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Optimal regularity for planar mappings of finite distortion ☆

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Dedicated to the memory of our dear friend and colleague Juha Heinonen (1960–2007)

Abstract

Let $f : \Omega \to \mathbb{R}^2$ be a mapping of finite distortion, where $\Omega \subset \mathbb{R}^2$. Assume that the distortion function $K(x, f)$ satisfies $e^{K(\cdot,f)} \in L^p_{loc}(\Omega)$ for some $p > 0$. We establish optimal regularity and area distortion estimates for f. In particular, we prove that $|Df|^2 \log^{\beta-1}(e + |Df|) \in L^1_{loc}(\Omega)$ for every $\beta < p$. This answers positively, in dimension $n = 2$, the well-known conjectures of Iwaniec and Sbordone [T. Iwaniec, C. Sbordone, Quasiharmonic fields, Ann. Inst. H. Poincaré Anal. Non Linéaire 18 (2001) 519–572, Conjecture 1.1] and of Iwaniec, Koskela and Martin [T. Iwaniec, P. Koskela, G. Martin, Mappings of BMO-distortion and Beltrami-type operators, J. Anal. Math. 88 (2002) 337–381, Conjecture 7.1].

Keywords: Mappings of finite distortion; Exponential distortion; Optimal regularity; Area distortion

1. Introduction

We say that a mapping $f = (u, v)$ defined in a domain $\Omega \subset \mathbb{R}^2$ is a *mapping of finite distortion* if

(i) $f \in W^{1,1}_{loc}(\Omega)$,
(ii) $J(\cdot, f) = u_x v_y - u_y v_x \in L^1_{loc}(\Omega)$, and

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(iii) there is a measurable function $K(z) \geqslant 1$, finite almost everywhere, such that

$$\left|Df(z)\right|^2 \leqslant K(z)J(z,f) \quad \text{almost everywhere in } \Omega. \tag{1}$$

The smallest such function is denoted by $K(z,f)$ and is called the distortion function of f.

In two dimensions the mappings of finite distortion are intimately related to elliptic PDE's. For equations with non-smooth coefficients the first condition, requiring that f has locally integrable distributional partial derivatives, is the smallest degree of smoothness where one can begin to discuss what it means to be a (weak) solution to such an equation.

The second condition is a (weak) regularity property which is automatically satisfied by all homeomorphisms f in the Sobolev class $W^{1,1}_{loc}(\Omega)$. The last condition, that the distortion function $1 \leqslant K(z,f) < \infty$, merely asks that the pointwise Jacobian $J(z,f) \geqslant 0$ almost everywhere and that the gradient $Df(z)$ vanishes at those points z where $J(z,f) = 0$. This is a minimal requirement for a mapping to carry geometric information.

In two dimensions the distortion inequality (1) can be reformulated with the complex notation as

$$\left|\frac{\partial f}{\partial \bar z}\right| \leqslant k(z)\left|\frac{\partial f}{\partial z}\right|, \quad \text{where } k(z) := \frac{K(z)-1}{K(z)+1} < 1,$$

using the identifications $Df = |f_z| + |f_{\bar z}|$ for the operator norm and $J(z,f) = |f_z|^2 - |f_{\bar z}|^2$ for the Jacobian. Yet another equivalent formulation is with the Beltrami equation

$$\frac{\partial f}{\partial \bar z} = \mu(z)\frac{\partial f}{\partial z}, \qquad \left|\mu(z)\right| = \frac{K(z)-1}{K(z)+1} < 1 \quad \text{a.e.}, \tag{2}$$

where the coefficient μ can be defined simply via the differential equation (2). Then at a given point z we have $K(z,f) < \infty$ if and only if $|\mu(z)| < 1$.

The classical theory of quasiconformal mappings considers precisely the mappings of bounded distortion, that is mappings with $K(z,f) \in L^\infty$ or, equivalently, mappings with complex dilation $|\mu(z)| \leqslant k$ for some constant $k < 1$. The classical measurable Riemann mapping theorem tells that if we have a coefficient $|\mu(z)| \leqslant k < 1$ for $z \in \mathbb{C}$, then Eq. (2) has a homeomorphic solution and all other solutions are obtained by post-composing with holomorphic functions, see [8,1], or [4, Chapter 5].

It is natural to ask how much the condition $K(z,f) \in L^\infty$ can be relaxed in order to obtain a useful theory. In the groundbreaking work [10] David generalized the measurable Riemann mapping theorem to maps of exponentially integrable distortion, where in general we have $\|\mu\|_\infty = 1$. Later, Iwaniec, Koskela, and Martin [16] generalized the corresponding regularity theory to higher dimensions, and for the two dimensions a systematic modern development was established by Iwaniec and Martin as part of their monograph [14]. During the last 10 years there has been an intensive study of finite distortion mappings, motivated also by the fact that these maps have natural applications e.g. to non-uniformly elliptic equations and elasticity theory. We refer the reader for instance to [4,9,12,14,15,20,23] or [24] and the references therein, for the basic literature on the subject.

As a typical example of a mapping with finite but unbounded distortion consider the following example.[1] Let

$$g_p(z) = \frac{z}{|z|}\left[\log\left(e + \frac{1}{|z|}\right)\right]^{-p/2}\left[\log\log\left(e + \frac{1}{|z|}\right)\right]^{-1/2} \quad \text{for } |z| < 1, \tag{3}$$

and for $|z| > 1$ set $g_p(z) = c_0 z$. For each value of the parameter $p > 0$ one easily computes that

$$e^{K(z,g_p)} \in L^p(\mathbb{D}),$$

while $e^{K(z,g_p)}$ is not locally integrable to any power $s > p$. Furthermore, we have the fundamental regularity $g_p \in W^{1,2}_{loc}(\mathbb{C})$ when $p > 1$. However, for $p = 1$ the mapping $g_1 \notin W^{1,2}_{loc}(\mathbb{C})$.

From example (3) we see that the integrability of $e^{K(z)}$ is in general *not* sufficient for the $W^{1,2}_{loc}$-regularity. Instead we have a refined logarithmic scale of regularity around the space $W^{1,2}_{loc}$. Namely, for each $p > 0$,

[1] The similar formula (20.75) in [4] has a slight misprint.

$$|Dg_p|^2 \log^{\beta-1} |Dg_p| \in L^1_{loc}(\mathbb{C}), \quad 0 < \beta < p,$$

while the inclusion fails for $\beta = p$.

The exponential integrability of the distortion function in this theory is no coincidence. Indeed, to ascertain the continuity of the mapping f of finite distortion, unless some additional regularity is required, the exponential integrability of K suffices but this condition has only little room for relaxation, see [14, Theorem 11.2.1] and [4, Section 20.5]. Hence the mappings of exponentially integrable distortion provide the most natural class where to look for a viable general theory.

Within this class of mappings one of the fundamental questions is their (optimal) regularity. That is, given a homeomorphism with *a priori* only $f \in W^{1,1}_{loc}(\Omega)$, we assume that

$$e^{K(z,f)} \in L^p_{loc}(\Omega), \quad \text{where } p > 0,$$

and then seek for the optimal improvement on regularity this condition brings, i.e. find in terms of p the best possible Sobolev class where the mapping f belongs to.

The example (3) suggests natural candidates for the extremal behavior. Indeed, in their paper Iwaniec and Sbordone [18] formulated a precise general conjecture on the regularity of mappings with exponentially integrable distortion. The main purpose of this paper is to give a proof to their conjecture, and establish the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a domain. Suppose that the distortion function $K(z, f)$ of a mapping of finite distortion $f \in W^{1,1}_{loc}(\Omega)$ satisfies*

$$e^{K(z,f)} \in L^p_{loc}(\Omega) \quad \text{for some } p > 0. \tag{4}$$

Then we have for every $0 < \beta < p$,

$$|Df|^2 \log^{\beta-1}\bigl(e + |Df|\bigr) \in L^1_{loc}(\Omega) \quad \text{and} \tag{5}$$

$$J(z, f) \log^{\beta}\bigl(e + J(z, f)\bigr) \in L^1_{loc}(\Omega). \tag{6}$$

Moreover, for every $p > 0$ there are examples that satisfy (4)*, yet fail* (5) *and* (6) *for $\beta = p$.*

Of particular interest is the question of when the derivatives of the mappings are L^2-integrable. As a special case of our theorem one has

Corollary 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a domain and $f \in W^{1,1}_{loc}(\Omega)$ a mapping of finite distortion. Suppose that*

$$e^{K(z,f)} \in L^p_{loc}(\Omega) \quad \text{for some } p > 1. \tag{7}$$

Then $f \in W^{1,2}_{loc}(\Omega)$.

The family of mappings (3) *shows that $W^{1,2}_{loc}$-regularity fails in general at $p = 1$.*

Previously [16,18] it was known that there exists some constant $p_0 > 1$ so that $e^{K(z,f)} \in L^p_{loc}$ with $p > p_0$ implies $f \in W^{1,2}_{loc}$. Our result identifies the precise regularity borderline $p_0 = 1$. Similarly [14] (see [17,12] for n-dimensional results) the conclusions (5) and (6) were only known to hold under the stronger assumption $\beta < c_1 p$, where $c_1 \in (0, 1)$ is some unspecified constant.

Our proof of optimal regularity employs the approach of David [10]. It turns out that it is practically impossible to describe our proof without delving in depth into David's mapping theorem. Hence, to make the exposition as clear as possible we will include a straightforward proof of this result, and present this in the modern framework developed by Iwaniec and Martin [14]:

Theorem 1.3 *(David, Iwaniec–Martin). Suppose the distortion function $K = K(z)$ is such that*

$$e^K \in L^p(\mathbb{D}) \quad \text{for some } p > 0.$$

Assume also that $\mu(z) \equiv 0$ for $|z| > 1$. Then the Beltrami equation $f_{\bar z}(z) = \mu(z) f_z(z)$ admits a unique principal solution f for which

$$f \in W^{1,Q}_{loc}(\mathbb{C}), \qquad Q(t) = t^2 \log^{-1}(e+t).$$

The principal solution is a homeomorphism. Moreover, every other $W^{1,Q}_{loc}$-solution h to this Beltrami equation in a domain $\Omega \subset \mathbb{C}$ admits the factorization

$$h = \phi \circ f,$$

where ϕ is a holomorphic function in the domain $f(\Omega)$.

By a principal solution we mean a homeomorphism $f \in W^{1,1}_{loc}(\mathbb{C})$ that satisfies

$$f(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \tag{8}$$

outside some compact set.

There is a simple explanation why it is convenient to develop the theory of mappings of exponential distortion using the space $W^{1,Q}_{loc}$ as a starting point. Namely, suppose $f \in W^{1,1}_{loc}(\mathbb{C})$ is an orientation preserving homeomorphism, that is $J(z,f) \geqslant 0$, whose distortion function $K(z)$ satisfies $e^K \in L^p_{loc}(\mathbb{C})$ for some $0 < p < \infty$. The Jacobian of any planar Sobolev homeomorphism is locally integrable. Hence we may use the elementary inequality

$$ab \leqslant a\log(1+a) + e^b - 1$$

to find that

$$\frac{|Df|^2}{\log(e+|Df|^2)} \leqslant \frac{KJ}{\log(e+KJ)} \leqslant \frac{1}{p}\frac{J}{\log(e+J)} pK \leqslant \frac{1}{p}\big(J + e^{pK} - 1\big)$$

for all $p > 0$. Thus

$$\int_\Omega \frac{|Df|^2}{\log(e+|Df|)} \leqslant \frac{2}{p}\int_\Omega J(z,f) + \frac{2}{p}\int_\Omega \big[e^{pK(z)} - 1\big]\,dz \tag{9}$$

for any bounded domain. This shows that f automatically belongs to the Orlicz–Sobolev class $W^{1,Q}_{loc}(\mathbb{C})$.

Conversely in [14, Theorem 7.2.1] it is shown that for an orientation preserving mapping f the $L^2\log^{-1}L$-integrability of the differential Df implies that $J(z,f) \in L^1_{loc}(\Omega)$. This slight gain in the regularity of the Jacobian determinant is precisely what makes the space $W^{1,Q}_{loc}$ the natural framework for the theory of mappings of exponentially integrable distortion.

The structure of this paper is the following. First in Section 2 we recall the basic modulus of continuity estimates for mappings of finite distortion. Section 3 presents the crucial sharp estimates for the rate of decay of the Neumann series of the solution to the Beltrami equation, in the case of an exponentially integrable distortion. Theorem 1.3 is then proven in Section 4, where our presentation is self-contained, apart from the generalized uniqueness and Stoilow factorization theorem due to Iwaniec and Martin (see Theorem 4.2 below). The proof of our main result, Theorem 1.1 is the content of Section 5. In that section we also obtain optimal area distortion estimates. The optimality of our results on the rate of convergence of the Neumann series is considered in Section 6. Finally, Section 7 gives an application of the optimal regularity results to degenerate elliptic equations.

2. Modulus of continuity

There is a particularly elegant geometric approach to obtaining modulus of continuity estimates for functions that are monotone. We will not consider the notion of monotonicity in its full generality, but just say that a continuous function u in a domain Ω is *monotone*, if for each relatively compact subdomain $\Omega' \subset \Omega$ we have

$$\max_{\partial\Omega'} u = \max_{\Omega'} u \quad \text{and} \quad \min_{\partial\Omega'} u = \min_{\Omega'} u,$$

that is, if u satisfies both the maximum and minimum principle. For instance, coordinates of open mappings are monotone.

The idea of the next well-known argument goes back to Gehring in his study of the Liouville theorem in space.

Lemma 2.1. *Let $u \in W^{1,2}(3\mathbb{D})$ be a continuous and monotone function. Then for all $a, b \in \mathbb{D}$ we have*

$$\left|u(a) - u(b)\right|^2 \leqslant \frac{\pi \int_{3\mathbb{D}} |\nabla u|^2}{\log(e + \frac{1}{|a-b|})}. \tag{10}$$

Proof. Consider the two concentric disks $\mathbb{D}(z, r)$ and $\mathbb{D}(z, 1)$, where $z = \frac{1}{2}(a + b)$ and $r = \frac{1}{2}|a - b|$. Note that all disks $\mathbb{D}(z, t)$ with $t \leqslant \rho_0 = \max\{1, 2r\}$ lie in $3\mathbb{D}$.

We have by monotonicity

$$\left|u(a) - u(b)\right| \leqslant \frac{1}{2} \int_{\partial\mathbb{D}(z,t)} |\nabla u|\, |dw|$$

for almost all $r < t < 1$. By Cauchy–Schwartz

$$\frac{|u(a) - u(b)|^2}{t} \leqslant \frac{\pi}{2} \int_{\partial\mathbb{D}(z,t)} |\nabla u|^2\, |dw|.$$

We integrate with respect to t, $r < t < \rho_0$, to get

$$\left|u(a) - u(b)\right|^2 \log(\rho_0/r) \leqslant \frac{\pi}{2} \int_{\mathbb{D}(z,\rho_0)} |\nabla u|^2\, dm \leqslant \frac{\pi}{2} \int_{3\mathbb{D}} |\nabla u|^2\, dm.$$

The estimate (10) is a quick consequence. $\square$

In view of this result uniform bounds for the L^2-derivatives become valuable. Within the (uniformly elliptic) Beltrami equation such bounds can be obtained for the *inverse* of the principal solution. Iwaniec and Šverák [19] were the first to make a systematic use of this phenomenon.

Theorem 2.2. *Let $f \in W^{1,2}_{loc}(\mathbb{C})$ be the principal solution to $f_{\bar z}(z) = \mu(z) f_z(z)$, $z \in \mathbb{C}$, where μ is supported in the unit disk $\mathbb{D}$ with $\|\mu\|_\infty < 1$. Let $g = f^{-1}$ be the inverse of f. Then*

$$\int_{\mathbb{C}} \left(|g_{\bar w}|^2 + |g_w - 1|^2\right) dw \leqslant 2 \int_{\mathbb{D}} K(z, f)\, dz.$$

Proof. Since the function $g(z) - z$ has derivatives in $L^2(\mathbb{C})$, an integration by parts shows that the integral of its Jacobian vanishes. This gives

$$\int_{\mathbb{C}} |g_w - 1|^2 - |g_{\bar w}|^2 = \int_{\mathbb{C}} J(w, g) = 0.$$

Therefore

$$\int_{\mathbb{C}} |g_w - 1|^2 = \int_{\mathbb{C}} |g_{\bar w}|^2 = \int_{f(\mathbb{D})} |g_{\bar w}|^2 \leqslant \int_{f(\mathbb{D})} |Dg|^2 = \int_{f(\mathbb{D})} K(w, g) J(w, g) = \int_{\mathbb{D}} K\big(f(z), g\big) = \int_{\mathbb{D}} K(z, f)$$

since quasiconformal mappings satisfy the usual rules of change of variables. $\square$

The key fact in the following modulus of continuity estimate, as in Theorem 2.2, is that the bounds do not depend on the value of $\|\mu\|_\infty$. It is obtained easily by combining Lemma 2.1 with Theorem 2.2.

Corollary 2.3. *Let f and $g = f^{-1}$ be as in Theorem 2.2, and let $R \geqslant 1$. Then*

$$\left|g(a) - g(b)\right|^2 \leqslant \frac{(4\pi)^2(R^2 + \int_{\mathbb{D}} K(z, f)\,dz)}{\log(e + \frac{1}{|a-b|})}$$

whenever $a, b \in \mathbb{D}(0, R)$.

3. Decay of the Neumann series

According to the classical measurable Riemann mapping theorem [8,1,4], given a compactly supported Beltrami coefficient with $|\mu(z)| \leqslant k < 1$ almost everywhere, the system

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z}, \quad f \in W^{1,2}_{loc}(\mathbb{C}),$$

always admits a homeomorphic solution f with the development (8). That is, f is a principal solution.

This uniformly elliptic equation is most conveniently solved by applying a Neumann-series argument using the Beurling operator $\mathcal{S}$. This is defined by the principal value integral

$$\mathcal{S}\varphi(z) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\varphi(\tau)}{(z-\tau)^2}\,d\tau.$$

The Beurling operator acts unitarily on $L^2(\mathbb{C})$. Also, we have $\mathcal{S}(h_{\bar{z}}) = h_z$ for every $h \in W^{1,2}(\mathbb{C})$.

We then look for the solution in the form

$$f = z + \mathcal{C}(\omega), \qquad C\omega(z) := \frac{1}{\pi}\int_{\mathbb{C}} \frac{\omega(\tau)}{(z-\tau)}\,d\tau,$$

where $\mathcal{C}$ is the Cauchy transform and $\omega = f_{\bar{z}}$ is to satisfy the identity

$$\omega(z) = \mu(z)\mathcal{S}\omega(z) + \mu(z) \quad \text{for almost every } z \in \mathbb{C}.$$

Such an ω is easy to find when $\|\mu\|_\infty \leqslant k < 1$. We define

$$\omega := (\mathbf{I} - \mu\mathcal{S})^{-1}\mu = \mu + \mu\mathcal{S}\mu + \mu\mathcal{S}\mu\mathcal{S}\mu + \mu\mathcal{S}\mu\mathcal{S}\mu\mathcal{S}\mu + \cdots. \tag{11}$$

The series converges in $L^2(\mathbb{C})$ since the operator norm of the nth iterate

$$\|\mu\mathcal{S}\mu\mathcal{S}\cdots\mu\mathcal{S}\|_{L^2(\mathbb{C})\to L^2(\mathbb{C})} \leqslant k^n, \quad n \in \mathbb{N}.$$

The converging series provides us with the solution f. However, showing that f is a homeomorphism takes some more effort. For this and other basic facts in the classical theory of quasiconformal maps we refer the reader to [1,4,21].

It is natural to study how far the above method carries in the situation of degenerate equations, even if now $\|\mu\|_\infty = 1$ and the Neumann series of $(\mathbf{I} - \mu\mathcal{S})^{-1}$ cannot converge in the operator norm. Hence the main question here is the rate of decay for the L^2-norms of the terms in the series (11). The next result, building on the approach initiated by David [10], gives an optimal answer.

Theorem 3.1. *Suppose $|\mu(z)| < 1$ almost everywhere, with $\mu(z) \equiv 0$ for $|z| > 1$. If the distortion function $K(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|}$ satisfies*

$$e^K \in L^p(\mathbb{D}), \quad p > 0,$$

then we have for every $0 < \beta < p$,

$$\int_{\mathbb{C}} \left|(\mu\mathcal{S})^n\mu\right|^2 \leqslant C_0(n+1)^{-\beta}, \quad n \in \mathbb{N},$$

where $C_0 = C_{p,\beta} \cdot \int_{\mathbb{D}} e^{pK}$ with $C_{p,\beta}$ depending only on β and p.

Proof. Since

$$K(z)+1=\frac{2}{1-|\mu(z)|},$$

with Chebychev's inequality we have the measure estimates

$$\left|\left\{z\in\mathbb{D}\colon\ \left|\mu(z)\right|\geqslant 1-\frac{1}{t}\right\}\right|\leqslant e^{-2pt}\int\limits_{\mathbb{D}} e^{p(K+1)}=Ce^{-2pt}$$

for each $t>1$. With this control on the Beltrami coefficient we will iteratively estimate the terms $(\mu\mathcal{S})^n\mu$ of the Neumann series. For this purpose we first fix the parameter $0<\beta<p$, and then for each $n\in\mathbb{N}$ divide the unit disk into the "bad" points

$$B_n=\left\{z\in\mathbb{D}\colon\ \left|\mu(z)\right|>1-\frac{\beta}{2n+\beta}\right\}$$

and the "good" ones, i.e. the complements

$$G_n=\mathbb{D}\setminus B_n.$$

The above bounds on area read now

$$|B_n|\leqslant C_1e^{-4np/\beta},\quad n\in\mathbb{N},$$

where $C_1=e^{-p}\int_{\mathbb{D}}e^{pK}$.

Next, let us consider the terms $\psi_n=(\mu\mathcal{S})^n\mu$ of the Neumann series, obtained inductively by

$$\psi_n=\mu\mathcal{S}(\psi_{n-1}),\quad \psi_0=\mu.$$

It is helpful to first look at the following auxiliary terms

$$g_n=\chi_{G_n}\mu\mathcal{S}(g_{n-1}),\quad g_0=\mu,$$

that is, in each iterative step we restrict μ to the corresponding good part of the disk.

The terms g_n are easy to estimate

$$\|g_n\|_{L^2}^2=\int\limits_{G_n}\left|\mu\mathcal{S}(g_{n-1})\right|^2\leqslant\left(1-\frac{\beta}{2n+\beta}\right)^2\|g_{n-1}\|_{L^2}^2.$$

Thus

$$\|g_n\|_{L^2}\leqslant\prod_{j=1}^{n}\left(1-\frac{\beta}{2j+\beta}\right)\|\mu\|_{L^2}\leqslant C_\beta n^{-\beta/2},$$

where $C_\beta=\sqrt{\pi}(1+\beta/2)^{\beta/2}$. For the true terms ψ_n decompose

$$\psi_n-g_n=\chi_{G_n}\mu\mathcal{S}(\psi_{n-1}-g_{n-1})+\chi_{B_n}\mu\mathcal{S}(\psi_{n-1}).$$

This gives the norm bounds

$$\|\psi_n-g_n\|_{L^2}^2\leqslant\left(1-\frac{\beta}{2n+\beta}\right)^2\|\psi_{n-1}-g_{n-1}\|_{L^2}^2+R(n),\tag{12}$$

where

$$R(n)=\left\|\chi_{B_n}\mu\mathcal{S}(\psi_{n-1})\right\|_{L^2}^2=\int\limits_{B_n}\left|(\mu\mathcal{S})^n\mu\right|^2.$$

A main step of the proof is to bound this last integral term. A natural approach is to apply Hölder's inequality and estimate the norms $\|(\mu\mathcal{S})^n\mu\|_p$ for $p>2$. This approach can be carried over by applying the spectral bounds on the operator μS obtained in [5]. On the other hand, these bounds are based on the area distortion results from [2]. To reveal more clearly the heart of the argument we will therefore directly use the area distortion. This can be done with a holomorphic representation as follows.

Recall that the principal solution $f = f^\lambda$ to the equation

$$f_{\bar z}(z) = \lambda\mu(z) f_z(z) \tag{13}$$

depends holomorphically on the parameter $\lambda \in \mathbb{D}$. This fact follows from the Neumann series representation for the derivative

$$f^\lambda_{\bar z} = \lambda\mu + \lambda^2\mu\mathcal{S}\mu + \cdots + \lambda^n(\mu\mathcal{S})^{n-1}\mu + \cdots. \tag{14}$$

The series converges absolutely in $L^2(\mathbb{C})$ since $\|\lambda\mu\|_\infty \leqslant |\lambda| < 1$ and $\mathcal{S}$ is an L^2-isometry.

The $L^2(\mathbb{C})$-valued holomorphic function $\lambda \to f^\lambda_{\bar z}$ can as well be represented by the Cauchy integral. From this we obtain the following integral representation

$$\chi_E(\mu\mathcal{S})^n\mu = \frac{1}{2\pi i}\int_{|\lambda|=\rho} \frac{1}{\lambda^{n+2}} f^\lambda_{\bar z}\chi_E\, d\lambda, \quad E \subset \mathbb{D}, \tag{15}$$

valid for any $0 < \rho < 1$. We are hence to estimate the norms

$$\|f^\lambda_{\bar z}\chi_E\|^2_{L^2} = \int_E |f^\lambda_{\bar z}|^2 \leqslant \frac{|\lambda|^2}{1-|\lambda|^2}\int_E J(z, f^\lambda) = \frac{|\lambda|^2}{1-|\lambda|^2}\big|f^\lambda(E)\big|,$$

and it is here that the need for a quasiconformal area distortion estimate arises. We have the uniform bounds

$$\big|f^\lambda(E)\big| \leqslant \pi M|E|^{1/M}, \qquad |\lambda| = \frac{M-1}{M+1}, \quad M > 1, \tag{16}$$

see [6, Theorem 1.6] or [4, Theorem 13.1.4].

If we now denote $\rho = |\lambda| = \frac{M-1}{M+1}$ and combine (15) with the above estimates, we end up with the result

$$\big\|\chi_E(\mu\mathcal{S})^n\mu\big\|_2 \leqslant \sqrt{\pi}\left(\frac{M+1}{M-1}\right)^n \frac{M+1}{2}|E|^{1/(2M)}. \tag{17}$$

The estimate is valid for every $M > 1$ and for any Beltrami coefficient with $|\mu| \leqslant \chi_{\mathbb{D}}$ almost everywhere.

For later purposes we recall that the ∂_z-derivative, too, has a power series representation

$$f^\lambda_z - 1 = \mathcal{S}f^\lambda_{\bar z} = \lambda\mathcal{S}\mu + \lambda^2\mathcal{S}(\mu\mathcal{S}\mu) + \cdots + \lambda^n\mathcal{S}(\mu\mathcal{S})^{n-1}\mu + \cdots.$$

With an analogous argument we obtain

$$\big\|\chi_E\mathcal{S}(\mu\mathcal{S})^n\mu\big\|_2 \leqslant \sqrt{\pi}\left(\frac{M+1}{M-1}\right)^{n+1} \frac{M+1}{2}|E|^{1/(2M)}, \tag{18}$$

an estimate which is similarly valid for every $M > 1$.

Let us then return to estimating the original Neumann series. Unwinding the iteration in (12) gives

$$\|\psi_n - g_n\|^2_{L^2} \leqslant \sum_{j=1}^n R(j)\prod_{k=j+1}^n\left(1 - \frac{\beta}{2k+\beta}\right)^2 \leqslant 2^\beta C^2_\beta n^{-\beta}\sum_{j=1}^n j^\beta R(j).$$

It remains to show that the remainder term $R(n)$ decays exponentially. But here we may use the bound (17) for the set $E = B_n$. We have $|B_n| \leqslant C_1 e^{-4np/\beta}$ and hence

$$R(n) \leqslant 4M^2\left(\frac{M+1}{M-1}\right)^{2n}|E|^{1/M} \leqslant 4C_1M^2\left(\frac{M+1}{M-1}\right)^{2n} e^{-\frac{4n}{M}\frac{p}{\beta}}.$$

Given $\beta < p$ we can choose $M > 1$ so that

$$\log\left(\frac{M+1}{M-1}\right) - \frac{2}{M}\frac{p}{\beta} < -\delta < 0$$

for some $\delta > 0$. With this choice $R(n) \leqslant Ce^{-2\delta n}$, $n \in \mathbb{N}$, where $C = 4M^2e^{-p}\int_{\mathbb{D}} e^{pK}$. This completes the proof. □

According to the theorem, the terms of the Neumann series decay with the rate

$$\left\|(\mu\mathcal{S})^n\mu\right\|_2 \leqslant C_\beta n^{-\beta/2} \quad \text{for any } \beta < p.$$

If the decay would be a little better, of the order $n^{-\beta}$, then for any $p > 1$ the series would be norm convergent in $L^2(\mathbb{C})$. In view of the example (3), this would immediately determine $p_0 = 1$ as the critical exponent for the $W^{1,2}$-regularity. However, as we will see later in Section 6, the order of decay in Theorem 3.1 cannot be improved. Hence further means are required for optimal regularity, to be discussed in Section 5.

Nevertheless, at the exponent $p_0 = 1$ there is an interesting interpretation of the above bounds, in terms of the vector-valued Hardy spaces. Namely, suppose μ is as in Theorem 3.1 and $p > \beta > 1$. Then

$$\sum_{n=0}^{\infty}\left\|(\mu\mathcal{S})^n\mu\right\|_{L^2}^2 < \infty. \tag{19}$$

On the other hand, as in (13) and (14), consider $\lambda \to f_{\bar z}^\lambda$ as a holomorphic vector-valued function $\mathbb{D} \to L^2(\mathbb{C})$. With the above bounds the function has square-summable Taylor coefficients $(\mu\mathcal{S})^n\mu$. In other words, it belongs to the vector-valued Hardy space $H^2(\mathbb{D}; L^2(\mathbb{C}))$. In particular, since the target Banach space $L^2(\mathbb{C})$ has the Radon–Nikodym property, the radial limits

$$\lim_{r\to 1}\frac{\partial f_{r\zeta}}{\partial \bar z} \in L^2(\mathbb{C}), \quad \zeta \in \mathbb{S}^1,$$

exist in $L^2(\mathbb{C})$ for almost every $\zeta \in \mathbb{S}^1$, see e.g. [7]. Similarly for the derivative $\partial_{\bar z} f_{r\zeta} - 1 \in L^2(\mathbb{C})$, and it is not difficult to see that the limiting derivatives are derivatives of a principal solution f_ζ, solving

$$\frac{\partial f_\zeta}{\partial \bar z} = \zeta\mu(z)\frac{\partial f_\zeta}{\partial z}. \tag{20}$$

Thus already here we obtain, for $p > 1$ and for almost every $\zeta \in \mathbb{S}^1$, the $W^{1,2}_{loc}$-regular solutions to (20).

For exponents $\beta > 2$, we have via the norm convergence (19) the next auxiliary result, which the reader may recognize as a distortion of area for mappings in the exponential class. The result here is rather weak but is a necessary step towards the optimal measure distortion bounds which will be established in Section 5.

Corollary 3.2. *Suppose μ and $0 < \beta < p$ are as in Theorem 3.1 and*

$$\sigma_\mu := \sum_{n=0}^{\infty}(\mu\mathcal{S})^n\mu.$$

If $\beta > 2$, then

$$\|\chi_E\sigma_\mu\|_2 + \|\chi_E\mathcal{S}\sigma_\mu\|_2 \leqslant C\log^{1-\beta/2}\left(e + \frac{1}{|E|}\right), \quad E \subset \mathbb{D},$$

where $C = C_0C_\beta < \infty$ with C_β depending only β.

Proof. Theorem 3.1 and its proof give us two different ways to estimate the terms. First,

$$\left\|\chi_E(\mu\mathcal{S})^n\mu\right\|_2 + \left\|\chi_E\mathcal{S}(\mu\mathcal{S})^n\mu\right\|_2 \leqslant 2\left\|(\mu\mathcal{S})^n\mu\right\|_2 \leqslant 2C_0(n+1)^{-\beta/2}.$$

Summing this up gives

$$\sum_{n=m+1}^{\infty}\left\|\chi_E(\mu\mathcal{S})^n\mu\right\|_2 + \left\|\chi_E\mathcal{S}(\mu\mathcal{S})^n\mu\right\|_2 \leqslant \frac{4C_0}{\beta-2}m^{1-\beta/2}, \quad m \in \mathbb{N}.$$

Secondly, let us choose, say, $M = 3$ in (17) and (18). Then for every $n \in \mathbb{N}$ we have the estimates $\|\chi_E(\mu\mathcal{S})^n\mu\|_2 + \|\chi_E\mathcal{S}(\mu\mathcal{S})^n\mu\|_2 \leqslant 6\sqrt{\pi}\cdot 2^n|E|^{1/6}$. This, in turn, leads to

$$\sum_{n=0}^{m}\left\|\chi_E(\mu\mathcal{S})^n\mu\right\|_2 + \left\|\chi_E\mathcal{S}(\mu\mathcal{S})^n\mu\right\|_2 \leqslant 6\sqrt{\pi}\cdot 2^{m+1}|E|^{1/6}.$$

Combining we arrive at

$$\|\chi_E \sigma_\mu\|_2 + \|\chi_E \mathcal{S}\sigma_\mu\|_2 \leqslant \frac{4C_0}{\beta - 2} m^{1-\beta/2} + 22 \cdot 2^m |E|^{1/6}.$$

The bound holds for any integer $m \in \mathbb{N}$, but of course it is best when the two terms on the right-hand side are roughly equal. We let

$$\frac{1}{10}\log\left(e + \frac{1}{|E|}\right) \leqslant m < 1 + \frac{1}{10}\log\left(e + \frac{1}{|E|}\right),$$

and with this choice the terms on the right-hand side of the above bound are both smaller than $C_0 C_\beta \log^{1-\beta/2}(e + \frac{1}{|E|})$, where the constant C_β depends only on the parameter $\beta > 2$. □

4. The degenerate Measurable Riemann Mapping Theorem

We will next give a simple proof of David's theorem, the generalization of the Measurable Riemann Mapping Theorem to the setting of exponentially integrable distortion. We give this presentation not only for the reader's convenience, but also since a few bits and pieces of the proof will be needed in the main result, the optimal regularity Theorem 1.1.

We start with the case where the exponential distortion $e^K \in L^p_{loc}$ for some $p > 2$. The key idea here is that for such large p, the Neumann series (11) converges in $L^2(\mathbb{C})$, and this quickly yields homeomorphic $W^{1,2}$-regular solutions to the corresponding Beltrami equation (2). From Section 2 we already know the usefulness of such regularity, providing us with general equicontinuity properties.

After the case of high exponential integrability, together with a few further consequences established, we then return to the general measurable Riemann mapping theorem and prove this towards the end of this section.

Theorem 4.1. *If μ is a Beltrami coefficient such that*

$$|\mu(z)| \leqslant \frac{K(z) - 1}{K(z) + 1}\chi_{\mathbb{D}},$$

where

$$e^K \in L^p(\mathbb{D}) \quad \textit{for some } p > 2,$$

then the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z} \quad \textit{for almost every } z \in \mathbb{C},$$

admits a unique principal solution $f \in W^{1,2}_{loc}(\mathbb{C})$.

Proof. Consider the following good approximations for the Beltrami coefficient $\mu(z)$,

$$\mu_m(z) = \begin{cases} \mu(z) & \text{if } |\mu(z)| \leqslant 1 - \frac{1}{m}, \\ (1 - \frac{1}{m})\frac{\mu(z)}{|\mu(z)|} & \text{otherwise,} \end{cases} \tag{21}$$

defined for $m = 1, 2, \ldots$. Of course $|\mu_m(z)| \leqslant 1 - \frac{1}{m} < 1$ and we also have the bound

$$|\mu_m(z)| \leqslant \frac{K(z) - 1}{K(z) + 1},$$

which is independent of m. Similarly the corresponding distortion functions satisfy

$$K_m(z) = \frac{1 + |\mu_m(z)|}{1 - |\mu_m(z)|} \leqslant \frac{1 + |\mu(z)|}{1 - |\mu(z)|} = K(z), \quad m = 1, 2, \ldots.$$

Note that each $K_m(z) \in L^\infty(\mathbb{C})$.

With the classical measurable Riemann mapping theorem, see [8,1,4] or [21], we have unique principal solutions $f^m : \mathbb{C} \to \mathbb{C}$ to the Beltrami equation

$$(f^m)_{\bar z} = \mu_m (f^m)_z, \quad m = 1, 2, \ldots.$$

Moreover,

$$f^m = z + \mathcal{C}(\omega_m),$$

where $\omega_m = (f^m)_{\bar z}$ satisfies the identity

$$\omega_m(z) = \mu_m(z)(\mathcal{S}\omega_m)(z) + \mu_m(z) \quad \text{almost every } z \in \mathbb{C}.$$

In order for these approximate solutions f^m to converge we need uniform L^2-bounds for their derivatives; once these have been established the proof follows quickly.

We use Theorem 3.1. This gives for any $2 < \beta < p$ the estimates

$$\|(\mu_m \mathcal{S})^k \mu_m\|_2 \leqslant C_m k^{-\beta/2}, \quad k \in \mathbb{N}.$$

The theorem also gives uniform bounds for the constant term C_m. Since the approximate solutions have distortion $K_m \leqslant K$ pointwise, we have

$$C_m \leqslant \sqrt{C_{p,\beta}} \left(\int_{\mathbb{D}} e^{pK_m} \right)^{1/2} \leqslant \sqrt{C_{p,\beta}} \left(\int_{\mathbb{D}} e^{pK} \right)^{1/2} = \sqrt{C_0},$$

where C_0 is as in Theorem 3.1. Hence if $e^K \in L^p(\mathbb{D})$ for some $p > 2$, we obtain L^2-estimates for the sums (11),

$$\|(f^m)_{\bar z}\|_2 \leqslant \sum_{k=0}^{\infty} \|(\mu_m \mathcal{S})^k \mu_m\|_2 \leqslant \sqrt{C_0} \sum_{k=0}^{\infty} (k+1)^{-\beta/2}. \tag{22}$$

Hence also

$$\|(f^m)_z - 1\|_2 = \|(f^m)_{\bar z}\|_2 \leqslant \sqrt{C_0} \sum_{k=0}^{\infty} (k+1)^{-\beta/2}.$$

This easily yields that

$$\int_{\mathbb{D}(R)} |Df^m|^2 \leqslant 4\pi R^2 + A, \quad R > 1, \tag{23}$$

where $A < \infty$ depends only on $\beta > 2$ and $\int_{\mathbb{D}} e^{pK}$.

Now Lemma 2.1 applies and shows that on compact subsets of the plane the sequence $\{f^m\}$ has a uniform modulus of continuity. Changing to a subsequence, and invoking (23) again we may assume that $f^m(z) \to f(z)$ locally uniformly with the derivatives Df^m converging weakly in $L^2_{loc}(\mathbb{C})$. Their weak limit necessarily equals Df. Furthermore, since for any $R > 1$ and $\varphi \in C_0^\infty(B(0,R))$

$$\int_{\mathbb{C}} \varphi(f^m_{\bar z} - \mu f^m_z) = \int_{\mathbb{C}} \varphi(\mu_m - \mu) f^m_z \leqslant \frac{\sqrt{\pi}\|\varphi\|_\infty}{m} \sqrt{4\pi R^2 + A} \to 0$$

by (23), we see that f is a solution to the Beltrami equation

$$f_{\bar z} = \mu f_z \quad \text{for almost every } z \in \mathbb{C}.$$

On the other hand, we can use Corollary 2.3 for the inverse mappings $g_m = (f^m)^{-1}$ and see that $g^m(z) \to g(z)$ uniformly on compact subsets of the plane, where $g : \mathbb{C} \to \mathbb{C}$ is continuous. From $g^m(f^m(z)) = z$ we have $g \circ f(z) = z$. That is, f is a homeomorphism, hence the principal solution we were looking for.

Last, for the uniqueness we refer to the following very general Stoilow factorization result from the monograph [14]; see also [4, Section 20.4.8]. The proof of Theorem 4.1 is now complete. □

Theorem 4.2. *(See [14, Theorem 11.5.1].) Let*

$$Q(t) = \frac{t^2}{\log(e+t)} \tag{24}$$

and suppose we are given a homeomorphic solution $f \in W^{1,Q}_{loc}(\Omega)$ to the Beltrami equation

$$f_{\bar z} = \mu(z) f_z, \quad z \in \Omega, \tag{25}$$

where $|\mu(z)| < 1$ almost everywhere. Then every other solution $h \in W^{1,Q}_{loc}(\Omega)$ to (25) takes the form

$$h(z) = \phi\big(f(z)\big), \quad z \in \Omega,$$

where $\phi : f(\Omega) \to \mathbb{C}$ is holomorphic.

In Theorem 4.1 we constructed the mapping f through a limiting process rather than directly by the Neumann series. Nevertheless, the representations

$$f_{\bar z} = \sum_{n=0}^{\infty} (\mu \mathcal{S})^n \mu, \qquad f_z - 1 = \sum_{n=0}^{\infty} \mathcal{S}(\mu \mathcal{S})^n \mu \tag{26}$$

are still valid. Namely, by Theorem 3.1 the sums are absolutely convergent in $L^2(\mathbb{C})$. Since the approximate coefficients $\mu_m \to \mu$ in $L^\infty(\mathbb{C})$, the terms $(\mu_m \mathcal{S})^k \mu_m \to (\mu \mathcal{S})^k \mu$ weakly in $L^2(\mathbb{C})$. As the derivatives of f^m are representable by the corresponding Neumann series, the identities (26) follow with the help of the uniform bounds (22).

In studying the general exponentially integrable distortion we need to know that the mappings of Theorem 4.1 preserve sets of Lebesgue measure zero. This follows easily from the above.

Corollary 4.3. *If μ and the principal solution $f \in W^{1,2}_{loc}(\mathbb{C})$ are as in Theorem 4.1, then f has the properties $\mathcal{N}$ and $\mathcal{N}^{-1}$, i.e. for any measurable set $E \subset \mathbb{C}$,*

$$\big|f(E)\big| = 0 \quad \Leftrightarrow \quad |E| = 0.$$

Proof. Any homeomorphism with the $W^{1,2}_{loc}$-regularity preserves Lebesgue null sets, see [22] or [4] for an elementary proof. We have shown that $f \in W^{1,2}_{loc}(\mathbb{C})$, and since $K(\cdot, f) \in L^1(\mathbb{D})$, Theorem 2.2 verifies that the inverse map $g = h^{-1}$ belongs to $W^{1,2}_{loc}(\mathbb{C})$. The claim follows. □

Concerning the existence of solutions, the situation is rather different in the general case where the integrability exponent of e^K is small. Recall that by example (3) the corresponding principal solutions need not be in $W^{1,2}_{loc}(\mathbb{C})$. Instead, inequality (9) leads us to look for principal solutions f in $W^{1,Q}_{loc}(\mathbb{C})$, where $Q(t)$ is given in (24). The idea of the following proof is then to reduce the argument to Theorem 4.1 via a suitable factorization.

Proof of Theorem 1.3. Given a Beltrami coefficient $\mu = \mu(z)$ with the distortion function $K(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|}$, assume that $e^K \in L^p(\mathbb{D})$ for some $0 < p \leqslant 2$. It is very suggestive to write

$$K(z) = \frac{3}{p} \cdot \frac{pK(z)}{3}.$$

Then $K_1(z) = pK(z)/3$ satisfies the hypotheses of Theorem 4.1 and the constant factor $K_2(z) = 3/p > 1$ could be represented as the distortion of a quasiconformal mapping. There is one problem here in that at points where $K(z)$ is already finite and perhaps small, the factor $K_1(z)$ might be less than 1 and so cannot be a distortion function. We will get around this point by constructing the related Beltrami coefficients using the hyperbolic geometry.

Let $M > 1$. For each $z \in \mathbb{D}$ choose a point $\nu = \nu(z)$ on the radial segment determined by $\mu(z)$, so that

$$\rho_{\mathbb{D}}(0, \nu) + \rho_{\mathbb{D}}(\nu, \mu) = \rho_{\mathbb{D}}(0, \mu) = \log \frac{1+|\mu|}{1-|\mu|}.$$

If $\rho_{\mathbb{D}}(0,\mu) > \log M$ we require $\rho_{\mathbb{D}}(\nu,\mu) = \log M$ and otherwise we set $\nu = 0$. In any case we always have

$$MK_\nu = e^{\log M} e^{\rho_{\mathbb{D}}(0,\nu)} \leqslant K_\mu + M.$$

It follows that

$$\int_{\mathbb{D}} e^{pMK_\nu} \leqslant e^{pM} \int_{\mathbb{D}} e^{pK_\mu} < \infty, \tag{27}$$

so that ν satisfies the hypotheses of Theorem 4.1 as soon as we choose $M = \frac{3}{p}$. We can therefore solve the Beltrami equation for $\nu = \nu(z)$ to get a principal mapping F of class $W^{1,2}(\mathbb{D})$. Next, set

$$\kappa(w) = \frac{\mu(z) - \nu(z)}{1 - \mu(z)\overline{\nu(z)}} \left(\frac{F_z}{|F_z|}\right)^2, \quad w = F(z),\ z \in \mathbb{C}. \tag{28}$$

According to Corollary 4.3 κ is well defined almost everywhere. We also see that

$$\frac{1+|\kappa|}{1-|\kappa|} = \frac{1 + |\frac{\mu-\nu}{1-\mu\bar{\nu}}|}{1 - |\frac{\mu-\nu}{1-\mu\bar{\nu}}|} = e^{\rho_{\mathbb{D}}(\nu,\mu)} \leqslant M = \frac{3}{p} < \infty.$$

Thus we may solve the uniformly elliptic Beltrami equation for κ to obtain an M-quasiconformal principal mapping g.

We next put $f = g \circ F$. The classical Gehring–Lehto result [13] on Sobolev homeomorphisms verifies that F is differentiable almost everywhere. Since F has the Lusin property $\mathcal{N}^{\pm 1}$, the same is true for f, and we have

$$\left|Df(z)\right|^2 \leqslant MJ(w,g)K(z,F)J(z,F) = MK_\nu(z)J(z,f), \quad w = F(z).$$

Arguing as in (9) we see that

$$\frac{|Df|^2}{\log(e + |Df|)} \leqslant \frac{2}{p}\left[J(\cdot, f) + e^{pMK_\nu}\right] \tag{29}$$

is locally integrable.

In the proof of Theorem 4.1 we constructed the approximants F^m of the principal mapping F. To obtain uniform bounds we may apply the classical Bieberbach area theorem [11]. This gives

$$\int_{B(0,r)} J(z,\phi) \leqslant \pi r^2, \quad r \geqslant 1,$$

for any quasiconformal principal solution, conformal outside the unit disc. In particular with (29) we see that the compositions $g \circ F^m$ are uniformly bounded in the space $W^{1,Q}_{loc}(\mathbb{C})$,

$$Q(t) = \frac{t^2}{\log(e+t)}.$$

Since $g \circ F^m(z) \to f(z)$ locally uniformly, we infer that f is a Sobolev mapping contained in the class $W^{1,Q}_{loc}(\mathbb{C})$. However, note that although the quasiconformal map g will lie [2] in the space $W^{1,s}_{loc}(\mathbb{C})$ for all

$$2 \leqslant s < \frac{6}{3-p},$$

yet the composition $f = g \circ F$ will not lie in $W^{1,2}_{loc}(\mathbb{C})$ in general.

We defined the Beltrami coefficient $\kappa = \mu_g$ through the formula (28), but as well one may identify $\mu_g = \mu_{f\circ F^{-1}}$ via the familiar formula for the dilatation of a composition of mappings

$$\mu_{f\circ F^{-1}}(w) = \frac{\mu_f(z) - \mu_F(z)}{1 - \mu_f(z)\overline{\mu_F(z)}} \left(\frac{F_z(z)}{|F_z(z)|}\right)^2, \quad w = F(z),$$

which is a direct consequence of the chain rule. Comparing the expressions shows that $\mu_f = \mu$, and hence f in fact solves the required Beltrami equation $f_{\bar{z}} = \mu f_z$. Clearly f is a principal solution. Therefore we have established the existence of the measurable Riemann mapping, for any exponentially integrable distortion.

Uniqueness and Stoilow factorization now follow from Theorem 4.2. Thus the proof of Theorem 1.3 is complete. □

Once we have the existence of the principal solution, with the equicontinuity properties provided by Lemma 2.1 and Corollary 2.3, the usual normal family arguments quickly give solutions to the global degenerate Beltrami equation, where the coefficient $\mu(z)$ is not necessarily compactly supported. For an overview, see [4, Section 20.5].

From the above proof and (27) we distill a powerful factorization, one of the key facts in obtaining the sharp regularity.

Corollary 4.4. *Suppose the distortion function $K = K(z)$ satisfies $e^K \in L^p(\mathbb{D})$ for some $p > 0$. Then for any $M \geqslant 1$ the principal solution to $f_{\bar z}(z) = \mu(z) f_z(z)$ admits a factorization*

$$f = g \circ F,$$

where both g and F are principal mappings, g is M-quasiconformal and F satisfies

$$\int_{\mathbb{D}} e^{pMK(z,F)} \leqslant C_0 < \infty.$$

Remark. Using the above factorization we obtain the properties $\mathcal{N}$ and $\mathcal{N}^{-1}$ for all maps of exponential distortion. Namely, assume that $\phi \in W^{1,Q}_{loc}(\mathbb{C})$ is a mapping of finite distortion with $e^{K(z,\phi)} \in L^p_{loc}(\Omega)$ for some $p > 0$. Then first use locally the Stoilow factorization $\phi = h \circ f$ where h is holomorphic and f is a principal solution as in Theorem 1.3. Applying Corollary 4.4 we can make a further factorization, $f = g \circ F$ where g is quasiconformal and F has exponential distortion $e^K \in L^p$ with $p > 2$. According to Corollary 4.3 each factor in $\phi = h \circ g \circ F$ has the properties $\mathcal{N}^{\pm 1}$.

5. Optimal regularity: Proof of Theorem 1.1

We start with general area distortion bounds that have independent interest, since they are optimal up to estimates at the borderline case. In fact, the factorization method described in Corollary 4.4 enables us to improve the distortion exponent of Corollary 3.2.

Theorem 5.1. *Let $|\mu(z)| < 1$ almost everywhere and $\mu(z) \equiv 0$ outside the unit disk. Suppose f is a principal solution to $f_{\bar z} = \mu f_z$. If*

$$e^{K(z,f)} \in L^p(\mathbb{D}) \quad \textit{for some } p > 0,$$

then for any $0 < \beta < p$ we have

$$\bigl|f(E)\bigr| \leqslant C \log^{-\beta}\left(e + \frac{1}{|E|}\right), \quad E \subset \mathbb{D}. \tag{30}$$

The constant C above depends on β, p and $\|e^{K(z,f)}\|_p$ only.

Proof. Choose $\beta_0 \in (\beta, p)$ and $M \geqslant 1$ so that

$$\frac{2}{M} < \beta < \beta_0 - \frac{2}{M}.$$

We will then use the factorization $f = g \circ F$ from Corollary 4.4. Since $pM > \beta_0 M > 2$, Corollary 3.2 applies to $\sigma_\mu = F_{\bar z}$ and $F_z = 1 + \mathcal{S}F_{\bar z} = 1 + \mathcal{S}\sigma_\mu$,

$$\bigl|F(E)\bigr| = \int_E |F_z|^2 - |F_{\bar z}|^2 \leqslant 2|E| + 2\int_E |F_z - 1|^2 \leqslant C \log^{2-\beta_0 M}\left(e + \frac{1}{|E|}\right).$$

Above $|F(E)|$ is legitimately obtained by integrating the Jacobian since F is a Sobolev homeomorphism that satisfies condition $\mathcal{N}$. On the other hand, since g is an M-quasiconformal principal mapping we can use the area distortion estimate (16). This gives

$$\left|f(E)\right| = \left|g\circ F(E)\right| \leqslant \pi M \left|F(E)\right|^{1/M} \leqslant \pi C\left[\log\left(e+\frac{1}{|E|}\right)\right]^{(2-\beta_0 M)/M}.$$

Since $\beta < \beta_0 - \frac{2}{M}$ the result follows. □

Again the family (3) shows that the measure distortion bounds are optimal in terms of the exponent of the logarithm in (30), up to the possible estimates at the borderline $\beta = p$.

We are ready for

Proof Theorem 1.1. Assume that we have a mapping of finite distortion $f \in W^{1,1}_{loc}(\Omega)$ with $e^{K(z,f)} \in L^p_{loc}(\Omega)$. From (9) it follows that $f \in W^{1,Q}_{loc}(\Omega)$, and hence the general Stoilow factorization Theorem 4.2 applies. With the factorization we may in fact assume that $\Omega = \mathbb{C}$ and that f is the principal solution of Theorem 1.3, conformal outside $\frac{1}{2}\mathbb{D}$. It is then enough to consider the regularity over $\mathbb{D}$.

We prove first

$$J(z,f)\log^\beta\bigl(e+J(z,f)\bigr) \in L^1_{loc}(\mathbb{D}), \quad 0<\beta<p. \tag{31}$$

In order to verify this it is enough to show that $J^*(x)\log^\beta(e+J^*(x)) \in L^1(0,\pi)$, where $J^*(x)$, $x \geqslant 0$, is the non-increasing rearrangement of $J(z,f)$.

For the rearrangement the estimate (30) holds in the form

$$\int_0^t J^*(x)\,dx \leqslant C\log^{-\beta}\left(e+\frac{1}{t}\right), \quad 0<\beta<p,\ 0<t\leqslant\pi. \tag{32}$$

Observe that we have applied here the remark after Corollary 4.4. Further, since J^* is non-increasing,

$$J^*(t) \leqslant \frac{1}{t}\int_0^t J^*(x)\,dx \leqslant \frac{C}{t}. \tag{33}$$

Let us now fix $0<\beta<p$ and choose $\alpha \in (\beta,p)$. If we write

$$\phi(t) := \int_0^t J^*(x)\,dx, \quad 0<t\leqslant\pi,$$

then $0 \leqslant \phi(t) \leqslant C\log^{-\alpha}(e+1/t)$, $0<t\leqslant\pi$. In addition, (33) gives

$$\int_0^\pi J^*(x)\log^\beta\bigl(e+J^*(x)\bigr) \leqslant C\int_0^\pi \phi'(x)\log^\beta(e+1/x)\,dx.$$

Now an integration by parts shows that the last integral is finite. We have thus proved the claim (31).

The second claim

$$|Df|^2\log^{\beta-1}\bigl(e+|Df|\bigr) \in L^1_{loc}(\Omega) \tag{34}$$

can be deduced from (31) by observing that for every $\beta, p>0$ there are positive constants C_1 and C_2 such that

$$xy\log^{\beta-1}(e+\sqrt{xy}) \leqslant C_1 x\log^\beta(e+x) + C_2 e^{py} \quad \text{for all } x,y>0.$$

For this consider separately the cases where $x < e^{\frac{p}{2}y}$ and where $x \geqslant e^{\frac{p}{2}y}$. Now, since $s \mapsto s^2\log^{\beta-1}(e+s)$ is increasing, we conclude

$$\begin{aligned}\left|Df(z)\right|^2 \log^{\beta-1}\left(e+\left|Df(z)\right|\right) &\leqslant K(z,f)J(z,f)\log^{\beta-1}\left(e+\sqrt{K(z,f)J(z,f)}\right)\\ &\leqslant C_1 J(z,f)\log^{\beta}\left(e+J(z,f)\right)+C_2 e^{pK(z,f)} \in L^1_{loc}(\mathbb{D}).\end{aligned}$$

Lastly, the family g_p from (3) shows that (31), (34) may fail at the borderline $\beta = p$. The proof of Theorem 1.1 is complete. □

6. Divergent Neumann series

We next verify that our decay estimate from Theorem 3.1 is essentially the best possible. One way to do this is simply to observe that if the decay of the L^2-norm of the nth term would be of the order $O((n+1)^{-\delta p})$ with a uniform $\delta > 1/2$, our proof above would yield an area distortion result which would be 'too good', contradicting example (3). We leave the details to the reader. Instead, we present here a more concrete approach that is of independent interest.

We start by constructing families of mappings. Suppose first that $\gamma : (0,1] \to \mathbb{C}$ is continuous with

$$\left|\gamma(s)\right| < 1 \quad \text{for } 0 < s \leqslant 1.$$

Then define

$$\rho(t) = \rho_\gamma(t) := \exp\left[-\int_t^1 \frac{1+\gamma(s)}{1-\gamma(s)}\,\frac{ds}{s}\right] \tag{35}$$

or equivalently,

$$\gamma(t) = \gamma_\rho(t) = \frac{t\rho'(t)-\rho(t)}{t\rho'(t)+\rho(t)}, \quad t \in (0,1].$$

In any case, we see from (35) that

$$t \mapsto \left|\rho(t)\right| \quad \text{is strictly increasing on (0,1]}.$$

If, in addition,

$$\int_0^1 \Re e\left(\frac{1+\gamma(s)}{1-\gamma(s)}\right)\frac{ds}{s} = \infty$$

then $\rho(t) \to 0$ as $t \to 0$. In particular, in this case the mapping

$$f(z) = \frac{z}{|z|}\rho\left(|z|\right), \quad |z| \leqslant 1, \quad \text{with} \quad f(z) = z, \quad |z| \geqslant 1,$$

defines a homeomorphism of $\mathbb{C}$.

Next, let $\alpha > 0$ with

$$\rho(t) = \left(\log\frac{5}{t}\right)^{-\alpha} \quad \text{that is,} \quad \gamma(t) = \frac{\alpha - \log(5/t)}{\alpha + \log(5/t)}.$$

Consider then the holomorphic family of mappings

$$f_\lambda(z) = \frac{z}{|z|}\rho_\lambda\left(|z|\right), \quad \rho_\lambda := \rho_{(\lambda\gamma)}, \text{ where } |\lambda| < 1.$$

In other words,

$$\rho_\lambda(t) = \exp\left[\int_1^t \frac{1+\lambda\gamma(s)}{1-\lambda\gamma(s)}\,\frac{ds}{s}\right], \quad 0 < t \leqslant 1, \quad \text{and} \quad \rho_\lambda(t) \equiv t, \quad t \geqslant 1.$$

The mappings f_λ are all quasiconformal in the entire plane. In fact, they define a holomorphic motion

$$\Phi : \mathbb{D} \times \mathbb{C} \to \mathbb{C}, \quad \Phi(\lambda, z) = f_\lambda(z).$$

At the boundary, when $\lambda \to \zeta \in \mathbb{S}^1$, we attain well-defined mappings f_ζ of finite distortion. If fact, calculating the complex dilatation shows that $|\mu(z)| = |\gamma(t)|$ for $|z| = t$. Hence the distortion function

$$K(z, f_\zeta) = \frac{1 + |\gamma(t)|}{1 - |\gamma(t)|}, \quad t = |z| \text{ and } |\zeta| = 1,$$

which is independent of ζ, with

$$K \simeq \frac{1}{\alpha} \log \frac{1}{|z|} \quad \text{as } |z| \to 0.$$

Thus given $\varepsilon > 0$ we can choose $\alpha < \frac{1}{2}$ so that $e^{K(z)} \in L^{1-\varepsilon}(\mathbb{D})$. On the other hand, one computes that

$$\frac{\partial f_\zeta}{\partial \bar{z}}(z) \simeq \frac{1}{|z|} \left(\log \frac{1}{|z|} \right)^{\frac{-4\alpha}{|1+\zeta|^2}} \quad \text{as } |z| \to 0.$$

Therefore $\partial_{\bar{z}} f_\zeta \notin L^2(\mathbb{C})$ whenever ζ belongs to the non-degenerate interval

$$\left\{ \zeta \in \mathbb{S}^1 \colon 2\alpha < \frac{|1+\zeta|^2}{4} \right\}.$$

In particular, $\lambda \to \partial_{\bar{z}} f_\lambda$ does not belong to the Hardy space $H^2(\mathbb{D}; L^2(\mathbb{C}))$, and hence the sequence of norms $\|(\mu\mathcal{S})^n \mu\|_2$ is not square summable, i.e.

$$\left\| (\mu\mathcal{S})^n \mu \right\|_2 \nleqslant C n^{-(1+\varepsilon)/2}$$

and the decay given by Theorem 3.1 cannot be improved at $p = 1$.

7. Applications to degenerate elliptic PDE's

The optimal regularity established in Theorem 1.1 obviously has a number of basic consequences, for instance towards removability of singularities. However, our aim here is not to consider these consequences systematically. We only indicate one such example, namely an application to degenerate elliptic equations. Degenerate elliptic PDE's arise naturally in hydrodynamics, non-linear elasticity, holomorphic dynamics and several other related areas. In two dimensions these equations are intimately related to the mappings of finite distortion.

Suppose that $u \in W^{1,1}_{loc}(\Omega)$ is a (distributional) solution to the equation

$$\operatorname{Div} A(z) \nabla u = 0, \quad \text{in } \Omega \subset \mathbb{R}^2. \tag{36}$$

Let us assume that u has *finite energy*, that is

$$\int_\Omega \bigl\langle \nabla u(z), A(z) \nabla u(z) \bigr\rangle \, dm(z) < \infty. \tag{37}$$

In case the domain Ω is simply connected, the solution u admits a conjugate function, defined by the condition

$$\nabla v = J A(z) \nabla u. \tag{38}$$

Here $J(x, y) = (-y, x)$ is the rotation by a right angle.

In the complex notation, the function

$$f = u + iv$$

solves the $\mathbb{R}$-linear equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z} + \nu(z) \overline{\frac{\partial f}{\partial z}}, \tag{39}$$

where the coefficients $\mu(z)$, $\nu(z)$ are explicit functions of the elements of $A(z)$, independent of the particular solution u. Conversely, for these coefficient μ and ν, the real part of any solution $f = u + iv$ to (39) solves the second order equation (36).

Next, let us assume that for almost all z, the matrix $A(z)$ is elliptic and symmetric. Then there is a function $1 \leqslant K(z) < \infty$, finite for almost all z, such that

$$\frac{1}{K(z)}|h|^2 \leqslant \langle h, A(z)h\rangle \leqslant K(z)|h|^2, \quad h \in \mathbb{R}^2. \tag{40}$$

In terms of the mapping $f = u + iv$, the conditions (38), (40) take the familiar form

$$\left|Df(z)\right|^2 \leqslant K(z)J(z,f),$$

where the Jacobian can alternatively be written as

$$J(z,f) = \langle \nabla u(z), A(z)\nabla u(z)\rangle.$$

In particular, $f = u + iv$ is a mapping of finite distortion if and only if the solution u has finite energy in the sense (37). As an immediate consequence of Corollary 1.2 we find optimal conditions guaranteeing the L^2-regularity of the derivatives of the solutions u.

Theorem 7.1. *Suppose $u \in W^{1,1}_{loc}(\Omega)$ is a solution to Eq.* (36) *with*

$$\int_\Omega \langle \nabla u(z), A(z)\nabla u(z)\rangle\, dm(z) < \infty.$$

If the ellipticity bound $K(z)$ of the coefficient matrix $A(z)$ satisfies

$$e^{K(z)} \in L^p_{loc}(\Omega) \tag{41}$$

for some $p > 1$, then the solution u has the regularity

$$|\nabla u|^2 + \left|A(z)\nabla u\right|^2 \in L^1_{loc}(\Omega). \tag{42}$$

Similarly, we have

$$|\nabla u|^2 \log^{\beta-1}\bigl(e + |\nabla u|\bigr) \in L^1_{loc}(\Omega), \quad 0 < \beta < p,$$

and

$$\left|A(z)\nabla u\right|^2 \log^{\beta-1}\bigl(e + \left|A(z)\nabla u\right|\bigr) \in L^1_{loc}(\Omega), \quad 0 < \beta < p,$$

whenever the ellipticity $K(z)$ satisfies $e^{K(z)} \in L^p_{loc}(\Omega)$, $p > 0$. Furthermore, via the Stoilow factorization many properties of the harmonic functions generalize to the solutions to these degenerate elliptic equations. There are even non-linear counterparts, see [4].

One of the fundamental features of the mappings of bounded distortion are their self-improving regularity properties. Similar phenomena occur also in the theory of mappings of finite distortion. A typical example is obtained by combining the Stoilow factorization Theorem 4.2 with our Theorem 1.1. There are factorization theorems slightly beyond the class $W^{1,Q}_{loc}(\mathbb{C})$, $Q(t) = t^2 \log^{-1}(e + t)$, see for instance [14, p. 280]. Both factorization methods lead to removability results for mappings of finite distortion. We refer the reader to [14,12,3] and their references for these and other related methods, such as Caccioppoli-type estimates. In view of the discussion in this section, one obtains corresponding removability results for the solutions of the elliptic PDE's.

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