^{2\pi} r^{n+1} \left(\rho_1^2 + \rho_2^2 \right) \mathbf{s} \left[\mathbf{P}^t \dot{\mathbf{A}} \mathbf{P} \right]_{pq} dt \\
&= r^{n+1} \left(\rho_1^2 + \rho_2^2 \right) \left\{ \int_0^{2\pi} \mathbf{s}\, dt \right\} \left[\mathbf{P}^t \dot{\mathbf{A}} \mathbf{P} \right]_{pq} = 0.
\end{aligned}$$

Thus as $\mathbf{s} > 0$ for $\rho_1^2 + \rho_2^2 \neq 0$ it follows that $[\mathbf{P}^t \dot{\mathbf{A}} \mathbf{P}]_{pq} = 0$. Since for the latter range of p, q we have that $\mathbf{D}_{pq} = 0$ referring to (5.8) it immediately that $\mathbf{F}_{pq} = 0$.

Hence *summarising* we have shown that in both cases (**1**) and (**2**) for *fixed* $r \in]a, b[$ we have $\mathbf{F}_{pq}(r, \cdot) = 0$ outside a copy of $\mathbb{S}^{n-3}$. By *continuity* of $\mathbf{F}_{pq}(r, \cdot)$ on $\mathbb{S}^{n-1}$ this gives (5.9) and as a result (5.2). The proof of Step 1 is therefore complete.

Step 2. [$\mathbf{A}^2 = -\sigma \mathbf{I}_n$] Here we establish the *remaining* part of (**2**) namely that $\mathbf{A}^2 = -\sigma \mathbf{I}_n$ for some $\sigma \in C^1]a, b[$ with $\sigma \geqslant 0$. To this end, we *first* observe that by utilising (5.2) the vector field $\mathbf{v}$ can be considerably simplified and *re-written* in the form [as in (5.3)]

$$\mathbf{v} = r\mathbf{s}\mathbf{A}^2\theta.$$

Now for $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ to be a *gradient* it is necessary that the differential 1-*form* $\omega = v_1\, dx_1 + \cdots + v_n\, dx_n$ be *closed*. In other words $d\omega = 0$ which in turn amounts to

$$\frac{\partial v_q}{\partial x_p} - \frac{\partial v_p}{\partial x_q} = 0,$$

for all $1 \leqslant p, q \leqslant n$. Setting $\mathbf{F} = \mathbf{A}^2$ we have that

$$\frac{\partial v_q}{\partial x_p} = r \frac{\partial \mathbf{s}}{\partial x_p} [\mathbf{F}\theta]_q + r\mathbf{s}[\dot{\mathbf{F}}\theta]_q \theta_p + \mathbf{s}\mathbf{F}_{qp}$$

[12] It can be easily shown that as a result of periodicity the following identities hold:

$$\int_0^{2\pi} \mathfrak{s}\left(\sin^2 t, \cos^2 t\right) \sin t\, dt = 0,$$

$$\int_0^{2\pi} \mathfrak{s}\left(\sin^2 t, \cos^2 t\right) \cos t\, dt = 0.$$

and in a similar way

$$\frac{\partial v_p}{\partial x_q} = r\frac{\partial \mathbf{s}}{\partial x_q}[\mathbf{F}\theta]_p + r\mathbf{s}[\dot{\mathbf{F}}\theta]_p\theta_q + \mathbf{s}\mathbf{F}_{pq}.$$

Thus in view of the *symmetry* of $\mathbf{F}$ for the latter range of p, q we have that

$$\begin{aligned}0 &= \frac{\partial v_q}{\partial x_p} - \frac{\partial v_p}{\partial x_q}\\ &= r\frac{\partial \mathbf{s}}{\partial x_p}[\mathbf{F}\theta]_q - r\frac{\partial \mathbf{s}}{\partial x_q}[\mathbf{F}\theta]_p + r\mathbf{s}\big\{[\dot{\mathbf{F}}\theta\otimes\theta]_{qp} - [\dot{\mathbf{F}}\theta\otimes\theta]_{pq}\big\}.\end{aligned}$$

Alternatively using *tensor* notation this can be *simplified* in the form

$$\begin{aligned}0 &= \nabla\mathbf{s}\otimes\mathbf{F}\theta - \mathbf{F}\theta\otimes\nabla\mathbf{s}\\ &\quad + \mathbf{s}(\theta\otimes\dot{\mathbf{F}}\theta - \dot{\mathbf{F}}\theta\otimes\theta)\\ &= \frac{1}{2}\beta r^2\langle\dot{\mathbf{F}}\theta,\theta\rangle(\mathbf{F}\theta\otimes\theta - \theta\otimes\mathbf{F}\theta)\\ &\quad + \mathbf{s}(\theta\otimes\dot{\mathbf{F}}\theta - \dot{\mathbf{F}}\theta\otimes\theta)\end{aligned} \tag{5.15}$$

where in concluding the *second* identity we have used

$$\begin{aligned}\nabla\mathbf{s} &= \nabla\big(n + r^2|\mathbf{A}\theta|^2\big)^{\frac{p-2}{2}}\\ &= \nabla\big(n - r^2\langle\mathbf{F}\theta,\theta\rangle\big)^{\frac{p-2}{2}}\\ &= -\beta\left[\frac{1}{2}r^2\langle\dot{\mathbf{F}}\theta,\theta\rangle\mathbf{I}_n + r\mathbf{F}\right]\theta,\end{aligned}$$

with $\beta = \beta(r,\theta,p) := (p-2)(n - r^2\langle\mathbf{F}\theta,\theta\rangle)^{\frac{p-4}{2}}$. Next a straight-forward calculation using (5.2) gives

$$\dot{\mathbf{F}} = -2\left(\frac{n+1}{r} + \frac{\mathbf{s}_r}{\mathbf{s}}\right)\mathbf{F}. \tag{5.16}$$

Therefore substituting this into (5.15) results in

$$\begin{aligned}0 &= \frac{1}{2}\beta r^2\langle\dot{\mathbf{F}}\theta,\theta\rangle(\mathbf{F}\theta\otimes\theta - \theta\otimes\mathbf{F}\theta)\\ &\quad - \mathbf{s}(\dot{\mathbf{F}}\theta\otimes\theta - \theta\otimes\dot{\mathbf{F}}\theta)\\ &= \left\{2\left(\frac{n+1}{r} + \frac{\mathbf{s}_r}{\mathbf{s}}\right)\left(\mathbf{s} - \frac{1}{2}\beta r^2\langle\mathbf{F}\theta,\theta\rangle\right)\right\}(\mathbf{F}\theta\otimes\theta - \theta\otimes\mathbf{F}\theta)\\ &= \gamma\times(\mathbf{F}\theta\otimes\theta - \theta\otimes\mathbf{F}\theta)\end{aligned} \tag{5.17}$$

where for the sake of convenience we have introduced

$$\begin{aligned}\gamma &= \gamma(r,\theta,p)\\ &=: 2\left(\frac{n+1}{r} + \frac{\mathbf{s}_r}{\mathbf{s}}\right)\left(\mathbf{s} - \frac{1}{2}\beta r^2\langle\mathbf{F}\theta,\theta\rangle\right).\end{aligned} \tag{5.18}$$

Claim 2. *Let* $p \in [1,\infty[$. *Then* $\gamma = \gamma(r,\theta,p) > 0$ *for all* $r \in\,]a,b[$ *and* $\theta \in \mathbb{S}^{n-1}$.

The proof of this claim follows by *direct* verification. Indeed here a straight-forward *differentiation* gives

$$\begin{aligned}\mathbf{s}_r = \frac{\partial\mathbf{s}}{\partial r} &= \frac{\partial}{\partial r}\big(n + r^2|\mathbf{A}\theta|^2\big)^{\frac{p-2}{2}}\\ &= \frac{\partial}{\partial r}\big(n - r^2\langle\mathbf{F}\theta,\theta\rangle\big)^{\frac{p-2}{2}}\\ &= -\beta\left[r\langle\mathbf{F}\theta,\theta\rangle + \frac{1}{2}r^2\langle\dot{\mathbf{F}}\theta,\theta\rangle\right].\end{aligned}$$

Now *eliminating* the term $\langle \dot{\mathbf{F}}\theta, \theta\rangle$ in the above expression with the aid of (5.16) results in

$$\mathbf{s}_r = \frac{nr\beta\mathbf{s}\langle \mathbf{F}\theta, \theta\rangle}{\mathbf{s} - r^2\beta\langle \mathbf{F}\theta, \theta\rangle}.$$

(*See* below for a justification that $\mathbf{s} - r^2\beta\langle \mathbf{F}\theta, \theta\rangle \neq 0$.) Hence referring to (5.18) we can write

$$\begin{aligned}\gamma &= 2\left(\frac{n+1}{r} + \frac{\mathbf{s}_r}{\mathbf{s}}\right)\left(\mathbf{s} - \frac{1}{2}\beta r^2\langle \mathbf{F}\theta, \theta\rangle\right)\\ &= \frac{(n+1)\mathbf{s} - r^2\beta\langle \mathbf{F}\theta, \theta\rangle}{r(\mathbf{s} - r^2\beta\langle \mathbf{F}\theta, \theta\rangle)}\big(2\mathbf{s} - r^2\beta\langle \mathbf{F}\theta, \theta\rangle\big)\\ &=: \frac{\mathbf{I}}{\mathbf{II}} \times \mathbf{III}.\end{aligned}$$

We now proceed by evaluating each term *separately*. Indeed with regards to the *first* term we have that

$$\begin{aligned}\mathbf{I} &= (n+1)\mathbf{s} - r^2\beta\langle \mathbf{F}\theta, \theta\rangle\\ &= \big(n - r^2\langle \mathbf{F}\theta, \theta\rangle\big)^{\frac{p-4}{2}}\big[n(n+1) - (n+p-1)r^2\langle \mathbf{F}\theta, \theta\rangle\big]\end{aligned}$$

and in a *similar* way

$$\begin{aligned}\mathbf{II} &= r\big(\mathbf{s} - r^2\beta\langle \mathbf{F}\theta, \theta\rangle\big)\\ &= r\big(n - r^2\langle \mathbf{F}\theta, \theta\rangle\big)^{\frac{p-4}{2}}\big[n - (p-1)r^2\langle \mathbf{F}\theta, \theta\rangle\big]\end{aligned}$$

and

$$\begin{aligned}\mathbf{III} &= \big(2\mathbf{s} - r^2\beta\langle \mathbf{F}\theta, \theta\rangle\big)\\ &= \big(n - r^2\langle \mathbf{F}\theta, \theta\rangle\big)^{\frac{p-4}{2}}\big[2n - pr^2\langle \mathbf{F}\theta, \theta\rangle\big].\end{aligned}$$

Now in view of $-\langle \mathbf{F}\theta, \theta\rangle = \langle \mathbf{A}^t\mathbf{A}\theta, \theta\rangle = |\mathbf{A}\theta|^2 \geqslant 0$ for all $r \in\,]a, b[$ and $\theta \in \mathbb{S}^{n-1}$ along with $p \in [1, \infty[$ it follows that *all* the terms **I**, **II** and **III** are *strictly* positive. As a result

$$\gamma > 0 \tag{5.19}$$

and so the claim is justified.

Now returning to the identity (5.17) it follows as a result of (5.19) that necessarily

$$\mathbf{F}\theta \otimes \theta - \theta \otimes \mathbf{F}\theta = 0 \tag{5.20}$$

for all $r \in\,]a, b[$ and $\theta \in \mathbb{S}^{n-1}$. The conclusion in Step 2 is now an immediate result of the following statement.

Claim 3. *Let* $\mathbf{F} \in \mathbb{M}_{n\times n}$. *Then* (5.20) *holds for all* $\theta \in \mathbb{S}^{n-1}$ *if and only if there exists* $-\sigma \in \mathbb{R}$ *such that* $\mathbf{F} = -\sigma\mathbf{I}_n$.

For a *proof* of Claim 3 we refer the interested reader to Proposition 7.1 in [10]. Finally $\sigma \in C^1]a, b[$ and $\sigma \geqslant 0$ are consequences of the representation above and the hypothesis of the theorem. With this the *proof* of Theorem 5.1 is complete. □

Theorem 5.2. *Let* $\Omega = \{\mathbf{x} \in \mathbb{R}^n\colon\ a < |\mathbf{x}| < b\}$ *and consider the vector field* $\mathbf{v}$ *as defined in Theorem* 5.1. *Then the following are equivalent.*

(1) $\mathbf{v}$ *is a gradient,*
(2) $\mathbf{A} = \mu\mathbf{J}$ *for some* $\mu \in C^1]a, b[$ *with* $\mu \geqslant 0$, $\mathbf{J} \in \mathbb{M}_{n\times n}$ *skew-symmetric with* $\mathbf{J}^2 = -\mathbf{I}_n$ *and*

$$\frac{d}{dr}\big(r^{n+1}\mathbf{s}\mu\big) = 0 \tag{5.21}$$

in $]a, b[$. *Here* $\mathbf{s} = (n + r^2\mu^2)^{\frac{p-2}{2}}$.

Proof. (**2**) ⇒ (**1**) The argument here is similar to that in Theorem 5.1 and so will be *abbreviated*.

(**1**) ⇒ (**2**) Let $\mathbf{v}$ be a *gradient*. Then according to (**2**) in Theorem 5.1, $\mathbf{A}^2 = -\sigma \mathbf{I}_n$ for some $\sigma \in C^1(]a,b[)$ with $\sigma \geqslant 0$ and so $\mathbf{A} = \sqrt{\sigma}\mathbf{J}$ where $\mathbf{J} = \mathbf{J}(r)$ and $\mathbf{J}^2 = -\mathbf{I}_n$. The aim is to show that $\mathbf{J}$ is *independent* of r.[13] To this end we proceed as follows. Indeed according to (**2**) in Theorem 5.1,

$$\frac{d}{dr}\big(r^{n+1}\mathbf{sA}\big) = 0.$$

Integrating the above equation gives $r^{n+1}\mathbf{sA} = \xi$ for some *constant* $\xi \in \mathbb{M}_{n\times n}$. Moreover,

$$-\big(r^{n+1}\mathbf{s}\big)^2\sigma \mathbf{I}_n = \big(r^{n+1}\mathbf{sA}\big)^2 = \xi^2 \tag{5.22}$$

giving $(r^{n+1}\mathbf{s})^2\sigma \equiv c$ for some *non-negative* constant c. Thus *either* $\sigma \equiv 0$ in which case $\mathbf{A} \equiv 0$ on $]a,b[$ and so the choice $\mu \equiv 0$ gives the conclusion *or* else $\sigma > 0$ on $]a,b[$ and so setting

$$\mathbf{J} := \frac{1}{\sqrt{c}}\xi$$

we have as a result of (5.22) that $\mathbf{J}^2 = -\mathbf{I}_n$. Furthermore setting

$$\mu := \frac{1}{\sqrt{c}} r^{n+1}\mathbf{s}\sigma$$

it follows that $\mu \in C^1]a,b[$, $\mu^2 = \sigma$ and by substitution $\mathbf{A} = \mu\mathbf{J}$. As a result μ also satisfies (5.21). The proof of the theorem is thus complete. □

Remark 5.1. Referring to the above *proof* it follows from $r^{n+1}\mathbf{s}\mu = c$ on $]a,b[$ that when $p > 1$ the function μ remains *bounded* on $]a,b[$.

Theorem 5.3. *Let $\Omega = \{\mathbf{x} \in \mathbb{R}^n\colon a < |\mathbf{x}| < b\}$ and $\mathbf{u} \in \mathcal{A}_p$ with $p \in]1,\infty[$ be a generalised twist whose corresponding twist loop $\mathbf{Q} \in C^2(]a,b[,\mathbf{SO}(n))$. Then the following are equivalent.*

(**1**) $\mathbf{u}$ *is a classical solution to the Euler–Lagrange equations associated with $\mathbb{F}_p$ over $\mathcal{A}_p$,*
(**2**) *depending on whether n is even or odd we have that*
  (**2a**) *($n = 2k$) there exist $g = g(r) \in C[a,b] \cap C^2]a,b[$ with $g(a), g(b) \in 2\pi\mathbb{Z}$ and $\mathbf{P} \in \mathbf{O}(n)$ such that*

$$\mathbf{Q} = \mathbf{P}\,\mathrm{diag}\big(\mathbf{R}(g),\ldots,\mathbf{R}(g)\big)\mathbf{P}^t$$

  whilst g is a solution on $]a,b[$ to

$$\frac{d}{dr}\left\{r^{n+1}\big(n + r^2 g'^2\big)^{\frac{p-2}{2}} g'\right\} = 0 \tag{5.23}$$

  or
  (**2b**) *($n = 2k+1$) necessarily $\mathbf{u} = \mathbf{x}$ on $\bar{\Omega}$.*

Proof. (**1**) ⇒ (**2**) Let $\mathbf{u} = \mathbf{Q}(r)\mathbf{x}$ be a *classical* solution to the stated Euler–Lagrange equations. Then setting $\mathbf{A} = \mathbf{Q}^t\dot{\mathbf{Q}}$ an application of Proposition 2.3 in conjunction with Theorem 5.2 gives

$$\frac{d}{dr}\mathbf{Q} = \mu\mathbf{QJ} \tag{5.24}$$

where $\mu \in C^1]a,b[$ satisfies (5.21) and $\mathbf{J}^2 = -\mathbf{I}_n$. Moreover *either* $\mu \equiv 0$ or else $\mu > 0$ and bounded on $]a,b[$. (*See* Remark 5.1.) We now consider the cases (**2a**) and (**2b**) *separately*.

(**2a**) ($n = 2k$) Let $g \in C[a,b] \cap C^2]a,b[$ be a *primitive* of μ satisfying $g(a) \in 2\pi\mathbb{Z}$. (The *continuity* of g on $[a,b]$ follows from g being *monotone* and $g' = \mu$ being bounded on $]a,b[$.) Next, a straight-forward calculation gives

[13] Note that in general there is *no* uniqueness or even *finiteness* associated with the choice of a *square* root of a matrix! Thus an argument *purely* based on continuity would not yield the *aforementioned* claim and it is *crucial* to additionally utilise (5.2).

$$\begin{aligned}\mathbf{s} &= \left(n + r^2|\mathbf{A}\theta|^2\right)^{\frac{p-2}{2}}\\ &= \left(n + r^2 g'^2|\mathbf{J}\theta|^2\right)^{\frac{p-2}{2}}\\ &= \left(n + r^2 g'^2\right)^{\frac{p-2}{2}}.\end{aligned}$$

Thus in view of (5.21) g satisfies (5.23) on $]a, b[$. An application of Tonelli and Hilbert–Weierstrass *differentiability* theorems (*see*, e.g., [5, pp. 57–61]) now gives $g \in C^2[a, b]$ and so in particular $\mu \in C^1[a, b]$.[14]

With this introduction now put $\mathbf{C} = g\mathbf{J}$. Then $\mathbf{A} = g'\mathbf{J} = \mu\mathbf{J}$. In particular $\mathbf{A}$ and $\mathbf{C}$ *commute* and so we have that

$$\frac{d}{dr}e^{\mathbf{C}} = e^{\mathbf{C}}\mathbf{A} = g' e^{\mathbf{C}}\mathbf{J} = \mu e^{\mathbf{C}}\mathbf{J}.$$

Thus $e^{\mathbf{C}}$ is a solution to (5.24). Moreover by bringing $\mathbf{C}$ into a block diagonal form we can write $\mathbf{C} = g\mathbf{P}\mathbf{J}_n\mathbf{P}^t$ where $\mathbf{P} \in \mathbf{O}(n)$ and $\mathbf{J}_n = \mathrm{diag}(\mathbf{J}_2, \dots, \mathbf{J}_2)$. As a result

$$\begin{aligned}e^{\mathbf{C}} &= e^{g\mathbf{P}\mathbf{J}_n\mathbf{P}^t}\\ &= \mathbf{P}e^{g\mathbf{J}_n}\mathbf{P}^t\\ &= \mathbf{P}\,\mathrm{diag}\big(\mathbf{R}(g), \dots, \mathbf{R}(g)\big)\mathbf{P}^t.\end{aligned}$$

Since $g(a) \in 2\pi\mathbb{Z}$ the above shows that $e^{\mathbf{C}}|_{r=a} = \mathbf{Q}(a) = \mathbf{I}_n$ and so by *uniqueness* of solutions to initial values problems $\mathbf{Q} = e^{\mathbf{C}}$ on $[a, b]$. Since $\mathbf{Q}(b) = \mathbf{I}_n$ it follows in a similar way that $g(b) \in 2\pi\mathbb{Z}$.

(**2b**) ($n = 2k + 1$) Here in view of the *skew-symmetry* of $\mathbf{Q}^t\dot{\mathbf{Q}}$, pre-multiplying (5.24) by $\mathbf{Q}^t$ and then taking *determinants* from both sides, $\mu \equiv 0$ and so $\dot{\mathbf{Q}} \equiv 0$ on $]a, b[$. As $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$ this gives $\mathbf{Q} \equiv \mathbf{I}_n$ on $[a, b]$ and so $\mathbf{u} = \mathbf{x}$ on $\bar{\Omega}$.

(**2**) ⇒ (**1**) For the case (**2b**) this is *trivial* and for (**2a**) it is enough to note that for such $\mathbf{u}$, (5.23) is equivalent to (5.2). □

6. A characterisation of all twist solutions

In Section 4 we proved the existence of *multiple* p-stationary loops by directly *minimising* the energy functional $\mathbb{E}_p$ over the homotopy classes $\mathfrak{c}_\star[\mathcal{E}_p]$ of the *loop* space $\mathcal{E}_p$. By contrast in this section we focus on the Euler–Lagrange equation itself and present a class of p-stationary loops that in turn will prove fruitful in discussing the *existence* of multiple solutions to the Euler–Lagrange equations associated with the energy functional $\mathbb{F}_p$ over the space $\mathcal{A}_p$.

To this end we consider the case of *even* dimensions ($n = 2k$) and for $p \in [1, \infty[$ and $m \in \mathbb{N}$ set

$$\mathcal{G}_p^m = \mathcal{G}_p^m(a, b) := \big\{g = g(r) \in W^{1,p}(a, b)\colon\ g(a) = 0,\ g(b) = 2\pi m\big\}. \tag{6.1}$$

Now for $g \in \mathcal{G}_p^m$ and $\mathbf{P} \in \mathbf{O}(n)$ set

$$\mathbf{Q} = \mathbf{P}\,\mathrm{diag}\big(\mathbf{R}(g), \dots, \mathbf{R}(g)\big)\mathbf{P}^t. \tag{6.2}$$

It is then evident that the path $\mathbf{Q}$ so defined forms an *admissible* loop, i.e., lies in $\mathcal{E}_p$. It is thus natural to set

$$\begin{aligned}\mathbb{G}_p[g] := \mathbb{E}_p[\mathbf{Q}] &= \int_a^b \int_{\mathbb{S}^{n-1}} \left(n + r^2|\dot{\mathbf{Q}}\theta|^2\right)^{\frac{p}{2}} r^{n-1}\, d\mathcal{H}^{n-1}(\theta)\, dr\\ &= n\omega_n \int_a^b \left(n + r^2 g'^2\right)^{\frac{p}{2}} r^{n-1}\, dr.\end{aligned} \tag{6.3}$$

An application of the direct methods of the calculus of variations and standard *regularity* theory (*see*, e.g., [5, pp. 57–61]) leads us to the following statement.

[14] As will be seen in the next section (5.23) is the Euler–Lagrange equation corresponding to the energy functional $\mathbb{G}_p$ over the space $\mathcal{G}_p^m$ [*see* (6.1), (6.3)]. In particular it follows that $g \in C^\infty[a, b]$.

Theorem 6.1. *Let $p \in]1, \infty[$ and consider the energy functional $\mathbb{G}_p$ over the space $\mathcal{G}_p^m$. Then for each $m \in \mathbb{N}$ there exists a unique $g = g(r; m, a, b) \in \mathcal{G}_p^m$ such that*

$$\mathbb{G}_p[g] = \inf_{\mathcal{G}_p^m} \mathbb{G}_p.$$

Moreover $g(r; m, a, b)$ satisfies the corresponding Euler–Lagrange equation

$$\frac{d}{dr}\left\{r^{n+1}\left(n + r^2 g'^2\right)^{\frac{p-2}{2}} g'\right\} = 0 \tag{6.4}$$

on $]a, b[$. Additionally $g \in C^\infty[a, b]$.

Remark 6.1. The Euler–Lagrange equation (6.4) for g is *equivalent* to Eq. (3.2) for the *twist* loop $\mathbf{Q}$ defined through (6.2) and implies the Euler–Lagrange equation (3.2) [or alternatively that given in Proposition 3.3 for $\mathbf{A} = \mathbf{Q}^t \dot{\mathbf{Q}}$]. Hence for every $\mathbf{P} \in \mathbf{O}(n)$ and every $m \in \mathbb{Z}$ the corresponding $\mathbf{Q}$ given by (6.2) with $g = g(r; m, a, b)$ is a p-stationary *loop*.

Theorem 6.2. *Let $\Omega = \{\mathbf{x} \in \mathbb{R}^n\colon a < |\mathbf{x}| < b\}$. Consider the energy functional $\mathbb{F}_p$ with $p \in]1, \infty[$ over the space $\mathcal{A}_p$. Then the set $\mathfrak{S}$ of all generalised twist solutions to the corresponding Euler–Lagrange equations can be characterised as follows.*

(1) $(n = 2k)$ *$\mathfrak{S}$ is infinite and any generalised twist $\mathbf{u} \in \mathfrak{S}$ can be described as*

$$\begin{aligned}\mathbf{u} &= r\mathbf{Q}(r; a, b, m)\theta \\ &= r\mathbf{P}\,\mathrm{diag}\big(\mathbf{R}(g), \ldots, \mathbf{R}(g)\big)(r)\mathbf{P}^t\theta\end{aligned}$$

where $\mathbf{P} \in \mathbf{O}(n)$ and $g \in C^\infty[a, b]$ satisfies

$$\frac{d}{dr}\left\{r^{n+1}\left(n + r^2 g'^2\right)^{\frac{p-2}{2}} g'\right\} = 0$$

with $g(a), g(b) \in 2\pi\mathbb{Z}$,

(2) $(n = 2k + 1)$ *$\mathfrak{S}$ consists of the single map $\mathbf{u} = \mathbf{x}$.*

Proof. This is an immediate consequence of Theorems 5.3 and 6.1. □

Remark 6.2. Is it possible to consider *generalised* twists $\mathbf{u}$ whose twist *loop* lies in other spaces [than $\mathbf{SO}(n)$ already considered] with the hope of finding *new* classes of classical solutions to the Euler–Lagrange equations associated with the energy functional $\mathbb{F}_p$ over $\mathcal{A}_p$?

Motivated by the requirement $\det \nabla\mathbf{u} = 1$ on such maps the choice of loops in $\mathbf{SL}(n) \supset \mathbf{SO}(n)$ seems a natural one.[15] However it turns out that the choice $\mathbf{SO}(n)$ is *no* less general than $\mathbf{SL}(n)$!

Claim. *Let $\Omega = \{\mathbf{x} \in \mathbb{R}^n\colon a < |\mathbf{x}| < b\}$. For $p \in [1, \infty[$ consider the map $\mathbf{u} \in C(\bar{\Omega}, \bar{\Omega})$ defined via*

$$\mathbf{u} = \mathbf{F}(r)\mathbf{x}$$

where $r = |\mathbf{x}|$ and $\mathbf{F} \in W^{1,p}([a, b], \mathbf{SL}(n))$. Then

$$\mathbf{u} \in \mathcal{A}_p(\Omega) \quad \Longrightarrow \quad \mathbf{F} \in W^{1,p}\big([a, b], \mathbf{SO}(n)\big).$$

Proof. A straight-forward calculation as in the proof of Proposition 2.1 gives

$$\begin{aligned}\nabla\mathbf{u} &= \mathbf{F} + r\dot{\mathbf{F}}\theta \otimes \theta \\ &= \mathbf{F}\big(\mathbf{I}_n + r\mathbf{F}^{-1}\dot{\mathbf{F}}\theta \otimes \theta\big).\end{aligned}$$

[15] Recall that for every non-negative integer n we have that

$$\mathbf{SL}(n) = \mathbf{SL}(\mathbb{R}, n) := \big\{\mathbf{F} \in \mathbb{M}_{n\times n}(\mathbb{R})\colon \det\mathbf{F} = 1\big\}.$$

Hence in view of $\det \mathbf{F} = 1$ we can write

$$\begin{aligned}\det \nabla \mathbf{u} &= \det(\mathbf{F} + r\dot{\mathbf{F}}\theta \otimes \theta) \\ &= \det\big(\mathbf{I}_n + r\mathbf{F}^{-1}\dot{\mathbf{F}}\theta \otimes \theta\big) \\ &= 1 + r\big\langle \mathbf{F}^{-1}\dot{\mathbf{F}}\theta, \theta\big\rangle.\end{aligned}$$

Evidently $\mathbf{u} \in \mathcal{A}_p(\Omega)$ provided that

(i) $\mathbf{u} = \mathbf{x}$ on $\partial\Omega$,
(ii) $\det \nabla \mathbf{u} = 1$ in Ω, and
(iii) $\|\mathbf{u}\|_{W^{1,p}(\Omega)} < \infty$.

Now again referring to the proof of Proposition 2.1 we have that

$$\text{(i)} \quad \Longleftrightarrow \quad \mathbf{F}(a) = \mathbf{F}(b) = \mathbf{I}_n,$$

whilst

$$\text{(ii)} \quad \Longleftrightarrow \quad \big\langle \mathbf{F}^{-1}\dot{\mathbf{F}}\theta, \theta\big\rangle = 0 \quad \text{for all } \theta \in \mathbb{S}^{n-1} \quad \Longleftrightarrow \quad \mathbf{F}^{1}\dot{\mathbf{F}} + \dot{\mathbf{F}}^t\mathbf{F}^{-t} = 0.$$

However, anticipating on the latter, we can write

$$\begin{aligned}\mathbf{F}^{-1}\dot{\mathbf{F}} + \dot{\mathbf{F}}^t\mathbf{F}^{-t} = 0 \quad &\Longleftrightarrow \quad \dot{\mathbf{F}} + \mathbf{F}\dot{\mathbf{F}}^t\mathbf{F}^{-t} = 0 \\ &\Longleftrightarrow \quad \dot{\mathbf{F}}\mathbf{F}^t + \mathbf{F}\dot{\mathbf{F}}^t = 0 \\ &\Longleftrightarrow \quad \frac{d}{dr}\big(\mathbf{F}\mathbf{F}^t\big) = 0.\end{aligned}$$

This together with (i) and the *continuity* of $\mathbf{F}$ on $[a, b]$ gives $\mathbf{F}\mathbf{F}^t = \mathbf{I}_n$ and so the conclusion follows. □

7. Limiting behaviour of the generalised twists as the inner hole shrinks to a point

In this section we consider the case where $b = 1$ and $a = \varepsilon > 0$ with the aim of discussing the *limiting* properties of the *generalised* twists from Theorem 6.2 as $\varepsilon \downarrow 0$. This is particularly interesting since in the limit (the *punctured* ball) *all* components of the function space collapse to a single one and so it is important to have a clear understanding as to how the twist solutions and their energies [for each *fixed* integer m] behave.[16]

To this end, let $\Omega_\varepsilon := \{\mathbf{x} \in \mathbb{R}^n\colon\ \varepsilon < |\mathbf{x}| < 1\}$ where $n = 2k$ and for each $m \in \mathbb{Z}$ let $\mathbf{u}_\varepsilon \in \mathcal{A}_p$ denote the *generalised* twist from (**1**) in Theorem 6.2, that is, with the notation $\mathbf{x} = r\theta$,

$$\begin{aligned}\mathbf{u}_\varepsilon &= r\mathbf{Q}(r; \varepsilon, 1, m)\theta \\ &= r\mathbf{P}_\varepsilon\big[\operatorname{diag}\big(\mathbf{R}(g_\varepsilon), \ldots, \mathbf{R}(g_\varepsilon)\big)\big]\mathbf{P}_\varepsilon^t\theta\end{aligned}$$

where $\mathbf{P}_\varepsilon \in \mathbf{O}(n)$ and $g_\varepsilon(r) = g(r; \varepsilon, 1, m)$.

In order to make the study of the *limiting* properties of $\mathbf{u}_\varepsilon$ more tractable, we fix the domain to be the unit ball and extend each map by *identity* off Ω_ε. [In what follows, unless otherwise stated, we speak of $\mathbf{u}_\varepsilon$ in this extended sense.] Thus, here, we have that

$$\mathbf{u}_\varepsilon : (r, \theta) \mapsto \big(r, \mathbf{G}_\varepsilon(r)\theta\big) \tag{7.1}$$

where

$$\mathbf{G}_\varepsilon(r) = \mathbf{P}_\varepsilon\big[\operatorname{diag}\big(\mathbf{R}(g_\varepsilon), \ldots, \mathbf{R}(g_\varepsilon)\big)\big]\mathbf{P}_\varepsilon^t$$

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[16] In the case of a *punctured* disk, say, $\Omega = \mathbb{B}\backslash\{0\}$, for any pair of maps $\phi_0, \phi_1 \in \mathfrak{A} := \{\phi \in C(\bar{\Omega}, \bar{\Omega})\colon\ \phi = \varphi$ on $\partial\Omega = \{0\} \cup \partial\mathbb{B}\}$, the *continuous* path $[0, 1] \ni t \mapsto \phi_t := (1 - t)\phi_0 + t\phi_1$ lies within $\mathfrak{A}$ and joins ϕ_0 to ϕ_1. Therefore, here, $\mathfrak{A}$ consists of a *single* component only! [Compare this with the discussion in the *footnote* at the end of Section 1.]

and

$$g_\varepsilon(r) = \begin{cases} 0, & r \leqslant \varepsilon, \\ g(r; \varepsilon, 1, m), & \varepsilon \leqslant r \leqslant 1. \end{cases}$$

In discussing the *limiting* properties of $\mathbf{u}_\varepsilon$ it is convenient to introduce a so-called *comparison* map. Indeed, *fix* $m \in \mathbb{Z}$ and consider the *generalised* twist

$$\mathbf{v}_\varepsilon : (r, \theta) \mapsto \big(r, \mathbf{H}_\varepsilon(r)\theta\big) \tag{7.2}$$

where

$$\mathbf{H}_\varepsilon(r) = \mathbf{P}_\varepsilon\big[\operatorname{diag}\big(\mathbf{R}(h_\varepsilon), \ldots, \mathbf{R}(h_\varepsilon)\big)\big]\mathbf{P}_\varepsilon^t$$

and

$$h_\varepsilon(r) := \begin{cases} 0, & r \in (0, \varepsilon), \\ 2m\pi(\frac{r}{\varepsilon} - 1), & r \in (\varepsilon, 2\varepsilon), \\ 2m\pi, & r \in (2\varepsilon, 1). \end{cases}$$

Proposition 7.1. *Let* $p \in\,]1, \infty[$. *The family of generalised twists* $(\mathbf{v}_\varepsilon)$ *enjoys the followings properties.*

(**1**) $\mathbf{v}_\varepsilon \to \mathbf{x}$ *in* $W^{1,p}(\mathbb{B}, \mathbb{R}^n)$,
(**2**) $\mathbf{v}_\varepsilon \to \mathbf{x}$ *uniformly on* $\bar{\mathbb{B}}$.

Proof. (**1**) Using (7.2) and a straight-forward calculation we have that

$$\|\mathbf{v}_\varepsilon - \mathbf{x}\|^p_{W_0^{1,p}} = \int_{\mathbb{B}} |\nabla \mathbf{v}_\varepsilon - \mathbf{I}|^p = \int_{\mathbb{B}_{2\varepsilon}\setminus\mathbb{B}_\varepsilon} |\nabla \mathbf{v}_\varepsilon - \mathbf{I}|^p \leqslant 2^{p-1} \int_{\mathbb{B}_{2\varepsilon}\setminus\mathbb{B}_\varepsilon} |\nabla \mathbf{v}_\varepsilon|^p + |\mathbf{I}|^p.$$

Furthermore, referring to Proposition 2.1 [*see* (2.4)] we can write

$$\int_{\mathbb{B}_{2\varepsilon}\setminus\mathbb{B}_\varepsilon} |\nabla \mathbf{v}_\varepsilon|^p = \int_\varepsilon^{2\varepsilon} \int_{\mathbb{S}^{n-1}} \big(n + r^2|\dot{\mathbf{H}}_\varepsilon\theta|^2\big)^{\frac{p}{2}} r^{n-1}\, d\mathcal{H}^{n-1}(\theta)\, dr = n\omega_n \int_\varepsilon^{2\varepsilon} \big(n + r^2 h_\varepsilon'^2\big)^{\frac{p}{2}} r^{n-1}\, dr \leqslant \omega_n\big(2^n - 1\big)\varepsilon^n\big[n + 4(2m\pi)^2\big]^{\frac{p}{2}}. \tag{7.3}$$

The above estimates when combined give (**1**) as a result of Poincaré inequality.

(**2**) By direct verification we have that

$$\begin{aligned} |\mathbf{v}_\varepsilon - \mathbf{x}|^2 &= \big|r\mathbf{H}_\varepsilon(r)\theta - r\theta\big|^2 \\ &= r^2\big|\mathbf{P}_\varepsilon \operatorname{diag}\big(\mathbf{R}(h_\varepsilon), \ldots, \mathbf{R}(h_\varepsilon)\big)\mathbf{P}_\varepsilon^t\theta - \theta\big|^2 \\ &= r^2\big|\mathbf{P}_\varepsilon\big(\operatorname{diag}\big(\mathbf{R}(h_\varepsilon), \ldots, \mathbf{R}(h_\varepsilon)\big) - \mathbf{I}_n\big)\mathbf{P}_\varepsilon^t\theta\big|^2 \\ &= r^2\big|\big(\operatorname{diag}\big(\mathbf{R}(h_\varepsilon), \ldots, \mathbf{R}(h_\varepsilon)\big) - \mathbf{I}_n\big)\omega_\varepsilon\big|^2 \quad \big(\omega_\varepsilon := \mathbf{P}_\varepsilon^t\theta\big) \\ &= \frac{1}{2}r^2\big|\mathbf{R}(h_\varepsilon) - \mathbf{I}_2\big|^2. \end{aligned} \tag{7.4}$$

However a straight-forward calculation gives

$$\big|\mathbf{R}(h_\varepsilon) - \mathbf{I}_2\big|^2 = 4(1 - \cos h_\varepsilon) = 8\sin^2\frac{h_\varepsilon}{2}.$$

Thus combining the above and referring to the definition of h_ε we arrive at the bound

$$\sup_{\mathbb{B}} |\mathbf{v}_\varepsilon - \mathbf{x}| = \sup_{[\varepsilon, 2\varepsilon]} 2r \left| \sin \frac{h_\varepsilon}{2} \right| \leqslant 4\varepsilon,$$

which gives the required conclusion. □

Let $p \in]1, \infty[$ and *fix* $m \in \mathbb{Z}$. Then $g_\varepsilon, h_\varepsilon \in \mathcal{G}_p^m(\varepsilon, 1)$ [*see* (6.1)] and so according to the minimising property of g_ε we have that

$$\mathbb{F}_p[\mathbf{u}_\varepsilon, \mathbb{B}] = \frac{1}{p}\mathbb{E}_p[\mathbf{G}_\varepsilon] = \frac{1}{p}\mathbb{G}_p[g_\varepsilon] \leqslant \frac{1}{p}\mathbb{G}_p[h_\varepsilon] = \frac{1}{p}\mathbb{E}_p[\mathbf{H}_\varepsilon] = \mathbb{F}_p[\mathbf{v}_\varepsilon, \mathbb{B}]. \tag{7.5}$$

This in conjunction with (**1**) in Proposition 7.1 implies the boundedness of $(\mathbf{u}_\varepsilon)$ in $W^{1,p}(\mathbb{B}, \mathbb{R}^n)$ and so as a result $(\mathbf{u}_\varepsilon)$ admits a *weakly* convergent subsequence. Indeed more is true!

Theorem 7.1. *Let* $\Omega_\varepsilon := \{\mathbf{x} \in \mathbb{R}^n \colon \varepsilon < |\mathbf{x}| < 1\}$. *For* $p \in]1, \infty[$ *and* $m \in \mathbb{Z}$ *let* $(\mathbf{u}_\varepsilon)_{\varepsilon > 0}$ *denote the family of generalised twists as in* (7.1). *Then,*

(**1**) $\mathbf{u}_\varepsilon \to \mathbf{x}$ *in* $W^{1,p}(\mathbb{B}, \mathbb{R}^n)$,
(**2**) $\mathbf{u}_\varepsilon \to \mathbf{x}$ *uniformly in* $\bar{\mathbb{B}}$.[17]

Proof. (**1**) Fix $m \in \mathbb{Z}$ and let $\mathbf{v}_\varepsilon$ be as in (7.2). Then referring to (7.5) it follows that by passing to a subsequence (not re-labeled) $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}$ in $W^{1,p}(\mathbb{B}, \mathbb{R}^n)$. Appealing to the *sequential* weak lower semicontintuity of $\mathbb{F}_p$ and (**1**) in Proposition 7.1 we can write

$$\begin{aligned}
\mathbb{F}_p[\mathbf{x}, \mathbb{B}] \leqslant \mathbb{F}_p[\mathbf{u}, \mathbb{B}] &\leqslant \liminf_{\varepsilon \searrow 0} \mathbb{F}_p[\mathbf{u}_\varepsilon, \mathbb{B}] \\
&\leqslant \limsup_{\varepsilon \searrow 0} \mathbb{F}_p[\mathbf{u}_\varepsilon, \mathbb{B}] \\
&\leqslant \lim_{\varepsilon \searrow 0} \mathbb{F}_p[\mathbf{v}_\varepsilon, \mathbb{B}] = \mathbb{F}_p[\mathbf{x}, \mathbb{B}].
\end{aligned}$$

This in view of the *strict* convexity of $\mathbb{F}_p$ (on $W^{1,p}$) gives $\mathbf{u} = \mathbf{x}$. As a result of the *uniform* convexity of the p-norm ($p > 1$) the *aforementioned* weak convergence can now be improved to *strong* convergence and this gives (**1**).

(**2**) By (**1**) we can assume without loss of generality that $\mathbf{u}_\varepsilon \to \mathbf{x}$ $\mathcal{L}^n$-a.e. in Ω. To justify the *uniform* convergence in (**2**) let g_ε be as that described in (7.1) and *fix* $\sigma \in (0, 1)$. Then we claim that $g_\varepsilon \to 2m\pi$ uniformly on $[\sigma, 1]$. Indeed, $(\mathbf{u}_\varepsilon)$ bounded in $W^{1,p}(\mathbb{B}, \mathbb{R}^n)$ gives $(\mathbf{u}_\varepsilon)$ bounded in $W^{1,p}(\mathbb{B} \backslash \bar{\mathbb{B}}_\sigma, \mathbb{R}^n)$ and so referring to (2.4) and using a calculation similar to that in (7.3) we have (g_ε) bounded in $W^{1,p}(\sigma, 1)$. Hence, there exists $f = f_\sigma \in W^{1,p}(\sigma, 1)$ so that passing to a subsequence (not re-labeled)

$$\begin{cases}
g_\varepsilon \rightharpoonup f, & \text{in } W^{1,p}(\sigma, 1), \\
g_\varepsilon \to f, & \text{in } L^\infty[\sigma, 1], \\
f(1) = 2m\pi.
\end{cases}$$

In addition referring again to (7.1) we can assume in view of $\mathbf{O}(n)$ being *compact*, that by passing to a *further* subsequence (again, not re-labeled) $\mathbf{P}_\varepsilon \to \mathbf{P}$ for some $\mathbf{P} \in \mathbf{O}(n)$. Hence for $\mathcal{L}^n$-a.e. $\mathbf{x} \in \Omega$ we can write

$$\begin{aligned}
\lim_{\varepsilon \searrow 0} \mathbf{u}_\varepsilon(\mathbf{x}) &= \lim_{\varepsilon \searrow 0} r\mathbf{G}_\varepsilon(r)\theta \\
&= \lim_{\varepsilon \searrow 0} r\mathbf{P}_\varepsilon \operatorname{diag}\big(\mathbf{R}(g_\varepsilon), \ldots, \mathbf{R}(g_\varepsilon)\big)\mathbf{P}_\varepsilon^t\theta \\
&= r\mathbf{P} \operatorname{diag}\big(\mathbf{R}(f), \ldots, \mathbf{R}(f)\big)\mathbf{P}^t\theta = r\theta \\
&= \mathbf{x},
\end{aligned}$$

[17] Note that here both convergences are in reference to the *entire* sequence and not merely a subsequence as was implied in discussing the *weak* convergence prior to the proposition. The argument is standard and will be abbreviated.

giving $\mathbf{R}(f) = \mathbf{I}_2$ and in turn that $f = 2\pi n(r)$ for some $n(r) \in \mathbb{Z}$. The continuity of f along with $f(1) = 2m\pi$ now gives $f = 2m\pi$ on $[\sigma, 1]$ justifying the assertion. Next, arguing as in (7.4) we can write

$$\begin{aligned}|\mathbf{u}_\varepsilon - \mathbf{x}|^2 &= \left|r\mathbf{G}_\varepsilon(r)\theta - r\theta\right|^2 \\ &= 2r^2(1 - \cos g_\varepsilon) \\ &= 4r^2 \sin^2 \frac{g_\varepsilon}{2}.\end{aligned}$$

Thus, to conclude [**2**] fix $\delta > 0$ and *first* take $\sigma \in (0, 2^{-1}\delta]$ and then ε_0 such that $|\sin(2^{-1}g_\varepsilon)| \leqslant 2^{-1}\delta$ on $[\sigma, 1]$ for $\varepsilon < \varepsilon_0$. Then $\sup_{\mathbb{B}} |\mathbf{u}_\varepsilon - \mathbf{x}| \leqslant \max(2\sigma, \delta) = \delta$.[18] □

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Appendix A

Recall from linear algebra that *all* eigen-values of a [real] *skew*-symmetric matrix have zero *real* parts. Hence they *either* appear as *purely* imaginary conjugate pairs *or* zero. In particular when n is *odd* there is necessarily a zero eigen-value. Thus distinguishing between the cases when n is *even* and *odd* respectively we can bring every *skew-symmetric* matrix to a *block* diagonal form. Let

$$\mathbf{J} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Proposition A.1. *Let* $\mathbf{A} \in \mathbb{M}_{n\times n}$ *be skew-symmetric. Then there exist* $\mathbf{P} \in \mathbf{SO}(n)$ *and* $(\lambda_j)_{j=1}^k \subset \mathbb{R}$ *such that*[19]

(**1**) $(n = 2k)$

$$\mathbf{A} = \mathbf{P}^t \operatorname{diag}(\lambda_1\mathbf{J}, \lambda_2\mathbf{J}, \ldots, \lambda_k\mathbf{J})\mathbf{P},$$

(**2**) $(n = 2k + 1)$

$$\mathbf{A} = \mathbf{P}^t \operatorname{diag}(\lambda_1\mathbf{J}, \lambda_2\mathbf{J}, \ldots, \lambda_k\mathbf{J}, 0)\mathbf{P}.$$

Proof. Indeed, here, $\mathbf{A}$ is *normal* [i.e., it commutes with its *transpose* $\mathbf{A}^t = -\mathbf{A}$] and so the conclusion follows from the well-known *spectral* theorem. □

With the aid of the above representation evaluating the exponential function for *skew-symmetric* matrices becomes remarkably convenient. Let

$$\mathbf{R}(s) := \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}.$$

Proposition A.2. *Let* $\mathbf{A} \in \mathbb{M}_{n\times n}$ *be skew-symmetric. Then using the notation in Proposition* A.1 *we have that*

(**1**) $(n = 2k)$

$$e^{s\mathbf{A}} = \mathbf{P}^t \operatorname{diag}\big(\mathbf{R}(s\lambda_1), \mathbf{R}(s\lambda_2), \ldots, \mathbf{R}(s\lambda_k)\big)\mathbf{P},$$

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[18] The *uniform* convergence in (**2**) above looks at *first* counter-intuitive, as, how can $\mathbf{u}_\varepsilon$ and $\mathbf{x}$ be *uniformly* close when $\mathbf{u}_\varepsilon$ twists m times while the *limit* $\mathbf{x}$ none? Indeed a *careful* consideration reveals that the latter twists occur at a distance ε from the origin and within a layer of thickness $O(\varepsilon)$ and this is in no conflict with the stated *uniform* convergence!

[19] Indeed by allowing $\mathbf{P} \in \mathbf{O}(n)$ we can additionally arrange for the *sequence* $(\lambda_j)_{j=1}^k$ to be *non-negative*.

(**2**) $(n = 2k + 1)$

$$e^{s\mathbf{A}} = \mathbf{P}^t \operatorname{diag}\big(\mathbf{R}(s\lambda_1), \mathbf{R}(s\lambda_2), \ldots, \mathbf{R}(s\lambda_k), 1\big)\mathbf{P}.$$

Proof. A straight-forward calculation gives

$$e^{s\mathbf{J}} = \sum_{n=0}^{\infty} \frac{1}{n!} s^n \mathbf{J}^n = \mathbf{R}(s).$$

The conclusion now follows by noting that $e^{\mathbf{A}} = e^{\mathbf{P}^t\mathbf{D}\mathbf{P}} = \mathbf{P}^t e^{\mathbf{D}}\mathbf{P}$. $\square$

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