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On the pullback equation $\varphi^*(g) = f$

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Abstract

We discuss the existence of a diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\varphi^*(g) = f$$

where $f, g : \mathbb{R}^n \to \Lambda^k$ are closed differential forms and $2 \leqslant k \leqslant n$. Our main results (the case $k = n$ having been handled by Moser [J. Moser, On the volume elements on a manifold, Trans. Amer. Math. Soc. 120 (1965) 286–294] and Dacorogna and Moser [B. Dacorogna, J. Moser, On a partial differential equation involving the Jacobian determinant, Ann. Inst. H. Poincaré Anal. Non Linéaire 7 (1990) 1–26]) are that

- when n is even and $k = 2$, under some natural non-degeneracy condition, we can prove the existence of such diffeomorphism satisfying Dirichlet data on the boundary of a bounded open set and the natural Hölder regularity; at the same time we get Darboux theorem with optimal regularity;
- we are also able to handle the degenerate cases when $k = 2$ (in particular when n is odd), $k = n - 1$ and some cases where $3 \leqslant k \leqslant n - 2$.

Résumé

Nous montrons l'existence d'un difféomorphisme $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ satisfaisant

$$\varphi^*(g) = f$$

où $f, g : \mathbb{R}^n \to \Lambda^k$ sont des formes différentielles fermées et $2 \leqslant k \leqslant n$. Nos résultats principaux (le cas $k = n$ a été discuté notamment dans Moser [J. Moser, On the volume elements on a manifold, Trans. Amer. Math. Soc. 120 (1965) 286–294] et Dacorogna et Moser [B. Dacorogna, J. Moser, On a partial differential equation involving the Jacobian determinant, Ann. Inst. H. Poincaré Anal. Non Linéaire 7 (1990) 1–26]) sont les suivants.

- Si n est pair, $k = 2$ et sous des conditions naturelles de non dégénérescence, nous montrons l'existence et la régularité dans les espaces de Hölder d'un tel difféomorphisme satisfaisant de plus une condition de Dirichlet. On obtient aussi le théorème de Darboux avec la régularité optimale.
- Par ailleurs quand $k = 2$ et n est impair ou $k = n - 1$, ainsi que quelques cas particuliers où $3 \leqslant k \leqslant n - 2$, nous montrons l'existence locale d'un tel difféomorphisme satisfaisant, en outre, des conditions de Cauchy.

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1. Introduction

In this article we discuss the existence of a diffeomorphism $\varphi:\mathbb{R}^n \to \mathbb{R}^n$ such that

$$\varphi^*(g) = f \tag{1}$$

where $f, g : \mathbb{R}^n \to \Lambda^k$ are closed differential forms (i.e. $df = dg = 0$), $2 \leqslant k \leqslant n$ (the case $k = 1$ is rather special and will also be discussed)

$$g = \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} g_{i_1\cdots i_k}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and similarly for f. The meaning of (1) is that

$$\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} g_{i_1\cdots i_k}\big(\varphi(x)\big)\, d\varphi^{i_1} \wedge \cdots \wedge d\varphi^{i_k} = \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} f_{i_1\cdots i_k}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

When $k = 2$ and $n = 2m$ is even, then the celebrated Darboux theorem (cf. for example, Abraham, Marsden and Ratiu [1], McDuff and Salamon [11] or Taylor [17]) states that if $g = \omega_0$ is the standard symplectic form, namely

$$\omega_0 = g = \sum_{i=1}^{m} dx^i \wedge dx^{m+i}$$

and $f : \mathbb{R}^n \to \Lambda^2$ with $df = 0$ and $f(p) = g(p)$ for a certain $p \in \mathbb{R}^n$, then there exists a diffeomorphism φ defined in the neighbourhood of p such that

$$\varphi^*(g) = f \quad \text{and} \quad \varphi(p) = p.$$

This fundamental result was generalized by Moser in his seminal article [13] for the case $k = n$ and $k = 2$ with n even, obtaining also a global result. He proposed to solve (1) by studying the flow associated to an appropriate vector field u_t, namely

$$\begin{cases} \frac{d}{dt}\varphi_t(x) = u_t(\varphi_t(x)), & t \in [0,1],\\ \varphi_0(x) = x. \end{cases}$$

A solution of (1) is then given by $\varphi = \varphi_1$. Now (for the notations see below) the vector field is recovered through two linear equations, namely

$$d\alpha = f - g \tag{2}$$

meaning that $\alpha : \mathbb{R}^n \to \Lambda^{k-1}$ and for every $t \in [0,1]$,

$$u_t \lrcorner \big[tg + (1-t)f\big] = \alpha. \tag{3}$$

The first one is a system of *linear* differential equations and is, at least locally, solvable since $df = dg = 0$. The second one is just a *linear* system of algebraic equations and Moser observed that it is well posed (in the sense that, whatever α is, there exists a unique u_t solving (3)), under some non-degeneracy conditions, only when $k = n$ or $k = 2$ and n even. Of course one could consider, with essentially no change, a more general closed homotopy f_t, with $f_0 = f$ and $f_1 = g$, and then the two equations read as

$$d\alpha_t = -\frac{d}{dt} f_t \quad \text{and} \quad u_t \lrcorner f_t = \alpha_t$$

but we will here, for the sake of simplicity, restrict our attention to the homotopy $f_t = tg + (1-t)f$.

The case $k = n$ after the paper of Moser received considerable attention notably by Banyaga [2], Dacorogna [4], Reimann [14], Tartar [16], Zehnder [19]. Eq. (1) takes then the following form

$$g\big(\varphi(x)\big) \det \nabla\varphi(x) = f(x).$$

The next important step, still for $k = n$, appeared in Dacorogna and Moser [7], where it was shown how to handle both the boundary value and regularity problems. It should be emphasized that the flow method, as elegant as it is, does not allow to handle regularity problems and in [7] it was necessary to combine a fixed point argument and an iteration procedure. The exact statement can be found in Theorem 12 below. Posterior contributions can also be found in Burago and Kleiner [3], McMullen [12], Rivière and Ye [15] and Ye [18].

Our purpose in the present report is twofold.

(1) In the non-degenerate case when $k = 2$ and n even, the other non-degenerate case being $k = n$ and already solved in [7], we can handle full boundary condition (meaning Dirichlet condition) as well as regularity in Hölder spaces. This is a delicate point and requires fine approximations of Hölder functions, a subtle fixed point argument and an iteration scheme. Our main results are Theorems 15 and 18. Although we will treat mainly contractible domains, we point out at each stage how our results can be extended to topologically more complex domains.

(2) We also show how to deal with degenerate problems. There we mostly obtain local results, though, in some particular cases, we can treat global problems. We are, however, able to impose Cauchy data (but, in general, neither Dirichlet data nor regularity) and not just the value at one point as in Darboux theorem. If we impose Cauchy data, then, of course, we need to assume that the tangential parts of f and g coincide. We achieve this goal by solving Eqs. (2) and (3) simultaneously and not separately as Moser did. This is indeed a more flexible procedure, since (2) is underdetermined while (3) is overdetermined. In particular, only under some minor non-degeneracy conditions, we can also handle the cases:
   - $k = 2$ and n odd with maximal rank (cf. Theorem 20),
   - $k = n - 1$ (cf. Theorem 21),
   - we finally suggest through simple examples that our method applies to higher degenerate case when $k = 2$ or to the case of k forms with $3 \leqslant k \leqslant n - 2$.

   A systematic study of these last cases will be undertaken elsewhere.

2. Preliminaries and notations

2.1. Notations

We will denote a k-form $g : \mathbb{R}^n \to \Lambda^k$ by

$$g = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} g_{i_1 \cdots i_k}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

or sometimes by

$$g = \sum_{I \in \mathcal{T}_k} g_I\, dx^I$$

where $\mathcal{T}_k = \{(i_1, \ldots, i_k):\ 1 \leqslant i_1 < \cdots < i_k \leqslant n\}$ is the set of strictly increasing k-indices. More generally we assign meaning to $g_{i_1 \cdots i_k}$ for any k-index by

$$g_{i_1 \cdots i_k} = (\operatorname{sgn} \sigma) g_{i_{\sigma(1)} \cdots i_{\sigma(k)}}$$

where σ is a permutation of $\{1, \ldots, k\}$.

(1) If $g \in \Lambda^k$, $1 \leqslant k \leqslant n$, and $u \in \mathbb{R}^n$ then $u \lrcorner g \in \Lambda^{k-1}$ (also denoted by some authors by $i_u(g)$) is defined as

$$\begin{aligned} u \lrcorner g &= \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} g_{i_1 \cdots i_k} \sum_{r=1}^{k} (-1)^{r-1} u^{i_r}\, dx^{i_1} \wedge \cdots \wedge dx^{i_{r-1}} \wedge dx^{i_{r+1}} \wedge \cdots \wedge dx^{i_k} \\ &= (-1)^{k-1} \sum_{1 \leqslant i_1 < \cdots < i_{k-1} \leqslant n} \sum_{i_k=1}^{n} g_{i_1 \cdots i_k} u^{i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}. \end{aligned}$$

Note that when $k = 2$, we have

$$u \lrcorner g = -\sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} u^j\, dx^i.$$

(2) It will often be convenient to represent $u \to u \lrcorner g$ as a matrix operating on a vector. We therefore introduce the antisymmetric representation of $g \in \Lambda^k$ as the matrix $\bar{g} \in \mathbb{R}^{\binom{n}{k-1} \times n}$ so that, by abuse of notations,

$$u \lrcorner g = (-1)^{k-1} \bar{g} u.$$

For example when $k = 2$, we have

$$\bar{g} = (g_{ij}) \in \mathbb{R}^{n \times n} \quad \text{with } g_{ij} = -g_{ji}.$$

Our terminology "non-degenerate" and "degenerate" refers to the fact that $\bar{g}$ is invertible or not.

We then consider

$$\Lambda_g^{k-1} = \{ w \in \Lambda^{k-1} \colon \exists u \in \mathbb{R}^n \text{ with } u \lrcorner g = w \}.$$

We easily find that if $g \neq 0$, then, for every $2 \leqslant k \leqslant n$,

$$k \leqslant \dim \Lambda_g^{k-1} \leqslant n$$

and in general, if $3 \leqslant k \leqslant n-1$, it can take any of the intermediate values, but $(k+1)$. So that when $k = n-1$, the dimension cannot be maximal, more precisely, if $g \neq 0$ we always have

$$\dim \Lambda_g^{n-2} = n - 1.$$

When $k = 2$, the dimension is necessarily even, this means that there exists an integer $1 \leqslant l \leqslant [n/2]$ such that

$$\dim \Lambda_g^1 = 2l.$$

Therefore when $k = 2$, the matrix $\bar{g} \in \mathbb{R}^{n \times n}$ is always singular when n is odd.

It will sometimes be more convenient to express the set Λ_g^1 in terms of the kernel of $\bar{g}$ the antisymmetric representation of g and its dimension by the rank of $\bar{g}$. Namely

$$\Lambda_g^1 = (\ker \bar{g})^{\perp} \quad \text{and} \quad \dim \Lambda_g^1 = \operatorname{rank} \bar{g}.$$

(3) If $g : \mathbb{R}^n \to \Lambda^k$ is such that

$$g = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} g_{i_1 \cdots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

then

$$dg = \sum_{i_1 < \cdots < i_{k+1}} \left(\sum_{\gamma=1}^{k+1} (-1)^{\gamma-1} \frac{\partial g_{i_1 \cdots i_{\gamma-1} i_{\gamma+1} \cdots i_{k+1}}}{\partial x^{i_\gamma}} \right) dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}$$

and, for $\nu \in \mathbb{R}^n$,

$$g \wedge \nu = \sum_{i_1 < \cdots < i_{k+1}} \left(\sum_{\gamma=1}^{k+1} (-1)^{\gamma-1} g_{i_1 \cdots i_{\gamma-1} i_{\gamma+1} \cdots i_{k+1}} \nu^{i_\gamma} \right) dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}.$$

If ν is the outward unit normal to the boundary of a set Ω, we call $g \wedge \nu$ the tangential part and $\nu \lrcorner g$ the normal part of the form g.

(4) For $f, g \in \Lambda^k$ we write inner product as

$$\langle f; g \rangle = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} f_{i_1 \cdots i_k} g_{i_1 \cdots i_k} \in \mathbb{R}.$$

(5) To avoid burdening the notations, we will sometimes, by abuse of notations, identify a 1-form with a vector field in $\mathbb{R}^n$ and a n-form with a function.

We now gather some simple algebraic and analytical formulas that follow from the above definitions and that we will use throughout.

(i) For every $f \in \Lambda^k$, $g \in \Lambda^{k-1}$ and $u \in \mathbb{R}^n$, then

$$\langle f; g \wedge u \rangle = (-1)^{k-1} \langle u \lrcorner f; g \rangle.$$

(ii) Let $f \in \Lambda^k$, $g \in \Lambda^l$ and $u \in \mathbb{R}^n$, then

$$u \lrcorner (f \wedge g) = (u \lrcorner f) \wedge g + (-1)^{kl}(u \lrcorner g) \wedge f.$$

(iii) Let $f \in C^1(\mathbb{R}^n; \Lambda^k)$ be closed

$$f = \sum_{I \in \mathcal{T}_k} f_I \, dx^I$$

and $u \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, then

$$d[u \lrcorner f] = \sum_{I \in \mathcal{T}_k} f_I d\big[u \lrcorner dx^I\big] + \sum_{I \in \mathcal{T}_k} \langle \operatorname{grad} f_I; u \rangle \, dx^I.$$

So, in particular, when $k = 2$

$$d[u \lrcorner f] = \sum_{1 \leqslant i < j \leqslant n} f_{ij}\big[du^i \wedge dx^j + dx^i \wedge du^j\big] + \sum_{1 \leqslant i < j \leqslant n} \langle \operatorname{grad} f_{ij}; u \rangle \, dx^i \wedge dx^j.$$

(iv) Let $0 \leqslant k \leqslant n$ be an integer, $\Omega \subset \mathbb{R}^n$ be a smooth domain and $f, g \in C^1(\overline{\Omega}; \Lambda^k)$ satisfying $f = g$ on $\partial\Omega$, then

$$df \wedge \nu = dg \wedge \nu \quad \text{on } \partial\Omega$$

where ν is a normal to $\partial\Omega$.

2.2. Function spaces and Hölder approximations

We will use the following functional notations. Let $\Omega \subset \mathbb{R}^n$ be an open set and r a non-negative integer.

(1) Let $0 < \alpha < 1$, we denote by $C^{r,\alpha}(\overline{\Omega})$ the usual set of Hölder functions and by $C^{r,\alpha}(\overline{\Omega}; \Lambda^k)$ the set of k-forms

$$g = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} g_{i_1 \cdots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

so that $g_{i_1 \cdots i_k} \in C^{r,\alpha}(\overline{\Omega})$.

(2) The set $C^{\omega}(\overline{\Omega})$ will denote the set of analytic functions and $C^{\omega}(\overline{\Omega}; \Lambda^k)$ the set of k-forms whose components are in $C^{\omega}(\overline{\Omega})$.

(3) The sets $\operatorname{Diff}^r(\overline{\Omega}; \mathbb{R}^n)$, $\operatorname{Diff}^{r,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ and $\operatorname{Diff}^{\omega}(\overline{\Omega}; \mathbb{R}^n)$, denote the sets of diffeomorphisms φ so that $\varphi \in C^r(\overline{\Omega}; \mathbb{R}^n)$ and $\varphi^{-1} \in C^r(\varphi(\overline{\Omega}); \mathbb{R}^n)$, $C^{r,\alpha}$ and C^{ω} respectively. When $\varphi(\overline{\Omega}) = \overline{\Omega}$, we just let $\operatorname{Diff}^r(\overline{\Omega})$, respectively $\operatorname{Diff}^{r,\alpha}(\overline{\Omega})$, $\operatorname{Diff}^{\omega}(\overline{\Omega})$.

In the sequel we will have to approximate closed forms in $C^{r,\alpha}(\overline{\Omega}; \Lambda^k)$ by smooth closed forms in a precise way and we will need the following theorem.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth contractible domain, $0 < \beta \leqslant \alpha < 1$, $p \geqslant q \geqslant r$, $q \geqslant 2$, $r \geqslant 1$ and $1 \leqslant k \leqslant n$ be integers and $g \in C^{q,\alpha}(\overline{\Omega}; \Lambda^k) \cap C^{p,\alpha}(\partial\Omega; \Lambda^k)$ with $dg = 0$. Then for every $\epsilon > 0$, there exist $g^{\epsilon} \in C^{\infty}(\Omega; \Lambda^k) \cap C^{p,\alpha}(\overline{\Omega}; \Lambda^k)$ and a constant $\gamma = \gamma(p, q, r, \alpha, \beta, \Omega) > 0$ such that $dg^{\epsilon} = 0$, $g^{\epsilon} \wedge \nu = g \wedge \nu$ on $\partial\Omega$ and*

$$\|g^{\epsilon} - g\|_{C^{r,\beta}(\overline{\Omega})} \leqslant \gamma \epsilon^{q-r+\alpha-\beta} \|g\|_{C^{q,\alpha}(\overline{\Omega})},$$
$$\|g^{\epsilon}\|_{C^{p,\alpha}(\overline{\Omega})} \leqslant \frac{\gamma}{\epsilon^{p-q}} \big[\|g\|_{C^{q,\alpha}(\overline{\Omega})} + \epsilon^{p-q} \|g\|_{C^{p,\alpha}(\partial\Omega)}\big].$$

Before starting the proof of the theorem, we need the equivalent of the theorem but for functions.

Lemma 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $0 < \beta \leqslant \alpha < 1$, $p \geqslant q \geqslant r \geqslant 2$ be integers and $u \in C^{q,\alpha}(\overline{\Omega}) \cap C^{p,\alpha}(\partial\Omega)$. Then for every $\epsilon > 0$, there exist $u^{\epsilon} \in C^{\infty}(\Omega) \cap C^{p,\alpha}(\overline{\Omega})$ and a constant $\gamma = \gamma(p, q, r, \alpha, \beta, \Omega) > 0$ such that $u^{\epsilon} = u$ on $\partial\Omega$ and*

$$\|u^{\epsilon} - u\|_{C^{r,\beta}(\overline{\Omega})} \leqslant \gamma \epsilon^{q-r+\alpha-\beta} \|u\|_{C^{q,\alpha}(\overline{\Omega})},$$
$$\|u^{\epsilon}\|_{C^{p,\alpha}(\overline{\Omega})} \leqslant \frac{\gamma}{\epsilon^{p-q}} \big[\|u\|_{C^{q,\alpha}(\overline{\Omega})} + \epsilon^{p-q} \|u\|_{C^{p,\alpha}(\partial\Omega)}\big].$$

Proof. We first find (see Hörmander [9]) $v^\epsilon \in C^\infty(\overline{\Omega})$ and a constant γ_1 such that

$$\|v^\epsilon - u\|_{C^{r,\beta}(\overline{\Omega})} \leqslant \gamma_1 \epsilon^{q-r+\alpha-\beta}\|u\|_{C^{q,\alpha}(\overline{\Omega})} \quad \text{and} \quad \|v^\epsilon\|_{C^{p,\alpha}(\overline{\Omega})} \leqslant \frac{\gamma_1}{\epsilon^{p-q}}\|u\|_{C^{q,\alpha}(\overline{\Omega})}.$$

We then fix the boundary data as follows. Let $u^\epsilon \in C^\infty(\Omega)\cap C^{p,\alpha}(\overline{\Omega})$ be the solution of

$$\begin{cases} \Delta u^\epsilon = \Delta v^\epsilon & \text{in } \Omega, \\ u^\epsilon = u & \text{on } \partial\Omega \end{cases} \quad \Leftrightarrow \quad \begin{cases} \Delta[u^\epsilon - v^\epsilon] = 0 & \text{in } \Omega, \\ u^\epsilon - v^\epsilon = u - v^\epsilon & \text{on } \partial\Omega. \end{cases}$$

The solution satisfies Schauder estimates

$$\|u^\epsilon\|_{C^{p,\alpha}(\overline{\Omega})} \leqslant \gamma_2\big[\|v^\epsilon\|_{C^{p,\alpha}(\overline{\Omega})} + \|u\|_{C^{p,\alpha}(\partial\Omega)}\big],$$
$$\|u^\epsilon - v^\epsilon\|_{C^{r,\beta}(\overline{\Omega})} \leqslant \gamma_2\|u - v^\epsilon\|_{C^{r,\beta}(\partial\Omega)} \leqslant \gamma_2\|u - v^\epsilon\|_{C^{r,\beta}(\overline{\Omega})}.$$

The combination of the estimates on u^ϵ and v^ϵ gives the result. □

We can now go back to the proof of Theorem 1.

Proof. *Step 1*. It is easy to see that, since Ω is contractible, we can find $G \in C^{q+1,\alpha}(\overline{\Omega}; \Lambda^{k-1}) \cap C^{p+1,\alpha}(\partial\Omega; \Lambda^{k-1})$ and a constant γ such that $dG = g$,

$$\|G\|_{C^{q+1,\alpha}(\overline{\Omega})} \leqslant \gamma\|g\|_{C^{q,\alpha}(\overline{\Omega})} \quad \text{and} \quad \|G\|_{C^{p+1,\alpha}(\partial\Omega)} \leqslant \gamma\|g\|_{C^{p,\alpha}(\partial\Omega)}.$$

This is easily obtained by an appropriate use of Theorem 3 below and we leave out the details.

Step 2. Applying Lemma 2 on each component of G, we get G^ϵ as in the lemma. Setting $g^\epsilon = dG^\epsilon$, we have the claim. □

3. Variations on the Poincaré lemma

3.1. The case without constraints

We start with the following theorem (cf. [5]).

Theorem 3 *(Dacorogna). Let $r \geqslant 0$, $2 \leqslant k \leqslant n$ be integers and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth contractible domain and ν denote the outward unit normal. The following two conditions are then equivalent.*

(i) • *Either $2 \leqslant k \leqslant n-1$ and $f \in C^{r,\alpha}(\overline{\Omega}; \Lambda^k)$ with*

$$df = 0 \quad \text{in } \Omega \quad \text{and} \quad f \wedge \nu = 0 \quad \text{on } \partial\Omega;$$

• *or $k = n$ and $f \in C^{r,\alpha}(\overline{\Omega})$ with*

$$\int_\Omega f(x)\,dx = 0.$$

(ii) *There exists $w \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^{k-1})$ satisfying*

$$\begin{cases} dw = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark 4.

(i) When $r = 0$, the constraint $df = 0$ has to be interpreted in the sense of distributions.

(ii) We now briefly discuss the case where Ω is not contractible, but only open and connected. To this aim we first define, for $0 \leqslant k \leqslant n$, the set of *harmonic k-forms* with Dirichlet boundary condition as the vector space

$$D_k(\Omega) := \big\{\psi \in C^0\big(\overline{\Omega}; \Lambda^k\big) \cap C^1\big(\Omega; \Lambda^k\big):\ d\psi = 0,\ \delta\psi = 0 \text{ in } \Omega \text{ and } \psi \wedge \nu = 0 \text{ on } \partial\Omega\big\}.$$

Note that we always have for connected Ω

$$D_0(\Omega) \simeq \{0\} \quad \text{and} \quad D_n(\Omega) \simeq \mathbb{R}.$$

Furthermore if $1 \leqslant k \leqslant n-1$ and if the set Ω is contractible, then

$$D_k(\Omega) \simeq \{0\} \subset \Lambda^k,$$

while for general sets we have

$$\dim D_k(\Omega) = B_{n-k}$$

where B_k are the Betti numbers of Ω (cf. Duff and Spencer [8] and Kress [10]). Theorem 3 remains valid for such general sets if we add the following necessary condition

$$\int_\Omega \langle f; \psi \rangle \, dx = 0, \quad \forall \psi \in D_k(\Omega).$$

Observe finally that when $k = n$ in Theorem 3 we therefore have no new condition.

(iii) Although there are infinitely many solutions w, the actual construction singles out a precise one and in fact we can construct a linear isomorphism of Banach spaces $L : X \to Y$ where (when $2 \leqslant k \leqslant n-1$ and similar ones when $k = n$)

$$X = \left\{ w \in C^{r+1,\alpha}\left(\overline{\Omega}; \Lambda^k\right): \ w = 0 \text{ on } \partial\Omega \right\},$$
$$Y = \left\{ f \in C^{r,\alpha}\left(\overline{\Omega}; \Lambda^{k+1}\right): \ df = 0 \text{ in } \Omega \text{ and } f \wedge \nu = 0 \text{ on } \partial\Omega \right\},$$

that associates in an isomorphic way to every $w \in X$ a unique $f = Lw \in Y$ such that $dw = f$.

(iv) The theorem, with an easier and more direct proof, is still valid when $k = 1$ and regularity holds even in C^r spaces.

(v) The theorem is also valid for unbounded smooth domains such as the half plane.

3.2. *The case with constraints*

We start with the simplest case of constant constraints. In the sequel we let

$$H = \left\{ x \in \mathbb{R}^n: \ x^n > 0 \right\}$$

and $\nu = -e_n$ will denote the outward unit normal.

Theorem 5. *Let $2 \leqslant k \leqslant n$, $b \in \mathbb{R}^n$ with $b^n \neq 0$ and $f \in C^r(\overline{H}; \Lambda^k)$, $r \geqslant 0$, with*

$$df = 0 \quad \text{in } H \quad \text{and} \quad f \wedge e_n = 0 \quad \text{on } \partial H.$$

Let $w \in C^r(\overline{H}; \Lambda^{k-1})$ be defined by

$$w(x) = \int_0^{x^n/b^n} [b \lrcorner f]\left(x + b\left(t - \frac{x^n}{b^n}\right)\right) dt.$$

Then w is the unique solution of

$$\begin{cases} dw = f & \text{in } H, \\ b \lrcorner w = 0 & \text{in } H, \\ w = 0 & \text{on } \partial H. \end{cases}$$

If, moreover, $\langle f; c \wedge b \rangle = 0$ for a certain $c \in \Lambda^{k-1}$, then w also satisfies

$$\langle w; c \rangle = 0 \quad \text{in } H.$$

Remark 6. The constraint $b \lrcorner w = 0$ can be seen as $\binom{n-1}{k-2}$ independent equations. When $k = 2$, there is only one independent equation and it is of the form $\langle w; b \rangle = 0$.

We now consider the case of non-constant constraints and, in general, one can expect only local solutions. We start with the case $k=2$, which requires less stringent smoothness assumptions than the cases $3 \leqslant k \leqslant n$ (cf. Theorem 8 for this last case). Although we will state the following results in the half plane H, both theorems mentioned below can be proved, by flattening out the boundary, for any smooth domain Ω.

Theorem 7. *Let $r \geqslant 0$, $a \in C^{r+1,\alpha}(\overline{H}; \Lambda^1)$ with $a_n(x) \neq 0$ for every $x \in \overline{H}$ and $f \in C^{r,\alpha}(\overline{H}; \Lambda^2)$ with*

$$df = 0 \quad \text{in } H \quad \text{and} \quad f \wedge e_n = 0 \quad \text{on } \partial H.$$

Then there exist $\epsilon > 0$ and $w \in C^{r,\alpha}(\overline{H} \cap B_\epsilon(0); \Lambda^1)$ satisfying

$$\begin{cases} dw = f & \text{in } H \cap B_\epsilon(0), \\ \langle w; a \rangle = 0 & \text{in } H \cap B_\epsilon(0), \\ w = 0 & \text{on } \partial H \cap B_\epsilon(0). \end{cases}$$

Proof. Set

$$w = u + dv$$

where (this has a solution $u \in C^{r+1,\alpha}(\overline{H}; \Lambda^1)$ by Theorem 3)

$$\begin{cases} du = f & \text{in } H, \\ u = 0 & \text{on } \partial H \end{cases}$$

and

$$\begin{cases} \langle dv(x); a(x) \rangle = \sum_{i=1}^n a_i(x) \frac{\partial v}{\partial x^i} = -\langle u(x); a(x) \rangle & \text{if } x^n > 0, \\ v(x) = 0 & \text{if } x^n = 0. \end{cases}$$

This last problem has a solution, $v \in C^{r+1,\alpha}(\overline{H} \cap B_\epsilon(0))$, and thus $w \in C^{r,\alpha}$.

Note that from $v(x) = 0$ when $x^n = 0$, we deduce that

$$\frac{\partial v}{\partial x^i}\left(x^1, \dots, x^{n-1}, 0\right) = 0 \quad \text{for every } i = 1, \dots, n-1.$$

Since $u = 0$ on ∂H and $a_n(x) \neq 0$, we deduce from the differential equation and the above identities that

$$\frac{\partial v}{\partial x^n}\left(x^1, \dots, x^{n-1}, 0\right) = 0$$

so that $dv = 0$ when $x^n = 0$. This concludes the proof of the theorem. $\square$

We now discuss the case $3 \leqslant k \leqslant n$. The result and the proof below are still valid if $k = 2$.

Theorem 8. *Let $a^\lambda \in C^\omega(\overline{H}; \Lambda^{k-1})$, $\lambda = 1, \dots, \binom{n-1}{k-2}$, with*

$$A = A(x) = \left(a^\lambda_{j_1 \cdots j_{k-2} n}(x)\right)^{\lambda = 1, \cdots, \binom{n-1}{k-2}}_{1 \leqslant j_1 < \cdots < j_{k-2} \leqslant n-1} \in \mathbb{R}^{\binom{n-1}{k-2} \times \binom{n-1}{k-2}}$$

invertible for every $x \in \overline{H}$ and $f \in C^\omega(\overline{H}; \Lambda^k)$ with

$$df = 0 \quad \text{in } H \quad \text{and} \quad f \wedge e_n = 0 \quad \text{on } \partial H.$$

Then there exist $\epsilon > 0$ and $w \in C^\omega(\overline{H} \cap B_\epsilon(0); \Lambda^{k-1})$ satisfying

$$\begin{cases} dw = f & \text{in } H \cap B_\epsilon(0), \\ \langle w; a^\lambda \rangle = 0 & \text{in } H \cap B_\epsilon(0),\ \lambda = 1, \dots, \binom{n-1}{k-2}, \\ w = 0 & \text{on } \partial H \cap B_\epsilon(0). \end{cases}$$

Proof. Set

$$w = u + dv$$

where (this has a solution $u \in C^{\omega}(\overline{H}; \Lambda^{k-1})$ by Theorem 3 or by Theorem 5)

$$\begin{cases} du = f & \text{in } H, \\ u = 0 & \text{on } \partial H. \end{cases}$$

The form $v : \mathbb{R}^n \to \Lambda^{k-2}$ is defined as follows. Choose first

$$v_{j_1 \cdots j_{k-3} n} = 0 \quad \text{for every } 1 \leqslant j_1 < \cdots < j_{k-3} \leqslant n-1$$

(if $k = 3$, one just reads $v_n = 0$ and if $k = 2$, the above constraints do not exist) and then solve, using Cauchy–Kowalewski theorem, the system of equations

$$\begin{cases} \langle dv(x); a^{\lambda}(x) \rangle = -\langle u(x); a^{\lambda}(x) \rangle & \text{if } x^n > 0,\ \lambda = 1, \ldots, \binom{n-1}{k-2}, \\ v(x) = 0 & \text{if } x^n = 0, \end{cases}$$

More precisely first observe that with our choice of v, we have

$$v = \sum_{1 \leqslant j_1 < \cdots < j_{k-2} \leqslant n-1} v_{j_1 \cdots j_{k-2}}\, dx^{j_1} \wedge \cdots \wedge dx^{j_{k-2}}$$

then

$$\begin{aligned} dv &= \sum_{1 \leqslant j_1 < \cdots < j_{k-2} \leqslant n-1} \left[\sum_{i=1}^{n} \frac{\partial v_{j_1 \cdots j_{k-2}}}{\partial x^i}\, dx^i \right] \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{k-2}} \\ &= \sum_{1 \leqslant j_1 < \cdots < j_{k-2} \leqslant n-1} (-1)^k \frac{\partial v_{j_1 \cdots j_{k-2}}}{\partial x^n}\, dx^{j_1} \wedge \cdots \wedge dx^{j_{k-2}} \wedge dx^n \\ &\quad + \sum_{1 \leqslant j_1 < \cdots < j_{k-2} \leqslant n-1} \left[\sum_{i=1}^{n-1} \frac{\partial v_{j_1 \cdots j_{k-2}}}{\partial x^i}\, dx^i \right] \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{k-2}}. \end{aligned}$$

We therefore obtain that

$$\begin{aligned} \langle dv; a^{\lambda} \rangle &= (-1)^k \sum_{1 \leqslant j_1 < \cdots < j_{k-2} \leqslant n-1} a^{\lambda}_{j_1 \cdots j_{k-2} n} \frac{\partial v_{j_1 \cdots j_{k-2}}}{\partial x^n} + g^{\lambda} \\ &= (-1)^k \left(A \frac{\partial v}{\partial x^n} \right)^{\lambda} + g^{\lambda} \end{aligned}$$

where A is as in the statement of the theorem and g^{λ} is a term that does not involve derivatives of v with respect to the variable x^n. Note that the system

$$\langle dv(x); a^{\lambda}(x) \rangle = -\langle u(x); a^{\lambda}(x) \rangle, \quad \lambda = 1, \ldots, \binom{n-1}{k-2}$$

can therefore be seen, since A is invertible, as

$$\begin{cases} \dfrac{\partial v}{\partial x^n} = (-1)^{k+1} A^{-1} \left[\begin{pmatrix} \langle u; a^1 \rangle \\ \vdots \\ \langle u; a^{\binom{n-1}{k-2}} \rangle \end{pmatrix} + \begin{pmatrix} g^1 \\ \vdots \\ g^{\binom{n-1}{k-2}} \end{pmatrix} \right] & \text{if } x^n > 0, \\ v = 0 & \text{if } x^n = 0 \end{cases}$$

(recall that we are considering only the $v_{j_1 \cdots j_{k-2}}$ with $1 \leqslant j_1 < \cdots < j_{k-2} \leqslant n-1$ and thus $\frac{\partial v}{\partial x^n} \in \mathbb{R}^{\binom{n-1}{k-2}}$). Since all the coefficients are analytic, we have locally a unique analytical solution in the neighbourhood of $x = 0$.

We also note that $\frac{\partial v}{\partial x^i} = 0$ at $x^n = 0$ for every $i = 1, \ldots, n-1$ and that also $\frac{\partial v}{\partial x^n} = 0$ at $x^n = 0$, since there $u = 0$ and $g^{\lambda} = 0$; so that $dv = 0$ when $x^n = 0$. $\square$

4. Abstract results

4.1. Necessary conditions

We give here some elementary necessary conditions, whose proofs are straightforward.

Theorem 9. *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain, $1 \leqslant k \leqslant n$, $f, g \in C^1(\overline{\Omega}; \Lambda^k)$ with $dg = 0$ in Ω and $\varphi \in \mathrm{Diff}^1(\overline{\Omega}; \mathbb{R}^n)$ be such that $\varphi^*(g) = f$ in Ω, then*

$$df = 0.$$

Moreover the two following results hold.

(i) *If Ω is bounded, $\varphi(\overline{\Omega}) = \overline{\Omega}$ and $n = mk$ with m an integer, then*

$$\int_\Omega f^m = \int_\Omega g^m$$

where $f^m = \underbrace{f \wedge \cdots \wedge f}_{m\text{-times}}$.

(ii) *If $\varphi(x) = x$ for $x \in \partial\Omega$, then*

$$f \wedge \nu = g \wedge \nu \quad \text{on } \partial\Omega.$$

4.2. The flow method

We have the following abstract result, which is the theorem of Moser [13] with the additional consideration on the boundary data. In several textbooks the flow method is also called the Lie transform method.

Theorem 10 *(Moser). Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain (ν denoting the outward unit normal), $r \geqslant 1$ and $1 \leqslant k \leqslant n$ be integers and $f_t \in C^1([0,1]; C^r(\overline{\Omega}; \Lambda^k))$ (respectively $f_t \in C^1([0,1]; C^{r,\alpha})$ with $0 < \alpha < 1$) satisfying, for every $t \in [0,1]$,*

$$df_t = 0 \quad \text{in } \Omega \quad \text{and} \quad f_t \wedge \nu = f_0 \wedge \nu \quad \text{on } \partial\Omega.$$

Assume that there exists $u_t \in C^1([0,1]; C^r(\overline{\Omega}; \mathbb{R}^n))$ (respectively $u_t \in C^1([0,1]; C^{r,\alpha})$) verifying, for every $t \in [0,1]$,

$$d[u_t \lrcorner f_t] = -\frac{d}{dt} f_t \quad \text{in } \Omega \quad \text{and} \quad u_t = 0 \quad \text{on } \partial\Omega.$$

Then there exists $\varphi \in \mathrm{Diff}^r(\overline{\Omega})$ (respectively $\varphi \in \mathrm{Diff}^{r,\alpha}(\overline{\Omega})$) such that

$$\varphi^*\big(f_1(x)\big) = f_0(x), \quad x \in \Omega. \tag{4}$$

and

$$\varphi(x) = x, \quad x \in \partial\Omega.$$

Proof. We solve (cf. [13])

$$\begin{cases} \frac{d}{dt}\varphi_t(x) = u_t(\varphi_t(x)), & t \in [0,1], \\ \varphi_0(x) = x. \end{cases}$$

The above system has a solution φ_t satisfying, for every $t \in [0,1]$,

$$\varphi_t^*(f_t) = f_0 \quad \text{in } \Omega \quad \text{and} \quad \varphi_t(x) = x \quad \text{on } \partial\Omega.$$

This concludes the proof. □

4.3. The fixed point method

The following theorem is particularly useful when dealing with non-linear problems, once good estimates are known for the linearized problem. We give it under a general form, because we will need it this way in Theorem 15. However in many instances, one can choose $X_1 = X_2 = X$ and $Y_1 = Y_2 = Y$ in the statement below (in which case the hypothesis (H_{XY}) below reduces to requiring that X is a Banach space and Y a normed space). Our theorem will lean on the following hypotheses.

(H_{XY}) Let $X_1 \supset X_2$ be Banach spaces and $Y_1 \supset Y_2$ be normed spaces such that the following property holds: if

$$u_\nu \xrightarrow{X_1} u \quad \text{and} \quad \|u_\nu\|_{X_2} \leqslant r,$$

then $u \in X_2$ and

$$\|u\|_{X_2} \leqslant \liminf_{\nu\to\infty} \|u_\nu\|_{X_2}.$$

(H_L) $L : X_2 \to Y_2$ is a linear isomorphism of Banach spaces and there exist $k_1, k_2 > 0$ such that for every $f \in Y_2$

$$\left\|L^{-1} f\right\|_{X_i} \leqslant k_i \|f\|_{Y_i}, \quad i = 1, 2.$$

(H_Q) $Q : X_2 \to Y_2$ is such that $Q(0) = 0$ and for every $u, v \in X_2$ with $\|u\|_{X_1}, \|v\|_{X_1} \leqslant 1$, the following two inequalities hold

$$\left\|Q(u) - Q(v)\right\|_{Y_1} \leqslant c\big(\|u\|_{X_1}, \|v\|_{X_1}\big)\|u - v\|_{X_1}, \tag{5}$$

$$\left\|Q(v)\right\|_{Y_2} \leqslant c\big(\|v\|_{X_1}, 0\big)\|v\|_{X_2}, \tag{6}$$

where $c : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, separately increasing and $c(0, 0) = 0$.

Theorem 11 *(Fixed point theorem). Let X_1, X_2, Y_1, Y_2, L, Q satisfy (H_{XY}), (H_L) and (H_Q). Then, for every $f \in Y_2$ verifying*

$$2 \max\{k_1, k_2\} c\big(2k_1\|f\|_{Y_1}, 2k_1\|f\|_{Y_1}\big) \leqslant 1 \quad \text{and} \quad 2k_1\|f\|_{Y_1} \leqslant 1, \tag{7}$$

there exists $u \in X_2$ such that

$$Lu = Q(u) + f \quad \text{and} \quad \|u\|_{X_i} \leqslant 2k_i\|f\|_{Y_i}, \quad i = 1, 2. \tag{8}$$

Proof. We set

$$N(u) = Q(u) + f.$$

We next define

$$B = \big\{u \in X_2 \colon \|u\|_{X_i} \leqslant 2k_i\|f\|_{Y_i},\ i = 1, 2\big\}.$$

We endow B with $\|\cdot\|_{X_1}$ norm; the property (H_{XY}) ensures that B is closed. We now want to show that $L^{-1}N : B \to B$ is a contraction mapping (cf. Claims 1 and 2 below). Applying Banach fixed point theorem we will have indeed found a solution verifying (8) and the proof will be complete.

Claim 1. Let us first show that $L^{-1}N$ is a contraction on B. To show this, let $u, v \in B$ and use (5), (7) to get that

$$\begin{aligned}
\left\|L^{-1}N(u) - L^{-1}N(v)\right\|_{X_1} &\leqslant k_1\left\|N(u) - N(v)\right\|_{Y_1} = k_1\left\|Q(u) - Q(v)\right\|_{Y_1} \\
&\leqslant k_1 c\big(\|u\|_{X_1}, \|v\|_{X_1}\big)\|u - v\|_{X_1} \\
&\leqslant k_1 c\big(2k_1\|f\|_{Y_1}, 2k_1\|f\|_{Y_1}\big)\|u - v\|_{X_1} \\
&\leqslant \frac{1}{2}\|u - v\|_{X_1}.
\end{aligned}$$

Claim 2. We next show $L^{-1}N : B \to B$ is well-defined. First, note that

$$\left\|L^{-1}N(0)\right\|_{X_1} \leqslant k_1 \left\|N(0)\right\|_{Y_1} = k_1\|f\|_{Y_1}.$$

Therefore, using Claim 1, we obtain

$$\begin{aligned}\left\|L^{-1}N(u)\right\|_{X_1} &\leqslant \left\|L^{-1}N(u) - L^{-1}N(0)\right\|_{X_1} + \left\|L^{-1}N(0)\right\|_{X_1} \\ &\leqslant \frac{1}{2}\|u\|_{X_1} + k_1\|f\|_{Y_1} \leqslant 2k_1\|f\|_{Y_1}.\end{aligned}$$

It remains to show that

$$\left\|L^{-1}N(u)\right\|_{X_2} \leqslant 2k_2\|f\|_{Y_2}.$$

Using (6), we have

$$\begin{aligned}\left\|L^{-1}N(u)\right\|_{X_2} &\leqslant k_2\left\|N(u)\right\|_{Y_2} \leqslant k_2\left\|Q(u)\right\|_{Y_2} + k_2\|f\|_{Y_2} \\ &\leqslant k_2 c\big(\|u\|_{X_1}, 0\big)\|u\|_{X_2} + k_2\|f\|_{Y_2} \leqslant 2k_2^2 c\big(\|u\|_{X_1}, 0\big)\|f\|_{Y_2} + k_2\|f\|_{Y_2} \\ &= k_2\big(2k_2 c\big(\|u\|_{X_1}, 0\big) + 1\big)\|f\|_{Y_2} \\ &\leqslant k_2\big(2k_2 c\big(2k_1\|f\|_{Y_1}, 2k_1\|f\|_{Y_1}\big) + 1\big)\|f\|_{Y_2} \\ &\leqslant 2k_2\|f\|_{Y_2}.\end{aligned}$$

This concludes the proof of Claim 2 and thus of the theorem. □

For the sake of illustration we give here an academic example loosely related to our problem.

Example. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth contractible domain and $0 < \alpha < 1$. Let $r \geqslant 1$ and $1 \leqslant k \leqslant n-2$ be integers. Consider the form $w : \mathbb{R}^n \to \Lambda^k$ where

$$w = \sum_{I \in \mathcal{T}_k} w_I \, dx^I$$

where $\mathcal{T}_k$ is the set of ordered k-indices. Let $I_1, \ldots, I_{k+1} \in \mathcal{T}_k$, then there exists $\epsilon > 0$ such that for every $f \in C^{r,\alpha}(\overline{\Omega}; \Lambda^{k+1})$ with

$$\|f\|_{C^{r,\alpha}} \leqslant \epsilon, \quad df = 0 \quad \text{and} \quad f \wedge \nu = 0 \quad \text{on } \partial\Omega$$

there exists $w \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^k)$ satisfying

$$\begin{cases} dw + \bigwedge_{r=1}^{k+1} dw_{I_r} = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

The proof is immediate if we set

$$\begin{aligned}X = X_1 = X_2 &= \left\{w \in C^{r+1,\alpha}\big(\overline{\Omega}; \Lambda^k\big):\ w = 0 \text{ on } \partial\Omega\right\}, \\ Y = Y_1 = Y_2 &= \left\{f \in C^{r,\alpha}\big(\overline{\Omega}; \Lambda^{k+1}\big):\ df = 0 \text{ in } \Omega \text{ and } f \wedge \nu = 0 \text{ on } \partial\Omega\right\},\end{aligned}$$

L equal to the operator constructed in Remark 4 ($Lw = f$ being equivalent to $dw = f$) and

$$Q(a) = \bigwedge_{r=1}^{k+1} da_{I_r}.$$

5. The non-degenerate cases

5.1. The case $k = n$

For the sake of completeness we recall here, without proof, the result of Dacorogna and Moser [7] (see also Dacorogna [6]).

Theorem 12 *(Dacorogna–Moser). Let $r \geqslant 0$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with a $C^{r+3,\alpha}$ boundary consisting of finitely many connected components. Let $f, g > 0$ in $\overline{\Omega}$. Then the two following statements are equivalent.*

(i) *$f, g \in C^{r,\alpha}(\overline{\Omega}; \Lambda^n)$ and*
$$\int_\Omega f(x)\,dx = \int_\Omega g(x)\,dx.$$
(ii) *There exists $\varphi \in \mathrm{Diff}^{r+1,\alpha}(\overline{\Omega})$ satisfying*
$$\begin{cases} \varphi^*(g(x)) = f(x), & x \in \Omega, \\ \varphi(x) = x, & x \in \partial\Omega. \end{cases}$$

5.2. The case $k = 2$ in even dimension

We now consider the case where $f, g : \mathbb{R}^n \to \Lambda^2$ with
$$f = \sum_{1 \leqslant i < j \leqslant n} f_{ij}\,dx^i \wedge dx^j \quad \text{and} \quad g = \sum_{1 \leqslant i < j \leqslant n} g_{ij}\,dx^i \wedge dx^j.$$
We denote by $\bar{f}, \bar{g} \in \mathbb{R}^{n\times n}$ their antisymmetric representations. As we have already seen the rank of these matrices is always even and therefore these matrices can be invertible only when the dimension n is even. The most favourable case, that we will discuss now, is therefore when n is even and the rank of these matrices is n. The other cases are studied in Section 6.2.

We give three theorems, the first one (Theorem 13) with the help of the flow method, which has the advantage of having a simple proof, the second one (Theorem 15) with the fixed point method which gives sharp regularity estimates and the last one is a version of Darboux theorem with optimal regularity (Theorem 18).

Theorem 13. *Let $n > 2$ be even and $\Omega \subset \mathbb{R}^n$ be a bounded smooth contractible domain (ν denoting the outward unit normal). Let $r \geqslant 1$ be an integer, $0 < \alpha < 1$ and let $f, g \in C^{r,\alpha}(\overline{\Omega}; \Lambda^2)$ satisfy*
$$df = dg = 0 \quad \text{in } \Omega \quad \text{and} \quad f \wedge \nu = g \wedge \nu \quad \text{on } \partial\Omega$$
and, for every $t \in [0, 1]$,
$$\operatorname{rank}\big[t\bar{g} + (1-t)\bar{f}\big] = n, \quad \text{in } \overline{\Omega}.$$
Then, there exists $\varphi \in \mathrm{Diff}^{r,\alpha}(\overline{\Omega})$ such that
$$\varphi^*(g) = f \quad \text{in } \Omega \quad \text{and} \quad \varphi(x) = x \quad \text{on } \partial\Omega.$$

Remark 14.

(i) As we mentioned in the introduction, with almost no change, we can consider a general homotopy f_t with $f_0 = f$, $f_1 = g$,
$$df_t = 0, \quad f_t \wedge \nu = f_0 \wedge \nu \quad \text{on } \partial\Omega \quad \text{and} \quad \operatorname{rank}[\overline{f_t}] = n \quad \text{in } \overline{\Omega}.$$
Note that the non-degeneracy condition $\operatorname{rank}[\overline{f_t}] = n$ implies
$$f^{n/2} \cdot g^{n/2} > 0 \quad \text{in } \overline{\Omega}.$$

(ii) The non-degeneracy condition

$$\operatorname{rank}\big[t\bar{g} + (1-t)\bar{f}\big] = n \quad \text{for every } t \in [0,1]$$

is equivalent to the condition that $\bar{g}\bar{f}^{-1}$ has no negative eigenvalues.

(iii) Although we have only considered contractible domains Ω, the theorem (with the boundary data) remains valid for smooth connected sets under the additional hypothesis (cf. Remark 4)

$$\int_\Omega \langle f;\psi\rangle\,dx = \int_\Omega \langle g;\psi\rangle\,dx, \quad \forall\psi \in D_2(\Omega).$$

Proof. We solve

$$\begin{cases} dw = f - g & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

with the help of Theorem 3. We then recover the vector field u_t (this is possible since $\operatorname{rank}[t\bar{g} + (1-t)\bar{f}] = n$) through

$$u_t \lrcorner \big[tg + (1-t)f\big] = w.$$

The result then follows at once from Theorem 10. Note, in passing, that, although w is smoother than f and g, the vector field u_t has the same smoothness as f and g. □

Theorem 15. *Let $n > 2$ be even and $\Omega \subset \mathbb{R}^n$ be a bounded smooth contractible domain. Let $r \geqslant 4$ be an integer and $0 < \alpha < 1$. Let $f, g \in C^{r,\alpha}(\overline{\Omega};\Lambda^2) \cap C^{r+2,\alpha}(\partial\Omega;\Lambda^2)$ satisfying*

$$df = dg = 0 \quad \text{in } \Omega \quad \text{and} \quad f \wedge \nu = g \wedge \nu \quad \text{on } \partial\Omega$$

and, for every $t \in [0,1]$,

$$\operatorname{rank}\big[t\bar{g} + (1-t)\bar{f}\big] = n \quad \text{in } \overline{\Omega}.$$

Then, there exists $\varphi \in \operatorname{Diff}^{r+1,\alpha}(\overline{\Omega})$ such that

$$\varphi^*(g) = f \quad \text{in } \Omega \quad \text{and} \quad \varphi(x) = x \quad \text{on } \partial\Omega.$$

Remark 16.

(i) The same remarks as in the previous theorem hold. Note also that the extra regularity on f and g holds only on the boundary.

(ii) With the same method, the regularity assumption on the boundary, namely $f, g \in C^{r+2,\alpha}(\partial\Omega;\Lambda^2)$, can be weakened and replaced by $f, g \in C^{s,\theta}(\partial\Omega;\Lambda^2)$ with $0 < \alpha, \theta < 1$ and

$$r + 2 + \alpha \geqslant s + \theta > r + 1 + \alpha.$$

The conclusion $\varphi \in \operatorname{Diff}^{r+1,\alpha}(\overline{\Omega})$ is still valid provided (denoting the integer part of $x > 0$ by $[x]$ and its fractional part by $\{x\} = x - [x]$)

$$r \geqslant 2 + \left[\frac{2}{s+\theta-r-1-\alpha}\right], \quad \text{when } \alpha > \left\{\frac{2}{s+\theta-r-1-\alpha}\right\},$$
$$r \geqslant 3 + \left[\frac{2}{s+\theta-r-1-\alpha}\right], \quad \text{when } \alpha \leqslant \left\{\frac{2}{s+\theta-r-1-\alpha}\right\}.$$

Thus $r \geqslant 4$, when $s = r + 2$ and $\theta = \alpha$.

(iii) Although our proof will use Theorem 13, we could avoid it by several applications of Lemma 17 that follows.

The proof of Theorem 15 relies on the following key lemma.

Lemma 17. *Let $n > 2$ be even and $\Omega \subset \mathbb{R}^n$ be a bounded smooth contractible domain. Let $r \geqslant 0$ be an integer and $0 < \beta \leqslant \alpha < 1$. Then there exists $\gamma = \gamma(r, \alpha, \beta, \Omega) > 0$ such that for every $g \in C^{r+2,\alpha}(\overline{\Omega}; \Lambda^2)$ and $f \in C^{r,\alpha}(\overline{\Omega}; \Lambda^2)$ satisfying the following hypotheses*:

$$df = dg = 0 \quad \text{in } \Omega, \quad f \wedge \nu = g \wedge \nu \quad \text{on } \partial\Omega \quad \text{and} \quad \operatorname{rank}[\bar{g}] = n \quad \text{in } \overline{\Omega},$$
$$\|f - g\|_{C^{0,\beta}} \leqslant \frac{\gamma}{\|(\bar{g})^{-1}\|_{C^{1,\beta}} \max\{\|g\|_{C^{r+2,\alpha}} \|(\bar{g})^{-1}\|_{C^{r+1,\alpha}}, 1\}}$$

there exists $\varphi \in \operatorname{Diff}^{r+1,\alpha}(\overline{\Omega})$ such that

$$\varphi^*(g) = f \quad \text{in } \Omega \quad \text{and} \quad \varphi(x) = x \quad \text{on } \partial\Omega. \tag{9}$$

With exactly the same proof of that of the lemma, we can obtain Darboux theorem with optimal regularity.

Theorem 18 *(Darboux theorem with optimal regularity). Let $r \geqslant 0$ be an integer and $0 < \alpha < 1$. Let $n = 2m \geqslant 4$, $\Omega \subset \mathbb{R}^n$ be a bounded open set and $p \in \Omega$. Let ω_0 be the standard symplectic form*

$$\omega_0 = \sum_{i=1}^{m} dx^i \wedge dx^{m+i}.$$

Let $\omega \in C^{r,\alpha}(\overline{\Omega}; \Lambda^2)$ be such that

$$d\omega = 0 \quad \text{and} \quad \omega(p) = \omega_0.$$

Then there exist a neighbourhood V of p and $\varphi \in \operatorname{Diff}^{r+1,\alpha}(\overline{V}; \mathbb{R}^n)$ such that

$$\varphi^*(\omega_0) = \omega \quad \text{in } V \quad \text{and} \quad \varphi(p) = p.$$

Proof of Theorem 18. *Step 1*. As we already mentioned the proof is almost identical to that of the lemma, which will be given below. We start by choosing V a sufficiently small neighbourhood of p and we define the sets

$$X_1 = C^{1,\beta}(\overline{V}; \mathbb{R}^n) \quad \text{and} \quad Y_1 = C^{0,\beta}(\overline{V}; \Lambda^2),$$
$$X_2 = C^{r+1,\alpha}(\overline{V}; \mathbb{R}^n) \quad \text{and} \quad Y_2 = \{b \in C^{r,\alpha}(\overline{V}; \Lambda^2): db = 0 \text{ in } V\}.$$

The remaining estimates and conclusions are exactly those of the lemma and in particular we get that there exists $\psi \in \operatorname{Diff}^{r+1,\alpha}(\overline{V}; \mathbb{R}^n)$ such that $\psi^*(\omega_0) = \omega$ in V, provided

$$\|\omega - \omega_0\|_{C^{0,\beta}} \leqslant \frac{\gamma}{\|(\overline{\omega_0})^{-1}\|_{C^{1,\beta}} \max\{\|\omega_0\|_{C^{r+2,\alpha}} \|(\overline{\omega_0})^{-1}\|_{C^{r+1,\alpha}}, 1\}}. \tag{10}$$

(Of course, here, $\|(\overline{\omega_0})^{-1}\|_{C^{1,\beta}} = \|(\overline{\omega_0})^{-1}\|_{C^0}$ and $\|\omega_0\|_{C^{r+2,\alpha}} = \|\omega_0\|_{C^0}$.) Setting $\varphi(x) = \psi(x) + p - \psi(p)$, we have indeed proved that $\varphi \in \operatorname{Diff}^{r+1,\alpha}(\overline{V}; \mathbb{R}^n)$ and

$$\varphi^*(\omega_0) = \omega \quad \text{in } V \quad \text{and} \quad \varphi(p) = p.$$

Step 2. Now it remains to check that under the hypotheses of Darboux theorem, namely $\omega \in C^{r,\alpha}$ with $d\omega = 0$ and $\omega(p) = \omega_0(p)$, we have automatically that $\|\omega - \omega_0\|_{C^{0,\beta}}$ is as small as we want and thus (10) is satisfied. Indeed choose $\epsilon > 0$ small and the neighbourhood V, smaller if necessary than in Step 1, such that

$$|x - y| \leqslant \epsilon \quad \text{for } x, y \in \overline{V}.$$

Let $0 < \beta < \alpha$ and $h = \omega - \omega_0$. Since $\omega(p) = \omega_0(p)$ and $\omega, \omega_0 \in C^{0,\alpha}$, then there exists $k > 0$ such that for every $x, y \in \overline{V}$,

$$|h(x) - h(y)| \leqslant k|x - y|^\alpha = k|x - y|^{\alpha-\beta}|x - y|^\beta \leqslant k\epsilon^{\alpha-\beta}|x - y|^\beta.$$

Since $h(p)=0$, we can indeed choose $\epsilon>0$ sufficiently small so that

$$\|\omega-\omega_0\|_{C^{0,\beta}}=\|h\|_{C^{0,\beta}}\leqslant\frac{\gamma}{\|(\overline{\omega_0})^{-1}\|_{C^{1,\beta}}\max\{\|\omega_0\|_{C^{r+2,\alpha}}\|(\overline{\omega_0})^{-1}\|_{C^{r+1,\alpha}},1\}}.$$

This concludes the proof of the theorem. □

We now turn to the proof of the lemma.

Proof of Lemma 17. The lemma will follow from Theorem 11. We divide the proof into four steps; the three first ones to verify the hypotheses of the theorem and the last one to check the conclusions of the lemma from the one of the theorem.

Step 1. We first extend g to $\mathbb{R}^n$ with the same regularity. We then define the spaces as follows

$$X_1=C^{1,\beta}(\overline{\Omega};\mathbb{R}^n)\quad\text{and}\quad Y_1=C^{0,\beta}(\overline{\Omega};\Lambda^2),$$
$$X_2=\{a\in C^{r+1,\alpha}(\overline{\Omega};\mathbb{R}^n):\ a=0\text{ on }\partial\Omega\},$$
$$Y_2=\{b\in C^{r,\alpha}(\overline{\Omega};\Lambda^2):\ db=0\text{ in }\Omega\text{ and }b\wedge\nu=0\text{ on }\partial\Omega\}.$$

It is easily seen that they satisfy hypothesis (H_{XY}) of Theorem 11 (see [6]).

Step 2. We next define a linear operator $L:X_2\to Y_2$ which associates in an isomorphic way any element $a\in X_2$ to a unique element $b\in Y_2$ through the equation

$$La=d[a\lrcorner g]=b.$$

This is indeed well defined according to Theorem 3 (cf. also Remark 4) and to the fact that $\operatorname{rank}[\bar{g}]=n$. Moreover we can find a constant $K_1>0$, independent of g, such that if

$$k_1:=K_1\|(\bar{g})^{-1}\|_{C^{1,\beta}}\quad\text{and}\quad k_2:=K_1\|(\bar{g})^{-1}\|_{C^{r+1,\alpha}}$$

then

$$\|L^{-1}b\|_{X_i}\leqslant k_i\|b\|_{Y_i}\quad\text{for }i=1,2;$$

so that (H_L) of Theorem 11 is satisfied.

Step 3. The central part of the lemma is to define the operator Q and to check the property (H_Q) of Theorem 11. This requires a more subtle linearization than the one in [7] and we divide the proof into five substeps. For this we let

$$c(r,s):=K_2\|g\|_{C^{r+2,\alpha}}(r+s)$$

where $K_2>0$ will be an appropriate constant that is independent of g.

Step 3.1. With the definition of L in hand, we now rewrite (9) as follows. Setting $\varphi=\mathrm{Id}+u$, we rewrite the equation $\varphi^*(g)=f$ in the equivalent form

$$\begin{aligned}Lu=d[u\lrcorner g]&=f-(\mathrm{Id}+u)^*g+d[u\lrcorner g]\\&=f-g+\big[g-(\mathrm{Id}+u)^*g+d[u\lrcorner g]\big]\\&=f-g+Q(u)\end{aligned}$$

where

$$Q(u):=g-(\mathrm{Id}+u)^*g+d[u\lrcorner g].$$

In order to get the right estimates, we rewrite $Q(u)$ in the following way

$$\begin{aligned}Q(u)(x)&=g(x)-\sum_{i<j}g_{ij}(x+u)\big(dx^i+du^i\big)\wedge\big(dx^j+du^j\big)+d[u\lrcorner g]\\&=g(x)-g(x+u)-\sum_{i<j}g_{ij}(x+u)\big[du^i\wedge dx^j+dx^i\wedge du^j\big]-\sum_{i<j}g_{ij}(x+u)\,du^i\wedge du^j+d[u\lrcorner g].\end{aligned}$$

We then appeal to the formula

$$d[u \lrcorner g] = \sum_{i<j} g_{ij}\big[du^i \wedge dx^j + dx^i \wedge du^j\big] + \sum_{i<j} \langle \operatorname{grad} g_{ij}; u\rangle\, dx^i \wedge dx^j$$

to obtain

$$\begin{aligned}Q(u)(x) &= \sum_{i<j}\big[g_{ij}(x) - g_{ij}(x+u)\big]\big[du^i \wedge dx^j + dx^i \wedge du^j\big] \\ &\quad - \sum_{i<j}\big[g_{ij}(x+u) - g_{ij}(x) - \big\langle \operatorname{grad} g_{ij}(x); u\big\rangle\big]\, dx^i \wedge dx^j - \sum_{i<j} g_{ij}(x+u)\, du^i \wedge du^j \\ &= Q_1(u)(x) - Q_2(u)(x) - Q_3(u)(x)\end{aligned}$$

where

$$\begin{aligned}Q_1(u)(x) &:= \sum_{i<j}\big[g_{ij}(x) - g_{ij}\big(x+u(x)\big)\big]\big[du^i \wedge dx^j + dx^i \wedge du^j\big], \\ Q_2(u)(x) &:= \sum_{i<j}\big[g_{ij}\big(x+u(x)\big) - g_{ij}(x) - \big\langle \operatorname{grad} g_{ij}(x); u(x)\big\rangle\big]\, dx^i \wedge dx^j, \\ Q_3(u)(x) &:= \sum_{i<j} g_{ij}\big(x+u(x)\big)\, du^i \wedge du^j.\end{aligned}$$

Step 3.2. We therefore have to check property (H_Q). Clearly $Q(0) = 0$. Moreover $Q : X_2 \to Y_2$. Indeed we have to check that $dQ(u) = 0$ in Ω and $Q(u) \wedge \nu = 0$ on $\partial\Omega$. The first condition follows immediately since $dg = 0$ and

$$dQ(u) = dg - (\mathrm{Id} + u)^*\, dg + dd[u \lrcorner g].$$

The second one is true since $u = 0$ on $\partial\Omega$. Indeed clearly $Q_1(u) = Q_2(u) = 0$ on $\partial\Omega$ and, since $u = 0$ on $\partial\Omega$, each of $\operatorname{grad} u^i$ and $\operatorname{grad} u^j$ is parallel to the normal ν. Thus, $du^i \wedge du^j = 0$ on $\partial\Omega$ for every $i < j$, which implies that $Q_3(u) = 0$ on $\partial\Omega$. Thus, we have, in fact, proved that $Q(u) = 0$ on $\partial\Omega$.

Step 3.3. Before starting our estimates, we recall some basic inequalities for Hölder functions (see Hörmander [9]). In the sequel γ will denote a generic constant. We have

$$\begin{aligned}&\|uv\|_{C^{r,\alpha}} \leqslant \gamma\big[\|u\|_{C^0}\|v\|_{C^{r,\alpha}} + \|u\|_{C^{r,\alpha}}\|v\|_{C^0}\big], \\ &\|g \circ u\|_{C^{r,\alpha}} \leqslant \gamma\|g\|_{C^{r,\alpha}}\big[1 + \|u\|_{C^{r,\alpha}} + \|u\|_{C^1}^{r+\alpha}\big]\end{aligned}$$

and

$$\begin{aligned}\|g \circ u - g \circ v\|_{C^{r,\alpha}} \leqslant \gamma[&\|g\|_{C^{r+1,\alpha}}\big(1 + \|u\|_{C^{r,\alpha}} + \|v\|_{C^{r,\alpha}} + \|u\|_{C^1}^{r+\alpha} + \|v\|_{C^1}^{r+\alpha}\big)\|u - v\|_{C^0} \\ &+ \|g\|_{C^1}\|u - v\|_{C^{r,\alpha}}\big].\end{aligned}$$

When $r = 0$, the second inequality should be replaced by

$$\|g \circ u\|_{C^{0,\alpha}} \leqslant \gamma\big[\|g\|_{C^{0,\alpha}}\|u\|_{C^1}^{\alpha} + \|g\|_{C^0}\big].$$

Step 3.4. We now show the estimate (5) in (H_Q). We, in fact, will prove the stronger estimate, namely that, for every $v, w \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ with $\|v\|_{C^{1,\beta}} \leqslant 1$ and $\|w\|_{C^{1,\beta}} \leqslant 1$,

$$\big\|Q(v) - Q(w)\big\|_{C^{0,\beta}} \leqslant K_2\|g\|_{C^{2,\beta}}\big(\|v\|_{C^{1,\beta}} + \|w\|_{C^{1,\beta}}\big)\|v - w\|_{C^{1,\beta}}.$$

Evidently it is enough to prove that, for each $p = 1, 2, 3$

$$\big\|Q_p(v) - Q_p(w)\big\|_{C^{0,\beta}} \leqslant K_2\|g\|_{C^{2,\beta}}\big(\|v\|_{C^{1,\beta}} + \|w\|_{C^{1,\beta}}\big)\|v - w\|_{C^{1,\beta}}.$$

In the sequel, K_2 will denote a generic constant that does not depend on g, v and w. We begin with Q_1, we have

$$\begin{aligned}
&\big\| Q_1(v) - Q_1(w) \big\|_{C^{0,\beta}} \\
&\quad = \bigg\| \sum_{i<j} \big[g_{ij}(\mathrm{Id}+w) - g_{ij}(\mathrm{Id})\big]\big[dw^i \wedge dx^j + dx^i \wedge dw^j\big] \\
&\qquad - \sum_{i<j} \big[g_{ij}(\mathrm{Id}+v) - g_{ij}(\mathrm{Id})\big]\big[dv^i \wedge dx^j + dx^i \wedge dv^j\big] \bigg\|_{C^{0,\beta}} \\
&\quad \leqslant \sum_{i<j} \big\| \big[g_{ij}(\mathrm{Id}+w) - g_{ij}(\mathrm{Id}+v)\big]\big[dw^i \wedge dx^j + dx^i \wedge dw^j\big] \big\|_{C^{0,\beta}} \\
&\qquad + \sum_{i<j} \big\| \big[g_{ij}(\mathrm{Id}+v) - g_{ij}(\mathrm{Id})\big]\big[\big(dv^i \wedge dx^j + dx^i \wedge dv^j\big) - \big(dw^i \wedge dx^j + dx^i \wedge dw^j\big)\big] \big\|_{C^{0,\beta}}
\end{aligned}$$

and hence (bearing in mind that $\|v\|_{C^{1,\beta}}, \|w\|_{C^{1,\beta}} \leqslant 1$)

$$\begin{aligned}
\big\| Q_1(v) - Q_1(w) \big\|_{C^{0,\beta}} &\leqslant K_2 \|g\|_{C^{1,\beta}} \|w - v\|_{C^{0,\beta}} \|w\|_{C^{1,\beta}} + K_2 \|g\|_{C^{1,\beta}} \|v\|_{C^{0,\beta}} \|w - v\|_{C^{1,\beta}} \\
&\leqslant K_2 \|g\|_{C^{2,\beta}} \big(\|v\|_{C^{1,\beta}} + \|w\|_{C^{1,\beta}}\big) \|v - w\|_{C^{1,\beta}}.
\end{aligned}$$

For Q_2 we proceed in the following way. We first observe that

$$\begin{aligned}
Q_2(v) &= \sum_{i<j} \int_0^1 \frac{d}{dt}\big[\big(g_{ij}(x+tv) - t\big\langle \operatorname{grad} g_{ij}(x); v\big\rangle\big)\, dx^i \wedge dx^j\big]\, dt \\
&= \sum_{i<j} \int_0^1 \big[\big\langle \operatorname{grad} g_{ij}(x+tv) - \operatorname{grad} g_{ij}(x); v\big\rangle dx^i \wedge dx^j\big]\, dt.
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
\big\| Q_2(v) - Q_2(w) \big\|_{C^{0,\beta}} &\leqslant \sum_{i<j} \int_0^1 \big\| \big\langle \operatorname{grad} g_{ij}(x+tv) - \operatorname{grad} g_{ij}(x); v\big\rangle \\
&\qquad - \big\langle \operatorname{grad} g_{ij}(x+tw) - \operatorname{grad} g_{ij}(x); w\big\rangle \big\|_{C^{0,\beta}}\, dt \\
&\leqslant \sum_{i<j} \int_0^1 \big\{ \big\| \big\langle \operatorname{grad} g_{ij}(x+tv) - \operatorname{grad} g_{ij}(x+tw); v\big\rangle \big\|_{C^{0,\beta}} \\
&\qquad + \big\| \big\langle \operatorname{grad} g_{ij}(x+tw) - \operatorname{grad} g_{ij}(x); v - w\big\rangle \big\|_{C^{0,\beta}} \big\}\, dt
\end{aligned}$$

and hence

$$\begin{aligned}
\big\| Q_2(v) - Q_2(w) \big\|_{C^{0,\beta}} &\leqslant K_2 \sum_{i<j} \int_0^1 \big\{ \big\| \operatorname{grad} g_{ij}(x+tv) - \operatorname{grad} g_{ij}(x+tw) \big\|_{C^{0,\beta}} \|v\|_{C^0} \\
&\qquad + \big\| \operatorname{grad} g_{ij}(x+tv) - \operatorname{grad} g_{ij}(x+tw) \big\|_{C^0} \|v\|_{C^{0,\beta}} \\
&\qquad + \big\| \operatorname{grad} g_{ij}(x+tw) - \operatorname{grad} g_{ij}(x) \big\|_{C^{0,\beta}} \|v - w\|_{C^0} \\
&\qquad + \big\| \operatorname{grad} g_{ij}(x+tw) - \operatorname{grad} g_{ij}(x) \big\|_{C^0} \|v - w\|_{C^{0,\beta}} \big\}\, dt.
\end{aligned}$$

This leads to (recall that $\|v\|_{C^{1,\beta}}, \|w\|_{C^{1,\beta}} \leqslant 1$)

$$\begin{aligned}
\big\| Q_2(v) - Q_2(w) \big\|_{C^{0,\beta}} &\leqslant K_2 \|g\|_{C^{2,\beta}} \|v - w\|_{C^{0,\beta}} \|v\|_{C^0} + K_2 \|g\|_{C^2} \|v - w\|_{C^0} \|v\|_{C^{0,\beta}} \\
&\qquad + K_2 \|g\|_{C^{2,\beta}} \|w\|_{C^{0,\beta}} \|v - w\|_{C^0} + K_2 \|g\|_{C^2} \|w\|_{C^0} \|v - w\|_{C^{0,\beta}}.
\end{aligned}$$

We therefore have the estimate

$$\big\| Q_2(v) - Q_2(w) \big\|_{C^{0,\beta}} \leqslant K_2 \|g\|_{C^{2,\beta}} \big(\|v\|_{C^{1,\beta}} + \|w\|_{C^{1,\beta}}\big) \|v - w\|_{C^{1,\beta}}.$$

It remains to prove the estimate for Q_3, namely

$$\begin{aligned}\left\|Q_3(v)-Q_3(w)\right\|_{C^{0,\beta}} &= \left\|\sum_{i<j} g_{ij}(\mathrm{Id}+w)\,dw^i\wedge dw^j-\sum_{i<j} g_{ij}(\mathrm{Id}+v)\,dv^i\wedge dv^j\right\|_{C^{0,\beta}}\\ &\leqslant \sum_{i<j}\left\|g_{ij}(\mathrm{Id}+w)\big(dw^i\wedge dw^j-dv^i\wedge dv^j\big)\right\|_{C^{0,\beta}}\\ &\quad+\sum_{i<j}\left\|\big(g_{ij}(\mathrm{Id}+v)-g_{ij}(\mathrm{Id}+w)\big)\,dv^i\wedge dv^j\right\|_{C^{0,\beta}}\end{aligned}$$

which leads to (recalling that $\|v\|_{C^{1,\beta}}, \|w\|_{C^{1,\beta}}\leqslant 1$)

$$\begin{aligned}\left\|Q_3(v)-Q_3(w)\right\|_{C^{0,\beta}} &\leqslant K_2\|g\|_{C^{0,\beta}}\big(\|v\|_{C^{1,\beta}}+\|w\|_{C^{1,\beta}}\big)\|v-w\|_{C^{1,\beta}}\\ &\quad+K_2\|g\|_{C^{1,\beta}}\|v-w\|_{C^{0,\beta}}\|\operatorname{grad} v\|_{C^{0,\beta}}\end{aligned}$$

and thus

$$\left\|Q_3(v)-Q_3(w)\right\|_{C^{0,\beta}}\leqslant K_2\|g\|_{C^{2,\beta}}\big(\|v\|_{C^{1,\beta}}+\|w\|_{C^{1,\beta}}\big)\|v-w\|_{C^{1,\beta}}$$

proving the estimate for Q_3.

Step 3.5. We finally establish the estimate (6) in (H_Q). We have to prove that, for every $v\in C^{r+1,\alpha}(\overline{\Omega};\mathbb{R}^n)$ with $\|v\|_{C^{1,\beta}}\leqslant 1$,

$$\left\|Q(v)\right\|_{C^{r,\alpha}}\leqslant K_2\|g\|_{C^{r+2,\alpha}}\|v\|_{C^{r+1,\alpha}}\|v\|_{C^{1,\beta}}.$$

The estimate is proved exactly as in Step 3.4 and we leave out the details.

Step 4. The hypotheses of Theorem 11 having been verified, we conclude that if

$$\begin{aligned}\|f-g\|_{C^{0,\beta}} &\leqslant \frac{\min\{1/(8K_1^2K_2K_3),\,1/(2K_1)\}}{\|(\bar g)^{-1}\|_{C^{1,\beta}}\max\{\|g\|_{C^{r+2,\alpha}}\|(\bar g)^{-1}\|_{C^{r+1,\alpha}},1\}}\\ &\leqslant \frac{1}{2K_1\|(\bar g)^{-1}\|_{C^{1,\beta}}}\min\left\{\frac{1}{4K_1K_2K_3\|g\|_{C^{r+2,\alpha}}\|(\bar g)^{-1}\|_{C^{r+1,\alpha}}},1\right\},\end{aligned}$$

where $K_3\geqslant 1$ is such that

$$\|\cdot\|_{C^{1,\beta}}\leqslant K_3\|.\|_{C^{r+1,\alpha}}.$$

Then there exists $\varphi\in C^{r+1,\alpha}(\overline{\Omega};\mathbb{R}^n)$ satisfying (9) and γ can be taken as

$$\gamma=\min\left\{\frac{1}{8K_1^2K_2K_3},\frac{1}{2K_1}\right\}.$$

By further restricting γ we can easily show that $\det(D\varphi(x))\neq 0$ for every $x\in\overline{\Omega}$ and $\|u\|_{C^{1,\beta}}\leqslant 1/2$. This last condition leads to the fact that $\varphi=\mathrm{Id}+u$ is globally one to one, maps $\overline{\Omega}$ onto $\overline{\Omega}$ and thus $\varphi\in\mathrm{Diff}^{r+1,\alpha}(\overline{\Omega})$. □

We can now conclude the proof of Theorem 15 by an iteration scheme involving appropriate regularization.

Proof. We divide the proof into three steps.

Step 1. We start with a preliminary computation. Let s be a positive integer and $h\in C^{s,\alpha}(\overline{\Omega})$ with $h\geqslant h_0>0$ in $\overline{\Omega}$, it is easy to see, by induction on s, that there exists a constant $\gamma_1>0$, independent of h, such that

$$\left\|\frac{1}{h}\right\|_{C^{s,\alpha}}\leqslant\gamma_1\sum_{j=0}^{s}\left[\frac{\|h\|_{C^{j,\alpha}}}{h_0^2}\frac{\|h\|_{C^1}^{s-j}}{h_0^{s-j}}\right].$$

Denoting by $\operatorname{adj}\bar g$ the transpose of the matrix of cofactors of $\bar g$ so as to have

$$(\bar g)^{-1}=\frac{\operatorname{adj}\bar g}{\det\bar g}$$

we find that there exists a constant $\gamma_2 > 0$, independent of g, such that

$$\left\|(\bar{g})^{-1}\right\|_{C^{s,\alpha}} \leqslant \gamma_2\left[\|g\|_{C^{s,\alpha}}\|g\|_{C^0}^{n-2}\left\|\frac{1}{\det\bar{g}}\right\|_{C^0} + \|g\|_{C^0}^{n-1}\left\|\frac{1}{\det\bar{g}}\right\|_{C^{s,\alpha}}\right],$$
$$\|\det\bar{g}\|_{C^{s,\alpha}} \leqslant \gamma_2\|g\|_{C^{s,\alpha}}\|g\|_{C^0}^{n-1}.$$

Combining the two estimates we obtain that there exists a constant γ_3 (depending also on $\min\det\bar{g}$ but not in any other way on g) such that

$$\left\|(\bar{g})^{-1}\right\|_{C^{r+1,\alpha}} \leqslant \gamma_3\left[\|g\|_{C^{r+1,\alpha}}\left(\|g\|_{C^0}^{2n-2} + \|g\|_{C^0}^{n-2}\right) + \operatorname{terms}\left(\|g\|_{C^{r,\alpha}}, \|g\|_{C^0}\right)\right].$$

Step 2. Choose $0 < \beta < \alpha$. We next regularize f and g with the help of Theorem 1 and construct for every $\epsilon > 0$, $f^\epsilon, g^\epsilon \in C^{r+2,\alpha}(\overline{\Omega}; \Lambda^2)$ such that $df^\epsilon = dg^\epsilon = 0$, $f^\epsilon \wedge \nu = g^\epsilon \wedge \nu = g \wedge \nu = f \wedge \nu$ on $\partial\Omega$ and (where $\gamma_4 = \gamma_4(r, \alpha, \beta, \Omega) > 0$ is a constant)

$$\|g^\epsilon - g\|_{C^{1,\beta}(\overline{\Omega})} \leqslant \gamma_4\epsilon^{r-1+\alpha-\beta}\|g\|_{C^{r,\alpha}(\overline{\Omega})},$$
$$\|g^\epsilon\|_{C^{0,\beta}(\overline{\Omega})} \leqslant \gamma_4\|g\|_{C^{0,\beta}(\overline{\Omega})}, \qquad \|g^\epsilon\|_{C^{1,\beta}(\overline{\Omega})} \leqslant \gamma_4\|g\|_{C^{1,\beta}(\overline{\Omega})},$$
$$\|g^\epsilon\|_{C^{r,\alpha}(\overline{\Omega})} \leqslant \gamma_4\|g\|_{C^{r,\alpha}(\overline{\Omega})},$$
$$\|g^\epsilon\|_{C^{r+1,\alpha}(\overline{\Omega})} \leqslant \frac{\gamma_4}{\epsilon}\left[\|g\|_{C^{r,\alpha}(\overline{\Omega})} + \epsilon\|g\|_{C^{r+1,\alpha}(\partial\Omega)}\right],$$
$$\|g^\epsilon\|_{C^{r+2,\alpha}(\overline{\Omega})} \leqslant \frac{\gamma_4}{\epsilon^2}\left[\|g\|_{C^{r,\alpha}(\overline{\Omega})} + \epsilon^2\|g\|_{C^{r+2,\alpha}(\partial\Omega)}\right]$$

and similarly for f and f^ϵ. Moreover by further restricting ϵ we can assume that

$$\operatorname{rank}\left[t\bar{g}^\epsilon + (1-t)\bar{f}^\epsilon\right] = n, \quad \text{for every } t \in [0,1].$$

Observe now that, for ϵ sufficiently small, the orders of magnitudes are

$$\left\|(\overline{g^\epsilon})^{-1}\right\|_{C^{r+1,\alpha}(\overline{\Omega})} \simeq \epsilon^{-1} \quad \text{and} \quad \|g^\epsilon\|_{C^{r+2,\alpha}(\overline{\Omega})} \simeq \epsilon^{-2},$$

so that

$$\frac{1}{\|g^\epsilon\|_{C^{r+2,\alpha}}\|(\overline{g^\epsilon})^{-1}\|_{C^{r+1,\alpha}}\|(\overline{g^\epsilon})^{-1}\|_{C^{1,\beta}}} \simeq \epsilon^3 \quad \text{and} \quad \|g^\epsilon - g\|_{C^{1,\beta}} \simeq \epsilon^{r-1+\alpha-\beta}.$$

Therefore, by still further restricting ϵ, we can assume, since $r \geqslant 4$, that (where γ is as in Lemma 17)

$$\|g^\epsilon - g\|_{C^{0,\beta}} \leqslant \frac{\gamma}{\|(\overline{g^\epsilon})^{-1}\|_{C^{1,\beta}}\max\{\|g^\epsilon\|_{C^{r+2,\alpha}}\|(\overline{g^\epsilon})^{-1}\|_{C^{r+1,\alpha}}, 1\}}$$

and similarly for f^ϵ.

Step 3. We then appeal to Lemma 17 to find $\varphi_1, \varphi_3 \in \operatorname{Diff}^{r+1,\alpha}(\overline{\Omega})$ such that

$$\begin{cases} \varphi_1^*(g^\epsilon(x)) = g(x), & x \in \Omega, \\ \varphi_1(x) = x, & x \in \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \varphi_3^*(f^\epsilon(x)) = f(x), & x \in \Omega, \\ \varphi_3(x) = x, & x \in \partial\Omega. \end{cases}$$

We finally use Theorem 13 to find $\varphi_2 \in \operatorname{Diff}^{r+1,\alpha}(\overline{\Omega})$ such that

$$\begin{cases} \varphi_2^*(g^\epsilon(x)) = f^\epsilon(x), & x \in \Omega, \\ \varphi_2(x) = x, & x \in \partial\Omega. \end{cases}$$

The claimed solution is then given by

$$\varphi = \varphi_1^{-1} \circ \varphi_2 \circ \varphi_3.$$

This achieves the proof of the theorem. □

6. The degenerate cases

6.1. The case $k = 1$

We discuss here the very elementary case where $k = 1$ and we can proceed in a direct way. We have

$$f = \sum_{i=1}^{n} f_i \, dx^i \quad \text{and} \quad g = \sum_{i=1}^{n} g_i \, dx^i .$$

Proposition 19. *Let $\Omega \subset \mathbb{R}^n$ be a smooth simply connected domain. Let $r \geqslant 1$ and $f, g \in C^r(\overline{\Omega}; \Lambda^1)$ with*

$$df = dg = 0 \quad \textit{and} \quad f_1, g_1 \neq 0.$$

Then there exists a diffeomorphism $\varphi \in \operatorname{Diff}^{r+1}(\overline{\Omega}; \mathbb{R}^n)$ satisfying

$$\varphi^*(g) = f.$$

If, in addition, $f \wedge \nu = g \wedge \nu$ on $\partial\Omega$, then φ can be chosen so as to verify

$$\varphi(x) = x \quad \textit{on } \partial\Omega.$$

Proof. By hypothesis, we can find $F, G \in C^{r+1}(\overline{\Omega})$ such that

$$dF = f \quad \text{and} \quad dG = g.$$

Therefore the equation $\varphi^*(g) = f$ will be satisfied if we can solve the equation $\varphi^*(G) = F$, which reads as

$$G(\varphi) = F$$

and is not anymore a differential equation. Note that if $f \wedge \nu = g \wedge \nu$ on $\partial\Omega$, then we can choose $F = G$ on $\partial\Omega$. We then let $\varphi^i = x^i$ for every $i = 2, \ldots, n$ and we recover φ^1 by solving

$$G\big(\varphi^1, x^2, \ldots, x^n\big) = F\big(x^1, \ldots, x^n\big).$$

This is possible if the function $t \to G(t, x^2, \ldots, x^n)$ is monotone for every $x^2, \ldots, x^n$, which happens if $g_1 \neq 0$ in $\overline{\Omega}$. Moreover the solution is a diffeomorphism if $f_1 \neq 0$ in $\overline{\Omega}$. The boundary data is satisfied since $F = G$ on $\partial\Omega$. □

6.2. The case $k = 2$

We now investigate the case

$$f = \sum_{1 \leqslant i < j \leqslant n} f_{ij} \, dx^i \wedge dx^j \quad \text{and} \quad g = \sum_{1 \leqslant i < j \leqslant n} g_{ij} \, dx^i \wedge dx^j .$$

We first recall that the rank of a 2-form is always even and therefore a 2-form f in odd dimension necessarily satisfies

$$\operatorname{rank}[\bar{f}] \leqslant n - 1 \quad \text{or equivalently} \quad \dim \ker[\bar{f}] \geqslant 1.$$

We only give a result concerning forms of maximal rank in odd dimension, meaning that $\operatorname{rank}[\bar{f}] = n - 1$ or equivalently $\dim \ker[\bar{f}] = 1$. Other more degenerate cases can be handled similarly, the details will be discussed elsewhere. Note that

$$\operatorname{rank}\big[t\bar{g} + (1-t)\bar{f}\big] = n - 1$$

implies that there exists $a = a(t, x)$ such that

$$\ker\big[t\bar{g} + (1-t)\bar{f}\big] = \operatorname{span}\{a\}.$$

We also let $H = \{x \in \mathbb{R}^n\colon\ x^n > 0\}$ and $\nu = -e_n$ be the outward unit normal. Although the theorem is stated for the half plane, it can be, in a straightforward way, adapted to smooth domains Ω.

Theorem 20. *Let $r \geqslant 1$, $0<\alpha<1$, n odd, $f, g \in C^{r,\alpha}(\overline{H}; \Lambda^2)$ with $df = dg = 0$ in H and*

$$\ker\left[t\bar{g} + (1-t)\bar{f}\right] = \operatorname{span}\{a\}$$

where $a \in C^{r+1,\alpha}([0,1]\times\overline{H}; \Lambda^1)$ with $a_n \neq 0$. Then there exist $\epsilon > 0$ and $\varphi \in \operatorname{Diff}^{r,\alpha}(\overline{H}\cap B_\epsilon(0); \mathbb{R}^n)$ verifying

$$\varphi^*(g) = f \quad \text{in } H \cap B_\epsilon(0).$$

Moreover if $f \wedge e_n = g \wedge e_n$ on ∂H, then φ can be chosen so that

$$\varphi(x) = x, \quad x \in \partial H \cap B_\epsilon(0).$$

Proof. We immediately deal with the Cauchy data problem. We solve the problem by the flow method, so that we only have to find the appropriate vector field. First use Theorem 7 to find $h_t \in C^{r,\alpha}(\overline{H}\cap B_\epsilon(0); \Lambda^1)$ such that

$$\begin{cases} dh_t = f - g & \text{in } H \cap B_\epsilon(0), \\ \langle h_t; a\rangle = 0 & \text{in } H \cap B_\epsilon(0), \\ h_t = 0 & \text{on } \partial H \cap B_\epsilon(0). \end{cases}$$

Then find u_t by solving

$$u_t \lrcorner \left[tg + (1-t)f\right] = h_t.$$

This is indeed possible since $\langle h_t; a\rangle = 0$.

Restricting, if necessary ϵ, we solve then the problem by the flow. □

6.3. The case $3 \leqslant k \leqslant n-2$

We here discuss two simple examples showing that our method may, in some cases, apply to the more difficult case $3 \leqslant k \leqslant n-2$. A systematic study will be undertaken elsewhere. The first example concerns the case k odd, while the second deals with the case k even.

Example. Let $n = 2m$ be even, $f, g : \mathbb{R}^n \to \Lambda^2$ and $b : \mathbb{R}^n \to \Lambda^1$ satisfying

$$df = dg = 0 \quad \text{and} \quad db = 0 \quad \text{in } H,$$

where $H = \{x \in \mathbb{R}^n:\ x^n > 0\}$ (in order to ensure Cauchy data we have to assume that $f \wedge e_n = g \wedge e_n$ on ∂H) as well as

$$tg + (1-t)f \quad \text{is non-degenerate for every } t \in [0,1].$$

Let $1 \leqslant l \leqslant m-1$ be an integer and consider the $k = (2l+1)$-forms

$$F = f^l \wedge b \quad \text{and} \quad G = g^l \wedge b.$$

We claim that there exists a diffeomorphism φ such that

$$\varphi^*(G) = F$$

(if $f \wedge e_n = g \wedge e_n$ on ∂H, then we can also guarantee that $\varphi(x) = x$ on ∂H). The result will be local. In some special cases, the result can be global, if, for example, we can apply below Theorem 5 instead of Theorem 7.

Step 1. We let

$$a_t = tg + (1-t)f \quad \text{and} \quad \bar{a}_t = t\bar{g} + (1-t)\bar{f}.$$

The matrix $\bar{a}_t$ is, by hypothesis, invertible. We then locally solve, by applying Theorem 7,

$$\begin{cases} dw_t = f - g, \\ \langle w_t; \bar{a}_t^{-T} b\rangle = 0 \end{cases}$$

(if $f \wedge e_n = g \wedge e_n$ on ∂H, we can also impose that $w_t = 0$ on ∂H, provided $(\bar{a}_t^{-T} b)^n \neq 0$) and we recover the vector field

$$u_t \lrcorner a_t = w_t \quad \text{or equivalently} \quad u_t = -\bar{a}_t^{-1} w_t$$

(if $f \wedge e_n = g \wedge e_n$ on ∂H, we obtain that $u_t = 0$ on ∂H).

Step 2. Observe that

$$u_t \lrcorner b = \langle u_t; b\rangle = -\big\langle \bar{a}_t^{-1} w_t; b\big\rangle = -\big\langle w_t; \bar{a}_t^{-T} b\big\rangle = 0.$$

We thus deduce that not only

$$d\big(u_t \lrcorner \big(tg + (1-t)f\big)\big) = f - g$$

but also

$$d\big(u_t \lrcorner b\big) = d\big(u_t \lrcorner \big(tb + (1-t)b\big)\big) = b - b = 0.$$

Therefore solving

$$\frac{d}{dt}\varphi_t = u_t(\varphi_t)$$

leads to

$$\varphi_1^*(g) = f \quad \text{and} \quad \varphi_1^*(b) = b$$

and hence to

$$\varphi_1^*\big(g^l \wedge b\big) = f^l \wedge b.$$

Example. We now discuss another example in the same spirit. Let $n = 2m$ be even, $1 \leqslant l \leqslant m-1$ and

$$\lambda = \sum_{1 \leqslant i<j \leqslant 2l} \lambda_{ij}\big(x^1, \ldots, x^{2l}\big)\, dx^i \wedge dx^j,$$

$$a = \sum_{2l+1 \leqslant i<j \leqslant n} a_{ij}\big(x^{2l+1}, \ldots, x^{n}\big)\, dx^i \wedge dx^j,$$

$$b = \sum_{2l+1 \leqslant i<j \leqslant n} b_{ij}\big(x^{2l+1}, \ldots, x^{n}\big)\, dx^i \wedge dx^j$$

with a and b closed and

$$\operatorname{rank}\big[t\bar{b} + (1-t)\bar{a}\big] = n - 2l = 2(m-l) \quad \text{for every } t \in [0,1].$$

Finally let $f, g : \mathbb{R}^n \to \Lambda^2$ be such that

$$f = \lambda + a \quad \text{and} \quad g = \lambda + b.$$

An easy computation shows that there exists a diffeomorphism φ such that

$$\varphi^*(G) = F \quad \text{where } F = f \wedge dx^1 \wedge \cdots \wedge dx^{2l} \text{ and } G = g \wedge dx^1 \wedge \cdots \wedge dx^{2l}.$$

6.4. *The case $k = n - 1$*

We introduce some notations that are more appropriate to the present context. It is convenient here to write any $g : \mathbb{R}^n \to \Lambda^{n-1}$ of the form

$$g = \sum_{1 \leqslant i_1 < \cdots < i_{n-1} \leqslant n} g_{i_1 \cdots i_{n-1}}\, dx^{i_1} \wedge \cdots \wedge dx^{i_{n-1}}$$

by the missing term. More precisely, we let

$$dx^{\hat{i}} := dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n \quad \text{and} \quad g_{\hat{i}} := g_{1 \cdots i-1 i+1 \cdots n}$$

so that we have

$$g = \sum_{i=1}^{n} g_{\hat{i}}\, dx^{\hat{i}}.$$

Recall also that if $g \neq 0$ then necessarily $\operatorname{rank}[\bar{g}] = n - 1$. We then have the following result.

Theorem 21. *Let* $H = \{x \in \mathbb{R}^n\colon\ x^n > 0\}$, $f, g \in C^\omega(\overline{H}; \Lambda^{n-1})$ *with*

$$df = dg = 0 \quad \text{in } H \quad \text{and} \quad f_{\hat{n}}, g_{\hat{n}} > 0 \quad \text{in } \overline{H}.$$

Then there exist $\epsilon > 0$ *and* $\varphi \in \operatorname{Diff}^\omega(\overline{H} \cap B_\epsilon(0); \mathbb{R}^n)$ *satisfying*

$$\varphi^*(g) = f \quad \text{in } H \cap B_\epsilon(0).$$

If, in addition, $f_{\hat{n}} = g_{\hat{n}}$ *on* ∂H, *then* φ *can be chosen so that*

$$\varphi(x) = x, \quad x \in \partial H \cap B_\epsilon(0).$$

Remark 22. With the proper adaptation, the theorem is valid if H is replaced by a smooth domain Ω.

Proof. We only discuss the case with Cauchy data, the other one being handled in exactly the same way.

Step 1. We start by fixing the notations. For $u \in \mathbb{R}^n$ and $g : \mathbb{R}^n \to \Lambda^{n-1}$ we have that

$$u \lrcorner g = \sum_{1 \leqslant i < j \leqslant n} \left[(-1)^{i-1} g_{\hat{j}} u^i + (-1)^j g_{\hat{i}} u^j\right] dx^{\widehat{ij}}$$

where, as before,

$$dx^{\widehat{ij}} = dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$$

and

$$dg = \left[\sum_{i=1}^{n} (-1)^{i-1} \frac{\partial g_{\hat{i}}}{\partial x^i}\right] dx^1 \wedge \cdots \wedge dx^n.$$

Similarly any $h : \mathbb{R}^n \to \Lambda^{n-2}$ of the form

$$h = \sum_{1 \leqslant i_1 < \cdots < i_{n-2} \leqslant n} h_{i_1 \cdots i_{n-2}} dx^{i_1} \wedge \cdots \wedge dx^{i_{n-2}}$$

will be written as

$$h = \sum_{1 \leqslant i < j \leqslant n} h_{\widehat{ij}} dx^{\widehat{ij}}$$

where

$$h_{\widehat{ij}} = h_{1,\ldots,i-1,i+1,\ldots,j-1,j+1,\ldots,n}.$$

Hence

$$dh = \sum_{1 \leqslant i < j \leqslant n} \left[(-1)^j \frac{\partial h_{\widehat{ij}}}{\partial x^j} dx^{\hat{i}} + (-1)^{i-1} \frac{\partial h_{\widehat{ij}}}{\partial x^i} dx^{\hat{j}}\right].$$

Now assume that $g : \mathbb{R}^n \to \Lambda^{n-1}$, with for example $g_{\hat{n}} \neq 0$, and $h : \mathbb{R}^n \to \Lambda^{n-2}$ are given and satisfy the following $\binom{n-1}{n-3} = \binom{n-1}{2}$ constraints

$$g_{\hat{n}} h_{\widehat{ij}} + g_{\hat{i}} h_{\widehat{jn}} - g_{\hat{j}} h_{\widehat{in}} = 0 \quad \text{for every } 1 \leqslant i < j \leqslant n.$$

(Note that when $j = n$, the equation is trivially satisfied, so that there are indeed only $\binom{n-1}{2}$ equations.) Then a solution $u \in \mathbb{R}^n$ of $u \lrcorner g = h$ is given by

$$u^i = \frac{(-1)^{i-1} h_{\widehat{in}} + (-1)^{n-i} g_{\hat{i}} u^n}{g_{\hat{n}}} \quad \text{for every } i = 1, \ldots, n-1.$$

Step 2. We may now proceed with the proof. First solve by Theorem 8, for every $t \in [0, 1]$

$$\begin{cases} dh^t = f - g & \text{in } H \cap B_\epsilon(0), \\ f^t_{\hat{n}} h^t_{\widehat{ij}} + f^t_{\hat{i}} h^t_{\widehat{jn}} - f^t_{\hat{j}} h^t_{\widehat{in}} = 0 & \text{in } H \cap B_\epsilon(0),\ 1 \leqslant i < j \leqslant n, \\ h^t = 0 & \text{on } \partial H \cap B_\epsilon(0) \end{cases}$$

where $f^t = tg + (1-t) f$.

Step 3. Then use the above computations to find u_t. For example, choose $u_t^n = 0$ and

$$u_t^i = \frac{(-1)^{i-1} h^t_{\widehat{in}}}{t g_{\hat{n}} + (1-t) f_{\hat{n}}} \quad \text{for every } i = 1, \ldots, n-1.$$

Step 4. Choosing ϵ smaller, if necessary, we then solve

$$\begin{cases} \frac{d}{dt}\varphi_t(x) = u_t(\varphi_t(x)), & t \in [0,1], \\ \varphi_0(x) = x. \end{cases}$$

This concludes the proof of the theorem. □

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