Homogenization of periodic semilinear parabolic degenerate PDEs

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Abstract

In this paper a second order semilinear parabolic PDE with rapidly oscillating coefficients is homogenized. The novelty of our result lies in the fact that we allow the second order part of the differential operator to be degenerate in some part of $\mathbf{R}^d$.

Our fully probabilistic method is based on the deep connection between PDEs and BSDEs and the weak convergence of a class of diffusion processes.

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1. Introduction

Our goal is to study by a probabilistic approach the homogenization property of a second order semilinear parabolic PDE with periodic coefficients. Namely, we deal with the semilinear parabolic PDE with Cauchy type condition

$$\begin{cases} \forall (t, x) \in \mathbf{R}_+ \times \mathbf{R}^d, \\ \partial_t u^\varepsilon(t, x) = L_\varepsilon u^\varepsilon(t, x) + \dfrac{1}{\varepsilon} e\left(\dfrac{x}{\varepsilon}, u^\varepsilon(t, x)\right) + f\left(\dfrac{x}{\varepsilon}, x, u^\varepsilon(t, x), \partial_x u^\varepsilon(t, x)\sigma\left(\dfrac{x}{\varepsilon}\right)\right), \\ u^\varepsilon(0, x) = g(x), \quad x \in \mathbf{R}^d. \end{cases} \tag{1.1}$$

The second order differential operator with rapidly oscillating coefficients L_ε is given by

$$L_\varepsilon(\cdot) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}\left(\frac{x}{\varepsilon}\right) \partial^2_{x_i x_j} + \sum_{i=1}^{d} \left[\frac{1}{\varepsilon} b_i\left(\frac{x}{\varepsilon}\right) + c_i\left(\frac{x}{\varepsilon}\right) \right] \partial_{x_i} \tag{1.2}$$

where a, b, c are periodic functions ($a = \sigma\sigma^*$ for some periodic function σ).

After the pioneer work of Freidlin [12] which is also presented in Chapter 3 of Bensoussan et al. [1], it is well known that a linear parabolic PDE can be homogenized by probabilistic arguments based on the Feynman–Kac formula, the

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ergodic theorem and the central limit theorem. Using the deep connection between backward stochastic differential equations and semilinear PDEs, several authors studied the extension of this approach to the case of non-linear equations with periodic coefficients and highly oscillating potential. The first scheme based on stability of BSDEs and a regularization procedure was developed by Buckdahn, Hu and Peng [3]. Briand and Hu [2] exploited this method and homogenized a system of semilinear elliptic PDEs using the stochastic representation of the solutions of such systems by BSDEs with random terminal time. The second way was initiated by Pardoux [19], who used weak convergence techniques. The results and the formulation of the limiting equation involve the solution u of the Poisson equation $Lu + f = 0$, where L is the infinitesimal generator of a Markov process on the d-dimensional torus induced by a nonrescaled version of (1.2). Pardoux and Veretennikov [24], using essentially probabilistic tools and some estimates from PDE theory, solved this Poisson equation for an elliptic and ergodic diffusion and provided some rather sharp estimates of the solution. This strong result has been extensively used for the study of the homogenization property of non-linear equations by means of probabilistic tools. For example Lejay [17] has treated the case of divergence form operators whereas Delarue [7], coupling this latter scheme with an efficiently controlled regularization procedure, has dealt with the case of quasilinear PDEs.

In all these results, a key assumption is the uniform non-degeneracy (also called uniform ellipticity) of the diffusion matrix a, that is $\lambda^{-1} \operatorname{Id} \leqslant a(x) \leqslant \lambda \operatorname{Id}$ for some strictly positive constant λ and any $x \in \mathbf{R}^d$. It implies irreducibility of the above Markov process and smoothness of the solution of the corresponding Poisson equation. More recently, some authors have been interested in weakening this non-degeneracy assumption, in other words in allowing the matrix a to vanish along some directions. Roughly speaking, the first idea was to investigate the case when a remains uniformly elliptic but the value of λ becomes very large (see for instance Heron and Mossino [14] on this topic). Afterwards, in a series of papers, De Arcangelis and Serra Cassano [5], Paronetto and Serra Cassano [26] and Paronetto [27,28] have investigated the periodic homogenization of a class of divergence form degenerate linear equations. Loosely speaking, the diffusion coefficient is controlled by the identity matrix $\lambda^{-1}(x) \operatorname{Id} \leqslant a(x) \leqslant \lambda(x) \operatorname{Id}$ where the scalar function λ satisfies a so-called Muckenhoupt condition, that is λ verifies suitable integrability conditions together with its inverse. In a similar spirit, Huang et al. [15] have considered non-linear equations with periodic coefficients and Engström et al. [10] have investigated homogenization of nonlinear random operators. However, the Muckenhoupt condition is rather close to the non-degenerate case. From the mathematical angle, the developed techniques are similar to the non-degenerate case (compactness methods based on Sobolev's type inequalities in appropriate weighted spaces). From the modelling angle, the geometry of the degeneracies of the matrix a are restrictive in the sense that, first, a may degenerate only on a subset of null Lebesgue measure and, second, when it does (at $x \in \mathbf{R}^d$), the matrix $a(x)$ can be nothing but the null matrix 0.

Thereafter, Rhodes [29,30] and Delarue and Rhodes [8] have worked under apparently minimal assumptions for the homogenization property to hold in the case of symmetric divergence form operators, respectively for linear and quasilinear random PDEs. Intuitively, their assumption on the matrix a could be expressed as follows (in the case of periodic coefficients): if a periodic function φ satisfies $a(x)\partial_x\varphi(x) = 0$ for Lebesgue almost every x then it is constant. For instance, the Muckenhoupt condition implies such a relation. The authors also give examples where the matrix a is everywhere degenerate, but the rank of a must be greater than 1 over a set of full Lebesgue measure. However, such a condition does not allow the matrix a to reduce to 0 over an open domain. The reason is simple: such a condition only relies on the matrix a. But if a reduces to 0 over an open domain, say D, it is plain to see that the leading term in (1.2) is b over D (up to a scaling factor). To improve the considered degeneracies of a, it is now clear that appropriate assumptions must be made both on the diffusion coefficient a and the drift term b. This is the underlying idea of our main assumption (H1) on L_ε: roughly speaking, we assume that the space can be divided in two parts, a regularizing area U where a is non-degenerate enough (i.e. a satisfies the strong Hörmander condition, see Definition 2.1), and its complementary U^c where a may degenerate (and even reduce to 0) but the drift term b compensates for the lack of non-degeneracy of a (mathematically speaking, we assume that $\forall x \in U^c$, $P_{t_0}^\varepsilon(\mathbf{1}_U)(x) > 0$ where P^ε is the semigroup associated to (1.2) and t_0 is a fixed time). This idea was first developed for linear parabolic PDEs by Hairer and Pardoux [13], to which the reader is referred for several illustrating examples (Section 7). The reader may wonder which comparison could be made between [8,29,30] and [13]. It turns out that these approaches are basically different and examples satisfying one condition but not the other one can be constructed and conversely.

The aim of the present paper is to extend the work [13] to semilinear PDEs. Unlike [3] or [2], the limiting equation may be degenerate so that it requires careful attention. Moreover, this difficulty is coupled with the oscillations of the

non-linear term $\frac{1}{\varepsilon}e(\frac{x}{\varepsilon}, u^\varepsilon(t, x))$ in (1.2) (e is not bounded with respect to u_ε). This raises the difficulty of controlling the gradient $\partial_x u_\varepsilon$.

The paper is organized as follows. Section 2 recalls the results obtained in the linear case. Our main assumptions and results are stated in Section 3. Section 4 is devoted to the main proofs.

2. Diffusions with periodic coefficients

In all what follows, we assume given a complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbf{P})$, where the filtration $(\mathcal{F}_t)_{t \geqslant 0}$ is generated by a d-dimensional Brownian motion $(B_t)_{t \geqslant 0}$, and the continuous functions

$$b, c : \mathbf{R}^d \longrightarrow \mathbf{R}^d, \qquad \sigma : \mathbf{R}^d \longrightarrow \mathbf{R}^d \times \mathbf{R}^d,$$

which are periodic of period 1 in each direction of $\mathbf{R}^d$. Given $\varepsilon > 0$ and $x \in \mathbf{R}^d$, let $\{X_s^{x, \varepsilon}\}_{s \geqslant 0}$ (which will be mostly written $(X_s^\varepsilon)_{s \geqslant 0}$) denote the solution of the stochastic differential equation

$$\forall t \geqslant 0, \quad X_t^\varepsilon = x + \int_0^t \left(\frac{1}{\varepsilon} b\left(\frac{X_s^\varepsilon}{\varepsilon} \right) + c\left(\frac{X_s^\varepsilon}{\varepsilon} \right) \right) ds + \sum_{j=1}^d \int_0^t \sigma_j\left(\frac{X_s^\varepsilon}{\varepsilon} \right) dB_s^j \tag{2.1}$$

and

$$L_\varepsilon(\cdot) = \frac{1}{2} \sum_{i, j=1}^d a_{ij}\left(\frac{x}{\varepsilon} \right) \partial_{x_i x_j}^2 + \sum_{i=1}^d \left[\frac{1}{\varepsilon} b_i\left(\frac{x}{\varepsilon} \right) + c_i\left(\frac{x}{\varepsilon} \right) \right] \partial_{x_i} \tag{2.2}$$

its infinitesimal generator, where $a = \sigma \sigma^*$. Considering the processes $(\tilde{X}_t^\varepsilon)_{t \geqslant 0}$ and $(\bar{X}_t^\varepsilon)_{t \geqslant 0}$ defined by

$$\forall t \geqslant 0, \quad \tilde{X}_t^\varepsilon = \frac{1}{\varepsilon} X_{\varepsilon^2 t}^\varepsilon; \qquad \bar{X}_t^\varepsilon = \frac{X_t^\varepsilon}{\varepsilon} = \tilde{X}_{t/\varepsilon^2}^\varepsilon,$$

then there exists a standard d-dimensional Brownian motion $(B_t)_{t \geqslant 0}$ depending on ε (in fact for $0 \leqslant s \leqslant t$, $B_s^\varepsilon = \frac{1}{\varepsilon} B_{\varepsilon^2 s}$ and we forget that dependence since it has no incidence on the law of the process), such that

$$\forall t \geqslant 0, \quad \tilde{X}_t^\varepsilon = \frac{x}{\varepsilon} + \int_0^t \left(b\left(\tilde{X}_s^\varepsilon \right) + \varepsilon c\left(\tilde{X}_s^\varepsilon \right) \right) ds + \sum_{j=1}^d \int_0^t \sigma_j\left(\tilde{X}_s^\varepsilon \right) dB_s^j. \tag{2.3}$$

We consider the Markov process $(\tilde{X}_t^\varepsilon)_{t \geqslant 0}$ solution of (2.3) as taking values in the d dimensional torus $\mathbf{T^d} = \mathbf{R}^d / \mathbf{Z}^d$ and $p^\varepsilon(t, x, A)$ its transition probability. We shall write $p(t, x, A)$ for $p_0(t, x, A)$. We will also consider the same equation starting from x but without the term εc, namely

$$\forall t \geqslant 0, \quad \tilde{X}_t^x = x + \int_0^t b\left(\tilde{X}_s^x \right) ds + \sum_{j=1}^d \int_0^t \sigma_j\left(\tilde{X}_s^x \right) dB_s^j \tag{2.4}$$

and $(J_t^x)_{t \geqslant 0}$ the Jacobian of the stochastic flow associated to $(\tilde{X}_t^x)_{t \geqslant 0}$, that is the $d \times d$ matrix valued stochastic process solving

$$d J_t^x = Db\left(\tilde{X}_t^x \right) J_t^x dt + \sum_{j=1}^d D\sigma_j\left(\tilde{X}_t^x \right) J_t^x dB_t^j, \quad J_0^x = I. \tag{2.5}$$

Moreover to the stochastic differential equation satisfied by $(\tilde{X}_t^x)_{t \geqslant 0}$, having in mind Stroock–Varadhan's support theorem, we associate the controlled ODE (where we use the convention of summation over repeated indices). For each $x \in \mathbf{T^d}$, $u \in L_{\text{loc}}^2(\mathbf{R}_+, \mathbf{R}^d)$, let $(z_u^{x, \varepsilon}(t), t \geqslant 0)$ denote the solution of

$$\begin{cases} \dfrac{dz_i}{dt}(t) = (b_i + \varepsilon c_i)\left(z(t) \right) - \dfrac{1}{2}(\partial_{x_k} \sigma_{ij} \sigma_{kj})\left(z(t) \right) + \sigma_{ij}\left(z(t) \right) u_j(t); \\ z(0) = x. \end{cases} \tag{2.6}$$

2.1. Assumptions and preliminary result

Let us recall the following

Definition 2.1. Let us denote by σ_j $(1 \leqslant j \leqslant d)$ the column vectors of σ. We will say that the strong Hörmander condition holds at some point $x \in \mathbf{T^d}$ if the Lie algebra generated by $\{\sigma_j\}_{1 \leqslant j \leqslant d}$ spans the whole space $\mathbf{R}^d$ at $x \in \mathbf{T^d}$.

We furthermore say that the parabolic Hörmander condition holds at $x \in \mathbf{T^d}$, if the Lie algebra generated by the $(d+1)$-dimensional vectors $(b, 1) \cup \{(\sigma_j, 0)\}_{1 \leqslant j \leqslant d}$ spans the whole space of $\mathbf{R}^{d+1}$ at $x \in \mathbf{T^d}$.

We say that the drift and the diffusion coefficients satisfy assumptions (H1) if the following holds (the same as in [13])

(H1.1) σ, b and c are of class C^∞ and periodic of period one in each direction.

(H1.2) There exists a non-empty, open and connected subset U of $\mathbf{T^d}$ on which the strong Hörmander conditions holds. Furthermore, there exists t_0 and ε_0 such that

$$\forall x \in \mathbf{T^d}, \ 0 \leqslant \varepsilon \leqslant \varepsilon_0, \qquad \inf_{u \in L^2(0, t_0, \mathbf{R}^d)} \left\{ \|u\|_{L^2}; \, z_u^{x, \varepsilon}(t_0) \in U \right\} < \infty. \tag{2.7}$$

(H1.3) The following holds

$$\inf_{t > 0} \sup_{x \in \mathbf{T^d}} \mathbf{E}\left(\left| J_t^x \right|, \left\{ \tau_V^x \geqslant t \right\} \right) < 1,$$

where V denotes the subset of $\mathbf{T^d}$ where the parabolic Hörmander condition holds, τ_V^x is the first hitting time of V by the process $\{\tilde{X}_t^x\}$.

Put in probabilistic words, (2.7) means that a particle X^ε driven by SDE (2.1) located at $x \in U^c$ at time $t = 0$ has a reasonable probability to reach U before the time t_0, namely that $\mathbf{P}_x(X_{t_0}^\varepsilon \in U) > 0$. Assumption (H1.3) ensures the semigroup associated to X^ε is regularizing enough.

Remark 2.1. Here is the simplest example of a situation where our assumptions are satisfied, with a degenerating matrix of diffusion coefficients. Let $\lambda \in C^\infty(\mathbf{T^d}, [0, 1])$ be such that $\{x, \ \lambda(x) > 0\}$ is connected and not empty. Let $U = V = \{x, \ \lambda(x) > 0\}$. For $x \in \mathbf{T^d} \setminus U$, let

$$t(x) := \inf\left\{ s > 0, \ z_s^x \in U, \ \text{where} \ \frac{dz_s^x}{ds} = b\left(z_s^x\right), \ z_0^x = x \right\}.$$

Let $a(x) = \lambda(x)I$, where I denotes the $d \times d$ identity matrix. Provided the smooth vector field $\{b(x), \ x \in \mathbf{T^d}\}$ is such that $\sup_{x \in \mathbf{T^d} \setminus U} t(x) < \infty$, the assumptions (H1.1), (H1.2) and (H1.3) are satisfied. Several precise examples of such coefficients $\{a(x), b(x), \ x \in \mathbf{T^d}\}$ are given in [13], which also satisfy the assumption (H1.4) below.

It is not difficult to verify that under (H1.1) and (H1.2) the following Doeblin condition is satisfied: there exists $t_1 > 0$, $0 < \varepsilon_1 < \varepsilon_0$, $\beta > 0$ and ν a probability measure on $\mathbf{T^d}$ which is absolutely continuous with respect to the Lebesgue measure, s.t. for all $0 < \varepsilon < \varepsilon_1$, $x \in \mathbf{T^d}$, A a Borel subset of $\mathbf{T^d}$,

$$p_\varepsilon(t_1; x, A) \geqslant \beta \nu(A).$$

This ensures existence and uniqueness of a unique invariant measure μ_ε of $(\tilde{X}_t^\varepsilon)_{t \geqslant 0}$ (let us denote $\mu = \mu_0$) and the following facts (see [13]).

Lemma 2.2 *(The spectral gap). There exists $\rho > 0$ such that for all $0 \leqslant \varepsilon \leqslant 1$, $t > 0$ and $f \in L^\infty(\mathbf{T^d})$,*

$$\left| \mathbf{E}\left[f\left(\tilde{X}_t^\varepsilon\right) \right] - \int_{\mathbf{T^d}} f(x) \mu_\varepsilon(dx) \right| \leqslant \|f\|_\infty e^{-\rho t}.$$

Lemma 2.3. *The following holds*

$$\mu_\varepsilon \xrightarrow{\varepsilon \to 0} \mu, \quad weakly.$$

We finally assume that

(H1.4) The crucial centering condition is satisfied: $\int_{\mathbf{T^d}} b(x)\mu(dx) = 0$.

2.2. The Poisson equation

Let us consider the infinitesimal generator L of the $\mathbf{T^d}$-valued diffusion process $(\tilde{X}^x)_{t \geqslant 0}$ given by

$$L = \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^*)_{ij}(x)\partial^2_{x_i x_j} + \sum_{i=1}^{d} b_i(x)\partial_{x_i} \tag{2.8}$$

and $\mathcal{P}_t$ the semigroup generated by $(\tilde{X}^x)_{t \geqslant 0}$.

For a function $f \in C^1(\mathbf{T^d})$ satisfying the centering condition

$$\int_{\mathbf{T^d}} f(x)\mu(dx) = 0, \tag{2.9}$$

we want to solve the PDE

$$L\hat{f}(x) + f(x) = 0, \quad x \in \mathbf{T^d}, \tag{2.10}$$

in order to get rid of the terms depending on ε^{-1} in the perturbed equations. For this purpose we recall the following result given in [13, Lemma 2.6] which will be useful in the sequel:

Lemma 2.4. *Under* (H1), $\mathcal{P}_t$ *maps* $C^1(\mathbf{T^d})$ *into itself and there exists two positive constants* $K > 0$ *and* $\rho > 0$ *such that for every* $f \in C^1(\mathbf{T^d})$ *satisfying* (2.9) *and for every* $t \geqslant 0$, *we have*

$$\|\mathcal{P}_t f\|_{C^1(\mathbf{T^d})} \leqslant K e^{-\rho t} \|f\|_{C^1(\mathbf{T^d})}. \tag{2.11}$$

It follows from Lemma 2.4 the

Lemma 2.5. *Under assumption* (H1) *if* $f \in C^1(\mathbf{T^d})$ *satisfies* (2.9), *then the function* $\hat{f}$ *defined by*

$$\hat{f}(x) = \int_0^{+\infty} \mathbf{E}_x\big[f(\tilde{X}_t)\big]\,dt, \quad x \in \mathbf{T^d},$$

belongs to $C^1(\mathbf{T^d})$ *and is the unique weak sense solution of Eq.* (2.10) *which is centered with respect to* μ.

(For the notion of weak sense solution to (2.10), see [25].)

3. Homogenization of a semilinear parabolic PDE

For each $\varepsilon > 0$, we consider the PDE with Cauchy type condition

$$\begin{cases} \forall (t,x) \in \mathbf{R}_+ \times \mathbf{R}^d, \\ \partial_t u^\varepsilon(t,x) = L_\varepsilon u^\varepsilon(t,x) + \frac{1}{\varepsilon} e\left(\frac{x}{\varepsilon}, u^\varepsilon(t,x)\right) + f\left(\frac{x}{\varepsilon}, x, u^\varepsilon(t,x), \partial_x u^\varepsilon(t,x)\sigma\left(\frac{x}{\varepsilon}\right)\right), \\ u^\varepsilon(0,x) = g(x), \quad x \in \mathbf{R}^d \end{cases} \tag{3.1}$$

where g belongs to $C(\mathbf{R}^d, \mathbf{R})$ and the measurable functions $f : \mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$, $e : \mathbf{R}^d \times \mathbf{R} \to \mathbf{R}$ satisfy the following assumptions (H2) (in what follows, keep in mind that y and z respectively stand for u^ε and $\partial_x u^\varepsilon$):

(H2.1) e and f are periodic of period 1 in each direction in the first argument and continuous.

(H2.2) e is twice continuously differentiable in y uniformly with respect to x and moreover there exists $c > 0$ such that $\forall y \in \mathbf{R}$,

(i) $e(\cdot, y)$, $\partial_y e(\cdot, y)$ and $\partial^2_{yy} e(\cdot, y)$ belong to $C^1(\mathbf{T^d})$.

(ii) $(1 + |y|)^{-1} \| e(\cdot, y) \|_{C^1(\mathbf{T^d})} + \| \partial_y e(\cdot, y) \|_{C^1(\mathbf{T^d})} + (1 + |y|) \| \partial^2_{yy} e(\cdot, y) \|_\infty \leqslant c$.

(H2.3) The following centering condition holds

$$\forall y \in \mathbf{R}, \quad \int_{\mathbf{T^d}} e(x, y) \mu(dx) = 0. \tag{3.2}$$

(H2.4) There exists $K' > 0$ such that for $x \in \mathbf{T^d}$, $(\tilde{x}, \tilde{x}') \in (\mathbf{R}^d)^2$, $(y, y') \in \mathbf{R}^2$ and $(z, z') \in (\mathbf{R}^d)^2$,

$$\big| g(\tilde{x}) \big| + \big| f(x, \tilde{x}, y, z) \big| \leqslant K'\big(1 + |y| + |z|\big),$$

$$\big| g(\tilde{x}) - g(\tilde{x}') \big| + \big| f(x, \tilde{x}, y, z) - f(x, \tilde{x}', y', z') \big| \leqslant K'\big(|\tilde{x} - \tilde{x}'| + |y - y'| + |z - z'|\big).$$

Remark 3.1. We first stress that the centering condition (H2.3) is classical (see [7,9,19] for instance). Moreover, our standing assumption on f with respect to $\tilde{x}$ and g can be weaken as follows. We may only assume continuity and sublinear growth. So in this case the homogenization property can be established under slight modifications. To prove existence of a unique bounded and continuous viscosity solution of the limit PDE (3.10), f and g must be at least locally Lipschitz in $\tilde{x}$.

Our assumption on e has the advantage of allowing e to grow linearly in y as $|y| \to \infty$. However, the assumption on the second derivative is rather restrictive. The arguments from [19] could be adapted here. They allow to treat a function e which is the sum of a linear function of y, and an element of $C^2_b(\mathbf{R})$, whose coefficients depend upon x.

Under the previous assumptions, for any fixed $y \in \mathbf{R}$ and $i = 1, \ldots, d$, we can consider the solutions of the following Poisson equations on the torus $\mathbf{T^d}$:

$$L\hat{e}(\cdot, y) + e(\cdot, y) = 0, \quad \text{and} \quad L\hat{b}_i(\cdot) + b_i(\cdot) = 0, \tag{3.3}$$

given for any $(x, y) \in \mathbf{T^d} \times \mathbf{R}$ by

$$\hat{e}(x, y) = \int_0^{+\infty} \mathbf{E}_x\big[e(\tilde{X}_t, y)\big] dt, \quad \text{and} \quad \hat{b}_i(x) = \int_0^{+\infty} \mathbf{E}_x\big[b_i(\tilde{X}_t)\big] dt. \tag{3.4}$$

Then we have (the proof is given in Section 4):

Proposition 3.1. *The functions $\hat{b}_i(\cdot)$ ($1 \leqslant i \leqslant d$) and $\hat{e}(\cdot, y)$ ($y \in \mathbf{R}$) belong to $C^1(\mathbf{T^d})$. Furthermore, for each $x \in \mathbf{T^d}$, the mapping $y \in \mathbf{R} \mapsto \hat{e}(x, y)$ is twice continuously differentiable and the derivatives are solutions of the following Poisson equations*

$$L\partial_y \hat{e}(\cdot, y) = -\partial_y e(\cdot, y); \qquad L\partial^2_{yy} \hat{e}(\cdot, y) = -\partial^2_{yy} e(\cdot, y).$$

Furthermore, $\partial_y \hat{e}(\cdot, y)$, $\partial^2_{yy} \hat{e}(\cdot, y)$ belong to $C^1(\mathbf{T^d})$ for any $y \in \mathbf{R}$ and there exists a constant c', only depending on K, ρ and c such that $\forall(x, y) \in \mathbf{T^d} \times \mathbf{R}$,

$$\forall y \in \mathbf{R}, \quad \big(1 + |y|\big)^{-1} \big\| \hat{e}(\cdot, y) \big\|_{C^1(\mathbf{T^d})} + \big\| \partial_y \hat{e}(\cdot, y) \big\|_{C^1(\mathbf{T^d})} + \big(1 + |y|\big) \big\| \partial^2_{yy} \hat{e}(\cdot, y) \big\|_\infty \leqslant c'. \tag{3.5}$$

We now aim at describing the limit PDE. Let us consider the following functions defined for every $(x, \tilde{x}, y, z) \in \mathbf{T^d} \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^d$ by

$$\Lambda(x) = (I + \partial_x \hat{b})(x) \sigma(x),$$

$$F(x, y) = (I + \partial_x \hat{b})(x)\big(c + a(x)\partial^2_{xy} \hat{e}(x, y)\big),$$

$$U_1(x, y) = \big\langle \partial_x \hat{e}(x, y), c \big\rangle - \partial_y \hat{e}(x, y) e(x, y) + \partial^2_{xy} \hat{e}^*(x, y) a(x) \partial_x \hat{e}(x, y),$$

$$U(x, \tilde{x}, y, z) = U_1(x, y) + f\big(x, \tilde{x}, y, z + \partial_x \hat{e}(x, y)\sigma(x)\big).$$

We should point out that there exists a positive constant $C > 0$ such that F and U satisfy for all $x \in \mathbf{T^d}$, $(\tilde{x}, \tilde{x}') \in (\mathbf{R}^d)^2$, $(y, y') \in \mathbf{R}^2$ and $(z, z') \in (\mathbf{R}^d)^2$

$$|U(x, \tilde{x}, y, z)| \leqslant C(1 + |y| + |z|),$$
$$|F(x, y) - F(x, y')| + |U(x, \tilde{x}, y, z) - U(x, \tilde{x}', y', z')| \leqslant C(|\tilde{x} - \tilde{x}'| + |y - y'| + |z - z'|). \tag{3.6}$$

We can then identify the coefficients of the limit PDE given for all $(\tilde{x}, y, z) \in \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^d$, by

$$A = \int_{\mathbf{T^d}} (\Lambda \Lambda^*)(x) \mu(dx), \tag{3.7}$$

$$\bar{F}(y) = \int_{\mathbf{T^d}} F(x, y) \mu(dx) \equiv D_1 + \bar{D}(y), \quad \text{where } D_1 = \int_{\mathbf{T^d}} (I + \partial_x \hat{b}) c(x) \mu(dx),$$

$$\bar{U}(\tilde{x}, y, z) = \int_{\mathbf{T^d}} U(x, \tilde{x}, y, \Lambda(x)z) \mu(dx), \tag{3.8}$$

and the second order operator

$$\bar{L}(\cdot) = \frac{1}{2} \sum_{i,j=1}^d A_{ij} \partial^2_{x_i x_j} + \sum_{i=1}^d \bar{F}_i(\cdot) \partial_{x_i}. \tag{3.9}$$

Then the equation satisfied by the limit of the solution of (3.1) can be formulated as

$$\begin{cases} \partial_t u(t, x) = \bar{L} u(t, x) + \bar{U}(x, u(t, x), \partial_x u(t, x)), & (t, x) \in [0, T] \times \mathbf{R}^d, \\ u(0, x) = g(x), & x \in \mathbf{R}^d. \end{cases} \tag{3.10}$$

We are in position to formulate our main result.

Theorem 3.2. *For all $t \geqslant 0$, $x \in \mathbf{R}^d$,*

$$u^\varepsilon(t, x) \xrightarrow{\varepsilon \longrightarrow 0} u(t, x) \quad pointwise,$$

where u^ε is the viscosity solution of (3.1) *and u the viscosity solution of* (3.10).

Because of the degeneracy allowed on the diffusion matrix, it is not obvious that the limit PDE is solvable under our standing assumptions. In the following section we discuss existence, uniqueness and regularity of the solution u of (3.10).

3.1. Analysis of the limit PDE

In what follows, we want to prove that this PDE admits a (unique) solution in some sense and that this solution can be approximated by a sequence of smooth functions, given by a regularization of the PDE (3.10). Namely, let us consider two smooth mollifiers $\rho : \mathbf{R}^d \to \mathbf{R}$ and $\varrho : \mathbf{R} \to \mathbf{R}$ and define, for $n \geqslant 1$, $\rho_n(\cdot) = n^d \rho(n \cdot)$ and $\varrho_n(\cdot) = n \varrho(n \cdot)$. The regularized coefficients are defined for any triple $(x, y, z) \in \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^d$ by

$$g^n(x) = (g * \rho_n)(x), \qquad \bar{D}^n(y) = (\bar{D} * \varrho_n)(y), \quad \text{and} \quad \bar{U}^n = [\bar{U} * (\rho_n \otimes \varrho_n \otimes \rho_n)](x, y, z),$$

where $*$ stands for the standard convolution operator and $(\rho_n \otimes \varrho_n \otimes \rho_n)(x, y, z) = \rho_n(x) \varrho_n(y) \rho_n(z)$. In what follows, $\bar{D}^0$ and $\bar{U}^0$ stand respectively for $\bar{D}$ and $\bar{U}$. Standard arguments of convolution techniques ensure that, for $n \geqslant 0$, with a constant still noted C that do not depend on $n \geqslant 0$, for every $(x, x') \in (\mathbf{R}^d)^2$, $(y, y') \in \mathbf{R}^2$ and $(z, z') \in (\mathbf{R}^d)^2$,

$$|\bar{D}^n(y)| + |\bar{U}^n(x, y, z)| + |g^n(x)| \leqslant C(1 + |y| + |z|),$$
$$|\bar{D}^n(y) - \bar{D}^n(y')| + |\bar{U}^n(x, y, z) - \bar{U}^n(x', y', z')| + |g^n(x) - g^n(x')|$$
$$\leqslant C(|x - x'| + |y - y'| + |z - z'|). \tag{3.11}$$

We can then consider the following regularized problem on $[0, T] \times \mathbf{R}^d$:

$$\begin{cases} \partial_t u^n(t, x) = \text{Trace}[A \partial^2_{xx} u^n](t, x) + \bar{D}^n(u^n(t, x)) \cdot \partial_x u^n(t, x) + D_1 \cdot \partial_x u^n(t, x) \\ \qquad\qquad + \bar{U}^n(x, u^n(t, x), \partial_x u^n(t, x)), \\ u^n(0, x) = g^n(x). \end{cases} \tag{3.12}$$

We shall prove

Theorem 3.3. *Assume that* (H1) *and* (H2) *are in force. Then the PDE* (3.10) *admits a unique bounded continuous viscosity solution u. Moreover, for every $n \geqslant 1$, there exists a unique classical solution $u^n \in C^{1,2}([0, T] \times \mathbf{R}^d; \mathbf{R})$ of PDE* (3.12) *satisfying*:

(i) *There exists a constant $C_{3.3}$ independent of n such that*

$$\forall (t, x) \in \mathbf{R}_+ \times \mathbf{R}^d, \quad |u^n(t, x)| + \|A^{1/2}\partial_x u_n(t, x)\| \leqslant C_{3.3}.$$

(ii) *There exists two constants $C_{3.3}^{(n)}$ and $\gamma(n)$ only depending on T, n and C such that*

$$\forall 1 \leqslant i, j \leqslant d, \ \forall (t, x) \in [0, T] \times \mathbf{R}^d, \quad |\partial_{x_i} u^n(t, x)| + |\partial_{x_i x_j}^2 u^n(t, x)| \leqslant C_{3.3}^{(n)}(1 + |x|)^{\gamma(n)}.$$

(iii) *u^n $(n \geqslant 1)$ converges pointwise towards u as n tends to infinity.*

Proof. Let us first say a word about the structure of the degeneracies of the coefficients. Note that, for a vector $X \in \mathbf{R}^d$, if $X \in \mathrm{Ker}(A)$ then $\bar{D}(y) \cdot X = 0$ and $\bar{U}(x, y, X) = \bar{U}(x, y, 0)$. Indeed, if $AX = 0$ then $\Lambda^*(x)X = 0$ for μ almost every $x \in \mathbf{T}^d$ (see (3.7)). It is then clear that $X \cdot \bar{D}(y) = \int_{\mathbf{T}^d} \partial_{xy}^2 \hat{e}^*(x, y)\sigma(x)\Lambda^*(x)X \, d\mu(x) = 0$. The same argument remains valid for $\bar{U}$. We can then express the matrix A as $A = M \mathrm{Diag}[\lambda_1, \dots, \lambda_r, 0, \dots, 0]M^*$, for r reals $\lambda_1, \dots, \lambda_r$ different from 0 and for an orthogonal matrix M (hence $r = \mathrm{Dim}(\mathrm{Im}(A))$), and define

$$\bar{D}_A(y) = \bar{D}(y)B, \qquad \bar{U}_A(x, y, z) = \bar{U}(x, y, zB)$$

where $B = M \mathrm{Diag}[\lambda_1^{-1/2}, \dots, \lambda_r^{-1/2}, 0, \dots, 0]M^*$. It is then readily seen that

$$\bar{D}_A(y)A^{1/2} = \bar{D}(y) \quad \text{and} \quad \bar{U}_A(x, y, zA^{1/2}) = \bar{U}(x, y, z).$$

Similarly, for $n \geqslant 1$, we can define $\bar{D}_A^n(y) = \bar{D}^n(y)B$, $\bar{U}_A^n(x, y, z) = \bar{U}^n(x, y, zB)$ and check that

$$\bar{D}_A^n(y)A^{1/2} = \bar{D}^n(y) \quad \text{and} \quad \bar{U}_A^n(x, y, zA^{1/2}) = \bar{U}^n(x, y, z).$$

Sticking with the spirit of the previous notations, $\bar{D}_A^0(y)$ and $\bar{U}_A^0$ respectively denote $\bar{D}_A(y)$ and $\bar{U}_A(y)$. With these notations, PDE (3.12) then reads

$$\begin{cases} \partial_t u^n(t, x) = \mathrm{Trace}[A\partial_{xx}^2 u^n](t, x) + \bar{D}_A^n(u^n(t, x)) \cdot A^{1/2}\partial_x u^n(t, x) + D_1 \cdot \partial_x u^n(t, x) \\ \qquad\qquad + \bar{U}_A^n(x, u^n(t, x), A^{1/2}\partial_x u^n(t, x)), \\ u^n(0, x) = g^n(x). \end{cases} \tag{3.13}$$

The reader can easily check that $u^n(t, x)$ is a continuous viscosity solution (resp. classical solution) of (3.13) if and only if $v_n(t, x) = u_n(t, x - D_1 t)$ is a continuous viscosity solution (resp. classical solution) of the PDE

$$\begin{cases} \partial_t u^n(t, x) = \mathrm{Trace}[A\partial_{xx}^2 u^n](t, x) + \bar{D}_A^n(u^n(t, x)) \cdot A^{1/2}\partial_x u^n(t, x) \\ \qquad\qquad + \bar{U}_A^n(x, u^n(t, x), A^{1/2}\partial_x u^n(t, x)), \\ u^n(0, x) = g^n(x). \end{cases} \tag{3.14}$$

For the definition of viscosity solution, the reader is referred to [4]. The main advantage of factorizing the coefficients $\bar{D}$ and $\bar{U}$ by $A^{1/2}$ is that we can now make use of the theory of BSDEs to solve (3.14) by means of the BSDE:

$$\begin{cases} X_s^x = x + A^{1/2}B_s, \quad x \in \mathbf{R}^d, \\ Y_s^{x,n} = g^n(X_t^x) + \int\limits_s^t [\bar{U}_A^n(X_r^x, Y_r^{x,n}, Z_r^{x,n}) + \bar{D}_A^n(Y_r^{x,n})Z_r^{x,n}] \, dr - \int\limits_s^t Z_r^{x,n} \, dB_r. \end{cases} \tag{3.15}$$

However, to solve this BSDE, we are faced with the term $\bar{D}_A^n(y)z$, which need not be Lipschitzian as required by the classical theory. To overcome this difficulty, we want to make use of the non-degeneracy of A along its image $\mathrm{Im}(A)$. Decompose the whole space $\mathbf{R}^d$ as the orthogonal sum $\mathbf{R}^d = \mathrm{Ker}(A) \oplus \mathrm{Im}(A)$ so that a vector $x \in \mathbf{R}^d$ can be

written as $x = x_K + x_I$ where $(x_K, x_I) \in \mathrm{Ker}(A) \times \mathrm{Im}(A)$. Fix $x_K \in \mathrm{Ker}(A)$. Following [6, Section 3], we can define $(Y^{x_I, x_K, n}, Z^{x_I, x_K, n})$ as the unique pair of processes solution of the BSDE

$$
\begin{cases}
X_s^{x_I} = x_I + A^{1/2} B_s, \quad x_I \in \mathrm{Im}(A), \\[2mm]
Y_s^{x_I, x_K, n} = g^n(X_t^{x_I} + x_K) + \displaystyle\int_s^t \bar{U}_A^n(X_r^{x_I} + x_K, Y_r^{x_I, x_K, n}, Z_r^{x_I, x_K, n}) \, dr \\[4mm]
\qquad\qquad + \displaystyle\int_s^t \bar{D}_A^n(Y_r^{x_I, x_K, n}) Z_r^{x_I, x_K, n} \, dr - \int_s^t Z_r^{x_I, x_K, n} \, dB_r.
\end{cases}
\tag{3.16}
$$

It is then easily checked that, for each $x = x_K + x_I \in \mathbf{R}^d$, the triple $(x_K + X^{x_I}, Y^{x_I, x_K, n}, Z^{x_I, x_K, n})$ solves the BSDE (3.15). Conversely, for each solution $(X^x, Y^{x,n}, Z^{x,n})$ of (3.15), then the triple of processes $(X^{x_I}, Y^{x_I, x_K, n}, Z^{x_I, x_K, n})$ is the unique solution of (3.16). As a consequence, (3.15) is uniquely solvable for $n \geqslant 0$. Furthermore (see [6, Theorem 3.1]), there exists a constant $\Gamma > 0$, which only depends on $d, \check{K}, T, A$, such that

$$
|Z_r^{x,n}| \leqslant \Gamma, \quad d\mathbf{P} \otimes dt \text{ a.e.}
$$

Considering a bounded smooth $h_\Gamma : \mathbf{R}^d \to \mathbf{R}^d$ such that $h_\Gamma(z) = z$ if $|z| \leqslant \Gamma + 1$ and $|h_\Gamma(z)| \leqslant |z|$ for $z \in \mathbf{R}^d$, the triple $(X^x, Y^{x,n}, Z^{x,n})$ $(n \geqslant 0)$ coincides with the unique solution of the following BSDE with standard Lipschitz assumption on the coefficients

$$
\begin{cases}
X_s^x = x + A^{1/2} B_s, \quad x \in \mathbf{R}^d, \\[2mm]
Y_s^{x,R,n} = g^n(X_t^x) + \displaystyle\int_s^t \left[\bar{U}_A^n(X_r^x, Y_r^{x,R,n}, Z_r^{x,R,n}) + \bar{D}_A^n(Y_r^{x,R,n}) h_R(Z_r^{x,R,n}) \right] dr - \int_s^t Z_r^{x,R,n} \, dB_r.
\end{cases}
\tag{3.17}
$$

Consequently, for each $n \geqslant 1$, the function $u^n(t, x) \equiv Y_0^{x,n} = Y_0^{x,R,n} \in C^{1,2}([0, T] \times \mathbf{R}^d; \mathbf{R})$ is the unique classical solution to (3.14) (see [22]). For $n = 0$, $u(t, x) \equiv Y_0^{x,0} = Y_0^{x,R,0}$ is a continuous bounded viscosity solution of (3.14) (see [21, Theorem 2.4 and Section 6.4]). Furthermore, for each $0 \leqslant t \leqslant T$ and $x \in \mathbf{R}^d$, $u^n(t, x) \to u(t, x)$ as n tends to ∞ (this follows from [21, Theorems 1.5, 2.4]. The fact that $|u^n(t, x)| \leqslant C_{3.3}$ for some constant $C_{3.3}$ independent of n is a consequence of [21, Proposition 1.1] and (3.11). Estimates of the derivatives up to order 2 of $\partial_{x_i} u^n$ and $\partial_{x_i x_j}^2 u^n$ are quite classical and can be established by iterating the scheme of the proof of [22, Theorem 2.9] (see also [6, Appendix B] for a more general framework).

Let us now tackle the uniqueness of the viscosity solution of (3.14) ($n = 0$). We already know (see [6, Theorem 3.1]) that u is bounded, continuous and Lipschitzian with respect to the variable $x_I \in \mathrm{Im}(A)$, namely

$$
\forall (x_I, x_I', x_K, t) \in \mathrm{Im}(A)^2 \times \mathrm{Ker}(A) \times [0, T], \quad |u(t, x_I + x_K) - u(t, x_I' + x_K)| \leqslant \Gamma |x_I - x_I'|.
$$

These properties are sufficient to ensure uniqueness among the viscosity solutions of (3.18) below that are continuous and bounded. Indeed, if v is such a solution then, for each fixed $x_K \in \mathrm{Ker}(A)$, the functions $x_I \mapsto u(t, x_I + x_K)$ and $x_I \mapsto v(t, x_I + x_K)$ are both viscosity solutions of the following PDE defined on $\mathrm{Im}(A)$:

$$
\begin{cases}
\partial_t u(t, x_I) = \mathrm{Trace}[A \partial_{x_I x_I}^2 u](t, x_I) + \bar{D}_A(u(t, x_I)) \cdot A^{1/2} \partial_{x_I} u(t, x_I) \\[2mm]
\qquad\qquad + \bar{U}_A(x_I + x_K, u(t, x_I), A^{1/2} \partial_{x_I} u(t, x_I)), \quad x_I \in \mathrm{Im}(A), \\[2mm]
u^n(0, x_I) = g^n(x_I + x_K), \quad x_I \in \mathrm{Im}(A),
\end{cases}
\tag{3.18}
$$

and satisfy the assumptions of [16, Theorem 3.2]. Hence they coincide. $\quad\square$

3.2. The homogenization property

Our approach is purely probabilistic and is based on BSDE techniques. The strategy consists in introducing the unique pair $(Y_s^\varepsilon, Z_s^\varepsilon)_{0 \leqslant s \leqslant t}$ of $\mathcal{F}_t$-progressively measurable processes solution of the BSDEs

$$
\forall 0 \leqslant s \leqslant t, \quad Y_s^\varepsilon = g(X_t^\varepsilon) + \int_s^t \left(\frac{1}{\varepsilon} e(\bar{X}_r^\varepsilon, Y_r^\varepsilon) + f(\bar{X}_r^\varepsilon, X_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon) \right) dr - \int_s^t Z_r^\varepsilon \, dB_r
\tag{3.19}
$$

satisfying the integrability condition

$$\mathbf{E}\left(\sup_{0\leqslant s\leqslant t}\left|Y_s^\varepsilon\right|^2+\int_0^t\left|Z_r^\varepsilon\right|^2dr\right)<\infty.$$

It is well known (see Pardoux [20]) that the solution of (3.1) admits the probabilistic representation

$$u^\varepsilon(t,x)=Y_0^\varepsilon,\quad\forall(t,x)\in\mathbf{R}_+\times\mathbf{R}^d$$

(of course $Y_.^\varepsilon$ depends on the starting point x of $X_.^\varepsilon$ and the final time t of the BSDE).

In order to get rid of the highly oscillating terms (depending on ε^{-1}), let us consider the following processes (recall that $\bar{X}_t^\varepsilon=\frac{X_t^\varepsilon}{\varepsilon}$) given by

$$\forall0\leqslant s\leqslant t,\quad\hat{X}_s^\varepsilon=X_s^\varepsilon+\varepsilon\left(\hat{b}\left(\bar{X}_s^\varepsilon\right)-\hat{b}\left(\frac{x}{\varepsilon}\right)\right),\qquad\hat{Y}_s^\varepsilon=Y_s^\varepsilon-\varepsilon\hat{e}\left(\bar{X}_s^\varepsilon,Y_s^\varepsilon\right).$$

Using Itô's formula (see Section 4.2), they both can be rewritten as

$$\forall0\leqslant s\leqslant t,\quad\hat{X}_s^\varepsilon=x+\int_0^s(I+\partial_x\hat{b})c\left(\bar{X}_r^\varepsilon\right)dr+\int_0^t\Lambda\left(\bar{X}_r^\varepsilon\right)dB_r,\tag{3.20}$$

$$\hat{Y}_s^\varepsilon=g\left(X_t^\varepsilon\right)-\varepsilon\hat{e}\left(\bar{X}_t^\varepsilon,Y_t^\varepsilon\right)+\int_s^t(U_1+f-\varepsilon\partial_y\hat{e}f)\left(\bar{X}_r^\varepsilon,X_r^\varepsilon,Y_r^\varepsilon,Z_r^\varepsilon\right)dr$$

$$-\int_s^t\tilde{Z}_r^\varepsilon\big[dB_r-(\sigma^*\partial_{xy}^2\hat{e})\left(\bar{X}_r^\varepsilon,Y_r^\varepsilon\right)dr\big]+\varepsilon\int_s^t\partial_y\hat{e}\left(\bar{X}_r^\varepsilon,Y_r^\varepsilon\right)Z_r^\varepsilon\,dB_r+\frac{\varepsilon}{2}\int_s^t\partial_{yy}^2\hat{e}\left(\bar{X}_r^\varepsilon,Y_r^\varepsilon\right)\left|Z_r^\varepsilon\right|^2dr\tag{3.21}$$

where for $0\leqslant s\leqslant t$,

$$\tilde{Z}_s^\varepsilon=Z_s^\varepsilon-\partial_x\hat{e}\left(\bar{X}_s^\varepsilon,Y_s^\varepsilon\right)\sigma\left(\bar{X}_s^\varepsilon\right).$$

By virtue of Girsanov's theorem, there exists a new probability $\tilde{\mathbf{P}}$ equivalent to $\mathbf{P}$ under which the process $(\tilde{B}_s)_{0\leqslant s\leqslant t}$ defined by

$$\forall0\leqslant s\leqslant t,\quad\tilde{B}_s=B_s-\int_0^s\left(\sigma^*\partial_{xy}^2\hat{e}\right)\left(\bar{X}_r^\varepsilon,Y_r^\varepsilon\right)dr\tag{3.22}$$

is a $\tilde{\mathbf{P}}$-Brownian motion. Then rewriting (3.20), we obtain for any $0\leqslant s\leqslant t$,

$$\hat{X}_s^\varepsilon=x+\int_0^sF\left(\bar{X}_r^\varepsilon,Y_r^\varepsilon\right)dr+\int_0^s\Lambda\left(\bar{X}_r^\varepsilon\right)d\tilde{B}_r.\tag{3.23}$$

and ($\Theta^\varepsilon(r)$ stands for $(\bar{X}_r^\varepsilon,X_r^\varepsilon,Y_r^\varepsilon,\tilde{Z}_r^\varepsilon)$)

$$\hat{Y}_s^\varepsilon=g\left(X_t^\varepsilon\right)-\varepsilon\hat{e}\left(\bar{X}_t^\varepsilon,Y_t^\varepsilon\right)+\int_s^tU\left(\Theta^\varepsilon(r)\right)dr-\int_s^t\tilde{Z}_r^\varepsilon\,d\tilde{B}_r+R_s^\varepsilon\tag{3.24}$$

where the process $R_s(\varepsilon)$ can be divided into two parts

$$R_s(\varepsilon) = \varepsilon \int\limits_s^t \left[\partial_y \hat{e}\big(\bar{X}_r^\varepsilon, Y_r^\varepsilon\big) Z_r^\varepsilon \sigma^* \partial_{xy}^2 \hat{e}\big(\bar{X}_r^\varepsilon, Y_r^\varepsilon\big) - \partial_y \hat{e}\big(\bar{X}_r^\varepsilon, Y_r^\varepsilon\big) f\big(\Theta^\varepsilon(r)\big) + \frac{1}{2} \partial_{yy}^2 \hat{e}\big(\bar{X}_r^\varepsilon, Y_r^\varepsilon\big) \big|Z_r^\varepsilon\big|^2 \right] dr$$

$$+ \varepsilon \int\limits_s^t \partial_y \hat{e}\big(\bar{X}_r^\varepsilon, Y_r^\varepsilon\big) Z_r^\varepsilon \, d\tilde{B}_r$$

$$= \int\limits_s^t R^\varepsilon(1, r) \, dr + \int\limits_s^t R^\varepsilon(2, r) \, d\tilde{B}_r.$$

Moreover let us consider the process

$$\forall 0 \leqslant s \leqslant t, \quad M_s^\varepsilon = -\int\limits_0^s \tilde{Z}_r^\varepsilon \, d\tilde{B}_r.$$

We intend to study the tightness property of the pair of processes $(Y_s^\varepsilon, M_s^\varepsilon)_{0 \leqslant s \leqslant t}$ indexed by $\varepsilon > 0$ in the space $\mathcal{D}([0, t]; \mathbf{R}^d)$ (the space of right continuous functions having left limits) equipped with the Meyer–Zheng topology (see [18] for further details).

It is well known that the sequence of quasi-martingales $\{U_s^n; 0 \leqslant s \leqslant t\}$ defined on the filtered probability space $\{\Omega; \mathcal{F}, (\mathcal{F}_s)_{0 \leqslant s \leqslant t}, \mathbf{P}\}$ is tight whenever

$$\sup_n \left[\sup_{0 \leqslant s \leqslant t} \mathbf{E}\big|U_s^n\big| + CV_t^0\big(U^n\big) \right] < \infty,$$

where $CV_t^0(U^n)$, the so-called "conditional variation of U^n on $[0, t]$", is defined as

$$CV_t^0\big(U^n\big) = \sup \mathbf{E}\left(\sum_{i=1} \big|\mathbf{E}\big(U_{t_{i+1}}^n - U_{t_i}^n / \mathcal{F}_{t_i}\big)\big| \right)$$

where the supremum is taken over all partitions of the interval $[0, t]$.

We claim that (the proof is given in Section 4.4).

Proposition 3.4. *There exists a positive constant $C_{3.4} > 0$ such that*

$$\forall \varepsilon > 0, \quad \tilde{\mathbf{P}}\Big(\sup_{0 \leqslant s \leqslant t} \big|Y_s^\varepsilon\big| \leqslant C_{3.4} \Big) = 1,$$

$$\sup_{\varepsilon > 0} \tilde{\mathbf{E}} \int\limits_0^t \big|\tilde{Z}_s^\varepsilon\big|^2 \, ds \leqslant C_{3.4}.$$

As a consequence, we deduce

Corollary 3.5. *For every $t \geqslant 0$, the following holds*

$$\lim_{\varepsilon \to 0} \tilde{\mathbf{E}}\left[\left(\int\limits_0^t \big|R_s(1, \varepsilon)\big| \, ds \right)^2 + \left(\int\limits_0^t \big|R_s(2, \varepsilon)\big|^2 \, ds \right)^2 \right] = 0.$$

In particular $\tilde{\mathbf{E}} \sup_{0 \leqslant s \leqslant t} |R_s(\varepsilon)|^2 \xrightarrow{\varepsilon \to 0} 0$.

Corollary 3.6. *The family of processes $(Y_\cdot^\varepsilon, M_\cdot^\varepsilon)$ indexed by ε is $\tilde{\mathbf{P}}$-tight as elements of $D([0, t], \mathbf{R}^2)$, equipped with the S-topology of Jakubowski.*

It is readily seen from (3.23), that the sequence of processes $\{X_s^\varepsilon, 0 \leqslant s \leqslant t, 0 \leqslant \varepsilon \leqslant 1\}$ is tight in the space $C([0, t], \mathbf{R}^d)$ endowed with the topology of uniform convergence. Moreover thanks to the martingale central limit theorem [11, Theorem 7.1.4], we have

$$\int_0^{\cdot} \Lambda\big(\bar{X}_s^\varepsilon\big)\, d\tilde{B}_s \quad \Longrightarrow \quad A^{1/2}\tilde{B}. \quad \text{in } C\big([0, T]; \mathbf{R}^d\big)$$

where $\Longrightarrow$ means "converges in law towards". Hence there exists a subsequence still denoted by $(X_s^\varepsilon, Y_s^\varepsilon, M_s^\varepsilon)$ such that

$$\big(X_s^\varepsilon, Y_s^\varepsilon, M_s^\varepsilon\big) \quad \Longrightarrow \quad \big(X^x, Y, M\big) \quad \text{in } D\big([0, T]; \mathbf{R}^{2d+1}\big).$$

Let us assume that the following extension of [13, Corollary 2.5] holds (the proof is given in Section 4.3),

Theorem 3.7. *Let* $\Psi : \mathbf{R}^d \times \mathbf{R}^N \to \mathbf{R}$ *be a measurable function, periodic with respect to its first variable, satisfying*:

(1) *for any $R > 0$, we can find $K_R > 0$ such that whenever $(x, v, v') \in \mathbf{R}^d \times \mathbf{R}^N \times \mathbf{R}^N$, $|v| \leqslant R$ and $|v'| \leqslant R$ then we have $|\Psi(x, v) - \Psi(x, v')| \leqslant K_R |v - v'|$;*
(2) *there exists $M > 0$ such that for any $x \in \mathbf{R}^d$, $v \in \mathbf{R}^N$, $|\Psi(x, v)| \leqslant M(1 + |v|)$.*

Suppose additionally that $(V^\varepsilon)_{\varepsilon>0}$ is a family of $\mathbf{R}^N$-valued processes, which is tight in $D([0, T]; \mathbf{R}^N)$ equipped with the S-topology of Jakubowski and satisfies $\sup_{\varepsilon>0} \mathbf{E}(\sup_{0 \leqslant s \leqslant t} |V_s^\varepsilon|^2) < \infty$.
Then the following convergence holds:

$$\mathbf{E}\left[\sup_{0 \leqslant s \leqslant t} \left| \int_s^t \Psi\big(\bar{X}_r^\varepsilon, V_r^\varepsilon\big)\, dr - \int_s^t \bar{\Psi}\big(V_r^\varepsilon\big)\, dr \right| \right] \to 0, \quad \text{as } \varepsilon \text{ tends to } 0, \tag{3.25}$$

where $\bar{\Psi}(v) = \int_{\mathbf{T}^d} \Psi(x, v)\mu(dx)$.

We can then apply Theorem 3.7 with the function $\Psi = F$ and $V^\varepsilon = Y^\varepsilon$, and deduce that $(X^x)_{t \geqslant 0}$ must solve the stochastic differential equation

$$\forall t \geqslant 0, \quad X_t^x = x + \int_0^t \bar{F}(Y_r)\, dr + A^{1/2}\tilde{B}_t.$$

Moreover thanks Corollary 3.5, the process $(Y_s^\varepsilon)_{0 \leqslant s \leqslant t}$ has the same asymptotic behaviour as the process $(\check{Y}_s^\varepsilon)_{0 \leqslant s \leqslant t}$ defined by

$$\forall 0 \leqslant s \leqslant t, \quad \check{Y}_s^\varepsilon = g\big(X_t^\varepsilon\big) + \int_s^t U\big(\Theta^\varepsilon(r)\big)\, dr - \int_s^t \tilde{Z}_r^\varepsilon\, d\tilde{B}_r. \tag{3.26}$$

From now on, our strategy consists in showing that the difference $\check{Y}_s^\varepsilon - u(t - s, \hat{X}_s^\varepsilon)$ tends to 0 as ε goes to 0. However, in the following computations, we are faced with the lack of smoothness of the function u. To overcome this difficulty, we approximate the function u with the help of the smooth approximating sequence $(u^n)_{n \in \mathbf{N}}$ defined in Theorem 3.3. Thus we consider, for every $n \in \mathbf{N}$, the pair of processes $(\tilde{Y}_s^{\varepsilon,n}, \tilde{Z}_s^{\varepsilon,n})_{0 \leqslant s \leqslant t}$ defined by

$$\forall 0 \leqslant s \leqslant t, \quad \tilde{Y}_s^{\varepsilon,n} = \check{Y}_s^\varepsilon - u^n\big(t - s, \hat{X}_s^\varepsilon\big), \qquad \tilde{Z}_s^{\varepsilon,n} = \tilde{Z}_s^\varepsilon - \partial_x u^n\big(t - s, \hat{X}_s^\varepsilon\big)\Lambda\big(\bar{X}_s^\varepsilon\big).$$

Then we claim

Theorem 3.8. *The following holds*

(i) *There exists a constant $C_{3.8} > 0$ such that for every $\varepsilon > 0$ and for every $n \in \mathbf{N}$, we have*
$$\big|\tilde{Y}_s^{\varepsilon,n}\big| \leqslant C_{3.8} \quad \text{a.s.}$$

(ii) *For all $\delta > 0$ there exists an integer $n(\delta)$ such that for all $n \geqslant n(\delta)$,*

$$\limsup_{\varepsilon \to 0} \big| \tilde{Y}_0^{\varepsilon,n} \big| \leqslant \delta. \tag{3.27}$$

Proof. Ito's formula yields for every $0 \leqslant s \leqslant t$ and $\alpha > 0$,

$$e^{\alpha s} |\check{Y}_s^\varepsilon|^2 + \int_s^t e^{\alpha r} |\tilde{Z}_r^\varepsilon|^2 \, dr = e^{\alpha t} |g(X_t^\varepsilon)|^2 + 2 \int_s^t e^{\alpha r} \check{Y}_r^\varepsilon U\big(\Theta^\varepsilon(r)\big) \, dr - \int_s^t \alpha e^{\alpha r} |\check{Y}_r^\varepsilon|^2 \, dr - 2 \int_s^t e^{\alpha r} \check{Y}_r^\varepsilon \tilde{Z}_r^\varepsilon \, d\tilde{B}_r.$$

Thanks to Proposition 3.4, and (3.6), there exists a constant still noted $C > 0$ (its value may change from line to line) s.t. for every $s \leqslant r \leqslant t$,

$$2 \check{Y}_r^\varepsilon U\big(\Theta^\varepsilon(r)\big) \leqslant C\big(1 + |\check{Y}_r^\varepsilon|^2\big) + \frac{1}{2} |\tilde{Z}_r^\varepsilon|^2.$$

Since g is bounded, this implies

$$e^{\alpha s} |\check{Y}_s^\varepsilon|^2 \leqslant C e^{\alpha t} + \int_s^t e^{\alpha r} (-\alpha + C) |\check{Y}_r^\varepsilon|^2 \, dr - 2 \int_s^t e^{\alpha r} \check{Y}_r^\varepsilon \tilde{Z}_r^\varepsilon \, d\tilde{B}_r.$$

Choosing $\alpha = C$ and taking the conditional expectation $\tilde{\mathbf{E}}^{\mathcal{F}_s}$, we deduce (i) from the boundedness of u^n.

Let us prove (3.27). Since $u^n \in C^{1,2}([0,T] \times \mathbf{R}^d)$, then Itô's formula yields for any $0 \leqslant s \leqslant t$,

$$u^n\big(t-s, \hat{X}_s^\varepsilon\big) = u^n\big(0, \hat{X}_t^\varepsilon\big) - \int_s^t \big(-\partial_r u^n\big(t-r, \hat{X}_r^\varepsilon\big) + \hat{L}_{\varepsilon,n}(r)\big) \, dr - \int_s^t \partial_x u^n\big(t-r, \hat{X}_r^\varepsilon\big) \Lambda\big(\bar{X}_r^\varepsilon\big) \, d\tilde{B}_r$$

where for every $0 \leqslant r \leqslant t$,

$$\hat{L}_{\varepsilon,n}(r) = \frac{1}{2} \sum_{i,j=1}^d \big[(\Lambda \Lambda^*)\big(\bar{X}_r^\varepsilon\big)\big]_{ij} \partial_{x_i x_j}^2 u^n\big(t-r, \hat{X}_r^\varepsilon\big) + \sum_{i=1}^d \big[F\big(\bar{X}_r^\varepsilon, Y_r^\varepsilon\big)\big]_i \partial_{x_i} u^n\big(t-r, \hat{X}_r^\varepsilon\big).$$

Hence putting

$$\forall (t,x) \in \mathbf{R}_+ \times \mathbf{R}^d, \quad L^n u^n(t,x) = \mathrm{Trace}\big[A \partial_{xx}^2 u^n(t,x)\big] + \bar{D}^n\big(u^n(t,x)\big) \cdot \partial_x u^n(t,x) + D_1 \cdot \partial_x u^n(t,x),$$

we deduce that for every $0 \leqslant s \leqslant t$,

$$u^n\big(t-s, \hat{X}_s^\varepsilon\big) = u^n\big(0, \hat{X}_t^\varepsilon\big) + \int_s^t \big(L^n u^n\big(t-r, \hat{X}_r^\varepsilon\big) - \hat{L}_{\varepsilon,n}(r)\big) \, dr - \int_s^t \partial_x u^n\big(t-r, \hat{X}_r^\varepsilon\big) \Lambda\big(\bar{X}_r^\varepsilon\big) \, d\tilde{B}_r$$

$$+ \int_s^t \bar{U}^{(n)}\big(\hat{X}_r^\varepsilon, u^n\big(t-r, \hat{X}_r^\varepsilon\big), \partial_x u^n\big(t-r, \hat{X}_r^\varepsilon\big)\big) \, dr$$

which implies that for every $0 \leqslant s \leqslant t$,

$$\tilde{Y}_s^{\varepsilon,n} = g(X_t^\varepsilon) - g^{(n)}\big(\hat{X}_t^\varepsilon\big) + \int_s^t \big(L^n u^n\big(t-r, \hat{X}_r^\varepsilon\big) - \hat{L}_{\varepsilon,n}(r)\big) \, dr$$

$$+ \int_s^t \big(U\big(\Theta^{\varepsilon,n}(r)\big) - \bar{U}^{(n)}\big(\hat{X}_r^\varepsilon, u^n\big(t-r, \hat{X}_r^\varepsilon\big), \partial_x u^n\big(t-r, \hat{X}_r^\varepsilon\big)\big)\big) \, dr - \int_s^t \tilde{Z}_r^{\varepsilon,n} \, d\tilde{B}_r$$

where $\Theta^{\varepsilon,n}(r) = (\bar{X}_r^\varepsilon, X_r^\varepsilon, \tilde{Y}_r^{\varepsilon,n} + u^n(t-r, \hat{X}_r^\varepsilon), \tilde{Z}_r^{\varepsilon,n} + \partial_x u^n(t-r, \hat{X}_r^\varepsilon) \Lambda(\bar{X}_r^\varepsilon))$. Itô's formula yields for any $0 \leqslant s \leqslant t$,

$$\left|\tilde{Y}_s^{\varepsilon,n}\right|^2 + \int\limits_s^t \left|\tilde{Z}_r^{\varepsilon,n}\right|^2 dr = \left|g\left(X_t^\varepsilon\right) - g^{(n)}\left(\hat{X}_t^\varepsilon\right)\right|^2 + 2\int\limits_s^t \tilde{Y}_r^{\varepsilon,n}\left(L^n u^n\left(t-r, \hat{X}_r^\varepsilon\right) - \hat{L}_{\varepsilon,n}(r)\right) dr$$

$$+ 2\int\limits_s^t \tilde{Y}_r^{\varepsilon,n}\left(\delta_{1,n}(\varepsilon,r) + \delta_{2,n}(\varepsilon,r) + \delta_{3,n}(\varepsilon,r) + \delta_{4,n}(\varepsilon,r)\right) dr - 2\int\limits_s^t \tilde{Y}_r^{\varepsilon,n} \tilde{Z}_r^{\varepsilon,n} d\tilde{B}_r$$

where

$$\delta_{1,n}(\varepsilon,r) = U\left(\Theta^{\varepsilon,n}(r)\right) - U\left(\bar{X}_r^\varepsilon, X_r^\varepsilon, \check{Y}_r^\varepsilon, \partial_x u^n\left(t-r, \hat{X}_r^\varepsilon\right) \Lambda\left(\bar{X}_r^\varepsilon\right)\right),$$

$$\delta_{2,n}(\varepsilon,r) = U\left(X_r^\varepsilon, \check{Y}_r^\varepsilon, \partial_x u^n\left(t-r, \hat{X}_r^\varepsilon\right) \Lambda\left(\bar{X}_r^\varepsilon\right)\right) - \bar{U}\left(X_r^\varepsilon, \check{Y}_r^\varepsilon, \partial_x u^n\left(t-r, \hat{X}_r^\varepsilon\right)\right),$$

$$\delta_{3,n}(\varepsilon,r) = \bar{U}\left(X_r^\varepsilon, \check{Y}_r^\varepsilon, \partial_x u^n\left(t-r, \hat{X}_r^\varepsilon\right)\right) - \bar{U}\left(X_r^\varepsilon, u^n\left(t-r, \hat{X}_r^\varepsilon\right), \partial_x u^n\left(t-r, \hat{X}_r^\varepsilon\right)\right),$$

$$\delta_{4,n}(\varepsilon,r) = \bar{U}\left(X_r^\varepsilon, u^n\left(t-r, \hat{X}_r^\varepsilon\right), \partial_x u^n\left(t-r, \hat{X}_r^\varepsilon\right)\right) - \bar{U}^{(n)}\left(\hat{X}_r^\varepsilon, u^n\left(t-r, \hat{X}_r^\varepsilon\right), \partial_x u^n\left(t-r, \hat{X}_r^\varepsilon\right)\right).$$

The Lipschitz property of U and $\bar{U}$, implies

$$\tilde{Y}_r^{\varepsilon,n}\left(\delta_{1,n}(\varepsilon,r) + \delta_{3,n}(\varepsilon,r)\right) \leqslant \tilde{K}\left(\left|\tilde{Y}_r^{\varepsilon,n}\right|\left|\tilde{Z}_r^{\varepsilon,n}\right| + \left|\tilde{Y}_r^{\varepsilon,n}\right|^2\right).$$

So we have for every $0 \leqslant s \leqslant t$,

$$\tilde{\mathbf{E}}\left|\tilde{Y}_s^{\varepsilon,n}\right|^2 + \tilde{\mathbf{E}}\int\limits_s^t \left|\tilde{Z}_r^{\varepsilon,n}\right|^2 dr \leqslant \tilde{\mathbf{E}}\left|g\left(X_t^\varepsilon\right) - g^{(n)}\left(\hat{X}_t^\varepsilon\right)\right|^2 + 2\tilde{\mathbf{E}}\sup_{0\leqslant s\leqslant t}\left|\int\limits_s^t \tilde{Y}_r^{\varepsilon,n}\left(Lu^n\left(t-r, \hat{X}_r^\varepsilon\right) - \hat{L}_{\varepsilon,n}(r)\right) dr\right|$$

$$+ C_{\tilde{K}}\tilde{\mathbf{E}}\int\limits_s^t \left|\tilde{Y}_r^{\varepsilon,n}\right|^2 dr + \frac{1}{2}\tilde{\mathbf{E}}\int\limits_s^t \left|\tilde{Z}_r^{\varepsilon,n}\right|^2 dr$$

$$+ \tilde{\mathbf{E}}\sup_{0\leqslant s\leqslant t}\left|\int\limits_s^t \tilde{Y}_r^{\varepsilon,n} \delta_{2,n}(\varepsilon,r) dr\right| + \tilde{\mathbf{E}}\int\limits_s^t \left|\delta_{4,n}(\varepsilon,r)\right|^2 dr$$

where the constant $C_{\tilde{K}}$ depends only on $\tilde{K}$. Then exploiting Gronwall's lemma, we deduce that

$$\left|\tilde{Y}_0^{\varepsilon,n}\right|^2 \leqslant C_n(\varepsilon) e^{C_{\tilde{K}} t} \tag{3.28}$$

where

$$C_n(\varepsilon) = \tilde{\mathbf{E}}\left|g\left(X_t^\varepsilon\right) - g^{(n)}\left(\hat{X}_t^\varepsilon\right)\right|^2 + 2\tilde{\mathbf{E}}\sup_{0\leqslant s\leqslant t}\left|\int\limits_0^t \tilde{Y}_r^{\varepsilon,n}\left(Lu^n\left(t-r, \hat{X}_r^\varepsilon\right) - \hat{L}_{\varepsilon,n}(r)\right) dr\right|$$

$$+ \tilde{\mathbf{E}}\sup_{0\leqslant s\leqslant t}\left|\int\limits_s^t \tilde{Y}_r^{\varepsilon,n} \delta_{2,n}(\varepsilon,r) dr\right| + \tilde{\mathbf{E}}\int\limits_s^t \left|\delta_{4,n}(\varepsilon,r)\right|^2 dr$$

$$:= C_n^1(\varepsilon) + C_n^2(\varepsilon) + C_n^3(\varepsilon) + C_n^4(\varepsilon).$$

It is easy to check that $C_n^1(\varepsilon)$ satisfies (3.27) thanks to the Lipschitz property of g.

We now are going to treat the terms $C_n^2(\varepsilon)$ and $C_n^3(\varepsilon)$. Fix $n \in \mathbf{N}$. Thanks to the tightness of the process $(X_s^\varepsilon, \hat{X}_s^\varepsilon, \tilde{Y}_s^{\varepsilon,n})_{0\leqslant s\leqslant t}$, we deduce from Theorem 3.7 that

$$C_n^2(\varepsilon) \xrightarrow{\varepsilon\to 0} 0, \quad \text{and} \quad C_n^3(\varepsilon) \xrightarrow{\varepsilon\to 0} 0.$$

Moreover the Lipschitz property of $\bar{U}$ (with a constant still noted C) with respect to its first argument yields

$$\left|\delta_{4,n}(\varepsilon,r)\right| \leqslant C\left|X_t^\varepsilon - \hat{X}_t^\varepsilon\right| + \sup_{x\in\mathbf{R}^d, |y|+|z|\leqslant C_{3.3}} \left|\left(\bar{U}_A - \bar{U}_A^{(n)}\right)(x,y,z)\right|,$$

which is enough to prove that $C_n^4(\varepsilon)$ satisfies (3.27). $\quad\square$

4. Proofs

4.1. Proof of Proposition 3.1

From Lemma 2.4, $\hat{b}_i$ ($1 \leqslant i \leqslant d$) and $\hat{e}(\cdot, y)$ ($y \in \mathbf{R}$) belong to $C^1(\mathbf{T^d})$. Furthermore, for all $x \in \mathbf{T^d}$, $y \in \mathbf{R}$, $T > 0$ and $\delta > 0$, we have

$$
\big|\hat{e}(x, y + \delta) - \hat{e}(x, y)\big| \leqslant \int_0^T \big|\mathbf{E}_x\big[e(\tilde{X}_t, y + \delta) - e(\tilde{X}_t, y)\big]\big| \, dt + \int_T^\infty \big|\mathbf{E}_x\big[e(\tilde{X}_t, y + \delta) - e(\tilde{X}_t, y)\big]\big| \, dt
$$

$$
\leqslant Tc|\delta| + (2/\rho)ce^{-\rho T}.
$$

The continuity of the function $y \mapsto \hat{e}(x, y)$ follows. Thanks to assumption (H2.2) and (H2.3), using similar techniques and Lemma 2.2, we conclude that the mapping $y \to \mathbf{E}_x[e(\tilde{X}_t, y)]$ is twice continuously differentiable with respect to y and satisfies with some positive constant $C > 0$,

$$
\big|\mathbf{E}_x\big[e(\tilde{X}_t, y)\big]\big| + \big|\mathbf{E}_x\big[\partial_y e(\tilde{X}_t, y)\big]\big| + \big|\mathbf{E}_x\big[\partial_{yy}^2 e(\tilde{X}_t, y)\big]\big| \leqslant Ce^{-\rho[t]}. \tag{4.1}
$$

Hence we deduce that for all $x \in \mathbf{T^d}$, the function $\hat{e}(x, \cdot)$ is twice differentiable with respect to y and the derivatives (by the same argument as before) $\partial_y \hat{e}$ and $\partial_{yy}^2 \hat{e}$ are continuous and bounded on $\mathbf{T^d} \times \mathbf{R}$ thanks to (4.1). Moreover Lemma 2.4 and assumption (H2.2) ensure that for every $y \in \mathbf{R}$, $\partial_y \hat{e}(\cdot, y) \in C^1(\mathbf{T^d})$.

4.2. Proof of the Itô formula

This section is devoted to establishing formula (3.21). This boils down to proving that we can apply the Itô formula to the function $(x, y) \mapsto \hat{e}(x, y)$ and to the couple of Itô processes $(\bar{X}^\varepsilon, Y^\varepsilon)$. We remind the reader that the Itô formula only holds for C^2-class functions and, obviously, $\hat{e}$ is not smooth enough. However, we have already proved the existence of the only derivatives of $\hat{e}$ involved in (3.21). So, as guessed by the reader, we just need to carry out a regularization procedure to establish formula (3.21). This is the guiding line of the following computations.

To begin with, let us establish the following result.

Lemma 4.1. *Let $\rho : \mathbf{R}^d \to \mathbf{R}_+$ be a smooth function with compact support s.t. $\int_{\mathbf{R}^d} \rho(x) \, dx = 1$. Then the sequence of mollifiers defined by for all $n \in \mathbf{N}$ and $x \in \mathbf{R}^d$, $\rho_n(x) = n^d \rho(nx)$ satisfies: For all function $v : \mathbf{R}^d \times \mathbf{R} \to \mathbf{R}$ such that*

$$
\forall y \in \mathbf{R}, \quad v(\cdot, y) \in C^1(\mathbf{T^d}) \quad and \quad Lv(\cdot, y) \in C^0(\mathbf{T^d}),
$$

we have

$$
L\big(v(\cdot, y) * \rho_n\big) \xrightarrow{n \to \infty} Lv(\cdot, y) \quad pointwise.
$$

Moreover if $\|v(\cdot, y)\|_{C^1(\mathbf{T^d})} + \|Lv(\cdot, y)\|_{C^0(\mathbf{T^d})} \leqslant C_R$ for every $|y| \leqslant R$, then

$$
\forall x \in \mathbf{T^d}, \; |y| \leqslant R, \quad \big|L(v * \rho_n)(x, y)\big| \leqslant C_R'.
$$

Proof. For every $n \in \mathbf{N}$, let us consider the function φ_n defined For every $(x, y) \in \mathbf{R}^d \times \mathbf{R}$ by

$$
\varphi_n(x, y) = L\big(v(x, y) * \rho_n\big) - \big[\big(Lv(\cdot, y)\big) * \rho_n\big](x)
$$

where using the convention of summation over repeated indices

$$
\varphi_n(x, y) = \frac{1}{2}\big[a_{ij}(x)\partial_{x_i}\big(\partial_{x_j} v(\cdot, y) * \rho_n\big)(x) - \partial_{x_i}\big(a_{ij}\partial_{x_j} v(\cdot, y) * \rho_n\big)(x) + \big(\partial_{x_i} a_{ij}\partial_{x_j} v(\cdot, y) * \rho_n\big)(x)\big]
$$

$$
+ \int_{\mathbf{R}^d} \big[b_i(x) - b_i(x - u)\big]\partial_{x_i} v(x - u, y)\rho_n(u) \, du.
$$

Then since $Lv(\cdot, y) \in C^0(\mathbf{T^d})$, we deduce that $(Lv(\cdot, y) * \rho_n) \xrightarrow{n \to +\infty} Lv(\cdot, y)$ and the sequence is uniformly bounded. Hence it remains to study the sequence φ_n.

Thanks the properties of convolution, this one can be rewritten as follows

$$\varphi_n(x,y) = \frac{1}{2}\left[a_{ij}(x)\left(\partial_{x_j}v(\cdot,y)*\partial_{x_i}\rho_n\right)(x) - \left(a_{ij}\partial_{x_j}v(\cdot,y)*\partial_{x_i}\rho_n\right)(x) + \left(\partial_{x_i}a_{ij}\partial_{x_j}v(\cdot,y)*\rho_n\right)(x)\right]$$
$$+ \int_{\mathbf{R}^d}\left[b_i(x) - b_i(x-u)\right]\partial_{x_i}v(x-u,y)\rho_n(u)\,du$$

which implies for all $x \in \mathbf{R}^d$,

$$\varphi_n(x,y) = \frac{1}{2}\left[\int_{\mathbf{R}^d}\left[a_{ij}(x) - a_{ij}(x-u)\right]\partial_{x_j}v(x-u,y)n^{d+1}\rho_i'(nu)\,du\right.$$
$$+ \int_{\mathbf{R}^d}\partial_{x_i}a_{ij}(x-u)\partial_{x_j}v(x-u,y)n^d\rho(nu)\bigg]$$
$$+ \int_{\mathbf{R}^d}\left[b_i(x) - b_i(x-u)\right]\partial_{x_j}v(x-u,y)n^d\rho(nu)\,du. \tag{4.2}$$

Obviously, the last term in the right-hand side of (4.2) is uniformly bounded and converges to 0 since it is equal to

$$b_i(x)\left[\partial_{x_i}v(\cdot,y)*\rho_n\right](x) - \left[\left(b_i\partial_{x_i}v(\cdot,y)\right)*\rho_n\right](x).$$

Note that

$$\int_{\mathbf{R}^d}\partial_{x_i}a_{ij}(x-u)\partial_{x_j}v(x-u,y)n^d\rho(nu) = \left[\left(\partial_{x_i}a_{ij}\partial_{x_j}v(\cdot,y)\right)*\rho_n\right](x)\xrightarrow[n\to\infty]{}\partial_{x_i}a_{ij}(x)\partial_{x_j}v(x,y).$$

We now prove that the remaining term converges to minus that last limit. Indeed, there exists u' satisfying $|u'| \leqslant |u|$ such that (using summation over repeated indices)

$$I_n = \int_{\mathbf{R}^d}\left[a_{ij}(x) - a_{ij}(x-u)\right]\partial_{x_j}(x-u,y)n^{d+1}\rho_i'(nu)\,du$$
$$= \int_{\mathbf{R}^d}u \cdot \partial_x a_{ij}(x-u')\partial_{x_j}v(x-u,y)n^{d+1}\rho_i'(nu)\,du$$
$$= \int_{\mathbf{R}^d}\partial_{x_k}a_{ij}(x)\partial_{x_j}v(x-u,y)u_k n^{d+1}\rho_i'(nu)\,du$$
$$+ \int_{\mathbf{R}^d}\partial_{x_k}\left(a_{ij}(x-u') - a_{ij}(x)\right)\partial_{x_j}v(x-u,y)u_k n^{d+1}\rho_i'(nu)\,du.$$

The first term of the above right-hand side exactly matches

$$\int_{\mathbf{R}^d}\partial_{x_k}a_{ij}(x)\partial_{x_j}v(x,y)u_k n^{d+1}\rho_i'(nu)\,du + \int_{\mathbf{R}^d}\partial_{x_k}a_{ij}(x)\left(\partial_{x_j}v(x-u,y) - \partial_{x_j}v(x,y)\right)u_k n^{d+1}\rho_i'(nu)\,du$$

and, using the change of variables $r = nu$, we deduce (where δ_{ik} denotes the Kronecker symbol)

$$I_n = -\partial_{x_k}a_{ij}(x)\partial_{x_j}v(x,y)\delta_{ik} + \partial_{x_k}a_{ij}(x)\int_{\mathbf{R}^d}\left(\partial_{x_j}v\left(x-\frac{r}{n},y\right) - \partial v_{x_j}(x,y)\right)r_k\rho_i'(r)\,dr$$
$$+ \int_{\mathbf{R}^d}\partial_{x_k}\left(a_{ij}(x-u') - a_{ij}(x)\right)\partial_{x_j}v\left(x-\frac{r}{n},y\right)r_k\rho_i'(r)\,dr.$$

Then using the fact that ρ_i' is null outside of the unit ball of radius 1, the continuity of the functions

$$x \longrightarrow \partial_{x_j} v(x, y) \quad \text{and} \quad x \longrightarrow \partial_{x_j} a_{ij}(x)$$

and the Lebesgue convergence dominated theorem, we prove that the last two integrals converge to 0 as $n \to +\infty$. This completes the proof. $\quad \square$

We are now in position to prove formula (3.21). To this purpose, we consider the mollifiers $(\rho_n)_{n \geqslant 1}$ and define for all $y \in \mathbf{R}$ the function $\hat{e}_n(\cdot, y)$ on $\mathbf{R}^d$ by

$$\forall x \in \mathbf{R}^d, \quad \hat{e}_n(x, y) = [\hat{e}(\cdot, y) * \rho_n](x).$$

The theorem of derivation under the integral sign implies easily $\forall n \in \mathbf{N}$, $\hat{e}_n \in C^2(\mathbf{T}^d \times \mathbf{R})$. Define for all $n \in \mathbf{N}$, the process

$$\forall 0 \leqslant s \leqslant t, \quad \hat{Y}_s^{\varepsilon,n} = Y_s^\varepsilon + \varepsilon \big(\hat{e}_n(\bar{X}_t^\varepsilon, Y_t^\varepsilon) - \hat{e}_n(\bar{X}_s^\varepsilon, Y_s^\varepsilon) \big).$$

Itô's formula, yields for all $0 \leqslant s \leqslant t$,

$$
\begin{aligned}
\hat{Y}_s^{\varepsilon,n} = {} & g(X_t^\varepsilon) + \int_s^t \big(\langle \partial_x \hat{e}_n, c \rangle - \partial_y \hat{e}_n e + f - \varepsilon \partial_y \hat{e}_n f \big)(X_r^\varepsilon, \bar{X}_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon)\, dr \\
& + \int_s^t \partial_{xy}^2 \hat{e}_n(\bar{X}_r^\varepsilon, Y_r^\varepsilon) \sigma(\bar{X}_r^\varepsilon) Z_r^\varepsilon \, dr + \int_s^t \big(\partial_x \hat{e}_n(\bar{X}_r^\varepsilon, Y_r^\varepsilon) \sigma(\bar{X}_r^\varepsilon) - Z_r^\varepsilon \big) dB_r \\
& + \varepsilon \int_s^t \partial_y \hat{e}_n(\bar{X}_r^\varepsilon, Y_r^\varepsilon) Z_r^\varepsilon \, dB_r + \frac{\varepsilon}{2} \int_s^t \partial_{yy}^2 \hat{e}_n(\bar{X}_r^\varepsilon, Y_r^\varepsilon) |Z_r^\varepsilon|^2 \, dr + \frac{1}{\varepsilon} \int_s^t (L\hat{e}_n + e)(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, dr. \quad (4.3)
\end{aligned}
$$

It just remains to explain how to pass to the limit as $n \to \infty$ in the above expression and get formula (3.21). Thanks to Lemma 4.1, we have

$$\forall y \in \mathbf{R}, \quad L\hat{e}_n(\cdot, y) \longrightarrow -e(\cdot, y) \quad \text{as } n \longrightarrow +\infty. \quad (4.4)$$

Moreover, since for all $y \in \mathbf{R}$,

$$\partial_y \hat{e}_n(\cdot, y) = \partial_y \hat{e}(\cdot, y) * \rho_n, \ \partial_{yy}^2 \hat{e}_n(\cdot, y) = \partial_{yy}^2 \hat{e}(\cdot, y) * \rho_n \quad \text{and} \quad \partial_{xy}^2 (\partial_y \hat{e}_n)(\cdot, y) = \partial_{xy}^2 \hat{e}(\cdot, y) * \rho_n,$$

we deduce that $\hat{e}_n$ and its derivatives $\partial_y \hat{e}_n, \partial_{yy}^2 \hat{e}_n, \partial_{xy}^2 \hat{e}_n$ are uniformly bounded on $\mathbf{T}^d \times \mathbf{R}$ and respectively converge towards $\hat{e}, \partial_y \hat{e}, \partial_{yy}^2 \hat{e}, \partial_{xy}^2 \hat{e}$. So, the Lebesgue dominated convergence theorem and Lemma 4.1 ensure we can pass to the limit as $n \to \infty$ in the various integrals in (4.3). We then obtain formula (3.21).

4.3. Theorem 3.7

Proof. *First step*: Suppose that Ψ is bounded and that K_R does not depend on R. In this case, the proof, without the sup, is quite classical and can readily be adapted from [23, Lemma 5]. The result (with the sup) then follows from the boundedness of Ψ and the following argument.

Fix $N \in \mathbf{N}^*$ and consider a fine enough equidistant subdivision of $[0, t]$ by means of points $(t_i)_{0 \leqslant i \leqslant N}$ s.t. for $0 \leqslant i \leqslant N$, $t_i = \frac{i}{N} t$. Then we have

$$
\tilde{\mathbf{E}} \left[\sup_{0 \leqslant s \leqslant t} \left| \int_s^t \Psi(\bar{X}_r^\varepsilon, V_r^\varepsilon)\, dr - \int_s^t \bar{\Psi}(V_r^\varepsilon)\, dr \right| \right] \leqslant \tilde{\mathbf{E}} \left[\sup_{0 \leqslant i \leqslant N-1} \left| \int_{t_i}^t \Psi(\bar{X}_r^\varepsilon, V_r^\varepsilon)\, dr - \int_{t_i}^t \bar{\Psi}(V_r^\varepsilon)\, dr \right| \right] + \frac{2t}{N} \|\Psi\|_\infty.
$$

It just remains to let ε go to 0 to make the first term in the above right-hand side vanish and then let N tend to ∞.

Second step: We no longer assume that Ψ is bounded and K_R does not depend on R. For each $R > 0$, let us consider a bounded Lipschitzian function $h_R : \mathbf{R}^d \to \mathbf{R}^d$ such that

$$
\begin{cases}
|h_R(v)| \leqslant \min(|v|, R+1) & \text{for } v \in \mathbf{R}^d, \\
h_R(v) = v & \text{if } |v| \leqslant R.
\end{cases}
\quad (4.5)
$$

It is easy to see that (3.25) holds for $\Psi_R(x, v) = \Psi(x, h_R(v))$.

To complete the proof, it just remains to estimate the difference

$$\tilde{\mathbf{E}}\left[\sup_{0 \leqslant s \leqslant t} \int_s^t \left|(\Psi - \Psi_R)(\bar{X}_r^\varepsilon, V_r^\varepsilon)\right| + \left|(\bar{\Psi} - \bar{\Psi}_R)(V_R^\varepsilon)\right| dr\right].$$

But this quantity is bounded by

$$4\tilde{\mathbf{E}}\left[\sup_{0 \leqslant s \leqslant t}\left|V_s^\varepsilon\right|; \left\{\sup_{0 \leqslant s \leqslant t}\left|V_s^\varepsilon\right| \geqslant R\right\}\right] \leqslant \frac{4}{R}\mathbf{E}\left(\sup_{0 \leqslant s \leqslant t}\left|V_s^\varepsilon\right|^2\right)$$

and thus converges to 0 as R goes to ∞ uniformly with respect to ε. The result follows. $\quad\square$

4.4. Proof of Proposition 3.4

Proposition 3.4 follows from the following proposition and its corollary

Proposition 4.2. *There exists a constant $C_{4.2}$, only depending on t, K', c', and $\varepsilon_0 > 0$ such that*

$$\forall 0 < \varepsilon < \varepsilon_0, \quad \sup_{0 \leqslant s \leqslant t} \mathbf{E}\left|Y_s^\varepsilon\right|^2 + \mathbf{E}\int_0^t \left|Z_r^\varepsilon\right|^2 dr \leqslant C_{4.2}.$$

Proof. From (3.21), we deduce thanks to Ito's formula applied to the function $y \mapsto y^2$

$$d\left|\hat{Y}_r^\varepsilon\right|^2 = -2\hat{Y}_r^\varepsilon\left(U_1 + f - \varepsilon\frac{\partial\hat{e}}{\partial y}f - \sigma^*\frac{\partial^2\hat{e}}{\partial x\,\partial y} - (\varepsilon/2)\frac{\partial^2\hat{e}}{\partial y^2}\left|Z_r^\varepsilon\right|^2\right)\left(X_r^\varepsilon, \bar{X}_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon\right) dr$$

$$+ 2\hat{Y}_r^\varepsilon\left[\tilde{Z}_r^\varepsilon - \varepsilon\frac{\partial\hat{e}}{\partial y}(\bar{X}_r^\varepsilon, Y_r^\varepsilon)Z_r^\varepsilon\right]dB_r + \left|\tilde{Z}_r^\varepsilon - \varepsilon\frac{\partial\hat{e}}{\partial y}(\bar{X}_r^\varepsilon, Y_r^\varepsilon)Z_r^\varepsilon\right|^2 dr.$$

We take the expectation. The martingale term vanishes and we obtain for every $0 \leqslant s \leqslant t$,

$$\mathbf{E}\left|\hat{Y}_s^\varepsilon\right|^2 + \mathbf{E}\int_s^t \left|\tilde{Z}_r^\varepsilon - \varepsilon\frac{\partial\hat{e}}{\partial y}(\bar{X}_r^\varepsilon, Y_r^\varepsilon)Z_r^\varepsilon\right|^2 dr$$

$$= \mathbf{E}\left|\hat{Y}_t^\varepsilon\right|^2 + 2\mathbf{E}\int_s^t \hat{Y}_r^\varepsilon\left(U_1 + f - \varepsilon\frac{\partial\hat{e}}{\partial y}f - \sigma^*\frac{\partial^2\hat{e}}{\partial x\,\partial y} - (\varepsilon/2)\frac{\partial^2\hat{e}}{\partial y^2}\left|Z_r^\varepsilon\right|^2\right)\left(X_r^\varepsilon, \bar{X}_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon\right) dr.$$

Recall that $\hat{Y}_r^\varepsilon = Y_r^\varepsilon + \varepsilon(\hat{e}(\bar{X}_r^\varepsilon, Y_r^\varepsilon) - \hat{e}(\bar{X}_t^\varepsilon, Y_t^\varepsilon))$ and $\tilde{Z}_r^\varepsilon = Z_r^\varepsilon - \partial_x\hat{e}(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\sigma(\bar{X}_r^\varepsilon)$. From the growth properties of the coefficients $U_1, \hat{e}, f, g$, there exists a constant $C_{4.2}$, only depending on t, K', c', such that for any $0 < \varepsilon \leqslant 1$ (the constant $C_{4.2}$ may change from line to line) and $0 \leqslant s \leqslant t$,

$$\mathbf{E}\left|Y_s^\varepsilon\right|^2 + \mathbf{E}\int_s^t \left|Z_r^\varepsilon\right|^2 dr \leqslant C_{4.2} + C_{4.2}\varepsilon\mathbf{E}\left|Y_s^\varepsilon\right|^2 + C_{4.2}\varepsilon\mathbf{E}\int_s^t \left|Z_r^\varepsilon\right|^2 dr + C_{4.2}\mathbf{E}\int_s^t \left(1 + \left|Y_r^\varepsilon\right|\right)\left(1 + \left|Y_r^\varepsilon\right| + \left|Z_r^\varepsilon\right|\right) dr$$

$$\leqslant C_{4.2} + C_{4.2}\varepsilon\mathbf{E}\left|Y_s^\varepsilon\right|^2 + \left(C_{4.2}\varepsilon + \frac{1}{2}\right)\mathbf{E}\int_s^t \left|Z_r^\varepsilon\right|^2 dr + C_{4.2}\mathbf{E}\int_s^t \left|Y_r^\varepsilon\right|^2 dr.$$

Hence for any $\varepsilon < (4C_{4.2})^{-1}$, we have $\mathbf{E}|Y_s^\varepsilon|^2 + \mathbf{E}\int_s^t |Z_r^\varepsilon|^2 dr \leqslant 4C_{4.2} + 4C_{4.2}\mathbf{E}\int_s^t |Y_r^\varepsilon|^2 dr$, so that the result follows from the Gronwall lemma. $\quad\square$

Corollary 4.3. *There exists a constant $C_{3.4}$ such that*

$$\forall \varepsilon_0 > \varepsilon > 0, \quad \tilde{\mathbf{P}}\Big(\sup_{0 \leqslant s \leqslant t} \big| Y_s^\varepsilon \big| \leqslant C_{3.4} \Big) = 1.$$

Proof. From Proposition 4.2, we have $\mathbf{E}|Y_0^\varepsilon|^2 \leqslant C_{4.2}$ for any $\varepsilon_0 > \varepsilon > 0$. Since Y_0^ε is $\mathcal{F}_0$-measurable, it is constant and hence $|Y_0^\varepsilon|^2 \leqslant C_{4.2}$. Note that the constant $C_{4.2}$ does not depend on the starting point $x \in \mathbf{R}^d$ of the diffusion process X^ε. Let us now reintroduce the starting point x in our notations and denote by $X^{\varepsilon,x}$ the process solution of (2.1) starting from x, and by $Y^{\varepsilon,t,x}$ the solution of (3.19). From uniqueness for BSDEs, it is not hard to see that $Y_s^{\varepsilon,t,x} = Y_0^{\varepsilon,t-s,X_s^{\varepsilon,x}}$ for $0 \leqslant s \leqslant t$. In particular, for any $0 \leqslant s \leqslant t$ and $\varepsilon_0 > \varepsilon > 0$, $|Y_s^{\varepsilon,t,x}|^2 \leqslant C_{4.2}$. $\quad\square$

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