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Ann. I. H. Poincaré – AN 26 (2009) 1265–1275

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

New isoperimetric estimates for solutions to Monge–Ampère equations

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Received 9 May 2008; accepted 26 September 2008

Available online 29 October 2008

Abstract

We prove some sharp estimates for solutions to Dirichlet problems relative to Monge–Ampère equations. Among them we show that the eigenvalue of the Dirichlet problem, when computed on convex domains with fixed measure, is maximal on ellipsoids. This result falls in the class of affine isoperimetric inequalities and shows that the eigenvalue of the Monge–Ampère operator behaves just the contrary of the first eigenvalue of the Laplace operator.

Keywords: Affine isoperimetric inequalities; Monge–Ampère equation

1. Introduction

In the pioneering papers [14,16] a deep connection between a priori estimates for solutions to Dirichlet problems relative to a class of linear elliptic equations and the classical isoperimetric inequality is established; nowadays the notion of rearrangement of a function is a standard tool when looking for sharp comparison results for solutions to elliptic and parabolic partial differential equations. Moreover, Pólya–Szegö type principles are certainly chief tools in this kind of results and, each time such principles appear, some isoperimetric inequality is tacitly used. The aim of this paper is somehow to show that, when dealing with the Monge–Ampère operator, affine isoperimetric inequalities are most likely the natural ones.

In order to introduce our results we consider two celebrated conjectures which belong to the folklore of applied mathematics:

Lord Rayleigh's conjecture Among all membranes with given area, the circle has the lowest fundamental frequency.
St. Venant's conjecture Among all solid bars with the same cross-sectional area, the circular shaft gives the maximal torsional rigidity.

An elegant proof of these conjectures based on Pólya–Szegö principle can be found in [10].

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2008.09.005

In the framework of partial differential equations, these conjectures lead to two well-known results: among all smooth, bounded, open set $\Omega \subset \mathbb{R}^n$ with fixed measure

(i) the first eigenvalue of the Laplace operator $\lambda_1(\Omega)$ is minimal just on balls, i.e.
$$\lambda_1(\Omega) \geqslant \lambda_1\big(\Omega^\sharp\big), \tag{1.1}$$
where $\Omega^\sharp$ is a ball having the same measure as Ω, and equality holds if and only if Ω is a ball;

(ii) the solution w to the following Poisson problem
$$\begin{cases} \Delta w = n & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$
satisfies
$$\|w\|_{L^1(\Omega)} \leqslant \frac{|\Omega^\sharp|^{\frac{n+2}{n}}}{(n+2)\omega_n^{\frac{2}{n}}} = \frac{|\Omega|^{\frac{n+2}{n}}}{(n+2)\omega_n^{\frac{2}{n}}}, \tag{1.3}$$
equality holding if and only if Ω is a ball. Here ω_n means, as usual, the measure of the unitary ball in $\mathbb{R}^n$.

These results still hold true when generalized to a wide class of quasilinear operators (see for instance [17,2] and the references quoted in). In this paper we are interested in considering the fully nonlinear Monge–Ampère operator. In particular, whenever $\Omega \subset \mathbb{R}^n$ is a smooth, bounded, strictly convex, open set, there exists a unique (see [7,22]) eigenvalue of the Monge–Ampère operator $\sigma_1(\Omega)$ and problem (1.2) finds its natural formulation in the following Dirichlet problem
$$\begin{cases} \det D^2 u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$
In [21,18] Pólya–Szegö type inequalities suitable for Monge–Ampère equations have been proved using Aleksandrov–Fenchel inequalities. As a consequence it holds
$$\sigma_1(\Omega) \geqslant \sigma_1\big(\Omega^\bigstar\big) \tag{1.5}$$
and moreover, if u is the solution to (1.4), then it satisfies
$$\|u\|_{L^1(\Omega)} \leqslant \frac{|\Omega^\bigstar|^{\frac{n+2}{n}}}{(n+2)\omega_n^{\frac{2}{n}}}. \tag{1.6}$$
Here, however, $\Omega^\bigstar$ denotes the ball having the same quermassintegral corresponding to the $(n-2)$-th mean curvature as Ω (note that $|\Omega| \leqslant |\Omega^\bigstar|$). In particular we remind that, for $n = 2$, such a quermassintegral is just the perimeter and (1.6) was first proved by Talenti in [15].

However, it was unclear whether (1.5) and (1.6) hold also when replacing $\Omega^\bigstar$ by $\Omega^\sharp$. To this aim it is crucial to observe that the Monge–Ampère operator is invariant under volume preserving affine transformations. In particular, if $\mathcal{A}$ is a volume preserving affine transformation, then $\sigma_1(\Omega) = \sigma_1(\mathcal{A}\Omega)$; this implies that the eigenvalue relative to a ball is equal to that one computed on any ellipsoid having the same measure. In addiction, if u is the solution to (1.4), then $u_{\mathcal{A}}(x) = u(\mathcal{A}x)$ is the solution to (1.4) in $\mathcal{A}^{-1}\Omega$ and $\|u\|_{L^1(\Omega)} = \|u_{\mathcal{A}}\|_{L^1(\mathcal{A}^{-1}\Omega)}$. These considerations suggested us to look for affine isoperimetric inequalities. The outcome is quite unexpected; we will prove the following facts:

among all smooth, bounded, convex, open sets $\Omega \subset \mathbb{R}^n$ with fixed measure

(I) the eigenvalue of the Monge–Ampère operator $\sigma_1(\Omega)$ is maximal just on ellipsoids, i.e.
$$\sigma_1(\Omega) \leqslant \sigma_1\big(\mathcal{E}(\Omega)\big), \tag{1.7}$$
where $\mathcal{E}(\Omega)$ is an ellipsoid having the same measure as Ω, and equality holds if and only if Ω is an ellipsoid;

(II) the solution u to (1.4) satisfies
$$\|u\|_{L^1(\Omega)} \geqslant \frac{|\Omega|^{\frac{n+2}{n}}}{(n+2)\omega_n^{\frac{2}{n}}}, \tag{1.8}$$
equality holding if and only if Ω is an ellipsoid.

The main idea underlying the proof of (1.7) and (1.8) consists in an a priori estimate of the energy of convex functions (according to the natural notion of energy associated to Monge–Ampère operator). In particular we restrict our attention to smooth convex functions whose level sets are all homothetic to the zero level set. Due to very special circumstances, the energy of a function, whose zero level set is a convex body Ω, happens to be related to the measure of the so-called polar body Ω^*.

We thank professor B. Kawohl for bringing to our attention the paper [5], where the use of functions with level sets mutually homothetic has been successfully applied to generalize to arbitrary dimension a well-known result by Pólya (see [11, §5.5–§5.6]).

2. Notation and preliminaries

We start by recalling some classical notions of convex analysis, following [3,13]. Let $\mathcal{K}_0^n$ be the class of convex bodies: nonempty, compact, convex sets of $\mathbb{R}^n$. A convex body Ω is the intersection of its supporting halfspaces, thus it can be conveniently described by its support function $h(\Omega,\cdot) = h_\Omega(\cdot)$, defined by

$$h_\Omega(x) = \sup\{\langle x, y\rangle:\ y \in \Omega\}, \quad x \in \mathbb{R}^n.$$

When $0 \in \operatorname{int}\Omega$, for a unit vector $\xi \in S^{n-1} \cap \operatorname{dom} h_\Omega(\cdot)$, $h_\Omega(\xi)$ means the distance of the support plane to Ω with exterior normal vector ξ from the origin. It immediately follows from the definition that $h_\Omega(\cdot)$ is a positive 1-homogeneous, subadditive, convex function. If Ω is of class C^2_+ (i.e. Ω is a convex body whose boundary is of class C^2 with nonvanishing Gaussian curvature k_Ω), the map $\nu : x \in \partial\Omega \to \xi = \nu(x) \in S^{n-1}$ has an inverse ν^{-1} of class C^1 and it holds

$$h_\Omega(\xi) = \langle \nu^{-1}(\xi), \xi\rangle$$

(clearly, $\nu^{-1}(\xi)$ is the only point belonging to $\partial\Omega$ having exterior normal unit vector equal to ξ).

Besides the support function, other functions can be used to describe a convex body analytically.

Let $\Omega \in \mathcal{K}_0^n$ be such that $0 \in \operatorname{int}\Omega$; the function

$$\rho(\Omega, x) = \max\{\lambda \geqslant 0:\ \lambda x \in \Omega\}, \quad \text{for } x \in \mathbb{R}^n - \{0\},$$

is called the radial function of Ω. It results that $\rho(\Omega,x)x \in \partial\Omega$ for every $x \in \mathbb{R}^n - \{0\}$ and

$$|\Omega| = \frac{1}{n}\int_{S^{n-1}} \rho(\Omega,\xi)^n \, d\mathcal{H}^{n-1}(\xi).$$

Now let us define the polar body of a convex body Ω as follows

$$\Omega^* = \{x \in \mathbb{R}^n:\ \langle x, y\rangle \leqslant 1\ \forall y \in \Omega\}.$$

It is well-known that $\Omega^* \in \mathcal{K}_0^n$ and

$$\rho(\Omega^*, \xi) = \frac{1}{h_\Omega(\xi)} \quad \text{for } \xi \in S^{n-1},$$

which expresses a simple way to find the boundary points of the polar body Ω^* from the support planes to Ω.

In order to recall the celebrated Blaschke–Santalò inequality we need some more definitions. Let $\Omega \in \mathcal{K}_0^n$; for $z \in \operatorname{int}\Omega$, let Ω^z be the polar body of its translation $\Omega - z$, i.e. $\Omega^z = (\Omega - z)^*$. As noted by Santalò in 1949, there exists a unique point $s \in \operatorname{int}\Omega$, named Santalò point, such that

$$|\Omega^s| \leqslant |\Omega^z| \quad \forall z \in \operatorname{int}\Omega.$$

The Blaschke–Santalò inequality states that (see, for instance, [8])

$$|\Omega||\Omega^s| \leqslant \omega_n^2, \tag{2.1}$$

where ω_n is the measure of the unitary ball in $\mathbb{R}^n$, equality holding in (2.1) if and only if Ω is an ellipsoid. We observe that the volume product $|\Omega||\Omega^s|$ is invariant under affine transformations; for this reason (2.1) is known as an affine isoperimetric inequality. Another affine isoperimetric inequality, equivalent to (2.1), is the following Petty inequality (see [9])

$$A_a(\Omega)^{n+1} = \left(\int_{S^{n-1}} k_\Omega(\xi)^{-\frac{n}{n+1}} \, d\mathcal{H}^{n-1}(\xi) \right)^{n+1} \leqslant n^{n+1}\omega_n^2|\Omega|^{n-1}, \tag{2.2}$$

where Ω is of class C^2_+ and equality holds if and only if Ω is an ellipsoid. The quantity $A_a(\Omega)$ is known as affine surface area of Ω and it is invariant under volume preserving affine transformations.

Now let $\Omega, \Theta \in \mathcal{K}^n_0$; Minkowski inequality states that the mixed volume $V(\Omega, \Theta, \dots, \Theta)$ bounds from above the product of the volumes of Ω and Θ; namely

$$V(\Omega, \Theta, \dots, \Theta) \geqslant |\Omega|^{\frac{1}{n}} |\Theta|^{\frac{n-1}{n}} \tag{2.3}$$

and equality holds if and only if Ω and Θ are homothetic. Here and in what follows, when $\Omega, \Theta \in C^2_+$, we will read

$$V(\Omega, \Theta, \dots, \Theta) = \frac{1}{n} \int_{S^{n-1}} \frac{h_\Omega(\xi)}{k_\Theta(\xi)} \, d\mathcal{H}^{n-1}(\xi).$$

It is possible to rewrite the quantities $A_a(\Omega)$ and $V(\Omega, \Theta, \dots, \Theta)$ as integrals over $\partial\Omega$; for the reader's convenience we quote here the changing of variables formulae. Whenever Ω is of class C^2_+, for every integrable real function f on $\partial\Omega$, we have

$$\int_{\partial\Omega} f(x) \, d\mathcal{H}^{n-1}(x) = \int_{S^{n-1}} \frac{f(\nu^{-1}(\xi))}{k_\Omega(\nu^{-1}(\xi))} \, d\mathcal{H}^{n-1}(\xi); \tag{2.4}$$

similarly, for every integrable real function g on S^{n-1}, we get

$$\int_{S^{n-1}} g(\xi) \, d\mathcal{H}^{n-1}(\xi) = \int_{\partial\Omega} g\big(\nu(x)\big) k_\Omega\big(\nu(x)\big) \, d\mathcal{H}^{n-1}(x).$$

In this way we can write

$$A_a(\Omega) = \int_{\partial\Omega} k_\Omega^{\frac{1}{n+1}} \, d\mathcal{H}^{n-1}(x), \qquad V(\Omega, \Theta, \dots, \Theta) = \frac{1}{n} \int_{\partial\Omega} h_\Omega \frac{k_\Omega}{k_\Theta} \, d\mathcal{H}^{n-1}(x).$$

In what follows, for practical reasons we will deal with open sets. To shorten notation, the notions introduced above (support and radial functions, polar body and affine isoperimetric inequalities, etc.) applied to an open set have to be understood applied to its closure.

For further applications we collect here some small things concerning the Monge–Ampère operator $\det D^2u$. It is elliptic only for functions $u \in C^2(\Omega)$ that are convex in Ω and, for this reason, from now on we consider only convex functions. Moreover, it can be written in the following divergence form

$$\det D^2u = \frac{1}{n} \big(S^{ij}\big(D^2u\big) u_j \big)_i, \tag{2.5}$$

where $S^{ij} = \frac{\partial}{\partial u_{ij}} (\det D^2u)$ is the cofactor of u_{ij}, and the following pointwise identity holds (see for instance [12,18])

$$k_{\{u \leqslant t\}} = \frac{S^{ij}(D^2u) u_i u_j}{|Du|^{n+1}} \bigg|_{\{u=t\}}. \tag{2.6}$$

It is well-known (see [7,22]) that, if $\Omega \subset \mathbb{R}^n$ is a smooth, bounded, strictly convex, open set, there exists a positive constant (depending only on Ω), denoted by $\sigma_1(\Omega)$, satisfying the following properties:

(i) there exists $\psi_1 \in C^{1,1}(\overline{\Omega}) \cap C^\infty(\Omega)$ such that $\psi_1 < 0$ in Ω and

$$\det D^2\psi_1 = \sigma_1(\Omega)(-\psi_1)^n \quad \text{in } \Omega, \qquad \psi_1 \text{ convex in } \overline{\Omega}, \qquad \psi_1 = 0 \text{ on } \partial\Omega; \tag{2.7}$$

(ii) if $(\psi,\mu)\in(C^{1,1}(\overline{\Omega})\cap C^{\infty}(\Omega))\times\,]0,+\infty[$ satisfies (2.7) with (ψ_1,σ_1) replaced by (ψ,μ), then $\mu=\sigma_1$ and $\psi=\theta\psi_1$ for some positive constant θ;

(iii) if $\Omega_1\subset\Omega_2$, then $\sigma_1(\Omega_1)\geqslant\sigma_1(\Omega_2)$.

All these features of σ_1 suggest the well-known properties of the first eigenvalue of a linear second order elliptic operator; this is why σ_1 is known as the first (and unique) eigenvalue of the Monge–Ampère operator. It is possible to characterize σ_1 in a variational way by

$$\sigma_1(\Omega)=\inf\left\{\frac{\int_\Omega(-u)\det D^2u\,dx}{\int_\Omega(-u)^{n+1}\,dx}\colon u\in C^2(\Omega),\ u\text{ convex in }\Omega,\ u=0\text{ on }\partial\Omega\right\}. \tag{2.8}$$

Finally we recall that a Pohožaev type identity holds true for the Monge–Ampère equation (see [21]).

Proposition 2.1. *Let $f\in C^1(\mathbb{R})$ be a nonnegative function and let $F(u)=\int_u^0 f(s)\,ds$. If u is a smooth convex solution to the problem*

$$\begin{cases}\det D^2u=f(u) & \text{in }\Omega,\\ u=0 & \text{on }\partial\Omega\end{cases}$$

in a bounded, convex, open set $\Omega\subset\mathbb{R}^n$ with smooth boundary, then

$$-\frac{n}{n+1}\int_\Omega(-u)\det D^2u\,dx+\frac{1}{n+1}\int_{\partial\Omega}k_\Omega\langle x,\nu\rangle|Du|^{n+1}\,d\mathcal{H}^{n-1}(x)=n\int_\Omega F(u)\,dx, \tag{2.9}$$

where ν is the exterior normal unit vector to $\partial\Omega$ at the point x.

We end this section by recalling some definitions and basic properties about rearrangements of functions. For an exhaustive treatment of these topics we refer the reader for instance to [6,16]. Let $G\subset\mathbb{R}^n$ be an open, bounded set and let $u:G\to\,]-\infty,0]$ be a measurable function. We define the distribution function μ_u of u as follows

$$\mu_u(t)=\left|\left\{x\in G\colon u(x)<t\right\}\right|,\quad t\leqslant 0.$$

μ_u is an increasing function; its generalized inverse function is called the increasing rearrangement $u^*$of u:

$$u^*(s)=\sup\left\{t\leqslant 0\colon \mu(t)<s\right\},\quad s\in\left[0,|G|\right].$$

The spherically symmetric increasing rearrangement of u is defined by

$$u^\sharp(x)=u^*\left(\omega_n|x|^n\right),\quad x\in G^\sharp,$$

where $G^\sharp$ is the ball centered at the origin, having the same measure as G. We explicitly observe that u and $u^\sharp$ are equidistributed, i.e. they have the same distribution function. This implies that

$$\|u\|_{L^p(G)}=\left\|u^\sharp\right\|_{L^p(G^\sharp)},\quad 1\leqslant p\leqslant+\infty.$$

The following proposition can be found in [1].

Proposition 2.2. *Let u and v be two nonpositive measurable functions defined on G such that*

$$\int_0^s\left(-u^*(r)\right)dr\geqslant\int_0^s\left(-v^*(r)\right)dr,\quad s\in\left[0,|G|\right].$$

Then for every $1\leqslant p\leqslant+\infty$

$$\|u\|_{L^p(G)}\geqslant\|v\|_{L^p(G)}.$$

Finally the following Pólya–Szegö principle holds.

Proposition 2.3. *Let $u \in W_0^{1,p}(G)$, $1 \leqslant p < +\infty$, be a nonpositive function. Then $u^\sharp \in W_0^{1,p}(G^\sharp)$ and*

$$\int_G |Du|^p\,dx \geqslant \int_{G^\sharp} \left|Du^\sharp\right|^p dx. \tag{2.10}$$

3. Main results

We begin by stating and proving the following "desymmetrization" lemma.

Lemma 3.1. *Let $K, L \subset \mathbb{R}^n$ be two C^2 bounded, convex, open sets with the same measure and let $w \in C^2(K)$ be a convex function, vanishing on ∂K. Then there exists a function w_L satisfying the following properties:*

(1) *w_L is a $C^2(L)$ convex function, vanishing on ∂L;*
(2) *w_L is equidistributed with w;*
(3) *w_L has level sets homothetic to L, all of them having the same Santalò point.*

Proof. Let s be the Santalò point of L; we want to show that the desired function is

$$w_L(x) = \sup\left\{t \leqslant 0\colon\ x - s \in \left(\frac{\mu_w(t)}{|L|}\right)^{1/n} L\right\}.$$

It is easy to see that, when $\tau \leqslant 0$, it holds $\{x \in L\colon\ w_L(x) \leqslant \tau\} = (\mu_w(\tau)/|L|)^{1/n} L$; then $|\{x \in L\colon\ w_L(x) \leqslant \tau\}| = \mu_w(\tau)$, that means w_L is equidistributed with w. Moreover, by definition, $w_L = 0$ on ∂L.

Since w is convex, $(1-\alpha)\{w < \tau_1\} + \alpha\{w < \tau_2\} \subseteq \{w < (1-\alpha)\tau_1 + \alpha\tau_2\}$. Then, by Brunn–Minkowski inequality,

$$\mu_w\big((1-\alpha)\tau_1 + \alpha\tau_2\big)^{1/n} \geqslant (1-\alpha)\mu_w(\tau_1)^{1/n} + \alpha\mu_w(\tau_2)^{1/n}.$$

This implies that w_L is convex. As a consequence of all these properties of w_L we have

$$\frac{1}{|Dw_L(x)|} = \langle x - s, \nu\rangle \frac{\mu_w'(w_L(x))}{\mu_w(w_L(x))n}, \tag{3.1}$$

where ν is the exterior normal unit vector to the level set of w_L passing through x. Using the regularity of w, a direct computation yields $w_L \in C^2(L)$. $\square$

The desymmetrization procedure yields the following reverse Pólya–Szegö type principle.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded, strictly convex, open set and let $\mathcal{E}(\Omega)$ be an ellipsoid having the same measure as Ω. Let $w \in C^2(\mathcal{E}(\Omega))$ be a negative convex function having elliptical symmetry and vanishing on the boundary (i.e. its sublevel sets $\{w \leqslant t\}$, for every $t \leqslant 0$, are ellipsoids concentric and homothetic to $\mathcal{E}(\Omega)$) and let w_Ω be the function given in Lemma 3.1 (with the choice $K = \mathcal{E}(\Omega)$ and $L = \Omega$). Then*

$$\int_\Omega (-w_\Omega)\det D^2 w_\Omega\,dx \leqslant \int_{\mathcal{E}(\Omega)} (-w)\det D^2 w\,dx, \tag{3.2}$$

equality holding if and only if Ω is an ellipsoid.

Proof. Without loss of generality we can suppose that Ω has 0 as Santalò point. By (3.1), (2.4) and (2.1) we get

$$\begin{aligned}\int_{\{w_\Omega = t\}} k_{\{w_\Omega \leqslant t\}} |Dw_\Omega|^n\,d\mathcal{H}^{n-1}(x) &= \frac{n^n \mu_w(t)^n}{(\mu_w'(t))^n} \int_{\{w_\Omega = t\}} \frac{k_{\{w_\Omega \leqslant t\}}}{(\langle x, \nu\rangle)^n}\,d\mathcal{H}^{n-1}(x)\\ &= \frac{n^n \mu_w(t)^n}{(\mu_w'(t))^n} \int_{S^{n-1}} \frac{1}{(h(\{w_\Omega \leqslant t\}, \xi))^n}\,d\mathcal{H}^{n-1}(\xi)\end{aligned}$$

$$
\begin{aligned}
&= \frac{n^n \mu_w(t)^n}{(\mu_w'(t))^n} \int_{S^{n-1}} \left(\rho\left(\{w_\Omega \leqslant t\}^*, \xi\right)\right)^n \, d\mathcal{H}^{n-1}(\xi) \\
&= \frac{n^{n+1} \mu_w(t)^n}{(\mu_w'(t))^n} \left|\{w_\Omega \leqslant t\}^*\right| \\
&\leqslant \omega_n^2 \frac{n^{n+1} \mu_w(t)^n}{(\mu_w'(t))^n} \left|\{w_\Omega \leqslant t\}\right|^{-1} \\
&= \omega_n^2 \frac{n^{n+1} \mu_w(t)^{n-1}}{(\mu_w'(t))^n} = \int_{\{w=t\}} k_{\{w \leqslant t\}} |Dw|^n \, d\mathcal{H}^{n-1}(x).
\end{aligned} \tag{3.3}
$$

By integrating on $\mathbb{R}^-$, recalling that $w_\Omega = 0$ on $\partial\Omega$ and using the co-area formula we get

$$
\begin{aligned}
\int_\Omega (-w_\Omega) \det\left(D^2 w_\Omega\right) dx &= \frac{1}{n} \int_{-\infty}^{0} dt \left(\int_{\{w_\Omega = t\}} k_{\{w_\Omega \leqslant t\}} |Dw_\Omega|^n \, d\mathcal{H}^{n-1}(x) \right) \\
&\leqslant \frac{1}{n} \int_{-\infty}^{0} dt \left(\int_{\{w=t\}} k_{\{w \leqslant t\}} |Dw|^n \, d\mathcal{H}^{n-1}(x) \right) \\
&= \int_{\mathcal{E}(\Omega)} (-w) \det\left(D^2 w\right) dx.
\end{aligned}
$$

If equality holds in (3.2), then it also holds in (3.3), thus in the Blaschke–Santalò inequality. This means that Ω is an ellipsoid. □

Now we are able to prove a reverse Faber–Krahn inequality for the Monge–Ampère eigenvalue and a Saint-Venant style principle.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded, strictly convex, open set and let $\mathcal{E}(\Omega)$ be an ellipsoid having the same measure as Ω. Then*:

(1) $$\sigma_1(\Omega) \leqslant \sigma_1\left(\mathcal{E}(\Omega)\right), \tag{3.4}$$
equality holding if and only if Ω is an ellipsoid;

(2) *the solution u to the problem*

$$
\begin{cases}
\det D^2 u = 1 & \textit{in } \Omega, \\
u = 0 & \textit{on } \partial\Omega, \\
u \textit{ convex in } \Omega,
\end{cases} \tag{3.5}
$$

satisfies

$$
\int_\Omega (-u)\, dx \geqslant \frac{|\Omega|^{\frac{n+2}{n}}}{(n+2)\omega_n^{\frac{2}{n}}}, \tag{3.6}
$$

equality holding if and only if Ω is an ellipsoid.

Proof. (1) Let v be a negative eigenfunction corresponding to $\sigma_1(\mathcal{E}(\Omega))$, that means v is a solution to the following problem

$$
\begin{cases}
\det D^2 v = \sigma_1(\mathcal{E}(\Omega))(-v)^n & \text{in } \mathcal{E}(\Omega), \\
v = 0 & \text{on } \partial\mathcal{E}(\Omega), \\
v \text{ convex in } \overline{\mathcal{E}(\Omega)}.
\end{cases} \tag{3.7}
$$

We explicitly observe that v has elliptical symmetry, since the Monge–Ampère operator is invariant under affine transformations and an eigenfunction in a ball is radially symmetric (see [4]). Now suppose that Ω has 0 as Santalò point; by Lemma 3.1 applied with the choice $K = \mathcal{E}(\Omega)$, $L = \Omega$ and $w = v$ we construct the convex C^2 function v_Ω, equidistributed with v, vanishing on $\partial\Omega$, having level sets homothetic to Ω with 0 as Santalò point. From the variational formulation of the eigenvalue (2.8) we deduce

$$\sigma_1\big(\mathcal{E}(\Omega)\big) = \frac{\int_{\mathcal{E}(\Omega)}(-v)\det D^2 v\,dx}{\int_{\mathcal{E}(\Omega)}(-v)^{n+1}\,dx} \geqslant \frac{\int_\Omega(-v_\Omega)\det D^2 v_\Omega\,dx}{\int_\Omega(-v_\Omega)^{n+1}\,dx} \geqslant \min_{\substack{u\in C^2(\bar\Omega)\\ u \text{ convex in } \Omega\\ u=0 \text{ on } \partial\Omega}} \frac{\int_\Omega(-u)\det D^2 u\,dx}{\int_\Omega(-u)^{n+1}\,dx} = \sigma_1(\Omega).$$

If equality holds in (3.4), then equality holds in (3.3), hence Ω is an ellipsoid.

(2) Let u be the unique solution to (3.5); following [20] we get

$$F_\Omega(u) = \frac{\int_\Omega(-u)\det D^2 u\,dx}{(\int_\Omega(-u)\,dx)^{n+1}} = \min_{\substack{z\in C^2(\Omega)\\ z \text{ convex in } \Omega\\ z=0 \text{ on } \partial\Omega}} \frac{\int_\Omega(-z)\det D^2 z\,dx}{(\int_\Omega(-z)\,dx)^{n+1}}.$$

Thus, if v is the solution to the following problem

$$\begin{cases} \det D^2 v = 1 & \text{in } \mathcal{E}(\Omega),\\ v = 0 & \text{on } \partial\mathcal{E}(\Omega),\\ v \text{ convex in } \mathcal{E}(\Omega), \end{cases} \tag{3.8}$$

arguing as before we get

$$\begin{aligned} F_{\mathcal{E}(\Omega)}(v) &= \frac{\int_\Omega(-v)\det D^2 v\,dx}{(\int_\Omega(-v)\,dx)^{n+1}} = \min_{\substack{w\in C^2(\mathcal{E}(\Omega))\\ w \text{ convex in } \mathcal{E}(\Omega)\\ w=0 \text{ on } \partial\mathcal{E}(\Omega)}} F_{\mathcal{E}(\Omega)}(w)\\ &\geqslant \frac{\int_\Omega(-v_\Omega)\det D^2 v_\Omega\,dx}{(\int_\Omega(-v_\Omega)\,dx)^{n+1}} \geqslant \min_{\substack{z\in C^2(\Omega)\\ z \text{ convex in } \Omega\\ z=0 \text{ on } \partial\Omega}} F_\Omega(z) = F_\Omega(u). \end{aligned}$$

From the fact that u and v solve (3.5) and (3.8), respectively, we deduce

$$\int_\Omega(-u)\,dx \geqslant \int_{\mathcal{E}(\Omega)}(-v)\,dx = \frac{|\Omega|^{\frac{n+2}{n}}}{(n+2)\omega_n^{\frac{2}{n}}}. \qquad \square$$

A second proof of (2). By Pohožaev identity (2.1) it holds

$$\int_\Omega(-u)\,dx = \frac{1}{n(n+2)}\int_{\partial\Omega} k_\Omega\langle x,\nu\rangle|Du|^{n+1}\,d\mathcal{H}^{n-1}(x). \tag{3.9}$$

Thus, by integrating the equation $\det D^2 u = 1$ over Ω, by using (2.5), (2.6), Hölder inequality and (3.9), we get

$$\begin{aligned} |\Omega| &= \frac{1}{n}\int_{\partial\Omega} k_\Omega|Du|^n\,d\mathcal{H}^{n-1}(x)\\ &\leqslant \frac{1}{n}\bigg(\int_{\partial\Omega}\frac{k_\Omega}{\langle x,\nu\rangle^n}\,d\mathcal{H}^{n-1}(x)\bigg)^{\frac{1}{n+1}}\bigg(\int_{\partial\Omega} k_\Omega\langle x,\nu\rangle|Du|^{n+1}\,d\mathcal{H}^{n-1}(x)\bigg)^{\frac{n}{n+1}}\\ &= \frac{1}{n}\big(n|\Omega^*|\big)^{\frac{1}{n+1}}\bigg(n(n+2)\int_\Omega(-u)\,dx\bigg)^{\frac{n}{n+1}}. \end{aligned}$$

The thesis immediately follows via Blaschke–Santalò inequality. $\square$

A third proof of (2). By Pohožaev identity (2.1) and (2.4) we have

$$\int_{\Omega}(-u)\,dx=\frac{1}{n(n+2)}\int_{\partial\Omega}k_{\Omega}\langle x,\nu\rangle|Du|^{n+1}\,d\mathcal{H}^{n-1}(x)$$
$$=\frac{1}{n(n+2)}\int_{S^{n-1}}h_{\Omega}|Du|^{n+1}\,d\mathcal{H}^{n-1}(\xi).$$

Now let us construct a convex set E such that $k_E(\nu^{-1}(\xi))=|Du|^{-(n+1)}(\nu^{-1}(\xi))$. We observe that $|Du|$ is the radial function of the starshaped set $Du(\Omega)$ and that $\int_{S^{n-1}}|Du|^{n+1}(\nu^{-1}(\xi))\xi\,d\mathcal{H}^{n-1}(\xi)=0$; then, by Minkowski existence theorem (see [13, p. 392]) there exists a convex set E, known as curvature image of $Du(\Omega)$, whose curvature function coincides with $|Du|^{n+1}$. By definition of mixed volume and (2.3) we have

$$\begin{aligned}\int_{\Omega}(-u)\,dx&=\frac{1}{n(n+2)}\int_{S^{n-1}}\frac{h_{\Omega}(\xi)}{k_E(\xi)}\,d\mathcal{H}^{n-1}(\xi)\\&=\frac{1}{n+2}V(\Omega,E,\dots,E)\\&\geqslant\frac{1}{n+2}|E|^{\frac{n-1}{n}}|\Omega|^{\frac{1}{n}}.\end{aligned}\tag{3.10}$$

On the other hand, by Petty inequality (2.2) we obtain

$$\begin{aligned}n|\Omega|=n\int_{\Omega}\det D^2u\,dx=\int_{\partial\Omega}k_{\Omega}|Du|^n\,d\mathcal{H}^{n-1}(x)=\int_{S^{n-1}}k_E^{-\frac{n}{n+1}}\,d\mathcal{H}^{n-1}(\xi)\\\leqslant n\omega_n^{\frac{2}{n+1}}|E|^{\frac{n-1}{n+1}}\end{aligned}$$

that is

$$|E|\geqslant\omega_n^{-\frac{2}{n-1}}|\Omega|^{\frac{n+1}{n-1}}.\tag{3.11}$$

Combining (3.10) and (3.11) we get the claim. □

Corollary 3.1. *Under the assumptions of Theorem 3.2, statement (2), the L^p-norm of u is minimal whenever Ω is an ellipsoid, for every $1\leqslant p\leqslant+\infty$.*

Proof. Estimate (3.6) can be rewritten as

$$\int_0^{|\Omega|}\big(-u^*(r)\big)\,dr\geqslant\int_0^{|\Omega|}\big(-v^*(r)\big)\,dr.$$

As a matter of fact (3.6) is valid on every sublevel set of u, that is

$$\int_{\{u<u^*(s)\}}(-u)\,dx\geqslant\int_{\{v<v^*(s)\}}(-v)\,dx,\quad s\in\big(0,|\Omega|\big),$$

or, equivalently,

$$\int_0^s\big(-u^*(r)\big)\,dr+su^*(s)\geqslant\int_0^s\big(-v^*(r)\big)\,dr+sv^*(s),\quad s\in\big(0,|\Omega|\big).\tag{3.12}$$

Let $W(s)=\int_0^s(v^*(r)-u^*(r))\,dr$; from (3.12) we can deduce

$$W(s)\geqslant sW'(s),\quad s\in\big(0,|\Omega|\big);\qquad W\big(|\Omega|\big)\geqslant 0;\qquad W(0)=0.$$

We get a contradiction if we suppose that $W(s)$ admits a negative minimum in $(0, |\Omega|)$. Then $W(s) \geqslant 0$ in the whole $(0, |\Omega|)$, that means

$$\int_0^s \left(-u^*(r)\right) dr \geqslant \int_0^s \left(-v^*(r)\right) dr. \tag{3.13}$$

This implies, via Proposition 2.2, that

$$\|u\|_{L^p(\Omega)} \geqslant \|v\|_{L^p(\mathcal{E}(\Omega))}, \quad 1 \leqslant p \leqslant +\infty. \qquad \square \tag{3.14}$$

Corollary 3.2. *Under the assumptions of Theorem* 3.2*, statement* (2)*, $\|u\|_{W_0^{1,p}(\Omega)}$ is minimal whenever Ω is a ball, for every $1 \leqslant p < +\infty$.*

Proof. Since

$$\int_\Omega |Du|^p \, dx = \int_\Omega |Du|^p \det D^2 u \, dx = \frac{1}{n} \int_{\partial\Omega} k_\Omega |Du|^{n+p} \, d\mathcal{H}^{n-1}(x) - \frac{p}{n} \int_\Omega |Du|^p \, dx$$

and

$$|\Omega| \leqslant \frac{1}{n} \left(\int_{\partial\Omega} k_\Omega |Du|^{n+p} \, d\mathcal{H}^{n-1}(x) \right)^{\frac{n}{n+p}} \left(\int_{\partial\Omega} k_\Omega \, d\mathcal{H}^{n-1}(x) \right)^{\frac{p}{n+p}},$$

by Gauss–Bonnet theorem we conclude

$$\int_\Omega |Du|^p \, dx \geqslant \frac{n}{(n+p)\omega_n^{\frac{p}{n}}} |\Omega|^{\frac{n+p}{n}}. \qquad \square$$

We end this section with our last application of the desymmetrization procedure to prove that a Poincaré constant is maximal on balls. In [19] the authors prove the following Poincaré type inequality for Hessian integrals.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a strictly convex, open set of class C^∞; then there exists a constant C, depending on n and Ω, such that*

$$C \left(\int_\Omega |Du|^2 \right)^{\frac{1}{2}} \leqslant \left(\int_\Omega (-u) \det D^2 u \right)^{\frac{1}{n+1}} \tag{3.15}$$

for all convex functions $u \in C^2(\overline{\Omega})$, vanishing on $\partial\Omega$. Moreover, the solution to the following Dirichlet problem

$$\begin{cases} \det D^2 u = \Delta u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \text{ convex in } \Omega \end{cases} \tag{3.16}$$

attains the best constant C_Ω, i.e.

$$\begin{aligned} C_\Omega &= \min \left\{ \frac{\int_\Omega (-w) \det D^2 w \, dx}{(\int_\Omega |Dw|^2 \, dx)^{\frac{n+1}{2}}} : \ w \in C^2(\overline{\Omega}), w \text{ convex in } \Omega, \ w = 0 \text{ on } \partial\Omega \right\} \\ &= \frac{\int_\Omega (-u) \det D^2 u \, dx}{(\int_\Omega |Du|^2 \, dx)^{\frac{n+1}{2}}}. \end{aligned} \tag{3.17}$$

We prove that the constant C_Ω in (3.17) satisfies the following isoperimetric inequality.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a strictly convex, open set of class C^∞; then $C_\Omega \leqslant C_{\Omega^\sharp}$, equality holding if and only if Ω is a ball.*

Proof. Let v be the solution to the symmetrized problem

$$\begin{cases} \det D^2 v = \Delta v & \text{in } \Omega^\sharp, \\ v = 0 & \text{on } \partial\Omega^\sharp, \\ v \text{ convex in } \Omega^\sharp. \end{cases}$$

Suppose that Ω has 0 as Santalò point; by Lemma 3.1 applied with the choice $K = \Omega^\sharp$, $L = \Omega$ and $w = v$ we construct the convex C^2 function v_Ω, equidistributed with v, vanishing on $\partial\Omega$, having level sets homothetic to Ω with 0 as Santalò point. Recalling Pólya–Szegö inequality (2.10) and observing that $(v_\Omega)^\sharp = v$, we obtain

$$\begin{aligned} C_{\Omega^\sharp} = \frac{\int_{\Omega^\sharp}(-v)\det D^2 v\,dx}{(\int_{\Omega^\sharp}|Dv|^2\,dx)^{\frac{n+1}{2}}} &\geqslant \frac{\int_{\Omega}(-v_\Omega)\det D^2 v_\Omega\,dx}{(\int_{\Omega}|Dv_\Omega|^2\,dx)^{\frac{n+1}{2}}} \\ &\geqslant \min_{\substack{z\in C^2(\overline{\Omega}) \\ z \text{ convex in } \Omega \\ z=0 \text{ on } \partial\Omega}} \frac{\int_{\Omega}(-z)\det D^2 z\,dx}{(\int_{\Omega}|Dz|^2\,dx)^{\frac{n+1}{2}}} \\ &= \frac{\int_{\Omega}(-u)\det D^2 u\,dx}{(\int_{\Omega}|Du|^2\,dx)^{\frac{n+1}{2}}} = C_\Omega. \end{aligned}$$

Finally a direct computation yields

$$C_\Omega \leqslant C_{\Omega^\sharp} = \frac{(n+2)^{\frac{n-1}{2}}\omega_n^{\frac{n-1}{n}}}{n^{\frac{n+1}{2}}|\Omega|^{\frac{(n+2)(n-1)}{2n}}}. \qquad \square$$

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