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Global boundary controllability of the Saint-Venant system for sloped canals with friction

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Abstract

We consider a sloped canal with friction that is governed by the Saint-Venant system with source term. We show that starting sufficiently close to a stationary constant subcritical initial state, we can control the system in finite time to a state in a C^1 neighbourhood of any other stationary constant subcritical state by boundary control at the ends of the canal in such a way that during the process the system state remains continuously differentiable.

Moreover, we show that if the derivative of the initial state is sufficiently small, it can be steered to every stationary constant subcritical state in finite time.

Résumé

On étudie un canal cylindrique incliné modélisé par les équations de Saint-Venant avec un terme source et un terme de frottement. On montre l'existence d'un voisinage pour la norme C^1 de toute partie bornée de la courbe des équilibres sous-critiques constants tel que, étant donnés deux éléments de ce voisinage, il est possible d'aller du premier élément au second en temps fini par des contrôles adéquats sur le bord du canal. En outre, on montre que, si la dérivée de l'état initial est suffisamment petite, on peut amener l'état du système à tout équilibre donné, constant et sous-critique, et ceci en temps fini.

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Mots-clés : Contrôlabilité exacte globale ; Équations de Saint-Venant ; Frottement ; Pente

1. Introduction

In hydraulic engineering, the Saint-Venant system is frequently used to model water flow through a sloped canal with friction. This system has been introduced in [4] and forms a quasilinear hyperbolic system of partial differential equations with source term. If the slope of the canal is not zero, there exists a family of constant stationary states where water height and velocity are such that the corresponding source term vanishes as stated by Saint-Venant: Le

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frottement est ... compensé par l'accélération due à la pente (The friction is compensated by the acceleration due to the slope). For each water height there exists a uniquely defined corresponding velocity, such that this equilibrium occurs.

In this paper we study the problem of exact boundary controllability in a C^1 neighbourhood of the set of subcritical stationary constant states of the type discussed above. We show that for each bounded part of the curve of constant subcritical equilibrium states there exists a C^1 neighbourhood of this set where exact controllability holds. Any subcritical initial state whose derivative is sufficiently small can be steered in finite time to this set where exact controllability holds. For a discussion of the concept of exact controllability see [12].

The Saint-Venant equations with zero source term can be used to model the flow through a horizontal frictionless canal. For this model, all constant states are stationary. The global exact boundary controllability between these states has been analysed in [8] and the corresponding result for the more general case of quasilinear hyperbolic systems of diagonal form has been discussed in [14]. The Saint-Venant equations with zero source term have also been considered in [3], where a boundary feedback control in networks of open canals is defined.

In practice, often networks of open canals appear. As a model, in [10], Saint-Venant systems on a graph coupled by interface conditions at the nodes have been analysed. The case of a star-shaped network of open canals with zero slope and friction modelled by the Saint-Venant system with a source term has been considered in [11]. In this case all constant states with zero velocity are stationary. In [11] exact controllability with continuously differentiable states locally around the constant subcritical equilibrium states has been shown. The result is based upon the semi-global existence result for C^1 solutions given in [16] where it is assumed that the source term vanishes at zero.

In this paper, in contrast to [11] we also admit the case of non-zero canal slopes, where for each water height we have exactly one non-zero velocity that yields a constant stationary state. We consider only subcritical flow that is during the control process, we do not wish to change the type of the flow. This means that from a given subcritical flow we want to go to another subcritical flow, so during the process the nature of the boundary conditions does not change. This is different from [6], where controllability between sub- and supercritical states is considered for the case without source terms.

The exact controllability of supercritical flow on tree shaped networks of not necessarily horizontal canals with friction has been considered in [9].

A general survey on controllability of partial differential equations is given in [18]. A sensitivity calculus for related problems of optimal boundary control has been given in [7].

2. The Saint-Venant equations

The dynamics of our system are described by the Saint-Venant equations, also called shallow water equations.

Let A denote the wetted cross section and V denote the velocity. The conservation of mass yields the equation

$$\frac{\partial}{\partial t}A + \frac{\partial}{\partial x}(VA) = 0. \tag{1}$$

The flux of momentum is modelled by the equation

$$\frac{\partial}{\partial t}V + \frac{\partial}{\partial x}\left(gH + \frac{V^2}{2}\right) = -S \tag{2}$$

where H denotes the water height corresponding to A and g is the gravity constant. The source term S has the form

$$S = S(A, V) = g\big(\gamma + \eta(A, V)\big), \tag{3}$$

where γ is the slope of the channel. If γ is negative, the channel proceeds downwards. The friction slope $\eta(A, V)$ models the friction effects due to walls. It is given by the Manning–Strickler relation

$$\eta(A, V) = \frac{C^2|V|V}{(A/P(A))^{4/3}} \tag{4}$$

with a roughness constant $C > 0$ where $P(A)$ is the wetted perimeter corresponding to A (see [13,5]). The quotient $A/P(A)$ is called the hydraulic radius.

We consider a prismatic canal with a constant rectangular cross section and width b.

Then $A = Hb$ and $P(A) = b + 2H$ and we can write the Saint-Venant equations in the form

$$\frac{\partial}{\partial t}\begin{pmatrix} A \\ V \end{pmatrix} + \begin{pmatrix} V & A \\ \frac{g}{b} & V \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} A \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ -S \end{pmatrix}. \tag{5}$$

We introduce the Riemann invariants

$$R_+(A, V) = V + \varphi(A), \tag{6}$$

$$R_-(A, V) = V - \varphi(A) \tag{7}$$

with $\varphi(A) = 2\sqrt{\frac{g}{b}A}$. Hence we have

$$V = \frac{R_+ + R_-}{2}, \qquad \varphi(A) = \frac{R_+ - R_-}{2},$$

so the change of coordinates to the Riemann invariants is a bijection.

In case of a continuously differentiable solution, we obtain a system in diagonal form with source term for the Riemann invariants, namely

$$\frac{\partial}{\partial t}\begin{pmatrix} R_+ \\ R_- \end{pmatrix} + \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} R_+ \\ R_- \end{pmatrix} = \begin{pmatrix} -S \\ -S \end{pmatrix} \tag{8}$$

where

$$\lambda_+ = \frac{3}{4}R_+ + \frac{1}{4}R_-, \qquad \lambda_- = \frac{1}{4}R_+ + \frac{3}{4}R_- \tag{9}$$

and S is the source term corresponding to the pair (R_+, R_-) that determines a unique state (A, V). A flow is called *subcritical* if $\lambda_-\lambda_+ < 0$. If $\lambda_-\lambda_+ > 0$ it is called supercritical. If $\lambda_-\lambda_+ = 0$, the state is critical.

3. Stationary states for subcritical flow

In [9], supercritical stationary solutions for sloped canals with friction have been analysed. These stationary solutions converge exponentially fast along the canal to constant stationary limit states for which the source term vanishes.

In this section, we analyse the behaviour of subcritical stationary solutions. For all $A > 0$, there exists a unique velocity V such that the corresponding source term vanishes, that is $S(A, V) = 0$, namely

$$V(A) = -\operatorname{sign}(\gamma), \frac{\sqrt{|\gamma|}}{C}\left(\frac{A}{P(A)}\right)^{2/3} = -\operatorname{sign}(\gamma)\frac{\sqrt{|\gamma|}}{C(1/H + 2/b)^{2/3}}. \tag{10}$$

For a horizontal canal with friction this yields $V(A) = 0$.

For all $A > 0$ the constant states $(A, V(A))$ are stationary, that is they satisfy the Saint-Venant system (1), (2). If $V(A)^2 < gA/b$, these stationary states are subcritical. The constant stationary solutions $(A, V(A))$ are well-defined for a channel of arbitrary length L. This is in contrast to a different type of stationary solutions, namely non-constant stationary solutions that develop a singularity after a finite canal length due to the source term.

For any stationary state (1) implies that the flow rate $Q = bHV$ is constant along the canal, hence $V = Q/(bH)$ and $V_x = -QH_x/(bH^2)$.

For the water height, (1), (2) yield along the canal the ordinary differential equation

$$H_x = -\frac{HS}{gH - V^2}$$

which can also be written as

$$H_x = -\frac{b^2H^3}{gb^2H^3 - Q^2}S = -\frac{gb^2H^3}{gb^2H^3 - Q^2}\left[\gamma + C^2\frac{|Q|Q}{b^2H^2}\left(\frac{1}{H} + \frac{2}{b}\right)^{4/3}\right]. \tag{11}$$

For a subcritical state we have $V^2 < gH$ which is equivalent to $gb^2H^3 - Q^2 > 0$.

Due to the source term, there exist stationary continuously differentiable solutions for which after a finite canal length x_0 a singularity occurs, since the right-hand side of (11) goes to minus infinity for $x \to x_0$ and $\lambda_-\lambda_+$ tends to zero.

Lemma 1. *Let real numbers $Q \in (0, \infty)$ and $H_0 \in (0, \infty)$ be given. Define $H_{\text{crit}} = (Q^2/gb^2)^{1/3}$. Assume that $H_0 > H_{\text{crit}}$ and that*

$$S_0 = g\left[\gamma + C^2 \frac{|Q|Q}{b^2 H_0^2}\left(\frac{1}{H_0} + \frac{2}{b}\right)^{4/3}\right] > 0.$$

Let $H(x)$ be the solution of the initial value problem with the initial condition $H(0) = H_0$, and the differential equation (11).

Then $(A, V) = (bH, Q/(bH))$ is a subcritical stationary solution of (1), (2) *that is defined on the finite interval*

$$I = \{x \geqslant 0: \ H(x) > H_{\text{crit}}\}.$$

If $I = [0, x_0)$ then $\lim_{x \to x_0 -} H'(x) = -\infty$ and $\lim_{x \to x_0 -} H(x) = H_{\text{crit}}$.

On the interval I, the source term S is strictly increasing, the function H is strictly decreasing and $V(x) = Q/(bH(x))$ is strictly increasing. Let

$$\alpha = \frac{g}{S_0}\left[H_0 - H_{\text{crit}} + \frac{H_{\text{crit}}^3}{2H_0^2}\right].$$

Then $I \subset [0, \alpha]$, that is for $x \geqslant \alpha$, the solution does not exist.

Define $S_{\text{crit}} = S(bH_{\text{crit}}, Q/(bH_{\text{crit}})) > S_0 > 0$ and

$$\beta = \frac{g}{S_{\text{crit}}}(H_0 - H_{\text{crit}})\left[1 - \frac{1}{2}\left(\frac{H_{\text{crit}}}{H_0} + \frac{H_{\text{crit}}^2}{H_0^2}\right)\right] > 0.$$

Then $[0, \beta] \subset I$, that is on the interval $[0, \beta]$ the solution exists.

Proof. Define $S(x) = S(bH(x), Q/(bH(x)))$. We have $\partial_V S(A, V) > 0$ and if $V > 0$ we have $\partial_A S(A, V) < 0$.

As long as $H(x) > H_{\text{crit}}$, (11) implies that

$$S'(x) = \frac{b^2 H^3}{gb^2 H^3 - Q^2}\left[\frac{Q}{bH^2}\partial_V S - b\partial_A S\right] S$$

thus $S'(x)/S(x) > 0$, hence S is strictly increasing which implies that H is strictly decreasing.

Our assumptions imply that I is a non-empty interval. For all $x \in I$, the right-hand side of (11) is well defined and since H is strictly decreasing, the solution exists on I. In H_{crit}, the right-hand side of (11) has a singularity, which is the reason that the solution cannot be extended beyond I.

For all $x \in I$, the ordinary differential equation (11) implies the inequality

$$\left(1 - \frac{Q^2}{gb^2 H^3}\right) H_x \leqslant -\frac{S_0}{g}.$$

Integration on the interval $(0, x)$ yields the inequality

$$\left[H(x) - H_0\right] + \frac{Q^2}{2gb^2}\left[\frac{1}{H^2(x)} - \frac{1}{H_0^2}\right] \leqslant -\frac{S_0}{g}x.$$

Hence we have

$$H(x) \leqslant H_0 + \frac{Q^2}{2gb^2 H_0^2} - \frac{S_0}{g}x$$

which implies that for $x \geqslant \alpha$, we have $H(x) \leqslant H_{\text{crit}}$, a contradiction, hence $x \notin I$. Thus we see that I is indeed a finite interval.

Now we prove that β is a lower bound for x_0. For all $x \in I$, the ordinary differential equation (11) implies the inequality

$$\left(1 - \frac{Q^2}{gb^2 H^3}\right) H_x \geqslant -\frac{S_{\text{crit}}}{g}.$$

Integration on the interval $(0, x)$ yields the inequality

$$\big[H(x) - H_0\big] + \frac{Q^2}{2gb^2}\left[\frac{1}{H^2(x)} - \frac{1}{H_0^2}\right] \geqslant -\frac{S_{\text{crit}}}{g}x.$$

Hence we have

$$H(x) + \frac{Q^2}{2gb^2 H_{\text{crit}}^2} > H_0 + \frac{Q^2}{2gb^2 H_0^2} - \frac{S_{\text{crit}}}{g}x$$

which implies that for $x \leqslant \beta$, we have $H(x) > H_{\text{crit}}$, hence $[0, \beta] \subset I$. □

Remark 2. Let $L > 0$ be given. Lemma 1 implies that a constant flow rate $Q > 0$ with $S_{\text{crit}} > 0$ and a water height $H_L > H_{\text{crit}}$ with $S(bH_L, Q/(bH_L)) > 0$ determine a unique subcritical continuously differentiable stationary solution of the Saint-Venant system (1), (2) on the interval $[0, L]$ with $H(L) = H_L$ if $L < x_0$. For this state we have $S(bH(0), Q/(bH(0))) > 0$.

Remark 3. Let $L > 0$ be given. Lemma 1 implies that a constant flow rate $Q > 0$ with $S_{\text{crit}} > 0$ and a water height $H_0 > H_{\text{crit}}$ with $S(bH_0, Q/(bH_0)) > 0$ do not determine a subcritical continuously differentiable stationary solution of the Saint-Venant system (1), (2) on the interval $[0, L]$ with $H(0) = H_0$ if $L \geqslant x_0$, for example if $L \geqslant \alpha$.

4. Boundary conditions for a single canal

Let L denote the length of the canal. For subcritical flow, at both ends $x = 0$ and $x = L$ of the canal a scalar boundary condition is prescribed. In terms of the Riemann invariants, the boundary conditions have the form

$$x = 0: R_+(0, t) = g_0(t), \tag{12}$$

$$x = L: R_-(L, t) = g_L(t). \tag{13}$$

The boundary controls g_0, g_L can only generate a continuously differentiable solution if they are continuously differentiable and satisfy the C^1 compatibility conditions with the initial values at $t = 0$.

5. Controllability results locally around the curve of constant stationary states

Now we consider the exact controllability of the system (1), (2) on a finite interval $[0, L]$ with boundary conditions (12), (13) from a given initial state in finite time to a desired target state such that during the process, the system has a continuously differentiable solution.

First we present our main result, which states that in a C^1-neighbourhood of the set of constant stationary subcritical states exact controllability holds. The distance between the initial and the final state can be arbitrarily large. Only the derivative of the initial and the target state must be sufficiently small. If the control time is chosen sufficiently large, exact controllability holds.

Theorem 4 *(Global exact controllability for sloped canals with friction). For a canal with a constant rectangular cross section, consider the Saint-Venant system* (1), (2) *with the source term given by* (3), (4) *and boundary conditions in characteristic form* (12), (13).

For real numbers $M > 0$, $\delta > 0$ and $\varepsilon > 0$ with $\delta < M$ define the finite time

$$T(\delta, M) = \max\left\{\max\left\{\frac{L}{|\lambda_+(A, V)|}, \frac{L}{|\lambda_-(A, V)|}\right\}: \tag{14}$$

$$(A, V) \in R^2\!: \delta \leqslant gA/b - V^2,\ |V| \leqslant M,\ A \leqslant M,\ S(A, V) = 0\right\} \tag{15}$$

and the set of functions

$$G(\delta, M, \varepsilon) = \{(A, V) \in C^1[0, L]\colon \textit{For all } x \in [0, L]\colon \\ \delta \leqslant gA(x)/b - V(x)^2,\ |V(x)| \leqslant M,\ A(x) \leqslant M,\ |S(A(x), V(x))| \leqslant \varepsilon, \\ |A'(x)| \leqslant \varepsilon,\ |V'(x)| \leqslant \varepsilon\}. \tag{16}$$

Then for all δ, M with $0 < \delta < M$ there exist a number $\varepsilon > 0$ and a time $T_0 > T(\delta, M)$ such that between all initial and final states in $G(\delta, M, \varepsilon)$ exact controllability in the time T_0 is possible, that is for all $(A_0, V_0) \in G(\delta, M, \varepsilon)$ and $(A_1, V_1) \in G(\delta, M, \varepsilon)$ there exist boundary controls g_0, g_L such that there exists a unique continuously differentiable solution of the initial boundary value problem

$$A(x, 0) = A_0(x), \quad V(x, 0) = V_0(x), \quad x \in [0, L] \tag{17}$$

(1), (2), (12), (13) ($(x, t) \in [0, L] \times [0, T_0]$) *that satisfies the final condition*

$$A(x, T_0) = A_1(x), \quad V(x, T_0) = V_1(x), \quad x \in [0, L]. \tag{18}$$

Remark 5. The set $G(\delta, M, \varepsilon)$ defined in Theorem 4 where exact controllability holds contains constant stationary states, but also non-constant stationary states of the type described in Lemma 1 and Remark 2.

The proof of Theorem 4 uses the following local exact controllability result that is a consequence of Theorem 3.1 from [17]. This result gives exact controllability in a C^1-neighbourhood of a stationary subcritical state. For this local exact controllability result, the control time can be fixed a priori.

Theorem 6 *(Local exact controllability for sloped canals with friction). For a canal with a constant rectangular cross section, consider the Saint-Venant system* (1), (2) *with the source term given by* (3), (4) *and boundary conditions in characteristic form* (12), (13).

Let $(\bar{A}(x), \overline{V}(x)) \in C^1[0, L] \times C^1[0, L]$ be given such that $0 < g\bar{A}(x)/b - \overline{V}(x)^2$ and $Q = \bar{A}(x)\overline{V}(x)$ is constant and

$$\frac{\partial}{\partial x}\left(g\frac{\bar{A}(x)}{b} + \frac{\overline{V}(x)^2}{2}\right) = -S(\bar{A}(x), \overline{V}(x))$$

for all $x \in [0, L]$. Define the finite time

$$T(\bar{A}, \overline{V}) = \max\left\{\max_{x\in[0,L]} \frac{L}{|\lambda_+(\bar{A}(x), \overline{V}(x))|}, \max_{x\in[0,L]} \frac{L}{|\lambda_-(\bar{A}(x), \overline{V}(x))|}\right\} \tag{19}$$

and the set of functions

$$\Gamma(\varepsilon) = \{(A, V) \in C^1[0, L]\colon \textit{For all } x \in [0, L]\colon\ 0 < gA(x)/b - V(x)^2, \\ \|V - \overline{V}\|_1 \leqslant \varepsilon,\ \|A - \bar{A}\|_1 \leqslant \varepsilon\} \tag{20}$$

where $\|\cdot\|_1$ stands for the C^1-norm.

Then for all $T_0 > T(\bar{A}, \overline{V})$ there exists a number $\varepsilon > 0$ such between all initial and final states in $\Gamma(\varepsilon)$ exact controllability in the time T_0 is possible, that is for all $(A_0, V_0) \in \Gamma(\varepsilon)$ and $(A_1, V_1) \in \Gamma(\varepsilon)$ there exist boundary controls g_0 g_L such that there exists a unique continuously differentiable solution of the initial boundary value problem (17), (1), (2), (12), (13) ($(x, t) \in [0, L] \times [0, T_0]$) *that satisfies the final condition* (18).

Proof of Theorem 6. For continuously differentiable states, we can write (1), (2) in the diagonal form (8). In order to apply the local exact controllability result Theorem 3.1 from [17], we transform the system to a system with a source term that vanishes at zero. Let $(\overline{R}_+, \overline{R}_-)$ denote the Riemann invariants corresponding to the subcritical stationary state $(\bar{A}, \overline{V})$ and $(\bar{\lambda}_+, \bar{\lambda}_-) = (\lambda_+(\bar{A}, \overline{V}), \lambda_-(\bar{A}, \overline{V}))$ the corresponding eigenvalues. We transform the system to a system of the functions (r_+, r_-) with $r_+ = R_+ - \overline{R}_+$, $r_- = R_- - \overline{R}_-$. Let $s(r_+, r_-) = S(\overline{R}_+ + r_+, \overline{R}_- + r_-)$ denote the source term S corresponding to the state $(\overline{R}_+ + r_+, \overline{R}_- + r_-)$. Define the transformed source term

$$F(x, t, r_+, r_-) = -\frac{\partial}{\partial t}\begin{pmatrix}\overline{R}_+\\ \overline{R}_-\end{pmatrix} - \begin{pmatrix}\bar{\lambda}_+ + \lambda_+(r_+, r_-) & 0\\ 0 & \bar{\lambda}_- + \lambda_-(r_+, r_-)\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\overline{R}_+\\ \overline{R}_-\end{pmatrix} - \begin{pmatrix}s(r_+, r_-)\\ s(r_+, r_-)\end{pmatrix}.$$

Then $F(x,t,0,0)=0$. We insert $R_+ = \overline{R}_+ + r_+$, $R_- = \overline{R}_- + r_-$ in (8) and due to the definition of F for (r_+, r_-) we obtain the quasi-linear hyperbolic system

$$\frac{\partial}{\partial t}\begin{pmatrix} r_+ \\ r_- \end{pmatrix} + \begin{pmatrix} \bar{\lambda}_+ + \lambda_+(r_+, r_-) & 0 \\ 0 & \bar{\lambda}_- + \lambda_-(r_+, r_-) \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} r_+ \\ r_- \end{pmatrix} = F(x,t,r_+,r_-). \tag{21}$$

For the representation of the system matrix we have used the linearity of λ_+ and λ_- as functions of the Riemann invariants, see (9).

Theorem 3.1 from [17] yields the following statement: For all $T_0 > T(\bar{A}, \overline{V})$ there exists a number $\varepsilon > 0$ such that for all states in $(C^1[0,L])^2$ with C^1-norm less than ε, exact controllability between these states in time T_0 with a unique continuously differentiable solution of the system equation (21) is possible by boundary controls in the boundary conditions (12), (13).

Due to our transformation of variables, this ε-neighbourhood of $(0,0)$ corresponds to an ε-neighbourhood with respect to the C^1-norm of the constant stationary state given by $(\overline{R}_+, \overline{R}_-)$. Since this neighbourhood contains only subcritical states, the condition $0 < gA(x)/b - V(x)^2$ holds which completes the proof. $\square$

Remark 7. For the proof of Theorem 6 it is important that Theorem 3.1 in [17] allows that the eigenvalues and the source term depend explicitly on x. The results in [15] and also in [1] assume that the eigenvalues and the source term do not depend explicitly on x. A result analogous to Theorem 6 for the case of one-sided control follows from Theorem 3.2 in [17].

Proof of Theorem 4. First we observe that for all (A, V) feasible in (15), we have the inequality

$$\begin{aligned} \min\big\{|\lambda_+|, |\lambda_-|\big\} &\geqslant \sqrt{gA/b} - |V| = \frac{(gA/b) - V^2}{\sqrt{gA/b} + |V|} \\ &\geqslant \frac{\delta}{\sqrt{gM/b} + M} \end{aligned}$$

hence

$$\max\left\{\frac{1}{|\lambda_+(A,V)|}, \frac{1}{|\lambda_-(A,V)|}\right\} \leqslant \frac{\sqrt{gM/b} + M}{\delta} < \infty,$$

thus the time $T(\delta, M)$ is finite.

The condition $T_0 > T(\delta, M)$ is necessary for the exact boundary controllability of our hyperbolic system on account of the finite speed of wave propagation.

Let $T_1 > T(\delta, M)$ be given.

Define the set

$$K = \big\{(A,V) \in R^2\colon\ \delta \leqslant gA/b - V^2,\ |V| \leqslant M,\ A \leqslant M,\ S(A,V) = 0\big\}. \tag{22}$$

Then the set K is compact. For each $(A, V) \in K$, we have $T_1 > T(A, V)$ with $T(A, V)$ as defined in (19). Hence for each $(A, V) \in K$, we can apply Theorem 6 that implies the existence of a number $\varepsilon(A, V) > 0$ such that between all initial and final states in $\Gamma(\varepsilon(A, V))$ exact controllability in the time T_1 is possible.

Define $\epsilon(A, V) = \min\{1/(3L), 1/3\}\varepsilon(A, V)$. For $(A, V) \in K$ define the open neighbourhood

$$\mathcal{U}(A,V) = \left\{(a,v) \in R^2\colon\ v^2 < g\frac{a}{b},\ |v - V| < \epsilon(A,V),\ |a - A| < \epsilon(A,V)\right\}.$$

Then due to the definition of $\Gamma(\varepsilon(A, V))$, between all constant states in $\mathcal{U}(A, V)$ exact controllability in the time T_1 is possible. We have the inclusion

$$K \subset \bigcup_{(A,V)\in K} \mathcal{U}(A,V)$$

so the sets $\mathcal{U}(A, V)$ form an open cover of K. Since the set K is compact, there exists a finite subcover $\{\mathcal{U}(A_i, V_i)\colon i \in \{1, \ldots, n\}\}$ such that for the open set

$$\hat{K} = \bigcup_{i=1}^{n} \mathcal{U}(A_i, V_i) \tag{23}$$

we have $K \subset \hat{K}$.

For $i \in \{1, \ldots, n\}$ let $\epsilon_i = \epsilon(A_i, V_i)$ and $\epsilon_0 = \min_{i\in\{1,\ldots,n\}} \epsilon_i$.

The equation $S(A, V) = 0$ in the definition of the set K implies that for all $(A, V) \in K$, we have $V = V(A)$ as defined in (10). In particular this yields for all $i \in \{1, \ldots, n\}$ the equation $V_i = V(A_i)$ and we have

$$K = \bigl\{\bigl(A, V(A)\bigr) \in R^2\colon\ \delta \leqslant gA/b - V(A)^2,\ \bigl|V(A)\bigr| \leqslant M,\ A \leqslant M\bigr\}. \tag{24}$$

Theorem 6 and the definition of $\Gamma(\varepsilon(A, V))$ imply that for all $\epsilon \in (0, \epsilon_0)$ we have exact controllability in the finite time $T_0 = nT_1$ between all initial and final states in the set

$$\begin{aligned} M_0 = \bigcup_{i=1}^{n} \bigl\{&(A, V) \in C^1[0, L]\colon \text{ For all } x \in [0, L]\colon\ 0 < gA(x)/b - V(x)^2,\ \bigl|V(x) - V_i\bigr| \leqslant \varepsilon(A_i, V_i),\\ &\bigl|A(x) - A_i\bigr| \leqslant \varepsilon(A_i, V_i),\ \bigl|A'(x)\bigr| \leqslant \epsilon,\ \bigl|V'(x)\bigr| \leqslant \epsilon\bigr\}. \end{aligned} \tag{25}$$

This follows since starting from any initial state in M_0, we can go in time T_1 to a constant state $(A, V) \in K$ that is contained in the intersection of the current neighbourhood $\mathcal{U}(A_i, V_i)$ and another neighbourhood in the finite subcover. By going at most $n - 2$ times from such a constant state to another constant state in K, in this way after at most $n - 1$ steps we can reach a constant state in K in the neighbourhood that contains the desired final state. In the last step we go from this constant state to the desired final state.

If the elements of $\hat{K}$ are interpreted as constant states, we have $\hat{K} \subset M_0$.

For all $(A, V) \in K$, we have $A \in [b\delta/g, M]$.

If $\gamma = 0$, we have $V(A) = 0$ and

$$S(A, V) = \frac{gC^2}{(A/P(A))^{4/3}} V|V|,$$

hence $|S(A, V)| \leqslant \varepsilon$ implies $|V| \leqslant \sqrt{\varepsilon}\frac{(A/P(A))^{2/3}}{\sqrt{g}C}$, thus $|V - V(A)| = \mathrm{O}(\sqrt{\varepsilon})$.

If $\gamma \neq 0$, we have $V(A) \neq 0$ and $\operatorname{sign}(V(A)) = -\operatorname{sign}(\gamma)$. In particular, all V_i have the same sign and

$$\kappa = \min_{A\in[b\delta/g,M]} \bigl|V(A)\bigr| > 0.$$

We assume that in this case $\varepsilon(A, V)$ is chosen less than or equal to $\kappa/2$. Then for all $(A, V) \in K$, the sign of V is the same and $|V| \geqslant \kappa/2$.

In this case by the mean value theorem we have

$$\bigl|S(A, V)\bigr| = \bigl|S(A, V) - S\bigl(A, V(A)\bigr)\bigr| = 2|W|\frac{gC^2}{(A/P(A))^{4/3}}\bigl|V - V(A)\bigr|$$

for some W between V and $V(A)$. Thus if $\gamma \neq 0$ the inequality $|S(A, V)| \leqslant \varepsilon$ implies the inequality

$$\bigl|V - V(A)\bigr| \leqslant \frac{\varepsilon}{\kappa}\frac{(A/P(A))^{4/3}}{gC^2} = \mathrm{O}(\varepsilon).$$

Let $(A, V) \in G(\delta, M, \varepsilon)$. Define the numbers $A_0 = A(0)$ and $V_0 = V(A_0)$. On account of $|S(A_0, V(0))| \leqslant \varepsilon$ we have $|V_0 - V(0)| = \mathrm{O}(\sqrt{\varepsilon})$ and we can choose $\varepsilon > 0$ so small that $|V_0| \leqslant \sup_{(a,v)\in\hat{K}} |v| > M$ and

$$(A_0, V_0) \in \hat{K} \tag{26}$$

with $\hat{K}$ as in (23). The definition of $\hat{K}$ implies that there exists an index $i \in \{1, \ldots, n\}$ with $|A_0 - A_i| \leqslant \epsilon_i$ and $|V_0 - V_i| \leqslant \epsilon_i$.

Hence if $\varepsilon \in (0, \epsilon_0)$ is chosen sufficiently small, using the definition of the set $G(\delta, M, \varepsilon)$ we obtain the inequalities

$$\bigl|A(x) - A_i\bigr| \leqslant \bigl|A(x) - A(0)\bigr| + \bigl|A(0) - A_i\bigr| \leqslant L\epsilon_i + \epsilon_i \leqslant \varepsilon(A_i, V_i)$$

and

$$\begin{aligned} \bigl|V(x) - V_i\bigr| &\leqslant \bigl|V(x) - V(0)\bigr| + \bigl|V(0) - V_0\bigr| + |V_0 - V_i|\\ &\leqslant L\epsilon_i + \mathrm{O}(\sqrt{\varepsilon}) + \epsilon_i \leqslant \varepsilon(A_i, V_i). \end{aligned}$$

With the other constraints in the definition of the set $G(\delta, M, \varepsilon)$ this implies $(A, V) \in M_0$, hence $G(\delta, M, \varepsilon) \subset M_0$, and the assertion follows. $\square$

6. Controllability results for states with small derivatives

Now we start from an arbitrary constant subcritical state and show that it is possible to reach in finite time the set $G(\delta, M, \varepsilon)$ from Theorem 4 with a continuously differentiable state. This implies that exact controllability in finite time from any constant subcritical state to a state in a neighbourhood of a constant stationary state is possible.

For this purpose we consider a canal of infinite length with a constant subcritical initial state. The corresponding Cauchy problem has a continuously differentiable solution for all $t \geqslant 0$, that can be computed explicitly (see the proof of Lemma 8). For this solution, with increasing t, the source term converges to zero. (For canals with non-zero slope this decay is exponential; for horizontal canals this is not the case.) This implies that in finite time, the source term becomes arbitrarily small. The boundary trace of the state yields control functions that drive the system to a state with arbitrarily small source terms.

Lemma 8. *Let $(\bar{A}, \overline{V}) \in R^2$ be given. Assume that $\gamma = 0$ or $\operatorname{sign}(\overline{V}) = -\operatorname{sign}(\gamma)$.*

Then there exists a unique continuously differentiable solution of the initial value problem

$$A(x,0) = \bar{A}, \quad V(x,0) = \overline{V}, \quad x \in R \tag{27}$$

(1), (2) ($(x,t) \in R \times [0,\infty)$). For all $(x,t) \in R \times [0,\infty)$ we have $A(x,t) = \bar{A}$ and $V_x(x,t) = 0$, that is V is independent of x and in finite time the absolute value of the source term $s(t) = S(\bar{A}, V(t))$ becomes arbitrarily small.

If $\gamma \neq 0$, the function $|s(t)|$ decays exponentially fast.

Proof. Let $V(t)$ be the solution of the initial value problem

$$V(0) = \overline{V}, \qquad \partial_t V(t) = -S\big(\bar{A}, V(t)\big) = -g\left(\gamma + \frac{C^2|V(t)|V(t)}{(\bar{A}/P(\bar{A}))^{4/3}}\right). \tag{28}$$

Then for $(\bar{A}, V)$ Eq. (1) holds and Eq. (2) is also valid. Since the initial condition (27) is also valid, this yields the desired solution of the initial value problem in Lemma 8.

Define the constant

$$\rho = \frac{C}{(\bar{A}/P(\bar{A}))^{2/3}} > 0.$$

Then (10) implies the equation $V_1 := V(\bar{A}) = -\operatorname{sign}(\gamma)\frac{\sqrt{|\gamma|}}{\rho}$ and we can write the ordinary differential equation in (28) in the form

$$V'(t) = g\rho^2\big[|V_1|V_1 - \big|V(t)\big|V(t)\big].$$

Case 1. If $\overline{V} = V_1$, we have $S(\bar{A}, \overline{V}) = 0$ and $V(t) = \overline{V}$ for all $t > 0$.

Case 2. Now we consider the case that $V_1 \neq 0$ which occurs if $\gamma \neq 0$.

Case 2a. If $\overline{V}/V_1 \in (1,\infty)$, let $c_1 > 0$ be such that $\coth(c_1) = \overline{V}/V_1$. Then for all $t \geqslant 0$ we have

$$V(t) = V_1 \coth\big(c_1 + g\rho^2|V_1|t\big).$$

Case 2b. If $\overline{V}/V_1 \in [0,1)$, let $c_1 \geqslant 0$ be such that $\tanh(c_1) = \overline{V}/V_1$. Then for all $t \geqslant 0$ we have

$$V(t) = V_1 \tanh\big(c_1 + g\rho^2|V_1|t\big).$$

Case 3. Now we consider the case that $V_1 = 0$ that is we have a horizontal canal where $\gamma = 0$. Then $V_1 = 0$ and $V'(t) = -g\rho^2|V(t)|V(t)$.

Case 3a. If $\overline{V} > 0$, we have

$$V(t) = \frac{1}{(1/\overline{V}) + g\rho^2 t}.$$

Case 3b. If $\overline{V} < 0$, we have

$$V(t) = \frac{1}{(1/\overline{V}) - g\rho^2 t}. \qquad \square$$

Theorem 9 *(Exact controllability for constant initial states). For a canal with a constant rectangular cross section, consider the Saint-Venant system* (1), (2) *with the source term given by* (3), (4) *and boundary conditions in characteristic form* (12), (13).

Let $(\bar{A}, \overline{V}) \in R^2$ be given such that $0 < g\bar{A}/b - \overline{V}^2$. Let $V_1 = V(\bar{A})$. Assume that $0 < g\bar{A}/b - V_1^2$.

Assume that $\gamma = 0$ or $\operatorname{sign}(\overline{V}) = -\operatorname{sign}(\gamma)$.

If $\gamma \neq 0$, for $\varepsilon > 0$, define the finite time

$$T(\bar{A}, \overline{V}, \varepsilon) = \frac{1}{g\rho^2 |V_1|} \sinh^{-1}\left(\frac{|V_1|\sqrt{g}\rho}{\sqrt{\varepsilon}}\right). \tag{29}$$

If $\gamma = 0$ and $\overline{V} \neq 0$, for $\varepsilon > 0$, define the finite time

$$T(\bar{A}, \overline{V}, \varepsilon) = \max\left\{0, \frac{1}{\sqrt{g}\rho\sqrt{\varepsilon}} - \frac{1}{g\rho^2|\overline{V}|}\right\}. \tag{30}$$

Then for all $T_0 > T(\bar{A}, \overline{V}, \varepsilon)$ there exist boundary controls g_0, g_L such that there exists a unique continuously differentiable solution $(\bar{A}, V(t))$ of the initial boundary value problem (27), (1), (2), (12), (13) *$((x,t) \in [0, L] \times [0, T_0])$ that satisfies the final condition $(\bar{A}, V(T_0)) \in \{(A, V) \in C^1[0, L]$: For all $x \in [0, L]$: $0 < gA(x)/b - V(x)^2$, $|S(A(x), V(x))| \leqslant \varepsilon$, $A'(x) = 0$, $V'(x) = 0\}$.*

Hence for $\delta = \min\{g\bar{A}/b - \overline{V}^2, g\bar{A}/b - V_1^2\}$ and $M = \max\{\bar{A}, |\overline{V}|, |V_1|\}$ we have $(\bar{A}, V(T_0)) \in G(\delta, M, \varepsilon)$ with the set of functions $G(\delta, M, \varepsilon)$ defined in Theorem 4.

Proof of Theorem 9. Lemma 8 implies that for the given constant initial state $(\bar{A}, \overline{V})$ the solution of the initial value problem (27), (1), (2) exists on $R \times [0, \infty)$. The proof of Lemma 8 shows that the values of $V(x,t)$ are between $\overline{V}$ and V_1, hence since $0 < g\bar{A}/b - \overline{V}^2$ and $0 < g\bar{A}/b - V_1^2$ the state remains subcritical for all $t \geqslant 0$.

If $\gamma \neq 0$ for all $T_0 > T(\bar{A}, \overline{V}, \varepsilon)$ and all $c_1 > 0$ we have

$$\frac{1}{\cosh^2(c_1 + g\rho^2|V_1|T_0)} \leqslant \frac{1}{\sinh^2(c_1 + g\rho^2|V_1|T_0)} \leqslant \frac{\varepsilon}{V_1^2 g\rho^2}$$

and with Cases 2a and 2b from the proof of Lemma 8, this implies $|V'(T_0)| < \varepsilon$.

If $\gamma = 0$ for all $T_0 > T(\bar{A}, \overline{V}, \varepsilon)$ we have

$$\frac{1}{(1/|\overline{V}|) + g\rho^2 T_0} < \frac{\sqrt{\varepsilon}}{\sqrt{g}\rho},$$

hence since this is Case 3 in the proof of Lemma 8,

$$\left|V'(T_0)\right| = \frac{g\rho^2}{((1/|\overline{V}|) + g\rho^2 T_0)^2} < \varepsilon.$$

On account of (28), we have $|V'(t)| = |S(\bar{A}, V(t))|$, hence $|S(\bar{A}, V(T_0))| < \varepsilon$.

Since $A(x,t) = \bar{A}$, we have $\partial_x A(x,t) = 0$ and since $V(x,t) = V(t)$, we have $\partial_x V(x,t) = 0$ With the control functions

$$g_0(t) = V(t) + \varphi(\bar{A}), \qquad g_L(t) = V(t) - \varphi(\bar{A})$$

the solution of the initial boundary value problem (27), (1), (2), (12), (13) satisfies the terminal conditions given in Theorem 9. $\square$

In Theorem 9 we have seen that it is possible to drive the system from a constant subcritical initial state to a set $G(\delta, M, \varepsilon)$ where exact controllability holds. In Theorem 10 we show that this can also be done if the derivatives of the initial state are sufficiently small.

Theorem 10 *(Exact controllability for initial states with small derivatives). For a canal with a constant rectangular cross section, consider the Saint-Venant system* (1), (2) *with the source term given by* (3), (4) *and boundary conditions in characteristic form* (12), (13).

Let $(\bar{A}, \overline{V}) \in R^2$ *be given such that* $0 < g\bar{A}/b - \overline{V}^2$. *Let* $V_1 = V(\bar{A})$. *Assume that* $0 < g\bar{A}/b - V_1^2$.

Assume that $\gamma = 0$ *or* $\operatorname{sign}(\overline{V}) = -\operatorname{sign}(\gamma)$.

For $\varepsilon > 0$, *define the finite time* $T(\bar{A}, \overline{V}, \varepsilon)$ *as in Theorem* 9. *Let the numbers* $\delta > 0$ *and* $M > 0$ *be defined as in Theorem* 9.

Then for all $T_0 > T(\bar{A}, \overline{V}, \varepsilon)$ *there exists a number* $\varepsilon_8 > 0$ *such that for all initial states* $(A_8(x), V_8(x))$ *with*

$$\max_{x\in[0,L]}\left\{\left|A_8(x) - \bar{A}\right|, \left|V_8(x) - \overline{V}\right|, \left|A_8'(x)\right|, \left|V_8'(x)\right|\right\} \leqslant \varepsilon_8 \tag{31}$$

there exist boundary controls g_0, g_L *that generate a unique continuously differentiable solution* $(A(x,t), V(x,t))$ *of the initial boundary value problem*

$$A(x,0) = A_8(x), \quad V(x,0) = V_8(x), \quad x \in [0, L] \tag{32}$$

(1), (2), (12), (13) $((x,t) \in [0,L] \times [0,T_0])$ *that satisfies the final condition* $(A(\cdot, T_0), V(\cdot, T_0)) \in G(\delta/2, M+1, 2\varepsilon)$ *with the set of functions G as defined in Theorem* 4.

Proof of Theorem 10. For continuously differentiable states, we can write (1), (2) in the diagonal form (8). Let $(\overline{R}_+(t), \overline{R}_-(t))$ denote the Riemann invariants corresponding to the subcritical state $(\bar{A}, V(t))$ described in Theorem 9 and $(\bar{\lambda}_+(t), \bar{\lambda}_-(t)) = (\lambda_+(\bar{A}, V(t)), \lambda_-(\bar{A}, V(t)))$ the corresponding eigenvalues. We transform the system (8) to a system of the functions (r_+, r_-) with $r_+ = R_+ - \overline{R}_+$, $r_- = R_- - \overline{R}_-$. Let $s(t, r_+, r_-) = S(\overline{R}_+(t) + r_+, \overline{R}_-(t) + r_-)$ denote the source term S corresponding to the state $(\overline{R}_+ + r_+, \overline{R}_- + r_-)$. Let

$$\overline{S}(t) = S\big(\overline{R}_+(t), \overline{R}_-(t)\big).$$

We have

$$\frac{\partial}{\partial t}\begin{pmatrix}\overline{R}_+(t)\\ \overline{R}_-(t)\end{pmatrix} = \begin{pmatrix}-\overline{S}(t)\\ -\overline{S}(t)\end{pmatrix}.$$

Define the transformed source term

$$F(t, r_+, r_-) = \begin{pmatrix}\overline{S}(t) - s(t, r_+, r_-)\\ \overline{S}(t) - s(t, r_+, r_-)\end{pmatrix}.$$

We insert $R_+ = \overline{R}_+ + r_+$, $R_- = \overline{R}_- + r_-$ in (8) and due to the definition of F for (r_+, r_-) we obtain the quasi-linear hyperbolic system

$$\frac{\partial}{\partial t}\begin{pmatrix}r_+\\ r_-\end{pmatrix} + \begin{pmatrix}\bar{\lambda}_+(t) + \lambda_+(r_+, r_-) & 0\\ 0 & \bar{\lambda}_-(t) + \lambda_-(r_+, r_-)\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}r_+\\ r_-\end{pmatrix} = F(t, r_+, r_-). \tag{33}$$

Since the source term does not depend explicitly on x, for the space derivative of the source term, we have

$$\partial_x F(t, r_+, r_-) = 0.$$

Hence we can apply Theorem 3.II from [2] which yields the following statement: There exists numbers $\varepsilon_7 > 0$ and $N_7 > 0$ such that if

$$\max\left\{\left|\partial_x r_+(x,0)\right|, \left|\partial_x r_-(x,0)\right|:\ x \in (-\infty, \infty)\right\} \leqslant \varepsilon_7,$$

the solution of the Cauchy problem with given $r_+(x,0)$, $r_-(x,0)$ for $x \in (-\infty, \infty)$, (33) exists in $(-\infty, \infty) \times [0, T_0]$ and satisfies the inequality

$$\max\left\{\left|\partial_x r_+(x,T_0)\right|, \left|\partial_x r_-(x,T_0)\right|:\ x \in (-\infty, \infty)\right\} \leqslant N_7\varepsilon_7.$$

Moreover, Theorem 3.IV implies that ε_7 can be chosen such that

$$\max\left\{\left|r_+(x,T_0)\right|, \left|r_-(x,T_0)\right|:\ x \in [0,L]\right\} \leqslant 2\max\left\{\left|r_+(x,0)\right|, \left|r_-(x,0)\right|:\ x \in (-\infty, \infty)\right\}.$$

So we see that if the C^1-norm of $(r_+(x,0), r_-(x,0))$ is sufficiently small, the solution of the Cauchy problem exists on $R \times [0, T_0]$ and the C^1-norm of $(r_+(x,T_0), r_-(x,T_0))$ on $[0,L]$ can be made arbitrarily small so $(r_+(\cdot,T_0)|_{[0,L]}, r_-(\cdot,T_0)|_{[0,L]})$ is in a C^1-neighbourhood of zero.

Due to our transformation of variables, this C^1-neighbourhood of $(0,0)$ corresponds to a C^1-neighbourhood of the constant state given by $(\overline{R}_+(T_0), \overline{R}_-(T_0))$, the Riemann invariants corresponding to the constant state $(\bar{A}, V(T_0)) \in G(\delta, M, \varepsilon)$. Transforming everything back to (A, V) variables we see that if the C^1-neighbourhood of $(\bar{A}, V(T_0))$ containing $(A(\cdot,T_0), V(\cdot,T_0))$ is small enough, we have $(A(\cdot,T_0), V(\cdot,T_0)) \in G(\delta/2, M+1, 2\varepsilon)$ which yields the assertion. We obtain the controls g_0 and g_L as the boundary traces

$$g_0(t) = R_+(0,t) = \overline{R}_+(t) + r_+(0,t),$$
$$g_L(t) = R_-(L,t) = \overline{R}_-(t) + r_-(L,t). \qquad \square$$

7. Example: Exact control to a state with small derivative

In this section we give an example where we have an initial state that can be chosen such that the maximum norm of its derivative is arbitrarily large. We give a number $\delta > 0$ such that for all $\varepsilon > 0$ we find a number M and boundary controls that steer the system in the finite time to a state in the set $G(\delta, M, \varepsilon)$ as defined in Theorem 4 where exact controllability holds. This example only works because for the source term $S(A,V)$ we have $S_A(A,V) > 0$ if $V > 0$.

Lemma 11. *Let real numbers $\bar{A}_0 \in (0,\infty)$ and $\overline{V}$ be given. Assume that $\overline{V} > 0$, $\gamma < 0$, $S_0 = S(\bar{A}_0, \overline{V}) > 0$ and $\delta = g\bar{a}_0/b - \overline{V}^2 > 0$. Let $a_0(z)$ be the solution of the initial value problem with the initial condition $a_0(0) = \bar{A}_0$ and the differential equation*

$$a_0'(z) = (b/g) S\big(a_0(z), \overline{V}\big).$$

Then a_0 is defined for all $z \geqslant 0$ and $\lim_{z\to\infty} a_0'(z) = 0$ and $\lim_{z\to\infty} S(a_0(z), \overline{V}) = 0$. Let $a_1 > 0$ be such that $S(a_1, \overline{V}) = 0$ and

$$\mu_0 = \frac{4}{3}\frac{b^2 C^2 \overline{V}^2}{a_1^2}\left(\frac{2}{b}\right)^{1/3}.$$

Then we have

$$0 \leqslant S\big(a_0(z), \overline{V}\big) \leqslant S_0 \exp(-\mu_0 z). \tag{34}$$

Let $g_0(t) = \overline{V} + \varphi(a_0(L + \overline{V}t))$, $g_L(t) = \overline{V} - \varphi(a_0(\overline{V}t))$. A unique continuously differentiable solution of the initial boundary value problem

$$A(x,0) = a_0(L-x), \qquad V(x,0) = \overline{V} \tag{35}$$

(1), (2), (12), (13) exists for $(x,t) \in [0,L] \times [0,\infty)$ and is given by

$$A(x,t) = a_0(L - x + \overline{V}t), \quad V(x,t) = \overline{V}.$$

Let $\varepsilon > 0$ be given. Then for all

$$T_0 \geqslant \max\left\{-\frac{1}{\overline{V}\mu_0}\ln\left(\frac{\varepsilon}{S_0}\right), -\frac{1}{\overline{V}\mu_0}\ln\left(\frac{g\,\varepsilon}{bS_0}\right)\right\}$$

we have $(A(\cdot,T_0), V(\cdot,T_0)) \in G(\delta, M, \varepsilon)$ where $M = \max\{\overline{V}, a_0(\overline{V}T_0 + L)\}$.

Proof. Let I denote the maximal interval where the solution exists. We have $S(a_0(0), \overline{V}) > 0$. Suppose that for some $z_0 \in I$, we have $S(a_0(z_0), \overline{V}) = 0$. Then from this point, the solution remains constant, $a_0'(z) = 0$ for $z \geqslant z_0$. This implies that for all $z \in I$, we have $S(a_0(z), \overline{V}) \geqslant 0$. Thus a_0 is increasing on I, hence $s(z) = S(a_0(z), \overline{V})$ is decreasing on I. Moreover, for all $z \in I$ we have $a_0(z) \geqslant \bar{A}_0$. Since $\gamma < 0$, the inequality $s(z) \geqslant 0$ implies that there exists $a_1 > 0$ with $S(a_1, \overline{V}) = 0$ and for all $z \geqslant 0$ we have $a_0(z) \leqslant a_1$. We have

$$S(a_0, \overline{V}) = g\left[\gamma + C^2\overline{V}^2\left(\frac{2}{b} + \frac{b}{a_0}\right)^{4/3}\right]$$

hence

$$S_A\big(a_0(z),\overline{V}\big) = -gC^2\overline{V}^2\frac{4}{3}\left(\frac{2}{b}+\frac{b}{a_0(z)}\right)^{1/3}\frac{b}{a_0(z)^2}$$
$$\leqslant -\frac{4}{3}gC^2\overline{V}^2\left(\frac{2}{b}\right)^{1/3}\frac{b}{a_1^2}.$$

Therefore for all $z\in I$ we have

$$s'(z) = S_A\big(a_0(z),\overline{V}\big)a_0'(z) = \frac{b}{g}S_A\big(a_0(z),\overline{V}\big)s(z)$$
$$\leqslant -\frac{4}{3}b^2C^2\overline{V}^2\left(\frac{2}{b}\right)^{1/3}\frac{s(z)}{a_1^2} = -\mu_0 s(z).$$

This inequality implies that for all $z\in I$, we have $0\leqslant s(z)\leqslant S_0\exp(-\mu_0 z)$. This in turn implies that $0\leqslant a_0'(z)\leqslant S_0\exp(-\mu_0 z)$. This implies that the solution exists for all $z\geqslant 0$, that is $I=[0,\infty)$ and that both $a_0'(z)$ and $s(z)$ converge to zero exponentially fast.

Now we consider the functions $A(x,t)=a_0(L-x+\overline{V}t)$, $V(x,t)=\overline{V}$. Then the initial conditions (35) and the boundary conditions (12), (13) with g_0, g_L as defined in Lemma 11 hold. We have

$$\frac{\partial}{\partial x}(\overline{V}A) = -\overline{V}a_0'(L-x+\overline{V}t), \qquad \frac{\partial}{\partial t}A = \overline{V}a_0'(L-x+\overline{V}t),$$

hence (1) holds. Now we check whether (2) is valid. We have

$$\frac{\partial}{\partial t}V + \frac{\partial}{\partial x}\left(gH+\frac{V^2}{2}\right) = g\frac{\partial}{\partial x}\frac{a_0(L-x+\overline{V}t)}{b} = -S\big(a_0(L-x+\overline{V}t),\overline{V}\big)$$

hence (2) holds. If T_0 is as stated in Lemma 11 we have

$$S_0\exp(-\mu_0\overline{V}t)\leqslant\varepsilon, \qquad (b/g)S_0\exp(-\mu_0\overline{V}t)\leqslant\varepsilon$$

and with (34) this implies $|S(A(x,T_0),\overline{V})|\leqslant\varepsilon$ and $|A_x(x,T_0)| = |a_0'(L-x+\overline{V}T_0)|\leqslant\varepsilon$.

Since a_0 is increasing, for all $x\in[0,L]$ we have $|A(x,T_0)|\leqslant|a_0(\overline{V}T_0+L)|\leqslant M$ and for all $z\geqslant 0$ we have $g\bar{A}_0(z)/b-\overline{V}^2\geqslant\delta$.

Hence $(A(\cdot,T_0),V(\cdot,T_0))\in G(\delta,M,\varepsilon)$. $\square$

8. Conclusion

In sloped canals, the influence of the boundary friction is essential for the dynamics of the water flow. In nature, often states that are close to constant equilibrium states are observed. In this paper, we have shown that between subcritical states of this type, exact controllability in finite time with a continuously differentiable solution of the system is possible. This result complements previous results for frictionless horizontal canals. In the analysis of the case with non-zero source term, the curve of constant stationary states where the source term vanishes plays a central role: In a C^1 neighbourhood of this curve, exact controllability is possible. This neighbourhood contains also stationary states that are not constant. Let us call this neighbourhood the set of exact controllability.

Subcritical states that are not in this set but have sufficiently small derivatives can be steered to this set of exact controllability in finite time. At the end of the paper, we give an example of states with arbitrarily large derivatives that can be steered to this set in finite time with a state of travelling wave type.

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