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A smoothing property for the L^2-critical NLS equations and an application to blowup theory

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Abstract

In this paper we prove a smoothing property for the L^2-critical nonlinear Schrödinger equation and we use it to study the blowup dynamics for singular solutions below the energy level.

Résumé

Dans cet article, on montre un effet régularisant pour l'équation de Schrödinger L^2-critique et on utilise ce résulat pour étudier la dynamique d'explosion pour les solutions singulières ayant des données initiales peu régulières.

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1. Introduction

Consider the L^2-critical nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + \kappa |u|^{\frac{4}{d}} u = 0; \quad u|_{t=0} = u_0 \in H^s(\mathbb{R}^d). \tag{1}$$

Here, $\Delta = \sum_{j=1}^d \partial_{x_j}^2$ is the Laplace operator on $\mathbb{R}^d$ and $u : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}$ is a complex-valued function. The parameter κ equal to 1 (resp. -1) corresponds to the focusing (resp. defocusing) NLS. It is well known (see [7] for instance) that the Cauchy problem (1) is locally well-posed in H^s for every $s \geqslant 0$.

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The unique solution satisfies the following conservation law

$$\int \left|u(t,x)\right|^2 dx = \int \left|u_0(x)\right|^2 dx.$$

Also, if $s \geqslant 1$, the energy

$$E(t) = \frac{1}{2}\int \left|\nabla u(t,x)\right|^2 dx - \frac{\kappa d}{4+2d}\int \left|u(t,x)\right|^{\frac{4}{d}+2} dx$$

is conserved as t varies. Here $\nabla = (\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_d})$ denotes the spatial gradient in $\mathbb{R}^d$.

For $s > 0$ Eq. (1) is subcritical: the lifespan of the solution depends only on the H^s norm of the data. Define $[0, T^*)$ to be the forward maximal lifespan of the solution of (1). We have the following blowup alternative: either $T^* = +\infty$ or $T^* < +\infty$ and

$$\lim_{t \uparrow T^*} \left\|D^s u(t)\right\|_{L^2} = +\infty.$$

The space L^2 and the equation have the same scaling. More precisely, if u solves (1), then, for every $\lambda > 0$, so it does $u_\lambda(x,t) = \lambda^{d/2} u(\lambda^2 t, \lambda x)$, with data $u_\lambda(0,x) = \lambda u_0(\lambda x)$. But $\|u_\lambda(0,\cdot)\|_{L^2(\mathbb{R}^2)} = \|u_0\|_{L^2(\mathbb{R}^2)}$ and from this point of view (1) is L^2-critical. In this case the situation is more subtle and the time of existence depends on shape of the data. More precisely, the blow up criterion becomes

$$T^* < +\infty \Longrightarrow \lim_{T \uparrow T^*} \iint\limits_{[0,T]\times\mathbb{R}^d} \left|u(t,x)\right|^{\frac{4}{d}+2} dx\, dt = +\infty.$$

The blowup or "wave collapse" corresponds to self-trapping of beams in laser propagation. A lot of theoretical and numerical works are dedicated to this subject when the initial data belongs to H^1 (see [7,21–26,31,38] and the references therein). This theory, which is based on energy arguments, is closely connected to the notion of ground state: the unique positive radial solution of the elliptic problem

$$\Delta Q - Q + |Q|^{\frac{4}{d}} Q = 0.$$

For a revisited H^1 blowup theory the reader is referred to [19].

For the case of initial data belonging to H^s, with $0 \leqslant s < 1$, the classical energy arguments do not work. Nevertheless, the general consensus is that, in this case too, the same phenomena happen (concentration, universality of blowup profile...).

The first result in this direction is due to J. Bourgain [5] for $d = 2$ and $s = 0$. In fact, by using a refined version of the Strichartz inequality proved in [29] and harmonic analysis techniques, this author proved that if a solution of (1), with initial data in L^2, blows up at finite time $T^* > 0$, then there is concentration of some part of its total mass in small balls of size $(T^* - t)^{1/2}$. Using this work by Bourgain, F. Merle and L. Vega [27] proved, among other things, an asymptotic compactness property in $L^2(\mathbb{R}^2)$ up to the invariance of the equation.

In [20] the first author defines the minimal mass δ_0 as the L^2 norm necessary to ignite a wave collapse and stresses its role in the blow up mechanism for *one* and *two* space dimensions (see also [6]). These results were generalized to higher dimensions by P. Bégout and A. Vargas [1].

For s close[2] to 1 and in dimension two, Colliander et al. [9] have proved that the blowup solutions, which are radially symmetric, concentrate at least the mass of the ground state. This result was extended by Tzirakis [35] to dimension 1 and by Visan and Zhang [37] to general dimension. Their proof is based on the so called I-method introduced in [8]. This type of concentration result was already known for $s = 1$. In [18] the first author and T. Hmidi have proved a refined compactness lemma adapted to the blowup theory of NLS and used it to improve the results of [9]: they have removed the assumption of radial symmetry of the initial data and proved that Q is a profile for the singular solutions with minimal mass.

The energy method also proves that in the defocusing case ($\kappa = -1$), for data in H^1, the solution is global. For data in $H^s(\mathbb{R}^2)$, $s > 2/3$, Bourgain [5] proved that the solution of $(1)_{\kappa=-1}$ is global. This was improved by Fang and

[2] More precisely, for $s > \frac{1+\sqrt{11}}{5}$.

Grillakis [14]. Related results for other dimensions have recently appeared in the work of Da Silva, Pavlovic, Staffilani and Tzirakis [12,13]. A global well posedness result for $s = 0$, and $d \geqslant 3$, with the additional assumption of radial symmetry, has been proved by Tao, Visan and Zhang [33] with different methods (namely, using the results of [20] and [1], see also [34]).

In this paper we prove the following theorem.

Theorem 1.1. *For $d = 1$, set $s_1 = 3/4$. For $2 \leqslant d \leqslant 4$, set $s_d = \frac{d}{d+2}$. Finally, for $d \geqslant 5$, set $s_d = \frac{d^2+2d-8}{d(d+2)}$. The solution of* (1) *with initial data $u_0 \in H^s(\mathbb{R}^d)$, $s > s_d$, can be written*

$$u(t) = e^{it\Delta}u_0 + w(t), \quad t \in [0, T^*[,$$

with $w \in C([0, T^*[, H^1(\mathbb{R}^d))$. Furthermore, if T^* is finite then there exists a constant $C > 0$ such that

$$\left\|\nabla w(t)\right\|_{L^2} \geqslant \frac{C}{\sqrt{T^* - t}}, \tag{2}$$

for every $t \in [0, T^*[$.

Remark 1.2. This type of result (i.e. the extra-regularity of the Duhamel part of the solution) was firstly discovered, and used, by Bourgain [5] in the context of the defocusing L^2-critical Schrödinger equation with initial data in $H^s(\mathbb{R}^2)$, $s > 2/3$.

Remark 1.3. Theorem 1.1 says that the blowup phenomenon has an H^1 mechanism. In fact, any singular solution can be split en two parts: an H^s part which is global (since it is linear) and an H^1 part, which blows up. This explains, in particular, why despite the fact that $u(t) \in H^s$, with $s < 1$, all the blowup profiles discovered in [18] belong to H^1 (see Remark 1.7 in [18]).

Combined with the well-known smoothing effects for the linear Schrödinger equation ([10,30] and [36]) Theorem 1.1 yields

Corollary 1.4. *Under the assumptions of Theorem* 1.1*, $u(t) \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ for almost every $t \in [0, T^*[$.*

For the rest of this section we take $\kappa = 1$ (the focusing case) and assume that, in the context of Theorem 1.1, the maximal time of existence T^* is finite. The energy $E(w(t))$ of $w(t)$ is, of course, not conserved. However, we believe that a weaker result is true: at blowup time, the potential part of $E(w(t))$ is asymptotically equal to the kinetic one. This result can be easily proved when the initial data u_0 lies in H^1 and we may conjecture that it remains true for initial data in H^s when $s > s_d$. More precisely we have

Conjecture. *Under the assumptions and notations of Theorem* 1.1*, assume that $\kappa = 1$ and T^* is finite. Then we have*

$$\frac{E(w(t))}{\|\nabla w(t)\|_{L^2}^2} \longrightarrow 0, \quad \text{as } t \to T^*. \tag{3}$$

Remark 1.5. If this conjecture is true, it would imply that global existence occurs for every $u_0 \in H^s$, $s > s_d$, such that $\|u_0\|_{L^2} < \|Q\|_{L^2}$. (See Appendix A for the proof of this claim and other discussions.)

The rest of this paper is organized as follows. In Section 2, we recall some function spaces and prove some results needed for the proof of our theorem which is given in Section 3. Some proof of auxiliary results are given in Appendix A.

2. Preliminaries

In this preliminary section, we are going to recall some definitions and prove some basic properties of the objects that will be used in our analysis.

In what follows positive constants will be denoted by C and will change from line to line. If necessary, by $C_{\star,\dots,\star}$ we denote positive constants depending only on the quantities appearing in the indices.

Let us first recall the dyadic decomposition of the full space $\mathbb{R}^d$. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d\backslash\{0\})$ two radially symmetric functions such that

- $\chi(\xi) + \sum_{j\geqslant 0}\varphi(2^{-j}\xi) = 1$, $\frac{1}{3} \leqslant \chi^2(\xi) + \sum_{j\geqslant 0}\varphi^2(2^{-j}\xi) \leqslant 1$,
- $\operatorname{supp}\varphi(2^{-j}\cdot) \cap \operatorname{supp}\varphi(2^{-k}\cdot) = \varnothing$, if $|j-k| \geqslant 2$,
- $j \geqslant 1 \Rightarrow \operatorname{supp}\chi \cap \operatorname{supp}\varphi(2^{-j}) = \varnothing$.

For every $v \in \mathcal{S}'$ one defines the inhomogeneous Littlewood–Paley operators

$$P_{-1}v = \chi(\mathrm{D})v; \quad \forall j \in \mathbb{N},\ P_j v = \varphi\big(2^{-j}\mathrm{D}\big)v \quad \text{and} \quad S_j = \sum_{-1\leqslant k\leqslant j-1} P_k.$$

From the paradifferential calculus introduced by J.-M. Bony [3] the product uv can be formally divided into three parts as follows:

$$uv = T_u v + T_v u + R(u,v),$$

where

$$T_u v = \sum_j S_{j-1}uP_j v, \quad \text{and} \quad R(u,v) = \sum_j P_j u\widetilde{P}_j v,$$

$$\text{with } \widetilde{P}_j = \sum_{i=-1}^{1} P_{j+i} = P_{j-1} + P_j + P_{j+1}.$$

$T_u v$ is called paraproduct of v by u and $R(u,v)$ the remainder term.

Our proof, which relies on ideas introduced in [5], uses the notion of Bourgain spaces $\mathbf{X}^{s,b}$.

Definition 2.1 *(Bourgain spaces).* Let I be an interval of $\mathbb{R}$. For every pair of real numbers (s,b), the space $\mathbf{X}^{s,b}[I]$ is the space of functions u from I to $\mathbb{C}$ such that

$$\|u\|_{\mathbf{X}^{s,b}[I]} = \inf\big\{\|\phi\|_{\mathbf{X}^{s,b}}, \phi_{|I} = u\big\} < \infty,$$

where

$$\|\phi\|_{\mathbf{X}^{s,b}} = \left[\iint \big(1+|\xi|^{2s}\big)\big(1+\big|\lambda+|\xi|^2\big|\big)^{2b}\big|\tilde{\phi}(\xi,\lambda)\big|^2\, d\xi\, d\lambda\right]^{\frac{1}{2}}.$$

Here, $\tilde{\phi}$ denotes the Fourier transform of ϕ in the variables (x,t).

We collect in the next proposition some properties of these spaces that will be needed in our proof.

Proposition 2.2. *Let I be an interval of $\mathbb{R}$ containing* 0.

(A) *For every $s \in \mathbb{R}$ and $b > 1/2$, $\mathbf{X}^{s,b}[I] \hookrightarrow C^0(I, H^s(\mathbb{R}^d))$.*
(B) *For every $s, b \in \mathbb{R}$, $(\mathbf{X}^{s,b})^* = \mathbf{X}^{-s,-b}$.*
(C) *For any Schwartz time cuttof $\eta \in \mathcal{S}(\mathbb{R})$ and $u_0 \in H^s(\mathbb{R}^d)$, we have*

$$\big\|\eta(t)e^{it\Delta}u_0\big\|_{\mathbf{X}^{s,b}(\mathbb{R}\times\mathbb{R}^d)} \leqslant C_{\eta,b}\|u_0\|_{H^s(\mathbb{R}^d)}.$$

(D) *For every $s \in \mathbb{R}$, $-\frac{1}{2} < b' \leqslant 0$, $0 \leqslant b \leqslant b'+1$, $|I| \leqslant 1$ and $f \in \mathbf{X}^{s,b'}[I]$ we have*

$$\|F\|_{\mathbf{X}^{s,b}[I]} \leqslant C|I|^{1-b+b'}\|f\|_{\mathbf{X}^{s,b'}[I]},$$

where $F(t,x) = \int_0^t e^{i(t-s)\Delta} f(s,x)\, ds$.

(E) *For every* $b > \frac{1}{2}$ *and every* $a > \frac{3}{4}$ *if* $d = 1$ *and every* $a > \frac{d}{d+2}$ *if* $d \geqslant 2$, *there exists* $C = C(a, b)$ *such that*

$$\|u \nabla v\|_{L_{x,t}^{\frac{d+2}{d}}} \leqslant C\|u\|_{\mathbf{X}^{a,b}}\|v\|_{\mathbf{X}^{a,b}}, \quad \forall u, v \in \mathbf{X}^{a,b}.$$

(F) *For every* $2 \leqslant p \leqslant \frac{2(d+2)}{d}$ *and* $\gamma > \frac{d+2}{2}(\frac{1}{2} - \frac{1}{p})$, *there exists* $C = C(p, \gamma)$ *such that*

$$\|u\|_{L_{x,t}^p} \leqslant C\|u\|_{\mathbf{X}^{0,\gamma}}, \quad \forall u \in \mathbf{X}^{0,\gamma}.$$

(G) *For every* $\frac{2(d+2)}{d} < p \leqslant +\infty$, $a > (d+2)(\frac{d}{2(d+2)} - \frac{1}{p})$ *and* $b > \frac{1}{2}$, *there exists* $C = C(p, a)$ *such that*

$$\|u\|_{L_{x,t}^p} \leqslant C\|u\|_{\mathbf{X}^{a,b}}, \quad \forall u \in \mathbf{X}^{a,b}.$$

For the proof of (A)–(D), see [5] and [17]. We will give proofs of (E), (F) and (G) in the next subsections.

2.1. Proof of Proposition 2.2(E)

The key estimate is the following improved Strichartz's inequality:

Proposition 2.3. *For* $\alpha = \frac{1}{4}$ *if* $d = 1$, *for* $\alpha = \frac{1}{2}$ *if* $d = 2$, *and for every* $\alpha < \frac{2}{d+2}$ *if* $d \geqslant 3$, *there exists* $C = C(\alpha)$, *such that the following estimate*

$$\left\|e^{it\Delta} f e^{it\Delta} g\right\|_{L^{\frac{d+2}{d}}(\mathbf{R}^{d+1})} \leqslant C\left(\frac{M}{N}\right)^{\alpha} \|f\|_{L^2}\|g\|_{L^2}$$

holds for all L^2 *functions* f *and* g *with* $\operatorname{supp} \hat{f} \subset \{\xi \in \mathbb{R}^d\colon |\xi| \leqslant 2M\}$ *and* $\operatorname{supp} \hat{g} \subset \{\xi \in \mathbb{R}^d\colon N \leqslant |\xi| \leqslant 2N\}$ *for all* $0 < M \leqslant N$.

Remark 2.4. For the case $d = 2$, this inequality was proved by Bourgain (see Lemma 111 in [5]).

Remark 2.5. The exponents appearing in this proposition are sharp.

For $d = 1$ take $\hat{f} = \mathbb{1}_{M \leqslant \xi \leqslant 2M}$, $\hat{g} = \mathbb{1}_{1 \leqslant \xi \leqslant 1+M^{1/2}}$ and

$$\Sigma = \left\{(t, x) \in \mathbb{R}^2\colon |x + 2t| \leqslant \frac{M^{-1/2}}{20},\ |t| \leqslant \frac{M^{-1}}{20}\right\}, \quad \text{with } M \ll 1.$$

It is easy to see (since M small) that for all $\xi \in [M, 2M]$ and $(t, x) \in \Sigma$, we have $|x\xi + t|\xi|^2| \leqslant \frac{1}{2}$. Hence, there is a constant $c > 0$, such that $\Re(e^{i(x\xi + t|\xi|^2)}) \geqslant c$. Thus,

$$\left|e^{it\Delta} f(x)\right| = \left|\int_M^{2M} e^{i(x\xi + t|\xi|^2)}\, d\xi\right| \sim M, \quad \forall (x, t) \in \Sigma.$$

Similarly, for every $(x, t) \in \Sigma$

$$\left|e^{it\Delta} g(x)\right| = \left|\int_1^{1+M^{1/2}} e^{i(x\xi + t|\xi|^2)}\, d\xi\right| = \left|\int_0^{M^{1/2}} e^{i(\eta(x+2t) + t|\eta|^2)}\, d\eta\right| \sim M^{1/2}.$$

But $|\Sigma| \sim M^{-3/2}$, and so

$$\left\|e^{it\Delta} f e^{it\Delta} g\right\|_{L^3(\mathbb{R}^2)} \geqslant CM.$$

Since $\|g\|_2 = M^{1/2}$ and $\|f\|_2 = M^{1/4}$ we get the result.

The example in higher dimensions $d \geqslant 2$, is quite different. Consider $\hat{f} = \mathbb{1}_A$, where, for some $M \ll 1$, $A = \{\xi = (\xi_1, \xi_2, \dots, \xi_N)\colon M^2 \leqslant \xi_1 \leqslant 2M^2, |\xi_k| \leqslant M, k = 2, 3, \dots, d\}$, and $\hat{g} = \mathbb{1}_B$, where $B = \{\xi = (\xi_1, \xi_2, \dots, \xi_d)\colon 1 \leqslant \xi_1 \leqslant$

$1+M^2$, $|\xi_k| \leqslant M$, $k=2,3,\dots,d\}$. Then, we see that, for $|t| \leqslant \frac{1}{20dM^2}$, $|x_1| \leqslant \frac{1}{20dM^2}$, $|x_2|, |, |x_3|, \dots, |x_d| \leqslant \frac{1}{20dM}$, $|e^{it\Delta} f(x)| \geqslant M^{d+1}$ and $|e^{it\Delta} g(x)| \geqslant M^{d+1}$. Therefore $\|e^{it\Delta} f e^{it\Delta} g\|_{L^{\frac{d+2}{d}}} \geqslant CM^{d+1+\frac{2}{d+2}}$, while $\|f\|_2 = \|g\|_2 = M^{\frac{d+1}{2}}$. This gives the desired result.

Proof of Proposition 2.3. By rescaling, it suffices to consider the case $N=1$. If $M \sim 1$, then, this proposition follows from Hölder and Strichartz and estimates. We will only deal with the case $M \ll 1$.

• *Case* $d \geqslant 2$. Proposition 2.3, for $d \geqslant 2$, follows from the following estimates and an interpolation argument.

Theorem 2.6. *(Cf. [32].) Assume that f and g are functions belonging to $L^2(\mathbb{R}^d)$, such that* $\operatorname{supp}\hat{f}$, $\operatorname{supp}\hat{g} \subset \{\xi$: $|\xi| \leqslant 2\}$, *and* $\operatorname{dist}(\operatorname{supp}\hat{f}, \operatorname{supp}\hat{g}) \sim 1$. *Then, for all $p > \frac{d+3}{d+1}$,*

$$\left\|e^{it\Delta} f e^{it\Delta} g\right\|_{L^p_{x,t}} \leqslant C\|f\|_{L^2}\|g\|_{L^2}.$$

Proposition 2.7. *Assume that f and g are functions belonging to $L^2(\mathbb{R}^d)$, such that* $\operatorname{supp}\hat{f} \subset \{\xi \in \mathbb{R}^d\colon |\xi| \leqslant 2M\}$ *and* $\operatorname{supp}\hat{g} \subset \{\xi \in \mathbb{R}^d\colon 1 \leqslant |\xi| \leqslant 2\}$ *for some $M \leqslant 1/4$. Then,*

$$\left\|e^{it\Delta} f e^{it\Delta} g\right\|_{L^2_{x,t}} \leqslant CM^{\frac{d-1}{2}}\|f\|_{L^2}\|g\|_{L^2}.$$

Proof. We follow the arguments introduced in [4] (see also [28] and [32]). Let $\psi \in \mathcal{S}(\mathbb{R})$ such that $\psi(0)=1$ and

$$\hat{\psi}(\tau) = 1 \quad \text{if } |\tau| < 1, \qquad \hat{\psi}(\tau) = 0 \quad \text{if } |\tau| > 2.$$

By the dominated convergence we have

$$\begin{aligned}\left\|e^{it\Delta} f e^{it\Delta} g\right\|^2_{L^2_{x,t}} &= \lim_{\varepsilon\to 0} \iint \psi(\varepsilon t) e^{it\Delta} f(x) \overline{e^{it\Delta} f(x)} e^{it\Delta} g(x) \overline{e^{it\Delta} g(x)}\, dx\, dt \\ &:= \lim_{\varepsilon\to 0} \mathrm{I}_\varepsilon.\end{aligned}$$

A straightforward calculus, using the Plancherel's identity in x, yields

$$\begin{aligned}\mathrm{I}_\varepsilon &= C\int_t \psi(\varepsilon t) \int_{\mathbb{R}^{3d}} \hat{f}(\xi)\bar{\hat{f}}(\eta)\hat{g}(\zeta)\bar{\hat{g}}(\xi+\zeta-\eta) e^{-2it(\langle \xi-\eta, \zeta-\eta\rangle)}\, d\xi\, d\eta\, d\zeta\, dt \\ &= C\int_{\mathbb{R}^{3d}} \hat{f}(\xi)\bar{\hat{f}}(\eta)\hat{g}(\zeta)\bar{\hat{g}}(\xi+\zeta-\eta)\frac{1}{\varepsilon}\hat{\psi}\left(\frac{2}{\varepsilon}\langle \xi-\eta, \zeta-\eta\rangle\right) d\xi\, d\eta\, d\zeta.\end{aligned}$$

Thus, we infer

$$|\mathrm{I}_\varepsilon| \leqslant \frac{C}{\varepsilon} \iiint_{\Gamma^\varepsilon_{\zeta,\eta}} \left|\hat{f}(\xi)\hat{f}(\eta)\hat{g}(\zeta)\hat{g}(\xi+\zeta-\eta)\right| d\xi\, d\eta\, d\zeta,$$

where

$$\Gamma^\varepsilon_{\zeta,\eta} := \left\{\xi \in \mathbb{R}^d\colon \left|\langle \xi-\eta, \zeta-\eta\rangle\right| \leqslant \varepsilon\right\}.$$

By Cauchy–Schwarz's inequality in ξ we get

$$|\mathrm{I}_\varepsilon| \leqslant C\iint \left[\int_{\Gamma^\varepsilon_{\zeta,\eta}} \frac{1}{\varepsilon}\left|\hat{g}(\xi+\zeta-\eta)\hat{f}(\xi)\right|^2 d\xi\right]^{\frac{1}{2}} \frac{1}{\sqrt{\varepsilon}}\left|\Gamma^\varepsilon_{\zeta,\eta} \cap \operatorname{supp}\hat{f}\right|^{\frac{1}{2}}\left|\hat{g}(\zeta)\hat{f}(\eta)\right| d\zeta\, d\eta.$$

By the assumptions on $\operatorname{supp}\hat{f}$ and $\operatorname{supp}\hat{g}$ we get easily that for every $\eta \in \operatorname{supp}\hat{f}$ and every $\zeta \in \operatorname{supp}\hat{g}$ we have $\frac{1}{2} \leqslant |\eta-\zeta| \leqslant 3$. Therefore, in this case $\Gamma^\varepsilon_{\zeta,\eta}$ is a $\frac{2\varepsilon}{|\eta-\zeta|}$ thick layer (which is orthogonal to the vector $\eta-\zeta$) and then

$$\left|\Gamma^\varepsilon_{\zeta,\eta} \cap \operatorname{supp}\hat{f}\right| \leqslant C\varepsilon M^{d-1}.$$

This implies

$$\mathrm{I}_\varepsilon \leqslant CM^{\frac{d-1}{2}} \iint \biggl(\frac{1}{\varepsilon} \int_{\Gamma^\varepsilon_{\zeta,\eta}} \bigl| \hat{g}(\xi+\zeta-\eta) \hat{f}(\xi) \bigr|^2 \, d\xi \biggr)^{\frac{1}{2}} \bigl| \hat{g}(\zeta) \hat{f}(\eta) \bigr| \, d\zeta \, d\eta.$$

We apply Cauchy–Schwarz's inequality in the variables ζ, η

$$\mathrm{I}_\varepsilon \leqslant CM^{\frac{d-1}{2}} \biggl(\int_{\eta \in \operatorname{supp} \hat{f}} \int \frac{1}{\varepsilon} \int_{\Gamma^\varepsilon_{\zeta,\eta}} \bigl| \hat{g}(\xi+\zeta-\eta) \hat{f}(\xi) \bigr|^2 \, d\xi \, d\zeta \, d\eta \biggr)^{\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2}.$$

For fixed ξ, η, we change variables $\zeta \to \xi + \zeta - \eta = u$. Note that, by the definition above, $\xi \in \Gamma^\varepsilon_{\zeta,\eta}$ if and only if $\eta \in \Gamma^\varepsilon_{\xi+\zeta-\eta,\xi} = \Gamma^\varepsilon_{u,\xi}$. Therefore,

$$\int_{\eta \in \operatorname{supp} \hat{f}} \int \int_{\Gamma^\varepsilon_{\zeta,\eta}} \bigl| \hat{g}(\xi+\zeta-\eta) \hat{f}(\xi) \bigr|^2 \, d\xi \, d\zeta \, d\eta = \int_{\eta \in \operatorname{supp} \hat{f}} \iint_{u:\ \eta \in \Gamma^\varepsilon_{u,\xi}} \bigl| \hat{g}(u) \hat{f}(\xi) \bigr|^2 \, du \, d\xi \, d\eta.$$

Finally, changing the order of integration,

$$\int_{\eta \in \operatorname{supp} \hat{f}} \int \int_{\Gamma^\varepsilon_{\zeta,\eta}} \bigl| \hat{g}(\xi+\zeta-\eta) \hat{f}(\xi) \bigr|^2 \, d\xi \, d\zeta \, d\eta = \iint \bigl| \hat{g}(u) \hat{f}(\xi) \bigr|^2 \bigl| \Gamma^\varepsilon_{u,\xi} \cap \operatorname{supp} \hat{f} \bigr| \, d\xi \, du.$$

As before this leads to

$$\int_{\eta \in \operatorname{supp} \hat{f}} \int \frac{1}{\varepsilon} \int_{\Gamma^\varepsilon_{\zeta,\eta}} \bigl| \hat{g}(\xi+\zeta-\eta) \hat{f}(\xi) \bigr|^2 \, d\xi \, d\zeta \, d\eta \leqslant CM^{d-1} \|f\|^2_{L^2} \|g\|^2_{L^2}.$$

The outcome is

$$\begin{aligned} \bigl\| e^{it\Delta} f e^{it\Delta} g \bigr\|^2_{L^2_{x,t}} &= \lim_{\varepsilon \to 0} \mathrm{I}_\varepsilon \\ &\leqslant CM^{d-1} \|f\|^2_{L^2} \|g\|^2_{L^2}. \qquad \square \end{aligned}$$

- *Case* $d = 1$. Proposition 2.3, for $d = 1$, can be proved by interpolation between the following estimates.

Theorem 2.8. *For every $p \geqslant 2$ there exists a constant $C = C(p)$ such that the following estimate*

$$\bigl\| e^{it\Delta} f e^{it\Delta} g \bigr\|_{L^p_{x,t}(\mathbb{R}^2)} \leqslant C \|\hat{f}\|_{L^{\frac{p}{p-1}}(\mathbb{R})} \|\hat{g}\|_{L^{\frac{p}{p-1}}(\mathbb{R})} \tag{4}$$

holds for all tempered distributions f and g such that their Fourier transforms $\hat{f}$ and $\hat{g}$ belong to $L^{\frac{p}{p-1}}(\mathbb{R})$ and are supported in $\{\xi\colon |\xi| \leqslant 2\}$, *with* $\operatorname{dist}(\operatorname{supp} \hat{f}, \operatorname{supp} \hat{g}) \sim 1$.

This theorem follows from a classical argument by Fefferman and Stein (see [15]). For sake of completeness we give the proof in Appendix A.

Proposition 2.9. *Assume that f and g are functions defined on $\mathbb{R}$ such that* $\operatorname{supp} \hat{f} \subset \{\xi \in \mathbb{R}\colon |\xi| \leqslant 2M\}$ *and* $\operatorname{supp} \hat{g} \subset \{\xi \in \mathbb{R}\colon 1 \leqslant |\xi| \leqslant 2\}$ *for some $M \leqslant 1/4$. Then,*

$$\bigl\| e^{it\Delta} f e^{it\Delta} g \bigr\|_{L^4_{x,t}} \leqslant CM^{3/8} \|f\|_{L^2} \|g\|_{L^2}.$$

Moreover, the exponent $3/8$ is sharp.

Proof of Proposition 2.9. Without loss of generality, we can assume that $\hat{g} \subset \{\xi \in \mathbb{R}\colon 1 \leqslant \xi \leqslant 2\}$. For the integers k, $M^{-1/2} - 1 \leqslant k \leqslant 2M^{-1/2}$, set $I_k = [kM^{1/2}, (k+1)M^{1/2}]$, and define functions g_k by $\hat{g}_k = \hat{g}\mathbb{1}_{I_k}$. Then, $g = \sum_k g_k$ and,

$$\left\| e^{it\Delta} f e^{it\Delta} g \right\|_{L^4_{x,t}} = \left\| \sum_k e^{it\Delta} f e^{it\Delta} g_k \right\|_{L^4_{x,t}}.$$

We will use the following well-known orthogonality lemma

Lemma 2.10 *(Orthogonality lemma).*

$$\left\| \sum_k e^{it\Delta} f e^{it\Delta} g_k \right\|_{L^4_{x,t}} \leqslant C \left(\sum_k \left\| e^{it\Delta} f e^{it\Delta} g_k \right\|^2_{L^4_{x,t}} \right)^{1/2}.$$

A similar orthogonality result was first observed by C. Fefferman [16] (see also A. Córdoba [11]). For the sake of completeness we will give a proof in Appendix A. Let us now finish the proof of Proposition 2.9. Using the orthogonality lemma, we are reduced to show that

$$\left(\sum_k \left\| e^{it\Delta} f e^{it\Delta} g_k \right\|^2_{L^4_{x,t}} \right)^{1/2} \leqslant C M^{3/8} \|f\|_2 \|g\|_2. \tag{5}$$

But, by (4) for $p = 4$, we have

$$\left(\sum_k \left\| e^{it\Delta} f e^{it\Delta} g_k \right\|^2_{L^4_{x,t}} \right)^{1/2} \leqslant C \left(\sum_k \|\hat{f}\|^2_{4/3} \|\hat{g}_k\|^2_{4/3} \right)^{1/2}.$$

By Hölder's inequality and using the size of the supports of $\hat{f}$ and $\hat{g}_k$ the last expression is bounded by

$$\begin{aligned} C M^{3/8} \left(\sum_k \|f\|_2^2 \|g_k\|_2^2 \right)^{1/2} &= C M^{3/8} \|f\|_2 \left(\sum_k \|g_k\|_2^2 \right)^{1/2} \\ &= C M^{3/8} \|f\|_2 \|g\|_2, \end{aligned}$$

which gives (5). For the sharpness of the exponent $3/8$ we can use the same example in Remark 2.5. □

Proposition 2.3 for $d = 1$ follows by interpolating between Theorem 2.8 and Proposition 2.9. □

Using Proposition 2.3, we now prove the following bilinear estimate.

Proposition 2.11. *Set $\beta = \beta(d) = \frac{2}{d+2}$ if $d \geqslant 2$ and $\beta = \frac{1}{4}$ if $d = 1$. For every $b \in]0, \beta[$, there is a constant C_b such that,*

$$\left\| e^{it\Delta} \psi_1 \nabla e^{it\Delta} \psi_2 \right\|_{L^{\frac{d+2}{d}}(\mathbb{R}^{d+1})} \leqslant C_b \|\psi_1\|_{H^b(\mathbb{R}^d)} \|\psi_2\|_{H^{1-b}(\mathbb{R}^d)},$$

for every $\psi_1 \in H^b(\mathbb{R}^d)$ and $\psi_2 \in H^{1-b}(\mathbb{R}^d)$.

Proof. Using Bony's decomposition and the fact that P_j commutes with the free propagator $e^{it\Delta}$ we write

$$\begin{aligned} \left\| e^{it\Delta} \psi_1 e^{it\Delta} \nabla \psi_2 \right\|_{L^{\frac{d+2}{d}}(\mathbb{R}^{d+1})} &\leqslant \sum_{j=1}^{+\infty} \left\| e^{it\Delta} S_{j-1} \psi_1 e^{it\Delta} \nabla P_j \psi_2 \right\|_{L^{\frac{d+2}{d}}(\mathbb{R}^{d+1})} \\ &\quad + \sum_{j=1}^{+\infty} \left\| e^{it\Delta} \nabla S_{j-1} \psi_2 e^{it\Delta} P_j \psi_1 \right\|_{L^{\frac{d+2}{d}}(\mathbb{R}^{d+1})} \\ &\quad + \sum_{j=-1}^{+\infty} \left\| e^{it\Delta} P_j \psi_1 e^{it\Delta} \nabla \tilde{P}_j \psi_2 \right\|_{L^{\frac{d+2}{d}}(\mathbb{R}^{d+1})} \\ &= I + II + III. \end{aligned}$$

• *About II*. By Cauchy–Schwarz and Strichartz's inequalities,

$$
\begin{aligned}
II &\leqslant \sum_{j=1}^{+\infty}\sum_{k=-1}^{j-2} 2^k \|P_k\psi_2\|_{L^2}\|P_j\psi_1\|_{L^2(\mathbb{R}^d)} \\
&\leqslant \sum_{j=1}^{+\infty}\sum_{k=-1}^{j-2} 2^{b(k-j)}2^{(1-b)k}\|P_k\psi_2\|_{L^2}2^{jb}\|P_j\psi_1\|_{L^2} \\
&\leqslant \left|\left\langle 2^{-b\cdot}\star 2^{(1-b)\cdot}\|P_\cdot\psi_2\|_{L^2}, 2^{\cdot b}\|P_\cdot\psi_1\|_{L^2}\right\rangle_{\ell^2}\right|,
\end{aligned}
$$

where $\star$ denotes the convolution in ℓ^2.

If we apply successively Cauchy–Schwarz and Young's estimates we get

$$
II \leqslant \|\psi_2\|_{H^{1-b}(\mathbb{R}^d)}\|\psi_1\|_{H^b(\mathbb{R}^d)}.
$$

• *About III*. By Hölder, Strichartz and Bernstein's inequalities

$$
\begin{aligned}
III &= \sum_{j=-1}^{+\infty}\left\|e^{it\Delta}(P_j\psi_1)e^{it\Delta}(\nabla\tilde{P}_j\psi_2)\right\|_{\mathrm{L}^{\frac{d+2}{d}}(\mathbb{R}^{d+1})} \\
&\leqslant \sum_{j=-1}^{+\infty} 2^j\|P_j\psi_1\|_{L^2(\mathbb{R}^d)}\|\tilde{P}_j\psi_2\|_{L^2(\mathbb{R}^d)} \\
&\leqslant \sum_{j=-1}^{+\infty} 2^{jb}\|P_j\psi_1\|_{L^2(\mathbb{R}^d)}2^{j(1-b)}\|\tilde{P}_j\psi_2\|_{L^2(\mathbb{R}^d)} \\
&\leqslant C\|\psi_1\|_{H^b(\mathbb{R}^d)}\|\psi_2\|_{H^{1-b}(\mathbb{R}^d)}.
\end{aligned}
$$

In the last line we have used the Cauchy–Schwartz inequality.

• *About I*. This is the nontrivial part of the proof. Let $\alpha\in\,]b,\beta(d)[$ fixed. By Proposition 2.3 we have

$$
\begin{aligned}
I &\leqslant \sum_{j=1}^{+\infty}\sum_{k=-1}^{j-2}\left\|e^{it\Delta}P_k\psi_1 e^{it\Delta}\nabla P_j\psi_2\right\|_{L^{\frac{d+2}{d}}(\mathbb{R}^{d+1})} \\
&\leqslant \sum_{j=1}^{+\infty}\sum_{k=-1}^{j-2} 2^{(k-j)\alpha}\|P_k\psi_1\|_{L^2(\mathbb{R}^d)}2^j\|P_j\psi_2\|_{\mathrm{L}^2(\mathbb{R}^d)}.
\end{aligned}
$$

We write it in an appropriate way,

$$
\begin{aligned}
&= \sum_{j=1}^{+\infty}\sum_{k=1}^{j-2} 2^{(j-k)(b-\alpha)}2^{kb}\|P_k\psi_1\|_{L^2}2^{j(1-b)}\|P_j\psi_2\|_{\mathrm{L}^2(\mathbb{R}^d)} \\
&= \left|\left\langle 2^{-(b-\alpha)\cdot}\star 2^{b\cdot}\|P_\cdot\psi_1\|_{L^2}, 2^{\cdot(1-b)}\|P_\cdot\psi_2\|_{\mathrm{L}^2}\right\rangle_{\ell^2}\right| \\
&\leqslant \|\psi_1\|_{H^b(\mathbb{R}^d)}\|\psi_2\|_{H^{1-b}(\mathbb{R}^d)},
\end{aligned}
$$

where we have applied successively Cauchy–Schwarz and Young's inequalities. □

Now, we are ready to prove Proposition 2.2(E). Write

$$
u_i(x,t) = \iint e^{i(x\xi+t\tau)}\tilde{u}_i(\xi,\tau)\,d\xi\,d\tau = \iint e^{i(x\xi+t|\xi|^2)}e^{it(\tau-|\xi|^2)}\tilde{u}_i(\xi,\tau)\,d\xi\,d\tau.
$$

Change variables, $\lambda=\tau-|\xi|^2$,

$$
= \iint e^{i(x\xi+t|\xi|^2)}e^{it\lambda}\tilde{u}_i\big(\xi,\lambda+|\xi|^2\big)\,d\xi\,d\lambda.
$$

By Fubini,

$$= \int e^{it\lambda} \int e^{i(x\xi + t|\xi|^2)} \tilde{u}_i\big(\xi, \lambda + |\xi|^2\big)\, d\xi\, d\lambda.$$

Define $u_{i,\lambda}$ by $\hat{u}_{i,\lambda}(\xi) = \tilde{u}_i(\xi, \lambda + |\xi|^2)$, where $\hat{}$ denotes the Fourier transform in the x-variable. Then,

$$u_i(x,t) = \int e^{it\lambda} e^{it\Delta} u_{i,\lambda}(x)\, d\lambda.$$

Therefore,

$$\|u_1 \nabla u_2\|_{L^{\frac{d+2}{d}}} = \left\| \int e^{it\lambda} \int e^{itr} e^{it\Delta} u_{1,\lambda}(x) e^{it\Delta} \nabla u_{2,r}(x)\, d\lambda\, dr \right\|_{L^{\frac{d+2}{d}}_{x,t}}.$$

By Minkowski's inequality,

$$\leqslant \iint \big\| e^{it\Delta} u_{1,\lambda}(x) e^{it\Delta} \nabla u_{2,r}(x) \big\|_{L^{\frac{d+2}{d}}_{x,t}}\, d\lambda\, dr.$$

By Proposition 2.11, this is bounded by

$$C \iint \|u_{1,\lambda}\|_{H^b} \|u_{2,r}\|_{H^{1-b}}\, d\lambda\, dr, \tag{6}$$

for all $b < \beta(d)$. Notice that

$$\int \|u_{i,\lambda}\|_{H^\alpha}\, d\lambda = \int \left[\int \big|\hat{u}_{i,\lambda}(\xi)\big|^2 \big(1 + |\xi|^{2\alpha}\big)\, d\xi \right]^{\frac{1}{2}} d\lambda.$$

Using Cauchy–Schwarz's inequality, for all $\varepsilon > 0$,

$$\begin{aligned}
&\leqslant \left(\iint \big|\hat{u}_{i,\lambda}(\xi)\big|^2 \big(1 + |\xi|^{2\alpha}\big)\, d\xi \big(1 + |\lambda|\big)^{1+\varepsilon}\, d\lambda \right)^{\frac{1}{2}} \left(\int \big(1 + |\lambda|\big)^{-1-\varepsilon}\, d\lambda \right)^{\frac{1}{2}} \\
&\leqslant C_\varepsilon \left(\iint \big|\tilde{u}_i\big(\xi, \lambda + |\xi|^2\big)\big| \big(1 + |\xi|^{2\alpha}\big) \big(1 + |\lambda|\big)^{1+\varepsilon}\, d\lambda\, d\xi \right)^{\frac{1}{2}} \\
&= C_\varepsilon \left(\iint \big|\tilde{u}_i(\xi, \tau)\big| \big(1 + |\xi|^{2\alpha}\big) \big(1 + \big|\tau - |\xi|^2\big|\big)^{1+\varepsilon}\, d\tau\, d\xi \right)^{\frac{1}{2}} \\
&= C_\varepsilon \|u_i\|_{\mathbf{X}^{\alpha, \frac{1}{2}+\varepsilon}}.
\end{aligned}$$

Thus, we estimate (6) by

$$C_\varepsilon \|u_1\|_{\mathbf{X}^{b, \frac{1}{2}+\varepsilon}} \|u_2\|_{\mathbf{X}^{1-b, \frac{1}{2}+\varepsilon}},$$

for all $\varepsilon > 0$. The outcome is

$$\|u_1 \nabla u_2\|_{L^{\frac{d+2}{d}}} \leqslant C_\varepsilon \|u_1\|_{\mathbf{X}^{a, \frac{1}{2}+\varepsilon}} \|u_2\|_{\mathbf{X}^{a, \frac{1}{2}+\varepsilon}},$$

for all $\varepsilon > 0$ and $a \geqslant \max(b, 1-b)$ with $b < \beta(d)$. This is Proposition 2.2(E).

2.2. Proof of Proposition 2.2(F)

Set $q = \frac{d+2}{d}$. Using Cauchy–Schwarz and Strichartz's inequalities, and using (6) as above, we see that, for all $\varepsilon > 0$,

$$\|u\|_{L^{2q}_{x,t}} \leqslant C_\varepsilon \|u\|_{\mathbf{X}^{0, \frac{1}{2}+\varepsilon}}.$$

Rename $v = \tilde{u}$. Then, the last inequality can be written as

$$\|\tilde{v}\|_{L^{2q}_{x,t}} \leqslant C_\varepsilon \left[\iint \big|v(\xi, \tau)\big|^2 \big(1 + \big|\tau + |\xi|^2\big|\big)^{1+\varepsilon}\, d\tau\, d\xi \right]^{\frac{1}{2}},$$

for all $\varepsilon > 0$. On the other hand

$$\|\tilde{v}\|_{L^2} = \left[\iint \left|v(\xi,\tau)\right|^2 d\tau\, d\xi \right]^{\frac{1}{2}}.$$

We can now use the theorem of interpolation with change of measure (see [2] Theorem 5.4.1) to obtain for $2 \leqslant p \leqslant 2q$ and for θ defined by $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{2q}$,

$$\|\tilde{v}\|_{L^p(I\times\mathbb{R}^2)} \leqslant C_\varepsilon \left[\iint \left|v(\xi,\tau)\right|^2 \left(1 + \left|\tau + |\xi|^2\right|\right)^{\theta(1+\varepsilon)} d\tau\, d\xi \right]^{\frac{1}{2}},$$

for all $\varepsilon > 0$. For $v = \tilde{u}$, we obtain

$$\|u\|_{L^p(I\times\mathbb{R}^2)} \leqslant C_\varepsilon \|u\|_{\mathbf{X}^{0,\theta(\frac{1}{2}+\varepsilon)}},$$

for all $\varepsilon > 0$. This gives Proposition 2.2(F).

2.3. *Proof of Proposition 2.2(G)*

As above, by interpolation with change of measure with the estimate (F) for $p = 2q$, it suffices to show that

$$\|w\|_{L^\infty_{x,t}} \leqslant C_{a,b} \|w\|_{\mathbf{X}^{a,b}},$$

for all $a > \frac{d}{2}$ and $b > \frac{1}{2}$. This follows form Proposition 2.2(A) and the Sobolev embedding.

3. Proof of Theorem 1.1

Using Proposition 2.2 we are going to prove Theorem 1.1. We will prove that for every $t_0 \in [0, T^*[$ there exists a small interval I containing t_0, such that $w \in \mathbf{X}^{1,\frac{1}{2}+\varepsilon}[I]$, for some $\varepsilon > 0$. This implies that $w \in \mathbf{X}^{1,\frac{1}{2}+\varepsilon}[J]$ for every compact interval $J \subset [0, T^*[$, which is, in view of Proposition 2.2(A), a stronger result than the first assertion in Theorem 1.1. Note also that, via a time translation, it is sufficient to consider $t_0 = 0$.

According to the integral formulation of the initial value problem (1) we have

$$w(t,x) = i\kappa \int_0^t e^{i(t-t')\Delta}\left[\left|u(t',x)\right|^{\frac{4}{d}} u(t',x)\right] dt'.$$

Let $\phi \in \mathcal{C}_0^\infty(\mathbb{R})$, such that

$$\phi(t) = 1 \quad \text{if } |t| < 1, \qquad \phi(t) = 0 \quad \text{if } |t| > 2.$$

For $T > 0$ we denote $\phi_T(t) = \phi(\frac{t}{T})$. On $[0, T]$ the function w is equal to

$$w(t,x) = i\kappa\phi_T(t) \int_0^t e^{i(t-t')\Delta}\left[\left|\phi_T(t')u(t',x)\right|^{\frac{4}{d}} \phi_T(t')u(t',x)\right] dt'.$$

Thus, by replacing u by $\phi_T u$ we can assume that u is globally defined (and compactly supported) in time. Meanwhile we keep the same notation u.

Note also that the value of T is not relevant for us in this part of proof. So, the dependence on T of the constants is not explicited.

Let $s > s_d$ fixed and $\varepsilon = \varepsilon(d,s)$ a sufficiently small positive number (it satisfies a finite number of smallness conditions). Applying Proposition 2.2(D) (with $b = \frac{1}{2} + \varepsilon$ and $b' = -\frac{1}{2} + \varepsilon$), one can estimate

$$\|\nabla w\|_{\mathbf{X}^{0,\frac{1}{2}+\varepsilon}} \leqslant C \left\|\nabla\left(|u|^{\frac{4}{d}} u\right)\right\|_{\mathbf{X}^{0,-\frac{1}{2}+\varepsilon}}. \tag{7}$$

Here and throughout this section the constants $C = C_{\varepsilon,s,d}$. For sake of shortness we drop the indices.

The main ingredient of our proof is the following nonlinear estimate.

Proposition 3.1. *For every $s > s_d$ there exists $\varepsilon > 0$, such that for all $b > \frac{1}{2}$ we have*

$$\big\|\nabla\big(|u|^{\frac{4}{d}}u\big)\big\|_{\mathbf{X}^{0,-\frac{1}{2}+\varepsilon}} \leqslant C_{s,d,b}\|u\|_{\mathbf{X}^{s,b}}^{\frac{4}{d}+1},$$

for all $u \in \mathbf{X}^{s,b}$.

Proof of Proposition 3.1. By duality (Proposition 2.2(B)) we have

$$\big\|\nabla\big(|u|^{\frac{4}{d}}u\big)\big\|_{\mathbf{X}^{0,-\frac{1}{2}+\varepsilon}} = \sup_{\|\psi\|_{\mathbf{X}^{0,\frac{1}{2}-\varepsilon}}\leqslant 1}\big|\big\langle\psi,\nabla\big(|u|^{\frac{4}{d}}u\big)\big\rangle\big|. \tag{8}$$

The remainder of the proof depends on the dimension d.

• *Case* $d = 1, 2, 3$. Obviously, Hölder's inequality yields

$$\big|\big\langle\psi,\nabla\big(|u|^{\frac{4}{d}}u\big)\big\rangle\big| \leqslant C\big\|\psi|u|^{\frac{4}{d}-1}\big\|_{L_{x,t}^{\frac{d+2}{2}}}\|u\nabla u\|_{L_{x,t}^{\frac{d+2}{d}}} \leqslant C\|\psi\|_{L_{x,t}^{p_1}}\|u\|_{L_{x,t}^{p_2}}^{\frac{4}{d}-1}\|u\nabla u\|_{L_{x,t}^{\frac{d+2}{d}}}$$

with

$$\frac{1}{p_1} = \frac{d}{2(d+2)} + \varepsilon \quad \text{and} \quad \frac{1}{p_2} = \frac{d}{2(d+2)} - \frac{d\varepsilon}{4-d}.$$

According to Proposition 2.2(F) we have

$$\|\psi\|_{L_{x,t}^{p_1}} \leqslant C\|\psi\|_{\mathbf{X}^{0,\gamma}}$$

for all $\gamma > \frac{d+2}{2}(\frac{1}{2} - \frac{d}{2(d+2)} - \varepsilon) = \frac{1}{2} - \frac{d+2}{2}\varepsilon$. In particular,

$$\|\psi\|_{L_{x,t}^{p_1}} \leqslant C\|\psi\|_{\mathbf{X}^{0,\frac{1}{2}-\varepsilon}}.$$

On the other hand Proposition 2.2(G) yields

$$\|u\|_{L_{x,t}^{p_2}} \leqslant C\|u\|_{\mathbf{X}^{a,b}}$$

for every $a > \frac{d(d+2)}{4-d}\varepsilon$, and $b > \frac{1}{2}$. For ε small we have $s > \frac{d(d+2)}{4-d}\varepsilon$, and then

$$\|u\|_{L_{x,t}^{p_2}} \leqslant \|u\|_{\mathbf{X}^{s,b}},$$

for all $b > \frac{1}{2}$. Finally, since $s > s_d$, Proposition 2.2(E) yields

$$\|u\nabla u\|_{L_{x,t}^{\frac{d+2}{d}}} \leqslant \|u\|_{\mathbf{X}^{s,b}}^2.$$

The outcome is

$$\big|\big\langle\psi,\nabla\big(|u|^{\frac{4}{d}}u\big)\big\rangle\big| \leqslant C\|\psi\|_{\mathbf{X}^{0,\frac{1}{2}-\varepsilon}}\|u\|_{\mathbf{X}^{s,b}}^{\frac{4}{d}+1} \leqslant C\|u\|_{\mathbf{X}^{s,b}}^{\frac{4}{d}+1},$$

for all $b > \frac{1}{2}$.

• *Case* $d = 4$. Coming back to (8), by Hölder's inequality,

$$\big|\big\langle\psi,\nabla\big(|u|u\big)\big\rangle\big| \leqslant C\|\psi\|_{L^{p_1}}\|u\nabla u\|_{L^{p_2}}$$

with

$$\frac{1}{p_1} = \frac{1}{3} + \varepsilon, \qquad \frac{1}{p_2} = \frac{2}{3} - \varepsilon.$$

Proposition 2.2(F) (for $d = 4$) yields

$$\|\psi\|_{L^{p_1}} \leqslant C\|\psi\|_{\mathbf{X}^{0,\frac{1}{2}-\varepsilon}}.$$

To estimate $\|u\nabla u\|_{L^{p_2}}$ we interpolate between Proposition 2.2(E) and the trivial fact

$$\|u\nabla v\|_{L^\infty} \leqslant \|u\|_{L^\infty}\|\nabla v\|_{L^\infty} \leqslant C\|u\|_{\mathbf{X}^{a,b}}\|v\|_{\mathbf{X}^{1+a,b}},$$

for $b > \frac{1}{2}$, $a > 2$, which follows from Proposition 2.2(G). This yields,

$$\|u\nabla u\|_{L^{p_2}} \leqslant C\|u\|^2_{\mathbf{X}^{\frac{2}{3}+\delta(\varepsilon),b}}$$

with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, we obtain (for ε small enough)

$$\|u\nabla u\|_{L^{p_2}} \leqslant C\|u\|^2_{\mathbf{X}^{s,b}},$$

for $s > s_4 = \frac{2}{3}$, $b > \frac{1}{2}$.

• *Case* $d \geqslant 5$. We go back to (8). Obviously, we have

$$\left|\left\langle \psi, \nabla\left(|u|^{\frac{4}{d}}u\right)\right\rangle\right| \leqslant C\left|\left\langle |\psi|, |u|^{\frac{4}{d}}|\nabla u|\right\rangle\right|.$$

Write $u = \sum_{k=-1}^{+\infty} u_k$ where $u_k = P_k u$. Then, by the triangular and Hölder's inequalities,

$$\left|\left\langle \psi, \nabla\left(|u|^{\frac{4}{d}}u\right)\right\rangle\right| \leqslant C\sum_k \|\psi\|_{L^{p_1}}\|u\nabla u_k\|^{\frac{4}{d}}_{L^{p_2}}\|\nabla u_k\|^{1-\frac{4}{d}}_{L^{p_3}}$$

with

$$\frac{1}{p_1} = \frac{d}{2(d+2)} + \varepsilon, \qquad \frac{1}{p_2} = \frac{d}{d+2} \quad \text{and} \quad \frac{1}{p_3} = \frac{d}{2(d+2)} - \frac{d}{d-4}\varepsilon.$$

According to Proposition 2.2(F), we have

$$\|\psi\|_{L^{p_1}_{x,t}} \leqslant C\|\psi\|_{\mathbf{X}^{0,\gamma}}$$

for all $\gamma > \frac{d+2}{2}(\frac{1}{2} - \frac{d}{2(d+2)} - \varepsilon) = \frac{1}{2} - \frac{d+2}{2}\varepsilon$. In particular,

$$\|\psi\|_{L^{p_1}_{x,t}} \leqslant C\|\psi\|_{\mathbf{X}^{0,\frac{1}{2}-\varepsilon}}.$$

On the other hand, Proposition 2.2(E) yields, for all $\varepsilon > 0$, $b > \frac{1}{2}$,

$$\|u\nabla u_k\|_{L^{p_2}} \leqslant C\|u\|_{\mathbf{X}^{\frac{d}{d+2}+\varepsilon,b}}\|u_k\|_{\mathbf{X}^{\frac{d}{d+2}+\varepsilon,b}}.$$

Finally, Proposition 2.2(G) gives

$$\|\nabla u_k\|_{L^{p_3}} \leqslant C\|\nabla u_k\|_{\mathbf{X}^{a,b}} \leqslant C\|u_k\|_{\mathbf{X}^{1+a,b}}$$

for every $a > \frac{d(d+2)}{d-4}\varepsilon$ and $b > \frac{1}{2}$.

Now, since $\|v_k\|_{\mathbf{X}^{s,b}} \sim 2^{ks}\|v_k\|_{\mathbf{X}^{0,b}}$, we get

$$\sum_k \|u\nabla u_k\|^{\frac{4}{d}}_{L^{p_2}}\|\nabla u_k\|^{1-\frac{4}{d}}_{L^{p_3}} \leqslant C\|u\|^{\frac{4}{d}}_{\mathbf{X}^{\frac{d}{d+2}+\varepsilon,b}}\sum_k \|u_k\|_{\mathbf{X}^{0,b}}2^{k(1-\frac{8}{d(d+2)}+\delta(\varepsilon))},$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Given $s > s_d = 1 - \frac{8}{d(d+2)}$, we choose $\varepsilon > 0$, so that $s > 1 - \frac{8}{d(d+2)} + \delta(\varepsilon) + \varepsilon$. Then, we estimate

$$\begin{aligned}
\sum_{k=-1}^{+\infty} \|u_k\|_{\mathbf{X}^{0,b}}2^{k(1-\frac{8}{d(d+2)}+\delta)} &\leqslant C\sum_{k=-1}^{+\infty}\|u_k\|_{\mathbf{X}^{0,b}}2^{ks}2^{-k\varepsilon} \\
&\leqslant C\sum_{k=-1}^{+\infty}\|u_k\|_{\mathbf{X}^{s,b}}2^{-k\varepsilon} \\
&\leqslant C\|u\|_{\mathbf{X}^{s,b}}\sum_k 2^{-k\varepsilon} \\
&\leqslant C\|u\|_{\mathbf{X}^{s,b}}.
\end{aligned}$$

(We have controlled $\|u_k\|_{\mathbf{X}^{s,\frac{1}{2}+\varepsilon}}$ by $\|u\|_{\mathbf{X}^{s,\frac{1}{2}+\varepsilon}}$ uniformly in $k \geqslant -1$.) Thus, for all $b > \frac{1}{2}$,

$$\sum_k \|u\nabla u_k\|_{L^{p_2}}^{\frac{4}{d}} \|\nabla u_k\|_{L^{p_3}}^{1-\frac{4}{d}} \leqslant C\, \|u\|_{\mathbf{X}^{\frac{d}{d+2}+\varepsilon,b}}^{\frac{4}{d}} \|u\|_{\mathbf{X}^{s,b}} \leqslant C\, \|u\|_{\mathbf{X}^{s,b}}^{\frac{4}{d}+1},$$

where we have used the fact that $s > \frac{d}{d+2} + \varepsilon$, for ε small enough.

Proposition 3.1 and (7) imply that

$$\|\nabla w\|_{\mathbf{X}^{0,\frac{1}{2}+\varepsilon}} \leqslant C\|u\|_{\mathbf{X}^{s,b}}^{\frac{4}{d}+1},$$

for all $b > \frac{1}{2}$. A similar reasoning shows that for all $b > \frac{1}{2}$,

$$\big\||u|^{\frac{4}{d}}u\big\|_{\mathbf{X}^{0,-\frac{1}{2}+\varepsilon}} \leqslant C\|u\|_{\mathbf{X}^{\delta,b}}^{\frac{4}{d}+1}, \tag{9}$$

for some $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence,

$$\|w\|_{\mathbf{X}^{0,\frac{1}{2}+\varepsilon}} \leqslant C\|u\|_{\mathbf{X}^{\delta,b}}^{\frac{4}{d}+1}.$$

The outcome is

$$\|w\|_{\mathbf{X}^{1,\frac{1}{2}+\varepsilon}} \leqslant C\|u\|_{\mathbf{X}^{s,b}}^{\frac{4}{d}+1},$$

for all $b > \frac{1}{2}$. Hence, in order to prove the first part of Theorem 1.1 we are reduced to show that $u \in \mathbf{X}^{s,\frac{1}{2}+\varepsilon}$ for some small $\varepsilon > 0$. Below we restate and prove (using the previous calculations) the local existence result for NLS equations in the Bourgain spaces which are relevant for us (i.e. for $s > s_d$).

Proposition 3.2. *Given $s > s_d$, there exists $\varepsilon > 0$, such that, for every $u_0 \in H^s$, the Cauchy problem* (1) *has unique local solution in $u \in \mathbf{X}^{s,\frac{1}{2}+\varepsilon}[I]$ for some interval $I = [0, T]$.*

Proof. Let $\phi \in \mathcal{C}_0^\infty(\mathbb{R})$, such that

$$\phi(t) = 1 \quad \text{if } |t| < 1, \qquad \phi(t) = 0 \quad \text{if } |t| > 2.$$

Fix $0 < T \leqslant 1$. On $[0, T]$ the initial value problem (1) is equivalent to the following integral equation

$$u(t,x) = \phi(t)e^{it\Delta}u_0(x) + i\kappa\phi\left(\frac{t}{T}\right)\int_0^t e^{i(t-s)\Delta}|u|^{\frac{4}{d}}u(s,x)\,ds, \quad \forall t \in [0,T].$$

We use the classical fixed point argument. Define, the functional A as

$$Au(t,x) = \phi(t)e^{it\Delta}u_0(x) + i\kappa\phi\left(\frac{t}{T}\right)\int_0^t e^{i(t-s)\Delta}|u|^{\frac{4}{d}}u(s,x)\,ds.$$

According to Proposition 2.2(C), (D), one has, for $\varepsilon' \geqslant \varepsilon$,

$$\|Au\|_{\mathbf{X}^{s,\frac{1}{2}+\varepsilon}} \leqslant C\|u_0\|_{H^s} + CT^{\varepsilon'-\varepsilon}\big\||u|^{\frac{4}{d}}u\big\|_{\mathbf{X}^{s,-\frac{1}{2}+\varepsilon'}}.$$

Let us now estimate the nonlinear term. If ε' is small enough,

$$\big\||u|^{4/d}u\big\|_{\mathbf{X}^{s,-\frac{1}{2}+\varepsilon'}} \leqslant \big\||u|^{4/d}u\big\|_{\mathbf{X}^{1,-\frac{1}{2}+\varepsilon'}} \leqslant C\|u\|_{\mathbf{X}^{s,\frac{1}{2}+\varepsilon}}^{4/d+1},$$

by Proposition 3.1 and (9).

The outcome is

$$\|Au\|_{\mathbf{X}^{s,\frac{1}{2}+\varepsilon}} \lesssim \|u_0\|_{H^s} + T^{\varepsilon'-\varepsilon}\|u\|_{\mathbf{X}^{s,\frac{1}{2}+\varepsilon}}^{4/d+1},$$

for some $\varepsilon' > \varepsilon$, small enough (depending on s).

In the same way we prove

$$\|Au - Av\|_{\mathbf{X}^{s,\frac{1}{2}+\varepsilon}} \lesssim \|u - v\|_{\mathbf{X}^{s,\frac{1}{2}+\varepsilon}} T^{\varepsilon'-\varepsilon}\big(\|u\|_{\mathbf{X}^{s,\frac{1}{2}+\varepsilon}} + \|v\|_{\mathbf{X}^{s,\frac{1}{2}+\varepsilon}}\big)^{4/d}.$$

These estimates are sufficient to prove the existence of a local solution in $\mathbf{X}^{s,\frac{1}{2}+\varepsilon}$. □

Let us now prove the lower bound (2). A well-known scaling argument yields the following lower bound on the blowup rate

$$\big\|u(t)\big\|_{\dot H^s} \geqslant \frac{C}{|T^* - t|^{s/2}}, \quad t \in [0, T^*[.$$

Since the H^s norm of the linear part is conserved then

$$\big\|w(t)\big\|_{\dot H^s} \geqslant \frac{C}{|T^* - t|^{s/2}}.$$

However, by interpolation, we can write

$$\big\|w(t)\big\|_{H^s} \leqslant \big\|w(t)\big\|_{L^2}^{1-s} \big\|w(t)\big\|_{H^1}^{s}.$$

Since the L^2 norm of u is conserved as t varies, we infer

$$\big\|w(t)\big\|_{L^2} \leqslant 2\|u_0\|_{L^2}.$$

This yields

$$\big\|w(t)\big\|_{\dot H^1} \geqslant \frac{C}{|T^* - t|^{\frac{1}{2}}}, \quad t \in [0, T^*[,$$

as claimed. □

Appendix A

A.1. Proof of the claim in Remark 1.5

Take $u_0 \in H^s$ such that the corresponding solution u of (1) blows up in finite time $T^* > 0$ and $t_n \uparrow T^*$ as $n \to +\infty$. We set

$$v_n(x) = (\lambda_n)^{\frac{d}{2}} w(t_n, \lambda_n x), \qquad \lambda_n = 1/\big\|\nabla w(t_n, .)\big\|_{L^2},$$

where w is as in Theorem 1.1. Using conservation of the L^2 norm for the linear and nonlinear Schrödinger equations, we get trivially that $\{v_n\}_{n=1}^\infty$ is a bounded sequence in L^2. Also, assuming the conjecture,

$$E(v_n) = \frac{E(w(t_n, \cdot))}{\|\nabla w(t_n, \cdot)\|_{L^2}^2} \longrightarrow 0, \quad \text{as } t_n \to T^*.$$

Since $\|\nabla v_n\|_{L^2} = 1$ this gives, in particular, $\|v_n\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \to \frac{d+2}{d}$.

The sequence $\{v_n\}_{n=1}^\infty$ satisfies the assumptions of Theorem 1.1 in [18] with $M = 1$ and $m^{\frac{4}{d}+2} = \frac{d+2}{d}$. Thus, there exists $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^d$ such that, up to a subsequence,

$$(\lambda_n)^{\frac{d}{2}} w(t_n, \lambda_n \cdot + x_n) \rightharpoonup V,$$

with $\|V\|_{L^2} \geqslant \|Q\|_{L^2}$.

On the other hand, since the linear part is continuous in time with values in L^2 then $(\lambda_n)^{\frac{d}{2}}(e^{it_n\Delta}u_0)(\lambda_n \cdot + x_n) \rightharpoonup 0$, which implies that

$$(\lambda_n)^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup V. \tag{A.1}$$

This yields, thanks to the conservation of the L^2 norm,

$$\|u_0\|_{L^2} = \liminf_{n\to+\infty} \|u(t_n,\cdot)\|_{L^2} \geqslant \|V\|_{L^2} \geqslant \|Q\|_{L^2}.$$

This implies that $\|Q\|_{L^2}$ is the critical mass for the formation of singularities, that is, if $u_0 \in H^s$, $s > s_d$, with

$$\|u_0\|_{L^2} < \|Q\|_{L^2}$$

then the corresponding solution is global.[3] Finally, notice that by using the asymptotic (A.1) and (2) we can easily prove that a concentration phenomenon in L^2 occurs at the blow up time. More precisely, for every solution u of (1) which blows up at finite time $T^* > 0$ we have

$$\liminf_{t\to T^*} \sup_{y\in\mathbb{R}^d} \int\limits_{|x-y|\leqslant\lambda(t)\sqrt{T^*-t}} |u(t,x)|^2\,dx \geqslant \int Q^2,$$

for any function $\lambda(t) > 0$, such that $\lambda(t) \to +\infty$ as $t \uparrow T^*$. (See, for instance, [20] Theorem 1.10 for a similar reasoning.)

A.2. *Proof of Theorem 2.8*

Write

$$\begin{aligned} e^{it\Delta} f(t,x) e^{it\Delta} g(t,x) &= \int e^{i(x\xi+t|\xi|^2)} \hat{f}(\xi)\,d\xi \int e^{i(x\eta+t|\eta|^2)} \hat{g}(\eta)\,d\eta \\ &= \iint e^{i(x(\xi+\eta)+t(|\xi|^2+|\eta|^2))} \hat{f}(\xi)\hat{g}(\eta)\,d\xi\,d\eta. \end{aligned}$$

Then, we change variables $(u,v) = (\xi+\eta, |\xi|^2+|\eta|^2)$ to get

$$e^{it\Delta} f(t,x) e^{it\Delta} g(t,x) = \iint e^{i(xu+tv)} H(u,v)\,du\,dv,$$

where $H(u,v) = \frac{\hat{f}(\xi)\hat{g}(\eta)}{2|\xi-\eta|}$ $(\xi = \xi(u,v),\ \eta = \eta(u,v))$. Hence,

$$e^{it\Delta} f(t,x) e^{it\Delta} g(t,x) = \widehat{H}(t,x)$$

is the Fourier transform of H. Therefore, by Hausdorff–Young inequality, for all $p \geqslant 2$,

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L^p_{x,t}} = \|\widehat{H}\|_{L^p_{x,t}} \leqslant \|H\|_{L^{p'}_{u,v}}.$$

To compute the $L^{p'}$ norm of H, we undo the change of variables. Notice that, since $\inf\{\operatorname{dist}(\xi,\eta)\colon\ \xi \in \operatorname{supp}\hat{f},\ \eta \in \operatorname{supp}\hat{g}\} \sim 1$, the factor $|\xi-\eta|$ is harmless, and therefore

$$\|H\|_{L^{p'}_{u,v}} \sim \|\hat{f}\|_{L^{p'}(\mathbb{R})} \|\hat{g}\|_{L^{p'}(\mathbb{R})},$$

which finishes the proof of the theorem.

A.3. *Proof of Lemma 2.10*

Write

$$\Big\|\sum_k e^{it\Delta} f e^{it\Delta} g_k\Big\|^2_{L^4_{x,t}} = \Big\|\Big(\sum_k e^{it\Delta} f e^{it\Delta} g_k\Big)^2\Big\|_{L^2_{x,t}} = \Big\|\sum_k e^{it\Delta} f e^{it\Delta} g_k \sum_j e^{it\Delta} f e^{it\Delta} g_j\Big\|_{L^2_{x,t}}.$$

[3] Notice that to get this result we need only (3) to be true for a time sequence $t_n \uparrow T^*$.

By Plancherel

$$= \left\| \sum_k \sum_j \widetilde{e^{it\Delta} f} * \widetilde{e^{it\Delta} f} * \widetilde{e^{it\Delta} g_k} * \widetilde{e^{it\Delta} g_j} \right\|_{L^2_{x,t}}.$$

Denote by $E_{k,j}$ the support of the function $\widetilde{e^{it\Delta} f} * \widetilde{e^{it\Delta} f} * \widetilde{e^{it\Delta} g_k} * \widetilde{e^{it\Delta} g_j}$. We will use

Claim. *The sets $E_{k,j}$ are essentially disjoints, i.e. there is a universal constant, independent of M, such that,*

$$\sum_k \sum_j \mathbb{1}_{E_{k,j}} \leqslant C.$$

Using this claim,

$$\leqslant \left\| \sum_k \sum_j \widetilde{e^{it\Delta} f} * \widetilde{e^{it\Delta} f} * \widetilde{e^{it\Delta} g_k} * \widetilde{e^{it\Delta} g_j} \right\|_{L^2_{x,t}}$$
$$\leqslant C \left(\sum_k \sum_j \left\| \widetilde{e^{it\Delta} f} * \widetilde{e^{it\Delta} f} * \widetilde{e^{it\Delta} g_k} * \widetilde{e^{it\Delta} g_j} \right\|^2_{L^2_{x,t}} \right)^{1/2}.$$

By Plancherel

$$= C \left(\sum_k \sum_j \left\| e^{it\Delta} f e^{it\Delta} g_k e^{it\Delta} f e^{it\Delta} g_j \right\|^2_{L^2_{x,t}} \right)^{1/2}$$
$$= C \left(\int \sum_k \sum_j \left| e^{it\Delta} f e^{it\Delta} g_k e^{it\Delta} f e^{it\Delta} g_j \right|^2 \right)^{1/2}$$
$$= C \left(\int \left(\sum_k \left| e^{it\Delta} f e^{it\Delta} g_k \right|^2 \right)^2 \right)^{1/2}$$
$$= C \left\| \sum_k \left| e^{it\Delta} f e^{it\Delta} g_k \right|^2 \right\|_{L^2}.$$

By the triangular inequality,

$$\leqslant C \sum_k \left\| \left| e^{it\Delta} f e^{it\Delta} g_k \right|^2 \right\|_{L^2}$$
$$= C \sum_k \left\| e^{it\Delta} f e^{it\Delta} g_k \right\|^2_{L^4}.$$

To prove the claim, notice that the support of $\widetilde{e^{it\Delta} g_k}$ is contained in the set $E_k = \{(\tau,\xi)\colon |\xi - kM^{1/2}| \leqslant M^{1/2}, \tau = |\xi|^2\}$, and the support of $\widetilde{e^{it\Delta} f}$ is contained in the set $\{(\tau,\xi)\colon |\xi| \leqslant 2M, \tau = |\xi|^2\}$. Hence, if $(\delta,\eta) \subset E_{j,k}$, then, there are ξ, ξ', $(|\xi|^2,\xi) \in E_k$, $(|\xi'|^2,\xi') \in E_j$, such that, $|\delta - (|\xi|^2 + |\xi'|^2)| \leqslant 4M$, and $|\eta - (\xi + \xi')| \leqslant 4M$.

Since $|\xi + \xi'|^2 + |\xi - \xi'|^2 = 2(|\xi|^2 + |\xi'|^2)$, and $M \ll M^{1/2}$, then, if $k \geqslant j$, we see that $E_{j,k} \subset H_{j,k}$, where

$$H_{j,k} = \left\{(\delta,\eta)\colon \left|\eta - (k+j)M^{1/2}\right| \leqslant 3M^{1/2},\ |k-j-4|^2 M \leqslant \left|2\delta - |\eta|^2\right| \leqslant |k-j+4|^2 M\right\}.$$

Now, from this expression, it is easy to see that if $(\delta,\eta) \in H_{j,k} \cap H_{i,l} \neq \emptyset$, $k \geqslant j$, $l \geqslant i$, then, $-6 \leqslant (k+j) - (i+l) \leqslant 6$ and $-8 \leqslant (k-j) - (i-l) \leqslant 8$. From this, $-14 \leqslant k - i \leqslant 14$ and $-14 \leqslant j - l \leqslant 14$, which proves the claim.

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