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Lyapunov control of a quantum particle in a decaying potential ☆

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Abstract

A Lyapunov-based approach for the trajectory generation of an N-dimensional Schrödinger equation in whole $\mathbb{R}^N$ is proposed. For the case of a quantum particle in an N-dimensional decaying potential the convergence is precisely analyzed. The free system admitting a mixed spectrum, the dispersion through the absolutely continuous part is the main obstacle to ensure such a stabilization result. Whenever, the system is completely initialized in the discrete part of the spectrum, a Lyapunov strategy encoding both the distance with respect to the target state and the penalization of the passage through the continuous part of the spectrum, ensures the approximate stabilization.

Keywords: Nonlinear control of PDEs; Approximate stabilization; Lyapunov techniques; Dispersive estimates; Pre-compactness

1. Introduction

1.1. Main results

We consider a quantum particle in an N-dimensional space, with a potential $V(x)$, and coupled to an external (laser) field $t \mapsto u(t) \in \mathbb{R}$ through its dipole moment $\mu(x)$. Under appropriate change of scales, the system's wavefunction evolves following the Schrödinger equation

$$i\frac{\partial \Psi}{\partial t}(t,x) = -\Delta\Psi(t,x) + \big(V(x) + u(t)\mu(x)\big)\Psi(t,x), \qquad x \in \mathbb{R}^N, \tag{1}$$

$$\Psi(0,x) = \Psi_0(x). \tag{2}$$

This is a bilinear control system, denoted by (Σ), where

- the control is the external field $u : \mathbb{R}_+ \to \mathbb{R}$,
- the state is the wave function $\Psi : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{C}$ with $\Psi(t) \in \mathbb{S}$ for every $t \geqslant 0$, with $\mathbb{S} = \{\varphi \in L^2(\mathbb{R}^N;\mathbb{C}) \mid \|\varphi\|_{L^2} = 1\}$.

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We distinguish between four different situations: dimension $N \geqslant 4$; dimension $N = 3$; dimension $N = 2$; and dimension $N = 1$. For each of these cases, we will assume some appropriate decay assumptions for the potential $V(x)$. Indeed, through this paper, we will assume the following assumption:

Decay assumption (A). We assume for the potential V that zero is neither an eigenvalue nor a resonance of the Hamiltonian $H_0 = -\Delta + V$. Furthermore, we assume one of the following assumptions (depending on the space dimension N)

- $N = 1$: $(1 + |x|)V \in L^1(\mathbb{R})$ [22];
- $N = 2$: $|V(x)| \leqslant C(1 + |x|)^{-3-\varepsilon}$ [41];
- $N = 3$: $V \in L^{\frac{3}{2}-\varepsilon}(\mathbb{R}^3) \cap L^{\frac{3}{2}+\varepsilon}(\mathbb{R}^3)$ [21];
- $N \geqslant 4$: $\widehat{V} \in L^1$ and $(1 + |x|^2)^{\gamma/2}V(x)$ is a bounded operator on the Sobolev space H^ν for some $\nu > 0$ and $\gamma > n + 4$ [29].

A brief discussion on the origin of the above assumption on the potential V is provided in Section 1.3. As one will see these decay assumptions are chosen to assure relevant dispersive estimates.

Furthermore, note that, under the decay assumption (A) on the potential V, the free Hamiltonian $H_0 = -\Delta + V(x)$ admits a mixed spectrum:

$$\sigma(H_0) = \sigma_{\mathrm{disc}}(H_0) \cup \sigma_{\mathrm{ac}}(H_0),$$

where the discrete spectrum $\sigma_{\mathrm{disc}}(H_0)$ contains a finite number of eigenvalues of finite multiplicities and the essential spectrum is actually an absolutely continuous spectrum $\sigma_{\mathrm{ac}}(H_0) = [0, \infty)$. Under the decay assumption (A), this decomposition of the spectrum for the 1D case is a classical result of the earliest days of quantum mechanics (in fact one only needs $V \in L^1(\mathbb{R})$, see e.g. [39], Section XIII.4). For the 2D case, one can find a proof in [43]. The 3D case has been proven in [23]. Finally, the decomposition for the N-dimensional case, with $N > 3$, is a classical result as the potential is a short range potential in the sense of Agmon [1].

Concerning the bound states, $\{\phi_j\}_{j=0}^M$, we know that $\phi_j \in H^2(\mathbb{R}^N, \mathbb{C})$. Moreover, the decay assumption (A) on the potential implies that $V \in L^1_{\mathrm{loc}}$ and $V_- \in M_{\mathrm{loc}}$ the local Stummel class (see [2], page 8, for a definition). This ensures the exponential decay of the eigenfunctions $\{\phi_j\}_{j=0}^M$ (see e.g. [2], p. 55, Corollary 4.2).

Let us recall the following classical existence and uniqueness result for the open-loop system (1)–(2). A proof of this result is given in Appendix A.

Proposition 1. *Let the potential $V(x)$ satisfy the decay assumption* (A) *and consider $\mu \in L^\infty(\mathbb{R}^N, \mathbb{R})$. Let $\Psi_0 \in \mathbb{S}$, $T > 0$ and $u \in \mathbb{C}^0([0, T], \mathbb{R})$. There exists a unique weak solution of* (1)–(2), *i.e. a function $\Psi \in C^0([0, T], \mathbb{S}) \cap C^1([0, T], H^{-2}(\mathbb{R}^N, \mathbb{C}))$ such that*

$$\Psi(t) = e^{-iH_0 t}\Psi_0 - i\int_0^t e^{-iH_0(t-s)}u(s)\mu(x)\Psi(s)\,ds \quad \text{in } L^2\big(\mathbb{R}^N, \mathbb{C}\big) \text{ for } t \in [0, T], \tag{3}$$

and then (1) *holds in $H^{-2}(\mathbb{R}^N, \mathbb{C})$.*

If, moreover, $\Psi_0 \in H^2(\mathbb{R}^N, \mathbb{C})$ and multiplication by $\mu(x)$ defines a bounded operator over $H^2(\mathbb{R}^N, \mathbb{R})$, then Ψ is a strong solution, i.e. $\Psi \in C^0([0, T], H^2(\mathbb{R}^N, \mathbb{C})) \cap C^1([0, T], L^2(\mathbb{R}^N, \mathbb{C}))$, Eq. (1) *holds in $L^2(\mathbb{R}^N, \mathbb{C})$ for $t \in [0, T]$ and the initial condition* (2) *holds in $H^2(\mathbb{R}^N, \mathbb{C})$.*

The weak (*resp. strong*) *solution is continuous with respect to the initial condition for the $C^0([0, T], L^2)$-topology* (*resp. $C^0([0, T], H^2)$-topology*).

Assuming the potential $V(x)$ such that the discrete spectrum $\sigma_{\mathrm{disc}}(H_0)$ is non-empty, we are interested here in stabilizing one of the eigenfunctions in this discrete part. Fixing $\varepsilon > 0$ to be a small positive constant and considering ϕ to be a normalized eigenfunction in this discrete part, we are interested in designing a feedback law $u_\varepsilon(\Psi)$ such that, the solution $\Psi(t, x)$ of (1)–(2) satisfies

$$\liminf_{t\to\infty} \big|\big\langle \Psi(t, x) \mid \phi(x)\big\rangle\big|^2 > 1 - \varepsilon. \tag{4}$$

Here

$$\langle \xi \mid \zeta \rangle = \int_{\mathbb{R}^N} \xi(x)\bar{\zeta}(x)\,dx,$$

denotes the Hermitian product of $L^2(\mathbb{R}^N, \mathbb{C})$. Note that, Ψ and ϕ living on the unit sphere $\mathbb{S}$ of $L^2(\mathbb{R}^N, \mathbb{C})$, the limit (4) denotes the approximate stabilization of the eigenfunction $\phi(x)$.

Note that, even though the feedback stabilization of a quantum system necessitates more complicated models taking into account the measurement back-action on the system (see e.g. [26,25,36]), the kind of strategy considered in this paper can be helpful for the open-loop control of closed quantum systems. Indeed, one can apply the stabilization techniques for the Schrödinger equation in simulation and retrieve the control signal that will be then applied in open-loop on the real physical system. As it will be detailed below, in the bibliographic overview, such kind of strategy has been widely used in the context of finite dimensional quantum systems.

The main result of this article is the following one.

Theorem 2. *Consider the Schrödinger equation* (1)–(2). *We suppose the potential $V(x)$ to satisfy the decay assumption* (A) *and we take $\mu \in \mathcal{L}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We assume the discrete spectrum σ_{disc} of $H_0 = -\Delta + V(x)$ to be non-empty. We consider moreover the following assumptions*:

A1 $\Psi_0 = \sum_{j=0}^M \alpha_j \phi_j$ *where $\{\phi_j\}_{j=0}^M$ are different normalized eigenfunctions in the discrete spectrum of H_0.*
A2 *The coefficient α_0 corresponding to the population of the eigenfunction ϕ_0 in the initial condition Ψ_0 is non-zero*: $\alpha_0 \neq 0$.
A3 *The Hamiltonian H_0 admits non-degenerate transitions*: $\lambda_{j_1} - \lambda_{k_1} \neq \lambda_{j_2} - \lambda_{k_2}$ *for* $(j_1, k_1) \neq (j_2, k_2)$ *and where* $\{\lambda_j\}_{j=0}^M$ *are different eigenvalues of the Hamiltonian H_0.*
A4 *The interaction Hamiltonian $\mu(x)$ ensures simple transitions between all eigenfunctions of H_0*:

$$\langle \mu\phi_j \mid \phi_k \rangle \neq 0 \quad \forall j \neq k \in \{0, 1, \ldots, M\}.$$

Then for any $\varepsilon > 0$, there exists a feedback law $u(t) = u_\varepsilon(\Psi(t))$ (that we will construct explicitly), such that the closed-loop system admits a unique weak solution in $C^0([0,\infty), \mathbb{S}) \cap C^1([0,\infty), H^{-2}(\mathbb{R}^N, \mathbb{C}))$. Moreover the state of the system ends up reaching a population more than $(1-\varepsilon)$ in the eigenfunction ϕ_0 (approximate stabilization):

$$\liminf_{t\to\infty} \left|\langle \Psi(t,x) \mid \phi_0(x) \rangle\right|^2 > 1 - \varepsilon.$$

If, moreover multiplication by $\mu(x)$ defines a bounded operator over $H^2(\mathbb{R}^N)$, then Ψ is a strong solution, i.e. $\Psi \in C^0([0,\infty), H^2(\mathbb{R}^N, \mathbb{C})) \cap C^1([0,\infty), L^2(\mathbb{R}^N, \mathbb{C}))$.

Remark 3. In this theorem $\mathcal{L}$ denotes $\bigcup_{p \geqslant 2} L^p(\mathbb{R}^N)$.

Remark 4. Note that, as the initial state is a linear combination of the bound states, we have in particular $\Psi_0 \in H^2$ and decays exponentially.

Remark 5. Note that, here a finite dimensional approximation of the system by removing the continuous part of the spectrum is not sufficient to treat the stabilization problem. In fact, even if the system is initialized in the discrete part of the spectrum (as assumed in A1), the interaction Hamiltonian μ will make the solution leave this discrete part. The state of the system will therefore leave this subspace just after the initial time.

The assumptions A1 through A4 can be relaxed significantly. However, as the final result with the relaxed assumptions may seem too complicated, we will discuss this relaxations, separately, in Section 5.

1.2. A brief bibliography

The controllability of a finite dimensional quantum system, $i\frac{d}{dt}\Psi = (H_0 + u(t)H_1)\Psi$ where $\Psi \in \mathbb{C}^N$ and H_0 and H_1 are $N \times N$ Hermitian matrices with coefficients in $\mathbb{C}$ has been completely explored [46,37,3,4,49]. However, this does not guarantee the simplicity of the trajectory generation. Very often the chemists formulate the task of the open-loop control as a cost functional to be minimized. Optimal control techniques (see e.g., [42]) and iterative stochastic techniques (e.g, genetic algorithms [32]) are then two classes of approaches which are most commonly used for this task.

When some non-degeneracy assumptions concerning the linearized system are satisfied, [34] provides another method based on Lyapunov techniques for generating trajectories. The relevance of such a method for the control of chemical models has been studied in [35]. Since measurement and feedback in quantum systems lead to much more complicated models and dynamics than the Schrödinger equation [26,36], the stabilization techniques presented in [34] are only used for generating open-loop control laws. Simulating the closed-loop system, we obtain a control signal which can be used in open-loop for the physical system. Such kind of strategy has already been applied widely in this framework [13,45].

The situation is much more difficult when we consider an infinite dimensional configuration. Concerning the controllability problem, very few results are available [48,8,10]. In [8,10] the controllability of a particle in a moving one-dimensional quantum box has been studied. A local controllability result is therefore provided using the return method [16]. In [12], applying some geometric control tools, the authors provide a quite general result concerning the controllability of discrete-spectrum Schrödinger equation. Finally, in [47], the authors consider the controllability of some particular Schrödinger equations with continuous spectra.

Concerning the trajectory generation problem for infinite dimensional systems still much less results are available. The very few existing literature is mostly based on the use of the optimal control techniques [6,7]. The simplicity of the feedback law found by the Lyapunov techniques in [34,9] suggests the use of the same approach for infinite dimensional configurations. However, an extension of the convergence analysis to the PDE configuration is not at all a trivial problem. Indeed, it requires the pre-compactness of the closed-loop trajectories, a property that is difficult to prove in infinite dimension. This strategy is used, for example in [14].

Let us mention some strategies for proving the stabilization of infinite dimensional control systems. One can try to build a feedback law for which one has a strict Lyapunov function. This strategy is used, for example, for hyperbolic systems of conservation laws in [15], for the 2D incompressible Euler equation in a simply connected domain in [17], see also [19] for the multi-connected case. For systems having a non-controllable linearized system around the equilibrium considered, the return method often provides good results, see for example [16] for controllable systems without drift and [20] for Camassa–Holm equation. In the end, we refer to [18] for a pedagogical presentation of strategies for the proof of stabilization of PDE control systems.

In this paper, we propose a Lyapunov-based method to approximately stabilize a particle in an N-dimensional decaying potential under some relevant assumptions. We assume that the system is initialized in the finite dimensional discrete part of the spectrum. Then, the idea consists in proposing a Lyapunov function which encodes both the distance with respect to the target state and the necessity of remaining in the discrete part of the spectrum. In this way, we prevent the possibility of the "mass lost phenomenon" at infinity. Finally, applying some dispersive estimates of Strichartz type, we ensure the approximate stabilization of an arbitrary eigenfunction in the discrete part of the spectrum.

The ideas of this paper (a short and simplified version is already published as a communication [33]) have been recently adapted to the case of a quantum particle in an infinite potential well [11]. In [11], as we are dealing with a pure discrete spectrum, much less restrictive assumptions are needed to ensure the approximate stabilization of the system.

As it can be remarked through the bibliography, except for a very few results [47], all the previous work on the control of the infinite dimensional quantum systems deal with discrete-spectrum Schrödinger equations. It seems that the techniques of this paper and the possibility of the relaxations, explained in Section 5, can open a new gateway to investigate this class of quantum systems.

1.3. Free dynamics and dispersive estimates

Before treating the control problem, let us have a look at the behavior of the system in the absence of the control field ($u(t) = 0$). We will denote by $S(t) = \exp(-itH_0)$ the C_0-semigroup on $L^2(\mathbb{R}^N, \mathbb{C})$ spanned by the infinitesimal generator $(-\Delta + V(x))/i$. Note in particular that, $S(t)$ induces an isometry over $L^2(\mathbb{R}^N, \mathbb{C})$: $\|S(t)\psi\|_{L^2} = \|\psi\|_{L^2}$.

Moreover, we denote by $\mathbb{P}_{\mathrm{disc}}$ the projection operator over the discrete subspace generated by the bound states and defined on $L^2(\mathbb{R}^N, \mathbb{C})$. Finally, $\mathbb{P}_{\mathrm{ac}}$ denotes the projection over the orthogonal subspace: $\mathbb{P}_{\mathrm{ac}} = \mathrm{Id} - \mathbb{P}_{\mathrm{disc}}$.

The discrete part of the freely evolving solution $\mathbb{P}_{\mathrm{disc}} S(t)\Psi_0$ represents a quasi-periodic behavior:

$$\Psi_{0,\mathrm{disc}} = \mathbb{P}_{\mathrm{disc}}\Psi_0 = \sum_{j=0}^{M} \alpha_j \phi_j(x) \quad \Rightarrow \quad \mathbb{P}_{\mathrm{disc}} S(t)\Psi_0 = \sum_{j=0}^{M} \alpha_j e^{-i\lambda_j t}\phi_j(x).$$

The continuous part, however, represents a dispersive behavior. In this subsection, we provide a very brief overview of the dispersive estimates and in particular the ones we use in this paper.

In heuristic terms, for the potential-free problem $V \equiv 0$, the explicit solution

$$\big(e^{it\Delta}\psi\big)(x) = C_N t^{-N/2} \int_{\mathbb{R}^N} e^{i\frac{|x-y|^2}{4t}} \psi(y)\,dy,$$

implies the dispersive estimate [44]

$$\sup_{t>0} |t|^{N/2} \big\| e^{it\Delta}\psi \big\|_{L^\infty(\mathbb{R}^N)} \leqslant \|\psi\|_{L^1(\mathbb{R}^N)}, \quad \forall \psi \in L^1\big(\mathbb{R}^N\big) \cap L^2\big(\mathbb{R}^N\big).$$

For general $V \neq 0$, no explicit solutions are available and therefore one needs to proceed differently. Consider the perturbed Hamiltonian $H_0 = -\Delta + V$, we seek to prove similar estimates on the time evolution operator $S(t)\mathbb{P}_{\mathrm{ac}} = e^{-itH_0}\mathbb{P}_{\mathrm{ac}}$. The projection onto the absolutely continuous spectrum of H_0 is needed to eliminate bound states which do not decay over any length of time. We, therefore, have the following dispersive estimate:

Theorem 6. *Under the decay assumption* (A) *on the potential V, we have*

$$\big\| S(t)\mathbb{P}_{\mathrm{ac}} \big\|_{1\to\infty} \leqslant |t|^{-\frac{N}{2}}. \tag{5}$$

Such dispersive estimates have a long history. For exponentially decaying potentials, Rauch [38] proved dispersive bounds in exponentially weighted L^2-spaces. Jensen and Kato [27] replaced exponential with polynomial decay and obtained asymptotic expansions of e^{-itH_0} (in terms of powers of t) in the usual weighted $L^{2,\sigma}$ spaces. The first authors to address a dispersive estimate of the form (5) were Journée, Soffer, and Sogge [29]. They were able to prove the dispersive estimate (5) under the fourth case of the decay assumption (A) for $N \geqslant 3$.

Concerning the case $N = 3$, following a large amount of results [51,40,22], finally Goldberg [21] proved the dispersive estimate (5) under the third case of the decay assumption (A). In contrast, trying to adapt these results to higher dimensions has lead Goldberg and Visan [24] to show that for $N \geqslant 4$, (5) fails unless V has some amount of regularity, i.e., decay alone is insufficient for (5) to hold if $N \geqslant 4$.

The one-dimensional case was open until recently. Weder [50] proved a version of Theorem 6 under the stronger assumption that $\int_{-\infty}^{\infty} |V(x)|(1+|x|)^{3/2+\varepsilon}\,dx < \infty$. Finally, in a similar way to [50], Goldberg and Schlag [22] were able to prove (5) under the first case of the decay assumption (A).

Finally, concerning the two-dimensional case, Yajima [52] and Jensen, Yajima [28] proved the $L^p(\mathbb{R}^2)$ boundedness of the wave operators under stronger decay assumptions on $V(x)$ (than the decay assumption (A)), but only for $1 < p < \infty$. Hence their result does not imply (5), but $L^p \to L^{p'}$ estimates for $1 < p \leqslant 2$. The first paper to provide an $L^1 \to L^\infty$ dispersive estimate of the form (5) in two dimensions was that of Schlag [41] that proves (5) under the second case of the decay assumption (A).

Note that, interpolating with the L^2-bound $\|e^{-itH_0}\mathbb{P}_{\mathrm{ac}}\psi\|_{L^2} \leqslant \|\psi\|_{L^2}$, we have

Corollary 7. *Under the decay assumption* (A) *on the potential V, we have*

$$\sup_{t>0} |t|^{N(\frac{1}{2}-\frac{1}{p})} \big\| S(t)\mathbb{P}_{\mathrm{ac}}\psi \big\|_{L^{p'}} \leqslant \|\psi\|_{L^p} \quad \text{for all } \psi \in L^1\big(\mathbb{R}^N\big) \cap L^2\big(\mathbb{R}^N\big), \tag{6}$$

where $1 \leqslant p \leqslant 2$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$.

Furthermore, through a T^*T argument, (6) leads to the class of Strichartz estimates (see e.g. [31]):

Theorem 8. *Under the decay assumption* (A) *on the potential* V, *we have*

$$\left\| S(t)\mathbb{P}_{\mathrm{ac}}\psi \right\|_{L_t^q(L_x^p)} \leqslant C\|\psi\|_{L^2}, \quad \text{for all } \frac{2}{q}+\frac{N}{p}=\frac{N}{2},\ 2<q\leqslant\infty. \tag{7}$$

1.4. Structure of the paper

The rest of the paper is organized as follows. In Section 2, we will provide the heuristic of the proof of Theorem 2. In this aim, we will first announce a new Theorem 9 providing the same result as Theorem 2 but under some more restrictive assumptions. After discussing heuristically the proof of this new theorem, we will give the elements to extend the proof to that of Theorem 2.

Theorem 9, will be proved in Section 3. Through a rather simple change in the feedback law, we will be able to extend this proof to that of Theorem 2. This will be addressed in Section 4.

As it can be seen, the assumptions of Theorem 2 may still seem too restrictive. However, through some arguments based on the analytic perturbation of linear operators and the quantum adiabatic theory, we are able to relax significantly these assumptions. This will be treated in Section 5.

2. Heuristic of the proof

From now on, we will assume that the system is initially prepared in a purely discrete state:

$$\Psi_0=\Psi_{0,\mathrm{disc}}\in\mathcal{E}_{\mathrm{disc}},$$

where $\mathcal{E}_{\mathrm{disc}}$ (resp. $\mathcal{E}_{\mathrm{ac}}$) denotes Range($\mathbb{P}_{\mathrm{disc}}$) (resp. Range($\mathbb{P}_{\mathrm{ac}}$)). The control task is to steer the system's state in the eigenspace corresponding to an eigenfunction ϕ_0 of the free Hamiltonian. Note that this eigenfunction ϕ_0 can be any eigenfunction in the discrete part of the spectrum and does not have to be the ground state. During the control process the system might and will cross the continuum $\mathcal{E}_{\mathrm{ac}}$.

Following the stabilization results for the finite dimensional systems [34,9], a first approach for this control problem might be to consider the simple Lyapunov function

$$\widetilde{\mathcal{V}}(\Psi)=1-\left|\langle\Psi\mid\phi_0\rangle\right|^2.$$

The fact that Ψ and ϕ_0 are both normalized, together with the Cauchy–Schwartz inequality, ensures that $\widetilde{\mathcal{V}}(\Psi)\geqslant 0$. The feedback law will be given by [9]:

$$\tilde{u}(\Psi)=\Im\big(\langle\mu\Psi\mid\phi_0\rangle\langle\phi_0\mid\Psi\rangle\big), \tag{8}$$

where $\Im$ denotes the imaginary part of a complex number. A deep analysis based on LaSalle type arguments shows that with such a feedback strategy, one cannot avoid phenomenons like mass lost at infinity. The population of the state ϕ_0 will surely keep increasing during the evolution. But, in order to be able to apply the LaSalle invariance principle for such infinite dimensional system, one needs to ensure the pre-compactness of the trajectories in $L^2(\mathbb{R}^N,\mathbb{C})$. In the particular case, of the Schrödinger equation with the decaying potential, considered in this paper and with the feedback law (8), one cannot even hope to have such a pre-compactness result. Indeed, as it has been said before, while the population of the state ϕ_0 keeps increasing through the application of the feedback law (8), during this same period some of the population might go through the continuous part of the spectrum. This population has then the possibility to disperse rapidly (cf. Section 1.3) and so we might have some un-controlled part of the L^2-norm which will be lost at infinity.

The approach of this paper consists in avoiding the population to go through the continuum while stabilizing the state around the target state ϕ_0. So, we consider a Lyapunov function $\mathcal{V}_\varepsilon(\Psi)$ which encodes these both tasks:

$$\mathcal{V}_\varepsilon(\Psi):=1-(1-\varepsilon)\sum_{j=0}^{M}\left|\langle\Psi\mid\phi_j\rangle\right|^2-\varepsilon\left|\langle\Psi\mid\phi_0\rangle\right|^2, \tag{9}$$

where $0<\varepsilon\ll 1$ is a small positive constant. Such a Lyapunov function clearly verifies:

$$0\leqslant\mathcal{V}_\varepsilon(\Psi)\quad\text{and}\quad\mathcal{V}_\varepsilon(\Psi)=0\Leftrightarrow\left|\langle\Psi\mid\phi_0\rangle\right|=1. \tag{10}$$

Here still, we have used the fact that Ψ and ϕ_j's are all normalized in $L^2(\mathbb{R}^N, \mathbb{C})$. Moreover, as the system is initially prepared in the discrete part of the spectrum, and as $|\langle \Psi_0 \mid \phi_0\rangle| > 0$,

$$\mathcal{V}_\varepsilon(\Psi_0) = 1 - (1-\varepsilon) - \varepsilon \big|\langle \Psi_0 \mid \phi_0\rangle\big|^2 < \varepsilon. \tag{11}$$

This Lyapunov function clearly encodes two tasks: 1 – it prevents the L^2-mass lost through the dispersion of the absolutely continuous population; 2 – it privileges the increase of the population in the eigenfunction ϕ_0.

By a simple computation we have,

$$\frac{d}{dt}\mathcal{V}_\varepsilon(\Psi) = -u(t)\Bigg[(1-\varepsilon)\sum_{j=0}^{M} \Im\big(\langle \mu\Psi \mid \phi_j\rangle\langle \phi_j \mid \Psi\rangle\big) + \varepsilon \Im\big(\langle \mu\Psi \mid \phi_0\rangle\langle \phi_0 \mid \Psi\rangle\big)\Bigg]. \tag{12}$$

A natural choice is therefore to consider the feedback law:

$$u(\Psi) = u_\varepsilon(\Psi) := c\Bigg[(1-\varepsilon)\sum_{j=0}^{M} \Im\big(\langle \mu\Psi \mid \phi_j\rangle\langle \phi_j \mid \Psi\rangle\big) + \varepsilon \Im\big(\langle \mu\Psi \mid \phi_0\rangle\langle \phi_0 \mid \Psi\rangle\big)\Bigg], \tag{13}$$

where $c > 0$ is a positive constant. Such a feedback law clearly ensures the decrease of the Lyapunov function $\mathcal{V}_\varepsilon$. Looking at the structure of the Lyapunov function (9), the feedback law (13) penalizes strongly exiting from the discrete part of the spectrum. Actually, as $\mathcal{V}_\varepsilon(\Psi_0) \leqslant \varepsilon$, the decrease in this Lyapunov function ensures that the population in the discrete part of the spectrum (the square of the L^2-norm of the discrete part) will always remain more than $1-\varepsilon$. Therefore in the worst case, we will only have an ε L^2-norm which will be lost by dispersing in the continuum.

At the same time, this feedback law (13) slightly encourages the increase in the population of the target state ϕ_0. It remains therefore to check whether this increase actually provides some kind of convergence toward this eigenfunction or not. This will be addressed in Section 3, where we prove the following theorem:

Theorem 9. *Consider the Schrödinger equation* (1)–(2)*. Assume the space dimension $N \geqslant 2$. We suppose the potential $V(x)$ to satisfy the decay assumption* (A) *and we take $\mu \in L^{2N-}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We assume the discrete spectrum σ_{disc} of $H_0 = -\Delta + V(x)$ to be non-empty. We consider moreover the following assumptions*:

- A1 $\Psi_0 = \sum_{j=0}^{M} \alpha_j \phi_j$ *where $\{\phi_j\}_{j=0}^{M}$ are different normalized eigenfunctions in the discrete spectrum of H_0.*
- A2 *The coefficient α_0 corresponding to the population of the eigenfunction ϕ_0 in the initial condition Ψ_0 is non-zero*: $\alpha_0 \neq 0$.
- A3 *The Hamiltonian H_0 admits non-degenerate transitions*: $\lambda_{j_1} - \lambda_{k_1} \neq \lambda_{j_2} - \lambda_{k_2}$ *for* $(j_1, k_1) \neq (j_2, k_2)$ *and where $\{\lambda_j\}_{j=0}^{M}$ are different eigenvalues of the Hamiltonian H_0.*
- A4 *The interaction Hamiltonian $\mu(x)$ ensures simple transitions between all eigenfunctions of H_0*:

$$\langle \mu\phi_j \mid \phi_k\rangle \neq 0 \quad \forall j \neq k \in \{0, 1, \ldots, M\}.$$

Then for any $\varepsilon > 0$, applying the feedback law $u(t) = u_\varepsilon(\Psi(t))$ given by (13)*, the closed-loop system admits a unique weak solution in $C^0([0,\infty), \mathbb{S}) \cap C^1([0,\infty), H^{-2}(\mathbb{R}^N, \mathbb{C}))$. Moreover the state of the system ends up reaching a population more than $(1-\varepsilon)$ in the eigenfunction ϕ_0* (*approximate stabilization*):

$$\liminf_{t\to\infty} \big|\langle \Psi(t,x) \mid \phi_0(x)\rangle\big|^2 > 1-\varepsilon.$$

If, moreover, the multiplication by $\mu(x)$ defines a bounded operator over $H^2(\mathbb{R}^N)$, then Ψ is a strong solution, i.e. $\Psi \in C^0([0,\infty), H^2\ (\mathbb{R}^N, \mathbb{C})) \cap C^1([0,\infty), L^2(\mathbb{R}^N, \mathbb{C}))$.

Remark 10. This theorem admits some more restrictive assumptions with respect to Theorem 2. In fact, we remove the 1D case and we assume the interaction Hamiltonian μ to be in a smaller space $L^{2N-}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, where

$$L^{2N-}\big(\mathbb{R}^N\big) = \bigcup_{2\leqslant p<2N} L^p,$$

and therefore, $L^{2N-}(\mathbb{R}^N) \subset \mathcal{L}(\mathbb{R}^N)$.

Theorem 9 will be proved by studying the L^2-weak limit of $\Psi(t)$ for $t \to \infty$. Namely, let $(t_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive real numbers such that $t_n \to \infty$. Since $\|\Psi(t_n)\|_{L^2} = 1$, there exists $\Psi_\infty \in L^2(\mathbb{R}^N, \mathbb{C})$ such that, up to a subsequence, $\Psi(t_n) \rightharpoonup \Psi_\infty$ weakly in $L^2(\mathbb{R}^N, \mathbb{C})$. Furthermore, through the dispersive estimates of Section 1.3, we will provide a strong convergence result with respect to the semi-norm $\|\psi\|_{\mathcal{H}} = \max(\|\mathbb{P}_{\mathrm{disc}}\psi\|_{L^2}, \|\mu\psi\|_{L^2})$. Through such a strong convergence and the assumptions A3 and A4 of the theorem, we will prove that $\Psi_\infty = \beta\phi_0$, where $\beta \in \mathbb{C}$ and $|\beta| \leqslant 1$. Through some further investigations and applying assumption A1 we will be able to show that $|\beta|^2 \geqslant 1 - \varepsilon$ and this will finish the proof of Theorem 9.

A deep study of the proof of Theorem 9, shows that the new restrictions (with respect to Theorem 2) may be removed if we could ensure the belonging of the feedback law $u_\varepsilon(\Psi(t))$ to the space $L_t^{1+\delta}$ for $\delta \in (0, 1]$. In fact, the feedback law (13) only belongs to the space L_t^2 as $\frac{d\mathcal{V}_\varepsilon}{dt} = -\frac{1}{c}u_\varepsilon^2$. However, we may improve this through the following change of the feedback law:

$$u_{\varepsilon,\alpha}(\Psi) = cf(\Psi)\big|f(\Psi)\big|^\alpha, \tag{14}$$

where

$$f(\Psi) := \left[(1-\varepsilon)\sum_{j=0}^{M} \Im\big(\langle\mu\Psi \mid \phi_j\rangle\langle\phi_j \mid \Psi\rangle\big) + \varepsilon\Im\big(\langle\mu\Psi \mid \phi_0\rangle\langle\phi_0 \mid \Psi\rangle\big)\right],$$

and $\alpha \geqslant 0$ and $c > 0$. This choice of the feedback law implies

$$\frac{d}{dt}\mathcal{V}_\varepsilon = -\frac{c^{1+\alpha}}{c^{2+\alpha}}|u_{\varepsilon,\alpha}|^{\frac{2+\alpha}{1+\alpha}},$$

and therefore $u_{\varepsilon,\alpha} \in L_t^{\frac{2+\alpha}{1+\alpha}}$. As $\alpha \to \infty$ this ensures that the feedback law $u_{\varepsilon,\alpha}$ belongs to $L_t^{1+\delta}$ for any $\delta \in (0, 1]$.

3. Proof of Theorem 9

We proceed the proof of Theorem 9 in 3 steps: 1 – we prove the well-posedness of the closed-loop system; 2 – we prove the existence of an asymptotic regime in some appropriate Hilbert space and we characterize the weak ω-limit set, i.e. the set of the functions ψ_∞ in $L^2(\mathbb{R}^N)$ such that there exists a sequence of times $\{t_n\}_{n=1}^\infty \nearrow \infty$ such that $\Psi(t_n) \rightharpoonup \Psi_\infty$ weakly in L^2; 3 – we finish the proof of the theorem through the application of the assumptions A1 through A4.

3.1. Solutions of the Cauchy problem

Proposition 11. *Let $\varepsilon > 0$ and $\Psi_0 \in \mathbb{S}$. There exists a unique weak solution Ψ of* (1)–(2) *with the feedback law $u(t) = u_\varepsilon(\Psi(t))$ given by* (13), *i.e. $\Psi \in C^0(\mathbb{R}^+, \mathbb{S}) \cap C^1(\mathbb{R}^+, H^{-2}(\mathbb{R}^N, \mathbb{C}))$,* (1) *holds in $H^{-2}(\mathbb{R}^N, \mathbb{C})$ for every $t \in \mathbb{R}^+$ and the equality* (2) *holds in $\mathbb{S}$.*

If, moreover, $\Psi_0 \in H^2(\mathbb{R}^N, \mathbb{C})$ and multiplication by $\mu(x)$ defines a bounded operator over $H^2(\mathbb{R}^N)$, then Ψ is a strong solution, i.e. $\Psi \in C^0([0, \infty), H^2\ (\mathbb{R}^N, \mathbb{C})) \cap C^1([0, \infty), L^2(\mathbb{R}^N, \mathbb{C}))$.

Proof. Let $M \in \mathbb{N}^*$ be the number of bound states of $H_0 = -\Delta + V(x)$ and $T > 0$ such that

$$2(M+1)c\|\mu\|_{L^\infty}^2 T e^{(M+1)c\|\mu\|_{L^\infty}^2 T} < 1. \tag{15}$$

In order to build solution on $[0, T]$, we apply the Banach fixed-point theorem to the following map

$$\Theta : C^0\big([0, T], \mathbb{S}\big) \to C^0\big([0, T], \mathbb{S}\big),$$
$$\xi \mapsto \Psi,$$

where Ψ is the solution of (1)–(2) with $u(t) = u_\varepsilon(\xi(t))$.

The map Θ is well defined and maps $C^0([0, T], \mathbb{S})$ into itself. Indeed, when $\xi \in C^0([0, T], \mathbb{S})$, $u : t \mapsto u_\varepsilon(\xi(t))$ is continuous and thus Proposition 1 ensures the existence of a unique weak solution Ψ. Notice that the map Θ takes values in $C^0([0, T], \mathbb{S}) \cap C^1([0, T], H^{-2})$.

Let us prove that Θ is a contraction of $C^0([0,T],\mathbb{S})$. Let $\xi_j \in C^0([0,T],\mathbb{S})$, $u_j := u_\varepsilon(\xi_j)$, $\Psi_j := \Theta(\xi_j)$, for $j=1,2$ and $\Delta := \Psi_1 - \Psi_2$. We have

$$\Delta(t) = -i\int_0^t e^{-i(t-s)H_0}\big[u_1\mu(x)\Delta(s) - (u_1-u_2)\mu(x)\Psi_2(s)\big]\,ds.$$

Thanks to (13), we have $\|u_j\|_{L^\infty(0,T)} \leqslant (M+1)c\|\mu\|_{L^\infty}$ for $j=1,2$ and $\|u_1-u_2\|_{L^\infty(0,T)} \leqslant 2(M+1)c\|\mu\|_{L^\infty} \times \|\xi_1-\xi_2\|_{C^0([0,T],L^2)}$. Thus

$$\big\|\Delta(t)\big\|_{L^2} \leqslant \int_0^t (M+1)c\|\mu\|_{L^\infty}^2\big\|\Delta(s)\big\|_{L^2} + 2(M+1)c\|\mu\|_{L^\infty}^2\|\xi_1-\xi_2\|_{C^0([0,T],L^2)}\,ds. \tag{16}$$

Therefore, the Gronwall Lemma implies

$$\big\|\Delta(t)\big\|_{C^0([0,T],L^2)} \leqslant 2(M+1)c\|\mu\|_{L^\infty}^2 T e^{(M+1)c\|\mu\|_{L^\infty}^2 T}\ \|\xi_1-\xi_2\|_{C^0([0,T],L^2)},$$

and so (15) ensures that Θ is a contraction of the Banach space $C^0([0,T],\mathbb{S})$. Therefore, there exists a fixed point $\Psi\in C^0([0,T],\mathbb{S})$ such that $\Theta(\Psi)=\Psi$. Since Θ takes values in $C^0([0,T],\mathbb{S})\cap C^1([0,T],H^{-2}(\mathbb{R}^N,\mathbb{C}))$, necessarily Ψ belongs to this space, thus, it is a weak solution of (1)–(2) on $[0,T]$.

If, moreover, $\Psi_0\in H^2(\mathbb{R}^N,\mathbb{C})$ and multiplication by $\mu(x)$ defines a bounded operator over $H^2(\mathbb{R}^N,\mathbb{R})$, then applying Proposition 1, the map Θ takes values in $C^0([0,T],H^2(\mathbb{R}^N,\mathbb{C}))\cap C^1([0,T],L^2(\mathbb{R}^N,\mathbb{C}))$ thus Ψ belongs to this space and it is a strong solution.

Finally, we have introduced a time $T>0$ and, for every $\Psi_0\in\mathbb{S}$, we have built a weak solution $\Psi\in C^0([0,T],\mathbb{S})$ of (1)–(2) on $[0,T]$. Thus, for a given initial condition $\Psi_0\in\mathbb{S}$, we can apply this result on $[0,T]$, $[T,2T]$, $[2T,3T]$, etc. This proves the existence and uniqueness of a global weak solution for the closed-loop system. □

Note that, by assumption A1, the initial state Ψ_0 is spanned by the exponentially decaying bound states and therefore $\Psi_0\in\mathbb{S}\cap H^2(\mathbb{R}^N)$. This, together with Proposition 11, terminates the proof of the well-posedness part of Theorem 9.

3.2. Weak ω-limit set

Before studying the weak ω-limit set of the closed-loop system, let us announce two simple and two rather complicated lemmas that we will need to characterize this asymptotic regime.

Lemma 12. *The feedback law $u=u_\varepsilon(\Psi)$ defined by* (13) *is a member of $L^2_t(\mathbb{R}^+,\mathbb{R})$. In particular, for any $\gamma>0$ there exists $T_\gamma>0$ large enough such that*:

$$\int_{T_\gamma}^\infty \big|u_\varepsilon\big(\Psi(s)\big)\big|^2\,ds \leqslant \gamma.$$

Proof. By definition, we have $\frac{d\mathcal{V}_\varepsilon}{dt} = -\frac{1}{c}|u_\varepsilon(\Psi)|^2$. The Lyapunov function $\mathcal{V}_\varepsilon(\Psi)$ being a decreasing non-negative function, there exists a positive constant ν such that $\mathcal{V}_\varepsilon(\Psi(t))\searrow\nu\geqslant 0$. Therefore, we have

$$\int_0^\infty \big|u_\varepsilon\big(\Psi(t)\big)\big|^2\,dt = -c\int_0^\infty \frac{d\mathcal{V}_\varepsilon}{dt} = c\big(\mathcal{V}_\varepsilon(\Psi_0)-\nu\big) < \infty. \qquad \square$$

Lemma 13. *Let $\Psi(t)$ denote the weak* (*or strong*) *solution of the closed-loop system. There exists a sequence of times $(t_n)_{n=1}^\infty \nearrow \infty$ and some function $\Psi_\infty\in L^2(\mathbb{R}^N,\mathbb{C})$* (*with $\|\Psi_\infty\|_{L^2}\leqslant 1$*) *such that*:

$$\Psi(t_n) \rightharpoonup \Psi_\infty \quad \text{weakly in } L^2\big(\mathbb{R}^N,\mathbb{C}\big), \tag{17}$$

$$\mathbb{P}_{\mathrm{disc}}\Psi(t_n) \to \mathbb{P}_{\mathrm{disc}}\Psi_\infty \quad \text{strongly in } L^2\big(\mathbb{R}^N,\mathbb{C}\big). \tag{18}$$

Proof. The solution Ψ belonging to $C^0(\mathbb{R}^+, \mathbb{S})$, we have $\|\Psi(t)\|_{L^2} = 1$ and therefore the existence of a subsequence $(t_n)_1^\infty \nearrow \infty$ and $\Psi_\infty \in L^2$ such that (17) holds true is trivial. Moreover,

$$\|\Psi_\infty\|_{L^2} \leqslant \liminf_{n\to\infty} \big\|\Psi(t_n)\big\|_{L^2} = 1.$$

The key to the proof of (18), is in the fact that $\mathcal{E}_{\mathrm{disc}} = \mathrm{Range}(\mathbb{P}_{\mathrm{disc}})$ is finite dimensional. Indeed, the weak convergence (17) implies

$$\big\langle \Psi(t_n) \mid \phi_j\big\rangle \to \langle \Psi_\infty \mid \phi_j\rangle, \quad j = 0, 1, \ldots, M,$$

and therefore

$$\mathbb{P}_{\mathrm{disc}}\Psi(t_n) = \sum_{j=0}^{M} \big\langle \Psi(t_n) \mid \phi_j\big\rangle \phi_j \to \sum_{j=0}^{M} \langle \Psi_\infty \mid \phi_j\rangle \phi_j = \mathbb{P}_{\mathrm{disc}}\Psi_\infty \quad \text{strongly in } L^2. \qquad \square$$

Lemma 14. *Let $\Psi(t)$ denote the weak (or strong) solution of the closed-loop system. Consider a sequence of times $(t_n)_{n=1}^\infty \nearrow \infty$ and some strictly positive time constant $\tau > 0$. We have*

$$S(\tau)\mathbb{P}_{\mathrm{ac}}\Psi(t_n) \to 0 \quad \textit{strongly in } L^2_{\mu^2}\big(\mathbb{R}^N, \mathbb{C}\big). \tag{19}$$

Proof. We have the Duhamel's formula:

$$S(\tau)\Psi(t_n) = S(t_n + \tau)\Psi_0 + \frac{1}{i}\int_0^{t_n} u_\varepsilon\big(\Psi(s)\big) S(t_n + \tau - s)\mu(x)\Psi(s)\, ds$$

and therefore

$$\big\|\mu S(\tau)\mathbb{P}_{\mathrm{ac}}\Psi(t_n)\big\|_{L^2_x} \leqslant \big\|\mu S(t_n + \tau)\mathbb{P}_{\mathrm{ac}}\Psi_0\big\|_{L^2_x} + \int_0^{t_n} \big|u_\varepsilon\big(\Psi(s)\big)\big| \big\|\mu(x) S(t_n + \tau - s)\mathbb{P}_{\mathrm{ac}}\mu(x)\Psi(s)\big\|_{L^2_x}\, ds, \tag{20}$$

where we have applied the fact that the semigroup operator $S(t)$ of H_0 commutes with the eigenprojection operator $\mathbb{P}_{\mathrm{ac}}$ of the same Hamiltonian.

We know by the assumption of Theorem 9 on μ that $\mu \in L^p(\mathbb{R}^N)$ where $p \in [2, 2N)$. Applying the Hölder inequality, we have

$$\big\|\mu(x) S(t)\mathbb{P}_{\mathrm{ac}}\psi\big\|_{L^2_x} \leqslant \|\mu\|_{L^p_x} \big\|S(t)\mathbb{P}_{\mathrm{ac}}\psi\big\|_{L^q_x}, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \tag{21}$$

where $\psi \in L^2(\mathbb{R}^N)$.

Moreover, applying the dispersive estimate of Corollary 7, we have

$$\big\|S(t)\mathbb{P}_{\mathrm{ac}}\psi\big\|_{L^q_x} \leqslant |t|^{-\frac{N}{p}} \|\psi\|_{L^{q'}_x}, \quad \frac{1}{q} + \frac{1}{q'} = 1, \tag{22}$$

for $\psi \in L^{q'} \cap L^2$.

Let us apply these estimates (21) and (22) to the inequality (20). For the first term in (20), we have

$$\big\|\mu\ S(t_n + \tau)\mathbb{P}_{\mathrm{ac}}\Psi_0\big\|_{L^2_x} \leqslant |t_n + \tau|^{-\frac{N}{p}} \|\mu\|_{L^p_x} \|\Psi_0\|_{L^{q'}_x}, \tag{23}$$

where we have used the fact that $\Psi_0 \in L^{q'}$ as it is a linear combination of the bound states and therefore decaying exponentially [2]. For the second term, we have

$$\begin{aligned}\big\|\mu S(t_n + \tau - s)\mathbb{P}_{\mathrm{ac}}\mu(x)\Psi(s)\big\|_{L^2_x} &\leqslant |t_n + \tau - s|^{-\frac{N}{p}} \|\mu\|_{L^p_x} \big\|\mu\Psi(s)\big\|_{L^{q'}_x} \\ &\leqslant |t_n + \tau - s|^{-\frac{N}{p}} \|\mu\|^2_{L^p_x} \big\|\Psi(s)\big\|_{L^2_x} \\ &= |t_n + \tau - s|^{-\frac{N}{p}} \|\mu\|^2_{L^p_x}.\end{aligned} \tag{24}$$

Here, to obtain the second line from the first one, we have applied a Hölder inequality noting that $\frac{1}{q'} = \frac{1}{2} + \frac{1}{p}$.

Furthermore, for any $\gamma > 0$ taking $t_n > T_\gamma$ (where T_γ is given by Lemma 12), we have

$$\begin{aligned}
&\int_0^{t_n} \left|u_\varepsilon\bigl(\Psi(s)\bigr)\right| \left\|\mu(x)S(t_n+\tau-s)\mathbb{P}_{\mathrm{ac}}\mu(x)\Psi(s)\right\|_{L^2_x} ds \\
&\quad = \int_0^{T_\gamma} \left|u_\varepsilon\bigl(\Psi(s)\bigr)\right| \left\|\mu(x)S(t_n+\tau-s)\mathbb{P}_{\mathrm{ac}}\mu(x)\Psi(s)\right\|_{L^2_x} ds \\
&\qquad + \int_{T_\gamma}^{t_n} \left|u_\varepsilon\bigl(\Psi(s)\bigr)\right| \left\|\mu(x)S(t_n+\tau-s)\mathbb{P}_{\mathrm{ac}}\mu(x)\Psi(s)\right\|_{L^2_x} ds.
\end{aligned} \tag{25}$$

Inserting the estimate (24) in the first integral of (25), we have

$$\begin{aligned}
&\int_0^{T_\gamma} \left|u_\varepsilon\bigl(\Psi(s)\bigr)\right| \left\|\mu(x)S(t_n+\tau-s)\mathbb{P}_{\mathrm{ac}}\mu(x)\Psi(s)\right\|_{L^2_x} ds \\
&\quad \leqslant \left\|u_\varepsilon\bigl(\Psi(t)\bigr)\right\|_{L^2_t} \|\mu\|^2_{L^p_x} \left(\int_0^{T_\gamma} (t_n+\tau-s)^{-\frac{2N}{p}} ds \right)^{1/2} \\
&\quad \leqslant \frac{\sqrt{p}}{\sqrt{|2N-p|}} \left\|u_\varepsilon\bigl(\Psi(t)\bigr)\right\|_{L^2_t} \|\mu\|^2_{L^p_x} \bigl(|t_n+\tau-T_\gamma|^{-\frac{2N-p}{p}} - |t_n+\tau|^{-\frac{2N-p}{p}}\bigr)^{1/2},
\end{aligned} \tag{26}$$

where we have applied the Cauchy–Schwartz inequality. Note, in particular that, p being strictly less than $2N$, $\frac{2N-p}{p}$ is strictly positive, and therefore the above integral (26) tends to 0 as $t_n \to \infty$.

Applying once again the Cauchy–Schwartz inequality, this time for the second integral in (25), we have

$$\begin{aligned}
&\int_{T_\gamma}^{t_n} \left|u_\varepsilon\bigl(\Psi(s)\bigr)\right| \left\|\mu(x)S(t_n+\tau-s)\mathbb{P}_{\mathrm{ac}}\mu(x)\Psi(s)\right\|_{L^2_x} ds \\
&\quad \leqslant \|\mu\|^2_{L^p_x} \left(\int_{T_\gamma}^{\infty} \left|u_\varepsilon\bigl(\Psi(t)\bigr)\right|^2 dt \right)^{1/2} \left(\int_{T_\gamma}^{t_n} (t_n+\tau-s)^{-\frac{2N}{p}} ds \right)^{1/2} \\
&\quad \leqslant \frac{\sqrt{p}}{\sqrt{|2N-p|}} \gamma^{1/2} \|\mu\|^2_{L^p_x} \bigl(|\tau|^{-\frac{2N-p}{p}} - |t_n+\tau-T_\gamma|^{-\frac{2N-p}{p}}\bigr)^{1/2},
\end{aligned} \tag{27}$$

where, we have used the fact that by definition of T_γ, $\int_{T_\gamma}^\infty |u_\varepsilon(\Psi(t))|^2 dt < \gamma$. In particular, this implies

$$\limsup_{t_n \to \infty} \int_{T_\gamma}^{t_n} \left|u_\varepsilon\bigl(\Psi(s)\bigr)\right| \left\|\mu(x)S(t_n+\tau-s)\mathbb{P}_{\mathrm{ac}}\mu(x)\Psi(s)\right\|_{L^2_x} ds \leqslant \frac{\sqrt{p}}{\sqrt{|2N-p|}} \gamma^{1/2} \|\mu\|^2_{L^p_x} |\tau|^{-\frac{2N-p}{2p}}. \tag{28}$$

Gathering (26) and (28), we have shown

$$\limsup_{t_n \to \infty} \int_0^{t_n} \left|u_\varepsilon\bigl(\Psi(s)\bigr)\right| \left\|\mu(x)S(t_n+\tau-s)\mathbb{P}_{\mathrm{ac}}\mu(x)\Psi(s)\right\|_{L^2_x} ds \leqslant \frac{\sqrt{p}}{\sqrt{|2N-p|}} \gamma^{1/2} \|\mu\|^2_{L^p_x} |\tau|^{-\frac{2N-p}{2p}}. \tag{29}$$

Note, however, that we can choose the constant $\gamma > 0$ as small as we want and therefore we have:

$$\lim_{t_n\to\infty}\int_0^{t_n}\big|u_\varepsilon\big(\Psi(s)\big)\big|\big\|\mu(x)S(t_n+\tau-s)\mathbb{P}_{\mathrm{ac}}\mu(x)\Psi(s)\big\|_{L^2_x}\,ds=0. \tag{30}$$

This, together with (23), finishes the proof of Lemma 14 and we have

$$\lim_{t_n\to\infty}\big\|\mu S(\tau)\mathbb{P}_{\mathrm{ac}}\Psi(t_n)\big\|_{L^2_x}=0. \qquad \square \tag{31}$$

Applying the above lemmas, we have the following lemma, proving the continuity of the solution of the closed-loop system with respect to its initial state in the L^2_{disc}-topology.

Lemma 15. *Let $\Psi(t)$ denote the weak (or strong) solution of the closed-loop system. Consider the time sequence $\{t_n\}_{n=1}^\infty \nearrow \infty$ and the weak limit Ψ_∞ as in Lemma 13 and define $\Psi_{\infty,\mathrm{disc}} = \mathbb{P}_{\mathrm{disc}}\Psi_\infty$. Consider the two closed-loop systems*

$$i\frac{d}{dt}\Psi_n=-\Delta\Psi_n+V(x)\Psi_n+u_\varepsilon(\Psi_n)\mu(x)\Psi_n,\qquad \Psi_n|_{t=0}=\Psi(t_n), \tag{32}$$

$$i\frac{d}{dt}\widetilde{\Psi}=-\Delta\widetilde{\Psi}+V(x)\widetilde{\Psi}+u_\varepsilon(\widetilde{\Psi})\mu(x)\widetilde{\Psi},\qquad \widetilde{\Psi}|_{t=0}=\widetilde{\Psi}_0=\Psi_{\infty,\mathrm{disc}}. \tag{33}$$

We have, for any $\tau > 0$, that

$$\mathbb{P}_{\mathrm{disc}}\Psi_n(\tau)\to\mathbb{P}_{\mathrm{disc}}\widetilde{\Psi}(\tau)\quad \textit{strongly in } L^2 \textit{ as } n\to\infty. \tag{34}$$

Proof. In this aim, we consider a stronger semi-norm than L^2_{disc} defined by $\|\psi\|_{\mathcal{H}} = \max(\|\mathbb{P}_{\mathrm{disc}}\psi\|_{L^2}, \|\mu\psi\|_{L^2})$. Note however that this semi-norm is weaker than the $L^2(\mathbb{R}^N)$-norm

$$\|\mathbb{P}_{\mathrm{disc}}\psi\|_{L^2}\leqslant\|\psi\|_{L^2}\quad\text{and}\quad \|\mu\psi\|_{L^2}\leqslant\|\mu\|_{L^\infty}\|\psi\|_{L^2},$$

and therefore

$$\|\psi\|_{\mathcal{H}}\leqslant\kappa\|\psi\|_{L^2(\mathbb{R}^N)},\quad \text{where } \kappa=\max\big(1,\|\mu\|_{L^\infty}\big). \tag{35}$$

It is clear that this is enough to prove

$$\big\|\Psi_n(\tau)-\widetilde{\Psi}(\tau)\big\|_{\mathcal{H}}\to 0\quad \text{as } n\to\infty. \tag{36}$$

We have by the Duhamel's formula

$$\Psi_n(\tau)=S(\tau)\Psi(t_n)+\frac{1}{i}\int_0^\tau u_\varepsilon\big(\Psi_n(s)\big)S(\tau-s)\mu(x)\Psi_n(s,x)\,ds,$$

$$\widetilde{\Psi}(\tau)=S(\tau)\widetilde{\Psi}_0+\frac{1}{i}\int_0^\tau u_\varepsilon\big(\widetilde{\Psi}(s)\big)S(\tau-s)\mu(x)\widetilde{\Psi}(s,x)\,ds.$$

Noting by $\delta\Psi_n(\tau)=\Psi_n(\tau)-\widetilde{\Psi}(\tau)$, we have

$$\delta\Psi_n(\tau)=S(\tau)\big(\Psi(t_n)-\widetilde{\Psi}_0\big)+\frac{1}{i}\int_0^\tau u_\varepsilon\big(\Psi_n(s)\big)S(\tau-s)\mu(x)\delta\Psi_n(s)\,ds$$
$$+\frac{1}{i}\int_0^\tau\big[u_\varepsilon\big(\Psi_n(s)\big)-u_\varepsilon\big(\widetilde{\Psi}(s)\big)\big]S(\tau-s)\mu(x)\widetilde{\Psi}(s)\,ds.$$

This implies

$$\begin{aligned}\left\|\delta\Psi_n(\tau)\right\|_{\mathcal{H}} \leqslant{}& \left\|S(\tau)\big(\Psi(t_n)-\widetilde{\Psi}_0\big)\right\|_{\mathcal{H}} + \kappa\int_0^\tau \left|u_\varepsilon\big(\Psi_n(s)\big)\right|\left\|S(\tau-s)\mu(x)\delta\Psi_n(s)\right\|_{L^2(\mathbb{R}^N)}\,ds\\ &+\kappa\int_0^\tau\left|u_\varepsilon\big(\Psi_n(s)\big)-u_\varepsilon\big(\widetilde{\Psi}(s)\big)\right|\left\|S(\tau-s)\mu(x)\widetilde{\Psi}(s)\right\|_{L^2(\mathbb{R}^N)}\,ds,\end{aligned}\tag{37}$$

where we have applied the inequality (35).

Furthermore, noting that $S(t)$ induces an isometry over the space $L^2(\mathbb{R}^N,\mathbb{C})$, we have

$$\begin{aligned}&\left\|S(\tau-s)\mu(x)\delta\Psi_n(s)\right\|_{L^2(\mathbb{R}^N)} = \left\|\mu(x)\delta\Psi_n(s)\right\|_{L^2(\mathbb{R}^N)} \leqslant \left\|\delta\Psi_n(s)\right\|_{\mathcal{H}},\\ &\left\|S(\tau-s)\mu(x)\widetilde{\Psi}(s)\right\|_{L^2(\mathbb{R}^N)} = \left\|\mu(x)\widetilde{\Psi}(s)\right\|_{L^2(\mathbb{R}^N)} \leqslant \|\mu\|_{L^\infty}\left\|\widetilde{\Psi}(s)\right\|_{L^2} \leqslant \|\mu\|_{L^\infty},\end{aligned}$$

where, for the second line, we have also applied

$$\left\|\widetilde{\Psi}(s)\right\|_{L^2} = \left\|\widetilde{\Psi}(0)\right\|_{L^2} \leqslant \|\Psi_\infty\|_{L^2} \leqslant 1.$$

Inserting the above inequalities in (37), we have

$$\begin{aligned}\left\|\delta\Psi_n(\tau)\right\|_{\mathcal{H}} \leqslant{}& \left\|S(\tau)\big(\Psi(t_n)-\widetilde{\Psi}_0\big)\right\|_{\mathcal{H}} + \kappa\|u_\varepsilon\|_{L_t^\infty}\int_0^\tau\left\|\delta\Psi_n(s)\right\|_{\mathcal{H}}\,ds\\ &+\kappa\|\mu\|_{L^\infty}\int_0^\tau\left|u_\varepsilon\big(\Psi_n(s)\big)-u_\varepsilon\big(\widetilde{\Psi}(s)\big)\right|\,ds.\end{aligned}\tag{38}$$

Note, in particular that, by the definition of the feedback law u_ε, $\|u_\varepsilon\|_{L_t^\infty} < c(M+1)\|\mu\|_{L^\infty}$. Let us study the second line of (38). We have

$$\begin{aligned}\left|u_\varepsilon\big(\Psi_n(s)\big)-u_\varepsilon\big(\widetilde{\Psi}(s)\big)\right| \leqslant{}& c(1-\varepsilon)\sum_{j=0}^{M}\left|\langle\mu\Psi_n(s)\mid\phi_j\rangle\langle\phi_j\mid\Psi_n(s)\rangle-\langle\mu\widetilde{\Psi}(s)\mid\phi_j\rangle\langle\phi_j\mid\widetilde{\Psi}(s)\rangle\right|\\ &+c\varepsilon\left|\langle\mu\Psi_n(s)\mid\phi_0\rangle\langle\phi_0\mid\Psi_n(s)\rangle-\langle\mu\widetilde{\Psi}(s)\mid\phi_0\rangle\langle\phi_0\mid\widetilde{\Psi}(s)\rangle\right|,\end{aligned}$$

and for all $j\in\{0,1,\dots,M\}$

$$\begin{aligned}&\left|\langle\mu\Psi_n(s)\mid\phi_j\rangle\langle\phi_j\mid\Psi_n(s)\rangle-\langle\mu\widetilde{\Psi}(s)\mid\phi_j\rangle\langle\phi_j\mid\widetilde{\Psi}(s)\rangle\right|\\ &\quad\leqslant\left|\langle\mu\delta\Psi_n(s)\mid\phi_j\rangle\langle\phi_j\mid\Psi_n(s)\rangle\right|+\left|\langle\mu\widetilde{\Psi}(s)\mid\phi_j\rangle\langle\phi_j\mid\delta\Psi_n(s)\rangle\right|\\ &\quad\leqslant\|\mu\delta\Psi_n\|_{L^2(\mathbb{R}^N)}+\|\mu\|_{L^\infty}\left\|\mathbb{P}_{\mathrm{disc}}\delta\Psi_n(s)\right\|_{L^2(\mathbb{R}_N)}\\ &\quad\leqslant\big(1+\|\mu\|_{L^\infty}\big)\left\|\delta\Psi_n(s)\right\|_{\mathcal{H}},\end{aligned}$$

where, for the last inequality, we have applied the Cauchy–Schwartz inequality, the facts that $\|\Psi_n(s)\|_{L^2}=1$ and $\|\widetilde{\Psi}(s)\|_{L^2}\leqslant 1$, and that $|\langle\psi\mid\phi_j\rangle|\leqslant\|\mathbb{P}_{\mathrm{disc}}\psi\|_{L^2}$.

The above inequality, together with (38), implies

$$\left\|\delta\Psi_n(\tau)\right\|_{\mathcal{H}} \leqslant \left\|S(\tau)\big(\Psi(t_n)-\widetilde{\Psi}_0\big)\right\|_{\mathcal{H}} + c\kappa(M+2)\|\mu\|_{L^\infty}\big(1+\|\mu\|_{L^\infty}\big)\int_0^\tau\left\|\delta\Psi_n(s)\right\|_{\mathcal{H}}\,ds.\tag{39}$$

Applying the Gronwall Lemma to (39), one only needs to prove

$$\left\|S(\tau)\big(\Psi(t_n)-\widetilde{\Psi}_0\big)\right\|_{\mathcal{H}}\to 0\quad\text{as } n\to\infty.\tag{40}$$

As a first step, we clearly have

$$\left\|\mathbb{P}_{\mathrm{disc}}S(\tau)\big(\Psi(t_n)-\widetilde{\Psi}_0\big)\right\|_{L^2} = \left\|\mathbb{P}_{\mathrm{disc}}\Psi(t_n)-\widetilde{\Psi}_0\right\|_{L^2}\to 0,\tag{41}$$

where we have used the fact that the semigroup $S(\tau)$ induces an isometry on $L^2(\mathbb{R}^N)$, and that the projection operator $\mathbb{P}_{\text{disc}}$ commutes with the evolution operator $S(\tau)$.

Moreover applying the fact that, $\mathbb{P}_{\text{disc}} + \mathbb{P}_{\text{ac}} = \mathrm{Id}_{L^2}$, we have

$$\big\|\mu S(\tau)\big(\Psi(t_n) - \widetilde{\Psi}_0\big)\big\|_{L^2_x} \leqslant \|\mu\|_{L^\infty}\big\|S(\tau)\big(\mathbb{P}_{\text{disc}}\Psi(t_n) - \widetilde{\Psi}_0\big)\big\|_{L^2} + \big\|\mu S(\tau)\mathbb{P}_{\text{ac}}\Psi(t_n)\big\|_{L^2}.$$

Applying (41), the first term, $\|S(\tau)(\mathbb{P}_{\text{disc}}\Psi(t_n) - \widetilde{\Psi}_0)\|_{L^2}$ converges toward 0 whenever $n \to \infty$. Moreover, applying Lemma 14

$$\big\|\mu S(\tau)\mathbb{P}_{\text{ac}}\Psi(t_n)\big\|_{L^2} \to 0 \quad \text{as } n \to \infty.$$

Therefore,

$$\big\|\mu S(\tau)\big(\Psi(t_n) - \widetilde{\Psi}_0\big)\big\|_{L^2} \to 0 \quad \text{as } n \to \infty. \tag{42}$$

The two limits (41) and (42) imply the limit (40) and therefore finish the proof of Lemma 15. □

We are now ready to characterize the weak ω-limit set.

Proposition 16. *Let $\Psi(t)$ denote the weak (or strong) solution of the closed-loop system. Assume for a sequence $(t_n)_{n=1}^\infty \nearrow \infty$ of times that $\Psi(t_n) \rightharpoonup \Psi_\infty \in L^2(\mathbb{R}^N, \mathbb{C})$ weakly in $L^2(\mathbb{R}^N, \mathbb{C})$ (with $\|\Psi_\infty\|_{L^2} \leqslant 1$). Define $\Psi_{\infty,\text{disc}} = \mathbb{P}_{\text{disc}}\Psi_\infty$. One necessarily has*

$$u_\varepsilon(\Psi_{\infty,\text{disc}}) = c\left[(1-\varepsilon)\sum_{j=0}^{M}\Im\big(\langle\mu\Psi_{\infty,\text{disc}} \mid \phi_j\rangle\langle\phi_j \mid \Psi_{\infty,\text{disc}}\rangle\big) + \varepsilon\Im\big(\langle\mu\Psi_{\infty,\text{disc}} \mid \phi_0\rangle\langle\phi_0 \mid \Psi_{\infty,\text{disc}}\rangle\big)\right] = 0.$$

Proof. Consider the Lyapunov function $\mathcal{V}_\varepsilon(\Psi)$ defined in (9). As it is shown in (12), the choice (13) of $u_\varepsilon(\Psi)$ ensures that the Lyapunov function $\mathcal{V}_\varepsilon(\Psi(t))$ is a decreasing function of time. The Lyapunov function $\mathcal{V}_\varepsilon$ being a positive function (10), we have

$$\lim_{t\to\infty}\mathcal{V}_\varepsilon\big(\Psi(t)\big) = \eta, \tag{43}$$

where $\eta \geqslant 0$ is a positive constant.

Consider now, the sequence $\{t_n\}_{n=1}^\infty \nearrow \infty$ of times. The Lyapunov function $\mathcal{V}_\varepsilon(\psi)$ is trivially continuous with respect to ψ for the L^2-weak topology. Therefore, as Ψ_∞ is the weak limit of $\Psi(t_n)$, we have

$$\mathcal{V}_\varepsilon(\Psi_\infty) = \lim_{n\to\infty}\mathcal{V}_\varepsilon\big(\Psi(t_n)\big) = \eta.$$

Furthermore, noting that the Lyapunov function $\mathcal{V}_\varepsilon$ only deals with the population of the bound states, we have

$$\mathcal{V}_\varepsilon(\psi) = \mathcal{V}_\varepsilon(\mathbb{P}_{\text{disc}}\psi),$$

and therefore

$$\mathcal{V}_\varepsilon(\Psi_{\infty,\text{disc}}) = \eta. \tag{44}$$

As in Lemma 15, let us consider the closed-loop Schrödinger equation with the wavefunction $\widetilde{\Psi}$ and the initial state $\widetilde{\Psi}_0 = \Psi_{\infty,\text{disc}}$. Applying Lemma 15, for any $\tau > 0$, we have

$$\mathbb{P}_{\text{disc}}\Psi(t_n + \tau) \to \widetilde{\Psi}(\tau) \quad \text{strongly in } L^2\big(\mathbb{R}^N, \mathbb{C}\big) \text{ as } n \to \infty.$$

As the Lyapunov function $\mathcal{V}_\varepsilon(\psi)$ is continuous with respect to ψ for the L^2_{disc} semi-norm, we have

$$\mathcal{V}_\varepsilon\big(\Psi(t_n + \tau)\big) \to \mathcal{V}_\varepsilon\big(\widetilde{\Psi}(\tau)\big) \quad \text{as } n \to \infty.$$

But, applying (43), we know that

$$\mathcal{V}_\varepsilon\big(\Psi(t_n + \tau)\big) \to \eta \quad \text{as } n \to \infty,$$

and therefore,

$$\mathcal{V}_\varepsilon\big(\widetilde{\Psi}(\tau)\big) = \eta = \mathcal{V}_\varepsilon(\Psi_{\infty,\text{disc}}) = \mathcal{V}_\varepsilon\big(\widetilde{\Psi}(0)\big). \tag{45}$$

Thus, the Lyapunov function $\mathcal{V}_\varepsilon$ remains constant on the closed-loop trajectory of $\widetilde{\Psi}(t)$. This, together with (12) and (13), implies

$$\frac{\partial}{\partial\tau}\mathcal{V}_\varepsilon\big(\widetilde{\Psi}(\tau)\big) = -\frac{1}{c}u_\varepsilon^2\big(\widetilde{\Psi}(\tau)\big) = 0, \tag{46}$$

and therefore by continuity of $u_\varepsilon(\widetilde{\Psi}(\tau))$ with respect to τ and passing to the limit at $\tau = 0$, we can finish the proof of Proposition 16. □

3.3. Non-degeneracy assumptions and the proof of Theorem 9

We have now all the elements to finish the proof of Theorem 9.

Proposition 17. *Let $\Psi(t)$ denote the weak (or strong) solution of the closed-loop system. Consider the sequence $(t_n)_{n=1}^\infty \nearrow \infty$, the weak limit Ψ_∞, and its discrete part $\Psi_{\infty,\mathrm{disc}}$ as in Proposition* 16*. Under the assumptions* A1 *through* A4 *of Theorem* 9*, we have*

$$\Psi_{\infty,\mathrm{disc}} = \varsigma\phi_0, \quad |\varsigma|^2 > 1-\varepsilon. \tag{47}$$

Proof. Define η as in (43). We now, in particular that,

$$\mathcal{V}_\varepsilon(\Psi_\infty) = \eta \leqslant \mathcal{V}_\varepsilon\big(\Psi(0)\big) < \varepsilon \tag{48}$$

where we have applied (11) (and therefore the assumptions A1 and A2).

Let us take

$$\Psi_{\infty,\mathrm{disc}} = \sum_{j=0}^{M}\varsigma_j\phi_j.$$

Taking the closed-loop system $\widetilde{\Psi}(t)$ as in the proof of Proposition 16, we have by (46) that $u_\varepsilon(\widetilde{\Psi}(\tau)) = 0$. Therefore the wavefunction $\widetilde{\Psi}(\tau)$ evolves freely with the Hamiltonian $H_0 = -\Delta + V(x)$ and so is given as follows

$$\widetilde{\Psi}(\tau) = \sum_{j=0}^{M}\varsigma_j e^{-i\lambda_j\tau}\phi_j.$$

By (46) we have

$$u(\widetilde{\Psi}) = c(1-\varepsilon)\sum_{j,k=0}^{M}\bar{\varsigma}_j\varsigma_k e^{i(\lambda_j-\lambda_k)\tau}\langle\mu\phi_k \mid \phi_j\rangle + c\varepsilon\sum_{j=0}^{M}\bar{\varsigma}_0\varsigma_j e^{i(\lambda_0-\lambda_j)\tau}\langle\mu\phi_j \mid \phi_0\rangle = 0 \quad \forall t \geqslant 0.$$

The assumption A3 of non-degenerate transitions applies now. As the above relation holds true for any $\tau \geqslant 0$, we can easily see that

$$\bar{\varsigma}_j\varsigma_k\langle\mu\phi_k \mid \phi_j\rangle = 0 \quad \forall j,k \in \{0,1,\dots,M\}.$$

This together with the assumption A4 of simple couplings imply

$$\bar{\varsigma}_j\varsigma_k = 0 \quad \forall j,k \in \{0,1,\dots,M\}.$$

Thus

$$\exists j \in \{0,1,\dots,M\} \quad \text{such that} \quad \varsigma_j = \varsigma \neq 0 \quad \text{and} \quad \varsigma_k = 0 \quad \forall k \neq j.$$

We show that the only possibility for this index j is to be 0. If this is not the case ($j \neq 0$) taking $\widetilde{\Psi} = \varsigma\phi_j$ with $|\varsigma| \leqslant 1$,

$$\eta = \mathcal{V}_\varepsilon(\Psi_\infty) = 1-(1-\varepsilon)|\varsigma|^2 \geqslant \varepsilon,$$

which is obviously in contradiction with (48). Thus

$$\Psi_{\infty,\mathrm{disc}} = \varsigma\phi_0$$

with $|\varsigma| \leqslant 1$. Therefore

$$\mathcal{V}_\varepsilon(\Psi_\infty) = 1 - (1-\varepsilon)|\varsigma|^2 - \varepsilon|\varsigma|^2 = 1 - |\varsigma|^2.$$

Apply once again (48), we have $1 - |\varsigma|^2 < \varepsilon$ and so we can finish the proof of Proposition 17. □

Let us now, finish the proof of Theorem 9.

Proof of Theorem 9. The well-posedness of the closed-loop system has been addressed in Proposition 11. In order to prove the approximate stabilization result, let us assume that there exists a sequence of times $\{\tilde{t}_n\}_{n=0}^\infty \nearrow \infty$ such that

$$\left|\langle \Psi(\tilde{t}_n) \mid \phi_0\rangle\right|^2 \leqslant 1 - \varepsilon \quad \forall n. \tag{49}$$

As in Proposition 17, we can extract from this sequence a subsequence (noted still by $\{\tilde{t}_n\}_{n=0}^\infty$ for simplicity sakes) such that

$$\mathbb{P}_{\mathrm{disc}}\Psi(\tilde{t}_n) \xrightarrow{L^2\text{-strong}} \tilde{\varsigma}\phi_0 \quad \text{as } n \to \infty,$$

with $|\tilde{\varsigma}|^2 > 1 - \varepsilon$. This obviously implies

$$\liminf_{t\to\infty}\left|\langle \Psi(\tilde{t}_n) \mid \phi_0\rangle\right|^2 > 1 - \varepsilon$$

and is in contradiction with (49). We have therefore finished the proof of Theorem 9. □

4. Proof of Theorem 2

Let us now get back to Theorem 2. Comparing to Theorem 9, the only difference is in the fact that, we are also considering the 1D case and that the interaction Hamiltonian $\mu \in \mathcal{L} \cap L^\infty$ instead of $L^{2N-} \cap L^\infty$ for Theorem 9. Therefore the only cases remaining to be treated are either the 1D case or the cases where $\mu \in L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $p \geqslant 2N$.

Considering these cases and following the same steps as in the proof of Theorem 9, the only place where we will have a problem to proceed the proof is the passage from (26) and (27) to (28) and (29). Indeed, as $2N - p$ is not strictly positive, we cannot ensure the convergence towards 0 of the terms in (26) and (27).

A deep study of the estimates (26) and (27) shows that they can be improved if one had $u_\varepsilon \in L_t^{1+\delta}$ for $\delta \in (0,1)$ instead of L_t^2 as in the proof of Theorem 9. Indeed, if one could show that

$$u_\varepsilon \in L_t^{(\frac{p}{p-N}-\varpi)} \quad \text{for some } \varpi > 0, \tag{50}$$

we could replace the estimates (26) and (27) with

$$\begin{aligned}
&\int_0^{T_\gamma} \left|u_\varepsilon(\Psi(s))\right| \left\|\mu(x)S(t_n+\tau-s)\mathbb{P}_{\mathrm{ac}}\mu(x)\Psi(s)\right\|_{L_x^2} ds \\
&\quad \leqslant \left\|u_\varepsilon(\Psi(t))\right\|_{L_t^{(\frac{p}{p-N}-\varpi)}} \|\mu\|_{L_x^p}^2 \left(\int_0^{T_\gamma} (t_n+\tau-s)^{-\frac{\zeta N}{p}} ds\right)^{1/2} \\
&\quad \leqslant \frac{\sqrt{p}}{\sqrt{|\zeta N - p|}} \left\|u_\varepsilon(\Psi(t))\right\|_{L_t^{(\frac{p}{p-N}-\varpi)}} \|\mu\|_{L_x^p}^2 \left(|t_n+\tau-T_\gamma|^{-\frac{\zeta N-p}{p}} - |t_n+\tau|^{-\frac{\zeta N-p}{p}}\right)^{1/2},
\end{aligned} \tag{51}$$

and

$$\int_{T_\gamma}^{t_n} \left|u_\varepsilon(\Psi(s))\right| \left\|\mu(x)S(t_n+\tau-s)\mathbb{P}_{\mathrm{ac}}\mu(x)\Psi(s)\right\|_{L_x^2} ds$$

$$\leqslant \|\mu\|^2_{L^p_x}\left(\int_{T_\gamma}^{\infty}\left|u_\varepsilon\big(\Psi(t)\big)\right|^{\frac{p}{p-N}-\varpi}\,dt\right)^{1/2}\left(\int_{T_\gamma}^{t_n}(t_n+\tau-s)^{-\frac{\zeta N}{p}}\,ds\right)^{1/2}$$

$$\leqslant \frac{\sqrt{p}}{\sqrt{|\zeta N-p|}}\gamma^{1/2}\|\mu\|^2_{L^p_x}\big(|\tau|^{-\frac{\zeta N-p}{p}}-|t_n+\tau-T_\gamma|^{-\frac{\zeta N-p}{p}}\big)^{1/2}, \tag{52}$$

where

$$\zeta=\frac{p-\varpi(p-N)}{N-\varpi(p-N)}, \tag{53}$$

noting that we have applied the Hölder inequality and the fact that

$$\frac{1}{\zeta}+\frac{1}{\frac{p}{p-N}-\varpi}=1.$$

Note that, as $p\geqslant 2N$ (this is also true for the 1D case as $p\geqslant 2$), $\frac{p}{p-N}\in(1,2]$ and therefore there exists some positive $\varpi>0$ such that $\frac{p}{p-N}-\varpi>1$. Furthermore, we have

$$\zeta=\frac{p-\varpi(p-N)}{N-\varpi(p-N)}>\frac{p}{N},$$

and therefore $\zeta N-p$ is positive. We can thus proceed the proof of Theorem 2 following the same steps as those of Theorem 9.

However, it seems that one cannot hope to prove an estimate of the form (50) for the feedback law u_ε of (13). We, therefore, need to change the feedback strategy. This might be done applying the feedback law (14).

The following proposition clearly implies Theorem 2.

Proposition 18. *Consider the Schrödinger equation* (1)–(2). *We suppose the assumptions of Theorem* 2 *on the potential $V(x)$ and we take $\mu\in L^p(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ for some $p\geqslant 2N$. We suppose moreover the assumptions* A1 *through* A4 *to hold true.*

Then for any $\varepsilon>0$, applying the feedback law $u(t)=u_{\varepsilon,\alpha}(\Psi(t))$ of (14) *with*

$$\alpha=\frac{p-2N+\varpi(p-N)}{N-\varpi(p-N)},\quad 0<\varpi<\frac{N}{N-p},$$

the closed-loop system admits a unique weak solution in $C^0([0,T],\mathbb{S})\cap C^1([0,T],H^{-2}(\mathbb{R}^N,\mathbb{C}))$. Moreover the state of the system ends up reaching a population more than $(1-\varepsilon)$ in the eigenfunction ϕ_0 (approximate stabilization):

$$\liminf_{t\to\infty}\left|\langle\Psi(t,x)\mid\phi_0(x)\rangle\right|^2>1-\varepsilon.$$

If, moreover multiplication by $\mu(x)$ defines a bounded operator over $H^2(\mathbb{R}^N)$, then Ψ is a strong solution, i.e. $\Psi\in C^0([0,T],H^2(\mathbb{R}^N,\mathbb{C}))\cap C^1([0,T],L^2(\mathbb{R}^N,\mathbb{C}))$.

Proof. Considering the Lyapunov function $\mathcal{V}_\varepsilon$ of (9), the choice of the feedback law implies

$$\frac{d}{dt}\mathcal{V}_\varepsilon=-\frac{c^{1+\alpha}}{c^{2+\alpha}}|u_{\varepsilon,\alpha}|^{\frac{2+\alpha}{1+\alpha}},$$

and therefore proceeding as in the proof of Lemma 12

$$u_{\varepsilon,\alpha}\in L_t^{\frac{2+\alpha}{1+\alpha}}=L_t^{\frac{p}{p-N}-\varpi}. \tag{54}$$

In particular, for $\gamma>0$, we will chose T_γ such that

$$\int_{T_\gamma}^{\infty}\left|u_\varepsilon\big(\Psi(s)\big)\right|^{\frac{p}{p-N}-\varpi}\,ds\leqslant\gamma. \tag{55}$$

One can then proceed the proof of Proposition 17, exactly as in the proof of Theorem 9, replacing only Lemma 12 with (54) and (55) and the estimates (26) and (27) by (51) and (52). □

5. Relaxations

As it has been proved in previous sections, the approximate stabilization of a quantum particle around the bound states of a decaying potential (satisfying the decay assumption (A)) may be investigated through explicit feedback laws (13) or (14). The assumptions on the potential V or the interaction Hamiltonian μ are not so restrictive and seem to be satisfied for a large class of physical systems. However, the assumptions A1 through A4 may seem to be too restrictive. In particular, the assumption A1 does not allow the approximate stabilization of an initial wavefunction with a non-zero population in the absolutely continuous part $\mathcal{E}_{\mathrm{ac}}$.

The aim of this section is to give some ideas to relax these assumptions and to consider some more general situations. Some discussions on the assumption A1 will be addressed in Section 5.1. Furthermore, a significant relaxation of the assumptions A3 and A4 will be addressed in Section 5.2. Concerning the assumption A2, we only give the following remark which states that this assumption is, actually, not at all restrictive in practice.

Remark 19. Physically, the assumption A2 in not really restrictive. Indeed, even if $\langle \Psi_0 \mid \phi_0 \rangle = 0$, a control field in resonance with the natural frequencies of the system (the difference between the eigenvalues corresponding to an eigenfunction whose population in the initial state is non-zero and the ground state) will, instantaneously, ensure a non-zero population of the ground state in the wavefunction. Then, one can just apply the feedback law of Theorems 9 or 2.

5.1. Assumption A1

Before discussing an idea which may result in a significant relaxation of this assumption, let us provide a remark which states that the result of Theorem 2 still holds true if we relax slightly the assumption A1.

Remark 20. Consider the Schrödinger equation (1)–(2) with the same assumptions on V and μ as in Theorem 2. We consider moreover Assumptions A2 through A4 and we replace A1 with

A1′ $\|\mathbb{P}_{\mathrm{ac}}\Psi_0\|^2_{L^2(\mathbb{R}^N)} < \frac{\varepsilon}{1-\varepsilon}|\langle \Psi \mid \phi_0\rangle|^2$.

The feedback law (14) still ensures the approximate stabilization of the closed-loop. One only needs to note that,

$$\mathcal{V}_\varepsilon(\Psi_0) < 1-(1-\varepsilon)\left(1-\frac{\varepsilon}{1-\varepsilon}\left|\langle \Psi_0 \mid \phi_0\rangle\right|^2\right) - \varepsilon\left|\langle \Psi_0 \mid \phi_0\rangle\right|^2 = \varepsilon,$$

where we have applied the assumptions A1′ and A2. The rest of the proof follows exactly as in Theorem 9.

Here, we have relaxed the assumption A1 by allowing a very small part of the population of the initial state to belong to the continuum. In fact, this allowed continuum population is bounded by an $\mathrm{O}(\varepsilon)$-proportion of the population in the target state ϕ_0.

The assumption A1′ of Remark 20 is still quite restrictive. The question is therefore to provide a strategy permitting us to approximately stabilize an important part of the continuum. Note that the controllability of this particular problem has never been treated. It seems that one cannot in general hope to have a strong controllability result. In fact, considering the potentials V and μ of compact supports and taking an initial state of support outside $\mathrm{supp}(V) \cup \mathrm{supp}(\mu)$, it seems that an important part of the population may be lost at infinity through the dispersion phenomena and this before the controller even has the time to see and to influence the state. However, one might be interested to control a part of the continuum.

Consider for example the potentials V and μ to be negative and of compact supports and moreover that $\mathrm{supp}(\mu) \subset \mathrm{supp}(V)$. Considering the Hamiltonian $H_\lambda = -\Delta + V + \lambda\mu$ in the strong coupling limit ($\lambda \to \infty$), this Hamiltonian admits more and more bound states. One can therefore cover a higher and higher dimensional subspace of $L^2(\mathbb{R}^N)$ through the discrete eigenspace of H_λ. Assume an initial state Ψ_0 which has a large population in the continuum of $H_0 = -\Delta + V$ but a small population in the continuum of H_λ for some $\lambda > 0$. Applying the strategy of Theorem 2, to the free Hamiltonian H_λ and the interaction Hamiltonian μ one may hope to reach an ε-neighborhood of an arbitrary bound state of H_λ. Note, in particular that while reaching this bound state the control field $u(t)$ has converged towards $-\lambda$. Letting now the control field $u(t) \sim -\lambda$ varying slowly towards zero and applying the quantum adiabatic theory (see e.g. [5]) the state of the system will follow closely a bound state of the Hamiltonian H_λ ($\lambda \to 0$). If the target bound state of H_λ is chosen to be on the analytic branch corresponding to the evolution of the ground state of H_0 (see e.g. [30]), as the control tends to 0, we may reach the ε-neighborhood of the desired target ϕ_0. This idea of applying the quantum adiabatic theory and the large coupling limit to ensure the control of a population in the continuum will be explored in future works.

5.2. Assumptions A3 and A4

After the above discussions on the assumptions A1 and A2, let us study the non-degeneracy assumptions A3 and A4. Similar assumptions to A3 and A4 have already been considered for the stabilization of finite dimensional quantum systems [34,9]. When dealing with finite dimensional systems, the assumptions A3 and A4 are equivalent to the controllability of the linearized system around various eigenstates of the system. For these finite dimensional systems, it was shown, in [34] through the quantum adiabatic theory, and in [9] through the implicit Lyapunov control techniques, that the non-degeneracy assumptions can be relaxed significantly. These relaxations have even been applied to the problem of the approximate stabilization of the quantum particle in an infinite potential well, being an infinite dimensional system (see [11]). In this subsection, we will see that such relaxations may also be considered for our control problem of the quantum particle in a decaying potential.

In this aim, we consider the potential V and the interaction Hamiltonian μ both to satisfy the decay assumption (A). Similarly to the previous works, we consider the family of the perturbed Hamiltonian $H_\sigma = -\Delta + V(x) + \sigma\mu(x)$ with $|\sigma| \ll 1$ a small real constant. The family $\{H_\sigma\}$ is a self-adjoint holomorphic family of type (A) in the sense of Kato (see [30], page 375). Thus, the eigenvalues and the bound states of H_σ are holomorphic functions of σ around zero.

The absence of zero energy eigenstate for $H_0 = -\Delta + V$ as been assumed in the decay assumption (A) implies the existence of a strictly positive threshold σ^* such that for σ evolving in $(-\sigma^*, \sigma^*)$ the bound states ϕ_j of H_0 stay bound states of H_σ and do not join the continuum. This ensures that, the perturbed eigenvalues $\{\lambda_{\sigma,j}\}_{j=0}^M$ of H_σ (with $\lambda_{0,j} = \lambda_j$) are well defined and remain less than zero. Note that, one might have the appearance of new bound states but the bound states of H_0 will not disappear while considering perturbations of amplitude $|\sigma| < \sigma^*$. We have therefore the following theorem:

Theorem 21. *Consider the Schrödinger equation* (1)–(2) *with the decay assumption* (A) *on both V and μ. We assume moreover that the space dimension $N \geqslant 2$ and $\mu \in L^{2N-} \cap L^\infty$. We consider the assumptions* A1 *and* A2 *and we replace* A3 *and* A4 *with*:

(A3–A4)′ *there exists $\bar\sigma \in (0, \sigma^*)$ such that the non-degeneracy assumptions* A3 *and* A4 *hold for the eigenvalues and the eigenstates of the perturbed Hamiltonian $H_{\bar\sigma} = -\Delta + V + \bar\sigma\mu$.*

There exists then a feedback law $u(\Psi)$, such that the closed-loop system admits a unique weak solution and that

$$\liminf_{t\to\infty} \left|\langle \Psi(t) \mid \phi_0\rangle\right|^2 > 1 - \varepsilon.$$

Remark 22. Roughly speaking the non-degeneracy assumptions A3 and A4 are always satisfied unless some kind of symmetry is admitted in the potential. Formally, the assumption (A3–A4)′ states that if we can break this symmetry through the addition of the interaction Hamiltonian, we are still able to ensure the approximate stabilization result.

Remark 23. Applying the same technique as in Proposition 18, one can extend the result of Theorem 21 to the case of dimensions $N \geqslant 1$ and $\mu \in \mathcal{L} \cap L^\infty$.

Proof. By the analyticity of the eigenvalues $\lambda_{\sigma,j}$ and $\phi_{\sigma,j}$ with respect to σ around zero, the assumption (A3–A4)′ ensures the existence of a strictly positive constant $\sigma^\sharp \in (0, \sigma^*)$ such that the non-degeneracy assumptions A3 and A4 hold true for the perturbed Hamiltonians H_σ with any σ in the interval $(0, \sigma^\sharp)$.

Applying once more the analyticity of the bound states $\{\phi_{j,\sigma}\}_{j=0}^M$ with respect to σ in $(0, \sigma^\sharp)$ implies that one can choose $\sigma^{\sharp\sharp} \in (0, \sigma^\sharp)$ such that

$$\|\phi_{j,\sigma}-\phi_j\|_{L^2}<\min\left(\frac{\varepsilon}{4},\frac{\varepsilon|\langle\Psi_0\mid\phi_0\rangle|^2}{2(M+1)(2-\varepsilon)+2\varepsilon}\right)\quad\forall j=0,1,\dots,M\text{ and }\forall\sigma\in\big(0,\sigma^{\sharp\sharp}\big).\tag{56}$$

Now, applying the inequality (56), the assumption A2 implies:

$$\begin{aligned}\big|\langle\Psi_0\mid\phi_{0,\sigma}\rangle\big|^2&\geqslant\big|\langle\Psi_0\mid\phi_0\rangle\big|^2-2\big|\langle\Psi_0\mid\phi_{0,\sigma}-\phi_0\rangle\big|\\&\geqslant\big|\langle\Psi_0\mid\phi_0\rangle\big|^2-2\|\phi_{j,\sigma}-\phi_j\|_{L^2}>0,\quad\forall\sigma\in\big(0,\sigma^{\sharp\sharp}\big),\end{aligned}\tag{57}$$

where we have applied the Cauchy–Schwartz inequality and the fact that

$$\frac{\varepsilon}{2(M+1)(2-\varepsilon)+2\varepsilon}<\frac{1}{2}.$$

Let us consider the Schrödinger equation (Σ_σ) characterized by the free Hamiltonian $H_\sigma=-\Delta+V+\sigma\mu$ and the interaction Hamiltonian μ for some $\sigma\in(0,\sigma^{\sharp\sharp})$. Applying the assumption (A3–A4)$'$, the assumptions A3 and A4 hold true for this system. Moreover, the inequality (57) implies the assumption A2 for this system. Finally, applying (56), we have ($M_\sigma+1\geqslant M+1$ is the number of the bound states of H_σ)

$$\begin{aligned}\mathbb{P}_{\mathrm{ac}}(H_\sigma)(\Psi_0)&=1-\sum_{j=0}^{M_\sigma}\big|\langle\Psi_0\mid\phi_{j,\sigma}\rangle\big|^2\leqslant1-\sum_{j=0}^{M}\big|\langle\Psi_0\mid\phi_{j,\sigma}\rangle\big|^2\\&\leqslant1-\sum_{j=0}^{M}\big(\big|\langle\Psi_0\mid\phi_{j,\sigma}\rangle\big|^2-2\|\phi_{j,\sigma}-\phi_j\|_{L^2}\big)\\&<\frac{2(M+1)\varepsilon|\langle\Psi_0\mid\phi_0\rangle|^2}{2(M+1)(2-\varepsilon)+2\varepsilon}<\frac{\varepsilon/2}{1-\varepsilon/2}\big|\langle\Psi_0\mid\phi_0\rangle\big|^2.\end{aligned}\tag{58}$$

This implies the assumption A1$'$ or the system (Σ_σ) (having replaced ε by $\varepsilon/2$).

Considering therefore the feedback law

$$v_{\frac{\varepsilon}{2},\sigma}\big(\Psi(t)\big):=c\left(1-\frac{\varepsilon}{2}\right)\sum_{j=0}^{M_\sigma}\big\langle\mu\Psi(t)\mid\phi_{j,\sigma}\big\rangle\big\langle\phi_{j,\sigma}\mid\Psi(t)\big\rangle+c\frac{\varepsilon}{2}\big\langle\mu\Psi(t)\mid\phi_{0,\sigma}\big\rangle\big\langle\phi_{0,\sigma}\mid\Psi(t)\big\rangle,\quad c>0,\tag{59}$$

and applying Theorem 9, we ensure the approximate stabilization result:

$$\liminf_{t\to\infty}\big|\big\langle\Psi(t)\mid\phi_{0,\sigma}\big\rangle\big|^2\geqslant1-\varepsilon/2.\tag{60}$$

Note that, the feedback law (59) means the application of the feedback $u:=-\sigma+v_{\frac{\varepsilon}{2},\sigma}(\Psi(t))$ for the main Schrödinger equation (1)–(2). Finally the limit (60) implies

$$\liminf_{t\to\infty}\big|\big\langle\Psi(t)\mid\phi_0\big\rangle\big|^2\geqslant\liminf_{t\to\infty}\big(\big|\big\langle\Psi(t)\mid\phi_{0,\sigma}\big\rangle\big|^2-2\|\phi_{0,\sigma}-\phi_0\|_{L^2}\big)\geqslant1-\varepsilon/2-\varepsilon/2=1-\varepsilon,\tag{61}$$

where once again we have applied (56). □

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Appendix A

This appendix is devoted to the proof of Proposition 1.

Proof of Proposition 1. Let $\Psi_0\in\mathbb{S}$, $T>0$ and $u\in C^0([0,T],\mathbb{R})$. Let $T_1\in(0,T)$ be such that

$$\|\mu\|_{L^\infty}\|u\|_{L^1(0,T_1)}<1.\tag{62}$$

We prove the existence of $\Psi\in C^0([0,T_1],L^2(\mathbb{R}^N,\mathbb{C}))$ such that (3) holds by applying the Banach fixed point theorem to the map

$$\Theta : C^0\big([0,T_1],L^2\big) \to C^0\big([0,T_1],L^2\big),$$
$$\xi \mapsto \Psi,$$

where Ψ is the weak solution of

$$i\frac{\partial \Psi}{\partial t} = H_0\Psi + u(t)\mu(x)\xi, \quad \Psi(0,x) = \Psi_0(x),$$

i.e. $\Psi \in C^0([0,T_1],L^2)$ and satisfies, for every $t \in [0,T_1]$,

$$\Psi(t) = e^{-iH_0t}\Psi_0 - i\int_0^t e^{-iH_0(t-s)}u(s)\mu(x)\xi(s)\,ds \quad \text{in } L^2\big(\mathbb{R}^N,\mathbb{C}\big).$$

Notice that Θ takes values in $C^1([0,T_1],H^{-2}(\mathbb{R}^N,\mathbb{C}))$.

For $\xi_1,\xi_2 \in C^0([0,T_1],L^2(\mathbb{R}^N,\mathbb{C}))$, $\Psi_1 := \Theta(\xi_1)$, $\Psi_2 := \Theta(\xi_2)$ we have

$$(\Psi_1 - \Psi_2)(t) = -i\int_0^t e^{-iH_0(t-s)}u(s)\mu(x)(\xi_1-\xi_2)(s)\,ds$$

thus

$$\big\|(\Psi_1-\Psi_2)(t)\big\|_{L^2} \leqslant \|\mu\|_{L^\infty}\int_0^t \big|u(s)\big|\,ds\,\|\xi_1-\xi_2\|_{C^0([0,T_1],L^2)}.$$

The assumption (62) guarantees that Θ is a contraction of $C^0([0,T_1],L^2)$, thus, Θ has a fixed point $\Psi \in C^0([0,T_1],L^2)$. Since Θ takes values in $C^1([0,T_1],H^{-2})$, then Ψ belongs to this space. Moreover, this function satisfies (3).

Finally, we have built weak solutions on $[0,T_1]$ for every Ψ_0, and the time T_1 does not depend on Ψ_0, thus, this gives solutions on $[0,T]$.

Let us prove that this solution is continuous with respect to the initial condition Ψ_0, for the $L^2(\mathbb{R}^N,\mathbb{C})$-topology. Let $\Psi_0,\Phi_0 \in \mathbb{S}$ and Ψ, Φ the associated weak solutions. We have

$$\big\|(\Psi-\Phi)(t)\big\|_{L^2} \leqslant \|\Psi_0-\Phi_0\|_{L^2} + \|\mu\|_{L^\infty}\int_0^t \big|u(s)\big|\big\|(\Psi-\Phi)(s)\big\|_{L^2}\,ds,$$

thus Gronwall Lemma gives

$$\big\|(\Psi-\Phi)(t)\big\|_{L^2} \leqslant \|\Psi_0-\Phi_0\|_{L^2} e^{\|\mu\|_{L^\infty}\|u\|_{L^1(0,T)}}.$$

This gives the continuity of the weak solutions with respect to the initial conditions.

Now, let us assume that $\Psi_0 \in H^2(\mathbb{R}^N,\mathbb{C})$. Take C to be the bound of the multiplication operator μ over H^2, i.e. C is a positive constant such that for every $\varphi \in H^2(\mathbb{R}^N,\mathbb{C})$, $\|\mu\varphi\|_{H^2} \leqslant C\|\varphi\|_{H^2}$. We consider, then, $T_2 > 0$ such that $C\|u\|_{L^1(0,T_2)} < 1$. By applying the fixed point theorem on

$$\Theta_2 : C^0\big([0,T_2],H^2\big) \to C^0\big([0,T_2],H^2\big)$$

defined by the same expression as Θ, and using the uniqueness of the fixed point of Θ, we get that the weak solution is a strong solution. The continuity with respect to the initial condition of the strong solution can also be proved applying the same arguments as in above.

Finally, let us justify that the weak solutions take values in $\mathbb{S}$. For $\Psi_0 \in H^2$, the solution belongs to $C^1([0,T],L^2) \cap C^0([0,T],H^2)$ thus, the following computations are justified

$$\frac{d}{dt}\big\|\Psi(t)\big\|_{L^2}^2 = 2\Re\left\langle \frac{\partial \Psi}{\partial t} \;\middle|\; \Psi \right\rangle = 0.$$

Thus $\Psi(t) \in \mathbb{S}$ for every $t \in [0,T]$.

For $\Psi_0 \in \mathbb{S}$, we get the same conclusion thanks to a density argument and the continuity for the $C^0([0,T],L^2)$-topology of the weak solutions with respect to the initial condition. $\square$

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