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A characterization of convex calibrable sets in $\mathbb{R}^N$ with respect to anisotropic norms

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Abstract

A set is called "calibrable" if its characteristic function is an eigenvector of the subgradient of the total variation. The main purpose of this paper is to characterize the "ϕ-calibrability" of bounded convex sets in $\mathbb{R}^N$ with respect to a norm ϕ (called *anisotropy* in the sequel) by the anisotropic mean ϕ-curvature of its boundary. It extends to the anisotropic and crystalline cases the known analogous results in the Euclidean case. As a by-product of our analysis we prove that any convex body C satisfying a ϕ-ball condition contains a convex ϕ-calibrable set K such that, for any $V \in [|K|, |C|]$, the subset of C of volume V which minimizes the ϕ-perimeter is unique and convex. We also describe the anisotropic total variation flow with initial data the characteristic function of a bounded convex set.

Résumé

On dit qu'un ensemble est « calibrable » si sa fonction est vecteur propre du sous-gradient de la variation totale. Le but de cet article est une caractérisation de la « ϕ-calibrabilité » des ensembles convexes bornés de $\mathbb{R}^N$, relativement à une norme ϕ (appelée *anisotropie*), en fonction de la ϕ-courbure moyenne anisotrope de leur frontière. Il s'agit donc d'une extension aux cas anisotropes et cristallins de résultats connus dans le cas euclidien. On démontre en particulier l'existence dans tout corps convexe régulier C d'un convexe $K \subseteq C$ ϕ-calibrable, tel que pour tout $V \in [|K|, |C|]$, l'ensemble de volume V de ϕ-périmètre minimal contenu dans C est unique et convexe. Nous étudions aussi le flot de la variation totale anisotrope à partir de la caractéristique d'un ensemble convexe borné.

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Mots-clés : Ensembles calibrables ; Ensembles convexes ; Courbure moyenne ; Variation totale

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1. Introduction

The purpose of this paper is to give a characterization of convex calibrable sets (with respect to an anisotropic perimeter) in $\mathbb{R}^N$ extending the corresponding result for $N = 2$ [17] and the corresponding results for the usual euclidean perimeter [27,14,2]. In the evolution of a set under anisotropic mean curvature flow, calibrable facets are those which do not bend or break during the evolution process, and they are characterized, in the convex case, in terms of the anisotropic curvature of the boundary [17].

The anisotropic perimeter P_ϕ in $\mathbb{R}^N$ is defined as

$$P_\phi(E) := \int\limits_{\partial E} \phi^\circ\big(\nu^E\big)\, d\mathcal{H}^{N-1}, \quad E \subseteq \mathbb{R}^N,$$

where ν^E is the outward unit normal to the boundary ∂E of E and ϕ° (the surface tension) is a norm on $\mathbb{R}^N$. We say that the anisotropy ϕ° is *crystalline* if $\{\phi^\circ \leqslant 1\}$ is a polyhedron.

Let F be a convex subset of $\mathbb{R}^2$. For any measurable set $X \subseteq \mathbb{R}^N$, $|X|$ denotes the Lebesgue measure of the set X. It has been proved in [17] that the following three assertions are equivalent.

(a) F is ϕ-calibrable, i.e., there is a vector field $\xi \in L^\infty(F, \mathbb{R}^2)$, with $\phi(\xi(x)) \leqslant 1$ a.e. in F (where ϕ is the dual norm of ϕ°), such that

$$\begin{aligned} &-\operatorname{div}\xi = \lambda_F^\phi := \frac{P_\phi(F)}{|F|} \quad \text{in } F,\\ &\xi\cdot\nu^F = -\phi^\circ\big(\nu^F\big) \quad \text{in } \partial F, \end{aligned} \tag{1.1}$$

where $\nu^F(x)$ denotes the outer unit normal to ∂F at the point $x \in \partial F$.

(b) F is a solution of the problem

$$\min_{X\subseteq F} P_\phi(X) - \lambda_F^\phi |X|. \tag{1.2}$$

(c) We have

$$\operatorname*{ess\,sup}_{x\in\partial F} \kappa_F^\phi(x) \leqslant \lambda_F^\phi, \tag{1.3}$$

where $\kappa_F^\phi(x)$ denotes the anisotropic curvature of ∂F at the point x.

The characterization of the calibrability of a convex set in $\mathbb{R}^2$, with respect to the euclidean perimeter, was proved by Giusti in [27], where he also proved that in a convex calibrable set the capillary problem in absence of gravity, with any prescribed contact angle at its boundary, has always a solution. In the euclidean case, this equivalence has been partly rederived in [14] where calibrable sets were used to construct explicit solutions of the denoising problem in image processing. A simple proof of the equivalence (b) $\Leftrightarrow$ (c) was given in [30] (where it was studied in connection which Cheeger sets, see Section 6). The extension of the above result for the euclidean perimeter and $N \geqslant 3$ was proved in [2]. In that case, the left-hand side of (1.3) has to be substituted by the sum of the principal curvatures at the point $x \in \partial F$. Our purpose in this paper is to extend the above set of equivalences to the anisotropic case, for a convex set in $\mathbb{R}^N$ which satisfies a ball condition (see Definition 2.7).

The proof of the equivalence (a) $\Leftrightarrow$ (b) is the same as in the euclidean case and it is independent of the dimension N (see [14,2]). We notice that the supremum of the curvature κ_C^ϕ in (1.3) has to be substituted with the number $(N-1)\|\mathbf{H}_C^\phi\|_\infty$, where $\|\mathbf{H}_C^\phi\|_\infty$ is defined in Section 2.5 and denotes the L^∞-norm of the anisotropic mean curvature of ∂C. To prove (b) $\Leftrightarrow$ (c) we follow the strategy used in [2] for the euclidean case, thus, we embed the variational problem (1.2) in a family of problems

$$\min_{X\subseteq C} P_\phi(X) - \lambda|X|, \quad \lambda > 0, \tag{1.4}$$

and we study the dependence of its solution on λ. In particular, we prove that C is a solution of (1.4) if and only if $\lambda \geqslant \max\{\lambda_C^\phi, (N-1)\|\mathbf{H}_C^\phi\|_\infty\}$. The solutions of (1.4) are related to the solution of the variational problem

$$\min_{u \in BV(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)} \int_{\mathbb{R}^N} \phi^\circ(Du) + \frac{\mu}{2} \int_{\mathbb{R}^N} (u - \chi_C)^2 \, dx, \quad \mu > 0. \tag{1.5}$$

Indeed, it turns out that the level sets of the solution of (1.5) embed the solutions of (1.4) for $\lambda \in [0, \mu]$. Since the solution u of (1.5) satisfies the equation

$$\begin{aligned} &v - \mu^{-1} \operatorname{div}\big(\partial\phi^\circ(Dv)\big) = 1 \quad \text{in } C, \\ &\partial\phi^\circ(Dv) \cdot \nu^C = -\phi^\circ\big(\nu^C\big) \quad \text{in } \partial C \end{aligned} \tag{1.6}$$

(the meaning of $\partial\phi^\circ(Dv)$ will be explained below) and the solutions of (1.6) can be approximated by the solutions u_ϵ of

$$\begin{aligned} &v - \mu^{-1} \operatorname{div}\left(\frac{T^\circ(Dv)}{\sqrt{\epsilon^2 + \phi^\circ(Dv)^2}}\right) = 1 \qquad \text{in } C, \\ &\frac{T^\circ(Dv)}{\sqrt{\epsilon^2 + \phi^\circ(Dv)^2}} \cdot \nu^C = -\phi^\circ\big(\nu^C\big) \quad \text{in } \partial C \end{aligned} \tag{1.7}$$

as $\epsilon \to 0$ (where $T^\circ(x) = \frac{1}{2}\partial(\phi^\circ)^2(x)$, $x \in \mathbb{R}^N$). We use the result of Korevaar [31] to conclude that u is concave in C, hence also continuous there. This implies the uniqueness and convexity of solutions of (1.4). Thus, by studying the dependence on λ of solutions of (1.4), we can prove that if C satisfies the curvature estimate (1.3) but is not a minimum of (1.2), then it can be approximated from inside by solutions C_λ of (1.4), with $\lambda \to \mu$ and $\mu > \lambda_C^\phi$. As we shall prove in Proposition 7.1, this implies that $(N-1)\|\mathbf{H}_C^\phi\|_\infty > \lambda_C^\phi$, a contradiction.

As an interesting by-product of our analysis we obtain that solutions of (1.4) are convex sets. Since (1.4) can be considered as the functional obtained by applying the Lagrange multiplier method to the area minimizing problem

$$\min_{X \subseteq C,\ |X| = V} P_\phi(X) \tag{1.8}$$

where $0 < V < |C|$, we obtain that, for some range of volumes, the solutions of this isoperimetric problem with fixed volume V are convex sets. The range of values of V for which the above result holds is $[|K|, |C|]$ where K is a convex ϕ-calibrable set contained in C obtained as solution of (1.4) for a certain value of λ (see Section 6). This extends the analogous result in [2]. In the euclidean case, a similar result has been also proved by E. Stredulinsky and W.P. Ziemer [39] in the case of a convex set C containing a ball B such that $\partial B \cap \partial C$ is a meridian of ∂B, and we mention the result of C. Rosales [36] when C is a rotationally symmetric convex body.

Finally, let us mention that our results enable us to describe the evolution of any convex set in $\mathbb{R}^N$, satisfying a ball condition, by the anisotropic total variation flow. The same result for the euclidean case was proved in [3] (for $N = 2$) and in [2]: as in those papers, it can be extended to unions of convex set which are far apart from each other. Other examples of evolution are given in [35].

Let us describe the plan of the paper. In Section 2 we collect some preliminary definitions and results about anisotropies, regularity conditions in the anisotropic case, functions of bounded variation and Green's formula. In Section 3 we recall the subdifferential of the anisotropic total variation in $\mathbb{R}^N$ and we define ϕ-calibrable sets. In Section 4 we relate the solution of the variational problem (1.4) with the solution of (1.5) and we study the basic properties of its minimizers. In Section 5 we prove the concavity of solutions of (1.5) for a certain range of values of μ. This will imply the convexity of the solutions of (1.4) for an interval of values of λ. In Section 6 we prove the convexity of solutions of (1.8) when $V \in [|K|, |C|]$ where K is a certain convex ϕ-calibrable set contained in C. Section 7 is devoted to the characterization of the ϕ-calibrability of a convex set in terms of the anisotropic mean curvature of its boundary. Finally, in Section 8 we characterize the ϕ-calibrability of the convex sets which satisfy a ball condition, and we describe the evolution of such sets by the minimizing anisotropic total variation flow.

2. Preliminaries

2.1. Notation

Given an open set $A \subseteq \mathbb{R}^N$ and a function $f : A \to \mathbb{R}$, we write $f \in \mathcal{C}^{1,1}(A)$ (resp. $f \in \mathcal{C}^{1,1}_{\mathrm{loc}}(A)$) if $f \in \mathcal{C}^1(A)$ and $\nabla f \in \mathrm{Lip}(A; \mathbb{R}^N)$ (resp. $\nabla f \in \mathrm{Lip}_{\mathrm{loc}}(A; \mathbb{R}^N)$). Let $B \subset \mathbb{R}^N$ be a set; we say that B (or ∂B) is of class $\mathcal{C}^{1,1}$ (resp. Lipschitz) if ∂B can be written, locally around each point, as the graph (with respect to a suitable orthogonal coordinate system) of a function f of class $\mathcal{C}^{1,1}$ (resp. Lipschitz).

Given two nonempty sets A, B, we denote the Hausdorff distance between A and B by $d_{\mathcal{H}}(A, B) = \max\{\sup_{a \in A} \mathrm{dist}(a, B), \sup_{b \in B} \mathrm{dist}(b, A)\}$. We denote by χ_A the characteristic function of A, and by $\bar{A}$ (resp. $\mathrm{int}(A)$) the closure (resp. the interior part) of A.

We let $S^{N-1} := \{\xi \in \mathbb{R}^N\colon |\xi| = 1\}$ and for $\rho > 0$ we let $B_\rho := \{x \in \mathbb{R}^N\colon |x| < \rho\}$. We denote by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$, and by $|\cdot|$ the Lebesgue measure. Given a function f defined on the boundary ∂C of a set C, we set $\|f\|_{L^\infty(\partial C)}$ to be the $\mathcal{H}^{N-1}$-essential supremum of $|f|$ on ∂C.

We shall use the notation $f(t) \in O(t)$ if $|\frac{f(t)}{t}|$ is bounded as $t \to 0$.

2.2. Anisotropies and distance functions

In the sequel of the paper, the function ϕ will always denote an anisotropy, i.e., a function $\phi : \mathbb{R}^N \to [0, \infty)$ such that

$$\phi(t\xi) = |t|\phi(\xi) \quad \forall \xi \in \mathbb{R}^N,\ \forall t \in \mathbb{R}, \tag{2.1}$$

and

$$m|\xi| \leqslant \phi(\xi) \quad \forall \xi \in \mathbb{R}^N, \tag{2.2}$$

for some $m > 0$. In particular $\phi(\xi) = \phi(-\xi)$ for any $\xi \in \mathbb{R}^N$. Observe that there exists $M \in [m, +\infty)$ such that $\phi(\xi) \leqslant M|\xi|$ for all $\xi \in \mathbb{R}^N$. We let $\mathcal{W}_\phi := \{\phi \leqslant 1\}$. The polar function ϕ° of ϕ (also called surface tension) is defined as $\phi^\circ(\xi) := \sup\{\eta \cdot \xi\colon \phi(\eta) \leqslant 1\}$ for any $\xi \in \mathbb{R}^N$. If ϕ is an anisotropy, then ϕ° is also an anisotropy and there holds $(\phi^\circ)^\circ = \phi$.

By a convex body we mean a compact convex set whose interior contains the origin. A convex body is said to be centrally symmetric if it is symmetric with respect to the origin. If ϕ is an anisotropy, then $\mathcal{W}_\phi := \{\xi\colon \phi(\xi) \leqslant 1\}$ (sometimes called Wulff shape) is a centrally symmetric convex body. If K is a convex body, the function $h_K(\xi) := \sup_{\eta \in K} \eta \cdot \xi$ is called the support function of K; notice that $\{(h_K)^\circ \leqslant 1\} = K$.

As usual, we shall denote by $\partial\phi(\xi)$ the subdifferential of ϕ at $\xi \in \mathbb{R}^N$. If ϕ is differentiable at ξ, we have $\partial\phi(\xi) = \{\nabla\phi(\xi)\}$. If Φ is a convex function defined on a Hilbert space, we still denote by $\partial\Phi$ the subdifferential of Φ.

Given a nonempty set $E \subseteq \mathbb{R}^N$, we let

$$d_\phi(x, E) := \inf_{y \in E} \phi(x - y), \quad x \in \mathbb{R}^N.$$

We denote by d^E_ϕ the signed ϕ-distance function to ∂E negative inside E, that is

$$d^E_\phi(x) := d_\phi(x, E) - d_\phi\big(x, \mathbb{R}^N \setminus E\big), \quad x \in \mathbb{R}^N. \tag{2.3}$$

Observe that $|d^E_\phi(x)| = d_\phi(x, \partial E)$.

It can be shown (the proofs are identical to the Euclidean case) that the function d^E_ϕ is Lipschitz, and at each point x where it is differentiable we have $\phi^\circ(\nabla d^E_\phi(x)) = 1$. We set

$$\nu^E_\phi := \nabla d^E_\phi \quad \text{on } \partial E, \tag{2.4}$$

at those points where ∇d^E_ϕ exists. When ϕ is the euclidean norm, i.e., $\phi(\xi) = |\xi|$, we set $\nu^E = \nu^E_{|\cdot|}$ and $B_1 = \mathcal{W}_{|\cdot|}$. We have

$$\nu^E_\phi(x) = \frac{\nu^E(x)}{\phi^\circ(\nu^E(x))}.$$

Let T° be the multivalued map in $\mathbb{R}^N$ defined by

$$T^\circ(x) = \frac{1}{2}\partial\big(\phi^\circ\big)^2(x), \quad x \in \mathbb{R}^N.$$

T° is a maximal monotone operator mapping $\mathcal{W}_{\phi^\circ}$ onto $\mathcal{W}_\phi$. If E is Lipschitz, at $\mathcal{H}^{N-1}$-a.e. $x \in \partial E$ we have

$$\big\langle \nu_\phi^E(x), p\big\rangle = 1 \quad \forall p \in T^\circ\big(\nu_\phi^E(x)\big).$$

Vector fields which are selections in $\partial\phi^\circ(\nabla d_\phi^E)$ are sometimes called Cahn–Hoffman vector fields, and we denote by $\mathrm{Nor}_\phi(\partial E, \mathbb{R}^N)$ the set of such fields.

Definition 2.1. We say that $\phi \in \mathcal{C}_+^{1,1}$ (resp. $\mathcal{C}_+^\infty$) if ϕ^2 is of class $\mathcal{C}^{1,1}(\mathbb{R}^N)$ (resp. $\mathcal{C}^\infty(\mathbb{R}^N \setminus \{0\})$) and there exists a constant $c > 0$ such that $\nabla^2(\phi^2) \geqslant c\,\mathrm{Id}$ almost everywhere. We say that a centrally symmetric convex body is of class $\mathcal{C}_+^{1,1}$ (resp. $\mathcal{C}_+^\infty$) if it is the unit ball of an anisotropy of class $\mathcal{C}_+^{1,1}$ (resp. $\mathcal{C}_+^\infty$).

Definition 2.2. We say that ϕ is crystalline if the unit ball $\mathcal{W}_\phi$ of ϕ is a polytope.

Remark 2.3. Observe that

(a) $\phi \in \mathcal{C}_+^{1,1}$ (resp. $\mathcal{C}_+^\infty$) if and only if $\phi^\circ \in \mathcal{C}_+^{1,1}$ (resp. $\mathcal{C}_+^\infty$) [37, p. 111];
(b) ϕ is crystalline if and only if ϕ° is crystalline.

2.3. ϕ-regularity and the $R\mathcal{W}_\phi$-condition

Following [15–18] we define the class of ϕ-regular sets and Lipschitz ϕ-regular sets (these latter are a generalization of sets of class $\mathcal{C}^{1,1}$ in the euclidean case).

Definition 2.4. Let $E \subset \mathbb{R}^N$ be a set. We say that E is ϕ-regular if ∂E is a compact Lipschitz hypersurface and there exist an open set $U \supset \partial E$ and a vector field $n \in L^\infty(U; \mathbb{R}^N)$ such that $\operatorname{div} n \in L^\infty(U)$, and $n \in \partial\phi^\circ(\nabla d_\phi^E)$ almost everywhere in U. We say that E is Lipschitz ϕ-regular if E is ϕ-regular and $n \in \mathrm{Lip}(U; \mathbb{R}^N)$.

It is clear that a Lipschitz ϕ-regular set is ϕ-regular. With a little abuse of notation, sometimes we will denote by (E, n), by (E, U) or by (E, U, n), a ϕ-regular set.

Observe that, in general, vector fields n are not unique, unless $\phi \in \mathcal{C}_+^{1,1}$. When $\phi \in \mathcal{C}_+^{1,1}$ the inclusion $n \in \partial\phi^\circ(\nabla d_\phi^E)$ becomes an equality; in this respect we give the following definition.

Definition 2.5. Let $\phi \in \mathcal{C}_+^{1,1}$ and (E, U) be a Lipschitz ϕ-regular set. Let $x \in U$ be a point where there exists $\nabla d_\phi^E(x)$. We set

$$n_\phi^E(x) := \nabla\phi^\circ\big(\nabla d_\phi^E(x)\big). \tag{2.5}$$

Remark 2.6. Observe that $(\mathcal{W}_\phi, n)$, with $n(x) := x/\phi(x)$, is Lipschitz ϕ-regular, and $\operatorname{div} n(x) = (N-1)/\phi(x)$ for almost every $x \in \mathbb{R}^N$.

The next definition will play an important role in the sequel.

Definition 2.7. Let $E \subset \mathbb{R}^N$ be a set with nonempty interior and $R > 0$. We say that E satisfies the $R\mathcal{W}_\phi$-condition if, for any $x \in \partial E$, there exists $y \in \mathbb{R}^N$ such that

$$R\mathcal{W}_\phi + y \subseteq \bar{E} \quad \text{and} \quad x \in \partial(R\mathcal{W}_\phi + y).$$

The first assertion of the following result is proved in [18, Lemmas 3.4, 3.5], and the second one is proved in [13, Proposition 3.9].

Lemma 2.8. *Let ϕ be any anisotropy.*

(i) *If E is a Lipschitz ϕ-regular set, then E and $\mathbb{R}^N \setminus E$ satisfy the $R\mathcal{W}_\phi$-condition for some $R > 0$.*
(ii) *A compact convex set satisfying the $R\mathcal{W}_\phi$-condition is ϕ-regular.*

If $\phi \in \mathcal{C}_+^{1,1}$, we list some relations between ϕ-regularity and the $R\mathcal{W}_\phi$-condition (see [13, Remark 4]).

Remark 2.9. Assume that $\phi \in \mathcal{C}_+^{1,1}$. The following assertions hold.

(a) E is Lipschitz ϕ-regular if and only if E is of class $\mathcal{C}^{1,1}$.
(b) Let C be a compact convex set which satisfies the $R\mathcal{W}_\phi$-condition for some $R > 0$. Then C is Lipschitz ϕ-regular (hence C is of class $\mathcal{C}^{1,1}$ by (a)).
(c) E is Lipschitz ϕ-regular if and only if E and $\mathbb{R}^N \setminus E$ satisfy the $R\mathcal{W}_\phi$-condition for some $R > 0$.

2.4. BV functions, ϕ-total variation and generalized Green formula

Let Ω be an open subset of $\mathbb{R}^N$. A function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a (vector valued) Radon measure with finite total variation $|Du|(\Omega)$ in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. We denote by $BV_{\mathrm{loc}}(\Omega)$ the space of functions $w \in L^1_{\mathrm{loc}}(\Omega)$ such that $w\varphi \in BV(\Omega)$ for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$. Concerning all properties and notation relative to functions of bounded variation we will follow [6].

A measurable set $E \subseteq \mathbb{R}^N$ is said to be of finite perimeter in Ω if $|D\chi_E|(\Omega) < \infty$. The (euclidean) perimeter of E in Ω is defined as $P(E, \Omega) := |D\chi_E|(\Omega)$, and we have $P(E, \Omega) = P(\mathbb{R}^N \setminus E, \Omega)$. We shall use the notation $P(E) := P(E, \mathbb{R}^N)$.

Let $u \in BV(\Omega)$. We define the anisotropic total variation of u with respect to ϕ in Ω [4] as

$$\int_\Omega \phi^\circ(Du) = \sup\left\{ \int_\Omega u \operatorname{div} \sigma \, dx \colon \sigma \in \mathcal{C}_c^1(\Omega; \mathbb{R}^N),\ \phi(\sigma(x)) \leqslant 1\ \forall x \in \Omega \right\}. \tag{2.6}$$

If $E \subseteq \mathbb{R}^N$ has finite perimeter in Ω, we set

$$P_\phi(E, \Omega) := \int_\Omega \phi^\circ(D\chi_E)$$

and we have [4]

$$P_\phi(E, \Omega) = \int_{\Omega \cap \partial^* E} \phi^\circ(\nu^E)\, d\mathcal{H}^{N-1}, \tag{2.7}$$

where $\partial^* E$ is the reduced boundary of E and ν^E the (generalized) outer unit normal to E at points of $\partial^* E$.

Recall that, since ϕ° is homogeneous, $\phi^\circ(Du)$ coincides with the nonnegative Radon measure in $\mathbb{R}^N$ given by

$$\phi^\circ(Du) = \phi^\circ(\nabla u(x))\, dx + \phi^\circ\left(\frac{D^s u}{|D^s u|}\right)|D^s u|,$$

where $\nabla u(x)\, dx$ is the absolutely continuous part of Du, and $D^s u$ its singular part.

Let Ω be an open subset of $\mathbb{R}^N$. Following [10], let

$$X_2(\Omega) := \{ z \in L^\infty(\Omega; \mathbb{R}^N) \colon \operatorname{div} z \in L^2(\Omega) \}.$$

If $z \in X_2(\Omega)$ and $w \in L^2(\Omega) \cap BV(\Omega)$ we define the distribution $(z, Dw) : \mathcal{C}_c^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (z, Dw), \varphi \rangle := -\int_\Omega w\varphi \operatorname{div} z \, dx - \int_\Omega wz \cdot \nabla\varphi \, dx \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega).$$

Then (z, Dw) is a Radon measure in Ω,

$$\int_\Omega (z, Dw) = \int_\Omega z \cdot \nabla w \, dx \quad \forall w \in L^2(\Omega) \cap W^{1,1}(\Omega),$$

and

$$\left| \int_B (z, Dw) \right| \leqslant \int_B \left|(z, Dw)\right| \leqslant \|z\|_\infty \int_B |Dw| \quad \forall B \subseteq \Omega \text{ Borel set.}$$

We recall the following result proved in [10].

Theorem 2.10. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Let $u \in BV(\Omega) \cap L^2(\Omega)$ and $z \in X_2(\Omega)$. Then there exists a function $[z \cdot \nu^\Omega] \in L^\infty(\partial\Omega)$ such that $\|[z \cdot \nu^\Omega]\|_{L^\infty(\partial\Omega)} \leqslant \|z\|_{L^\infty(\Omega;\mathbb{R}^N)}$, and*

$$\int_\Omega u \operatorname{div} z \, dx + \int_\Omega (z, Du) = \int_{\partial\Omega} \left[z \cdot \nu^\Omega\right] u \, d\mathcal{H}^{N-1}.$$

When $\Omega = \mathbb{R}^N$ we have the following integration by parts formula [10], for $z \in X_2(\mathbb{R}^N)$ and $w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} w \operatorname{div} z \, dx + \int_{\mathbb{R}^N} (z, Dw) = 0. \tag{2.8}$$

Remark 2.11. Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz open set, and let $z_{\mathrm{inn}} \in L^\infty(\Omega; \mathbb{R}^N)$ with $\operatorname{div} z_{\mathrm{inn}} \in L^2_{\mathrm{loc}}(\Omega)$, and $z_{\mathrm{out}} \in L^\infty(\mathbb{R}^N \setminus \overline{\Omega}; \mathbb{R}^N)$ with $\operatorname{div} z_{\mathrm{out}} \in L^2_{\mathrm{loc}}(B_R \setminus \overline{\Omega})$, for all $R > 0$. Assume that

$$\left[z_{\mathrm{inn}} \cdot \nu^\Omega\right](x) = -\left[z_{\mathrm{out}} \cdot \nu^{\mathbb{R}^2 \setminus \overline{\Omega}}\right](x) \quad \text{for } \mathcal{H}^{N-1} - \text{a.e } x \in \partial\Omega.$$

Then if we define $z := z_{\mathrm{inn}}$ on Ω and $z := z_{\mathrm{out}}$ on $\mathbb{R}^N \setminus \overline{\Omega}$, we have $z \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ and $\operatorname{div} z \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$.

2.5. The anisotropic mean curvature

Let (E, U, n) be a ϕ-regular set. For any $p \in [1, +\infty]$, we define

$$\tilde{H}_\phi^{\mathrm{div},p}\left(U, \mathbb{R}^N\right) := \left\{N \in L^\infty\left(U; \mathbb{R}^N\right):\ N \in T^\circ\left(\nabla d_\phi^E\right), \operatorname{div} N \in L^p(U)\right\}.$$

Fix now $\delta_0 > 0$ be such that $U_t := \{|d_\phi^E| < t\} \subseteq U$ for $t \in [0, \delta_0]$. Then, following [18] (see also Theorem 2.12 below) there exists a vector field $\tilde{z}_t \in L^\infty(U_t, \mathbb{R}^N)$ such that $\tilde{z}_t \in T^\circ(\nabla d_\phi^E)$ a.e. in U_0, $\operatorname{div} \tilde{z}_t \in L^2(U_0)$ and

$$\|\operatorname{div} \tilde{z}_t\|_{L^2(U_t)} \leqslant \|\operatorname{div} Z\|_{L^2(U_t)} \quad \forall Z \in \tilde{H}_\phi^{\mathrm{div},2}\left(U_t, \mathbb{R}^N\right). \tag{2.9}$$

We point out that, even if the minimizer $\tilde{z}_t$ may be nonunique, its divergence is always uniquely defined. In particular, it follows that

$$\operatorname{div} \tilde{z}_s = \operatorname{div} \tilde{z}_t \quad \text{a.e. in } U_s, \tag{2.10}$$

for all $0 < s < t$.

Theorem 2.12. *Let (E, U, n) be a ϕ-regular set. Let $0 < \delta_0 \leqslant R$ be such that $U_0 := \{|d_\phi^E| < \delta_0\} \subseteq U$, and let (u^h, z^h), $u^h \in BV_{\mathrm{loc}}(\mathbb{R}^N) \cap L^2_{\mathrm{loc}}(\mathbb{R}^N)$, be the solution of*

$$u^h - h \operatorname{div} z^h = d_\phi^E \quad \text{in } \mathbb{R}^N, \tag{2.11}$$

where $z^h \in \partial\phi^\circ(\nabla u^h)$ and $(z^h, Du^h) = \phi(Du^h)$ in $\mathcal{D}'(\mathbb{R}^N)$. Then, there exists $\tilde{z} \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$, and a subsequence $h_j \to 0^+$ such that $z^{h_j} \to \tilde{z}$ weakly in $\mathbb{R}^N$, where $\tilde{z}$ is such that $\tilde{z} \in T^\circ(\nabla d_\phi^E)$ in U_0 and*

$$\|\operatorname{div} \tilde{z}\|_{L^q(U_0)} \leqslant \|\operatorname{div} n\|_{L^q(U_0)} \quad \forall q \in [1, \infty]. \tag{2.12}$$

More generally, $\tilde z$ satisfies the following inequality

$$\|\operatorname{div}\tilde z\|_{L^q(U_\delta)} \leqslant \|\operatorname{div} Z\|_{L^q(U_\delta)} \quad \forall Z \in \tilde H_\phi^{\mathrm{div},\infty}(U_\delta, \mathbb{R}^N), \tag{2.13}$$

for all $q \in [1,\infty]$ and for all $0 < \delta < \delta_0$, where $U_\delta := \{|d_\phi^E| < \delta\}$. Finally, if E is convex, then $\operatorname{div}\tilde z \geqslant 0$ in U_0.

Let us recall that (2.11) has a unique solution $u^h \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$ [23]. Moreover $u^h \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ [23] and $\|u^h\|_{L^\infty(B_R)} \leqslant \|d_\phi^E\|_{L^\infty(B_{2R})} + C$ for some constant C which does not depend on h. Let us also point out that u^h is Lipschitz with a Lipschitz constant depending only on the Lipschitz constant of d_ϕ^E. Indeed, by the results in [23] u^h can be obtained as limit in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ of the solutions $u_j^h \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ of

$$u - h \operatorname{div} \partial\phi(\nabla u) \ni \min\{d_\phi^E, j\} \quad \text{in } \mathbb{R}^N, \tag{2.14}$$

and, for any $y \in \mathbb{R}^N$, $u_j^h(\cdot + y)$ is the solution of (2.14) with right-hand side $\min\{d_\phi^E(\cdot + y), j\}$. As in [23], Corollary C.2, we prove that

$$\|(u_j^h - u_j^h(\cdot + y))^+\|_\infty \leqslant \|\min\{d_\phi^E, j\} - \min\{d_\phi^E(\cdot + y), j\}\|_\infty \leqslant \|d_h^E - d_h^E(\cdot + y)\|_\infty.$$

This implies that

$$\|(u^h - u^h(\cdot + y))^+\|_\infty \leqslant \|d_h^E - d_h^E(\cdot + y)\|_\infty.$$

Interchanging the role of u^h and $u^h(\cdot + y)$ we deduce that

$$\|\nabla u^h\|_\infty \leqslant \|\nabla d_\phi^E\|_\infty. \tag{2.15}$$

We may also prove this along the lines of the proof of Theorem 3 in [23] which uses another approximation of (2.11) and viscosity solution theory.

Proof. For simplicity, let us denote $d := d_\phi^E$. By the remarks previous to the proof we have that $|u^h| \leqslant c_\lambda$ on $\{d \leqslant \lambda\}$ where c_λ is a constant depending on λ for any $\lambda > 0$. Multiplying (2.11) by $u^h - d$ and integrating by parts in $\{d \leqslant \lambda\}$ we obtain

$$\int_{\{d\leqslant\lambda\}} (u^h - d)^2\, dx = -h \int_{\{d\leqslant\lambda\}} z^h \cdot (\nabla u^h - \nabla d)\, dx + h \int_{\partial\{d\leqslant\lambda\}} z^h \cdot \nu^{\{d\leqslant\lambda\}} (u^h - d)\, d\mathcal{H}^{N-1},$$

hence $u^h \to d$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ as $h \to 0^+$. By the estimate (2.15), we have that the convergence takes place also locally uniformly in $\mathbb{R}^N$. Moreover, modulo a subsequence, we may assume that $z^h \to \tilde z$ weakly* in $L^\infty(\mathbb{R}^N)$ as $h \to 0^+$. Let $a < b$ and $Q_{a,b}^h := \{u^h \geqslant a\} \cap \{d \leqslant b\}$ be such that $Q_{a,b}^h \subseteq U_0$. Let us assume that h varies along a sequence converging to 0. Since $u^h \in BV_{\mathrm{loc}}(\mathbb{R}^N)$ we may assume that a is such that $\{u^h < a\}$ is a set of finite perimeter in $\mathbb{R}^N$. Since u^h converges to d locally uniformly in $\mathbb{R}^N$ we may assume h small enough so that $\{u^h < a\} \subseteq \{d \leqslant b\}$ and $\{u^h = a\} \cap \{d = b\} = \emptyset$. Let $P : \mathbb{R} \to [0,\infty)$ be a smooth, increasing and nonnegative function. Then

$$\begin{aligned}\int_{Q_{a,b}^h} (u^h - d) P(u^h - d)\, dx &= h \int_{Q_{a,b}^h} \operatorname{div} z^h P(u^h - d)\, dx \\ &= h \int_{Q_{a,b}^h} (\operatorname{div} z^h - \operatorname{div} n) P(u^h - d)\, dx + h \int_{Q_{a,b}^h} \operatorname{div} n\, P(u^h - d)\, dx.\end{aligned}$$

The first term can be written as

$$\begin{aligned}&\int_{Q_{a,b}^h} (\operatorname{div} z^h - \operatorname{div} n) P(u^h - d)\, dx \\ &\quad = -\int_{Q_{a,b}^h} (z^h - n) \cdot \nabla P(u^h - d)\, dx - \int_{\mathbb{R}^N} \big((z^h, D\chi_{Q_{a,b}^h}) - (n, D\chi_{Q_{a,b}^h})\big) P(u^h - d).\end{aligned}$$

First, observe that

$$\int_{Q^h_{a,b}} \big(z^h - n\big)\cdot\nabla P\big(u^h - d\big)\,dx = \int_{Q^h_{a,b}} P'\big(u^h - d\big)\big(z^h - n\big)\cdot\nabla\big(u^h - d\big)\,dx$$

$$= \int_{Q^h_{a,b}} P'\big(u^h - d\big)\big(\phi^\circ\big(\nabla u^h\big) - n\cdot\nabla u^h + \phi^\circ(\nabla d) - z^h\cdot\nabla d\big)\,dx \geqslant 0.$$

To prove that the second term is negative, we observe that

$$-\int_{\mathbb{R}^N} \big(\big(z^h, D\chi_{Q^h_{a,b}}\big) - \big(n, D\chi_{Q^h_{a,b}}\big)\big)P\big(u^h - d\big)$$

$$= \int_{\mathbb{R}^N} \big(\big(z^h, D\chi_{\{u^h<a\}}\big) - \big(n, D\chi_{\{u^h<a\}}\big)\big)P\big(u^h - d\big) - \int_{\mathbb{R}^N} \big(\big(z^h, D\chi_{\{d\leqslant b\}}\big) - \big(n, D\chi_{\{d\leqslant b\}}\big)\big)P\big(u^h - d\big).$$

Now, by the proof of [23, Lemma 5.1] (see also [13, Lemma 4]), we have that $-(z^h, D\chi_{\{u^h<s\}}) = \phi^\circ(D\chi_{\{u^h<s\}})$, where the equality means the equality of both measures, for almost every $s\in\mathbb{R}$ and we may assume that a has been chosen to satisfy this equality. On the other hand, since $\phi(n)\leqslant 1$, we have that $|(n, D\chi_{\{u^h<a\}})| \leqslant \phi^\circ(D\chi_{\{u^h<a\}})$. This implies that

$$\int_{\mathbb{R}^N} \big(\big(z^h, D\chi_{\{u^h<a\}}\big) - \big(n, D\chi_{\{u^h<a\}}\big)\big)P\big(u^h - d\big) \leqslant 0.$$

By the same arguments we could have also chosen $b > a$ from the beginning so that $(n, D\chi_{\{d\leqslant b\}}) = -\phi^\circ(D\chi_{\{d\leqslant b\}})$, and, again, we have $|(z^h, D\chi_{\{d\leqslant b\}})| \leqslant \phi^\circ(D\chi_{\{d\leqslant b\}})$. Hence

$$\int_{\mathbb{R}^N} \big(\big(z^h, D\chi_{\{d\leqslant b\}}\big) - \big(n, D\chi_{\{d\leqslant b\}}\big)\big)P\big(u^h - d\big) \geqslant 0.$$

Combining all these inequalities we obtain that

$$\int_{Q^h_{a,b}} \big(u^h - d\big)P\big(u^h - d\big)\,dx \leqslant h\int_{Q^h_{a,b}} \operatorname{div} n\, P\big(u^h - d\big)\,dx. \tag{2.16}$$

If $q < \infty$, let $\tilde q = q$. If $q = \infty$, let $\tilde q < \infty$. Let P_j be a sequence of increasing nonnegative functions such that $P_j(r) \to r^{+(\tilde q - 1)}$ locally uniformly as $j\to\infty$. Using $P = P_j$ in (2.16) we obtain

$$\frac{1}{h}\int_{Q^h_{a,b}} \big(\big(u^h - d\big)^+\big)^{\tilde q}\,dx \leqslant \int_{Q^h_{a,b}} \operatorname{div} n\big(\big(u^h - d\big)^+\big)^{\tilde q - 1}\,dx.$$

Applying Young's inequality we obtain

$$\frac{1}{h}\big\|\big(u^h - d\big)^+\big\|_{L^{\tilde q}(Q^h_{a,b})} \leqslant \|\operatorname{div} n\|_{L^{\tilde q}(Q^h_{a,b})}.$$

Hence, we have

$$\big\|\big(\operatorname{div} z^h\big)^+\big\|_{L^{\tilde q}(Q^h_{a,b})} \leqslant \|\operatorname{div} n\|_{L^{\tilde q}(Q^h_{a,b})}.$$

Letting $h\to 0$ and $\tilde q\to\infty$ if $q = \infty$, we obtain

$$\big\|(\operatorname{div}\tilde z)^+\big\|_{L^q(Q_{a,b})} \leqslant \|\operatorname{div} n\|_{L^q(Q_{a,b})} \quad \forall q\in[1,\infty], \tag{2.17}$$

where $Q_{a,b} := \{a\leqslant d\leqslant b\}$. Letting $a\to -\delta_0$, $b\to\delta_0$, we deduce that

$$\big\|(\operatorname{div}\tilde z)^+\big\|_{L^q(U_0)} \leqslant \|\operatorname{div} n\|_{L^q(U_0)} \quad \forall q\in[1,\infty]. \tag{2.18}$$

I a similar way we obtain

$$\left\|(\operatorname{div}\tilde z)^-\right\|_{L^q(U_0)} \leqslant \|\operatorname{div} n\|_{L^q(U_0)} \quad \forall q\in[1,\infty]. \tag{2.19}$$

Indeed it suffices to change u^h into $-u^h$, n into $-n$ and to integrate in $\{u^h\leqslant b\}\cap\{d\geqslant a\}$ to obtain (2.19). Both inequalities (2.18) and (2.19) prove (2.12).

Now, we observe that $u^h\to d$ locally uniformly in $\mathbb{R}^N$, $z^h\to\tilde z$ and $\operatorname{div} z^h\to\operatorname{div}\tilde z$ weakly in $L^2_{\mathrm{loc}}(U_0)$. From this it follows that $\tilde z(x)\in\partial T^\circ(\nabla d)$ a.e. in U_0. To prove it, observe that since $\phi(z^h)\leqslant 1$ we deduce that $\phi(\tilde z)\leqslant 1$. Let ψ be a nonnegative test function with support contained in U_0. Then

$$\begin{aligned}
\int_{U_0}\phi^\circ(\nabla d)\psi\,dx &\leqslant \liminf_{h\to 0}\int_{U_0}\phi^\circ\big(\nabla u^h\big)\psi\,dx = \liminf_{h\to 0}\int_{U_0} z^h\cdot\nabla u^h\psi\,dx\\
&= \liminf_{h\to 0} -\int_{U_0}\operatorname{div} z^h u^h\psi\,dx - \int_{U_0} z^h\cdot\nabla\psi u^h\,dx\\
&= -\int_{U_0}\operatorname{div}\tilde z\, d\psi\,dx - \int_{U_0}\tilde z\cdot\nabla\psi d\,dx\\
&= \int_{U_0}\tilde z\cdot\nabla d\psi\,dx \leqslant \int_{U_0}\phi^\circ(\nabla d)\psi\,dx.
\end{aligned}$$

Hence

$$\int_{U_0}\tilde z\cdot\nabla d\psi\,dx = \int_{U_0}\phi^\circ(\nabla d)\psi\,dx.$$

Since this is true for any test function ψ with compact support in U_0 we obtain that $\tilde z\cdot\nabla d=\phi^\circ(\nabla d)$ in U_0, hence $\tilde z\in T^\circ(\nabla d)$ in U_0.

To prove the inequality (2.13) we observe that if $0<\delta<\delta_0$ and $Z\in\tilde H_\phi^{\mathrm{div},\infty}(U_\delta,\mathbb{R}^N)$, then (E,U_δ,Z) is ϕ-regular and, by repeating the computations that lead to (2.12), we deduce that (2.13) holds.

Finally, if E is convex, the inequality $\operatorname{div}\tilde z\geqslant 0$ follows from the inequality $d\leqslant u^h$, proved in [23, Theorem 3]. □

From (2.10) and (2.13) it follows that, if E satisfies the assumptions of Theorem 2.12, the function $t\to \|\operatorname{div}\tilde z_t\|_{L^\infty(U_t)}=\|\operatorname{div}\tilde z\|_{L^\infty(U_t)}$ is nondecreasing, hence we may take the limit

$$\left\|\mathbf{H}_E^\phi\right\|_\infty := \lim_{t\to 0^+}\|\operatorname{div}\tilde z_t\|_{L^\infty(U_t)}. \tag{2.20}$$

Let (E,n) be Lipschitz ϕ-regular and let $\mathrm{N}\in\mathrm{Nor}_\phi(\partial E,\mathbb{R}^N)\cap\mathrm{lip}(\partial E,\mathbb{R}^N)$. By [18, Lemmas 3.4, 3.5, 4.5], we have that

(i) there exists a neighborhood U of ∂E and $\delta>0$ such that the map $F_{\mathrm{N}}:\partial E\times(-\delta,\delta)\to\mathbb{R}^N$ defined by

$$F_{\mathrm{N}}(x,t)=x+t\mathrm{N}(x)$$

is bilipschitz, moreover

$$d_\phi^E\big(x+t\mathrm{N}(x)\big)=t,\quad x\in\partial E,$$

and $\nabla d_\phi^E(x+t\mathrm{N}(x))=\nu_\phi^E(x)$ for any $t\in(-\delta,\delta)$ and $\mathcal{H}^{N-1}$-a.e. $x\in\partial E$;

(ii) given $y\in U$, there is a unique $x\in\partial E$ such that $y=F_{\mathrm{N}}(x,t)$ where $t=d_\phi^E(x)$. We shall denote this point x by $\pi_{\mathrm{N}}(y)$. This permits to extend the vector field N to a vector field N^e on U by the formula

$$\mathrm{N}^e(x)=\mathrm{N}\big(\pi_{\mathrm{N}}(x)\big),\quad x\in U.$$

Using π_{N}, any vector field η can be extended from ∂E to U. Hence, from now on we shall write η instead of η^e, i.e. we shall assume that η is defined on a neighborhood of ∂E;

(iii) the trace of $\operatorname{div} \mathrm{N}^e$ (denoted by $\operatorname{div} \mathrm{N}$) is defined $\mathcal{H}^{N-1}$-almost everywhere on ∂E and coincides on ∂E with the tangential divergence of N to be defined below.

Finally, if (E, n) is a Lipschitz ϕ-regular set and $\mathrm{N} \in \mathrm{Nor}_\phi(\partial E, \mathbb{R}^N)$, we may define the (weak) tangential divergence $\operatorname{div}_\tau \mathrm{N} : \mathrm{Lip}(\partial E) \to \mathbb{R}$ as follows

$$\int_{\partial E} \operatorname{div}_\tau \mathrm{N} \psi \phi^\circ(\nu^E)\, d\mathcal{H}^{N-1} := \int_{\partial E} \mathrm{N} \cdot n \psi \operatorname{div}_\tau n \phi^\circ(\nu^E)\, d\mathcal{H}^{N-1} - \int_{\partial E} \big[(\mathrm{Id} - n \otimes n)\nabla_\tau \psi\big] \cdot \mathrm{N} \phi^\circ(\nu^E)\, d\mathcal{H}^{N-1},$$

where $\psi \in \mathrm{Lip}(\partial E)$. As proved in [18], this divergence does not depend on the vector field n. Letting

$$H_\phi^{\mathrm{div},p}(\partial E, \mathbb{R}^N) := \big\{ \mathrm{N} \in \mathrm{Nor}_\phi(\partial E, \mathbb{R}^N) \colon \operatorname{div}_\tau \mathrm{N} \in L^p(\partial E) \big\}, \quad p \in [1, +\infty],$$

we define $\mathrm{N}_{\min} \in H_\phi^{\mathrm{div},2}(\partial E, \mathbb{R}^N)$ to be a minimizer (possibly nonunique) of the functional

$$\int_{\partial E} (\operatorname{div}_\tau \mathrm{N})^2 \phi^\circ(\nu^E)\, d\mathcal{H}^{N-1}, \quad \mathrm{N} \in H_\phi^{\mathrm{div},2}(\partial E, \mathbb{R}^N). \tag{2.21}$$

As proved in [18], the function $\operatorname{div}_\tau \mathrm{N}_{\min}$ does not depend on the choice of the minimizer $\mathrm{N}_{\min}$ of (2.21). Moreover, by [18, Theorem 6.7] we have that $\operatorname{div}_\tau \mathrm{N}_{\min} \in L^\infty(\partial E)$ and

$$\|\operatorname{div}_\tau \mathrm{N}_{\min}\|_\infty = \min\big\{ \|\operatorname{div}_\tau \mathrm{N}\|_\infty \colon \mathrm{N} \in H_\phi^{\mathrm{div},\infty}(\partial E, \mathbb{R}^N) \big\}. \tag{2.22}$$

Remark 2.13. Let $\phi \in \mathcal{C}_+^{1,1}$ and E be a Lipschitz ϕ-regular set. Then

$$\operatorname{div}_\tau \mathrm{N}_{\min} = \operatorname{div} n_\phi^E, \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial E \quad \text{and} \quad (N-1)\big\|\mathbf{H}_E^\phi\big\|_\infty = \|\operatorname{div}_\tau \mathrm{N}_{\min}\|_{L^\infty(\partial E)}. \tag{2.23}$$

We do not know if the second equality in (2.23) holds for all Lipschitz ϕ-regular set $E \subset \mathbb{R}^N$. However, we can prove it under the additional assumption that the anisotropy ϕ is crystalline and E is a polyhedron.

Let us first observe that a polyhedron $E \subset \mathbb{R}^N$ is Lipschitz ϕ-regular if and only if for all vertices v of E there holds

$$C(v) := \bigcap_{F \text{ facet of } E \colon v \in F} \partial\phi^\circ(\nu^F) \neq \emptyset, \tag{2.24}$$

where ν^F is the outer unit normal to ∂E at the facet F.

Proposition 2.14. *Assume that ϕ is crystalline and let $E \subset \mathbb{R}^N$ be a Lipschitz ϕ-regular polyhedron. Then*

$$(N-1)\big\|\mathbf{H}_E^\phi\big\|_\infty = \|\operatorname{div}_\tau \mathrm{N}_{\min}\|_{L^\infty(\partial E)}.$$

Proof. Given a vertex v of E, we shall denote by $N(v)$ a generic element of the set $C(v)$, defined by (2.24).

Letting $E_t := \{d_\phi^E \leqslant t\}$, we know from [18] that there exists $\delta_0 > 0$ such that E_t is a Lipschitz ϕ-regular polyhedron for all $|t| \leqslant \delta_0$. Let also $\mathrm{N}_{\min}^t : \partial E_t \to \mathbb{R}^N$ be a minimizer of $\|\operatorname{div}_\tau \mathrm{N}\|_{L^2(\partial E_t)}$, which is equivalently a minimizer of $\|\operatorname{div}_\tau \mathrm{N}\|_{L^\infty(\partial E_t)}$ by [18]. Letting $H_t := \|\operatorname{div}_\tau \mathrm{N}_{\min}^t\|_{L^\infty(\partial E_t)}$, it is enough to prove that the function $t \in [-\delta_0, \delta_0] \to H_t$ is continuous at $t = 0$ (hence it is also continuous on the whole interval). Indeed, letting $\tilde{z}$ as in Theorem 2.12 and differentiating the equality $\phi(\tilde{z}) = 1$, we obtain $\nabla\tilde{z} \cdot \nabla d_\phi^E = 0$ in a neighborhood of ∂E. As a consequence, we get that $\operatorname{div}_\tau \tilde{z} = \operatorname{div} \tilde{z}$ a.e. in that neighborhood, where the tangential divergence (which, in this case, is an euclidean divergence) is computed with respect to ∂E_t at a point $x \in \partial E_t$. It follows that the field $\tilde{z}$ can be obtained by patching together the minimizing vector fields $\mathrm{N}_{\min}^t$, which are defined on ∂E_t.

Letting now F_t be the facet of E_t corresponding to the facet F of E, we shall prove the equivalent statement that the function

$$t \to H_t^F := \big\|\operatorname{div}_\tau \mathrm{N}_{\min}^t\big\|_{L^\infty(F_t)}$$

is continuous at $t=0$ (notice that $H_t = \max_F H_t^F$). To simplify the notation we shall identify F_t with its orthogonal projection on the hyperplane spanned by F. Notice that, for t small enough, the facet F_t can be obtained by parallelly translating the edges of F of a distance proportional to t (with a constant depending on the edge) and possibly inserting new edges, with length of order t, near the vertices of F. As a consequence, to a vertex v of F will correspond some vertices (at least one) of F_t which lie at a distance of order t from v. Notice that, for all the vertices v' of F_t corresponding to v, we still have $N(v) \in C(v')$. Moreover, there exists a constant $C>0$, depending on F, such that $d_{\mathcal{H}}(\partial F_t, \partial F) \leqslant C|t|$, for all t small enough. Let us also denote by F_t^- the facet obtained by parallelly translating the edges of F of a distance of $2C|t|$, in the direction $-\nu^F$. We then have $F_t^- \subset F_t$, for all t small enough. Notice that, in this case, to a vertex v of F corresponds only one vertex v^- of F_t^- respectively, and we have $N(v) \in C(v^-)$. It follows that, to any vertex v' of $F_t \setminus F_t^-$, we can uniquely associate a vertex v of F, and we set $N(v') := N(v)$.

In order to prove the result, it is enough to construct a vector field $\widetilde{\mathrm{N}}_t$ on F_t, with the property

$$\|\mathrm{div}_\tau \widetilde{\mathrm{N}}_t\|_{L^\infty(F_t)} = \|\mathrm{div}_\tau \mathrm{N}_{\min}\|_{L^\infty(F)} + \mathrm{O}(t).$$

Let $\psi_t : \mathbb{R}^N \to \mathbb{R}^N$ be a one-parameter family of Lipschitz diffeomorphisms such that $\psi_t(F_t^-) = F$ and $\|\psi_t - \mathrm{Id}\|_{W^{1,\infty}} \in \mathrm{O}(t)$. We define the field $\widetilde{\mathrm{N}}_t$ to be equal to $\mathrm{N}_{\min} \circ \psi_t$ on F_t^-, and to the linear interpolation of $N(v')$ on $F_t \setminus F_t^-$ (in order to do this we first perform a triangulation of $F_t \setminus F_t^-$, without adding new vertices). The thesis now follows by observing that

$$\|\mathrm{div}_\tau \widetilde{\mathrm{N}}_t\|_{L^\infty(F_t \setminus F_t^-)} = \mathrm{O}(t). \qquad \square$$

3. The ϕ-anisotropic total variation and ϕ-calibrable sets

Let $\phi : \mathbb{R}^N \to \mathbb{R}$ be an anisotropy and let ϕ° be its polar function. Since ϕ° is homogeneous of degree 1, for any $\eta \in \partial\phi^\circ(\xi)$ we have $\phi^\circ(\xi) = \eta \cdot \xi$. We also observe that

$$\chi \cdot \eta \leqslant \phi^\circ(\eta) \quad \text{for any } \chi \in \partial\phi^\circ(\xi), \text{ and any } \xi, \eta \in \mathbb{R}^N. \tag{3.1}$$

Consider the energy functional $\Psi_\phi : L^2(\mathbb{R}^N) \to (-\infty, +\infty]$ defined by

$$\Psi_\phi(u) := \begin{cases} \int_{\mathbb{R}^N} \phi^\circ(Du) & \text{if } u \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N), \\ +\infty & \text{if } u \in L^2(\mathbb{R}^N) \setminus BV(\mathbb{R}^N). \end{cases} \tag{3.2}$$

Since the functional Ψ_ϕ is convex, lower semicontinuous and proper, then $\partial\Psi_\phi$ is a maximal monotone operator with dense domain, generating a contraction semigroup in $L^2(\mathbb{R}^N)$ (see [21]). The next lemma gives the characterization of the subdifferential $\partial\Psi_\phi$ (the proof is the same as the proof of Proposition 1.10 in [9], see also [23], or [35] for more general cases).

Lemma 3.1. *Let $u \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$. The following assertions are equivalent*:

(a) $v \in \partial\Psi_\phi(u)$;

(b)

$$v \in L^2(\mathbb{R}^N) \quad \textit{and} \quad \exists z \in X_2(\mathbb{R}^N),\ \phi(z(x)) \leqslant 1 \textit{ a.e., such that } v = -\mathrm{div}\, z \textit{ in } \mathcal{D}'(\mathbb{R}^N) \tag{3.3}$$

and

$$\int_{\mathbb{R}^N} (z, Du) = \int_{\mathbb{R}^N} \phi^\circ(Du). \tag{3.4}$$

From now on we shall sometimes write $v = \mathrm{div}(\partial\phi^\circ(Du))$ instead of $v \in \partial\Psi_\phi(u)$.

Under the rest of conditions of (b), condition $(*)$ $\phi(z(x)) \leqslant 1$ is equivalent to say that $(**)$ $z(x) \in \partial\phi^\circ(\nabla u(x))$ a.e. Obviously, by (3.1), $(**)$ implies $(*)$. Assume now that $\phi(z(x)) \leqslant 1$. Then (3.4) implies that $(z, Du) = \phi^\circ(Du)$ as measures in $\mathbb{R}^N$. Hence $z(x) \cdot \nabla u(x) = \phi^\circ(\nabla u(x))$ a.e. Then

$$\phi^\circ(\eta) - \phi^\circ(\nabla u(x)) \geqslant \langle z(x), \eta - \nabla u(x) \rangle \quad \forall \eta \in \mathbb{R}^N,$$

is equivalent to

$$\phi^\circ(\eta) \geqslant \langle z(x), \eta \rangle \quad \forall \eta \in \mathbb{R}^N$$

and this follows from $\phi(z(x)) \leqslant 1$. We deduce that $z(x) \in \partial\phi^\circ(\nabla u(x))$ a.e.

Given a function $g \in L^2(\mathbb{R}^N)$, we define

$$\|g\|_{\phi,*} := \sup\left\{ \int_{\mathbb{R}^N} g(x)u(x)\,dx\colon\ u \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N),\ \int_{\mathbb{R}^N} \phi^\circ(Du) \leqslant 1 \right\}.$$

Note that $\|g\|_{\phi,*}$ may be infinite. Let us recall the following result [14,34].

Lemma 3.2. *Let $f \in L^2(\mathbb{R}^N)$ and $\lambda > 0$. The following assertions hold.*

(a) *the function u is the solution of*

$$\min_{w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)} D(w) := \int_{\mathbb{R}^N} \phi^\circ(Dw) + \frac{\lambda}{2} \int_{\mathbb{R}^N} (w - f)^2\,dx \tag{3.5}$$

if and only if there exists $z \in X_2(\mathbb{R}^N)$ satisfying (3.4) *such that $\phi(z(x)) \leqslant 1$ a.e. and* $\operatorname{div} z = \lambda(u - f)$.

(b) *The function $u \equiv 0$ is the solution of* (3.5) *if and only if $\|f\|_{\phi,*} \leqslant \frac{1}{\lambda}$.*

(c) *We have $\partial\Psi_\phi(0) = \{f \in L^2(\mathbb{R}^N)\colon\ \|f\|_{\phi,*} \leqslant 1\}$.*

Obviously, part (a) follows from Lemma 3.1 since $\partial\Psi_\phi(u) + \lambda(u - f) \ni 0$ is the Euler–Lagrange equation for (3.5). Part (b) can be found in [14,34], and it is easily deduced from (a). Part (c) follows from (a) and (b), or as an immediate consequence of duality.

Definition 3.3. Let E be a bounded set of finite perimeter in $\mathbb{R}^N$. We say that E is ϕ-calibrable if there exists a vector field $\xi \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$ with $\phi(\xi(x)) \leqslant 1$ a.e. such that $(\xi, D\chi_E) = \phi^\circ(D\chi_E)$ as measures in R^N, and

$$-\operatorname{div} \xi = \lambda_E \chi_E \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{3.6}$$

for some constant λ_E.

Notice that, a set of finite perimeter E is ϕ-calibrable if and only if it exists $\lambda_E \in \mathbb{R}$ such that $\lambda_E \chi_E \in \partial\Psi_\phi(\chi_E)$. Observe that if E is ϕ-calibrable, then $\lambda_E = \frac{P_\phi(E)}{|E|} := \lambda_E^\phi$. Indeed, multiplying (3.6) by χ_E and integrating in $\mathbb{R}^N$ we obtain

$$\lambda_E |E| = -\int_{\mathbb{R}^N} \operatorname{div} \xi\, \chi_E\,dx = \int_{\mathbb{R}^N} (\xi, D\chi_E) = \int_{\mathbb{R}^N} \phi^\circ(D\chi_E) = P_\phi(E).$$

The following result was proved in [17,19] (see also [14]). For the proof we refer to [2, Proposition 2] and we skip the details. Lemma 3.6 below is used in the proof of Proposition 3.4.

Proposition 3.4. *Let E be a bounded set of finite perimeter in $\mathbb{R}^N$. Assume E to be convex. The following assertions are equivalent*

(i) *E is ϕ-calibrable*;

(ii) *E minimizes the functional*

$$P_\phi(X) - \lambda_E |X| \tag{3.7}$$

among the sets of finite perimeter $X \subseteq E$.

For the proof of the following result we refer to [14,9,2].

Proposition 3.5. *Let* $\lambda > 0$. *The solution of*

$$u - \lambda^{-1} \operatorname{div}\big(\partial\phi^\circ(Du)\big) = \chi_{\mathcal{W}_\phi} \quad \text{in } \mathbb{R}^N \tag{3.8}$$

is $u = (1 - \frac{\lambda^\phi_{\mathcal{W}_\phi}}{\lambda})^+ \chi_{\mathcal{W}_\phi}$.

Finally, the following result can be proved as in [5].

Lemma 3.6. *For any set of finite perimeter* E *in* $\mathbb{R}^N$ *and any convex set* C *we have*

$$P_\phi(E \cap C) \leqslant P_\phi(E). \tag{3.9}$$

4. The level sets of the solution of a variational problem

Proposition 4.1. *Let* C *be a bounded convex domain in* $\mathbb{R}^N$. *Let* $u \in BV(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ *be the solution of the variational problem*

$$(Q)_\lambda: \quad \min_{u \in BV(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} \phi^\circ(Du) + \frac{\lambda}{2} \int_{\mathbb{R}^N} (u - \chi_C)^2\, dx \right\}. \tag{4.1}$$

Then $0 \leqslant u \leqslant 1$. *Let* $E_s := \{u \geqslant s\}$, $s \in (0,1]$. *Then* $E_s \subseteq C$, *and, for any* $s \in (0,1]$, *we have*

$$P_\phi(E_s) - \lambda(1-s)|E_s| \leqslant P_\phi(F) - \lambda(1-s)|F| \quad \forall F \subseteq C. \tag{4.2}$$

Proof. Recall that u satisfies the following partial differential equation

$$u - \lambda^{-1} \operatorname{div}\big(\partial\phi^\circ(Du)\big) = \chi_C \quad \text{in } \mathbb{R}^N. \tag{4.3}$$

Let $u^- = \min(u, 0)$. Multiplying (4.3) by u^- and integrating by parts, we deduce that $u^- = 0$. Similarly, multiplying (4.3) by $(u-1)^+$ we deduce that $u \leqslant 1$. Let us prove that $u = 0$ outside C. Let H be a half-plane containing C. Since $\chi_C \leqslant \chi_H$, and $v = \chi_H$ is the solution of (4.3) with right-hand side equal to v (indeed it suffices to take $z(x) = \eta \in \partial\phi^\circ(\nu^H)$, ν^H being the euclidean unit normal to H pointing towards H), by the comparison principle proved in [23] (see also [14]) we have that $u \leqslant \chi_H$. This implies that $u = 0$ outside C, hence $E_s \subseteq C$ for all $s \in (0,1]$.

Let $F \subseteq C$ be a set of finite perimeter. By the proof of Lemma 5.1 in [23] (see also Lemma 4 in [13]), we have that $(z, D\chi_{E_s}) = \phi^\circ(D\chi_{E_s})$ for almost all $s \in (0,1]$. Hence, for such an $s \in (0,1]$, we have

$$-\int_{\mathbb{R}^N} \operatorname{div} z\,(\chi_F - \chi_{E_s})\, dx = \int_{\mathbb{R}^N} (z, D\chi_F) - \int_{\mathbb{R}^N} (z, D\chi_{E_s}) = \int_{\mathbb{R}^N} (z, D\chi_F) - P_\phi(E_s) \leqslant P_\phi(F) - P_\phi(E_s)$$

and we deduce

$$P_\phi(F) - P_\phi(E_s) \geqslant \lambda \int_{\mathbb{R}^N} (\chi_C - u)(\chi_F - \chi_{E_s}) = \lambda \int_{\mathbb{R}^N} \big((\chi_C - s) + (s - u)\big)(\chi_F - \chi_{E_s}).$$

Since $(s-u)(\chi_F - \chi_{E_s}) \geqslant 0$ we have

$$P_\phi(F) - P_\phi(E_s) \geqslant \lambda \int_{\mathbb{R}^N} (\chi_C - s)(\chi_F - \chi_{E_s}) = \lambda(1-s)\big(|F| - |E_s|\big).$$

Since all sets E_s are contained in C and P_ϕ is lower semicontinuous in the L^1-topology, we deduce that (4.2) holds for any $s \in (0,1]$. □

Lemma 4.2. *Let* C *be a bounded convex domain in* $\mathbb{R}^N$. *Let* u_λ *be the solution of* $(Q)_\lambda$, $\lambda > 0$.

(i) $u_\lambda \neq \chi_C$ *for any* $\lambda > 0$.
(ii) $u_\lambda \to \chi_C$ *in* $L^2(\mathbb{R}^N)$ *as* $\lambda \to \infty$.

(iii) *Assume that C satisfies the $R\mathcal{W}_\phi$-condition, for some $R > 0$. Then for any $\lambda > 0$, we have*

$$u_\lambda \geqslant \left(1 - \frac{N}{R\lambda}\right)^+ \chi_C.$$

(iv) $u_\lambda \neq 0$ *if and only if* $\lambda > \frac{1}{\|\chi_C\|_{\phi,*}}$.

(v) *Assume that C is not ϕ-calibrable* (*i.e., there is no vector field $z \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$, $\phi(z(x)) \leqslant 1$ a.e. such that $-\operatorname{div} z = \lambda_C^\phi \chi_C$*)*. For any $\lambda > \frac{1}{\|\chi_C\|_{\phi,*}}$ u_λ cannot be a multiple of χ_C. Thus, for any such λ, there is some $s \in [0,1]$ such that $\{u_\lambda \geqslant s\} \neq C$.*

Proof. (i) Suppose that there is $\lambda > 0$ such that $u_\lambda = \chi_C$. Then, by Lemma 3.2 there is a vector field $z_\lambda \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$, $\phi(z_\lambda(x)) \leqslant 1$ a.e., such that $(z_\lambda, D\chi_C) = \phi^\circ(D\chi_C)$ and

$$\operatorname{div} z_\lambda = 0.$$

Multiplying this equation by χ_C and integrating in $\mathbb{R}^N$, we obtain

$$0 = -\int_{\mathbb{R}^N} \operatorname{div} z_\lambda \chi_C \, dx = \int_{\mathbb{R}^N} (z_\lambda, D\chi_C) = \int_{\mathbb{R}^N} \phi^\circ(D\chi_C) = P_\phi(C).$$

This contradiction proves that $u_\lambda \neq \chi_C$.

(ii) Since

$$\int_{\mathbb{R}^N} \phi^\circ(Du_\lambda) + \frac{\lambda}{2} \int_{\mathbb{R}^N} (u_\lambda - \chi_C)^2 \, dx \leqslant \int_{\mathbb{R}^N} \phi^\circ(D\chi_C) = P_\phi(C),$$

we deduce that

$$\int_{\mathbb{R}^N} (u_\lambda - \chi_C)^2 \, dx \leqslant \frac{2}{\lambda} P_\phi(C),$$

i.e. $u_\lambda \to \chi_C$ in $L^2(\mathbb{R}^N)$ as $\lambda \to \infty$. Moreover, u_λ is bounded in $BV(\mathbb{R}^N)$.

(iii) Let $p \in \partial C$ and let W_p be the translation of $R\mathcal{W}_\phi$ which osculates from inside ∂C at p. Let us compare u_λ with the solution u_p of

$$u - \lambda^{-1} \operatorname{div}\big(\partial\phi^\circ(Du)\big) = \chi_{W_p}.$$

Since $\chi_{W_p} \leqslant \chi_C$, by the comparison principle [14] we deduce that $u_p \leqslant u_\lambda$. The solution u_p is given explicitly by

$$u_p = \left(1 - \frac{\lambda_{W_p}^\phi}{\lambda}\right)^+ \chi_{W_p}.$$

But

$$\lambda_{W_p}^\phi = \frac{P_\phi(W_p)}{|W_p|} = \frac{1}{R} \frac{P_\phi(\mathcal{W}_\phi)}{|\mathcal{W}_\phi|} = \frac{1}{R} \lambda_{\mathcal{W}_\phi}^\phi = \frac{N}{R}.$$

Hence

$$u_p = \left(1 - \frac{N}{R\lambda}\right)^+ \chi_{W_p}.$$

Since this is true for any $p \in \partial C$, and since also any p in the interior of C lies in some translation of $R\mathcal{W}_\phi$, we deduce that

$$u_\lambda \geqslant \left(1 - \frac{N}{R\lambda}\right)^+ \chi_C.$$

(iv) By Lemma 3.2, we know that u_λ is characterized by the equation

$$u_\lambda - \lambda^{-1} \operatorname{div} z_\lambda = \chi_C$$

where $z_\lambda \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$, $\phi(z_\lambda(x)) \leqslant 1$ a.e., with $(z_\lambda, Du_\lambda) = \phi^\circ(Du_\lambda)$. Thus $u_\lambda = 0$ if and only if $-\operatorname{div} z_\lambda = \lambda\chi_C$, i.e. if and only if $\|\lambda\chi_C\|_{\phi,*} \leqslant 1$.

(v) Suppose that for some $\lambda > \frac{1}{\|\chi_C\|_{\phi,*}}$, we have $u_\lambda = c_\lambda\chi_C$ for some constant $0 \leqslant c_\lambda \leqslant 1$. Observe that, by (i) and (iv), we have $c_\lambda \in (0,1)$. Then

$$-\operatorname{div} z_\lambda = \lambda(1-c_\lambda)\chi_C.$$

Since $(z_\lambda, Du_\lambda) = \phi^\circ(Du_\lambda)$, and $c_\lambda > 0$, we have that $(z_\lambda, D\chi_C) = \phi^\circ(D\chi_C) = P_\phi(C)$. Multiplying the equation by χ_C and integrating by parts we deduce

$$\lambda(1-c_\lambda) = \lambda_C.$$

Hence

$$-\operatorname{div} z_\lambda = \lambda_C\chi_C,$$

and therefore C is ϕ-calibrable, a contradiction. The final assertion is a simple consequence of the first one. □

Lemma 4.3. *Let C be a bounded convex domain in $\mathbb{R}^N$. For any $\lambda > 0$, let us consider the problem*

$$(P)_\lambda: \quad \min_{F\subseteq C} P_\phi(F) - \lambda|F|. \tag{4.4}$$

Then

(i) *Let C_λ, C_μ be minimizers of $(P)_\lambda$ and $(P)_\mu$, respectively. If $\lambda < \mu$, then $C_\lambda \subseteq C_\mu$.*
(ii) *Let $\mu > \lambda$. Assume that C is a minimizer of $(P)_\lambda$. Then C is also the unique minimizer of $(P)_\mu$.*
(iii) *Let $\lambda_n \uparrow \lambda$. Then $C_\lambda^\cup := \bigcup_n C_{\lambda_n}$ is a minimizer of $(P)_\lambda$. Moreover $P_\phi(C_{\lambda_n}) \to P_\phi(C_\lambda^\cup)$. Similarly, if $\lambda_n \downarrow \lambda$, then $C_\lambda^\cap := \bigcap_n C_{\lambda_n}$ is a minimizer of $(P)_\lambda$, and $P_\phi(C_{\lambda_n}) \to P_\phi(C_\lambda^\cap)$.*
(iv) *Assume that C satisfies the $R\mathcal{W}_\phi$-condition, for some $R > 0$. Then C is a minimizer of $(P)_\lambda$ for any $\lambda \geqslant \frac{\lambda^\phi_{\mathcal{W}_\phi}}{R}$.*

Proof. The proof is similar to the proof of Lemma 4 in [13] and we only give the proof of (iv).

By (ii), it suffices to prove that C is a solution of $(P)_{\lambda^\phi_{\mathcal{W}_\phi}/R}$. Let $\eta > \frac{\lambda^\phi_{\mathcal{W}_\phi}}{R}$ and take $0 < s_n < 1 - \frac{\lambda^\phi_{\mathcal{W}_\phi}}{R\eta}$ such that $\eta(1-s_n) \downarrow \frac{\lambda^\phi_{\mathcal{W}_\phi}}{R}$. We observe that, by Lemma 4.2(iii), we have $\{u_\eta \geqslant s_n\} = C$ and, by Proposition 4.1, C is a minimizer of

$$P_\phi(F) - \eta(1-s_n)|F|. \tag{4.5}$$

Now, by assertion (iii) in the present lemma, we deduce that C is also a minimizer of

$$P_\phi(F) - \frac{\lambda^\phi_{\mathcal{W}_\phi}}{R}|F|. \qquad \square \tag{4.6}$$

Remark 4.4. In Proposition 4.1 we have proved that for any $s \in (0,1]$, the level set $\{u_\lambda \geqslant s\}$ is a minimizer of $(P)_{\lambda(1-s)}$. Moreover, by Lemma 4.3, the sets $\{u_\lambda \geqslant s\}^\cup := \bigcup_{\epsilon>0}\{u_\lambda \geqslant s+\epsilon\}$, $s \in [0,1)$, and $\{u_\lambda \geqslant s\}^\cap := \bigcap_{\epsilon>0}\{u_\lambda \geqslant s-\epsilon\}$, $s \in (0,1]$, are also minimizers of $(P)_{\lambda(1-s)}$ (obviously $\{u_\lambda \geqslant 1\}^\cup = \emptyset$ is also a minimizer of $(P)_0$). Notice that, except on countably many values of s, they both coincide with $\{u_\lambda \geqslant s\}$.

5. The concavity of solutions of $(Q)_\lambda$

Our purpose is to prove the following result.

Theorem 5.1. *Let C be a bounded convex domain in $\mathbb{R}^N$ satisfying the $R\mathcal{W}_\phi$-condition, for some $R > 0$. If $\lambda \geqslant \frac{2N}{R}$, then the solution u_λ of $(Q)_\lambda$ is concave in C. In particular $\{u_\lambda \geqslant s\}$ is convex for any $s \in [0,1]$.*

Before going into the proof, we observe that, being concave in C, u_λ is continuous in C. In particular $\{u_\lambda \geqslant s\}^\cap = \{u_\lambda \geqslant s\}$ and $\{u_\lambda \geqslant s\}^\cup = \{u_\lambda > s\}$, and $\{u_\lambda \geqslant s\} = \{u_\lambda > s\}$ (modulo a null set) for any $s \in (0, \max\{u_\lambda\})$.

The result is a consequence of Korevaar's concavity result [31]. First we need to recall some approximation results which reduce the proof of Theorem 5.1 to the case of a smooth anisotropy.

5.1. The approximation of a generic anisotropy ϕ with smooth ones

The following result is proved in [37, Theorem 3.3.1 and p. 111].

Theorem 5.2. *Let $\epsilon > 0$ and let $\eta : [0,\infty) \to [0,\infty)$ be a function of class $\mathcal{C}^\infty$ with support in $[\frac{\epsilon}{2}, \epsilon]$ and with $\int_{\mathbb{R}^N} \eta(|x|)\,dx = 1$. If $\phi^\circ : \mathbb{R}^N \to [0,+\infty)$ is an anisotropy, then the function $\widetilde{\phi^\circ}$ defined by*

$$\widetilde{\phi^\circ}(\xi) := \int_{\mathbb{R}^N} \phi^\circ\big(\xi + |\xi| z\big)\eta\big(|z|\big)\,dz, \quad \xi \in \mathbb{R}^N, \tag{5.1}$$

is an anisotropy of class $\mathcal{C}^\infty(\mathbb{R}^N \setminus \{0\})$.

Similarly, given a convex body K, define the map $K \mapsto \mathcal{T}(K)$ as follows: let $\tilde{h}_K(\xi) := \int_{\mathbb{R}^N} h_K(\xi + |\xi| z)\eta(|z|)\,dz$ for any $\xi \in \mathbb{R}^N$: then, $\tilde{h}_K$ is the support function $h_{\mathcal{T}(K)}$ of $\mathcal{T}(K)$. The map $\mathcal{T}$ has the following properties: if K_1 and K_2 are two convex bodies, then

(a) $\mathcal{T}(K_1 + K_2) = \mathcal{T}(K_1) + \mathcal{T}(K_2)$ *and* $\mathcal{T}(\alpha K_1) = \alpha \mathcal{T}(K_1)$ *for any* $\alpha > 0$;
(b) *if K_1 is contained in B_R, then* $d_{\mathcal{H}}(K_1, \mathcal{T}(K_1)) \leqslant R\epsilon$;
(c) $d_{\mathcal{H}}(\mathcal{T}(K_1), \mathcal{T}(K_2)) \leqslant (1+\epsilon) d_{\mathcal{H}}(K_1, K_2)$;
(d) $\mathcal{T}(K_1) + B_\epsilon$ *is of class* $\mathcal{C}^\infty_+$.

Theorem 5.2 provides a way to approximate at the same time a generic anisotropy with $\mathcal{C}^\infty_+$ anisotropies and a convex set with $\mathcal{C}^\infty_+$ convex sets. Indeed, the following result holds [13].

Lemma 5.3. *Let ϕ be an anisotropy, and let C be a convex body in $\mathbb{R}^N$. Then there exist a sequence $\{\phi_\epsilon\}$ of anisotropies and a sequence $\{C_\epsilon\}$ of compact convex sets satisfying the following properties:*

(i) *$\{\phi_\epsilon\}$ converges to ϕ uniformly on $\mathbb{R}^N$ as $\epsilon \to 0$;*
(ii) *$\{C_\epsilon\}$ converges to C in the Hausdorff distance as $\epsilon \to 0$;*
(iii) *$\phi_\epsilon, \phi_\epsilon^\circ \in \mathcal{C}^\infty_+$ and C_ϵ is of class $\mathcal{C}^\infty_+$ for any $\epsilon > 0$;*
(iv) *if C satisfies the $r\mathcal{W}_\phi$-condition, $r > 0$, then C_ϵ satisfies the $r\mathcal{W}_{\phi_\epsilon}$-condition for any $\epsilon > 0$.*

Proof. Let $\mathcal{T}$ be the map defined in Theorem 5.2. Let ϕ_ϵ be the anisotropy such that $\mathcal{W}_{\phi_\epsilon} = \mathcal{T}(\mathcal{W}_\phi) + B_\epsilon$; then $\phi_\epsilon \in \mathcal{C}^\infty_+$ by (d) of Theorem 5.2, hence also $\phi_\epsilon^\circ \in \mathcal{C}^\infty_+$ by (a) of Remark 2.3. Then (b) of Theorem 5.2 yields (i). Let $C_\epsilon := \mathcal{T}(C) + B_{r\epsilon}$. It is clear that (ii) is satisfied. From Theorem 5.2(d) we have that C_ϵ is of class $\mathcal{C}^\infty_+$. Assume that C satisfies the $r\mathcal{W}_\phi$-condition, thus there exists $C' \subset C$ such that $C = C' + r\mathcal{W}_\phi$. By (a) in Theorem 5.2 we have

$$\begin{aligned} C_\epsilon &= \mathcal{T}(C) + B_{r\epsilon} = \mathcal{T}(C') + r\mathcal{T}(\mathcal{W}_\phi) + B_{r\epsilon} \\ &= \mathcal{T}(C') + r\big(\mathcal{T}(\mathcal{W}_\phi) + B_\epsilon\big) = \mathcal{T}(C') + r\mathcal{W}_{\phi_\epsilon}, \end{aligned}$$

hence (iv) follows. □

Observe that

$$\phi_\epsilon^\circ(\xi) = \sup_{x \in \mathcal{T}(\mathcal{W}_\phi) + B_\epsilon} x \cdot \xi = \sup_{y \in \mathcal{T}(\mathcal{W}_\phi)} \sup_{z \in B_\epsilon} (y + z) \cdot \xi = \widetilde{\phi^\circ}(\xi) + \epsilon|\xi|. \tag{5.2}$$

We also observe that, from (5.1) we get

$$\big|\widetilde{\phi^\circ}(\xi) - \phi^\circ(\xi)\big| \leqslant \epsilon|\xi| \quad \forall \xi \in \mathbb{R}^N. \tag{5.3}$$

5.2. The Dirichlet problem

Let Ω be an open bounded subset of $\mathbb{R}^N$ with boundary of class C^1, let $h \in L^1(\partial\Omega)$, and let $\Psi_{\phi,h} : L^2(\Omega) \to (-\infty, +\infty]$ be the functional defined by

$$\Psi_{\phi,h}(u) := \begin{cases} \int_\Omega \phi^\circ(Du) + \int_{\partial\Omega} |u-h|\phi^\circ(\nu^\Omega)\, d\mathcal{H}^{N-1} & \text{if } u \in L^2(\Omega) \cap BV(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega). \end{cases} \tag{5.4}$$

The functional $\Psi_{\phi,h}$ is convex and lower semicontinuous in $L^2(\Omega)$, hence $\partial\Psi_{\phi,h}$ is a maximal monotone operator in $L^2(\Omega)$. Let us recall the characterization of $\partial\Psi_{\phi,h}$.

Proposition 5.4. *The following conditions are equivalent*

(i) $v \in \partial\Psi_{\phi,h}(u)$;
(ii) $u, v \in L^2(\Omega)$, $u \in BV(\Omega)$ *and there exists* $z \in X_2(\Omega)$ *with* $\phi(z(x)) \leqslant 1$ *a.e.,* $v = -\operatorname{div}(z)$ *in* $\mathcal{D}'(\Omega)$ *such that* $(z, Du) = \phi^\circ(Du)$ *and* $[z \cdot \nu^\Omega] \in \operatorname{sign}(h-u)\phi^\circ(\nu^\Omega(x))\ \mathcal{H}^{N-1}$ *a.e. on* $\partial\Omega$.

Proof. In the case $h = 0$, which is the case we need below, the result follows as in [9, Proposition 1.10], since $\Psi_{\phi,0}$ is positively homogeneous of degree 1. The general case is contained in [35]. Since we need some intermediate results, we shall sketch a direct proof of it.

Assume first that ϕ is a smooth anisotropy and fix $\epsilon > 0$. Let

$$\Psi^\epsilon_{\phi,h}(u) := \begin{cases} \int_\Omega \sqrt{\epsilon^2 + \phi^\circ(Du)^2} + \int_{\partial\Omega} |u-h|\phi^\circ(\nu^\Omega)\, d\mathcal{H}^{N-1} & \text{if } u \in L^2(\Omega) \cap BV(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega). \end{cases} \tag{5.5}$$

By [9, Theorem 6.7] (see also [33,7,35]) we know that $\partial\Psi^\epsilon_{\phi,h}$ is a maximal monotone operator which can be characterized as Proposition 5.4. Since, as $\epsilon \to 0$, the solutions of

$$u + \lambda\partial\Psi^\epsilon_{\phi,h}(u) \ni f,$$

where $f \in L^2(\Omega)$ converge to the solution of $u + \lambda\partial\Psi_{\phi,h}(u) \ni f$, the thesis follows. The case of a general anisotropy also follows by approximating it with smooth ones. □

The following comparison principle can be easily deduced by an integration by parts.

Proposition 5.5. *Let* $f_i \in L^2(\Omega)$, $h_i \in L^1(\partial\Omega)$, $i = 1, 2$. *Assume that* $f_1 \leqslant f_2$ *and* $h_1 \leqslant h_2$. *Let* u_i, $i = 1, 2$, *be the solution of*

$$u + \lambda\partial\Psi_{\phi,h_i}(u) \ni f_i. \tag{5.6}$$

Then $u_1 \leqslant u_2$.

The same result also holds for $\partial\Psi^\epsilon_{\phi,h}$ [9, Theorem 6.14], [35].

5.3. Some technical results

We recall two auxiliary results. The following theorem was proved in [31].

Theorem 5.6. *Assume that* $\phi \in \mathcal{C}^\infty_+$ *and* $\mathcal{W}_\phi$ *is strictly convex. Let* Ω *be a strictly convex bounded domain in* $\mathbb{R}^N$ *of class* $\mathcal{C}^{1,1}$. *Let* $b : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *be such that*

$$\frac{\partial b}{\partial u} > 0, \qquad \frac{\partial^2 b}{\partial u^2} \geqslant 0.$$

Assume that $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ *satisfies*

$$\operatorname{div}\left(\frac{T^\circ(Du)}{\sqrt{1+\phi^\circ(Du)^2}}\right) = b(u, Du)$$

and the graph of u is a C^1 surface above $\overline{\Omega}$ having zero contact angle with $\partial\Omega \times \mathbb{R}$, i.e.

$$\frac{T^\circ(Du)}{\sqrt{1+\phi^\circ(Du)^2}} \cdot \nu^\Omega = -\phi^\circ\big(\nu^\Omega\big). \tag{5.7}$$

Then u is a concave function.

The sense of the boundary condition (5.7) will be made precise during the proof of Theorem 5.1. Let us recall the following result which was proved in [2] using the results by Atkinson and Peletier in [11].

Lemma 5.7. *Assume that $\lambda \geqslant \frac{2N}{R}$, $R > 0$. Then there is a radius $\tilde{R} \leqslant R$ and a radial solution $u_{\tilde{B}}(x) = U_{\tilde{B}}(|x|)$ of*

$$\begin{aligned} &u - \lambda^{-1}\operatorname{div}\left(\frac{Du}{\sqrt{\epsilon^2+|Du|^2}}\right) = 1 \quad \text{in } \tilde{B} = B(0,\tilde{R}),\\ &u = 0 \quad \text{on } \partial\tilde{B}, \end{aligned} \tag{5.8}$$

such that

$$\begin{aligned} &0 > U'_{\tilde{B}}(r) > -\infty, \qquad U < U_{\tilde{B}}(r) < \gamma \quad \text{for } 0 < r < \tilde{R},\\ &U'_{\tilde{B}}(r) \to -\infty, \qquad U_{\tilde{B}}(r) \to U \quad \text{as } r \uparrow \tilde{R}, \end{aligned}$$

for some values $\gamma > 0$, $U > 0$. Hence $u_{\tilde{B}} \geqslant U > 0$ on $\tilde{B}$, and has zero contact angle with $\partial\tilde{B} \times \mathbb{R}$. In particular, we have

$$\frac{Du_{\tilde{B}}}{\sqrt{\epsilon^2+|Du_{\tilde{B}}|^2}} \cdot \nu^{\tilde{B}} = -1 = \operatorname{sign}(-u_{\tilde{B}}) \quad \text{on } \partial\tilde{B}.$$

Let us recall that the solution $u_{\tilde{B}}(x) = U_{\tilde{B}}(|x|)$ of (5.8) can be characterized as a minimizer of

$$\mathcal{E}_e(u) := \int_{\tilde{B}} \sqrt{\epsilon^2+|Du|^2} + \lambda \int_{\tilde{B}} F(u)\,dx + \int_{\partial\tilde{B}} |u|\,d\mathcal{H}^{N-1} \tag{5.9}$$

and $U_{\tilde{B}}(r)$ can be characterized as a minimizer of

$$\mathcal{E}_{er}(v) := \int_0^{\tilde{R}} \sqrt{\epsilon^2+v'^2}s^{N-1}\,ds + \lambda \int_0^{\tilde{R}} F(v)s^{N-1}\,ds + \tilde{R}^{N-1}\big|v(\tilde{R})\big|. \tag{5.10}$$

Lemma 5.8. *Assume that $\phi \in C^\infty_+$. Let $u_{\tilde{B}}$ be the solution of (5.8) given by Lemma 5.7. Let $u_{\tilde{W}}(x) = U_{\tilde{B}}(\phi(x))$, $x \in \tilde{W} := \tilde{R}\mathcal{W}_\phi$. Then $u_{\tilde{W}}$ is a solution of*

$$\begin{aligned} &u - \lambda^{-1}\operatorname{div}\left(\frac{T^\circ(Du)}{\sqrt{\epsilon^2+\phi^\circ(Du)^2}}\right) = 1 \quad \text{in } \tilde{W} = R\mathcal{W}_\phi,\\ &u = 0 \quad \text{on } \partial\tilde{W}. \end{aligned} \tag{5.11}$$

Before going into the proof let us make the following observation. If ϕ is an smooth anisotropy, then $\nabla\phi(x) = \nu_\phi^{\mathcal{W}_\phi}(x) = \frac{\nu(x)}{\phi^\circ(\nu(x))}$ on $\partial\mathcal{W}_\phi$ where $\nu(x)$ the euclidean unit normal to $\partial\mathcal{W}_\phi$, since $\phi(x) = d_\phi(x,\partial\mathcal{W}_\phi) - 1$. We also have $|\nabla\phi(x)| = \frac{1}{\phi^\circ(\nu(x))}$.

Proof. Let us write $F(u) = \frac{1}{2}(u-1)^2$. Recall that $u_{\tilde{W}}$ is a solution of (5.11) if and only if is a minimizer of

$$\mathcal{E}(u) := \int_{\tilde{W}} \sqrt{\epsilon^2+\phi^\circ(Du)^2} + \lambda \int_{\tilde{W}} F(u)\,dx + \int_{\partial\tilde{W}} |u|\phi^\circ\big(\nu^{\partial\tilde{W}}(x)\big)\,d\mathcal{H}^{N-1}. \tag{5.12}$$

Let $w \in W^{1,1}(\tilde{W})$. Then

$$\int_{\tilde{W}} \sqrt{\epsilon^2 + \phi^\circ(\nabla w)^2} = \int_0^{\tilde{R}} \int_{\{\phi=s\}} \sqrt{\epsilon^2 + \phi^\circ\big(\nabla w(x)\big)^2}\, \frac{d\mathcal{H}^{N-1}(x)}{|\nabla\phi(x)|}\, ds$$
$$= \int_0^{\tilde{R}} \int_{\{\phi=1\}} \sqrt{\epsilon^2 + \phi^\circ\big(\nabla w(sy)\big)^2}\, \frac{d\mathcal{H}^{N-1}(y)}{|\nabla\phi(y)|} s^{N-1}\, ds$$
$$= \int_{\{\phi=1\}} \frac{d\mathcal{H}^{N-1}(y)}{|\nabla\phi(y)|} \int_0^{\tilde{R}} \sqrt{\epsilon^2 + \phi^\circ\big(\nabla w(sy)\big)^2} s^{N-1}\, ds.$$

Let $w_y(s) = w(sy)$, $\phi(y) = 1$. Since $w_y'(s) = y\nabla w(sy) \leqslant \phi^\circ(\nabla w(sy))$ for any $y \in \{\phi = 1\}$, we obtain

$$\int_{\tilde{W}} \sqrt{\epsilon^2 + \phi^\circ(\nabla w)^2} \geqslant \int_{\{\phi=1\}} \frac{d\mathcal{H}^{N-1}(y)}{|\nabla\phi(y)|} \int_0^{\tilde{R}} \sqrt{\epsilon^2 + \big|w_y'(s)\big|^2} s^{N-1}\, ds.$$

In a similar way we have

$$\int_{\tilde{W}} F(u)\, dx = \int_{\{\phi=1\}} \frac{d\mathcal{H}^{N-1}(y)}{|\nabla\phi(y)|} \int_0^{\tilde{R}} F\big(w_y(s)\big) s^{N-1}\, ds$$

and, using that $|\nabla\phi(x)| = |\nu_\phi(x)| = 1/\phi^\circ(\nu(x))$, we have

$$\int_{\partial\tilde{W}} |u|\phi^\circ\big(\nu^{\partial\tilde{W}}(x)\big)\, d\mathcal{H}^{N-1} = \tilde{R}^{N-1} \int_{\{\phi=1\}} \frac{d\mathcal{H}^{N-1}(y)}{|\nabla\phi(y)|} \big|w_y(\tilde{R})\big|.$$

Since $U_{\tilde{B}}$ is a minimizer of $\mathcal{E}_{er}$, by the above inequalities, we have

$$\mathcal{E}(w) \geqslant \int_{\{\phi=1\}} \frac{d\mathcal{H}^{N-1}(y)}{|\nabla\phi(y)|} \int_0^{\tilde{R}} \sqrt{\epsilon^2 + \big|w_y'(s)\big|^2} s^{N-1}\, ds + \int_{\{\phi=1\}} \frac{d\mathcal{H}^{N-1}(y)}{|\nabla\phi(y)|} \int_0^{\tilde{R}} F\big(w_y(s)\big) s^{N-1}\, ds$$
$$+ \tilde{R}^{N-1} \int_{\{\phi=1\}} \frac{d\mathcal{H}^{N-1}(y)}{|\nabla\phi(y)|} \big|w_y(\tilde{R})\big|$$
$$= \int_{\{\phi=1\}} \frac{d\mathcal{H}^{N-1}(y)}{|\nabla\phi(y)|} \mathcal{E}_{er}(w_y) \geqslant \int_{\{\phi=1\}} \frac{d\mathcal{H}^{N-1}(y)}{|\nabla\phi(y)|} \mathcal{E}_{er}(U_{\tilde{B}}).$$

Now, we have $\nabla u_{\tilde{W}}(x) = U_{\tilde{B}}(\phi(x))\nabla\phi(x)$, hence $\phi^\circ(\nabla u_{\tilde{W}}(x)) = |U_{\tilde{B}}(\phi(x))|\phi^\circ(\nabla\phi(x)) = |U_{\tilde{B}}(\phi(x))|$. With the same computations as above we obtain

$$\mathcal{E}(u_{\tilde{W}}) = \int_{\{\phi=1\}} \frac{d\mathcal{H}^{N-1}(y)}{|\nabla\phi(y)|} \mathcal{E}_{er}(U_{\tilde{B}})$$

and we deduce that

$$\mathcal{E}(w) \geqslant \mathcal{E}(u_{\tilde{W}})$$

for any $w \in W^{1,1}(\tilde{W})$. This implies that $u_{\tilde{W}}$ is a minimizer of $\mathcal{E}$, hence, a solution of (5.11). □

Theorem 5.9. *Assume that $\phi \in \mathcal{C}_+^\infty$. Let C be a bounded convex domain in $\mathbb{R}^N$ satisfying the $R\mathcal{W}_\phi$-condition, for some $R > 0$. Let $\lambda \geqslant \frac{2N}{R}$. Let us consider the following problem*

$$\begin{cases} u - \lambda^{-1} \operatorname{div}\left(\frac{T^\circ(Du)}{\sqrt{\epsilon^2 + \phi^\circ(Du)^2}}\right) = 1 & \text{in } C, \\ u = 0 & \text{on } \partial C. \end{cases} \tag{5.13}$$

Then, there is a unique solution u^ϵ of (5.13) *such that $0 \leqslant u^\epsilon \leqslant 1$. Moreover $u^\epsilon \geqslant \alpha > 0$ in a neighborhood of ∂C for some $\alpha > 0$. Hence, u satisfies*

$$\left[\frac{T^\circ(Du^\epsilon)}{\sqrt{\epsilon^2 + \phi^\circ(Du^\epsilon)^2}} \cdot \nu^C\right] = \operatorname{sign}\left(-u^\epsilon\right)\phi^\circ\left(\nu^C\right) = -\phi^\circ\left(\nu^C\right) \quad \text{on } \partial C. \tag{5.14}$$

Proof. Existence and uniqueness of a solution u^ϵ of (5.13) satisfying the Dirichlet boundary condition in the generalized sense follows by the results in [8,33]. Multiplying (5.13) by test functions as in the proof of Proposition 4.1 and integrating by parts we deduce that $0 \leqslant u^\epsilon \leqslant 1$.

Let us prove that $u^\epsilon \geqslant \alpha > 0$ for some $\alpha > 0$. For that we shall use Lemmas 5.7 and 5.8. Recall that at each point $p \in \partial C$, there is a ball W_p of radius $\tilde{R} \leqslant R$ such that $W_p \subseteq C$ and $p \in \partial W_p$. Since the solution u^ϵ of (5.13) in C satisfies $u^\epsilon \geqslant 0$ in W_p, by applying the comparison principle for the problem (5.11) in W_p instead of $\tilde{W}$ (see Section 5.2) we deduce that $u^\epsilon \geqslant u_{W_p} \geqslant U$. Since this is true for all balls W_p, we deduce that $u^\epsilon \geqslant U$ on a neighborhood of ∂C. Finally, by Proposition 5.4 in Section 5.2 we get (5.14). □

5.4. The proof of Theorem 5.1

Let ϕ be any anisotropy, and assume that C satisfies the $R\mathcal{W}_\phi$-condition. Let $\phi_\delta \in \mathcal{C}_+^\infty$, C_δ be the regularization of ϕ and C given by Lemma 5.3. We know that C_δ satisfies the $R\mathcal{W}_{\phi_\delta}$-condition, hence is Lipschitz ϕ_δ-regular by Remark 2.9(b). By Theorem 5.9, for any $\lambda \geqslant \frac{2N}{R}$ there is a solution u^ϵ of

$$\begin{cases} u - \lambda^{-1} \operatorname{div}\left(\frac{T^\circ_{\phi_\delta}(Du)}{\sqrt{\epsilon^2 + \phi^\circ_\delta(Du)^2}}\right) = 1 & \text{in } C_\delta, \\ u = 0 & \text{on } \partial C_\delta. \end{cases} \tag{5.15}$$

Let $v^\epsilon(x) = u^\epsilon(\frac{x}{\epsilon})$. We know that $v^\epsilon(x)$ is a solution of

$$\operatorname{div}\left(\frac{T^\circ_{\phi_\delta}(Dv)}{\sqrt{1 + \phi^\circ_\delta(Dv)^2}}\right) + \frac{\lambda}{\epsilon}(1 - v) = 0 \quad \text{in } \epsilon C_\delta, \tag{5.16}$$

satisfying

$$\left[\frac{T^\circ_{\phi_\delta}(Dv^\epsilon)}{\sqrt{1 + \phi^\circ_\delta(Dv^\epsilon)^2}} \cdot \nu^{\epsilon C_\delta}\right] = \operatorname{sign}\left(-v^\epsilon\right)\phi^\circ_\delta\left(\nu^{\epsilon C_\delta}\right) = -\phi^\circ_\delta\left(\nu^{\epsilon C_\delta}\right) \quad \text{on } \partial(\epsilon C). \tag{5.17}$$

Moreover, by the results of Korevaar and Simon [32, Theorems 2, 3 and Section 3] (see also [38]), since C_δ is a bounded convex domain of class C^∞, we have that $v^\epsilon \in C^2(\epsilon C_\delta) \cap C(\overline{\epsilon C_\delta})$. Indeed, by the results in [32] (Theorems 2, 3 and Section 3), there is a solution $w^\epsilon \in C^2(\epsilon C_\delta) \cap C(\overline{\epsilon C_\delta})$ of (5.16) satisfying the boundary condition in a classical sense, that is, $\frac{T^\circ_{\phi_\delta}(Dw^\epsilon)}{\sqrt{1+\phi^\circ_\delta(Dw^\epsilon)^2}} \in C(\overline{\epsilon C_\delta})$ (even more, is a Lipschitz function on the graph of w^ϵ) and (5.17) holds. Since the solution of (5.16)–(5.17) is unique [8,33], we have that $w^\epsilon = v^\epsilon$. Hence $v^\epsilon \in C^2(\epsilon C_\delta) \cap C(\overline{\epsilon C_\delta})$.

From Korevaar's Theorem 5.6 [31], we then deduce that v^ϵ is concave, hence also u^ϵ is concave. Since, as $\epsilon \to 0$, u^ϵ converges to the solution w_δ of

$$\begin{aligned} &u - \lambda^{-1} \operatorname{div}\left(\partial\phi^\circ_\delta(Du)\right) = 1 \quad \text{in } C_\delta, \\ &u = 0 \quad \text{on } \partial C_\delta \end{aligned} \tag{5.18}$$

we deduce that w_δ is also concave. Moreover, from Theorem 5.9 and Lemma 5.7 we also know that $w_\delta \geqslant \beta > 0$ (which comes also by a comparison with balls). Thus the vector field ξ_δ satisfies $\phi_\delta(\xi_\delta(x)) \leqslant 1$ a.e., $(\xi_\delta, Dw_\delta) = \phi^\circ_\delta(Dw_\delta)$,

$w_\delta - \lambda^{-1} \operatorname{div} \xi_\delta = 1$ on C_δ, and $[\xi_\delta \cdot \nu^{C_\delta}] = -\phi_\delta^\circ(\nu^{C_\delta})$. Hence, if we define $w_\delta = 0$ outside C_δ (see Remark 2.11), we have that w_δ is a solution of

$$u - \lambda^{-1} \operatorname{div}\big(\partial \phi_\delta^\circ(Du)\big) = \chi_{C_\delta} \quad \text{in } \mathbb{R}^N. \tag{5.19}$$

Finally, letting $\delta \to 0^+$, we have that w_δ converges in $L^2(\mathbb{R}^N)$ to a solution w_λ of

$$u - \lambda^{-1} \operatorname{div}\big(\partial \phi^\circ(Du)\big) = \chi_C \quad \text{in } \mathbb{R}^N, \tag{5.20}$$

which is concave in C. Hence $w_\lambda = u_\lambda$. We conclude that u_λ is concave in C. The theorem is proved.

6. A partial result on the convexity of the minima of the anisotropic perimeter with fixed volume

As in [2], using Lemma 4.3 and Theorem 5.1 we prove the following result.

Proposition 6.1. *Assume that C is a bounded convex domain in $\mathbb{R}^N$ satisfying the $R\mathcal{W}_\phi$-condition, $R > 0$. For $\alpha > 0$, let u_α be the solution of $(Q)_\alpha$. Let $\alpha, \beta \geqslant \frac{2N}{R}$.*

(i) *If $\lambda > \alpha(1 - \|u_\alpha\|_\infty)$, then problem $(P)_\lambda$ has a unique solution. Moreover, the solution is a convex set.*
(ii) *We have $\alpha(1 - \|u_\alpha\|_\infty) = \beta(1 - \|u_\beta\|_\infty)$. Let λ^* denote this common value.*
(iii) *We have $\{u_\alpha \geqslant \|u_\alpha\|_\infty\} = \{u_\beta \geqslant \|u_\beta\|_\infty\}$, and*

$$\lambda^* = \frac{P_\phi(\{u_\alpha \geqslant \|u_\alpha\|_\infty\})}{|\{u_\alpha \geqslant \|u_\alpha\|_\infty\}|}. \tag{6.1}$$

As a consequence, we obtain that the set $\{u_\alpha \geqslant \|u_\alpha\|_\infty\}$ is ϕ-calibrable.

Let us denote the ϕ-calibrable set $\{u_\alpha \geqslant \|u_\alpha\|_\infty\}$ constructed in Proposition 6.1 by K. Then $\lambda_K^\phi = \lambda^*$ and K minimizes

$$\min_{F \subseteq C} P_\phi(F) - \lambda_K^\phi |F|. \tag{6.2}$$

Now, extending the usual concept in the euclidean case, let us call the Cheeger ϕ-constant of C the quantity

$$h_\phi(C) := \min_{F \subseteq C} \frac{P_\phi(F)}{|F|}. \tag{6.3}$$

In a similar was as in the euclidean case, we call a Cheeger ϕ-set of C any set G which minimizes (6.3). Notice that for any Cheeger ϕ-set G of C, $\lambda_G^\phi = h_\phi(C)$. Observe that G is a Cheeger ϕ-set of C if and only if G minimizes

$$\min_{F \subseteq C} P_\phi(F) - \lambda_G^\phi |F|. \tag{6.4}$$

In particular, if G is a Cheeger ϕ-set of C which is convex, then G is ϕ-calibrable. Thus, C is a Cheeger ϕ-set of C if and only if C is ϕ-calibrable. On the other hand, we have that K is a Cheeger ϕ-set of C. Moreover, if G is any other Cheeger ϕ-set of C, then it minimizes (6.4), and using that $\lambda_K^\phi = \lambda_G^\phi = h_\phi(C)$ we have that $G \subseteq C_\lambda$ for any $\lambda > \lambda_K^\phi$. By Lemma 4.3, this implies that $G \subseteq C_\lambda$ for any $\lambda > \lambda_K^\phi$. Since $K = \bigcap_{\lambda > \lambda_K^\phi} C_\lambda$, we have that $G \subseteq K$. In other words, K is the largest Cheeger ϕ-set of C.

Remark 6.2. In the euclidean case, a convex set $C \subseteq \mathbb{R}^2$ is a Cheeger set of C if and only if $\max_{x \in \partial C} \kappa_C(x) \leqslant \lambda_C := \frac{P(C)}{|C|}$. This has been proved in [27,14,30] (see also [3]) though it was stated in terms of calibrability in [14,3]. This result was extended to any dimension in [2] by replacing the curvature of the boundary by the sum of principal curvatures. Moreover, when $C \subseteq \mathbb{R}^2$ is convex, the convexity and uniqueness of the Cheeger set of C was proved in [30] (see also [29]) and can be deduced from the results in [3,2] which were stated in terms of calibrable sets. In higher dimension, uniqueness (hence convexity) of the Cheeger set of a convex set $C \subseteq \mathbb{R}^N$ has been recently proved by [24,1].

Observe that the empty set is also a solution of (6.2). Collecting the above results and using Lemma 4.3 we obtain the following theorem.

Theorem 6.3. *Let C be a bounded convex domain in $\mathbb{R}^N$ satisfying the $R\mathcal{W}_\phi$-condition, for some $R > 0$. Then there is a set $K \subseteq C$ which is the largest Cheeger ϕ-set of C. Moreover, K is convex and minimizes*

$$\min_{F \subseteq C} P_\phi(F) - \lambda_K^\phi |F|, \tag{6.5}$$

thus K is ϕ-calibrable. For any $\lambda \neq \lambda_K^\phi$, $\lambda > 0$, there is a unique minimizer C_λ of $(P)_\lambda$, which is convex, and the function $\lambda \to C_\lambda$ is increasing and continuous (hence also the function $\lambda \to P_\phi(C_\lambda)$ is increasing and continuous). Moreover, $C_\lambda = \emptyset$ for all $\lambda \in (0, \lambda_K^\phi)$.

Let us state without proof the following observation.

Lemma 6.4. *Let C be a bounded convex subset of $\mathbb{R}^N$. Let $\mu \geqslant 0$ and let E be a solution of the variational problem*

$$\min_{F \subseteq C} P_\phi(F) - \mu |F|. \tag{6.6}$$

Let $V = |E|$. Then E is a solution of

$$\min_{F \subseteq C,\, |F| = V} P_\phi(F). \tag{6.7}$$

Theorem 6.5. *Let C be a bounded convex domain in $\mathbb{R}^N$ satisfying the $R\mathcal{W}_\phi$-condition for some $R > 0$. For any $V \in [|K|, |C|]$ there is a unique convex solution of the constrained isoperimetric problem* (6.7).

Proof. Any solution of (6.7) corresponding to a value $V \in [|K|, |C|]$ coincides with the solution obtained from the corresponding problem $(P)_\lambda$ for some $\lambda \in [\lambda_K^\phi, \infty)$. Indeed, if $V \in [|K|, |C|]$, there is a value of $\lambda \in [\lambda_K^\phi, \infty)$ such that, if C_λ is the minimum of $(P)_\lambda$, then $|C_\lambda| = V$. By Lemma 6.4 we know that C_λ is a solution of (6.7). Now, let Q be another solution of (6.7). We have that $P_\phi(Q) = P_\phi(C_\lambda)$, and $|Q| = |C_\lambda|$. Hence

$$P_\phi(Q) - \lambda |Q| = P(C_\lambda) - \lambda |C_\lambda| \leqslant P_\phi(F) - \lambda |F|$$

for any $F \subseteq C$. Thus, Q is a minimum of $(P)_\lambda$, hence $Q = C_\lambda$. □

Remark 6.6. Thanks to Lemma 6.4 and Proposition 4.1, the algorithm described in [26,25], permits to compute the solution of (6.7) for any $V \in [|K|, |C|]$.

7. A characterization of a class of convex ϕ-calibrable sets by its anisotropic mean curvature

Proposition 7.1. *Let C be a bounded convex subset of $\mathbb{R}^N$ which satisfies the $R\mathcal{W}_\phi$-condition for some $R > 0$. Let $\mu_n \to \mu$. Let C_n be a minimizer of $(P)_{\mu_n}$. Assume that C_n is a sequence of convex sets converging to C, and $C_n \neq C$. Then $\mu \leqslant (N-1)\|\mathbf{H}_C^\phi\|_\infty$.*

Proof. Let $\mathrm{N} \in \mathrm{Nor}_\phi(U, \mathbb{R}^N)$ be the vector field $\tilde{z}$ given by Theorem 2.12 applied to the set C, where $U := \{|d_\phi^C| < \delta\}$ for some $\delta > 0$. We know that $\mathrm{N} \in T^\circ(\nabla d_\phi^C)$ a.e. in U.

Assume by contradiction that $(N-1)\|\mathbf{H}_C^\phi\|_\infty < \mu$. We may assume that $\delta > 0$ is small enough so that $\|\operatorname{div} \mathrm{N}\|_{L^\infty(U)} < \mu$. Then, for n large enough, we may also assume that $\|\operatorname{div} \mathrm{N}\|_{L^\infty(U)} < \mu_n$ and $C_n \subseteq U$. Now, we integrate $\operatorname{div} \mathrm{N}$ on $C \setminus C_n$. We have

$$\mu_n |C \setminus C_n| > \int_{C \setminus C_n} \operatorname{div} \mathrm{N}\, dx$$

$$= \int_{\partial C \setminus \partial C_n} \mathrm{N} \cdot \nu^C \, d\mathcal{H}^{N-1} - \int_{\partial C_n \setminus \partial C} \mathrm{N} \cdot \nu^{C_n} \, d\mathcal{H}^{N-1}$$
$$\geqslant \int_{\partial C \setminus \partial C_n} \phi^\circ(\nu^C) \, d\mathcal{H}^{N-1} - \int_{\partial C_n \setminus \partial C} \phi^\circ(\nu^{C_n}) \, d\mathcal{H}^{N-1}$$
$$= \int_{\partial C} \phi^\circ(\nu^C) \, d\mathcal{H}^{N-1} - \int_{\partial C_n} \phi^\circ(\nu^{C_n}) \, d\mathcal{H}^{N-1}$$
$$= P_\phi(C) - P_\phi(C_n).$$

Hence

$$P_\phi(C) - \mu_n|C| < P_\phi(C_n) - \mu_n|C_n|.$$

This contradiction proves that $\mu \leqslant (N-1)\|\mathbf{H}_C^\phi\|_\infty$. □

Theorem 7.2. *Let $C \subset \mathbb{R}^N$ be a bounded convex domain of class $\mathcal{C}^\infty$ and $\phi \in C_+^\infty$. If E is the minimizer of $(P)_\lambda$, with $\lambda > \lambda_K^\phi$, then E is of class $C^{1,1}$.*

Proof. Observe that, by Remark 2.3(a), $\phi^\circ \in \mathcal{C}_+^\infty$. By the regularity results in [32], $\partial E \cap C$ is smooth. Following the ideas in [39] we prove that $\partial E \in C^{1,1}$ in some neighborhood of ∂C. Since E is convex by Theorem 6.3, then near each point $x \in \partial E \cap \partial C$, we may represent both $\partial E \cap \partial C$ as graphs of functions u and β, respectively, defined on an open set $U' \subset \mathbb{R}^{N-1}$ containing x' where $x = (x', y'')$, $y'' \in \mathbb{R}$. We will assume u and β chosen in such a way that $u \geqslant \beta$, $u = 0$ on $\partial U'$ and $\beta \leqslant 0$ on $\partial U'$. Now select $v \in K := \{w : U' \to \mathbb{R}\colon\ v \geqslant \beta \text{ in } U' \text{ and } v = 0 \text{ on } \partial U'\}$. For $0 < \varepsilon < 1$, define u_ε on U' as $u_\varepsilon = u + \varepsilon(v - u)$. We will assume ε chosen small enough so that the graph of u_ε remains in $\bar{C}$. Select a point $z \in (\partial E) \cap C$ at which ∂E is regular. Then, there is a neighborhood of z where the anisotropic mean curvature of ∂E is constant and in which we can represent ∂E as the graph of a function w defined on some open set $V' \subset \mathbb{R}^{N-1}$ containing z' where $z = (z', z'')$. Note that we can take the sets U' and V' to be disjoint. Let $\varphi \in C_0^\infty(V')$ denote a function which satisfies

$$\int_{V'} \varphi \, d\mathcal{H}^{N-1} = \int_{U'} (v - u) \, d\mathcal{H}^{N-1} \tag{7.1}$$

and define $w_\varepsilon = w - \varepsilon\varphi$. The graphs of the functions u_ε and w_ε produce a perturbation of the set E, say E_ε. Because of (7.1) we have that $|E| = |E_\varepsilon|$. Taking

$$F(\varepsilon) = \int_{U'} \phi^\circ(\nabla u_\varepsilon, -1) \, d\mathcal{H}^{N-1} + \int_{V'} \phi^\circ(\nabla w_\varepsilon, -1) \, d\mathcal{H}^{N-1},$$

the minimizing property of ∂E implies that $F(0) \leqslant F(\varepsilon)$ for all small ε and therefore, $F'(0) \geqslant 0$. Thus,

$$\int_{U'} \nabla\phi^\circ(\nabla u, -1) \cdot \nabla(v - u) \, d\mathcal{H}^{N-1} - \int_{V'} \nabla\phi^\circ(\nabla w, -1) \cdot \nabla\varphi \, d\mathcal{H}^{N-1} \geqslant 0.$$

Since w has constant anisotropic mean curvature K, we obtain

$$\int_{V'} \nabla\phi^\circ(\nabla w, -1) \cdot \nabla\varphi = -K \int_{V'} \varphi \, d\mathcal{H}^{N-1} = -K \int_{U'} (v - u) \, d\mathcal{H}^{N-1}$$

and therefore

$$\int_{U'} \nabla\phi^\circ(\nabla u, -1) \cdot \nabla(v - u) \geqslant -K \int_{U'} (v - u) \, d\mathcal{H}^{N-1}.$$

Finally, applying a regularity result due to Brézis and Kinderlehrer [22], we conclude that $u \in C^{1,1}(V)$ on any domain V with $\bar{V} \subset U'$. □

Theorem 7.3. *Let C be a bounded convex domain $\mathbb{R}^N$ which satisfies the $R\mathcal{W}_\phi$-condition for some $R > 0$. Let $\Lambda := (N-1)\|\mathbf{H}_C^\phi\|_\infty$. Let C_μ be the solution of $(P)_\mu$, $\mu > 0$. Then $C_\mu = C$ if and only if $\mu \geqslant \max\{\lambda_C^\phi, \Lambda\}$.*

Proof. Assume that C is a solution of $(P)_\mu$, and let us prove that $\mu \geqslant \max\{\lambda_C^\phi, \Lambda\}$. First of all, notice that $P_\phi(C) - \mu|C| \leqslant P_\phi(\emptyset) - \mu|\emptyset| = 0$, i.e. $\mu \geqslant \lambda_C^\phi$. If K denotes the ϕ-calibrable set contained in C defined by Theorem 6.3, then $K = \arg\min_{X\subset C} P(X) - \lambda_K^\phi|X|$, and we have $P(C) - \lambda_K^\phi|C| \geqslant P(K) - \lambda_K|K| = 0$, that is, $\lambda_C^\phi \geqslant \lambda_K^\phi$.

The proof of $\mu \geqslant \Lambda$ requires an approximation argument. Let $\phi_\epsilon \in C_+^\infty$ and $C_\epsilon \in C_+^\infty$ be the anisotropies and convex sets satisfying (i)–(iii) in Lemma 5.3, in particular, they converge to ϕ and C respectively. We recall the construction: ϕ_ϵ is the anisotropy such that $\mathcal{W}_{\phi_\epsilon} = \mathcal{T}_\epsilon(\mathcal{W}_\phi) + B_\epsilon$, where $\mathcal{T}_\epsilon$ is given by Theorem 5.2, and $C_\epsilon := \mathcal{T}_\epsilon(C) + B_{R\epsilon}$. Let $\lambda_{C_\epsilon}^\epsilon := \frac{P_{\phi_\epsilon}(C_\epsilon)}{|C_\epsilon|}$, $\lambda_{K_\epsilon}^\epsilon := \frac{P_{\phi_\epsilon}(K_\epsilon)}{|K_\epsilon|}$, where K_ϵ is the largest ϕ_ϵ-calibrable set contained in C_ϵ obtained in Theorem 6.3. As in the last paragraph, we also deduce that $\lambda_{C_\epsilon}^\epsilon \geqslant \lambda_{K_\epsilon}^\epsilon$.

Using (5.2), (5.3) and the Lipschitz local continuity of ϕ° we have that $|\phi_\epsilon^\circ(\xi) - \phi^\circ(\xi)| \leqslant 2\epsilon$ for any $\xi \in \mathbb{R}^N$, $|\xi| = 1$. This implies that

$$\left|P_{\phi_\epsilon}(X) - P_\phi(X)\right| \leqslant 2\epsilon P(X) \tag{7.2}$$

for any set of finite perimeter $X \subseteq \mathbb{R}^N$. Hence, since $P_\phi(C_\epsilon) \to P_\phi(C)$ we deduce that $P_{\phi_\epsilon}(C_\epsilon) \to P_\phi(C)$. Since we also have that $|C_\epsilon| \to |C|$ [37], then $\lambda_{C_\epsilon}^\epsilon \to \lambda_C^\phi$.

Let $\delta > 0$, from the last argument we know that $\mu + \delta > \lambda_{C_\epsilon}^\epsilon \geqslant \lambda_{K_\epsilon}^\epsilon$. Now, we consider the problem

$$(P)_{\mu,\epsilon,\delta}: \quad \min_{F\subseteq C_\epsilon} P_{\phi_\epsilon}(F) - (\mu+\delta)|F|. \tag{7.3}$$

Let $D_{\epsilon,\delta}$ be a minimizer of $(P)_{\mu,\epsilon,\delta}$. By Theorem 6.3 we know that the minimum is unique and it is a convex set.

Now, as the sets $D_{\epsilon,\delta}$ are uniformly bounded in ϵ, by extracting a subsequence if necessary, we may assume that $D_{\epsilon,\delta}$ converge to a convex set D_δ in the Hausdorff distance. Using (7.2) and the lower semicontinuity of P_ϕ, we obtain that D_δ is a minimizer of $(P)_{\mu+\delta}$. By applying (ii) of Lemma 4.3, we obtain that $D_\delta = C$ for every $\delta > 0$.

By Theorem 7.2, we know that $D_{\epsilon,\delta}$ is of class $C^{1,1}$ and, as $\phi_\epsilon \in \mathcal{C}_+^\infty$, from Remark 2.9(a) (see also [13, Remark 4(a)]) it follows that $D_{\epsilon,\delta}$ is Lipschitz ϕ_ϵ-regular. Hence, by Lemma 2.8 $D_{\epsilon,\delta}$ satisfies the $\tau\mathcal{W}_{\phi_\epsilon}$-condition for some $\tau > 0$. Let $n_{\epsilon,\delta}$ be the Cahn–Hofmann vector field of $D_{\epsilon,\delta}$. Now, by applying the first variation formula for the perimeter P_{ϕ_ϵ} [20,18], we deduce that $(N-1)\mathbf{H}_{D_{\epsilon,\delta}}^{\phi_\epsilon} = \operatorname{div} n_{\epsilon,\delta} \leqslant \mu + \delta$. Let $d_{\epsilon,\delta} := d_{\phi_\epsilon}^{D_{\epsilon,\delta}}$. By [13, Theorem 4] we have that $d_{\epsilon,\delta} \in C_{\mathrm{loc}}^{1,1}(\{|d_{\epsilon,\delta}| < (\mu+\delta)^{-1}\})$ and

$$0 \leqslant \operatorname{div} n_{\epsilon,\delta} \leqslant \frac{\mu+\delta}{1 - |d_{\epsilon,\delta}|(\mu+\delta)} \quad \text{in } |d_{\epsilon,\delta}| < (\mu+\delta)^{-1}.$$

By [13, Corollary 1], we know that $D_{\epsilon,\delta}$ satisfies the $(\mu+\delta)^{-1}\mathcal{W}_{\phi_\epsilon}$-condition. By the stability result proved in [13, Lemma 2], we know that C satisfies the $(\mu+\delta)^{-1}\mathcal{W}_\phi$-condition. Moreover, we may assume that $n_{\epsilon,\delta} \to n$ and $\operatorname{div} n_{\epsilon,\delta} \to \operatorname{div} n$ weakly* in $L_{\mathrm{loc}}^\infty(\{|d_{\epsilon,\delta}| < (\mu+\delta)^{-1}\})$. As in the proof of Theorem 2.12 this implies that $n \in T^\circ(\nabla d_\phi^C)$ a.e. in $\{|d_\phi^C| < (\mu+\delta)^{-1}\}$. Moreover

$$0 \leqslant \operatorname{div} n \leqslant \frac{\mu+\delta}{1 - |d_\phi^C|(\mu+\delta)} \quad \text{in } \left\{\left|d_\phi^C\right| < (\mu+\delta)^{-1}\right\}. \tag{7.4}$$

By Theorem 2.12, there exists a vector field $\tilde{z} \in T^\circ(\nabla d_\phi^C)$ a.e. in $\{|d_\phi^C| < (\mu+\delta)^{-1}\}$ which minimizes (2.13) and such that

$$\|\operatorname{div} \tilde{z}\|_{L^\infty(U_t)} \leqslant \|\operatorname{div} n\|_{L^\infty(U_t)}, \tag{7.5}$$

for any $t < (\mu+\delta)^{-1}$, where $U_t := \{|d_\phi^C| < t\}$. Using (7.4) and (7.5) we then get

$$(N-1)\left\|\mathbf{H}_C^\phi\right\|_\infty \leqslant \mu + \delta.$$

Letting $\delta \to 0^+$, we obtain that $(N-1)\|\mathbf{H}_C^\phi\|_\infty \leqslant \mu$.

Assume now that $\mu \geqslant \max\{\lambda_C^\phi, \Lambda\}$, but C is not a minimizer of $(P)_\mu$. In particular, by Proposition 3.4 and Lemma 4.3(ii), C is not ϕ-calibrable. We shall construct a sequence of sets $E_\lambda \neq C$ each one being a solution of $(P)_{\mu_\lambda}$ with $\mu_\lambda \to \beta$, $\beta > \mu$. Let $\lambda > \max\{\frac{\lambda^\phi_{\mathcal{W}_\phi}}{R}, \frac{1}{\|\chi_C\|_{\phi,*}}, \mu\}$. By Lemma 4.2(iii), we know that $u_\lambda \geqslant (1 - \frac{\lambda^\phi_{\mathcal{W}_\phi}}{R\lambda})^+ \chi_C$. Let us define

$$\beta_\lambda := \inf\left\{\gamma\colon\ u_\lambda \geqslant \left(1 - \frac{\gamma}{\lambda}\right)^+ \chi_C\right\}.$$

Obviously, we have $\beta_\lambda \leqslant \frac{\lambda^\phi_{\mathcal{W}_\phi}}{R}$, and

$$u_\lambda \geqslant \left(1 - \frac{\beta_\lambda}{\lambda}\right)^+ \chi_C. \tag{7.6}$$

Case $\beta_\lambda \leqslant \mu$. Take $s = 1 - \frac{\mu}{\lambda}$. Then, by Proposition 4.1, $\{u_\lambda \geqslant s\}$ is a solution of $(P)_{\lambda(1-s)} = (P)_\mu$. Finally we observe that $\{u_\lambda \geqslant s\} = C$. Thus C is a solution of $(P)_\mu$.

Case $\mu < \beta_\lambda \leqslant \frac{\lambda_{\mathcal{W}_\phi}^\phi}{R}$. For each $\lambda > \max\{\frac{\lambda^\phi_{\mathcal{W}_\phi}}{R}, \frac{1}{\|\chi_C\|_{\phi,*}}\}$, take $s_\lambda \in (1 - \frac{\beta_\lambda}{\lambda}, 1 - \frac{\beta_\lambda}{\lambda} + \frac{\epsilon_\lambda}{\lambda}]$, $\epsilon_\lambda > 0$, being a sequence converging to 0. Then

$$\beta_\lambda - \epsilon_\lambda \leqslant \lambda(1 - s_\lambda) < \beta_\lambda.$$

Let $E_\lambda = \{u_\lambda \geqslant s_\lambda\}$. Since $\lambda(1 - s_\lambda) < \beta_\lambda$, and by Lemma 4.2(v), we know that u_λ is not constant, by an appropriate choice of s_λ we may assume that $E_\lambda \neq \emptyset$, $E_\lambda \neq C$. By Lemma 4.2(ii), choosing s_λ sufficiently near $1 - \frac{\beta_\lambda}{\lambda}$, i.e. ϵ_λ sufficiently small, we have that $E_\lambda \to C$ as $\lambda \to \infty$. Without loss of generality me may assume that $\beta_\lambda \to \beta$ where $\mu \leqslant \beta \leqslant \frac{\lambda^\phi_{\mathcal{W}_\phi}}{R}$. If $\beta = \mu$, then $\lambda(1 - s_\lambda) \to \mu$. Since E_λ is a solution of $(P)_{\lambda(1-s_\lambda)}$, then C would be a solution of $(P)_\mu$, and this would conclude. Therefore we may assume that $\mu < \beta \leqslant \frac{\lambda^\phi_{\mathcal{W}_\phi}}{R}$.

To summarize, we proved that E_λ is a solution of $(P)_{\mu_\lambda}$ with $\mu_\lambda := \lambda(1 - s_\lambda) \to \beta$ with $\mu < \beta \leqslant \frac{\lambda^\phi_{\mathcal{W}_\phi}}{R}$, and $E_\lambda \neq C$, $E_\lambda \to C$.

Moreover, since E_λ is an upper level set of u_λ and λ can be taken $\geqslant \frac{2N}{R}$ (recall that $\lambda \to \infty$), by Theorem 5.1, we know that u_λ is concave, hence E_λ is convex. By Proposition 7.1, we have that

$$\beta \leqslant (N-1)\left\|\mathbf{H}_C^\phi\right\|_\infty = \Lambda \leqslant \mu,$$

and we obtain a contradiction. We have proved that C minimizes $(P)_\mu$. □

Corollary 7.4. *Let C be a bounded convex domain in $\mathbb{R}^N$ which satisfies the $R\mathcal{W}_\phi$-condition for some $R > 0$. Then $E = C$ is a solution of*

$$\min_{F \subseteq C} P_\phi(F) - \lambda_C^\phi |F| \tag{7.7}$$

if and only if $(N-1)\|\mathbf{H}_C^\phi\|_\infty \leqslant \lambda_C^\phi$.

Remark 7.5. Corollary 7.4 extends to the anisotropic case the analogous results proved in [27,14,30] when $N = 2$ and in [13] when $N \geqslant 2$. In terms of Cheeger sets, it characterizes those convex sets C (satisfying the $R\mathcal{W}_\phi$-condition for some $R > 0$) which are Cheeger ϕ-sets in themselves.

8. The evolution of a convex set by the anisotropic total variation flow

8.1. The minimizing anisotropic total variation flow

We are interested in computing the solution of the minimizing anisotropic total variation flow

$$\frac{\partial u}{\partial t} = \operatorname{div} \partial\phi^\circ(Du) \quad \text{in } Q_T :=]0, T[\times \mathbb{R}^N, \tag{8.1}$$

coupled with the initial condition

$$u(0) = u_0 \in L^2(\mathbb{R}^N), \tag{8.2}$$

when $u_0 = \chi_C$, C being a bounded convex domain in $\mathbb{R}^N$ satisfying a ball condition.

The following notion of strong solution is adapted from the notion of strong solution in the semigroup sense [21] (see also [35,9,14]).

In the following definition, we denote by $L^1_w(0, T; BV(\mathbb{R}^N))$ the space of functions $w : [0, T] \to BV(\mathbb{R}^N)$ such that $w \in L^1((0, T) \times \mathbb{R}^N)$, the maps $t \in [0, T] \to \int_{\mathbb{R}^N} \psi \, dDw(t)$ are measurable for every $\psi \in C^1_0(\mathbb{R}^N; \mathbb{R}^N)$ and $\int_0^T |Dw(t)|(\mathbb{R}^N)\, dt < \infty$.

Definition 8.1. A function $u \in C([0, T]; L^2(\mathbb{R}^N))$ is called a strong solution of (8.1) if

$$u \in W^{1,2}_{\text{loc}}(0, T; L^2(\mathbb{R}^N)) \cap L^1_w(0, T; BV(\mathbb{R}^N))$$

and there exists $z \in L^\infty t(]0, T[\times \mathbb{R}^N; \mathbb{R}^N)$ with $\varphi(z(x)) \leqslant 1$ a.e. such that

$$u_t = \operatorname{div} z \quad \text{in } \mathcal{D}'(]0, T[\times \mathbb{R}^N)$$

and

$$\int_{\mathbb{R}^N} (z(t), Du(t)) = \int_{\mathbb{R}^N} \phi^\circ(Du(t)) \quad t > 0 \text{ a.e.} \tag{8.3}$$

Theorem 8.2. *Let $u_0 \in L^2(\mathbb{R}^N)$. Then there exists a unique strong solution in the semigroup sense u of* (8.1) *in* $[0, T]$ *for every $T > 0$. Moreover, if u and v are strong solutions of* (8.1) *corresponding to the initial conditions $u_0, v_0 \in L^2(\mathbb{R}^N)$, then*

$$\|u(t) - v(t)\|_2 \leqslant \|u_0 - v_0\|_2 \quad \textit{for any } t \geqslant 0. \tag{8.4}$$

8.2. The evolution of a convex ϕ-calibrable set

Let Ω be a set of finite perimeter in $\mathbb{R}^N$. We shall say that the set Ω decreases at constant speed λ if

$$u(t, x) := (1 - \lambda t)^+ \chi_\Omega(x) \tag{8.5}$$

is the strong solution of (8.1) with initial condition $u_0 = \chi_\Omega$. It can be easily checked (see [14]) that Ω decreases at speed λ if and only if the function $v := \chi_\Omega$ satisfies the equation

$$-\operatorname{div} \partial\phi^\circ(Du) = \lambda v, \tag{8.6}$$

i.e. if and only if there exists a vector field $\xi \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ such that $\phi(\xi) \leqslant 1$,

$$-\operatorname{div} \xi = \lambda v \tag{8.7}$$

and

$$\int_{\mathbb{R}^N} (\xi, Dv) = \int_{\mathbb{R}^N} \phi^\circ(Dv). \tag{8.8}$$

In other words, the set decreases at constant speed if and only if it is ϕ-calibrable. Using Theorem 7.3 we obtain a characterization of the convex sets which decrease at constant speed.

Theorem 8.3. *Let C be a bounded convex subset of $\mathbb{R}^N$ which satisfies the $R\mathcal{W}_\phi$-condition for some $R > 0$. The following conditions are equivalent*:

(i) *C decreases at constant speed*;
(ii) *C is ϕ-calibrable*;
(iii) $(N-1)\|\mathbf{H}_C^\phi\|_\infty \leqslant \lambda_C^\phi$.

8.3. The evolution of a bounded convex domain satisfying a ball condition

Let us assume that C is a bounded convex domain in $\mathbb{R}^N$ satisfying the $R\mathcal{W}_\phi$-condition for some $R > 0$. Let K be the largest ϕ-calibrable set contained in C, as in Theorem 6.3. For each $\lambda > 0$ let C_λ be the solution of $(P)_\lambda$. By Theorems 6.3 and 7.3 we have that $C_\lambda = \emptyset$ for any $\lambda < \lambda_K^\phi$, and $C_\lambda = C$ for any $\lambda \geqslant \max\{\lambda_C^\phi, (N-1)\|\mathbf{H}_C^\phi\|_\infty\}$. Following [12,28,2], and recalling the monotonicity of C_λ, we define

$$H_C(x) := \begin{cases} -\inf\{\lambda \colon x \in C_\lambda\} & \text{on } x \in C, \\ 0 & \text{on } \mathbb{R}^N \setminus C. \end{cases} \tag{8.9}$$

Observe that $H_C \leqslant 0$ on C, and $H_C(x) = -\lambda_K^\phi$ for all $x \in K$.

Definition 8.4. Let $H \in L^1(\mathbb{R}^N)$ and let $\mathcal{F}_H$ be the functional defined as

$$\mathcal{F}_H(X) := P_\phi(X) + \int_X H(x)\,dx,$$

for all $X \subseteq \mathbb{R}^N$ of finite perimeter. Let E be a set of finite perimeter in $\mathbb{R}^N$. We say that H is a ϕ-variational mean curvature of E if

$$\mathcal{F}_H(E) \leqslant \mathcal{F}_H(X) \quad \forall X \text{ of finite perimeter in } \mathbb{R}^N.$$

The following result can be proved arguing as in [12,28].

Proposition 8.5. *We have*

(i) *H_C is a ϕ-variational mean curvature of C and* $\int_C H_C(x)\,dx = -P_\phi(C)$.
(ii) *$H_C\chi_{C_\lambda}$ is a ϕ-variational mean curvature of C_λ and* $\int_{C_\lambda} H_C(x)\,dx = -P_\phi(C_\lambda)$.

Lemma 8.6. *We have $\|H_C\|_{\phi,*} = 1$. In particular, there exists a vector field $\xi_C \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$, such that $\phi(\xi_C) \leqslant 1$ and $\operatorname{div}\xi_C = H_C$ in $\mathbb{R}^N$. Moreover,*

$$(\xi, D\chi_{C_\lambda}) = \phi^\circ(D\chi_{C_\lambda}) \quad \text{for any } \lambda > 0.$$

Proof. Since $\mathcal{F}_H(C) = 0$, we have $-\int_X H_C(x)\,dx \leqslant P_\phi(X)$ for any set $X \subseteq \mathbb{R}^N$ of finite perimeter. This inequality, as in the proof of Lemma 3.2, implies that $\|H_C\|_{\phi,*} \leqslant 1$. Since $\int_C H_C(x)\,dx = -P_\phi(C)$, we deduce that $\|H_C\|_{\phi,*} = 1$. Hence, by Lemma 3.2 there exists a vector field ξ_C such that $\phi(\xi_C) \leqslant 1$ and $\operatorname{div}\xi_C = H_C$ in $\mathbb{R}^N$.

Now, multiplying $\operatorname{div}\xi_C = H_C$ by χ_{C_λ} and integrating on $\mathbb{R}^N$, we obtain

$$-\int_{\mathbb{R}^N} (\xi_C, D\xi_{C_\lambda}) = \int_{C_\lambda} H_C(x)\,dx = -P_\phi(C_\lambda) = -\int_{R^N} \phi^\circ(D\chi_{C_\lambda}).$$

Since $\phi(\xi_C) \leqslant 1$, we deduce that $(\xi_C, D\chi_{C_\lambda}) = \phi^\circ(D\chi_{C_\lambda})$. $\square$

Theorem 8.7. *Let C be a bounded convex domain in $\mathbb{R}^N$ satisfying the $R\mathcal{W}_\phi$-condition for some $R > 0$, and let H_C be the variational curvature of C defined by* (8.9). *Then, $u(t,x) = (1 + H_C(x)t)^+\chi_C(x)$ is the solution of* (8.1) *corresponding to the initial condition $u_0 = \chi_C$.*

Proof. Let $t > 0$. We have $u_t(t,x) = \text{sign}^+(1 + H_C(x)t)H_C(x)$, where $\text{sign}^+(1 + H_C(x)t) = 1$ if and only if $t < -\frac{1}{H_C(x)}$, otherwise $\text{sign}^+(1 + H_C(x)t) = 0$. In particular, we observe that for $t \geqslant \|\frac{1}{H_C}\|_{L^\infty(C)} = \frac{1}{\lambda_K^\phi}$ we have $u_t = u = 0$. Thus

$$u_t(t,x) = H_C(x)\chi_{C_{1/t}}(x)\chi_{[0,T)}(t),$$

where $T := \frac{1}{\lambda_K^\phi}$. Let ξ_C be the vector field given by Lemma 8.6. We have

$$\left[\xi_C \cdot \nu^{C_{1/s}}\right] = -\phi^\circ\left(\nu^{C_{1/s}}\right) \quad \text{on } \partial C_{1/s},$$

for all $s > 0$. Arguing as in [14,17], we now modify the vector field ξ_C in such a way that its modification $\xi(t,x)$ satisfies $\xi(t,x) \in X_2(\mathbb{R}^N)$ and $\text{div}(\xi(t,x)) = 0$ in $\mathbb{R}^N \setminus C_{1/t}$. If $t \geqslant \frac{1}{\lambda_K^\phi}$, we set $\xi(t,x) := 0$. By Lemma 8.6, we have $(\xi(t), Du(t)) = \phi^\circ(Du(t))$ and

$$\text{div}\,\xi(t) = H_C(x)\chi_{C_{1/t}} = u_t \quad \forall t \in (0,T).$$

By the characterization of $\partial\Psi_\phi$ given in Lemma 3.1 and recalling Theorem 8.2, we get that $u(t)$ is the unique strong solution of (8.1), corresponding to the initial condition $u_0 = \chi_C$. □

Remark 8.8. The proof also shows that actually, for a convex initial data, $u(t,x)$ is the solution of the problem $(Q)_\lambda$ defined in (4.1) for $\lambda = 1/t$.

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References

[1] F. Alter, V. Caselles, Uniqueness of the Cheeger set of a convex body, Preprint CVGMT, Scuola Normale di Pisa, 2007.

[2] F. Alter, V. Caselles, A. Chambolle, A characterization of convex calibrable sets in $\mathbb{R}^N$, Math. Ann. 332 (2) (2005) 329–366.

[3] F. Alter, V. Caselles, A. Chambolle, Evolution of characteristic functions of convex sets in the plane by the minimizing total variation flow, Interfaces Free Bound. 7 (1) (2005) 29–53.

[4] M. Amar, G. Bellettini, A notion of total variation depending on a metric with discontinuous coefficients, Ann. Inst. H. Poincaré Anal. Non Linéaire 11 (1) (1994) 91–133.

[5] L. Ambrosio, Movimenti minimizzanti, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) 19 (1995) 191–246.

[6] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[7] F. Andreu, C. Ballester, V. Caselles, J.M. Mazón, The Dirichlet problem for the total variation flow, J. Funct. Anal. 180 (2) (2001) 347–403.

[8] F. Andreu, V. Caselles, J.M. Mazón, A parabolic quasilinear problem for linear growth functionals, Rev. Mat. Iberoamericana 18 (1) (2002) 135–185.

[9] F. Andreu-Vaillo, V. Caselles, J.M. Mazón, Parabolic Quasilinear Equations Minimizing Linear Growth Functionals, Progress in Mathematics, vol. 223, Birkhäuser Verlag, Basel, 2004.

[10] G. Anzellotti, Pairings between measures and bounded functions and compensated compactness, Ann. Mat. Pura Appl. (4) 135 (1984) 293–318, 1983.

[11] F.V. Atkinson, L.A. Peletier, Bounds for vertical points of solutions of prescribed mean curvature type equations. I, Proc. Roy. Soc. Edinburgh Sect. A 112 (1–2) (1989) 15–32.

[12] E. Barozzi, The curvature of a set with finite area, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 5 (2) (1994) 149–159.

[13] G. Bellettini, V. Caselles, A. Chambolle, M. Novaga, Crystalline mean curvature flow of convex sets, Arch. Ration. Mech. Anal. 179 (1) (2006) 109–152.

[14] G. Bellettini, V. Caselles, M. Novaga, The total variation flow in $\mathbb{R}^N$, J. Differential Equations 184 (2) (2002) 475–525.

[15] G. Bellettini, M. Novaga, Approximation and comparison for nonsmooth anisotropic motion by mean curvature in $\mathbf{R}^N$, Math. Models Methods Appl. Sci. 10 (1) (2000) 1–10.

[16] G. Bellettini, M. Novaga, M. Paolini, Facet-breaking for three-dimensional crystals evolving by mean curvature, Interfaces Free Bound. 1 (1) (1999) 39–55.

[17] G. Bellettini, M. Novaga, M. Paolini, Characterization of facet breaking for nonsmooth mean curvature flow in the convex case, Interfaces Free Bound. 3 (4) (2001) 415–446.

[18] G. Bellettini, M. Novaga, M. Paolini, On a crystalline variational problem. I. First variation and global L^∞ regularity, Arch. Ration. Mech. Anal. 157 (3) (2001) 165–191.

[19] G. Bellettini, M. Novaga, M. Paolini, On a crystalline variational problem. II. BV regularity and structure of minimizers on facets, Arch. Ration. Mech. Anal. 157 (3) (2001) 193–217.

[20] G. Bellettini, M. Paolini, Anisotropic motion by mean curvature in the context of Finsler geometry, Hokkaido Math. J. 25 (3) (1996) 537–566.

[21] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland Mathematics Studies, vol. 5, North-Holland Publishing Co., Amsterdam, 1973, Notas de Matemática (50).

[22] H. Brézis, D. Kinderlehrer, The smoothness of solutions to nonlinear variational inequalities, Indiana Univ. Math. J. 23 (1973/1974) 831–844.

[23] V. Caselles, A. Chambolle, Anisotropic curvature-driven flow of convex sets, Nonlinear Anal. 65 (8) (2006) 1547–1577.

[24] V. Caselles, A. Chambolle, M. Novaga, Uniqueness of the Cheeger set of a convex body, Pacific J. Math. 232 (1) (2007) 77–90.

[25] A. Chambolle, An algorithm for mean curvature motion, Interfaces Free Bound. 6 (2) (2004) 195–218.

[26] A. Chambolle, An algorithm for total variation minimization and applications, J. Math. Imaging Vision 20 (1–2) (2004) 89–97. Special issue on mathematics and image analysis.

[27] E. Giusti, On the equation of surfaces of prescribed mean curvature. Existence and uniqueness without boundary conditions, Invent. Math. 46 (2) (1978) 111–137.

[28] E.H.A. Gonzalez, U. Massari, Variational mean curvatures, in: Partial Differential Equations, II, Turin, 1993, Rend. Sem. Mat. Univ. Politec. Torino 52 (1) (1994) 1–28.

[29] B. Kawohl, V. Fridman, Isoperimetric estimates for the first eigenvalue of the p-Laplace operator and the Cheeger constant, Comment. Math. Univ. Carolin. 44 (4) (2003) 659–667.

[30] B. Kawohl, T. Lachand-Robert, Characterization of Cheeger sets for convex subsets of the plane, Pacific J. Math. 225 (1) (2006) 103–118.

[31] N. Korevaar, Capillary surface convexity above convex domains, Indiana Univ. Math. J. 32 (1) (1983) 73–81.

[32] N. Korevaar, L. Simon, Equations of mean curvature type with contact angle boundary conditions, in: Geometric Analysis and the Calculus of Variations, Internat. Press, Cambridge, MA, 1996, pp. 175–201.

[33] A. Lichnewsky, R. Temam, Pseudosolutions of the time-dependent minimal surface problem, J. Differential Equations 30 (3) (1978) 340–364.

[34] Y. Meyer, Oscillating Patterns in Image Processing and Nonlinear Evolution Equations, University Lecture Series, vol. 22, American Mathematical Society, Providence, RI, 2001. The fifteenth Dean Jacqueline B. Lewis memorial lectures.

[35] J.S. Moll, The anisotropic total variation flow, Math. Ann. 332 (1) (2005) 177–218.

[36] C. Rosales, Isoperimetric regions in rotationally symmetric convex bodies, Indiana Univ. Math. J. 52 (5) (2003) 1201–1214.

[37] R. Schneider, Convex Bodies: The Brunn–Minkowski Theory, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1993.

[38] L. Simon, J. Spruck, Existence and regularity of a capillary surface with prescribed contact angle, Arch. Ration. Mech. Anal. 61 (1) (1976) 19–34.

[39] E. Stredulinsky, W.P. Ziemer, Area minimizing sets subject to a volume constraint in a convex set, J. Geom. Anal. 7 (4) (1997) 653–677.