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Hydrodynamic limits: some improvements of the relative entropy method

Laure Saint-Raymond

Université Paris 6 & Ecole Normale Supérieure, Département de Mathématiques et Applications, 45, rue d'Ulm, 75005 Paris, France

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Abstract

The present paper is devoted to the study of the incompressible Euler limit of the Boltzmann equation via the relative entropy method. It extends the convergence result for well-prepared initial data obtained by the author in [L. Saint-Raymond, Convergence of solutions to the Boltzmann equation in the incompressible Euler limit, Arch. Ration. Mech. Anal. 166 (2003) 47–80]. It explains especially how to take into account the acoustic waves and relaxation layer, and thus to obtain convergence results under weak assumptions on the initial data.

The study presented here requires in return some nonuniform control on the large tails of the distribution, which is satisfied for instance by the classical solutions close to a Maxwellian built by Guo [Y. Guo, The Vlasov–Poisson–Boltzmann system near Maxwellians, Comm. Pure Appl. Math. 55 (2002) 1104–1135].
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Résumé

Cet article est consacré à l'étude – par la méthode d'entropie relative – de l'asymptotique de l'équation de Boltzmann conduisant aux équations d'Euler incompressibles. Il étend le résultat établi par l'auteur dans [L. Saint-Raymond, Convergence of solutions to the Boltzmann equation in the incompressible Euler limit, Arch. Ration. Mech. Anal. 166 (2003) 47–80] pour des données bien préparées. Le problème est de prendre en compte les ondes acoustiques et la couche initiale de relaxation, pour obtenir un résultat de convergence sous des hypothèses peu contraignantes sur la donnée initiale.

L'étude présentée ici requiert en contrepartie un contrôle (non uniforme) sur la distribution des grandes vitesses, qui est satisfait par exemple par les solutions classiques construites par Guo dans [Y. Guo, The Vlasov–Poisson–Boltzmann system near Maxwellians, Comm. Pure Appl. Math. 55 (2002) 1104–1135].
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The subject matter of this article is to develop new tools for the study of hydrodynamic limits, and more precisely to understand how the relative entropy method (to be described in the next section) can be adapted in domains where the distribution is expected to present rapid variations. The main idea is that, in such domains, the formal hydrodynamic approximation is not relevant, and that correctors have to be added in order to obtain the convenient asymptotics.

E-mail address: Laure.Saint-Raymond@ens.fr.

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The main point is indeed to obtain a refined description of the asymptotics taking into account both the relaxation in the initial layer and the acoustic waves.

The subsequent modifications of the stability inequality are then very similar to those arising when taking into account fast oscillations of small amplitudes in energy methods (see for instance [17] or [30]).

1. The relative entropy method

1.1. Strategy of the relative entropy method

The main idea behind energy and entropy methods is to compare the distribution under consideration and its formal asymptotics in some appropriate metrics, and then to prove that this quantity tends to zero at the limit.

For the study of hydrodynamic limits of the Boltzmann equation, this program is applied as follows

• *Step 1*: The *functional which measures the stability* for the solutions f_ϵ to the scaled Boltzmann equation is obtained naturally from the relative entropy

$$H(f_\epsilon|M) = \iint \left(f_\epsilon \log \frac{f_\epsilon}{M} - f_\epsilon + M \right) dx\, dv$$

where M is some reference global equilibrium, for instance the reduced centered Gaussian:

$$M(v) = \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{|v|^2}{2}\right).$$

This functional is indeed a nonnegative Lyapunov functional for the Boltzmann equation. It further controls the size of the fluctuation in incompressible regimes

$$H(f_\epsilon|M) \geqslant 2 \int (\sqrt{f_\epsilon} - \sqrt{M})^2\, dv\, dx$$

(which is not the case of the L^1-norm).

Note that the idea of using the notion of relative entropy for this kind of problems goes back to C. Bardos, F. Golse and C.D. Levermore who introduce the notion of entropic convergence in [3], and independently to Yau for his elegant derivation of the hydrodynamic limit of the Ginzburg–Landau lattice model [31].

• *Step 2*: The *approximate solution* is then determined by formal arguments, coupling expansions such as the Hilbert or Chapman–Enskog expansions [21,8], filtering methods [30,17] and study of the boundary layers [25,10]. It depends of course of the scaling of the Boltzmann equation.

In the regime we consider here, i.e. in the regime leading to the incompressible Euler equations, and assuming specular reflection at the boundary, we expect the approximate solutions to be decomposed as the sum of

- a purely kinetic part (determined by the relaxation process in the initial layer);
- a fast oscillating hydrodynamic part (governed by the acoustic equations);
- a nonoscillating hydrodynamic part (obtained by formal expansion of Hilbert or Chapman–Enskog's type) satisfying the incompressible Euler equations, supplemented by some suitable equation for the temperature.

For a formal derivation of this asymptotics including a brief justification of the suitable scaling, we refer to Appendix C and the references therein.

• *Step 3*: The convergence is then obtained in the form of some stability inequality on the modulated entropy defined by

$$H(f_\epsilon|f_{app}) = \iint \left(f_\epsilon \log \frac{f_\epsilon}{f_{app}} - f_\epsilon + f_{app} \right) dx\, dv. \tag{1.1}$$

This is of course the main difficult step, which requires many technical computations and estimates, and where the theory of the Boltzmann equation under consideration plays a crucial role.

1.2. Convergence results for the renormalized solutions to the scaled Boltzmann equation

Given the state of the art about renormalized solutions to the Boltzmann equation, we are actually able to establish only a very partial convergence result.

Before stating more precisely that convergence result, let us introduce the usual notations and assumptions regarding the Boltzmann equation:

$$\partial_t f + v \cdot \nabla_x f = Q(f, f),$$
$$f_{|t=0} = f^{in} \quad \text{with } H(f^{in}|M) < +\infty. \tag{1.2}$$

The collision integral is given by

$$Q(f, f)(v) = \iint\limits_{\mathbf{R}^3 \times \mathbf{S}^2} \big(f(v')f(v_1') - f(v)f(v_1)\big)b(v - v_1, \omega)\, dv_1\, d\omega \tag{1.3}$$

with

$$v' \equiv v'(v, v_1, \omega) = v - (v - v_1) \cdot \omega\omega,$$
$$v_1' \equiv v_1'(v, v_1, \omega) = v_1 + (v - v_1) \cdot \omega\omega, \tag{1.4}$$

and where the function $b \equiv b(v - v_1, \omega)$, called the collision kernel, is measurable, a.e. positive and satisfies Grad's cutoff assumption, i.e. the bounds

$$0 < b(z, \omega) \leqslant C_b\big(1 + |z|\big)^\beta \quad \text{a.e. on } \mathbf{R}^3 \times \mathbf{S}^2,$$
$$\iint\limits_{\mathbf{S}^2} b(z, \omega)\, d\omega \geqslant \frac{1}{C_b} \frac{|v|}{1 + |v|} \quad \text{a.e. on } \mathbf{R}^3 \tag{1.5}$$

for some $C_b > 0$ and $\beta \in [0, 1]$. In most of the literature on the Boltzmann equation, the values of f at v_1, v' and v_1' are denoted respectively

$$f_1 := f(v_1), \quad f' := f(v'), \quad f_1' := f(v_1').$$

Throughout this paper, we shall henceforth follow this well-established usage.

The theory of renormalized solutions in this framework is due to Lions [22] following the fundamental paper by Di Perna and Lions [13]. In order to study hydrodynamic limits, we also need some later improvements of the theory due to Lions and Masmoudi [23] (introduction of a defect measure to recover the local conservation of momentum) and to Mischler [26] (extension to spatial domains with boundaries). A compendium of these results can be found for instance in [29]:

Theorem 1.1. *Let Ω be some smooth domain. Given any initial data f^{in} satisfying*

$$H\big(f^{in}\big|M\big) < +\infty,$$

there exists a renormalized solution $f \in C(\mathbf{R}^+, L^1_{\mathrm{loc}}(\Omega \times \mathbf{R}^3))$ relatively to M to the Boltzmann equation (1.2) *with initial data f^{in}, supplemented with the condition of specular reflection on $\Sigma_- = \{(x, v) \in \partial\Omega \times \mathbf{R}^3 / v \cdot n(x) < 0\}$:*

$$f(t, x, v) = f\big(t, x, v - 2\big(v \cdot n(x)\big)n(x)\big).$$

Moreover, f satisfies

– *the continuity equation*

$$\partial_t \int f\, dv + \nabla_x \cdot \int f v\, dv = 0; \tag{1.6}$$

– *the momentum equation with defect measure*

$$\partial_t \int f v\,dv + \nabla_x \cdot \int f v \otimes v\,dv + \nabla_x \cdot m = 0 \tag{1.7}$$

where m is a Radon measure on $\mathbf{R}^+ \times \Omega$ with values in the nonnegative symmetric matrices;

– *the entropy inequality*

$$H(f|M)(t) + \int \operatorname{trace} m(t) + \int_0^t \int_\Omega D(f)(s,x)\,ds\,dx \leqslant H\big(f^{in}\big|M\big) \tag{1.8}$$

where the entropy dissipation $D(f)$ is defined by

$$D(f) = \frac{1}{4} \iiint (f' f_1' - f f_1) \log \frac{f' f_1'}{f f_1} b\,dv\,dv_1\,d\omega.$$

In such a framework, the incompressible Euler asymptotics has been established for *well-prepared initial data*, that is in the case when the purely kinetic part, the fast oscillating hydrodynamic part and the nonoscillating part of both the density and temperature vanish asymptotically. Up to some standard change of variables, we can always assume that the macroscopic density and the temperature are identically equal to 1. With the notations of Section 1.1, we then choose f_{app} to be the Maxwellian distribution with unit mass and variance, centered at ϵu

$$f_{app}(t,x,v) = \mathcal{M}_{1,\epsilon u(t,x),1}(v) = \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{|v-\epsilon u|^2}{2}\right),$$

where ϵ is the order of magnitude of the Mach number and u is the limiting bulk velocity.

Furthermore we only consider asymptotics leading to *smooth solutions u* of the incompressible Euler equations without Prandtl boundary layers.

More precisely the convergence result established in [28] can be stated as follows

Theorem 1.2. *Let Ω be some smooth domain of $\mathbf{R}^3$ (possibly the whole space $\mathbf{R}^3$), or the three-dimensional torus $\mathbf{T}^3$. Let $f_\epsilon^{in} \in L^1_{\mathrm{loc}}(\Omega \times \mathbf{R}^3)$ be a family of initial fluctuations around some global equilibrium M (for instance the centered reduced Gaussian), i.e. satisfying*

$$\frac{1}{\epsilon^2} H\big(f_\epsilon^{in}|M\big) \leqslant C_{in}, \tag{1.9}$$

and such that

$$\frac{1}{\epsilon^2} H\big(f_\epsilon^{in}\big|\mathcal{M}_{1,\epsilon u^{in},1}\big) \to 0 \quad \text{as } \epsilon \to 0, \tag{1.10}$$

for some given divergence-free smooth vector field $u^{in} \in L^2(\Omega)$.

Let f_ϵ be a family of renormalized solutions to the scaled Boltzmann equation

$$\begin{aligned} &\epsilon \partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = \frac{1}{\epsilon^q} Q(f_\epsilon, f_\epsilon) \quad \text{on } \mathbf{R}^+ \times \Omega \times \mathbf{R}^3, \\ &f_\epsilon(0,x,v) = f_\epsilon^{in}(x,v) \quad \text{on } \Omega \times \mathbf{R}^3, \end{aligned} \tag{1.11}$$

where $q > 1$, supplemented with the condition of specular reflection on $\Sigma_- = \{(x,v) \in \partial\Omega \times \mathbf{R}^3 / v \cdot n(x) < 0\}$:

$$f_\epsilon(t,x,v) = f_\epsilon\big(t,x,v - 2\big(v \cdot n(x)\big)n(x)\big).$$

Then the family of fluctuations g_ϵ defined by $f_\epsilon = M(1+\epsilon g_\epsilon)$ is relatively weakly compact in $L^1_{\mathrm{loc}}(dt\,dx, L^1(dv))$, and any limit point g of (g_ϵ) is an infinitesimal Maxwellian

$$g = u \cdot v.$$

Furthermore u coincides with the Lipschitz solution to the incompressible Euler equations

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = 0, \quad \nabla_x \cdot u = 0 \quad \textit{on } \mathbf{R}^+ \times \Omega,$$
$$u(0,x) = u^{in}(x) \quad \textit{on } \Omega,$$
$$u \cdot n = 0 \quad \textit{on } \mathbf{R}^+ \times \partial\Omega \tag{1.12}$$

as long as the latter does exist.

• Let us just mention that the first result obtained in this framework is due to Golse: in [5], the convergence of renormalized solutions of the scaled Boltzmann equation to solutions of the incompressible Euler equations is established for well-prepared data assuming further

(i) the local conservation of momentum which is not guaranteed for renormalized solutions of the Boltzmann equation; and
(ii) some nonlinear estimate, namely

$$\big(1+|v|^2\big)\frac{g_\epsilon^2}{1+\frac{\epsilon}{2}g_\epsilon} \text{ relatively weakly compact in } L^1_{\mathrm{loc}}\big(dt\,dx, w - L^1(M\,dv)\big)$$

which provides both a control on large velocities, and some equiintegrability with respect to space variables.

The proof is based on the following stability inequality

$$\begin{aligned}
&\frac{1}{\epsilon^2}H(f_\epsilon|\mathcal{M}_{1,\epsilon w,1})(t) + \frac{1}{\epsilon^{q+3}}\iint_0^t D(f_\epsilon)(s,x)\,dx\,ds \\
&\quad\leqslant \frac{1}{\epsilon^2}H(f_{\epsilon,in}|\mathcal{M}_{1,\epsilon w_{in},1}) + \frac{1}{\epsilon}\iint_0^t A(w)\cdot\int(\epsilon w - v)f_\epsilon(s,x,v)\,dv\,dx\,ds \\
&\qquad - \frac{1}{2\epsilon^2}\iint_0^t \big(\nabla_x w + (\nabla_x w)^T\big) : \int (v-\epsilon w)^{\otimes 2} f_\epsilon(s,x,v)\,dv\,dx\,ds
\end{aligned} \tag{1.13}$$

satisfied under assumption (i), for all $t \in [0,T)$ and all smooth solenoidal vector field $w \in C_c^\infty([0,T]\times\Omega)$, where A is the acceleration operator defined by

$$A(w) = \partial_t w + w \cdot \nabla_x w. \tag{1.14}$$

The control of the last term in the right-hand side of (1.13) is then obtained from assumption (ii).

• Assumption (i) was removed by Lions and Masmoudi in [23]; their argument uses the local momentum conservation with nonnegative matrix-valued defect measure satisfied by renormalized solutions of the Boltzmann equation. Inequality (1.13) is then replaced by

$$\begin{aligned}
&\frac{1}{\epsilon^2}H(f_\epsilon|\mathcal{M}_{1,\epsilon w,1})(t) + \frac{1}{\epsilon^2}\int \operatorname{trace}(m_\epsilon)(t) + \frac{1}{\epsilon^{q+3}}\iint_0^t D(f_\epsilon)(s,x)\,dx\,ds \\
&\quad\leqslant \frac{1}{\epsilon^2}H(f_{\epsilon,in}|\mathcal{M}_{1,\epsilon w_{in},1}) + \frac{1}{\epsilon}\iint_0^t A(w)\cdot\int(\epsilon w - v)f_\epsilon(s,x,v)\,dv\,dx\,ds \\
&\qquad - \frac{1}{2\epsilon^2}\iint_0^t \big(\nabla_x w + (\nabla_x w)^T\big) : \left(m_\epsilon(s) + \int (v-\epsilon w)^{\otimes 2} f_\epsilon(s,x,v)\,dv\,dx\right) ds.
\end{aligned} \tag{1.15}$$

That the defect measure $\frac{1}{\epsilon^2}m_\epsilon$ vanishes in the incompressible Euler limit follows from the strong convergence result to be proved.

• Assumption (ii) was removed by the author first in the framework of the BGK equation [27], then in the case of the original Boltzmann equation [28] using refined dissipation estimates. The main idea is to introduce a suitable decomposition of the momentum flux, and estimate each term in that decomposition either by the modulated entropy, or by the entropy dissipation. One therefore proves

$$-\frac{1}{2\epsilon^2}\int_0^t\iint \big(\nabla_x w + (\nabla_x w)^T\big) : (v-\epsilon w)^{\otimes 2}(f_\epsilon - \mathcal{M}_{1,\epsilon w,1})(s,x,v)\,dv\,dx\,ds$$
$$\leqslant \frac{C}{\epsilon^2}\int_0^t \big\|\big(\nabla_x w + (\nabla_x w)^T\big)\big\|_{L^2\cap L^\infty(\Omega)} H\big(f_\epsilon|\mathcal{M}_{1,\epsilon w,1}\big)(s)\,ds + \mathrm{o}(1). \tag{1.16}$$

In other words, the argument is based on loop estimates instead of a priori estimates, and the conclusion follows from Gronwall's inequality.

1.3. About the assumptions on the initial data

The aim of this paper is to provide answers and tools to current limitations on the understanding of the incompressible Euler limit of the Boltzmann equation. More precisely, our goal is to isolate problems of technical order (which come from the lack of physical estimates for the renormalized solutions to the Boltzmann equation) from difficulties linked to the physics of the system.

Note that the condition (1.10) in Theorem 1.2 is a very strong assumption on the family of initial data, meaning that "well-prepared" has to be understood in the following sense.

- We first require that the initial distribution has a *velocity profile* close to local thermodynamic equilibrium, or in other words that
$$g_\epsilon^{in} = \rho^{in} + u^{in}\cdot v + \theta^{in}\frac{|v|^2-3}{2} + \mathrm{o}(\epsilon),$$
in entropic sense, in order that there is no relaxation layer.
- We then ask the limiting *initial thermodynamic fields* to satisfy the incompressibility and Boussinesq constraints
$$\nabla\cdot u^{in} = 0, \quad \nabla\big(\rho^{in}+\theta^{in}\big) = 0,$$
which ensures that there is no acoustic wave. We further require that the initial temperature fluctuation (and thus mass fluctuation) is negligible
$$\rho^{in} = \theta^{in} = 0.$$
We therefore expect the temperature fluctuation to remain negligible.
- We finally need some *spatial regularity* on the limiting bulk velocity, more precisely we require some Lipschitz continuity.

We are thus able to consider very general initial data (satisfying only the physical estimate (1.9)), but in the vicinity of a small set of asymptotic distributions.

A natural question is then to know whether or not it is possible to get rid of these restrictions on the asymptotic distribution. We will see that the first two assumptions come actually from the poor understanding of the Boltzmann equation, in particular from the fact that renormalized solutions to the Boltzmann equation are not known to satisfy the local conservation of energy (the heat flux is not even defined), whereas the last assumption concerning the regularity of the limiting distribution is inherent to the modulated entropy method.

Considering solutions to the Boltzmann equation satisfying rigorously the basic physical properties, we can expect to *control the energy flux* and extend the convergence result to take into account acoustic waves. In order to also deal with the relaxation layer, we further need to understand the *dissipation mechanism*, which should be done by slight modifications of the method.

To close estimates in both cases, we will need an additional assumption on large velocities and large tails of the solutions of the Boltzmann equation. We will therefore consider a stronger notion of solutions (for instance the solutions built by Guo [19,20]) which ensures the (nonuniform) required controls, and we will establish in this framework a convergence result without restriction on the initial data, i.e. without assuming any profile condition or thermodynamic constraints.

However, obtaining full proofs valid in all physical configurations remain a challenging open problem, due in particular to our limited knowledge concerning the solutions of the 3D incompressible Euler equations.

The first restriction is due to the *apparition of Prandtl boundary layers* (coming from the incompatibility between the braking boundary condition and the inviscid motion equation) which are generally unstable [18], meaning that the formal asymptotics is not expected to provide a good approximation.

The second restriction is inherent to the relative entropy method in its present form: flux terms are estimated in terms of the modulated entropy, which requires a *Lipschitz bound on the solution of the target equations*. In order to extend the method to the study of discontinuous asymptotics, one should probably modulate both the entropy and the entropy dissipation, and use the local version (in t and x) of the entropy inequality:

$$\partial_t \frac{1}{\epsilon^2} \int \left(f_\epsilon \log \frac{f_\epsilon}{M} - f_\epsilon + M \right) dv + \nabla_x \cdot \frac{1}{\epsilon^3} \int \left(f_\epsilon \log \frac{f_\epsilon}{M} - f_\epsilon + M \right) v \, dv \quad = -\frac{1}{\epsilon^{q+3}} D(f_\epsilon).$$

Note that such a generalization of the relative entropy method, combined for instance with the weak notion of solution to the one-dimensional Boltzmann equation [7,4], should offer perspectives for compressible hydrodynamic limits.

2. Description of the main results

2.1. Control of large tails and large velocities

Before stating our main theorem, let us briefly explain the difficulties encountered in extending the previous convergence result.

We have seen that dealing with the well-prepared case requires to control the momentum flux, that is more or less to obtain a bound on the quantity arising in assumption (ii). When considering more general initial data, we will have to control the energy flux, that is a moment of order 3 of the distribution. Indeed we know that, even if the initial temperature fluctuation is negligible, acoustic waves will couple all thermodynamic fields.

Such a control on $g_\epsilon |v|^3$ cannot come from the relative entropy (since Young's inequality is saturated with a factor $|v|^2$). However, it is clear that, for Maxwellian distributions, the relative entropy controls all moments. The basic idea (as in [28]) is therefore to use a decomposition of Chapman–Enskog's type and to control the distance to local equilibrium by the entropy dissipation, which requires to have some (nonuniform) control on large tails at our disposal. That control will appear in our statement as an additional assumption, which can be removed by considering a suitable notion of solution to the Boltzmann equation.

2.2. The incompressible Euler limit: convergence result

Theorem 2.1. *Let Ω be some smooth bounded domain of $\mathbf{R}^3$, or the three-dimensional torus $\mathbf{T}^3$. Let f_ϵ^{in} be a family of measurable nonnegative functions on $\Omega \times \mathbf{R}^3$ satisfying the scaling condition* (1.9)

$$\frac{1}{\epsilon^2} H\big(f_\epsilon^{in} \big| M\big) \leqslant C_{in}.$$

Assume furthermore that the fluctuations g_ϵ^{in} defined by $f_\epsilon^{in} = M(1+\epsilon g_\epsilon^{in})$ converge entropically to some g^{in}, i.e.

$$\begin{aligned} & g_\epsilon^{in} \rightharpoonup g^{in} \quad \text{weakly in } L^1_{\mathrm{loc}}(dt\,dx\,dv), \\ & \frac{1}{\epsilon^2} H\big(f_\epsilon^{in} \big| M\big) \to \frac{1}{2} \iint M\big(g_\epsilon^{in}\big)^2 \, dv\,dx. \end{aligned} \tag{2.1}$$

Let f_ϵ be some family of solutions to the scaled Boltzmann equation

$$\epsilon\partial_t f_\epsilon + v\cdot\nabla_x f_\epsilon = \frac{1}{\epsilon^q} Q(f_\epsilon, f_\epsilon) \quad on\ \mathbf{R}^+\times\Omega\times\mathbf{R}^3,$$
$$f_\epsilon(0,x,v) = f_\epsilon^{in}(x,v) \quad on\ \Omega\times\mathbf{R}^3, \tag{2.2}$$

with $q>1$, supplemented with the condition of specular reflection on $\Sigma_- = \{(x,v)\in\partial\Omega\times\mathbf{R}^3/v\cdot n(x)<0\}$:

$$f_\epsilon(t,x,v) = f_\epsilon\big(t,x,v-2\big(v\cdot n(x)\big)n(x)\big).$$

Assume that there exists some nonnegative constant C such that, for all ϵ

$$\int M\left(\frac{f_\epsilon - M}{M}\right)^2 dv \leqslant C \quad a.e.\ on\ \mathbf{R}^+\times\Omega. \tag{2.3}$$

Then the family of fluctuations (g_ϵ) defined by $f_\epsilon = M(1+\epsilon g_\epsilon)$ is relatively weakly compact in $L^1_{\mathrm{loc}}(dt\,dx, L^1(dv))$, and any limit point g of (g_ϵ) as $\epsilon\to 0$ is an infinitesimal Maxwellian

$$g = u\cdot v + \theta\left(\frac{|v|^2-5}{2}\right),$$

where (u,θ) coincides with the Lipschitz solution to the incompressible Euler equations

$$\partial_t u + u\cdot\nabla_x u + \nabla_x p = 0, \quad \nabla_x\cdot u = 0 \quad on\ \mathbf{R}^+\times\Omega,$$
$$\partial_t\theta + u\cdot\nabla_x\theta = 0 \quad on\ \mathbf{R}^+\times\Omega,$$
$$u(0,x) = Pu^{in}(x), \quad \theta(0,x) = \frac{1}{5}\big(3\theta^{in} - 2\rho^{in}\big) \quad on\ \Omega,$$
$$u\cdot n = 0 \quad on\ \mathbf{R}^+\times\partial\Omega \tag{2.4}$$

as long as the latter does exist, say on $[0,T^*)$.

Furthermore the difference $g_\epsilon - g$ behaves asymptotically as

$$g_{osc}\left(\frac{t}{\epsilon},x,v\right) = (\rho_{osc},u_{osc},\theta_{osc})\left(\frac{t}{\epsilon},x\right)\cdot\left(1,v,\frac{1}{2}\big(|v|^2-3\big)\right)$$

where $(\rho_{osc},u_{osc},\theta_{osc})$ is the oscillating solution of the acoustic system (3.11) with initial data $(\rho^{in}+\frac{1}{5}(3\theta^{in}-2\rho^{in}), u^{in}-Pu^{in}, \theta^{in}-\frac{1}{5}(3\theta^{in}-2\rho^{in}))$. More precisely,

$$g_\epsilon(t,x,v) - g(t,x,v) - g_{osc}\left(\frac{t}{\epsilon},x,v\right) \to 0$$

strongly in $L^1_{\mathrm{loc}}([0,T^*)\times\Omega, L^1(M\,dv))$ as $\epsilon\to 0$.

Remark 2.2.

(i) The estimate (2.3) giving some control on large tails is satisfied for instance by the classical solutions of the Boltzmann equation built by Guo when $\Omega=\mathbf{T}^3$ [19] or $\Omega=\mathbf{R}^3$ [20]. Of course for such solutions the local conservation laws are satisfied.

(ii) The acoustic system (3.11) governing the oscillating part of the distribution becomes linear (with coefficients depending on the nonoscillating part g) if there is no other resonance between acoustic waves than the trivial ones.

(iii) In the case of unbounded domains, the linear penalization in (3.11) admits also continuous spectrum $\mathfrak{S}_c$: we therefore expect the corresponding part of the solution $\int_{\lambda\in\mathfrak{S}_c}\Pi_\lambda(\rho,u,\theta)\,d\mu(\lambda)$ to satisfy some dispersion property, and to converge strongly to 0 by Strichartz estimates (see [14] for a review on that topic). The oscillating part is then obtained by considering the projection of (ρ,u,θ) on the subspace generated by the eigenmodes of the acoustic penalization $(\rho_{osc},u_{osc},\theta_{osc}) = \sum_{\lambda\in\mathfrak{S}_p}\Pi_\lambda(\rho,u,\theta)$.

For the sake of simplicity, we restrict here our attention to the case when $\mathfrak{S}_c=\emptyset$. However a similar result can be proved in any smooth domain (see Remark 3.4 in the next section for the case of $\mathbf{R}^3$).

(iv) Note that the purely kinetic part does not appear in that convergence statement since its contribution to the $L^1_{\mathrm{loc}}([0,T^*)\times\Omega, L^1(M(1+|v|^2)\,dv))$ norm is negligible. The entropic convergence we will establish is actually stronger.

3. Taking into account acoustic waves

3.1. The modulated entropy inequality

Since acoustic waves only contribute to the hydrodynamic part of the distribution, relaxing the constraints on the initial thermodynamic fields does not require strong modifications of the method.

Outside from the initial layer, the strategy consists then in modulating the entropy by any fluctuation of Maxwellian (without any restriction on the moments). We then expect the modulated entropy inequality to differ from the usual one by some penalization arising in the acceleration operator. More precisely, we have the following

Proposition 3.1. *Denote by f_{app} the fluctuation of Maxwellian defined by*

$$\log f_{app} = -\frac{3}{2}\log(2\pi) + \epsilon\left(\rho - \frac{3}{2}\theta\right) - \frac{1}{2}e^{-\epsilon\theta}|v-\epsilon u|^2. \tag{3.1}$$

Then, any solution to the scaled Boltzmann equation (2.2) *such that* (2.3) *holds satisfies the following modulated entropy inequality*

$$\begin{aligned}
H(f_\epsilon|f_{app})(t) &+ \frac{1}{\epsilon^{q+1}}\iint_0^t D(f_\epsilon)\,ds\,dx \leqslant H\left(f_\epsilon^{in}\middle|f_{app}^{in}\right) + \iint_0^t \partial_t \exp(\epsilon\rho)\,dx\,ds \\
&- \epsilon\iiint_0^t f_\epsilon\left(1, e^{-\epsilon\theta}(v-\epsilon u), \frac{1}{2}\left(e^{-\epsilon\theta}|v-\epsilon u|^2 - 3\right)\right)\cdot A_\epsilon(\rho,u,\theta)\,dv\,dx\,ds \\
&- \iiint_0^t f_\epsilon \nabla_x u : \Phi_\epsilon\,dx\,dv + \iint f_\epsilon e^{\frac{1}{2}\epsilon\theta}\nabla_x\theta\cdot\Psi_\epsilon\,dx\,dv\,ds
\end{aligned} \tag{3.2}$$

for some acceleration operator $A_\epsilon(\rho,u,\theta)$ to be defined by (3.10).

Proof. Start from the entropy inequality satisfied by the solution of the scaled Boltzmann equation with specular reflection at the boundary:

$$H\big(f_\epsilon(t)|M\big) + \frac{1}{\epsilon^{q+1}}\iiint_0^t D(f_\epsilon)(s,x)\,ds\,dx \leqslant H\big(f_\epsilon^{in}|M\big). \tag{3.3}$$

By definition of the modulated entropy (1.1) and of the approximate solution (3.1), we then have

$$\begin{aligned}
H(f_\epsilon|f_{app})(t) &+ \frac{1}{\epsilon^{q+1}}\iint_0^t D(f_\epsilon)\,ds\,dx \leqslant H\left(f_\epsilon^{in}\middle|f_{app}^{in}\right) + \iint_0^t \partial_t\left(\int f_{app}\,dv\right)dx\,ds \\
&- \int_0^t \frac{d}{dt}\iint\left(\epsilon\left(\rho - \frac{3}{2}\theta\right) - \frac{1}{2}e^{-\epsilon\theta}|v-\epsilon u|^2 + \frac{1}{2}|v|^2\right) f_\epsilon\,dv\,dx\,ds
\end{aligned} \tag{3.4}$$

with

$$\int f_{app}\,dv = \exp(\epsilon\rho).$$

Using the continuity equation

$$\partial_t\int f_\epsilon\,dv + \nabla_x\cdot\frac{1}{\epsilon}\int v f_\epsilon\,dv = 0, \tag{3.5}$$

the conservation of momentum

$$\partial_t \int v f_\epsilon \, dv + \nabla_x \cdot \frac{1}{\epsilon} \int v \otimes v f_\epsilon \, dv = 0, \tag{3.6}$$

and the conservation of energy

$$\partial_t \int \frac{1}{2}|v|^2 f_\epsilon \, dv + \nabla_x \cdot \frac{1}{\epsilon} \int \frac{1}{2}|v|^2 v f_\epsilon \, dv = 0, \tag{3.7}$$

as well as the boundary condition on Σ_-

$$f_\epsilon(t,x,v) = f_\epsilon\big(t,x,v - 2\big(v\cdot n(x)\big)n(x)\big),$$

and integrating by parts, we obtain

$$\begin{aligned}
&\frac{1}{\epsilon}\frac{d}{dt}\iint \left(\epsilon\left(\rho - \frac{3}{2}\theta\right) - \frac{1}{2}e^{-\epsilon\theta}|v-\epsilon u|^2 + \frac{1}{2}|v|^2\right) f_\epsilon \, dv\, dx \\
&\quad = \iint f_\epsilon\left(\partial_t\left(\rho - \frac{3}{2}\theta\right) + (u\cdot\nabla_x)\left(\rho - \frac{3}{2}\theta\right) + \frac{1}{\epsilon}(v-\epsilon u)\cdot\nabla_x\left(\rho - \frac{3}{2}\theta\right)\right) dv\, dx \\
&\qquad + \iint f_\epsilon e^{-\epsilon\theta}(v-\epsilon u)\cdot\left(\partial_t u + (u\cdot\nabla_x)u + \frac{1}{\epsilon}(v-\epsilon u)\cdot\nabla_x u\right) dv\, dx \\
&\qquad + \frac{1}{2}\iint f_\epsilon e^{-\epsilon\theta}|v-\epsilon u|^2\left(\partial_t\theta + (u\cdot\nabla_x)\theta + \frac{1}{\epsilon}(v-\epsilon u)\cdot\nabla_x\theta\right) dv\, dx
\end{aligned}$$

provided that $u\cdot n = 0$ on $\partial\Omega$.

Let us then introduce the kinetic momentum and energy fluxes

$$\begin{aligned}
\Phi &= \left(v^{\otimes 2} - \frac{1}{3}|v|^2 \mathrm{Id}\right), \\
\Psi &= \frac{1}{2}v\big(|v|^2 - 5\big)
\end{aligned} \tag{3.8}$$

and their scaled translated variants

$$\begin{aligned}
\Phi_\epsilon &= e^{-\epsilon\theta}\left((v-\epsilon u)^{\otimes 2} - \frac{1}{3}|v-\epsilon u|^2\mathrm{Id}\right), \\
\Psi_\epsilon &= \frac{1}{2}e^{-\frac{3}{2}\epsilon\theta}(v-\epsilon u)\big(|v-\epsilon u|^2 - 5\big)
\end{aligned} \tag{3.9}$$

and recall that Φ and Ψ belong to the orthogonal complement of the kernel $\operatorname{Ker}\mathcal{L}$ where $\mathcal{L}$ is the linearized collision operator at M. We have

$$\begin{aligned}
&e^{-\epsilon\theta}\nabla_x u : (v-\epsilon u)^{\otimes 2} = \nabla_x u : \Phi_\epsilon + \frac{1}{3}e^{-\epsilon\theta}\nabla_x\cdot u|v-\epsilon u|^2, \\
&\frac{1}{2}e^{-\epsilon\theta}\nabla_x\theta\cdot(v-\epsilon u)|v-\epsilon u|^2 = e^{\frac{1}{2}\epsilon\theta}\nabla_x\theta\cdot\Psi_\epsilon + \frac{5}{2}e^{-\epsilon\theta}\nabla_x\theta\cdot(v-\epsilon u)
\end{aligned}$$

so that

$$\begin{aligned}
&\frac{1}{\epsilon}\frac{d}{dt}\iint \left(\epsilon\left(\rho - \frac{3}{2}\theta\right) - \frac{1}{2}e^{-\epsilon\theta}|v-\epsilon u|^2 + \frac{1}{2}|v|^2\right) f_\epsilon \, dv\, dx \\
&\quad = \iint f_\epsilon\left(\partial_t\left(\rho - \frac{3}{2}\theta\right) + (u\cdot\nabla_x)\left(\rho - \frac{3}{2}\theta\right)\right) dv\, dx \\
&\qquad + \iint f_\epsilon e^{-\epsilon\theta}(v-\epsilon u)\cdot\left(\partial_t u + (u\cdot\nabla_x)u + \frac{1}{\epsilon}e^{\epsilon\theta}\nabla_x\left(\rho - \frac{3}{2}\theta\right) + \frac{5}{2\epsilon}\nabla_x\theta\right) dv\, dx \\
&\qquad + \frac{1}{2}\iint f_\epsilon e^{-\epsilon\theta}|v-\epsilon u|^2\left(\partial_t\theta + (u\cdot\nabla_x)\theta + \frac{2}{3}\nabla_x\cdot u\right) dv\, dx \\
&\qquad + \frac{1}{\epsilon}\iint f_\epsilon\nabla_x u : \Phi_\epsilon \, dx\, dv + \frac{1}{\epsilon}\iint f_\epsilon e^{\frac{1}{2}\epsilon\theta}\nabla_x\theta\cdot\Psi_\epsilon \, dx\, dv.
\end{aligned}$$

It is then natural to define the acceleration operator

$$A_\epsilon(\rho,u,\theta)=\begin{pmatrix}\partial_t\rho+(u\cdot\nabla_x)\rho+\frac{1}{\epsilon}\nabla_x\cdot u\\ \partial_t u+(u\cdot\nabla_x)u+\big(\frac{e^{\epsilon\theta}-1}{\epsilon}\big)\nabla_x\big(\rho-\frac{3}{2}\theta\big)+\frac{1}{\epsilon}\nabla_x(\rho+\theta)\\ \partial_t\theta+(u\cdot\nabla_x)\theta+\frac{2}{3\epsilon}\nabla_x\cdot u\end{pmatrix}\tag{3.10}$$

so that the inequality can be recasted in suitable form

$$\begin{aligned}&\frac{1}{\epsilon}\frac{d}{dt}\iint\left(\epsilon\left(\rho-\frac{3}{2}\theta\right)-\frac{1}{2}e^{-\epsilon\theta}|v-\epsilon u|^2+\frac{1}{2}|v|^2\right)f_\epsilon\,dv\,dx\\ &\quad=\iint f_\epsilon\left(1,e^{-\epsilon\theta}(v-\epsilon u),\frac{1}{2}\big(e^{-\epsilon\theta}|v-\epsilon u|^2-3\big)\right)\cdot A_\epsilon(\rho,u,\theta)\,dv\,dx\\ &\qquad+\frac{1}{\epsilon}\iint f_\epsilon\nabla_x u:\Phi_\epsilon\,dx\,dv+\frac{1}{\epsilon}\iint f_\epsilon e^{\frac{1}{2}\epsilon\theta}\nabla_x\theta\cdot\Psi_\epsilon\,dx\,dv.\end{aligned}$$

Plugging this last inequality in (3.4) leads to the announced result.

Note that the acceleration operator defined by (3.10) differs from the usual one (1.14) (defined for well-prepared initial data) by some penalization forcing the weak limit to satisfy the constraints

$$\nabla_x\cdot u=0,\qquad\nabla_x(\rho+\theta)=0.\qquad\square$$

Remark 3.2. Note that the proof of Proposition 3.1 does not require any assumption on the spatial domain Ω.

3.2. Construction of the approximate solution

The next step is to construct a sequence of approximate solutions $(\rho_\epsilon,u_\epsilon,\theta_\epsilon)$ to the systems

$$\begin{aligned}&\partial_t\rho+(u\cdot\nabla_x)\rho+\frac{1}{\epsilon}\nabla_x\cdot u=0,\\ &\partial_t u+(u\cdot\nabla_x)u+\left(\frac{e^{\epsilon\theta}-1}{\epsilon}\right)\nabla_x\left(\rho-\frac{3}{2}\theta\right)+\frac{1}{\epsilon}\nabla_x(\rho+\theta)=0,\\ &\partial_t\theta+(u\cdot\nabla_x)\theta+\frac{2}{3\epsilon}\nabla_x\cdot u=0\end{aligned}\tag{3.11}$$

or in other words to the systems

$$A_\epsilon(\rho,u,\theta)=0.$$

More precisely, we will require that

$$A_\epsilon(\rho_\epsilon,u_\epsilon,\theta_\epsilon)\to 0\quad\text{in }L^2(dt\,dx)\text{ as }\epsilon\to 0.\tag{3.12}$$

One of the difficulty here (in comparison with classical penalization problems) is that we further need that these approximate solutions conserve the total mass at higher order

$$\frac{1}{\epsilon^2}\int\partial_t\exp(\epsilon\rho_\epsilon)\,dx\to 0\quad\text{in }L^1(dt)\text{ as }\epsilon\to 0.\tag{3.13}$$

(Note that, for exact solutions, the total mass is exactly conserved.)

Such a construction is done by a filtering method, i.e. considering the family $\mathcal{W}(\frac{t}{\epsilon})(\rho_\epsilon,u_\epsilon,\theta_\epsilon)$ where $\mathcal{W}$ is the semigroup generated by the linear penalization operator W defined by

$$W(\rho,u,\theta)=\left(\nabla_x\cdot u,\nabla_x(\rho+\theta),\frac{2}{3}\nabla_x\cdot u\right).$$

The first order approximation is then obtained by taking (strong) limits in the filtered system. Nevertheless, because of the high frequency oscillations, we do not expect the error in this first order approximation to converge strongly to 0.

We therefore have to add some correctors (i.e. the second order approximation) in order to establish the convergence statement (3.12).

More precisely, we have the following

Proposition 3.3. *Let $(\rho^{in}, u^{in}, \theta^{in})$ belong to $H^s(\Omega)$ for some $s > \frac{5}{2}$.*

Then there exist some $T > 0$, and some family $(\rho_\epsilon^N, u_\epsilon^N, \theta_\epsilon^N)$ such that

$$\sup_{N\in\mathbf{N}} \lim_{\epsilon\to 0} \big\| \big(\rho_\epsilon^N, u_\epsilon^N, \theta_\epsilon^N\big) \big\|_{L^1([0,T],H^s(\Omega))} \leqslant C_T, \tag{3.14}$$

$$\big(\rho_\epsilon^{in,N}, u_\epsilon^{in,N}, \theta_\epsilon^{in,N}\big) \to \big(\rho^{in}, u^{in}, \theta^{in}\big) \quad \text{in } H^s(dx) \text{ as } \epsilon \to 0 \text{ then } N \to \infty, \tag{3.15}$$

$$A_\epsilon\big(\rho_\epsilon^N, u_\epsilon^N, \theta_\epsilon^N\big) \to 0 \quad \text{in } L^2(dt\,dx) \text{ as } \epsilon \to 0 \text{ then } N \to \infty, \tag{3.16}$$

and

$$\frac{1}{\epsilon^2} \int \partial_t \exp\big(\epsilon\rho_\epsilon^N\big)\,dx \to 0 \quad \text{in } L^1(dt) \text{ as } \epsilon \to 0 \text{ then } N \to \infty. \tag{3.17}$$

Proof. • Let us first introduce some notations to recast the system

$$A_\epsilon(\rho, u, \theta) = 0$$

in a suitable form. For any $V = (\rho, u, \theta)$ we define the symmetric bilinear form $\mathcal{B}$ by

$$\mathcal{B}(V, V) = \begin{pmatrix} (u\cdot\nabla_x)\rho \\ (u\cdot\nabla_x)u + \theta\nabla_x\big(\rho - \frac{3}{2}\theta\big) \\ (u\cdot\nabla_x)\theta \end{pmatrix}.$$

We are therefore interested in the (approximate) solutions to

$$\partial_t V + \frac{1}{\epsilon} W V + \mathcal{B}(V, V) = -\begin{pmatrix} 0 \\ \frac{1}{\epsilon}(e^{\epsilon\theta} - 1 - \epsilon\theta)\nabla_x\big(\rho - \frac{3}{2}\theta\big) \\ 0 \end{pmatrix}$$

which are also approximate solutions (in the sense of (3.16)) to

$$\partial_t V + \frac{1}{\epsilon} W V + \mathcal{B}(V, V) = 0$$

provided that V is uniformly bounded in $L^\infty([0, T], W^{1,\infty} \cap L^2(\Omega))$. Let us also recall that we further need that these approximate solutions satisfy some global conservation of mass (3.17).

Let us then conjugate the system by the semi-group $\mathcal{W}(\frac{t}{\epsilon})$ generated by W

$$\partial_t\left(\mathcal{W}\left(\frac{t}{\epsilon}\right)V\right) + \mathcal{W}\left(\frac{t}{\epsilon}\right)\mathcal{B}(V, V) = 0,$$

or equivalently

$$\partial_t \tilde{V} + \mathcal{W}\left(\frac{t}{\epsilon}\right)\mathcal{B}\left(\mathcal{W}\left(-\frac{t}{\epsilon}\right)\tilde{V}, \mathcal{W}\left(-\frac{t}{\epsilon}\right)\tilde{V}\right) = 0 \tag{3.18}$$

denoting by $\tilde{V}$ the filtered field $\tilde{V} = \mathcal{W}(\frac{t}{\epsilon})V$.

We therefore expect the solutions (and approximate solutions) to (3.18) to have a very different behaviour depending on the nature of the spectrum of W. In the case when Ω is a smooth bounded domain, $(\mathrm{Id} - \Delta)^{-1}$ is a compact operator with discrete spectrum, from which we deduce that W has discrete spectrum. The (formal) limit system depends therefore on the resonances between acoustic modes. In the case when Ω is an exterior domain, the Laplacian has continuous spectrum and one can prove using dispersion properties that the corresponding acoustic waves converge strongly to 0.

• Let us focus on the case of bounded domains. Let $(i\lambda_k)$ be the sequence of eigenvalues of W corresponding to the boundary condition of Neumann type

$$u \cdot n = 0 \quad \text{on } \partial\Omega,$$

(see Appendix A.1 for a detailed study of the spectrum of W). Denote by Π_λ the projection on $\mathrm{Ker}(W - \lambda\,\mathrm{Id})$.

Note that a similar diagonalization result holds in the case of the torus $\mathbf{T}^3$, so that all that follows can be extended in that case.

At leading order, we then obtain

$$\partial_t \tilde V_0 + \mathcal{B}_W(\tilde V_0, \tilde V_0) = 0, \tag{3.19}$$

denoting by $\mathcal{B}_W$ the limiting quadratic operator defined by

$$\mathcal{B}_W = \sum_k \sum_{\lambda_{k_1}+\lambda_{k_2}=\lambda_k} \Pi_{\lambda_k} \mathcal{B}(\Pi_{\lambda_{k_1}} \cdot, \Pi_{\lambda_{k_2}} \cdot). \tag{3.20}$$

An algebraic computation (which is the basic argument in the compensated compactness method, see [24] for instance) shows that, for all $\lambda, \mu \neq 0$

$$\Pi_0 \mathcal{B}(\Pi_\lambda \tilde V_0, \Pi_\mu \tilde V_0) = 0.$$

Indeed we have the following formula for Π_0

$$\Pi_0(\rho, u, \theta) = \left(\frac{2\rho - 3\theta}{5}, Pu, \frac{3\theta - 2\rho}{5}\right).$$

Then, with the notations $\Pi_\lambda \tilde V_0 = (\rho_\lambda, u_\lambda, \theta_\lambda)$ and $\Pi_\mu \tilde V_0 = (\rho_\mu, u_\mu, \theta_\mu)$, we get

$$\begin{aligned}
\Pi_0 \mathcal{B}(\Pi_\lambda \tilde V_0, \Pi_\mu \tilde V_0) &= \frac{1}{2} \Pi_0 \begin{pmatrix} (u_\lambda \cdot \nabla_x)\rho_\mu + (u_\mu \cdot \nabla_x)\rho_\lambda \\ (u_\lambda \cdot \nabla_x)u_\mu + (u_\mu \cdot \nabla_x)u_\lambda \\ (u_\lambda \cdot \nabla_x)\theta_\mu + (u_\mu \cdot \nabla_x)\theta_\lambda \end{pmatrix} \\
&= \frac{1}{10} \begin{pmatrix} (u_\lambda \cdot \nabla_x)(2\rho_\mu - 3\theta_\mu) + (u_\mu \cdot \nabla_x)(2\rho_\lambda - 3\theta_\lambda) \\ 5P((u_\lambda \cdot \nabla_x)u_\mu + (u_\mu \cdot \nabla_x)u_\lambda) \\ (u_\lambda \cdot \nabla_x)(3\theta_\mu - 2\rho_\mu) + (u_\mu \cdot \nabla_x)(3\theta_\lambda - 2\rho_\lambda) \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 0 \\ P(\nabla_x(u_\lambda \cdot u_\mu) - u_\mu \wedge (\nabla_x \wedge u_\lambda) - u_\lambda \wedge (\nabla_x \wedge u_\mu)) \\ 0 \end{pmatrix}
\end{aligned} \tag{3.21}$$

since $\nabla_x \wedge u_\lambda = 0$ and $3\theta_\lambda - 2\rho_\lambda = 0$.

In other words the equation governing the nonoscillating part can be decoupled from the rest of the system

$$\partial_t \Pi_0 \tilde V_0 + \Pi_0 \mathcal{B}(\Pi_0 \tilde V_0, \Pi_0 \tilde V_0) = 0,$$

which can be rewritten

$$\begin{aligned}
&\partial_t \bar\rho + (\bar u \cdot \nabla_x)\bar\rho = 0, \qquad \nabla_x(\bar\rho + \bar\theta) = 0, \\
&\partial_t \bar u + (\bar u \cdot \nabla_x)\bar u + \nabla_x p = 0, \qquad \nabla_x \cdot \bar u = 0,
\end{aligned} \tag{3.22}$$

where $(\bar\rho, \bar u, \bar\theta) = \Pi_0 \tilde V_0 = \Pi_0 V_0$. Note in particular that

$$\partial_t \int \bar\rho \, dx = 0.$$

A classical study based on harmonic analysis (see [14] for instance, and Appendix A.3 for a brief summary) allows to prove that (3.19) has a unique strong solution $V_0 \in L^\infty_{\mathrm{loc}}([0, T^*), H^s(\Omega))$ provided that $V^{in} \in H^s(\Omega)$ for $s > \frac{5}{2}$. The point is to check that

$$\|V\|^2_{H^s(\Omega)} \sim \|\Pi_0 V\|^2_{H^s(\Omega)} + \sum_k \left(1 + \lambda_k^2\right)^s \|\Pi_k V\|^2_{L^2(\Omega)}$$

(see Appendix A.2 for more details). Note that

$$\partial_t \tilde V_0 \in L^\infty_{\mathrm{loc}}\left([0, T^*), H^{s-1}(\Omega)\right).$$

Remarking that, for all $\lambda \neq 0$,

$$\int \rho_\lambda \, dx = \frac{1}{i\lambda} \int (\nabla_x \cdot u_\lambda) \, dx = \int_{\partial\Omega} u_\lambda \cdot n \, d\sigma_x = 0, \tag{3.23}$$

we get, denoting $(\rho_0, u_0, \theta_0) = V_0 = \mathcal{W}(-\frac{t}{\epsilon})\tilde{V}_0$,

$$\partial_t \int \rho_0 \, dx = \partial_t \int \bar{\rho} \, dx = 0.$$

Furthermore, the projection J_N defined by (A.2) (which is a L^2 projection on a finite dimensional subset of $C^\infty(\Omega)$) is such that

$$\tilde{V}_0 - J_N \tilde{V}_0 \to 0 \quad \text{as } N \to \infty \text{ in } L^\infty_{\mathrm{loc}}\big([0, T^*), H^s(\Omega)\big) \cap W^{1,\infty}_{\mathrm{loc}}\big([0, T^*), H^{s-1}(\Omega)\big),$$

and

$$\int \rho_0^N \, dx = \int \rho_0 \, dx.$$

Note however that $\tilde{V}_0$ (and consequently $\tilde{V}_0^N = J_N \tilde{V}_0$) is not an approximate solution to (3.18) in the sense of (3.16). We have indeed

$$\begin{aligned}
&\partial_t \tilde{V}_0^N + \mathcal{W}\left(\frac{t}{\epsilon}\right)\mathcal{B}\left(\mathcal{W}\left(-\frac{t}{\epsilon}\right)\tilde{V}_0^N, \mathcal{W}\left(-\frac{t}{\epsilon}\right)\tilde{V}_0^N\right) \\
&\quad = (\mathrm{Id} - J_N)\mathcal{B}_W(\tilde{V}_0, \tilde{V}_0) + \mathcal{B}_W\big(\tilde{V}_0^N - \tilde{V}_0, \tilde{V}_0^N + \tilde{V}_0\big) \\
&\qquad + \mathcal{W}\left(\frac{t}{\epsilon}\right)\mathcal{B}\left(\mathcal{W}\left(-\frac{t}{\epsilon}\right)\tilde{V}_0^N, \mathcal{W}\left(-\frac{t}{\epsilon}\right)\tilde{V}_0^N\right) - \mathcal{B}_W\big(\tilde{V}_0^N, \tilde{V}_0^N\big)
\end{aligned}$$

where the last term is expected to be an oscillating term, that is to converge weakly but not strongly to 0. We therefore need to build the second order approximation.

The second order approximation V_1^N is defined by

$$\tilde{V}_1^N = J_N \sum_{\lambda_{k_1} + \lambda_{k_2} \neq \lambda_k} \frac{\exp(\frac{it}{\epsilon}(\lambda_k - \lambda_{k_1} - \lambda_{k_2}))}{i(\lambda_{k_1} + \lambda_{k_2} - \lambda_k)} \Pi_{\lambda_k}\mathcal{B}\big(\Pi_{\lambda_{k_1}}\tilde{V}_0^N, \Pi_{\lambda_{k_2}}\tilde{V}_0^N\big) \tag{3.24}$$

so that

$$\begin{aligned}
\partial_t\big(\epsilon \tilde{V}_1^N\big) = {} & -J_N\left(\mathcal{W}\left(\frac{t}{\epsilon}\right)\mathcal{B}\left(\mathcal{W}\left(-\frac{t}{\epsilon}\right)\tilde{V}_0^N, \mathcal{W}\left(-\frac{t}{\epsilon}\right)\tilde{V}_0^N\right) - \mathcal{B}_W\big(\tilde{V}_0^N, \tilde{V}_0^N\big)\right) \\
& + 2\epsilon J_N \sum_{\lambda_{k_1} + \lambda_{k_2} \neq \lambda_k} \frac{\exp(\frac{it}{\epsilon}(\lambda_k - \lambda_{k_1} - \lambda_{k_2}))}{i(\lambda_{k_1} + \lambda_{k_2} - \lambda_k)} \Pi_{\lambda_k}\mathcal{B}\big(\Pi_{\lambda_{k_1}}\tilde{V}_0^N, \partial_t \Pi_{\lambda_{k_2}}\tilde{V}_0^N\big).
\end{aligned}$$

From the bounds on $\tilde{V}_0$ (which are clearly inherited by $\tilde{V}_0^N$) and the definition of $\mathcal{B}$, we deduce that, for all $N > 0$ and all $\sigma > 0$

$$\tilde{V}_1^N \text{ in } L^\infty_{\mathrm{loc}}\big([0, T^*), H^\sigma(\Omega)\big)$$

(with an estimate depending on N), and that, for all $T < T^*$

$$\sup_{N \in \mathbf{N}} \lim_{\epsilon \to 0} \big\|\tilde{V}_0^N + \epsilon \tilde{V}_1^N\big\|_{L^\infty([0,T], H^s(\Omega))} \leqslant C\|\tilde{V}_0\|_{L^\infty([0,T^*], H^s(\Omega))}.$$

It remains then to check that $\tilde{V}_0^N + \epsilon \tilde{V}_1^N$ is an approximate solution to (3.18) in a strong sense.

$$\begin{aligned}
&\partial_t\big(\tilde{V}_0^N + \epsilon \tilde{V}_1^N\big) + \mathcal{W}\left(\frac{t}{\epsilon}\right)\mathcal{B}\left(\mathcal{W}\left(-\frac{t}{\epsilon}\right)\big(\tilde{V}_0^N + \epsilon \tilde{V}_1^N\big), \mathcal{W}\left(-\frac{t}{\epsilon}\right)\big(\tilde{V}_0^N + \epsilon \tilde{V}_1^N\big)\right) \\
&\quad = (\mathrm{Id} - J_N)\mathcal{B}_W(\tilde{V}_0, \tilde{V}_0) + \mathcal{B}_W\big(\tilde{V}_0^N - \tilde{V}_0, \tilde{V}_0 + \tilde{V}_0^N\big) \\
&\qquad + (\mathrm{Id} - J_N) \sum_{\lambda_{k_1} + \lambda_{k_2} \neq \lambda_k} \exp\left(i\frac{t}{\epsilon}(\lambda_k - \lambda_{k_1} - \lambda_{k_2})\right)\Pi_{\lambda_k}\mathcal{B}\big(\Pi_{\lambda_{k_1}}\tilde{V}_0^N, \Pi_{\lambda_{k_2}}\tilde{V}_0^N\big)
\end{aligned}$$

$$+\epsilon\mathcal{W}\left(\frac{t}{\epsilon}\right)\mathcal{B}\left(\mathcal{W}\left(-\frac{t}{\epsilon}\right)\tilde V_1^N,\mathcal{W}\left(-\frac{t}{\epsilon}\right)\left(2\tilde V_0^N+\epsilon\tilde V_1^N\right)\right)$$
$$+2\epsilon J_N\sum_{\lambda_{k_1}+\lambda_{k_2}\neq\lambda_k}\frac{\exp(-\frac{it}{\epsilon}(\lambda_{k_1}+\lambda_{k_2}-\lambda_k))}{i(\lambda_k-\lambda_{k_1}-\lambda_{k_2})}\Pi_{\lambda_k}\mathcal{B}\left(\Pi_{\lambda_{k_1}}\partial_t\tilde V_0^N,\Pi_{\lambda_{k_2}}\tilde V_0^N\right). \tag{3.25}$$

From the uniform bound on $\tilde V_0^N$ and the convergence $\tilde V_0^N\to\tilde V_0$ we deduce that the first three terms in the right-hand side of the previous identity go to zero as $N\to\infty$. For fixed N, using the (nonuniform) estimates on $\tilde V_0^N$ and $\tilde V_1^N$ and the bound from below on $|\lambda_k-\lambda_{k_1}-\lambda_{k_2}|$, we obtain that the three other terms go to zero as $\epsilon\to 0$. We conclude that

$$\partial_t\left(\tilde V_0^N+\epsilon\tilde V_1^N\right)+\mathcal{W}\left(\frac{t}{\epsilon}\right)\mathcal{B}\left(\mathcal{W}\left(-\frac{t}{\epsilon}\right)\left(\tilde V_0^N+\epsilon\tilde V_1^N\right),\mathcal{W}\left(-\frac{t}{\epsilon}\right)\left(\tilde V_0^N+\epsilon\tilde V_1^N\right)\right)$$

goes to zero in $L^\infty_{\mathrm{loc}}([0,T^*),H^{s-1}(\Omega))$ as $\epsilon\to 0$ then $N\to\infty$.

In other words $(\tilde V_0^N+\epsilon\tilde V_1^N)$ is an approximate solution to (3.18) in the sense of (3.16). However it does not satisfy the (approximate) global conservation of mass (3.17).

Indeed we have proved that the oscillating modes have zero total mass (see (3.23)) and that the equation governing the nonoscillating part of $\tilde V_0$ is conservative, from which we deduce that the first three terms in the right-hand side of (3.25) have no contribution to the variation of the total mass. But the last two terms are expected to give some variation of the total mass of order ϵ, which is not admissible for (3.17) to hold.

We therefore need to build some third order approximation.

The third order approximation $\tilde V_2^N$ is then defined by

$$\tilde V_2^N=\sum_{k\in\mathbf{Z}^*}\frac{\exp(-i\frac{t}{\epsilon}\lambda_k)}{i\lambda_k}\Pi_0\mathcal{B}\left(\Pi_{\lambda_k}\tilde V_1^N,2\Pi_0\tilde V_0^N\right)+\sum_{k\in\mathbf{Z}^*}\frac{\exp(-i\frac{t}{\epsilon}\lambda_k)}{i\lambda_k}\Pi_0\mathcal{B}\left(\Pi_0\tilde V_1^N,2\Pi_{\lambda_k}\tilde V_0^N\right)$$
$$-2\sum_{\lambda_k\neq 0}\frac{\exp(-\frac{it}{\epsilon}\lambda_k)}{(\lambda_k)^2}\Pi_0\mathcal{B}\left(\Pi_{\lambda_k}\partial_t\tilde V_0^N,\Pi_0\tilde V_0^N\right)-2\sum_{\lambda_k\neq 0}\frac{\exp(-\frac{it}{\epsilon}\lambda_k)}{(\lambda_k)^2}\Pi_0\mathcal{B}\left(\Pi_0\partial_t\tilde V_0^N,\Pi_{\lambda_k}\tilde V_0^N\right). \tag{3.26}$$

From the bounds on $\tilde V_0^N$ and $\tilde V_1^N$, we get the required regularity estimate on $\tilde V_2^N$ (depending on N).

The same type of computations as (3.25) shows that, for fixed N,

$$\partial_t\left(\epsilon^2\tilde V_2^N\right)=\mathrm{O}(\epsilon)_{L^\infty_{\mathrm{loc}}([0,T^*),H^{s-1}(\Omega))}$$

so that $\tilde V_\epsilon^N=\tilde V_0^N+\epsilon\tilde V_1^N+\epsilon^2\tilde V_2^N$ is an approximate solution to (3.18) in the sense of (3.16).

It remains then to check that the total mass is approximatively conserved in the sense of (3.17). Let us first recall that the only contribution to the total mass comes from the nonoscillating part. We can prove that

$$\partial_t\Pi_0\tilde V_\epsilon^N+\mathcal{W}\left(\frac{t}{\epsilon}\right)\mathcal{B}\left(\mathcal{W}\left(-\frac{t}{\epsilon}\right)\tilde V_\epsilon^N,\mathcal{W}\left(-\frac{t}{\epsilon}\right)\tilde V_\epsilon^N\right)$$
$$=(\Pi_0-\Pi_{0,N})\mathcal{B}_W(\tilde V_0,\tilde V_0)+\Pi_0\mathcal{B}_W\left(\tilde V_0^N-\tilde V_0,\tilde V_0+\tilde V_0^N\right)$$
$$+(\Pi_0-\Pi_{0,N})\sum_{\lambda_{k_1}+\lambda_{k_2}\neq\lambda_k}\exp\left(i\frac{t}{\epsilon}(\lambda_k-\lambda_{k_1}-\lambda_{k_2})\right)\Pi_{\lambda_k}\mathcal{B}\left(\Pi_{\lambda_{k_1}}\tilde V_0^N,\Pi_{\lambda_{k_2}}\tilde V_0^N\right)+\mathrm{O}\left(\epsilon^2\right)$$

using the fact (see (3.21)) that for $\lambda_{k_1}\neq 0$ and $\lambda_{k_2}\neq 0$,

$$\Pi_0\mathcal{B}\left(\Pi_{\lambda_{k_1}}\tilde V_0^N,\Pi_{\lambda_{k_2}}\tilde V_0^N\right)=0.$$

Integrating the first component with respect to x, we obtain that the total mass is conserved up to order 2.

In particular, denoting $(\rho_\epsilon^N,u_\epsilon^N,\theta_\epsilon^N)=\mathcal{W}(-\frac{t}{\epsilon})\tilde V_\epsilon^N$, we get

$$\frac{1}{\epsilon^2}\int\partial_t\exp\left(\epsilon\rho_\epsilon^N\right)dx=\frac{1}{\epsilon}\int\exp\left(\epsilon\rho_\epsilon^N\right)\left(\partial_t\rho_\epsilon^N+u_\epsilon^N\cdot\nabla_x\rho_\epsilon^N+\frac{1}{\epsilon}\nabla_x\cdot u_\epsilon^N\right)dx$$
$$=\frac{1}{\epsilon}\int\left(\partial_t\rho_\epsilon^N+u_\epsilon^N\cdot\nabla_x\rho_\epsilon^N+\frac{1}{\epsilon}\nabla_x\cdot u_\epsilon^N\right)dx$$

$$+\int \frac{\exp(\epsilon\rho_\epsilon^N)-1}{\epsilon}\left(\partial_t\rho_\epsilon^N + u_\epsilon^N\cdot\nabla_x\rho_\epsilon^N + \frac{1}{\epsilon}\nabla_x\cdot u_\epsilon^N\right)dx$$

which tends to zero as $\epsilon\to 0$ for any fixed N. □

Remark 3.4. In the case when $\Omega=\mathbf{R}^3$, the construction of the sequence of approximate solutions is actually much simpler. The point is to see that the acoustic waves generated by the linear penalization satisfy some dispersion property, and consequently converge strongly to zero by Strichartz estimates.

In particular, one does not need to consider precisely the coupling resulting from the nonlinearity. Using Duhamel's formula, one can indeed prove that the corresponding contribution converges strongly to zero. For a detailed study regarding that case, we refer for instance to [12].

Combining that result with our general method of proof (Proposition 3.1 and final arguments in Section 3.3) should lead to the following convergence statement in $\mathbf{R}^3$: the family of fluctuations (g_ϵ) defined by $f_\epsilon = M(1+\epsilon g_\epsilon)$ converges entropically on $]0,T^*[$ to the infinitesimal Maxwellian

$$g = u\cdot v + \theta\left(\frac{|v|^2-5}{2}\right),$$

where (u,θ) is the Lipschitz solution on $[0,T^*[$ to the incompressible Euler equations (2.4).

3.3. Convergence proof

The previous results on the sequence of approximate solutions to

$$A_\epsilon(\rho,u,\theta)=0$$

should allow to control the acceleration term in the relative entropy inequality (3.2). In order to obtain a stability inequality, we need then to obtain some control on the flux term.

Lemma 3.5. *Let (f_ϵ) be some solution to the scaled Boltzmann equation* (2.2) *satisfying* (2.3). *Denote by Φ_ϵ and Ψ_ϵ the kinetic momentum and energy fluxes defined by* (3.9).

Then the flux term occurring in the modulated entropy inequality can be controlled by both the modulated entropy and the entropy dissipation:

$$-\frac{1}{\epsilon^2}\int_0^t\iint f_\epsilon\nabla_x u:\Phi_\epsilon\,dx\,dv\,ds - \frac{1}{\epsilon^2}\int_0^t\iint f_\epsilon e^{\frac12\epsilon\theta}\nabla_x\theta\cdot\Psi_\epsilon\,dx\,dv\,ds$$
$$\leqslant \frac{C}{\epsilon^2}\int_0^t \left\|D_x(u,\theta)(s)\right\|_{L^2\cap L^\infty(\Omega)} H(f_\epsilon|f_{app})(s)\,ds + \mathrm{o}(1) \tag{3.27}$$

where the constant C depends only on the L^∞ norm of (ρ,u,θ).

Proof. Such an estimate relies on some suitable decomposition of the flux term which is very similar to Chapman–Enskog's expansion, and which is also used to study the weak Navier–Stokes limit of the Boltzmann equation (see [15]).

• More precisely we introduce the renormalized fluctuation

$$\hat g_\epsilon = \frac{2}{\epsilon}\left(\frac{\sqrt{f_\epsilon}-\sqrt{f_{app}}}{\sqrt{f_{app}}}\right)$$

so that

$$f_\epsilon = f_{app}\left(1+\epsilon\hat g_\epsilon + \frac{\epsilon^2}{4}\hat g_\epsilon^2\right). \tag{3.28}$$

The interest in considering such a renormalized fluctuation is that its L^2 norm is controlled by the modulated entropy:

$$\iint f_{app}\hat g_\epsilon^2\,dv\,dx \leqslant \frac{2}{\epsilon^2}H(f_\epsilon|f_{app}).$$

It therefore makes sense to use the Hilbertian theory of the linearized collision operator $\mathcal{L}_\epsilon$ defined by

$$\mathcal{L}_\epsilon g = \int f_{app1}(g + g_1 - g' - g_1')b(v - v_1, \omega)\,dv_1\,d\omega.$$

In particular, $\mathcal{L}_\epsilon$ is known to satisfy the Fredholm alternative, from which we deduce that there exists $\tilde{\Phi}_\epsilon$ and $\tilde{\Psi}_\epsilon$ in $(\operatorname{Ker}\mathcal{L}_\epsilon)^\perp$ such that

$$\mathcal{L}_\epsilon\tilde{\Phi}_\epsilon = \Phi_\epsilon, \qquad \mathcal{L}_\epsilon\tilde{\Psi}_\epsilon = \Psi_\epsilon.$$

One has therefore

$$\begin{aligned}\frac{1}{\epsilon}\int f_{app}\hat{g}_\epsilon\Phi_\epsilon\,dv &= \int f_{app}\tilde{\Phi}_\epsilon\frac{1}{\epsilon}\mathcal{L}_\epsilon\hat{g}_\epsilon\,dv\\ &= -\frac{1}{2}\iiint f_{app}f_{\mathrm{app}1}\tilde{\Phi}_\epsilon(\hat{g}_\epsilon\hat{g}_{\epsilon1} - \hat{g}_\epsilon'\hat{g}_{\epsilon1}')b(v - v_1, \omega)\,d\omega\,dv_1\,dv\\ &\quad - \frac{2}{\epsilon^2}\iiint\tilde{\Phi}_\epsilon\sqrt{f_{app}f_{app1}}\bigl(\sqrt{f_\epsilon'f_{\epsilon1}'} - \sqrt{f_\epsilon f_{\epsilon1}}\bigr)b(v - v_1, \omega)\,d\omega\,dv_1\,dv\end{aligned}$$

where the last term can be controlled by the entropy dissipation.

The suitable decomposition is more precisely

$$\begin{aligned}\frac{1}{\epsilon^2}\int f_\epsilon\Phi_\epsilon\,dv &= \frac{1}{4}\int f_{app}\Phi_\epsilon\hat{g}_\epsilon^2\,dv - \frac{1}{2}\iiint f_{app}f_{app1}\tilde{\Phi}_\epsilon(\hat{g}_\epsilon\hat{g}_{\epsilon1} - \hat{g}_\epsilon'\hat{g}_{\epsilon1}')b\,d\omega\,dv_1\,dv\\ &\quad - \frac{2}{\epsilon^2}\iiint\tilde{\Phi}_\epsilon\sqrt{f_{app}f_{app1}}\bigl(\sqrt{f_\epsilon'f_{\epsilon1}'} - \sqrt{f_\epsilon f_{\epsilon1}}\bigr)b\,d\omega\,dv_1\,dv\end{aligned}\tag{3.29}$$

and similarly

$$\begin{aligned}\frac{1}{\epsilon^2}\int f_\epsilon\Psi_\epsilon\,dv &= \frac{1}{4}\int f_{app}\Psi_\epsilon\hat{g}_\epsilon^2\,dv - \frac{1}{2}\iiint f_{app}f_{app1}\tilde{\Psi}_\epsilon(\hat{g}_\epsilon\hat{g}_{\epsilon1} - \hat{g}_\epsilon'\hat{g}_{\epsilon1}')b\,d\omega\,dv_1\,dv\\ &\quad - \frac{2}{\epsilon^2}\iiint\tilde{\Psi}_\epsilon\sqrt{f_{app}f_{app1}}\bigl(\sqrt{f_\epsilon'f_{\epsilon1}'} - \sqrt{f_\epsilon f_{\epsilon1}}\bigr)b\,d\omega\,dv_1\,dv.\end{aligned}\tag{3.30}$$

• In both decompositions (3.29) and (3.30) the last term is controlled by the entropy dissipation in the following way.

From the functional inequality $(z - y)\log\frac{z}{y} \geqslant 4(\sqrt{z} - \sqrt{y})^2$, we deduce that

$$\iiint f_{app}f_{app1}\biggl(\sqrt{\frac{f_\epsilon'f_{\epsilon1}'}{f_{app}'f_{app1}'}} - \sqrt{\frac{f_\epsilon f_{\epsilon1}}{f_{app}f_{app1}}}\biggr)^2 b\,dv\,dv_1\,d\omega \leqslant D(f_\epsilon).$$

Thus, by the Cauchy–Schwarz inequality,

$$\int f_{app}\biggl(\iint f_{app1}\biggl(\sqrt{\frac{f_\epsilon'f_{\epsilon1}'}{f_{app}'f_{app1}'}} - \sqrt{\frac{f_\epsilon f_{\epsilon1}}{f_{app}f_{app1}}}\biggr)b\,dv_1\,d\omega\biggr)^2\nu_\epsilon^{-1}\,dv \leqslant D(f_\epsilon)$$

denoting as usual by ν_ϵ the collision frequency defined by

$$\nu_\epsilon = \int f_{app1}b\,dv_1\,d\omega.$$

The classical theory of the linearized Boltzmann operator $\mathcal{L}_\epsilon$ established by Grad [16] implies on the other hand that

$$\begin{aligned}&\int f_{app}(\tilde{\Phi}_\epsilon)^2\nu_\epsilon\,dv \leqslant C\int f_{app}\bigl(\Phi_\epsilon^2\bigr)\,dv,\\ &\int f_{app}(\tilde{\Psi}_\epsilon)^2\nu_\epsilon\,dv \leqslant C\int f_{app}\bigl(\Psi_\epsilon^2\bigr)\,dv\end{aligned}$$

(see Appendix B.1 for a review of the main results).

Then

$$-\frac{2}{\epsilon^2}\iiint \tilde\Phi_\epsilon\sqrt{f_{app}f_{app1}}\big(\sqrt{f'_\epsilon f'_{\epsilon 1}}-\sqrt{f_\epsilon f_{\epsilon 1}}\big)b\,d\omega\,dv_1\,dv = \mathrm{O}\big(\epsilon^{(q-1)/2}\big)_{L^2(dt\,dx)},$$

$$-\frac{2}{\epsilon^2}\iiint \tilde\Psi_\epsilon\sqrt{f_{app}f_{app1}}\big(\sqrt{f'_\epsilon f'_{\epsilon 1}}-\sqrt{f_\epsilon f_{\epsilon 1}}\big)b\,d\omega\,dv_1\,dv = \mathrm{O}\big(\epsilon^{(q-1)/2}\big)_{L^2(dt\,dx)}.$$

• The other terms are expected to be controlled by the modulated entropy since

$$\int f_{app}\hat g_\epsilon^2\,dx\,dv \leqslant \frac{2}{\epsilon^2}H(f_\epsilon|f_{app}).$$

Nevertheless, as $\Psi_\epsilon = O(|v|^3)$ as $|v|\to\infty$, we need some additional integrability with respect to the v-variable.

The main idea to obtain such refined estimates is to use the control on the relaxation, and more precisely the decomposition

$$\hat g_\epsilon = \Pi_\epsilon\hat g_\epsilon + (\hat g_\epsilon - \Pi_\epsilon\hat g_\epsilon)$$

where Π_ϵ is the projection on hydrodynamic modes, i.e. on $\operatorname{Ker}\mathcal{L}_\epsilon$. The explicit formula for Π_ϵ shows that

$$\|\Pi_\epsilon\hat g_\epsilon\|^2_{L^2(dx,L^p(f_{app}\,dv))} \leqslant \frac{C_p}{\epsilon^2}H(f_\epsilon|f_{app}) \tag{3.31}$$

for all $p<+\infty$.

On the other hand, using a truncated cross-section, we can obtain a control on the purely kinetic part $\Pi_\epsilon^\perp\hat g_\epsilon = (\hat g_\epsilon - \Pi_\epsilon\hat g_\epsilon)$. Let $\bar{\mathcal{L}}_\epsilon$ be the linear collision operator associated with the truncated cross-section $\bar b$ defined by

$$\bar b(v-v_1,\omega) = \frac{b(v-v_1,\omega)}{1+\int f_{app1}b(v-v_1,\omega)\,dv_1\,d\omega},$$

so that

$$\bar\nu_\epsilon = \int f_{app1}\bar b\,dv_1\,d\omega \leqslant 1.$$

Then start from the identity

$$\begin{aligned}\frac{1}{\epsilon}\bar{\mathcal{L}}_\epsilon\hat g_\epsilon = {}& -\frac12\iint f_{app1}(\hat g_\epsilon\hat g_{\epsilon1}-\hat g'_\epsilon\hat g'_{\epsilon1})\bar b(v-v_1,\omega)\,d\omega\,dv_1\\ &-\frac{2}{\epsilon^2}\iint\sqrt{\frac{f_{app1}}{f_{app}}}\big(\sqrt{f'_\epsilon f'_{\epsilon1}}-\sqrt{f_\epsilon f_{\epsilon1}}\big)\bar b(v-v_1,\omega)\,d\omega\,dv_1.\end{aligned}$$

Using the fact that $\bar{\mathcal{L}}_\epsilon$ is coercive on $(\operatorname{Ker}\bar{\mathcal{L}}_\epsilon)^\perp$, we easily obtain that

$$\Pi_\epsilon^\perp\hat g_\epsilon = \mathrm{O}\bigg(\epsilon^{(q+1)/2}\sqrt{\frac{1}{\epsilon^{q+3}}D(f_\epsilon)}\bigg)_{L^2(f_{app}\,dv)} + \mathrm{O}\bigg(\epsilon\frac{1}{\epsilon^2}H(f_\epsilon|f_{app})\bigg)_{L^1(dx,L^2(f_{app}\,dv))}. \tag{3.32}$$

Using the control of large tails (2.3), we get for all $p<1$

$$\begin{aligned}\int f_{app}\bigg(\frac{f_\epsilon}{f_{app}}\bigg)^{2p}dv &\leqslant \bigg(\int\frac{f_\epsilon^2}{M}\,dv\bigg)^p\bigg(\int\bigg(\frac{M^{2p}}{f_{app}^{2p-1}}\bigg)^{1/(1-p)}dv\bigg)^{1-p}\\ &\leqslant C_p\end{aligned}$$

provided that ϵ is sufficiently small (depending on p), since the moments of f_{app} differ from those of M by quantities of order ϵ. We then have

$$\epsilon\hat g_\epsilon = \mathrm{O}(1)_{L^\infty_{t,x}(L^{4p}(f_{app}\,dv))}\quad\text{and thus}\quad \epsilon\Pi_\epsilon^\perp\hat g_\epsilon = \mathrm{O}(1)_{L^\infty_{t,x}(L^{4p}(f_{app}\,dv))}. \tag{3.33}$$

Coupling this last estimate with (3.32) leads finally to

$$\big(\Pi_\epsilon^\perp\hat g_\epsilon\big)^2 = \mathrm{O}\bigg(\epsilon^{(q-1)/2}\sqrt{\frac{1}{\epsilon^{q+3}}D(f_\epsilon)}\bigg)_{L^{4p/(2p+1)}(f_{app}\,dv)} + \mathrm{O}\bigg(\frac{1}{\epsilon^2}H(f_\epsilon|f_{app})\bigg)_{L^1(dx,L^{4p/(2p+1)}(f_{app}\,dv))}. \tag{3.34}$$

Using (3.31)–(3.34) to control the first and second terms in decompositions (3.29) and (3.30) leads to the announced estimate of the fluxes. □

Remark 3.6. Note that the control of large tails (2.3) is a very strong assumption which is not really necessary to conclude. It indeed provides an $L^q(f_{app}\,dv)$ $(q>2)$ estimate on $\hat{g}_\epsilon$, while a weighted $L^2((1+|v|^3)f_{app}\,dv)$ should be sufficient.

Proof of Theorem 2.1. Equipped with these preliminary results, we are now able to achieve the proof of Theorem 2.1 in the case when the initial velocity profile is close to local thermodynamic equilibrium, i.e. when

$$\frac{1}{\epsilon^2}H\big(f_\epsilon^{in}\big|\mathcal{M}_{\exp(\epsilon\rho^{in}),\epsilon u^{in},\exp(\epsilon\theta_{in})}\big)\to 0. \tag{3.35}$$

Denote as previously by f_{app} the fluctuation of Maxwellian defined by

$$\log f_{app}=-\frac{3}{2}\log(2\pi)+\epsilon\left(\rho-\frac{3}{2}\theta\right)-\frac{1}{2}e^{-\epsilon\theta}|v-\epsilon u|^2.$$

Combining Proposition 3.1 and Lemma 3.5 leads to the following inequality

$$\begin{aligned}
&\frac{1}{\epsilon^2}H(f_\epsilon|f_{app})(t)+\frac{1}{\epsilon^{q+3}}\iint_0^t D(f_\epsilon)\,ds\,dx\leqslant\frac{1}{\epsilon^2}H\big(f_\epsilon^{in}\big|f_{app}^{in}\big)+\frac{1}{\epsilon^2}\iint_0^t\partial_t\exp(\epsilon\rho)\,dx\,ds\\
&\quad-\frac{1}{\epsilon}\iiint_0^t f_\epsilon\left(1,e^{-\epsilon\theta}(v-\epsilon u),\frac{1}{2}\big(e^{-\epsilon\theta}|v-\epsilon u|^2-3\big)\right)\cdot A_\epsilon(\rho,u,\theta)\,dv\,dx\,ds\\
&\quad+\frac{C}{\epsilon^2}\int_0^t\big\|D_x(u,\theta)(s)\big\|_{L^2\cap L^\infty(\Omega)}H(f_\epsilon|f_{app})(s)\,ds+\mathrm{o}(1).
\end{aligned}$$

Integrating next this differential inequality leads to

$$\begin{aligned}
&\frac{1}{\epsilon^2}H(f_\epsilon|f_{app})(t)+\frac{1}{\epsilon^{q+3}}\iint_0^t D(f_\epsilon)\,dv\,dx\,ds\\
&\quad\leqslant\frac{1}{\epsilon^2}H\big(f_\epsilon^{in}\big|f_{app}^{in}\big)\exp\left(C\int_0^t\big\|D_x(u,\theta)(s)\big\|_{L^2\cap L^\infty(\Omega)}\,ds\right)+\mathrm{o}(1)\\
&\quad+\frac{1}{\epsilon^2}\int_0^t\exp\left(C\int_s^t\big\|D_x(u,\theta)(\tau)\big\|_{L^2\cap L^\infty(\Omega)}\,d\tau\right)\int\partial_t\exp(\epsilon\rho)\,dx\,ds\\
&\quad-\frac{1}{\epsilon}\int_0^t\exp\left(C\int_s^t\big\|D_x(u,\theta)(\tau)\big\|_{L^2\cap L^\infty(\Omega)}\,d\tau\right)\\
&\quad\times\iint f_\epsilon\left(1,e^{-\epsilon\theta}(v-\epsilon u),\frac{1}{2}\big(e^{-\epsilon\theta}|v-\epsilon u|^2-3\big)\right)\cdot A_\epsilon(\rho,u,\theta)\,dv\,dx\,ds.
\end{aligned}\tag{3.36}$$

Plugging the approximate solution $(\rho_\epsilon^N,u_\epsilon^N,\theta_\epsilon^N)$ built in Proposition 3.3 in Gronwall's inequality (3.36) leads then to

$$\frac{1}{\epsilon^2}H(f_\epsilon|f_{app})\to 0\quad\text{in }L^\infty_{\mathrm{loc}}\big([0,T^*)\big). \tag{3.37}$$

Indeed the first term in the right-hand side is proved to converge to 0 as $\epsilon\to 0$ then $N\to\infty$ using the identity

$$H\big(f_\epsilon^{in}\big|f_{app}^{in}\big) = H\big(f_\epsilon^{in}\big|\mathcal{M}_{\exp(\epsilon\rho^{in}),\epsilon u^{in},\exp(\epsilon\theta_{in})}\big) + H\big(\mathcal{M}_{\exp(\epsilon\rho^{in}),\epsilon u^{in},\exp(\epsilon\theta_{in})}\big|f_{app}^{in}\big) + \iint \big(f_\epsilon^{in} - \mathcal{M}_{\exp(\epsilon\rho^{in}),\epsilon u^{in},\exp(\epsilon\theta_{in})}\big)\log\frac{\mathcal{M}_{\exp(\epsilon\rho^{in}),\epsilon u^{in},\exp(\epsilon\theta_{in})}}{f_{app}^{in}}\,dv\,dx$$

together with the assumption on the initial data (3.35) and the convergence statement (3.15)

$$\big(\rho_\epsilon^{in,N}, u_\epsilon^{in,N}, \theta_\epsilon^{in,N}\big) \to \big(\rho^{in}, u^{in}, \theta^{in}\big)$$

which implies in particular that

$$\frac{1}{\epsilon^2}H\big(\mathcal{M}_{\exp(\epsilon\rho^{in}),\epsilon u^{in},\exp(\epsilon\theta_{in})}\big|f_{app}^{in}\big) \to 0.$$

The convergence of the two other terms in the right-hand side of (3.36) is obtained by combining the uniform bound (with respect to ϵ and N)

$$\big\|D_x\big(u_\epsilon^N, \theta_\epsilon^N\big)\big\|_{L^\infty([0,T],L^2\cap L^\infty(\Omega))} \leqslant C_T$$

with the convergence statements (3.16) and (3.17)

$$\frac{1}{\epsilon^2}\int \partial_t \exp\big(\epsilon\rho_\epsilon^N\big)\,dx \to 0, \qquad A_\epsilon\big(\rho_\epsilon^N, u_\epsilon^N, \theta_\epsilon^N\big) \to 0.$$

The entropic convergence (3.37) implies the strong convergence of the fluctuation $\frac{1}{\epsilon}(f_\epsilon - f_{app})$ in $L^\infty_{\rm loc}([0,T^*), L^1(dx\,dv))$, in particular the convergence result stated in the theorem. Indeed we have seen that the L^2 norm of

$$\sqrt{f_{app}}\hat{g}_\epsilon = \frac{2}{\epsilon}(\sqrt{f_\epsilon} - \sqrt{f_{app}})$$

is bounded by the square-root of the modulated entropy $\frac{1}{\epsilon^2}H(f_\epsilon|f_{app})$. Thus, using the decomposition (3.28), we get

$$g_\epsilon - \frac{1}{\epsilon}\frac{f_{app} - M}{M} \to 0 \quad \text{in } L^\infty_{\rm loc}\big([0,T^*), L^1(dx\,dv)\big)$$

as $\epsilon \to 0$ then $N \to \infty$, and finally

$$g_\epsilon - g - g_{osc} \to 0 \quad \text{in } L^\infty_{\rm loc}\big([0,T^*), L^1(dx\,dv)\big)$$

as $\epsilon \to 0$. □

Remark 3.7. As mentioned in Remark 3.4, the final argument presented in this paragraph can be extended to the case of unbounded domains – replacing $L^1(dx)$ by $L^1_{\rm loc}(dx)$.

4. Taking into account the Knudsen layer

4.1. The modulated entropy inequality

For general initial data, the purely kinetic part of the solution to the Boltzmann equation is expected to converge to 0 exponentially in time, in particular in $L^1_{\rm loc}([0,T^*)\times\Omega\times\mathbf{R}^3)$, but not in $L^\infty_{\rm loc}([0,T^*), L^1_{\rm loc}(\Omega\times\mathbf{R}^3))$. In order to consider the relaxation process in the relative entropy method, one thus has to construct a refined approximation f_{app}, and then to introduce it in the modulated entropy inequality (3.2). This requires in particular to also modulate the entropy dissipation.

Proposition 4.1. *Denote by f_{app} some smooth function on $[0,T]\times\Omega\times\mathbf{R}^3$ satisfying the boundary condition*

$$f_{app}(t,x,v) = f_{app}\big(t,x,v - 2\big(v\cdot n(x)\big)n(x)\big) \quad \text{on } \Sigma^-.$$

Then, the solution to the scaled Boltzmann equation (2.2) *satisfies the following modulated entropy inequality*

$$
\begin{aligned}
&H(f_\epsilon|f_{app})(t)+\frac{1}{\epsilon^{q+1}}\iint_0^t D(f_\epsilon|f_{app})\,ds\,dx\leqslant H\big(f_\epsilon^{in}\big|f_{app}^{in}\big)\\
&\quad-\epsilon\int_0^t\!\!\iiint g_\epsilon\left(\partial_t f_{app}+\frac{1}{\epsilon}v\cdot\nabla_x f_{app}-\frac{1}{\epsilon^{q+1}}Q(f_{app},f_{app})\right)dv\,dx\,ds\\
&\quad+\frac{1}{4\epsilon^{q-1}}\int_0^t\!\!\iiiint (g_\epsilon g_{\epsilon 1}-g'_\epsilon g'_{\epsilon 1})(f'_{app}f'_{app1}-f_{app}f_{app1})b\,dv\,dv_1\,d\omega\,dx\,ds
\end{aligned}
\tag{4.1}
$$

where g_ϵ denotes the fluctuation

$$
g_\epsilon=\frac{1}{\epsilon}\frac{f_\epsilon-f_{app}}{f_{app}},
$$

and $D(f_\epsilon|f_{app})$ is the modulated entropy dissipation defined by

$$
\begin{aligned}
D(f_\epsilon|f_{app})=\frac{1}{4}\iint &(f'_\epsilon f'_{\epsilon 1}-f_\epsilon f_{\epsilon 1})\log\left(\frac{f'_\epsilon f'_{\epsilon 1}f_{app}f_{app1}}{f_\epsilon f_{\epsilon 1}f'_{app}f'_{app1}}\right)\\
&-(f'_{app}f'_{app1}-f_{app}f_{app1})\left(\frac{f'_\epsilon f'_{\epsilon 1}}{f'_{app}f'_{app1}}-\frac{f_\epsilon f_{\epsilon 1}}{f_{app}f_{app1}}\right)b\,dv\,dv_1\,d\omega.
\end{aligned}
\tag{4.2}
$$

Remark 4.2. Note that the integrand arising in the definition of the modulated entropy dissipation is always nonnegative, which is crucial to get some stability. We have indeed

$$
\begin{aligned}
D(f_\epsilon|f_{app})=\frac{1}{4}\iiint f_\epsilon f_{\epsilon 1}\Bigg(&k\left(\frac{f'_\epsilon f'_{\epsilon 1}}{f_\epsilon f_{\epsilon 1}}\right)-k\left(\frac{f'_{app}f'_{app1}}{f_{app}f_{app1}}\right)\\
&-\left(\frac{f'_\epsilon f'_{\epsilon 1}}{f_\epsilon f_{\epsilon 1}}-\frac{f'_{app}f'_{app1}}{f_{app}f_{app1}}\right)k'\left(\frac{f'_{app}f'_{app1}}{f_{app}f_{app1}}\right)\Bigg)b\,dv\,dv_1\,d\omega
\end{aligned}
$$

where k is the convex function defined by $k(z)=(z-1)\log z$.

This has naturally to be compared with the definition (1.1) of the modulated entropy

$$
H(f_\epsilon|f_{app})=\iint f_{app}\big(h(f_\epsilon-1)-h(f_{app}-1)-(f_\epsilon-f_{app})h'(f_{app}-1)\big)\,dv\,dx
$$

with $h(z)=(1+z)\log(1+z)-z$.

Proof. Start from the entropy inequality (3.3) satisfied by the solution of the scaled Boltzmann equation with specular reflection at the boundary:

$$
H\big(f_\epsilon(t)\big|M\big)+\frac{1}{\epsilon^{q+1}}\iint_0^t D(f_\epsilon)(s,x)\,ds\,dx\leqslant H\big(f_\epsilon^{in}\big|M\big).
$$

By definition of the modulated entropy (1.1), we then have

$$
\begin{aligned}
&H(f_\epsilon|f_{app})(t)+\frac{1}{\epsilon^{q+1}}\iint_0^t D(f_\epsilon)\,ds\,dx\\
&\quad\leqslant H\big(f_\epsilon^{in}\big|f_{app}^{in}\big)-\int_0^t\!\!\iiint \log f_{app}\left(\partial_t+\frac{1}{\epsilon}v\cdot\nabla_x\right)f_\epsilon\,dv\,dx\,ds\\
&\qquad-\int_0^t\!\!\iiint\left(\frac{f_\epsilon}{f_{app}}-1\right)\left(\partial_t+\frac{1}{\epsilon}v\cdot\nabla_x\right)f_{app}\,dv\,dx\,ds
\end{aligned}
\tag{4.3}
$$

using the specular reflection on the boundary for f_ϵ and f_{app}.

Now, for solutions of the Boltzmann equation satisfying (2.3), the collision term $Q(f_\epsilon, f_\epsilon)$ makes sense, and the kinetic equation (2.2) holds in the sense of distributions. We thus have

$$-\int_0^t\iint \log f_{app}\left(\partial_t + \frac{1}{\epsilon}v\cdot\nabla_x\right)f_\epsilon\, dv\, dx\, ds$$
$$= -\frac{1}{\epsilon^{q+1}}\int_0^t\iint (\log f_{app})Q(f_\epsilon, f_\epsilon)\, dv\, dx\, ds$$
$$= -\frac{1}{4\epsilon^{q+1}}\int_0^t\iiiint (f'_\epsilon f'_{\epsilon 1} - f_\epsilon f_{\epsilon 1})\log\left(\frac{f_{app}f_{app1}}{f'_{app}f'_{app1}}\right)b\, dv\, dv_1\, d\omega\, dx\, ds$$

using the classical symmetries of the collision integrand.

In the same way, we have

$$-\int_0^t\iint\left(\frac{f_\epsilon}{f_{app}}-1\right)\left(\partial_t + \frac{1}{\epsilon}v\cdot\nabla_x\right)f_{app}\, dv\, dx\, ds$$
$$= -\int_0^t\iint \frac{f_\epsilon - f_{app}}{f_{app}}\left(\partial_t f_{app} + \frac{1}{\epsilon}v\cdot\nabla_x f_{app} - \frac{1}{\epsilon^{q+1}}Q(f_{app}, f_{app})\right)dv\, dx\, ds$$
$$+ \frac{1}{4\epsilon^{q+1}}\int_0^t\iiiint (f'_{app}f'_{app1} - f_{app}f_{app1})\left(\frac{f'_\epsilon}{f'_{app}} + \frac{f'_{\epsilon 1}}{f'_{app1}} - \frac{f_\epsilon}{f_{app}} - \frac{f_{\epsilon 1}}{f_{app1}}\right)b\, dv\, dv_1\, d\omega\, dx\, ds.$$

Plugging both identities in (4.3) leads then to

$$H(f_\epsilon|f_{app})(t) + \frac{1}{\epsilon^{q+1}}\int_0^t\int D(f_\epsilon|f_{app})\, ds\, dx \leqslant H\left(f_\epsilon^{in}\middle|f_{app}^{in}\right)$$
$$-\int_0^t\iint \frac{f_\epsilon - f_{app}}{f_{app}}\left(\partial_t f_{app} + \frac{1}{\epsilon}v\cdot\nabla_x f_{app} - \frac{1}{\epsilon^{q+1}}Q(f_{app}, f_{app})\right)dv\, dx\, ds$$
$$+ \frac{1}{4\epsilon^{q+1}}\int_0^t\iiiint (f'_{app}f'_{app1} - f_{app}f_{app1})$$
$$\times\left(\left(\frac{f_\epsilon}{f_{app}}-1\right)\left(\frac{f_{\epsilon 1}}{f_{app1}}-1\right) - \left(\frac{f'_\epsilon}{f'_{app}}-1\right)\left(\frac{f'_{\epsilon 1}}{f'_{app1}}-1\right)\right)b\, dv\, dv_1\, d\omega\, dx\, ds$$

which is the expected inequality. □

4.2. *Construction of the approximate solution*

The next step is to construct a sequence of approximate solutions f_{app} to the family of (homogeneous) relaxation equations

$$\partial_t f = \frac{1}{\epsilon^{q+1}}Q(f, f),$$
$$f(x)_{|t=0} = f^{in}(x). \tag{4.4}$$

More precisely, we will require that

$$\partial_t f - \frac{1}{\epsilon^{q+1}} Q(f, f) \to 0 \quad \text{in as } \epsilon \to 0. \tag{4.5}$$

One of the difficulty here is that we further need to control the dependence with respect to the spatial variable x, to get some uniform bound on

$$\frac{1}{\epsilon^2} \int_0^{\tau_\epsilon} \left\| v \cdot \nabla_x \log f_{app}(s) \right\|_{L^\infty(\Omega, L^{p'}(f_{app}\, dv))} ds \to 0 \quad \text{as } \epsilon \to 0 \tag{4.6}$$

for some τ_ϵ characterizing the size of the initial layer and some $p' > 2$ to be determined later.

More precisely, we have the following

Proposition 4.3. *Let f_ϵ^{in} be some sequence of initial data such that the fluctuations $g_\epsilon^{in} = \frac{1}{\epsilon} \frac{f_\epsilon^{in} - M}{M}$ converge entropically to some $g^{in} \in L^2(M\, dv\, dx)$, i.e. such that*

$$\begin{aligned} &\frac{1}{\epsilon^2} H\big(f_\epsilon^{in} \big| M\big) \to \iint M\big(g^{in}\big)^2 \, dx\, dv, \\ &\frac{1}{\epsilon} \frac{f_\epsilon^{in} - M}{M} \to g^{in} \quad \text{in } L^1_{\mathrm{loc}}(M\, dv\, dx). \end{aligned} \tag{4.7}$$

Then there exists some family f_{app} of nonnegative functions satisfying approximatively the homogeneous Boltzmann equation as $\epsilon \to 0$ then $N \to \infty$

$$\partial_t f_{app} - \frac{1}{\epsilon^{q+1}} Q(f_{app}, f_{app}) \to 0 \quad \text{in } L^2\big(dt\, dx, L^{p'}(f_{app}\, dv)\big) \tag{4.8}$$

with suitable initial data

$$\frac{1}{\epsilon^2} H\big(f_\epsilon^{in} \big| f_{app}^{in}\big) \to 0. \tag{4.9}$$

It furthermore satisfies the relaxation estimate

$$\frac{1}{\epsilon^{q+1}} \int_0^t \left\| \frac{f'_{app} f'_{app1}}{f_{app} f_{app1}} - 1 \right\|_{L^\infty(\Omega, L^{p'}(f_{app} f_{app1} b\, dv\, dv_1\, d\omega))} ds \to 0, \tag{4.10}$$

and the regularity estimate

$$\| v \cdot \nabla_x \log f_{app} \|_{L^2(dx, L^{p'}(f_{app}\, dv))} \leqslant C\epsilon \quad \text{uniformly in time}, \tag{4.11}$$

for some $p' > 4$.

Proof. In order to build the approximate solution we need, we will start from the solution f_ϵ^N of the homogeneous Boltzmann equation (4.4) with some smooth initial data $f_\epsilon^{in,N}$, then truncate the contribution of large velocities to get a bound from below on f_{app}.

Step 1: we have therefore to *choose some suitable initial data $f_\epsilon^{in,N}$*.

In view of Proposition B.2 and the remarks below, in order that f_ϵ^N has good decay properties with respect to v and a smooth dependence with respect to x (which is now a simple parameter), it is enough to impose such conditions on the initial data.

We first define

$$\tilde{f}_\epsilon^{in,N} = \chi_N * f_\epsilon^{in}$$

where χ_N is a regularizing kernel acting on the space variable x. Note that, because of the entropic convergence of the initial data (4.7), for all fixed N,

$$\frac{1}{\epsilon^2} H\big(f_\epsilon^{in} \big| \tilde{f}_\epsilon^{in,N}\big) \to \frac{1}{2} \iint M\big(g^{in} - g^{in,N}\big)^2 \, dv\, dx \quad \text{as } \epsilon \to 0,$$

and thus

$$\frac{1}{\epsilon^2} H\big(f_\epsilon^{in} \big| \tilde{f}_\epsilon^{in,N}\big) \to 0 \quad \text{as } \epsilon \to 0 \text{ then } N \to \infty.$$

We then define $f_\epsilon^{in,N}$ such that

- it coincides with $\tilde{f}_\epsilon^{in,N}$ if $|v| \leqslant N$;
- it coincides with the Maxwellian M_ϵ^N of same moments as $\tilde{f}_\epsilon^{in,N}$ if $|v| \geqslant 2N$;
- it has the same moments as $\tilde{f}_\epsilon^{in,N}$.

We further require that $f_\epsilon^{in,N}$ depends smoothly on x, which can be realized by some suitable fluctuation of the local Maxwellian M_ϵ^N for $N < |v| < 2N$.

Using again the entropic convergence of the initial data (4.7), it is then easy to see that

$$\frac{1}{\epsilon^2} H\big(f_\epsilon^{in} \big| f_\epsilon^{in,N}\big) \to 0 \quad \text{as } \epsilon \to 0 \text{ then } N \to \infty. \tag{4.12}$$

With that choice of $f_\epsilon^{in,N}$, using (4.7), we get

$$\left\| \frac{f_\epsilon^{in,N} - M_\epsilon^N}{M_\epsilon^N} \right\|_{\alpha,r} \leqslant C_N \epsilon$$

uniformly in x and ϵ, for all $\alpha < \frac{1}{2}$ and all $r \geqslant 0$, where

$$\|g\|_{\alpha,r} \stackrel{\text{def}}{=} \sup_v \big(1+|v|\big)^r e^{(\alpha-\frac{1}{2})|v|^2} |g|.$$

In particular, for ϵ sufficiently small, f_ϵ^N is defined globally in time (see Proposition B.2 and remarks below in Appendix B) and satisfies the uniform relaxation estimate

$$\left\| \frac{f_\epsilon^N - M_\epsilon^N}{M_\epsilon^N}(t) \right\|_{\alpha,r} \leqslant C_N \epsilon \exp\left(-\gamma \frac{t}{\epsilon^{q+1}}\right). \tag{4.13}$$

We have now to check that f_ϵ^N depends smoothly on the space variable x. Start from the equation satisfied by $\nabla_x f_\epsilon^N$

$$\begin{aligned} \partial_t \nabla_x f_\epsilon^N &= \frac{2}{\epsilon^{q+1}} Q\big(\nabla_x f_\epsilon^N, f_\epsilon^N\big) \\ &= \frac{2}{\epsilon^{q+1}} Q\big(\nabla_x f_\epsilon^N, \big(f_\epsilon^N - M_\epsilon^N\big)\big) + \frac{2}{\epsilon^{q+1}} Q\big(\nabla_x f_\epsilon^N, M_\epsilon^N\big), \end{aligned}$$

which can be rewritten using the semi-group U_ϵ^N generated by the collision operator linearized at M_ϵ^N, and Duhamel's formula

$$\frac{\nabla_x f_\epsilon^N}{M_\epsilon^N}(t) = U_\epsilon^N\left(\frac{t}{\epsilon^{q+1}}\right) \frac{\nabla_x f_\epsilon^{in,N}}{M_\epsilon^N} + \int_0^t U_\epsilon^N\left(\frac{t-s}{\epsilon^{q+1}}\right) \frac{2}{\epsilon^{q+1}} \frac{1}{M_\epsilon^N} Q\left(M_\epsilon^N \frac{\nabla_x f_\epsilon^N}{M_\epsilon^N}, M_\epsilon^N \frac{f_\epsilon^N - M_\epsilon^N}{M_\epsilon^N}\right)(s)\, ds.$$

From the relaxation estimate (4.13), we deduce that

$$\frac{1}{\epsilon^{q+1}} \int_0^t \left\| \frac{f_\epsilon^N - M_\epsilon^N}{M_\epsilon^N}(s) \right\|_{\alpha,r} ds \leqslant C\epsilon.$$

Extending – by a simple change of variables – the continuity estimates of Proposition B.1 to the semi-group U_ϵ^N, and using standard continuity estimates for the quadratic operator (see for instance the lemma in Appendix B.2), we then obtain by Gronwall's lemma

$$\left\| \frac{\nabla_x f_\epsilon^N}{M_\epsilon^N}(t) \right\|_{\alpha,r} \leqslant C_N \epsilon, \tag{4.14}$$

with C_N depending on N, α and r but not on ϵ.

Step 2: in order to have a bound from below on f_{app}, we then need to *truncate large velocities* as follows

$$f_{app} = \left(f_\epsilon^N - M_\epsilon^N\right)\mathbf{1}_{|v|^2\leqslant K|\ln\epsilon|} + M_\epsilon^N$$

so that, using the bound (4.13)

$$\frac{f_{app}}{M_\epsilon^N} = 1 + \epsilon\frac{f_\epsilon^N - M_\epsilon^N}{\epsilon M_\epsilon^N}\mathbf{1}_{|v|^2\leqslant K|\ln\epsilon|} \geqslant 1 - C_N\epsilon e^{(\frac{1}{2}-\alpha)K|\ln\epsilon|} \geqslant \frac{1}{2}$$

for ϵ sufficiently small, provided that

$$\left(\frac{1}{2}-\alpha\right)K < 1. \tag{4.15}$$

Of course, the same argument provides also a bound from above, so that

$$\frac{1}{2}M_\epsilon^N \leqslant f_{app} \leqslant \frac{3}{2}M_\epsilon^N. \tag{4.16}$$

Note that, for ϵ sufficiently small,

$$f_{app}^{in} = f_\epsilon^{in,N},$$

so that

$$\frac{1}{\epsilon^2}H\left(f_\epsilon^{in}\middle|f_{app}\right) \to 0 \quad \text{as } \epsilon \to 0 \text{ then } N \to \infty. \tag{4.17}$$

The regularity estimate (4.14) is also clearly inherited by f_{app}:

$$\left\|\frac{\nabla_x f_{app}}{M_\epsilon^N}(t)\right\|_{\alpha,r} \leqslant C_N\epsilon,$$

which implies in particular (4.11).

In the same way, we deduce from (4.13) the following relaxation estimate for f_{app}:

$$\left\|\frac{f_{app} - M_\epsilon^N}{M_\epsilon^N}(t)\right\|_{\alpha,r} \leqslant C_N \exp\left(-\gamma\frac{t}{\epsilon^{q+1}}\right).$$

Combined with the L^∞ bound (4.16), it leads to

$$\begin{aligned}\left\|\frac{f'_{app}f'_{app1}}{f_{app}f_{app1}} - 1\right\|_{L^{p'}(f_{app}f_{app1}b\,dv\,dv_1\,d\omega)} &\leqslant C_{p'}\epsilon\left\|\frac{(f_{app}-M_\epsilon^N)^{p'}}{\epsilon(M_\epsilon^N)^{p'-1}}\right\|_{L^1(dv)}^{1/p'}\\ &\leqslant C_{p'}\epsilon\exp\left(-\gamma\frac{t}{\epsilon^{q+1}}\right)\left\|\frac{f_\epsilon^{in,N}-M_\epsilon^N}{M_\epsilon^N}\right\|_{\alpha,r}\end{aligned}$$

provided that $\alpha p' > \frac{1}{2}$. In particular,

$$\frac{1}{\epsilon^{q+1}}\int_0^t\left\|\frac{f'_{app}f'_{app1}}{f_{app}f_{app1}} - 1\right\|_{L^\infty(\Omega,L^{p'}(f_{app}f_{app1}b\,dv\,dv_1\,d\omega))} ds \leqslant C_N\epsilon$$

tends to 0 as $\epsilon \to 0$, then $N \to \infty$. We have therefore proved the relaxation statement (4.10).

However the homogeneous Boltzmann equation (4.4) is no longer satisfied: it remains then to check that it is satisfied in some approximate sense.

We have

$$\begin{aligned}&\partial_t f_{app} - \frac{1}{\epsilon^{q+1}}Q(f_{app}, f_{app})\\ &\quad= -\frac{1}{\epsilon^{q+1}}Q\left(f_\epsilon^N, f_\epsilon^N\right)\mathbf{1}_{|v|^2>K|\ln\epsilon|} + \frac{1}{\epsilon^{q+1}}Q\left(f_\epsilon^N - f_{app}, f_\epsilon^N + f_{app}\right)\\ &\quad= -\frac{1}{\epsilon^{q+1}}Q\left(f_\epsilon^N, f_\epsilon^N\right)\mathbf{1}_{|v|^2>K|\ln\epsilon|} + \frac{1}{\epsilon^{q+1}}Q\left(\left(f_\epsilon^N - M_N^\epsilon\right)\mathbf{1}_{|v|^2>K|\ln\epsilon|}, f_\epsilon^N + f_{app}\right).\end{aligned} \tag{4.18}$$

Using the bound from below (4.16) and the decay estimate (4.13), we get

$$\begin{aligned}
&\left\| \frac{1}{\epsilon^{q+1}} \frac{1}{f_{app}} Q(f_\epsilon^N, f_\epsilon^N) \mathbf{1}_{|v|^2 > K|\ln\epsilon|} \right\|_{L^{p'}(f_{app}\,dv)} \\
&\quad \leqslant \frac{C}{\epsilon^{q+1}} \left\| \frac{1}{M_\epsilon^N} Q\big(f_\epsilon^N, f_\epsilon^N\big) \right\|_{\alpha,r} \left(\int \exp\left(\left(\frac{1}{2}(p'-1) - \alpha' p' \right) \mathbf{1}_{|v|^2 > K|\ln\epsilon|} \right) dv \right)^{1/p'} \\
&\quad \leqslant \frac{C}{\epsilon^{q+1}} \left(\|M_\epsilon^N\|_{\alpha,r} + \epsilon \left\| \frac{f_\epsilon^N - M_\epsilon^N}{\epsilon M_\epsilon^N} \right\|_{\alpha,r} \right)^2 \left(\int \exp\left(\left(\frac{1}{2}(p'-1) - \alpha' p' \right) \mathbf{1}_{|v|^2 > K|\ln\epsilon|} \right) dv \right)^{1/p'}
\end{aligned}$$

for some $\alpha' < \alpha$, which proves that the first term in the right-hand side of (4.18) goes to zero at least like $\mathrm{O}(\epsilon^3)$ as $\epsilon \to 0$ for any fixed N, provided that

$$K\left(\frac{1}{2p'} + \alpha' - \frac{1}{2} \right) > q + 4. \tag{4.19}$$

In the same way, we have

$$\begin{aligned}
&\left\| \frac{1}{\epsilon^{q+1}} \frac{1}{f_{app}} Q\big((f_\epsilon^N - M_\epsilon^N) \mathbf{1}_{|v|^2 > K|\ln\epsilon|}, f_\epsilon^N + f_{app} \big) \right\|_{L^{p'}(f_{app}\,dv)} \\
&\quad \leqslant \frac{C}{\epsilon^q} \left\| \frac{(f_\epsilon^N - M_\epsilon^N)}{\epsilon (M_\epsilon^N)^{1-\frac{1}{p'}+}} \mathbf{1}_{|v|^2 > K|\ln\epsilon|} \right\|_{L^{p'}(dv)} \left\| \frac{(f_\epsilon^N + f_{app})}{(M_\epsilon^N)^{1-\frac{1}{p'}+}} \right\|_{L^{p'}(dv)} \\
&\quad \leqslant \frac{C}{\epsilon^q} \left\| \frac{f_\epsilon^N - M_\epsilon^N}{\epsilon M_\epsilon^N} \right\|_{\alpha,r} \left(\|M_\epsilon^N\|_{\alpha,r} + \epsilon \left\| \frac{f_\epsilon^N - M_\epsilon^N}{\epsilon M_\epsilon^N} \right\|_{\alpha,r} \right) \left(\int \exp\left(\left(\frac{1}{2}(p'-1) - \alpha' p' \right) \mathbf{1}_{|v|^2 > K|\ln\epsilon|} \right) dv \right)^{1/p'}
\end{aligned}$$

for some $\alpha' < \alpha$, which proves that the second term in the right-hand side of (4.18) goes to zero at least like $\mathrm{O}(\epsilon^3)$ as $\epsilon \to 0$ for any fixed N.

We therefore choose some K satisfying both conditions (4.15) and (4.19), which is possible provided that α', and thus α are sufficiently close to $\frac{1}{2}$. The statement (4.8) is then satisfied. □

4.3. Convergence proof

The previous estimates on the sequence of approximate solutions to

$$\partial_t f - \frac{1}{\epsilon^{q+1}} Q(f, f) = 0$$

should allow to control the different terms in the entropy inequality (4.1) provided that the fluctuation g_ϵ can be estimated in terms of the scaled relative entropy $\frac{1}{\epsilon^2} H(f_\epsilon | f_{app})$ in a suitable norm.

We therefore establish the following

Lemma 4.4. *Let (f_ϵ) be some solution to the scaled Boltzmann equation* (2.2) *satisfying* (2.3). *Denote by f_{app} the sequence of approximate solutions to the relaxation equation built in Proposition* 4.3.

Then the fluctuation g_ϵ defined by

$$g_\epsilon = \frac{1}{\epsilon} \frac{f_\epsilon - f_{app}}{f_{app}},$$

is controlled by the modulated entropy as follows: *for all $p < \frac{4}{3}$, there exists $C_p > 0$ such that*

$$\|g_\epsilon\|^2_{L^2(\Omega, L^p(f_{app}\,dv))} \leqslant \frac{C_p}{\epsilon^2} H(f_\epsilon | f_{app}).$$

Proof. We first recall from the previous section that the renormalized fluctuation $\hat{g}_\epsilon$ defined by

$$\hat{g}_\epsilon = \frac{2}{\epsilon} \frac{\sqrt{f_\epsilon} - \sqrt{f_{app}}}{\sqrt{f_{app}}}$$

is controlled in L^2 norm by the modulated entropy

$$\iint f_{app}\hat{g}_\epsilon^2\,dv\,dx \leqslant \frac{2}{\epsilon^2}H(f_\epsilon|f_{app}).$$

From the control on large tails (2.3), we get for all $q<4$

$$\int f_{app}\left(\frac{f_\epsilon}{f_{app}}\right)^{q/2}dv \leqslant \left(\int \frac{f_\epsilon^2}{M}\,dv\right)^{\frac{q}{4}}\left(\int \frac{M^{4/(4-q)}}{f_{app}^{(2q-4)/(4-q)}}\,dv\right)^{1-\frac{q}{4}} \leqslant C_q$$

provided that ϵ is sufficiently small (depending on q): indeed f_{app} is bounded from below by M_ϵ^N and the moments of M_ϵ^N differ from those of M by quantities of order ϵ. Therefore,

$$\epsilon\hat{g}_\epsilon = 2\left(\sqrt{\frac{f_\epsilon}{f_{app}}}-1\right) = \mathrm{O}(1)_{L^\infty(\Omega,L^q(f_{app}\,dv))}.$$

Using the identity

$$g_\epsilon = \hat{g}_\epsilon + \frac{\epsilon}{4}\hat{g}_\epsilon^2,$$

and

$$\left\|\frac{\epsilon}{4}\hat{g}_\epsilon^2\right\|^2_{L^2(\Omega,L^p(f_{app}\,dv))} \leqslant \frac{1}{4}\|\epsilon\hat{g}_\epsilon\|_{L^\infty(\Omega,L^{2p/(2-p)}(f_{app}\,dv))}\|\hat{g}_\epsilon\|_{L^2(\Omega,L^2(f_{app}\,dv))}$$

we get the expected inequality, meaning that the relative entropy controls some spatial L^2 norm of the fluctuation. □

Proof of Theorem 2.1. Equipped with these preliminary results, we are now able to achieve the proof of Theorem 2.1 in the general case. Actually we will prove that on a thin time layer, the distribution becomes close to local thermodynamic equilibrium in the following sense

$$\frac{1}{\epsilon^2}H(f_\epsilon|f_{app})(\tau_\epsilon)\to 0 \quad \text{for some } \tau_\epsilon\to 0 \text{ as } \epsilon\to 0$$

with

$$\log f_{app}(\tau_\epsilon) = -\frac{3}{2}\log(2\pi) + \epsilon\left(\rho^{in}-\frac{3}{2}\theta^{in}\right) - \frac{1}{2}e^{-\epsilon\theta^{in}}\left|v-\epsilon u^{in}\right|^2.$$

Then, we will use the results of the previous section, combined with the continuity with respect to time of the solutions to the system (3.11), to obtain the convergence on the time interval $[\tau_\epsilon, T^*)$.

Step 1: combining Proposition 4.1 and Lemma 4.4 leads to the following inequality

$$\begin{aligned}
&\frac{1}{\epsilon^2}H(f_\epsilon|f_{app})(t) + \frac{1}{\epsilon^{q+3}}\int_0^t\!\!\iint D(f_\epsilon|f_{app})\,ds\,dx\\
&\quad\leqslant \frac{1}{\epsilon^2}H\left(f_\epsilon^{in}\middle|f_{app}^{in}\right) - \frac{1}{\epsilon}\int_0^t\!\!\iiint g_\epsilon\left(\partial_t f_{app} + \frac{1}{\epsilon}v\cdot\nabla_x f_{app} - \frac{1}{\epsilon^{q+1}}Q(f_{app},f_{app})\right)dv\,dx\,ds\\
&\qquad + \frac{C_p}{2\epsilon^{q+1}}\int_0^t\left\|\frac{f'_{app}f'_{app1}}{f_{app}f_{app1}}-1\right\|_{L^\infty(\Omega,L^{p'}(f_{app}f_{app}^1 b\,dv\,dv_1\,d\omega))}\frac{1}{\epsilon^2}H(f_\epsilon|f_{app})(s)\,ds.
\end{aligned}$$

Integrating next this differential inequality leads to

$$
\begin{aligned}
&\frac{1}{\epsilon^2}H(f_\epsilon|f_{app})(t)+\frac{1}{\epsilon^{q+3}}\int_0^t\iint D(f_\epsilon|f_{app})\,dv\,dx\,ds\leqslant\frac{1}{\epsilon^2}H\big(f_\epsilon^{in}\big|f_{app}^{in}\big)\exp\big(\chi_{app}(0,t)\big)\\
&\quad-\frac{1}{\epsilon}\int_0^t\exp\big(\chi_{app}(s,t)\big)\iint g_\epsilon\left(\partial_t f_{app}+\frac{1}{\epsilon}v\cdot\nabla_x f_{app}-\frac{1}{\epsilon^{q+1}}Q(f_{app},f_{app})\right)dv\,dx\,ds
\end{aligned}
\tag{4.20}
$$

where χ_{app} is the function defined by

$$
\chi_{app}(s,t)=\frac{C_p}{2\epsilon^{q+1}}\int_s^t\left\|\frac{f'_{app}f'_{app1}}{f_{app}f_{app1}}-1\right\|_{L^\infty(\Omega,L^{p'}(f_{app}f_{app1}b\,dv\,dv_1\,d\omega))}ds.
$$

Plugging the approximate solution f_{app} built in Proposition 4.3 in Gronwall's inequality (4.20) leads then to

$$
\sup_{t\in[0,\tau_\epsilon]}\frac{1}{\epsilon^2}H(f_\epsilon|f_{app})(t)\to 0 \tag{4.21}
$$

for any τ_ϵ such that

$$
\frac{\epsilon^{q+1}}{\tau_\epsilon}\to 0\quad\text{and}\quad\frac{\tau_\epsilon}{\epsilon}\to 0.
$$

Indeed the first term in the right-hand side of (4.20) is proved to converge to 0 as $\epsilon\to 0$ using the convergence of the initial data (4.17) and the uniform bound (4.10):

$$
\frac{1}{\epsilon^2}H\big(f_\epsilon^{in}\big|f_{app}^{in}\big)\exp\big(\chi_{app}(0,t)\big)\to 0.
$$

The convergence of the other term in the right-hand side of (4.20) is obtained by combining the uniform bound (4.10) with the convergence statement (4.8) and the regularity estimate (4.11)

$$
\begin{aligned}
&\frac{1}{\epsilon}\int_0^t\exp\big(\chi_{app}(s,t)\big)\iint g_\epsilon\left(\partial_t f_{app}-\frac{1}{\epsilon^{q+1}}Q(f_{app},f_{app})\right)dv\,dx\,ds\to 0,\\
&\frac{1}{\epsilon}\int_0^t\exp\big(\chi_{app}(s,t)\big)\iint g_\epsilon\left(\frac{1}{\epsilon}v\cdot\nabla_x f_{app}\right)dv\,dx\,ds\to 0.
\end{aligned}
$$

Step 2: we choose for instance

$$
\tau_\epsilon=\epsilon^q
$$

and we define $\tilde f_{app}$ by

$$
\tilde f_{app}=f_{app}\quad\text{on }[0,\tau_\epsilon)
$$

and

$$
\log\tilde f_{app}=-\frac{3}{2}\log(2\pi)+\epsilon\left(\rho-\frac{3}{2}\theta\right)-\frac{1}{2}e^{-\epsilon\theta}|v-\epsilon u|^2\quad\text{for }t\geqslant\tau_\epsilon,
$$

with the following continuity condition for the moments

$$
(\rho,u,\theta)(\tau_\epsilon)=\lim_{t\to\tau_\epsilon^-}(\rho,u,\theta)(t)=\big(\rho^{in},u^{in},\theta^{in}\big).
$$

Combining Proposition 3.1 and Lemma 3.5, and integrating the resulting differential inequality on $[\tau_\epsilon,t]$ leads to

$$\begin{aligned}
&\frac{1}{\epsilon^2} H(f_\epsilon|\tilde{f}_{app})(t) + \frac{1}{\epsilon^{q+3}} \int_{\tau_\epsilon}^{t} \iint D(f_\epsilon)\,dv\,dx\,ds \\
&\quad \leqslant \frac{1}{\epsilon^2} H(f_\epsilon|\tilde{f}_{app})(\tau_\epsilon) \exp\left(C \int_{\tau_\epsilon}^{t} \left\| D_x(u,\theta)(s) \right\|_{L^2\cap L^\infty(\Omega)} ds \right) + \mathrm{o}(1) \\
&\qquad + \frac{1}{\epsilon^2} \int_{\tau_\epsilon}^{t} \exp\left(C \int_{s}^{t} \left\| D_x(u,\theta)(\tau) \right\|_{L^2\cap L^\infty(\Omega)} d\tau \right) \int \partial_t \exp(\epsilon\rho)\,dx\,ds \\
&\qquad + \frac{1}{\epsilon} \int_{\tau_\epsilon}^{t} \exp\left(C \int_{s}^{t} \left\| D_x(u,\theta)(\tau) \right\|_{L^2\cap L^\infty(\Omega)} d\tau \right) \\
&\qquad \times \iint f_\epsilon \left(1, e^{-\epsilon\theta}(v-\epsilon u), \frac{1}{2}\big(e^{-\epsilon\theta}|v-\epsilon u|^2 - 3\big) \right) \cdot A_\epsilon(\rho,u,\theta)\,dv\,dx\,ds. \qquad (4.22)
\end{aligned}$$

Translating (with respect to time) the approximate solution $(\rho_\epsilon^N, u_\epsilon^N, \theta_\epsilon^N)$ built in Proposition 3.3 on $[\tau_\epsilon, T^* + \tau_\epsilon)$, and plugging it in Gronwall's inequality (4.22) leads then to

$$\sup_{t\in[\tau_\epsilon,T]} \frac{1}{\epsilon^2} H(f_\epsilon|\tilde{f}_{app}) \to 0 \quad \text{for all } T < T^*, \qquad (4.23)$$

provided that the first term in the right-hand side converges to 0 as $\epsilon \to 0$ then $N \to \infty$.

It remains therefore to check that

$$\frac{1}{\epsilon^2} H(f_\epsilon|\tilde{f}_{app})(\tau_\epsilon) \to 0$$

which results from the estimate obtained in the first step. We indeed have

$$\frac{1}{\epsilon^2} H(f_\epsilon|f_{app})(\tau_\epsilon) \to 0$$

and, by Proposition 4.3,

$$\frac{f_{app}}{\tilde{f}_{app}}(\tau_\epsilon) = 1 + \mathbf{1}_{|v|^2\leqslant K|\ln\epsilon|} \frac{f_\epsilon^N - M_\epsilon^N}{M_\epsilon^N}(\tau_\epsilon) = 1 + \mathrm{O}\left(\epsilon \exp\left(-\frac{\tau_\epsilon}{\epsilon^{q+1}} \right) \right)_{L^\infty},$$

from which we deduce that

$$\frac{1}{\epsilon^2} H(f_\epsilon|\tilde{f}_{app})(\tau_\epsilon) = \frac{1}{\epsilon^2} H(f_\epsilon|f_{app})(\tau_\epsilon) + \frac{1}{\epsilon^2} \iint f_\epsilon \log \frac{f_{app}}{\tilde{f}_{app}}(\tau_\epsilon)\,dx\,dv \to 0$$

as $\epsilon \to 0$.

Step 3: the entropic convergences (4.21), (4.23) imply the strong convergence of the fluctuation g_ϵ in $L^\infty_{\mathrm{loc}}([0,T^*), L^1(dx\,dv))$, in particular the convergence result stated in the theorem. Indeed we have seen that the L^2 norm of the renormalized fluctuation $\hat{g}_\epsilon$ is bounded by the square-root of the modulated entropy $\frac{1}{\epsilon^2} H(f_\epsilon|f_{app})$. Thus, using the decomposition (3.28), we get

$$g_\epsilon - \frac{1}{\epsilon} \frac{\tilde{f}_{app} - M}{M} \to 0 \quad \text{in } L^\infty_{\mathrm{loc}}\big([0,T^*), L^1(dx\,dv)\big)$$

as $\epsilon \to 0$ then $N \to \infty$.

Finally, using the fact that the purely kinetic part of the approximate solution is equal to 0 on $[\tau_\epsilon, T^*)$ and converges to 0 in $L^1_{\mathrm{loc}}([0,T^*), L^1(dx\,dv))$, we get

$$g_\epsilon - g - g_{osc} \to 0 \quad \text{in } L^\infty_{\mathrm{loc}}\big((0,T^*), L^1(dx\,dv)\big) \cap L^1_{\mathrm{loc}}\big([0,T^*), L^1(dx\,dv)\big)$$

as $\epsilon \to 0$. □

Appendix A. Spectral study of the acoustic operator

A.1. Spectrum of W

The acoustic waves are governed by the pseudodifferential operator W defined by

$$W(\rho, u, \theta) = \left(\nabla_x \cdot u, \nabla_x(\rho + \theta), \frac{2}{3}\nabla_x \cdot u\right),$$

which is skew-symmetric on $L^2(\Omega)$ equipped with the norm

$$\left\|(\rho, u, \theta)\right\|^2_{L^2(\Omega)} = \|\rho\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + \frac{3}{2}\|\theta\|^2_{L^2(\Omega)}.$$

Let us first describe its spectrum:

$$W(\rho, u, \theta) = i\lambda(\rho, u, \theta) \quad \text{with } u \cdot n = 0$$

if and only if

- either $\lambda = 0$ and

$$\nabla_x \cdot u = 0 \quad \text{with } u \cdot n = 0 \text{ on } \partial\Omega,$$
$$\nabla_x(\rho + \theta) = 0$$

- or $\lambda \neq 0$ and

$$\theta = \frac{2}{3}\rho,$$
$$i\lambda u = \nabla_x(\rho + \theta),$$
$$\frac{5}{3}\Delta_x(\rho + \theta) = -\lambda^2(\rho + \theta) \quad \text{with } n \cdot \nabla_x(\rho + \theta) = 0 \text{ on } \partial\Omega.$$

That means that – on the orthogonal of its kernel – W has the same spectral structure as the Laplacian with Neumann boundary condition.

In the case when Ω is a smooth bounded domain, there exists an Hilbertian basis $(\varphi_n)_{n\in I}$ of $L^2(\Omega)$ such that

$$-\Delta_x\varphi_n = \mu_n\varphi_n \quad \text{with } n \cdot \nabla_x\varphi_n = 0 \text{ on } \partial\Omega$$

with $\mu_n > 0$ for $n \in \mathbf{N}^*$, and

$$-\Delta_x\varphi_n = 0 \quad \text{with } n \cdot \nabla_x\varphi_n = 0 \text{ on } \partial\Omega$$

for $n \in I \setminus \mathbf{N}^*$.

It is then easy to check that

Proposition A.1. *The family* $(\psi_n)_{n\in\mathbf{Z}}$ *defined by*

$$\psi_n = \left(\frac{3}{5}\varphi_{|n|}, i\,\mathrm{sgn}(n)\sqrt{\frac{3}{5\mu_{|n]}}}\nabla_x\varphi_{|n|}, \frac{2}{5}\varphi_{|n|}\right)$$

is an Hilbertian basis of $(\operatorname{Ker} W)^\perp$.

Remark. In the case when Ω is a smooth unbounded domain, the Laplacian has also continuous spectrum, and the following spectral decomposition (of Fourier type) holds

$$\varphi = \bar\varphi + \int_0^{+\infty} \hat\varphi(\mu)\, dm(\mu),$$
$$-\Delta\varphi = \int_0^{+\infty} \mu\hat\varphi(\mu)\, dm(\mu)$$

where m is the spectral measure of the Laplacian on $]0, +\infty[$, and $\bar\varphi$ denotes the harmonic component of φ.

We could then have an explicit formula for W in terms of the spectral measure m.

A.2. Suitable functional spaces

An explicit computation shows that

$$\Pi_0(\rho, u, \theta) = \left(\frac{2\rho - 3\theta}{5}, Pu, \frac{3\theta - 2\rho}{5}\right)$$

where P denotes the Leray projection on divergence-free vector fields (satisfying Neumann's boundary condition). In particular, Π_0 is continuous on $H^s(\Omega)$, and we have

$$\frac{1}{C}\big\|(\rho, u, \theta)\big\|_{H^s(\Omega)} \leqslant \big\|\Pi_0(\rho, u, \theta)\big\|_{H^s(\Omega)} + \big\|(\mathrm{Id} - \Pi_0)(\rho, u, \theta)\big\|_{H^s(\Omega)},$$
$$\big\|\Pi_0(\rho, u, \theta)\big\|_{H^s(\Omega)} + \big\|(\mathrm{Id} - \Pi_0)(\rho, u, \theta)\big\|_{H^s(\Omega)} \leqslant C\big\|(\rho, u, \theta)\big\|_{H^s(\Omega)}.$$

On $(\mathrm{Ker}\, W)^\perp$, as the spectral decomposition according to W corresponds to the spectral decomposition according to the Laplacian, we have

$$\|\psi\|^2_{H^s(\Omega)} \sim \sum_{n\in\mathbf{Z}^*} \big(1 + \lambda_n^2\big)^s \big|\langle\psi, \psi_n\rangle_{L^2(\Omega)}\big|^2.$$

Both results imply that

$$\big\|(\rho, u, \theta)\big\|^2_{H^s(\Omega)} \sim \big\|\Pi_0(\rho, u, \theta)\big\|^2_{H^s(\Omega)} + \sum_{n\in\mathbf{Z}^*} \big(1 + \lambda_n^2\big)^s \big|\langle\psi, \psi_n\rangle_{L^2(\Omega)}\big|^2 \tag{A.1}$$

A.3. Well-posedness and stability of (3.19) in the case of bounded domains

Let us now state the main result regarding the local existence of solutions to the filtered acoustic system.

Proposition A.2. *Let $V^{in} \in H^s(\Omega)$ for $s > \frac{5}{2}$. Then the filtered acoustic system*

$$\partial_t V_0 + \mathcal{B}_W(V_0, V_0) = 0,$$
$$V_{0|t=0} = V^{in}$$

where the quadratic operator $\mathcal{B}_W$ is defined by (3.20)*, has a unique strong solution $V_0 \in L^\infty_{\mathrm{loc}}([0, T^*), H^s(\Omega))$.*

Proof. We recall briefly the strategy to build solutions to (3.19), and more generally to hyperbolic equations such as the incompressible Euler equations (see [9] for instance for a detailed proof).

The first step is to define a suitable approximation scheme by projecting the equation on a finite-dimensional subset of $C^\infty(\bar\Omega)$. Here we will consider the projection

$$J_N = \sum_{n\leqslant N} \Pi_{\lambda_n} + \Pi_{0,N} \tag{A.2}$$

where $\Pi_{0,N}$ is the composition of Π_0 and of the projection on the N first harmonics (note that both operators commute). The Cauchy problem for

$$\partial_t J_N\psi + J_N\mathcal{B}_W(J_N\psi, J_N\psi) = 0$$

is then solved using the Cauchy–Lipschitz theorem. We will denote by ψ^N the solution to that Nth approximate system.

The next step is to establish some stability inequality (uniform with respect to N) , which comes actually from a precised energy estimate. For the sake of simplicity we will not mention the superscript N of ψ^N in the intermediate computations. We actually prove

$$\begin{aligned}\frac{d}{dt}\|\Pi_{0,N}\psi\|^2_{H^s(\Omega)} &\leqslant -\big\langle \Pi_{0,N}\psi \big| \mathcal{B}(\Pi_{0,N}\psi, \Pi_{0,N}\psi)\big\rangle_{H^s(\Omega)}\\ &\leqslant C_s\|\Pi_{0,N}\psi\|^2_{H^s(\Omega)}\|\Pi_{0,N}\psi\|_{W^{1,\infty}(\Omega)}\end{aligned}$$

since the equation on the nonoscillating part $\Pi_0\psi$ can be decoupled (see the computation (3.21) in Section 3.2). Thus, if the initial data belongs to $H^s(\Omega)$ for $s > \frac{5}{2}$, there exists some T^* (which does not depend on N) such that

$$\|\Pi_{0,N}\psi\|^2_{L^\infty([0,T],H^s(\Omega))} \leqslant C_T$$

for all $T < T^*$. In the same way, we have

$$\begin{aligned}\frac{d}{dt}\sum_{k\in\mathbf{Z}^*}\big(1+\lambda_k^2\big)^s\|\Pi_{\lambda_k}\psi\|^2_{L^2(\Omega)} &\leqslant -\sum_{k\in\mathbf{Z}^*}\sum_{\lambda_{k_1}+\lambda_{k_2}=\lambda_k}\big(1+\lambda_k^2\big)^s\big\langle \Pi_{\lambda_k}\big|\mathcal{B}(\Pi_{\lambda_{k_1}}\psi, \Pi_{\lambda_{k_2}}\psi)\big\rangle_{L^2(\Omega)}\\ &\leqslant C_s\sum_{k\in\mathbf{Z}^*}\sum_{\lambda_{k_1}+\lambda_{k_2}=\lambda_k}\big(1+\lambda_k^2\big)^{s/2}\big(1+\lambda_{k_1}^2\big)^{s/2}\|\Pi_{\lambda_k}\psi\|_{L^2(\Omega)}\\ &\quad\times\|\Pi_{\lambda_{k_1}}\psi\|_{L^2(\Omega)}\|\Pi_{\lambda_{k_2}}\psi\|_{W^{1,\infty}(\Omega)}.\end{aligned}$$

(Note that the higher order terms disappear due to the conservative properties of $\mathcal{B}$ and to the symmetries of the resonance condition $\lambda_{k_1}+\lambda_{k_2}=\lambda_k$, which extends a classical result on the commutator in pseudo-differential calculus.) Using the Cauchy–Schwarz inequality, we then obtain

$$\frac{d}{dt}\sum_{k\in\mathbf{Z}^*}\big(1+\lambda_k^2\big)^s\|\Pi_{\lambda_k}\psi\|^2_{L^2(\Omega)} \leqslant C_s\bigg(\sum_{k\in\mathbf{Z}^*}\big(1+\lambda_k^2\big)^{s/2}\|\Pi_{\lambda_k}\psi\|_{L^2(\Omega)}\bigg)^2\sup_\lambda\|\Pi_\lambda\psi\|_{W^{1,\infty}(\Omega)}.$$

Since, for $\lambda\neq 0$

$$\|\Pi_\lambda\psi\|_{W^{1,\infty}(\Omega)} \sim \big(1+\lambda^2\big)^{1/2}\|\Pi_\lambda\psi\|_{L^\infty(\Omega)},$$

the crucial point to conclude is the summability condition

$$\sum_{n\in\mathbf{Z}^*}\lambda_n^{-2\beta} < +\infty \quad \text{for all } \beta > \frac{3}{2},$$

which implies that, for all $s > \frac{5}{2}$,

$$\sup_\lambda\|\Pi_\lambda\psi\|_{W^{1,\infty}(\Omega)} \leqslant C_s\|\psi\|_{H^s(\Omega)}.$$

Thus, if the initial data belongs to $H^s(\Omega)$ for $s > \frac{5}{2}$, there exists some T^* (which does not depend on N) such that

$$\big\|\psi^N\big\|^2_{L^\infty([0,T],H^s(\Omega))} \leqslant C_T.$$

The last step is to take limits as $N\to\infty$ using the previous uniform bounds. As the lifespan T^* of the approximate solutions and the local (in time) bound in $H^s(\Omega)$ depend only on the H^s norm of the initial data, the sequence of approximate solutions admits some weak limit.

By Sobolev embeddings, we further get some strong compactness on the sequence of approximate solutions, which allows to take limits in the evolution equation and gives the existence of a solution ψ to

$$\partial_t\psi + \mathcal{B}_W(\psi,\psi) = 0.$$

The uniqueness is obtained by rewriting the precised energy estimate for the difference between two solutions, which leads to some stability inequality in $H^s(\Omega)$. $\square$

Appendix B. Study of the homogeneous relaxation equation

In order to derive rigorously hydrodynamic limits of the Boltzmann equation, we need to understand the effects of the relaxation process, especially to establish quantitative variants of the mechanism of decreasing of the entropy.

As we restrict our attention to fluctuations around some global equilibrium (say around the centered reduced Gaussian M), the relaxation towards local thermodynamic equilibrium is essentially governed by the linearized collision operator $\mathcal{L}$ defined by

$$\mathcal{L}g = -\frac{2}{M}Q(M, Mg).$$

The quadratic part of the collision operator $\mathcal{Q}$ defined by

$$\mathcal{Q}g = \frac{1}{M}Q(Mg, Mg).$$

is indeed expected to act as a corrector.

B.1. *Refined estimates on the linearized collision operator $\mathcal{L}$*

We first recall that $\mathcal{L}$ is a compact perturbation of some multiplication operator:

Lemma. *(See [21].) Assume that b satisfies Grad's cut-off assumption* (1.5) *for some $\beta \in [0, 1]$. Then the linearized collision operator $\mathcal{L}$ can be decomposed as*

$$\mathcal{L}g(v) = \nu\big(|v|\big)g(v) - \mathcal{K}g(v)$$

where $\mathcal{K}$ is a compact integral operator on $L^2(M\,dv)$ and $\nu = \nu(|v|)$ is a scalar called the collision frequency that satisfies, for some nonnegative ν_-, ν_+,

$$\nu_-\big(1 + |v|\big)^\beta \leqslant \nu\big(|v|\big) \leqslant \nu_+\big(1 + |v|\big)^\beta.$$

The proof of that result is based on a clever change of variables sometimes called "Carleman's collision parametrization" although it goes back to Hilbert [21]. The compactness of $\mathcal{K}$ is then obtained by proving that its iterated kernel belongs to L^2.

As a consequence of that result, we get in particular the following coercivity estimate:

Corollary. *Assume that b satisfies Grad's cut-off assumption* (1.5) *for some $\beta \in [0, 1]$. Then the linear collision operator $\mathcal{L}$ is a nonnegative unbounded self-adjoint operator on $L^2(M\,dv)$ with domain*

$$\mathcal{D}(\mathcal{L}) = \big\{g \in L^2(M\,dv) \;\big|\; |v|^\beta g \in L^2(M\,dv)\big\} = L^2\big(\mathbf{R}^3; \nu M(v)\,dv\big)$$

and nullspace

$$\mathrm{Ker}(\mathcal{L}) = \mathrm{span}\big\{1, v_1, v_2, v_3, |v|^2\big\}.$$

Moreover the following coercivity estimate holds: *there exists $c > 0$ such that, for each $g \in \mathcal{D}(\mathcal{L}) \cap (\mathrm{Ker}(\mathcal{L}))^\perp$*

$$\int g\mathcal{L}g(v)M(v)\,dv \geqslant c\|g\|^2_{L^2(M\nu\,dv)}.$$

In order to obtain the required regularity estimates on the solution to the linear relaxation equation

$$\partial_\tau g = -\mathcal{L}g, \tag{B.1}$$

we need some additional properties of the compact part $\mathcal{K}$ of $\mathcal{L}$. These properties have been studied in detail by Grad in his fundamental paper [16], and later improved by Caflisch [6].

Lemma. *(See [16].) Assume that b satisfies Grad's cut-off assumption* (1.5) *for some* $\beta \in [0, 1]$. *Then the compact part* $\mathcal{K}$ *of the linear collision operator* $\mathcal{L}$ *satisfies the following continuity estimates*

$$\|\mathcal{K}g\|_{\frac{1}{4},\frac{3}{2}(1-\beta)} \leqslant C_0 \|g\|_{L^2(M\,dv)},$$
$$\|\mathcal{K}g\|_{\alpha,r+2-\beta} \leqslant C_{\alpha,r}\|g\|_{\alpha,r}$$

for any $\alpha \in [0, \frac{1}{2})$ *and* $r \geqslant 0$, *where*

$$\|f\|_{\alpha,r} = \sup_{v\in\mathbf{R}^3} \big(1+|v|^r\big)e^{(\alpha-\frac{1}{2})|v|^2}\big|f(v)\big|.$$

Using these continuity statements, we are then able to establish the exponential trend to equilibrium in suitable functional spaces.

Proposition B.1. *Assume that b satisfies Grad's cut-off assumption* (1.5) *for some* $\beta \in [0, 1]$. *Let* g^{in} *be some function such that*

$$\|g^{in}\|_{\alpha,r} < +\infty$$

for some $\alpha \in [\frac{1}{4}, \frac{1}{2})$, $r \geqslant 0$, *and*

$$\int Mg^{in}\,dv = \int Mg^{in}v_i\,dv = \int Mg^{in}|v|^2\,dv = 0.$$

Then, the solution g to the linear relaxation equation (B.1) *converges exponentially to* 0: *there exist some nonnegative constants* C, γ *(depending only on* α *and* r*) such that*

$$\|g(\tau)\|_{\alpha,r} \leqslant C\exp(-\gamma\tau)\|g^{in}\|_{\alpha,r}.$$

Proof. Since $\mathcal{L}$ is a Fredholm operator on $L^2(Mdv)$, it generates a semi-group that we will denote U in all the sequel. In particular $g(\tau) = U(\tau)g^{in}$ is well-defined for all $\tau \geqslant 0$.

• From the energy inequality

$$\int Mg^2(\tau,v)\,dv + \int_0^\tau\!\!\int Mg\mathcal{L}g(s,v)\,dv\,ds \leqslant \int M(g^{in})^2(v)\,dv$$

and the coercivity estimate, we deduce by Gronwall's lemma

$$\int M(\Pi_\perp g)^2(\tau,v)\,dv \leqslant \left(\int M\big(g^{in}\big)^2(v)\,dv\right)\exp(-c\tau). \tag{B.2}$$

But we also have

$$\Pi g(\tau) = \Pi g^{in} = 0.$$

We therefore get the expected exponential decay in the space $L^2(M\,dv)$.

• It remains then to obtain some exponential decay with respect to v when $\alpha < \frac{1}{2}$.

Using Hilbert's decomposition of $\mathcal{L}$ and Duhamel's formula, we obtain

$$g(\tau) = \exp(-\nu\tau)g^{in} + \int_0^\tau \exp\big(-\nu(\tau-s)\big)\mathcal{K}g(s)\,ds.$$

We then have

$$\big\|\exp(\gamma\tau)g(\tau)\big\|_{\alpha,r} \leqslant \big\|g^{in}\big\|_{\alpha,r} + \int_0^\tau \big\|\exp\big(-(\nu-\gamma)(\tau-s)\big)\mathcal{K}\exp(\gamma s)g(s)\big\|_{\alpha,r}\,ds, \tag{B.3}$$

provided that $\gamma < \nu_-$. Now, splitting the contributions of large and moderate velocities, we get

$$\begin{aligned}\left\|\exp(\gamma s)\mathcal{K}g(s)\right\|_{\alpha,r} \leqslant{}& \left(1+|v_0|^r\right)\exp\left(\left(\alpha-\frac{1}{4}\right)|v_0|^2\right)\left\|\exp(\gamma s)\mathcal{K}g(s)\right\|_{\frac{1}{4},0}\\ &+\frac{1}{(1+|v_0|^{2-\beta})}\left\|\exp(\gamma s)\mathcal{K}g(s)\right\|_{\alpha,r+2-\beta}\end{aligned}$$

and thus, using the continuity results stated in the previous lemma, we deduce

$$\begin{aligned}\left\|\exp(\gamma s)\mathcal{K}g(s)\right\|_{\alpha,r} \leqslant{}& C_0\left(1+|v_0|^r\right)\exp\left(\left(\alpha-\frac{1}{4}\right)|v_0|^2\right)\left\|\exp(\gamma s)g(s)\right\|_{L^2(M\,dv)}\\ &+\frac{C_{\alpha,r}}{(1+|v_0|^{2-\beta})}\left\|\exp(\gamma s)g(s)\right\|_{\alpha,r}.\end{aligned}$$

Plugging this last estimate in (B.3), using (B.2) and choosing $\gamma < \min(\nu_-, c)$ we obtain

$$\begin{aligned}\left\|\exp(\gamma\tau)g(\tau)\right\|_{\alpha,r} \leqslant{}& \left\|g^{in}\right\|_{\alpha,r}+C\left(1+|v_0|^r\right)\exp\left(\left(\alpha-\frac{1}{4}\right)|v_0|^2\right)\left\|g^{in}\right\|_{L^2(M\,dv)}\\ &+\frac{C}{(1+|v_0|^{2-\beta})}\sup_{s\in[0,\tau]}\left\|\exp(\gamma s)g(s)\right\|_{\alpha,r}.\end{aligned}$$

Choose $|v_0|^2$ large enough so that $C/(1+|v_0|^{2-\beta}) \leqslant 1/2$. Then

$$\sup_{s\in[0,\tau]}\left\|\exp(\gamma s)g(s)\right\|_{\alpha,r} \leqslant C_{\alpha,r}\left\|g^{in}\right\|_{\alpha,r}. \qquad \square$$

B.2. Well-posedness and stability of the homogeneous Boltzmann equation

For the study of the nonlinear relaxation equation

$$\partial_\tau f = Q(f,f) \tag{B.4}$$

which can be rewritten in perturbative form

$$\begin{aligned}&\partial_\tau g = -\mathcal{L}g + \epsilon\mathcal{Q}(g,g),\\ &f = M(1+\epsilon g),\end{aligned}$$

we further need to control the contribution of the nonlinear effects. A simple computation which can be found for instance in [5] leads to the following statement:

Lemma. *Assume that b satisfies Grad's cut-off assumption* (1.5) *for some $\beta \in [0,1]$. Then the symmetric bilinear operator $\mathcal{Q}$ defined by polarization satisfies the following continuity estimates*

$$\left\|\nu^{-1}\mathcal{Q}(g,h)\right\|_{\alpha,r} \leqslant C\|g\|_{\alpha,r}\|h\|_{\alpha,r}.$$

Combining these nonlinear estimates with the results on the linear relaxation equation stated in Proposition B.1 leads then to

Proposition B.2. *Assume that b satisfies Grad's cut-off assumption* (1.5) *for some $\beta \in [0,1]$. Let $\alpha \in [0,\frac{1}{2})$ and $r \geqslant \beta$. Then there exists some $\delta > 0$ (depending only on α and r) such that, for all nonnegative initial data f^{in} such that*

$$\begin{aligned}&\int f^{in}\,dv = \frac{1}{3}\int f^{in}|v|^2\,dv = 1,\\ &\int f^{in}v\,dv = 0,\end{aligned}$$

and

$$\left\|\frac{f^{in}-M}{M}\right\|_{\alpha,r} \leqslant \delta$$

the homogeneous nonlinear Boltzmann equation (B.4) *admits a unique global solution* f.

Furthermore, that solution converges exponentially to M: *there exists some nonnegative constants* C *and* γ (*depending only on* α *and* r) *such that*

$$\left\| \frac{f-M}{M}(\tau)\right\|_{\alpha,r} \leqslant C\exp(-\gamma\tau)\left\| \frac{f^{in}-M}{M}\right\|_{\alpha,r}.$$

Proof. The global existence result is based on Duhamel's formula and on Picard's fixed point theorem. Indeed the Boltzmann equation (B.4) can be reduced to the integral equation

$$g = N[g], \tag{B.5}$$

where the functional N is defined by

$$N[g](\tau) = U(\tau)g^{in} + \int_0^\tau U(\tau-s)\mathcal{Q}(g,g)(s)\,ds. \tag{B.6}$$

The global well-posedness of the Cauchy problem for (B.4) is then established by proving that N is a contraction in a ball of $L^\infty(\mathbf{R}^+\times\mathbf{R}^3)\cap C(\mathbf{R}^+, L^\infty(\mathbf{R}^3))$. The relaxation estimate is obtained in the same way, considering some weighted spaces with exponential decay with respect to time.

More precisely, we consider the following norm

$$\|g\| = \sup_{s\in\mathbf{R}^+} e^{\gamma s}\left\|g(s)\right\|_{\alpha,r}$$

for some nonnegative γ to be fixed later.

• Applying Proposition B.1 gives directly an estimate for the first term

$$\left\|U(t)g^{in}\right\| \leqslant C_{\alpha,r}\left\|g^{in}\right\|_{\alpha,r},$$

provided that γ is smaller than the linear relaxation constant $\gamma_{\alpha,r}$.

• To control the second term is much more difficult since an inverse power of ν appears in the continuity estimate on $\mathcal{Q}$. In order to improve the integrability with respect to v, we therefore go back to Hilbert's decomposition:

$$\begin{aligned}\int_0^\tau U(\tau-s)\mathcal{Q}(g,h)(s)\,ds &= \int_0^\tau \exp\bigl(-\nu(\tau-s)\bigr)\mathcal{Q}(g,h)(s)\,ds\\ &\quad+\int_0^\tau\int_0^{\tau-s}\exp\bigl(-\nu(\tau-s-\sigma)\bigr)\mathcal{K}U(\sigma)\mathcal{Q}(g,h)(s)\,d\sigma\,ds.\end{aligned}$$

From the continuity estimate on $\mathcal{Q}$ we then deduce

$$\begin{aligned}\left\|\int_0^\tau \exp\bigl(-\nu(\tau-s)\bigr)\mathcal{Q}(g,h)(s)\,ds\right\| &\leqslant C_\alpha\left(\sup_{\tau,v}\int_0^\tau \nu\exp\bigl(-\nu(\tau-s)\bigr)\exp(\gamma\tau-2\gamma s)\,ds\right)\|g\|\|h\|\\ &\leqslant C\|g\|\|h\|,\end{aligned}$$

provided that $2\gamma<\nu_-$.

Using further the continuity of U and the continuity of $\mathcal{K}$ we get, provided that $r-\beta\geqslant 0$,

$$\begin{aligned}\left\|\mathcal{K}U(\sigma)\mathcal{Q}(g,h)(s)\right\|_{\alpha,r} &\leqslant C\left\|U(\sigma)\mathcal{Q}(g,h)(s)\right\|_{\alpha,r-\beta}\\ &\leqslant C\exp(-\gamma_{\alpha,r-\beta}\sigma)\left\|\mathcal{Q}(g,h)(s)\right\|_{\alpha,r-\beta}\\ &\leqslant C\exp(-\gamma_{\alpha,r-\beta}\sigma)\left\|\nu^{-1}\mathcal{Q}(g,h)(s)\right\|_{\alpha,r}\\ &\leqslant C\exp(-2\gamma s-\gamma_{\alpha,r-\beta}\sigma)\|g\|\|h\|,\end{aligned}$$

with $\gamma_{\alpha,r-\beta} < \nu_-$ (see the proof of Proposition B.1), and consequently

$$\left\| \int_0^\tau \int_0^{\tau-s} \exp\bigl(-\nu(\tau-s-\sigma)\bigr)\mathcal{K}U(\sigma)\mathcal{Q}(g,h)(s)\,d\sigma\,ds \right\| \leqslant C\|g\|\|h\|$$

for some nonnegative constant C depending on r, ν_-, α, β and γ, provided that $2\gamma < \gamma_{\alpha,r-\beta}$.

We therefore obtain

$$\left\| \int_0^\tau U(\tau-s)\mathcal{Q}(g,h)(s)\,ds \right\| \leqslant C'_{\alpha,r}\|g\|\|h\|.$$

• Finally we have

$$\bigl\|N[g]\bigr\| \leqslant C_{\alpha,r}\bigl\|g^{in}\bigr\|_{\alpha,r} + C'_{\alpha,r}\|g\|^2$$

and

$$\bigl\|N[g]-N[h]\bigr\| \leqslant C'_{\alpha,r}\bigl(\|g\|+\|h\|\bigr)\|g-h\|.$$

Choosing a_0 and a_1 such that

$$2C'_{\alpha,r}a_1 < 1 \quad \text{and} \quad C_{\alpha,r}a_0 + C'_{\alpha,r}a_1^2 \leqslant a_1,$$

we get that N is a contraction on the ball of radius a_1 as soon as

$$\bigl\|g^{in}\bigr\|_{\alpha,r} \leqslant a_0.$$

We then conclude by Picard's fixed point theorem. □

Remarks.

(i) Note that, as a consequence of that proof, we also get that the solution to the nonlinear homogeneous Boltzmann equation (B.4) depends Lipschitz continuously of the initial data provided that it belongs to the ball of radius a_0.

(ii) As the mass, momentum and kinetic energy are conserved by the homogeneous Boltzmann equation, up to some Galilean change of variables, we can extend the previous result to the case when

$$R = \int f^{in}\,dv \neq 1,$$
$$U = \frac{1}{R}\int f^{in}v\,dv \neq 0,$$
$$\Theta = \frac{1}{3R}\int f^{in}|v-U|^2\,dv \neq 1.$$

All the constants involved in the estimates we have established will depend smoothly on the thermodynamic parameters R, U and Θ, and thus it will be possible to make them uniform for (R, U, Θ) in a ball around $(1, 0, 1)$.

Appendix C. Formal derivation of the incompressible Euler asymptotics

For the sake of completeness, we recall here the formal expansions leading to the incompressible Euler limit. Note that it has been only recently observed ([1] for time independent problems and [2,11] for time dependent problems) that, under convenient scaling, incompressible equations could be directly derived from the Boltzmann equation.

Our starting point is the nondimensional Boltzmann equation:

$$Ma\,\partial_t f + v\cdot\nabla_x f = \frac{1}{Kn}Q(f,f) \tag{C.1}$$

where Ma denotes the Mach number (measuring the compressibility of the fluid) and Kn is the Knudsen number, defined as the ratio between the mean free path and the observation length scale. The precise formulation of the collision operator Q will not be useful for the sequel of our presentation. What is needed is the physics encoded in this mathematical operator.

Because collisions are assumed to be elastic, Q has some symmetry properties leading to the following identities

$$\int Q(f,f)(v)\,dv = \int Q(f,f)v_i\,dv = \int Q(f,f)|v|^2\,dv = 0. \tag{C.2}$$

In particular, integrating the kinetic equation (C.1) against 1, v and $\frac{1}{2}|v|^2$, we recover the local conservations of mass, momentum and energy, or in other words the *first principle of thermodynamics*.

Using the same symmetries, we also obtain

$$D(f) \stackrel{\text{def}}{=} -\int Q(f,f)\ln f(v)\,dv \geqslant 0 \tag{C.3}$$

and thus the local decay of entropy (note that the mathematical entropy is the opposite of the physical entropy!):

$$Ma\,\partial_t \int f \ln f\,dv + \nabla_x \cdot \int f \ln f(v) v\,dv \leqslant 0. \tag{C.4}$$

We therefore obtain a Lyapunov functional for the Boltzmann equation, which expresses the irreversibility predicted by the *second principle of thermodynamics*.

Collisions are responsible of that relaxation process: each elementary process looses some information on the precise microscopic configuration that is realized, so that the global effect of collisions is to increase the uncertainty. The asymptotic distribution, which minimizes the entropy, and cancels the entropy dissipation

$$\int Q(f,f)\ln f(v)\,dv = 0 \quad \Leftrightarrow \quad \forall v \in \mathbf{R}^3,\ Q(f,f)(v) = 0,$$

is the Gaussian having the same mass $R = \int f\,dv$, momentum $RU = \int f v\,dv$ and energy $\frac{1}{2}R(U^2 + 3\Theta) = \frac{1}{2}\int f|v|^2\,dv$. In other words, the thermodynamic equilibrium obeys *Maxwell's statistics*.

C.1. The incompressible inviscid regime

In the fast relaxation limit, i.e. when the Knudsen number *Kn* tends to zero, we thus expect the gas to be at local thermodynamic equilibrium. The distribution f is therefore completely determined by the macroscopic quantities $R = R(t,x)$, $U = U(t,x)$ and $\Theta = \Theta(t,x)$.

Let us first recall that, at leading order with respect to the Knudsen number *Kn*, the hydrodynamic equations, obtained from the local conservation laws replacing f by the corresponding local Maxwellian

$$f(t,x,v) \sim \frac{R(t,x)}{(2\pi\Theta(tx,x))^{3/2}} \exp\left(-\frac{|v-U(t,x)|^2}{2\Theta(t,x)}\right),$$

are, up to terms of order O(*Kn*),

$$\begin{aligned} &Ma\,\partial_t R + \nabla_x \cdot (RU) = 0,\\ &Ma\,\partial_t (RU) + \nabla_x \cdot (RU \otimes U + R\Theta \mathrm{Id}) = 0,\\ &Ma\,\partial_t\left(\frac{1}{2}RU^2 + \frac{3}{2}R\Theta\right) + \nabla_x \cdot \left(\frac{1}{2}RU^2 U + \frac{5}{2}R\Theta U\right) = 0, \end{aligned} \tag{C.5}$$

known as the compressible Euler system for perfect gases.

Of course such an asymptotics does not remain relevant if the Mach number *Ma* also goes to zero in the regime to be considered. The Mach number *Ma* – defined as the ratio between the bulk velocity of the fluid and the speed of sound – measures indeed the compressibility of the fluid: if $Ma \to 0$ the first equation in (C.5) is nothing else than the incompressibility constraint $\nabla \cdot (RU) = 0$. The equations of motion are then obtained by a systematic multiscale expansion. Their precise formulation depends further on another important feature of the fluid, namely on its viscosity, which is measured by the Reynold's number *Re*.

Note that, for perfect gases (i.e. for all gases which can be described by Boltzmann's equation), the Von Karmann's relation states

$$Re = \frac{Ma}{Kn}$$

so that all features of the fluid are completely determined by the two nondimensional parameters Ma and Kn (see [1] for more details).

C.2. Taking limits as $\epsilon \to 0$

In all the sequel we are interested in the regimes leading to the incompressible Euler equations: we will thus choose

$$Ma = \epsilon, \qquad Kn = \epsilon^q \quad \text{with } q > 1,$$

meaning in particular that $U/\sqrt{\Theta} = O(\epsilon)$, and we consider the limit $\epsilon \to 0$.

We will further restrict our attention to the homogeneous case, in the sense that the density R and temperature Θ will be assumed to be fluctuations of order ϵ around their equilibrium values, say without loss of generality around 1. Denoting by ρ, u and θ the fluctuations of mass, momentum and temperature, and plugging the expansions

$$R = 1 + \epsilon\rho, \qquad U = \epsilon u, \qquad \Theta = 1 + \epsilon\theta,$$

in the previous hydrodynamic equations (C.5), we get at leading order with respect to ϵ

$$\begin{aligned} &\nabla \cdot u = 0, \\ &\nabla(\rho + \theta) = 0, \end{aligned} \tag{C.6}$$

which are the macroscopic constraints (incompressibility and Boussinesq relations), then at second order the equations of motion

$$\begin{aligned} &\partial_t u + (u \cdot \nabla_x)u + \nabla p = 0, \\ &\partial_t \theta + \nabla_x \cdot (\theta u) = 0. \end{aligned} \tag{C.7}$$

Remarks. Dealing with the more general case when R and Θ have variations of order 1 is not really more difficult from a formal point of view. We actually get the following asymptotics

$$\begin{aligned} &\nabla \cdot u = 0, \\ &\nabla(R\Theta) = 0, \end{aligned}$$

and

$$\begin{aligned} &\partial_t R + \nabla_x \cdot (Ru) = 0, \\ &\partial_t u + (u \cdot \nabla_x)u + \frac{1}{R}\nabla p = 0. \end{aligned}$$

The point is that the mathematical theory of that system is not so well understood as for (C.6), (C.7). Furthermore the asymptotic analysis would require to control quantities of different sizes (namely $R = O(1)$, $\Theta = O(1)$ and $U = O(\epsilon)$), and thus to introduce new mathematical tools.

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