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Anti-symmetric Hamiltonians (II): Variational resolutions for Navier–Stokes and other nonlinear evolutions

Hamiltoniens anti-symmétriques : Résolution variationnelle des équations de Navier–Stokes et d'autres évolutions nonlinéaires

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Abstract

The nonlinear selfdual variational principle established in a preceding paper [N. Ghoussoub, Anti-symmetric Hamiltonians: Variational resolution of Navier–Stokes equations and other nonlinear evolutions, Comm. Pure Appl. Math. 60 (5) (2007) 619–653] – though good enough to be readily applicable in many stationary nonlinear partial differential equations – did not however cover the case of nonlinear evolutions such as the Navier–Stokes equations. One of the reasons is the prohibitive coercivity condition that is not satisfied by the corresponding selfdual functional on the relevant path space. We show here that such a principle still hold for functionals of the form

$$I(u)=\int_0^T\left[L\big(t,u(t),\dot u(t)+\Lambda u(t)\big)+\big\langle \Lambda u(t),u(t)\big\rangle\right]dt+\ell\left(u(0)-u(T),\frac{u(T)+u(0)}{2}\right)$$

where L (resp., ℓ) is an anti-selfdual Lagrangian on state space (resp., boundary space), and Λ is an appropriate nonlinear operator on path space. As a consequence, we provide a variational formulation and resolution to evolution equations involving nonlinear operators such as the Navier–Stokes equation (in dimensions 2 and 3) with various boundary conditions. In dimension 2, we recover the well-known solutions for the corresponding initial-value problem as well as periodic and anti-periodic ones, while in dimension 3 we get Leray solutions for the initial-value problems, but also solutions satisfying $u(0)=\alpha u(T)$ for any given α in $(-1,1)$. Our approach is quite general and does apply to many other situations.

Résumé

Le principe variationnel auto-dual nonlinéaire établi par le premier auteur dans un article antérieur – quoique suffisant pour les équations nonlinéaires stationnaires – ne couvrait pas le cas des équations d'évolution de Navier–Stokes. Celà est dû aux

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hypothèses de coercivité forte requises, qui sont rarement satisfaites par les fonctionnelles auto-duales une fois définies sur les espaces de trajectoires. Dans cet article, on établit un nouveau principe variationnel qui s'applique à des fonctionnelles de la forme

$$I(u)=\int_0^T \left[L\left(t,u(t),\dot u(t)+\Lambda u(t)\right)+\left\langle \Lambda u(t),u(t)\right\rangle\right]dt+\ell\left(u(0)-u(T),\frac{u(T)+u(0)}{2}\right)$$

où L (resp., ℓ) est un Lagrangien anti-autodual sur l'espace des états (resp., sur la frontière), et Λ est un opérateur convenable sur un espace de trajectoires. Comme application, on retrouve variationellement entre autres, les solutions de Leray pour les équations de Navier–Stokes en dimension 2 et 3 avec, soit des conditions initiales, ou soit des conditions au bord de type périodiques. L'approche est assez générale pour s'appliquer à d'autres équations d'évolution non linéaire.

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1. Introduction

This paper is a continuation of [5] where the first-named author established a general nonlinear selfdual variational principle, that yields a variational formulation and resolution for several nonlinear partial differential equations which are not normally of Euler–Lagrange type. Applications included nonlinear transport equations, the stationary Navier–Stokes equations, and the generalized Choquard–Pekar Schrödinger equations with certain nonlocal potentials. The principle did not however cover Leray's existence results for Navier–Stokes evolutions in low dimensions [7,8]. The primary objective of this paper is to develop a sharper selfdual variational principle to be able to deal with this shortcoming, and to encompass a larger class of nonlinear evolution equations in its scope of applications.

We first recall the basic concept of *selfduality*. It relates to the following class of Lagrangians which play a significant role in our proposed variational formulation. If X is a reflexive Banach space, and $L: X\times X^*\to \mathbb{R}\cup\{+\infty\}$ is a convex lower semi-continuous function, that is not identically equal to $+\infty$, we say that L is an *anti-selfdual Lagrangian* (*ASD*) on $X\times X^*$ if

$$L^*(p,x)=L(-x,-p)\quad \text{for all } (p,x)\in X^*\times X, \tag{1}$$

where L^* is the Legendre–Fenchel dual (in both variables) of L, defined on $X^*\times X$ as:

$$L^*(q,y)=\sup\left\{\langle q,x\rangle+\langle p,y\rangle-L(x,p);\ x\in X,\ p\in X^*\right\}.$$

We shall frequently use the following basic properties of an ASD Lagrangian:

$$L(x,p)+\langle x,p\rangle\geqslant 0\quad \text{for every } (x,p)\in X\times X^*, \tag{2}$$

and the fact that

$$L(x,p)+\langle x,p\rangle=0\quad \text{if and only if}\quad (-p,-x)\in\partial L(x,p). \tag{3}$$

We therefore define the *derived vector fields of* L at $x\in X$ to be the – possibly empty – sets

$$\bar\partial L(x):=\left\{p\in X^*;\ L(x,-p)-\langle x,p\rangle=0\right\}=\left\{p\in X^*;\ (p,-x)\in\partial L(x,-p)\right\}. \tag{4}$$

These *anti-selfdual vector fields* are natural extensions of subdifferentials of convex lower semi-continuous functions. Indeed, the most basic anti-selfdual Lagrangians are of the form $L(x,p)=\varphi(x)+\varphi^*(-p)$ where φ is such a function in X, and φ^* is its Legendre conjugate on X^*, in which case $\bar\partial L(x)=\partial\varphi(x)$. More interesting examples of anti-selfdual Lagrangians are of the form $L(x,p)=\varphi(x)+\varphi^*(-\Gamma x-p)$ where φ is a convex and lower semi-continuous function on X, and $\Gamma: X\to X^*$ is a skew adjoint operator. The corresponding anti-selfdual vector field is then $\bar\partial L(x)=\Gamma x+\partial\varphi(x)$. Actually, it turned out that every *maximal monotone operator* is an *anti-selfdual vector field* (see for example [6]). This means that ASD-Lagrangians can be seen as the *potentials* of maximal monotone operators, in the same way as the Dirichlet integral is the potential of the Laplacian operator (and more generally as any convex lower semi-continuous energy is a potential for its own subdifferential), leading to a variational formulation and resolution of most equations involving maximal monotone operators.

In this article, we develop further the approach – introduced in [5] – to allow for a variational resolution of nonlinear PDEs of the form

$$\Lambda u + \bar{\partial} L(u) = 0, \tag{5}$$

and nonlinear evolution equations of the form

$$\dot{u}(t) + \Lambda u(t) + \bar{\partial} L\big(u(t)\big) = 0 \quad \text{starting at } u(0) = u_0, \tag{6}$$

where L is an anti-selfdual Lagrangian and $\Lambda : D(\Lambda) \subset X \to X^*$ is a nonlinear *regular map*, that is if

$$\Lambda \text{ is weak-to-weak continuous and } u \to \langle \Lambda u, u\rangle \text{ is weakly lower semi-continuous on } D(\Lambda). \tag{7}$$

We note that positive linear operators are necessarily *regular maps*, but that there is also a wide class of nonlinear regular operators, such as those appearing in the basic equations of hydrodynamics and magnetohydrodynamics (see below and [9]).

Our approach is based on the following simple observation: If L is an anti-selfdual Lagrangian on $X \times X^*$, then for any map $\Lambda : D(\Lambda) \subset X \to X^*$, we have from (2) and (3) above that

$$I(x) := L(x, \Lambda x) + \langle x, \Lambda x\rangle \geqslant 0 \quad \text{for all } x \in D(\Lambda), \tag{8}$$

and that Eq. (5) is satisfied by $\bar{x} \in X$ provided the infimum of I is equal to zero and that it is attained at $\bar{x}$. The following theorem established in [5] provides conditions under which such an existence result holds.

Theorem 1.1. *Let L be an anti-selfdual Lagrangian on a reflexive Banach space X and let H_L be its Hamiltonian. If $\Lambda : D(\Lambda) \subset X \to X^*$ is a regular map such that $\overline{\mathrm{Dom}_1(L)} \subset D(\Lambda)$ and*

$$\lim_{\|x\| \to +\infty} H_L(0, -x) + \langle \Lambda x, x\rangle = +\infty, \tag{9}$$

then the functional $I(x) = L(x, \Lambda x) + \langle \Lambda x, x\rangle$ attains its minimum at $\bar{x} \in D(\Lambda)$ in such a way that:

$$I(\bar{x}) = \inf_{x \in D(\Lambda)} I(x) = 0, \tag{10}$$

$$0 \in \Lambda \bar{x} + \bar{\partial} L(\bar{x}). \tag{11}$$

We have denoted here the *effective domain* of L by $\mathrm{Dom}(L) = \{(x, p) \in X \times X^*;\ L(x, p) < +\infty\}$, and by $\mathrm{Dom}_1(L)$ its projection on X, that is $\mathrm{Dom}_1(L) = \{x \in X;\ L(x, p) < +\infty \text{ for some } p \in X^*\}$.

The Hamiltonian $H_L : X \times X \to \bar{\mathbb{R}}$ of L is defined by:

$$H_L(x, y) = \sup\big\{\langle y, p\rangle - L(x, p);\ p \in X^*\big\},$$

which is the Legendre transform in the second variable.

As shown in [5], Theorem 1.1 applies readily to many nonlinear stationary equations giving variational proofs of existence of solutions. For example, one can obtain (weak) solutions of the incompressible stationary Navier–Stokes equation on a smooth bounded domain Ω of $\mathbb{R}^3$

$$\begin{cases} (u \cdot \nabla)u + f = \nu \Delta u - \nabla p & \text{on } \Omega, \\ \operatorname{div} u = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{12}$$

where $\nu > 0$ and $f \in L^p(\Omega; \mathbb{R}^3)$, as follows. Letting

$$\Phi(u) = \frac{\nu}{2} \int_\Omega \sum_{j,k=1}^{3} \left(\frac{\partial u_j}{\partial x_k}\right)^2 dx + \int_\Omega \sum_{j=1}^{3} f_j u_j \, dx \tag{13}$$

be the convex continuous function on the space $X = \{u \in H_0^1(\Omega; \mathbf{R}^3);\ \operatorname{div} v = 0\}$, and Φ^* be its Legendre transform on X^*, Eq. (12) can then be reformulated as

$$\begin{cases} \Lambda u = -\partial \Phi(u) = \nu \Delta u - f - \nabla p, \\ u \in X, \end{cases} \tag{14}$$

where $\Lambda : X \to X^*$ is the regular nonlinear operator defined as

$$\langle \Lambda u, v\rangle = \int_\Omega \sum_{j,k=1}^{3} u_k \frac{\partial u_j}{\partial x_k} v_j \, dx = \big\langle (u\cdot\nabla)u, v\big\rangle. \tag{15}$$

Theorem 1.1 then readily yields that if $p > \frac{6}{5}$, then the infimum of the functional

$$I(u) = \Phi(u) + \Phi^*\big(-(u\cdot\nabla)u\big) \tag{16}$$

on X is equal to zero, and is attained at a solution of (12). Theorem 1.1 does not however cover the case of nonlinear evolutions such as the Navier–Stokes equations. This is because of the prohibitive coercivity condition (9) that is not satisfied by the corresponding selfdual functional on the relevant path space. We shall therefore prove a similar result under a more relaxed coercivity condition that will allow us to prove a selfdual variational principle that is more appropriate to nonlinear evolution equations. The concept can be seen as a selfdual version of the classical Palais–Smale condition in standard variational problems. Indeed, if I is a selfdual functional of the form $I(u) = L(u, \Lambda u) + \langle u, \Lambda u\rangle$, then its *stationary states* correspond to those points u where $I(u) = \inf I = 0$, in which case they satisfy the equation $\bar\partial L(u) + \Lambda u = 0$. So by analogy to classical variational theory, we introduce the following.

Assume J to be a duality map from X to X^*, i.e., for every $u \in X$, Ju is the element of the dual X^* that is uniquely determined by the relation

$$\langle Ju, u\rangle = \|u\|_X^2 \quad \text{and} \quad \|Ju\|_{X^*} = \|u\|_X. \tag{17}$$

It is well known that if X is a reflexive Banach space equipped with a strictly convex norm, then J is one-to-one and onto X^*, while being monotone and continuous from X (with its strong topology) to X^* equipped with its weak topology.

Definition 1.2. Given a map $\Lambda : D(\Lambda) \subset X \to X^*$, and a Lagrangian L on $X \times X^*$.

1. Say that $(u_n)_n$ is a *selfdual Palais–Smale sequence* for the functional $I_{L,\Lambda}(u) = L(u, \Lambda u) + \langle u, \Lambda u\rangle$, if for some $\epsilon_n \to 0$ it satisfies
$$\Lambda u_n + \bar\partial L(u_n) = -\epsilon_n J u_n. \tag{18}$$
2. The functional $I_{L,\Lambda}$ is said to satisfy the *selfdual Palais–Smale condition* (selfdual-PS), if every selfdual Palais–Smale sequence for $I_{L,\Lambda}$ is bounded in X.
3. The functional $I_{L,\Lambda}$ is said to be *weakly coercive* if
$$\lim_{\|x_n\|\to+\infty} L\left(x_n, \Lambda x_n + \frac{1}{n} J x_n\right) + \langle x_n, \Lambda x_n\rangle + \frac{1}{n}\|x_n\|^2 = +\infty. \tag{19}$$

It is clear that a weakly coercive functional necessarily satisfies the selfdual Palais–Smale condition. On the other hand, a strongly coercive selfdual functional (i.e., if it satisfies (9)) is necessarily weakly coercive.

In the dynamic case, one considers an evolution triple $X \subset H \subset X^*$ where H is a Hilbert space equipped with $\langle\,,\rangle$ as scalar product, and where X is a dense vector subspace of H, that is a reflexive Banach space once equipped with its own norm $\|\cdot\|$. Let $[0,T]$ be a fixed real interval and consider for $p, q > 1$, the Banach space L^p_X as well as the space $\mathcal{X}_{p,q}$ of all functions in L^p_X such that $\dot u \in L^q_{X^*}$, equipped with the norm

$$\|u\|_{\mathcal{X}_{p,q}} = \|u\|_{L^p_X} + \|\dot u\|_{L^q_{X^*}}.$$

Let now L be a time-dependent anti-selfdual Lagrangian on $[0,T]\times X\times X^*$, ℓ an anti-selfdual Lagrangian on $H\times H$, and let $\Lambda : \mathcal{X}_{p,q} \to L^q_{X^*}$ be a given map. We shall make use of the selfdual Palais–Smale property for the following type of selfdual functionals on path space.

$$I_{L,\ell,\Lambda}(u) = \int_0^T \big[L\big(t, u(t), \dot u(t) + \Lambda u(t)\big) + \big\langle \Lambda u(t), u(t)\big\rangle\big]\, dt + \ell\left(u(0) - u(T), \frac{u(T)+u(0)}{2}\right). \tag{20}$$

In this case, $I_{L,\ell,\Lambda}$ is said to satisfy the *selfdual Palais–Smale condition on* $\mathcal{X}_{p,q}$ if any sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{X}_{p,q}$ satisfying

$$\begin{cases} \dot{x}_n(t) + \Lambda x_n(t) + \epsilon_n \|x_n\|^{p-2} J x_n(t) \in -\bar{\partial} L(t, x_n(t)) \quad \text{a.a. } t \in [0,T], \\ \frac{x_n(0)+x_n(T)}{2} \in \bar{\partial}\ell(x_n(0) - x_n(T)) \end{cases} \tag{21}$$

for some $\epsilon_n \to 0$, is necessarily bounded in $\mathcal{X}_{p,q}$.

Similarly, $I_{L,\ell,\Lambda}$ is said to be *weakly coercive* if for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{X}_{p,q}$ we have

$$\begin{aligned} \lim_{\|x_n\|_{\mathcal{X}_{p,q}} \to +\infty} \int_0^T \Big[L\Big(t, x_n(t), \dot{x}_n(t) + \Lambda x_n(t) + \frac{1}{n}\|x_n\|^{p-2} J x_n(t)\Big) + \big\langle x_n(t), \Lambda x_n(t)\big\rangle + \frac{1}{n}\big\|x_n(t)\big\|^p \Big]\, dt \\ + \ell\Big(x_n(0) - x_n(T), \frac{x_n(T) + x_n(0)}{2}\Big) = +\infty. \end{aligned} \tag{22}$$

Here is one useful corollary of the variational principle we establish for nonlinear evolutions in Section 3.

Theorem 1.3. *Let $X \subset H \subset X^*$ be an evolution triple where X is a reflexive Banach space, and H is a Hilbert space. For $p > 1$ and $q = \frac{p}{p-1}$, assume that $\Lambda : \mathcal{X}_{p,q} \to L^q_{X^*}$ is a regular map such that for some nondecreasing continuous real function w, and $0 \leqslant k < 1$, it satisfies*

$$\|\Lambda x\|_{L^q_{X^*}} \leqslant k\|\dot{x}\|_{L^q_{X^*}} + w\big(\|x\|_{L^p_X}\big) \quad \textit{for every } x \in \mathcal{X}_{p,q}, \tag{23}$$

and

$$\left| \int_0^T \big\langle \Lambda x(t), x(t)\big\rangle\, dt \right| \leqslant w\big(\|x\|_{L^p_X}\big) \quad \textit{for every } x \in \mathcal{X}_{p,q}. \tag{24}$$

Let ℓ be an anti-selfdual Lagrangian on $H \times H$ that is bounded below with $0 \in \mathrm{Dom}(\ell)$, and let L be a time dependent anti-selfdual Lagrangian on $[0,T] \times X \times X^*$ such that for some $C > 0$ and $r > 1$, we have

$$\int_0^T L\big(t, u(t), 0\big)\, dt \leqslant C\big(1 + \|u\|^r_{L^p_X}\big) \quad \textit{for every } u \in L^p_X. \tag{25}$$

The functional

$$I(u) = \int_0^T \big[L\big(t, u(t), \dot{u}(t) + \Lambda u(t)\big) + \big\langle \Lambda u(t), u(t)\big\rangle\big]\, dt + \ell\Big(u(0) - u(T), \frac{u(T) + u(0)}{2}\Big) \tag{26}$$

is then selfdual on $\mathcal{X}_{p,q}$, and if in addition it satisfies the selfdual Palais–Smale condition, then it attains its minimum at $v \in \mathcal{X}_{p,q}$ in such a way that $I(v) = \inf_{u \in \mathcal{X}_{p,q}} I(u) = 0$ and

$$\begin{cases} -\Lambda v(t) - \dot{v}(t) \in \bar{\partial} L(t, v(t)) \quad \textit{a.e on } [0,T], \\ -\frac{v(0)+v(T)}{2} \in \bar{\partial}\ell(v(0) - v(T)). \end{cases} \tag{27}$$

Now while the main Lagrangian L is expected to be smooth and hence its subdifferential coincides with its gradient, and the differential inclusion is often an equation, it is crucial that the boundary Lagrangian ℓ be allowed to be degenerate so that its subdifferential can cover the various boundary conditions discussed below.

As a consequence of the above theorem, we provide a variational resolution to evolution equations involving nonlinear operators such as the Navier–Stokes equation with various boundary conditions. Indeed, by considering

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + f = \nu \Delta u - \nabla p & \text{on } \Omega \subset \mathbb{R}^n, \\ \operatorname{div} u = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

where $f \in L^2_{X^*}([0,T])$, $X = \{u \in H^1_0(\Omega; \mathbf{R}^n);\ \operatorname{div} v = 0\}$, and $H = L^2(\Omega)$, we can associate the nonlinear operator equation

$$\begin{cases} \frac{\partial u}{\partial t} + \Lambda u \in -\partial \Phi(t,u) \quad \text{on } [0,T], \\ \frac{u(0)+u(T)}{2} \in -\bar{\partial}\ell(u(0)-u(T)). \end{cases} \tag{29}$$

where ℓ is any anti-selfdual Lagrangian on $H \times H$, while Φ and Λ are defined in (13) and (15), respectively.

Note that Λ maps X into its dual X^* as long as the dimension $N \leqslant 4$. On the other hand, if we lift Λ to path space by defining $(\Lambda u)(t) = \Lambda(u(t))$, we have the following facts:

- If $N = 2$, then Λ is a regular operator from $\mathcal{X}_{2,2}[0,T]$ into $L^2_{X^*}[0,T]$.
- However, if $N = 3$, we then have that Λ is a regular operator from $\mathcal{X}_{2,2}[0,T]$ into $L^{4/3}_{X^*}[0,T]$.

We therefore distinguish the two cases.

Corollary 1.4. *Assuming $N = 2$, f in $L^2_{X^*}([0,T])$, and ℓ to be an anti-selfdual Lagrangian on $H \times H$ that is bounded from below, then the infimum of the functional*

$$I(u) = \int_0^T \left[\Phi\big(t,u(t)\big) + \Phi^*\big(t, -\dot{u}(t) - (u\cdot\nabla)u(t)\big)\right] dt + \ell\left(u(0)-u(T), \frac{u(0)+u(T)}{2}\right)$$

on $\mathcal{X}_{2,2}$ is zero and is attained at a solution u of (28) *that satisfies the following time-boundary condition*:

$$-\frac{u(0)+u(T)}{2} \in \bar{\partial}\ell\big(u(0)-u(T)\big). \tag{30}$$

Moreover, u verifies the following "energy identity":

$$\big\|u(t)\big\|_H^2 + 2\int_0^t \left[\Phi\big(t,u(t)\big) + \Phi^*\big(t,-\dot{u}(t) - (u\cdot\nabla)u(t)\big)\right] dt = \big\|u(0)\big\|_H^2 \quad \textit{for every } t \in [0,T]. \tag{31}$$

In particular, with appropriate choices for the boundary Lagrangian ℓ, the solution u can be chosen to verify either one of the following boundary conditions:

- *an initial value problem*: $u(0) = u_0$ *where u_0 is a given function in X*;
- *a periodic orbit*: $u(0) = u(T)$;
- *an anti-periodic orbit*: $u(0) = -u(T)$.

However, in the three dimensional case, we have to settle for the following result.

Corollary 1.5. *Assume $N = 3$, f in $L^2_{X^*}([0,T])$, and consider ℓ to be an anti-selfdual Lagrangian on $H \times H$ that is now coercive in both variables. Then, there exists $u \in \mathcal{X}_{2,\frac{4}{3}}$ such that*

$$I(u) = \int_0^T \left[\Phi\big(t,u(t)\big) + \Phi^*\big(t,-\dot{u}(t) - (u\cdot\nabla)u(t)\big)\right] dt + \ell\left(u(0)-u(T), \frac{u(0)+u(T)}{2}\right) \leqslant 0,$$

and u is a weak solution of (28) *that satisfies the time-boundary condition* (30). *Moreover, u verifies the following "energy inequality"*:

$$\frac{\|u(T)\|_H^2}{2} + \int_0^T \left[\Phi\big(t,u(t)\big) + \Phi^*\big(t,-\dot{u}(t) - (u\cdot\nabla)u(t)\big)\right] dt \leqslant \frac{\|u(0)\|_H^2}{2}. \tag{32}$$

In particular, with appropriate choices for the boundary Lagrangian ℓ, the solution u will verify either one of the following boundary conditions:

- *an initial value problem*: $u(0) = u_0$;
- *a periodicity condition of the form*: $u(0) = \alpha u(T)$, *for any given* α *with* $-1 < \alpha < 1$.

The above results are actually particular cases of a much more general nonlinear selfdual variational principle which applies to both the stationary and to the dynamic case. It will be stated and established in full generality in the next section.

2. Basic properties of selfdual functionals

Consider the Hamiltonian $H = H_L$ associated to an ASD Lagrangian L on $X \times X^*$. It is easy to check that $H : X \times X \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ then satisfies:

- for each $y \in X$, the function $H_y : x \to -H(x, y)$ from X to $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ is convex;
- the function $x \to H(-y, -x)$ is the convex lower semi-continuous envelope of H_y.

It readily follows that for such a Hamiltonian, the function $y \to H(x, y)$ is convex and lower semi-continuous for each $x \in X$, and that the following inequality holds:

$$H(-y, -x) \leqslant -H(x, y) \quad \text{for every } (x, y) \in X \times X. \tag{33}$$

In particular, we have

$$H(x, -x) \leqslant 0 \quad \text{for every } x \in X. \tag{34}$$

Note that H_L is always concave in the first variable, however, it is not necessarily upper semi-continuous in the first variable (see [3], p. 55).

Another property of ASD Lagrangians that will be used in the sequel is the following: If we define the following operation on two ASD Lagrangians L and M on $X \times X^*$,

$$L \oplus M(x, p) = \inf\bigl\{L(x, r) + M(x, p - r);\ r \in X^*\bigr\}, \tag{35}$$

then we have for any $(x, p) \in X \times X^*$,

$$L \oplus M(x, p) = \sup\bigl\{\langle y, -p\rangle + H_L(y, -x) + H_M(y, -x);\ y \in X\bigr\}. \tag{36}$$

As in [5], we consider the following notion which extends considerably the class of Hamiltonians associated to selfdual Lagrangians.

Definition 2.1. Let E be a convex subset of a reflexive Banach space X.

1. A functional $M : E \times E \to \mathbb{R}$ is said to be *an anti-symmetric Hamiltonian* on $E \times E$ if it satisfies the following conditions:

$$\text{For every } x \in E, \text{ the function } y \to M(x, y) \text{ is concave on } E, \tag{37}$$

$$M(x, x) \leqslant 0 \quad \text{for every } x \in E. \tag{38}$$

2. It is said to be a *regular anti-symmetric Hamiltonian* if in addition it satisfies:

$$\text{For every } y \in E, \text{ the function } x \to M(x, y) \text{ is weakly lower semi-continuous on } E. \tag{39}$$

The class of *regular anti-symmetric Hamiltonians* on a given convex set E – denoted $\mathcal{H}^{\mathrm{asym}}(E)$ – is an interesting class of its own. It contains the "Maxwellian" Hamiltonians $H(x, y) = \varphi(y) - \varphi(-x) + \langle Ay, x\rangle$, where φ is convex and A is skew-adjoint. More generally,

1. If L is an anti-selfdual Lagrangian on a Banach space X, then the Hamiltonian $M(x, y) = H_L(y, -x)$ is in $\mathcal{H}^{\mathrm{asym}}(X)$.

2. If $\Lambda : D(\Lambda) \subset X \to X^*$ is a – nonnecessarily linear – regular map, then the Hamiltonian $H(x, y) = \langle x - y, \Lambda x \rangle$ is in $\mathcal{H}^{\text{asym}}(D(\Lambda))$.

Since $\mathcal{H}^{\text{asym}}(X)$ is obviously a convex cone, we can therefore superpose certain nonlinear operators with anti-selfdual Lagrangians, via their corresponding anti-symmetric Hamiltonians, to obtain a remarkably rich family that generates nonconvex selfdual functionals as follows.

Definition 2.2. A functional $I : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *selfdual on a convex set $E \subset X$* if it is nonnegative and if there exists a regular anti-symmetric Hamiltonian $M : E \times E \to \mathbb{R}$ such that for every $x \in E$,

$$I(x) = \sup_{y \in E} M(x, y). \tag{40}$$

A key aspect of our variational approach is that solutions of many nonlinear PDEs can be obtained by minimizing properly chosen selfdual functionals in such a way that the infimum is actually zero. This is indeed the case in view of the following immediate application of a fundamental min–max theorem of Ky–Fan (see [1] or [2]).

Proposition 2.1. *Let $I : E \to \mathbb{R} \cup \{+\infty\}$ be a selfdual functional on a closed convex subset E of a reflexive Banach space X, with M being its corresponding anti-symmetric Hamiltonian on $E \times E$. If M is coercive in the following sense,*

$$\lim_{\|x\| \to +\infty} M(x, x_0) = +\infty \quad \text{for some } x_0 \in E, \tag{41}$$

then there exists $\bar{x} \in E$ such that $I(\bar{x}) = \sup_{y \in E} M(\bar{x}, y) = 0$.

The following was also proved in [5].

Proposition 2.2. *Let $X \subset H \subset X^*$ be an evolution triple and consider a time-dependent anti-selfdual Lagrangian L on $[0, T] \times X \times X^*$ such that*

$$\text{For each } r \in L^q_{X^*}, \text{ the map } u \to \int_0^T L\big(t, u(t), r(t)\big)\, dt \text{ is continuous on } L^p_X. \tag{42}$$

$$\text{The map } u \to \int_0^T L\big(t, u(t), 0\big)\, dt \text{ is bounded on the unit ball of } L^p_X. \tag{43}$$

Let ℓ be an anti-selfdual Lagrangian on $H \times H$ such that:

$$-C \leqslant \ell(a, b) \leqslant C\big(1 + \|a\|^2_H + \|b\|^2_H\big) \quad \text{for all } (a, b) \in H \times H. \tag{44}$$

Then the Lagrangian

$$\mathcal{L}(u, r) = \begin{cases} \int_0^T L(t, u(t), r(t) + \dot{u}(t))\, dt + \ell(u(0) - u(T), \frac{u(T)+u(0)}{2}) & \text{if } u \in \mathcal{X}_{p,q}, \\ +\infty & \text{otherwise} \end{cases}$$

is anti-selfdual on $L^p_X \times L^q_{X^*}$.

Consider now the following convex lower semi-continuous function on L^p_X:

$$\psi(u) = \begin{cases} \frac{1}{q} \int_0^T \|\dot{u}(t)\|^q_{X^*}\, dt & \text{if } u \in \mathcal{X}_{p,q}, \\ +\infty & \text{if } u \in L^p_X \setminus \mathcal{X}_{p,q}, \end{cases} \tag{45}$$

and for any $\mu > 0$, we let Ψ_μ be the anti-selfdual Lagrangian on $L^p_X \times L^q_{X^*}$ defined by

$$\Psi_\mu(u, r) = \mu\psi(u) + \mu\psi^*\bigg(-\frac{r}{\mu}\bigg). \tag{46}$$

Now for each $(u,r)\in L^p_X\times L^q_{X^*}$, define

$$\mathcal{L}\oplus\Psi_\mu(u,r):=\inf_{s\in L^q_{X^*}}\big\{\mathcal{L}(u,s)+\Psi_\mu(u,r-s)\big\}. \tag{47}$$

Lemma 2.3. *Let L and ℓ be two anti-selfdual Lagrangians verifying the hypothesis of Proposition 2.2, and let $\mathcal{L}$ be the corresponding anti-selfdual Lagrangian on path space $L^p_X\times L^q_{X^*}$. Suppose Γ is a regular operator from $\mathcal{X}_{p,q}$ into $L^q_{X^*}$ then,*

1. *The functional*

$$I_\mu(u)=\mathcal{L}\oplus\Psi_\mu(u,\Gamma u)+\int_0^T\big\langle\Gamma u(t),u(t)\big\rangle\,dt$$

is then selfdual on $\mathcal{X}_{p,q}$, and its corresponding anti-symmetric Hamiltonian on $\mathcal{X}_{p,q}\times\mathcal{X}_{p,q}$ is

$$M_\mu(u,v):=\int_0^T\big\langle\Gamma u(t),u(t)-v(t)\big\rangle\,dt+H_{\mathcal{L}}(v,-u)+\mu\psi(u)-\mu\psi(v),$$

where $H_{\mathcal{L}}(v,u)=\sup_{r\in L^q_{X^*}}\{\int_0^T\langle r,u\rangle\,dt-\mathcal{L}(v,r)\}$ is the Hamiltonian of $\mathcal{L}$ on $L^p_X\times L^p_X$.

2. *If in addition $\lim_{\|u\|_{\mathcal{X}_{p,q}}\to+\infty}\int_0^T\langle\Gamma u(t),u(t)\rangle\,dt+H_{\mathcal{L}}(0,-u)+\mu\psi(u)=+\infty$, then there exists $u\in\mathcal{X}_{p,q}$ with $\partial\psi(u)\in L^q_{X^*}$ such that*

$$\dot u(t)+\Gamma u(t)+\mu\partial\psi\big(u(t)\big)\in-\bar\partial L\big(t,u(t)\big)\quad\text{on }[0,T], \tag{48}$$

$$\frac{u(T)+u(0)}{2}\in-\bar\partial\ell\big(u(0)-u(T)\big), \tag{49}$$

$$\dot u(T)=\dot u(0)=0. \tag{50}$$

Proof. First note that since $\mathcal{L}$ and Ψ_μ are anti-selfdual, we have that $\mathcal{L}\oplus\Psi_\mu(u,r)+\langle u,r\rangle\geqslant 0$ for all $(u,r)\in L^p_X\times L^q_{X^*}$, and therefore $I(u)\geqslant 0$ on $\mathcal{X}_{p,q}$.

Now by (36), we have for any $(u,r)\in L^p_X\times L^q_{X^*}$,

$$\mathcal{L}\oplus\Psi_\mu(u,r)=\sup_{v\in L^p_X}\left\{\int_0^T\langle -r,v\rangle\,dt+H_{\mathcal{L}}(v,-u)+\mu\psi(-u)-\mu\psi(v)\right\}.$$

But for $u\in\mathcal{X}_{p,q}$ and $v\in L^p_X\setminus\mathcal{X}_{p,q}$, we have $H_{\mathcal{L}}(v,-u)=\sup_{r\in L^q_{X^*}}\{\int_0^T-\langle r,u\rangle\,dt-\mathcal{L}(v,r)\}=-\infty$, and therefore for any $u\in\mathcal{X}_{p,q}$, we have

$$\begin{aligned}\sup_{v\in\mathcal{X}_{p,q}}M_\mu(u,v)&=\sup_{v\in L^p_X}M_\mu(u,v)\\&=\int_0^T\big\langle\Gamma u(t),u(t)\big\rangle\,dt+\sup_{v\in L^p_X}\int_0^T\big\langle\Gamma u(t),-v(t)\big\rangle\,dt+H_{\mathcal{L}}(v,-u)+\mu\psi(u)-\mu\psi(v)\\&=\int_0^T\big\langle\Gamma u(t),u(t)\big\rangle\,dt+\mathcal{L}\oplus\Psi_\mu(u,\Gamma u)\\&=I(u).\end{aligned}$$

It follows from Proposition 2.1 that there exists $u_\mu\in\mathcal{X}_{p,q}$ such that

$$I_\mu(u_\mu) = \mathcal{L} \oplus \Psi_\mu(u_\mu, \Gamma u_\mu) + \int_0^T \langle \Gamma u_\mu(t), u_\mu(t)\rangle\, dt = 0. \tag{51}$$

Since $\mathcal{L} \oplus \Psi_\mu$ is convex and coercive in the second variable, there exists $r \in L^q_{X^*}$ such that

$$\mathcal{L} \oplus \Psi_\mu(u_\mu, \Gamma u_\mu) = \mathcal{L}(u_\mu, r) + \Psi_\mu(u_\mu, \Gamma u_\mu - r). \tag{52}$$

It follows that

$$\begin{aligned}
0 &= \mathcal{L}(u_\mu, r) + \Psi_\mu(u_\mu, \Gamma u_\mu - r) + \int_0^T \langle \Gamma u_\mu(t), u_\mu(t)\rangle\, dt \\
&= \int_0^T \big[L\big(t, u_\mu(t), \dot u_\mu(t) + r(t)\big) + \langle u_\mu(t), r(t)\rangle\big]\, dt + \ell\left(u_\mu(T) - u_\mu(0), \frac{u_\mu(T) + u_\mu(0)}{2}\right) \\
&\quad + \Psi_\mu(u_\mu, \Gamma u_\mu - r) + \int_0^T \langle \Gamma u_\mu(t) - r(t), u_\mu(t)\rangle\, dt \\
&= \int_0^T \big[L\big(t, u_\mu(t), \dot u_\mu(t) + r(t)\big) + \langle u_\mu(t), \dot u_\mu(t) + r(t)\rangle\big]\, dt - \frac{1}{2}\|u_\mu(T)\|^2 + \frac{1}{2}\|u_\mu(0)\|^2 \\
&\quad + \ell\left(u_\mu(0) - u_\mu(T), \frac{u_\mu(T) + u_\mu(0)}{2}\right) + \Psi_\mu(u_\mu, \Gamma u_\mu - r) + \int_0^T \langle \Gamma u_\mu - r, u_\mu(t)\rangle\, dt.
\end{aligned}$$

Since this is the sum of three nonnegative terms, we get the following three identities,

$$\int_0^T \big[L\big(t, u_\mu(t), \dot u_\mu(t) + r(t)\big) + \langle u_\mu, \dot u_\mu + r\rangle\big]\, dt = 0, \tag{53}$$

$$\Psi_\mu(u_\mu, \Gamma u_\mu - r) + \int_0^T \langle \Gamma u_\mu - r, u_\mu(t)\rangle\, dt = 0, \tag{54}$$

$$\ell\left(u_\mu(0) - u_\mu(T), \frac{u_\mu(T) + u_\mu(0)}{2}\right) - \frac{1}{2}\|u_\mu(T)\|^2 + \frac{1}{2}\|u_\mu(0)\|^2 = 0. \tag{55}$$

It follows from the limiting case of Fenchel duality that

$$\begin{aligned}
&\dot u_\mu(t) + \Gamma u_\mu(t) + \mu\partial\psi\big(u_\mu(t)\big) \in -\bar\partial L\big(t, u_\mu(t)\big) \quad \text{on } [0, T], \\
&\frac{u_\mu(T) + u_\mu(0)}{2} \in -\bar\partial\ell\big(u_\mu(0) - u_\mu(T)\big).
\end{aligned}$$

Since $u := u_\mu \in \mathcal{X}_{p,q}$, we have that $-\mu\partial\psi(u(t)) = \dot u(t) + \Gamma u(t) + \bar\partial L(t, u(t)) \in L^q_{X^*}$.

It follows that $\partial\psi(u(t)) = -\frac{d}{dt}(\|\dot u\|_*^{q-2} J^{-1}\dot u)$, where J is the duality map between L^p_X and $L^q_{X^*}$. Hence, for each $v \in \mathcal{X}_{p,q}$ we have

$$\begin{aligned}
0 &= \int_0^T \big[\langle \dot u(t) + \Gamma u(t) + \bar\partial L\big(t, u(t)\big), v\rangle + \mu\langle \|\dot u\|_*^{q-2} J^{-1}\dot u, \dot v\rangle\big]\, dt \\
&= \int_0^T \left\langle \dot u(t) + \Gamma u(t) - \mu\frac{d}{dt}\big(\|\dot u\|_*^{q-2} J^{-1}\dot u\big) + \bar\partial L\big(t, u(t)\big), v\right\rangle dt
\end{aligned}$$

$$+ \mu\langle \|\dot u(T)\|_*^{q-2} J^{-1}\dot u(T), v(T)\rangle - \mu\langle \|\dot u(0)\|_*^{q-2} J^{-1}\dot u(0), v(0)\rangle$$

from which we deduce that

$$\dot u(t) + \Gamma u(t) - \frac{d}{dt}\big(\|\dot u\|^{q-2} J^{-1}\dot u(t)\big) \in -\bar\partial L\big(t, u(t)\big) \quad \text{on } [0,T],$$
$$\dot u(T) = \dot u(0) = 0. \qquad \square$$

We shall make repeated use of the following lemma which describes three ways of regularizing an anti-selfdual Lagrangian by way of λ-convolution. It is an immediate consequence of the calculus of anti-selfdual Lagrangians developed in [4] to which we refer the reader.

Lemma 2.4. *For a Lagrangian $L : X \times X^* \to \mathbb{R} \cup \{+\infty\}$, define for every $(x, r) \in X \times X^*$*

$$L^1_\lambda(x,r) = \inf\left\{ L(y,r) + \frac{\|x-y\|_X^p}{\lambda p} + \frac{\lambda^{q-1}\|r\|_{X^*}^q}{q};\ y \in X\right\}$$

and

$$L^2_\lambda(x,r) = \inf\left\{ L(x,s) + \frac{\|r-s\|_{X^*}^q}{\lambda q} + \frac{\lambda^{p-1}\|x\|_X^p}{p};\ s \in X^*\right\}$$

and

$$L^{1,2}_\lambda(x,r) = \inf\left\{ L(y,s) + \frac{1}{2\lambda}\|x-y\|_X^2 + \frac{\lambda}{2}\|r\|_{X^*}^2 + \frac{1}{2\lambda}\|s-r\|_{X^*}^2 + \frac{\lambda}{2}\|y\|_X^2;\ y \in X,\ s \in X^*\right\}.$$

If L is anti-selfdual then the following hold:

1. *L^1_λ, L^2_λ and $L^{1,2}_\lambda$ are also anti-selfdual Lagrangians on $X \times X^*$.*
2. *L^1_λ (resp., L^2_λ) (resp., $L^{1,2}_\lambda$) is continuous in the first variable (resp., in the second variable) (resp., in both variables). Moreover, $\|\bar\partial L^1_\lambda(x)\| \leqslant \frac{\|x\|}{\lambda}$ for every $x \in X$.*
3. *$\bar\partial L^2_\lambda(x) = \bar\partial L(x) + \lambda^{p-1}\|x\|^{p-2} Jx$ for every $x \in X$.*
4. *$\bar\partial L^1_\lambda(x) = \bar\partial L(x + \lambda^{q-1}\|r\|^{q-2} J^{-1} r)$ for every $x \in X$ where $r = \bar\partial L(x)$.*
5. *Suppose L is bounded from below. If $x_\lambda \rightharpoonup x$ and $p_\lambda \rightharpoonup p$ weakly in X and X^* respectively as $\lambda \to 0$, and if $L^{1,2}_\lambda(x_\lambda, p_\lambda)$ is bounded from above, then*

$$L(x,p) \leqslant \liminf_{\lambda\to 0} L^{1,2}_\lambda(x_\lambda, p_\lambda).$$

Proof. It suffices to notice that $L^1_\lambda = L \star M_\lambda$ and $L^2_\lambda = L \oplus M_\lambda$ where $M_\lambda(x,r) = \psi_\lambda(x) + \psi^*_\lambda(r)$ with $\psi_\lambda(x) = \frac{1}{\lambda p}\|x\|^p$. Note that $L^{1,2}_\lambda = (L \oplus M_\lambda) \star M_\lambda$ with $M_\lambda(x,r) = \frac{1}{2\lambda}\|x\|^2 + \frac{\lambda}{2}\|r\|^2$. The rest follows from the calculus of selfdual Lagrangians developed in [4]. $\square$

3. A selfdual variational principle for nonlinear evolutions

This section is dedicated to the proof of the following general variational principle for nonlinear evolutions.

Theorem 3.1. *Let $X \subset H \subset X^*$ be an evolution triple where X is a reflexive Banach space, and H is a Hilbert space. Let L be a time dependent anti-selfdual Lagrangian on $[0,T] \times X \times X^*$ such that for some $C > 0$ and $r > 0$, we have*

$$\int_0^T L\big(t, u(t), 0\big)\, dt \leqslant C\big(1 + \|u\|^r_{L^p_X}\big) \quad \textit{for every } u \in L^p_X. \tag{56}$$

Let ℓ be an anti-selfdual Lagrangian on $H \times H$ that is bounded below with $0 \in \mathrm{Dom}(\ell)$, and consider $\Lambda : \mathcal{X}_{p,q} \to L^q_{X^*}$ to be a regular map such that for some $q > 1$:

$$\|\Lambda u\|_{L^q_{X^*}} \leqslant k\|\dot{u}\|_{L^q_{X^*}} + w\big(\|u\|_{L^p_X}\big) \quad \text{for every } u \in \mathcal{X}_{p,q}, \tag{57}$$

where w is a nondecreasing continuous real function and $0 < k < 1$. Assume that one of the following two conditions hold:

(A) $|\int_0^T \langle \Lambda u(t), u(t)\rangle\, dt| \leqslant w(\|u\|_{L^p_X})$ *for every* $u \in \mathcal{X}_{p,q}$.

(B) *For each $p \in L^q_{X^*}$, the functional $u \to \int_0^T L(t, u(t), p(t))\, dt$ is continuous on L^p_X, and there exists $C > 0$ such that for every $u \in L^p_X$ we have:*

$$\big\|\bar{\partial} L(t,u)\big\|_{L^q_{X^*}} \leqslant w\big(\|u\|_{L^p_X}\big), \tag{58}$$

$$\int_0^T \big\langle \bar{\partial} L\big(t, u(t)\big) + \Lambda u(t), u(t)\big\rangle\, dt \geqslant -C\big(\|u\|_{L^p_X} + 1\big). \tag{59}$$

Then the functional

$$I(u) = \int_0^T \big[L\big(t, u(t), \dot{u}(t) + \Lambda u(t)\big) + \big\langle \Lambda u(t), u(t)\big\rangle\big]\, dt + \ell\bigg(u(0) - u(T), \frac{u(T) + u(0)}{2}\bigg) \tag{60}$$

is selfdual on $\mathcal{X}_{p,q}$, and if in addition it is weakly coercive on that space, then it attains its minimum at $v \in \mathcal{X}_{p,q}$ in such a way that $I(v) = \inf_{u\in\mathcal{X}_{p,q}} I(u) = 0$ and

$$\begin{cases} -\Lambda v(t) - \dot{v}(t) = \bar{\partial} L(t, v(t)) & \text{on } [0,T], \\ -\frac{v(0)+v(T)}{2} \in \bar{\partial}\ell(v(0) - v(T)). \end{cases} \tag{61}$$

For the proof of Theorem 3.1, we start with the following proposition in which we consider a regularization (coercivization) of the anti-selfdual Lagrangian $\mathcal{L}$ by the ASD Lagrangian Ψ_μ, and also a perturbation of Λ by the operator

$$Ku = w\big(\|u\|_{L^p_X}\big) Ju + \|u\|^{p-1}_{L^p_X} Ju \tag{62}$$

which is regular from $\mathcal{X}_{p,q}$ into $L^q_{X^*}$.

Lemma 3.2. *Let Λ be a regular map from $\mathcal{X}_{p,q}$ into $L^q_{X^*}$ satisfying* (57). *Let L to be a time-dependent anti-selfdual Lagrangian on $[0,T] \times X \times X^*$, satisfying conditions* (42) *and* (43) *and let ℓ be an anti-selfdual Lagrangian on $H \times H$ satisfying condition* (44). *Then for any $\mu > 0$, the functional*

$$I_\mu(u) = \mathcal{L} \oplus \Psi_\mu(u, \Lambda u + Ku) + \int_0^T \big\langle \Lambda u(t) + Ku(t), u(t)\big\rangle\, dt$$

is selfdual on $\mathcal{X}_{p,q}$. Moreover, there exists $u_\mu \in \{u \in \mathcal{X}_{p,q};\ \partial\psi(u) \in L^q_{X^*},\ \dot{u}(T) = \dot{u}(0) = 0\}$ such that

$$\dot{u}_\mu(t) + \Lambda u_\mu(t) + K u_\mu(t) + \mu\partial\psi\big(u_\mu(t)\big) \in -\bar{\partial} L\big(t, u_\mu(t)\big) \quad \text{on } [0,T], \tag{63}$$

$$\ell\bigg(u_\mu(0) - u_\mu(T), \frac{u_\mu(T) + u_\mu(0)}{2}\bigg) = \int_0^T \big\langle \dot{u}_\mu(t), u_\mu(t)\big\rangle\, dt. \tag{64}$$

Proof. It suffices to apply Lemma 2.3 to the regular operator $\Gamma = \Lambda + K$, provided we show the required coercivity condition $\lim_{\|u\|_{\mathcal{X}_{p,q}} \to +\infty} M(u,0) = +\infty$ where

$$M(u,0) = \int_0^T \langle \Lambda u(t) + Ku(t), u(t)\rangle\,dt + H_{\mathcal{L}}(0,-u) + \mu\psi(u).$$

Note first that it follows from (57) that for $\epsilon < \frac{\mu}{q}$, there exists $C(\epsilon) > 0$ such that

$$\begin{aligned}\int_0^T \langle \Lambda u(t), u(t)\rangle\,dt &\leqslant k\|u\|_{L^p_X}\|\dot u\|_{L^q_{X^*}} + w\big(\|u\|_{L^p_X}\big)\|u\|_{L^p_X}\\ &\leqslant \epsilon\|\dot u\|^q_{L^q_{X^*}} + C(\epsilon)\|u\|^p_{L^p_X} + w\big(\|u\|_{L^p_X}\big)\|u\|_{L^p_X}.\end{aligned}$$

On the other hand, by the definition of K, we have

$$\int_0^T \langle Ku(t), u(t)\rangle\,dt = w\big(\|u\|_{L^p_X}\big)\|u\|^2_{L^p_X} + \|u\|^{p+1}_{L^p_X}.$$

Therefore the coercivity follows from the following estimate:

$$\begin{aligned}M(u,0) &= \int_0^T \big[\langle \Lambda u(t) + Ku(t), u(t)\rangle\big]\,dt + H_{\mathcal{L}}(u,0) + \mu\frac{1}{q}\|\dot u\|^q_{L^q_{X^*}}\\ &\geqslant -\epsilon\|\dot u\|^q_{L^q_{X^*}} - C(\epsilon)\|u\|^p_{L^p_X} - w\big(\|u\|_{L^p_X}\big)\|u\|_{L^p_X} + w\big(\|u\|_{L^p_X}\big)\|u\|^2_{L^p_X} + \|u\|^{p+1}_{L^p_X}\\ &\quad - \mathcal{L}(0,0) + \mu\frac{1}{q}\|\dot u\|^q_{L^q_{X^*}}\\ &\geqslant \left(\frac{\mu}{q} - \epsilon\right)\|\dot u\|^q_{L^q_{X^*}} + \|u\|^{p+1}_{L^p_X}\big(1 + o\big(\|u\|_{L^p_X}\big)\big). \qquad \square\end{aligned}$$

In the following lemma, we get rid of the regularizing diffusive term $\mu\psi(u)$ and prove the theorem with Λ replaced by the operator $\Lambda + K$, and under the additional assumption that ℓ satisfies the boundedness condition (44).

Lemma 3.3. *Let L be a time dependent anti-selfdual Lagrangian as in Theorem* 3.1 *satisfying either one of conditions* (A) *or* (B)*, and assume that ℓ is an anti-selfdual Lagrangian on $H \times H$ that satisfies condition* (44)*. Then there exists $u \in \mathcal{X}_{p,q}$ such that*

$$\int_0^T \big[L\big(t, u(t), \dot u(t) + \Lambda u(t) + Ku(t)\big)\,dt + \langle \Lambda u(t) + Ku(t), u(t)\rangle\big]\,dt + \ell\left(u(0) - u(T), \frac{u(T) + u(0)}{2}\right) = 0.$$

Proof under condition (B). Note first that in this case L satisfies both conditions (42) and (43) of Lemma 2.3, which then yields for every $\mu > 0$ an element $u_\mu \in \mathcal{X}_{p,q}$ satisfying

$$\dot u_\mu(t) + \Lambda u_\mu(t) + Ku_\mu(t) + \mu\partial\psi\big(u_\mu(t)\big) \in -\bar\partial L\big(t, u_\mu(t)\big) \quad \text{on } [0,T] \tag{65}$$

and

$$\ell\left(u_\mu(0) - u_\mu(T), \frac{u_\mu(T) + u_\mu(0)}{2}\right) = \int_0^T \langle \dot u_\mu(t), u_\mu(t)\rangle\,dt. \tag{66}$$

We now establish upper bounds on the norm of u_μ in $\mathcal{X}_{p,q}$. Multiplying (65) by u_μ and integrating over $[0,T]$ we obtain

$$\int_0^T \big\langle \dot u_\mu(t) + \Lambda u_\mu(t) + K u_\mu(t) + \mu\partial\psi\big(u_\mu(t)\big), u_\mu(t)\big\rangle\,dt = -\int_0^T \big\langle \bar\partial L\big(t, u_\mu(t)\big), u_\mu(t)\big\rangle\,dt. \tag{67}$$

It follows from (59) and the above equality that

$$\int_0^T \big\langle \dot u_\mu(t) + K u_\mu(t) + \mu\partial\psi\big(u_\mu(t)\big), u_\mu(t)\big\rangle \leqslant C\big(1 + \|u_\mu\|_{L^p_X}\big). \tag{68}$$

Taking into account (66) and the fact that $\int_0^T \langle \partial\psi(u_\mu(t)), u_\mu(t)\rangle \geqslant 0$, it follows from (84) that

$$\ell\left(u_\mu(0) - u_\mu(T), \frac{u_\mu(T) + u_\mu(0)}{2}\right) + \int_0^T \big\langle K u_\mu(t), u_\mu(t)\big\rangle \leqslant C\big(1 + \|u_\mu\|_{L^p_X}\big).$$

Since ℓ is bounded from below (say by C_1), the above inequality implies that $\|u_\mu\|_{L^p_X}$ is bounded, since we have

$$C_1 + w\big(\|u_\mu\|_{L^p_X}\big)\|u_\mu\|^2_{L^p_X} + \|u_\mu\|^{p+1}_{L^p_X} \leqslant C\|u_\mu\|_{L^p_X}.$$

Now we show that $\|\dot u_\mu\|_{L^q_{X^*}}$ is also bounded. For that, we multiply (65) by $J^{-1}\dot u_\mu$ to get that

$$\|\dot u_\mu\|^2_{L^q_{X^*}} + \int_0^T \big[\big\langle \Lambda u_\mu(t) + K u_\mu(t) + \mu\partial\psi\big(u_\mu(t)\big) + \bar\partial L\big(t, u_\mu(t)\big), J^{-1}\dot u_\mu(t)\big\rangle\big]\,dt = 0. \tag{69}$$

The last identity and the fact that $\int_0^T \langle \partial\psi(u_\mu(t)), J^{-1}\dot u_\mu(t)\rangle\,dt = 0$ imply that

$$\|\dot u_\mu\|^2_{L^q_{X^*}} \leqslant \|\Lambda u_\mu\|_{L^q_{X^*}}\|\dot u_\mu\|_{L^q_{X^*}} + \|K u_\mu\|_{L^q_{X^*}}\|\dot u_\mu\|_{L^q_{X^*}} + C\|\dot u_\mu\|_{L^q_{X^*}}.$$

It follows from the above inequality and (57) that

$$\|\dot u_\mu\|_{L^q_{X^*}} \leqslant \|\Lambda u_\mu\|_{L^q_{X^*}} + \|K u_\mu\|_{L^q_{X^*}} + C \leqslant k\|\dot u_\mu\|_{L^q_{X^*}} + w\big(\|u\|_{L^p_X}\big) + \|K u_\mu\|_{L^q_{X^*}}$$

from which we obtain that $(1-k)\|\dot u_\mu\|_{L^q_{X^*}} \leqslant w(\|u_\mu\|_{L^p_X}) + \|K u_\mu\|_{L^q_{X^*}}$, which means that $\|\dot u_\mu\|_{L^q_{X^*}}$ is bounded.

Consider now $u \in \mathcal{X}_{p,q}$ such that $u_\mu \rightharpoonup u$ weakly in L^p_X and $\dot u_\mu \rightharpoonup \dot u$ in $L^q_{X^*}$. From (65) and (66) we have

$$\begin{aligned}
J_\mu(u_\mu) :=& \int_0^T \big[\big\langle \Lambda u_\mu(t) + K u_\mu(t), u_\mu(t)\big\rangle + L\big(t, u_\mu(t), \dot u_\mu(t) + \Lambda u_\mu(t) + K u_\mu(t) + \mu\partial\psi\big(u_\mu(t)\big)\big)\big]\,dt \\
&+ \ell\left(u_\mu(0) - u_\mu(T), \frac{u_\mu(T) + u_\mu(0)}{2}\right) \\
\leqslant& \int_0^T \big[\big\langle \Lambda u_\mu(t) + K u_\mu(t) + \mu\partial\psi\big(u_\mu(t)\big), u_\mu(t)\big\rangle \\
&+ L\big(t, u_\mu(t), \dot u_\mu(t) + \Lambda u_\mu(t) + K u_\mu(t) + \mu\partial\psi\big(u_\mu(t)\big)\big)\big]\,dt \\
&+ \ell\left(u_\mu(0) - u_\mu(T), \frac{u_\mu(T) + u_\mu(0)}{2}\right) \\
=& I_\mu(u_\mu) = 0.
\end{aligned}$$

Since $\Lambda + K$ is regular, $\partial\psi(u_\mu)$ is uniformly bounded and L is weakly lower semi-continuous on $X \times X^*$, we get by letting $\mu \to 0$ that

$$\ell\left(u(T)-u(0),\frac{u(T)+u(0)}{2}\right)+\int_0^T\left[\left\langle \Lambda u(t)+Ku(t),u(t)\right\rangle+L\left(t,u(t),\dot u(t)+\Lambda u(t)+Ku(t)\right)\right]dt\leqslant 0.$$

The reverse inequality is true for any $u\in\mathcal{X}_{p,q}$ since L and ℓ are anti-selfdual Lagrangians. □

Proof of Lemma 3.3 under condition (A). Note first that condition (56) implies that there is a $D>0$ such that

$$\int_0^T L\left(t,u(t),p(t)\right)dt\geqslant D\left(\|p\|^s_{L^q_{X^*}}-1\right)\quad\text{for every } p\in L^q_{X^*},\tag{70}$$

where $\frac{1}{r}+\frac{1}{s}=1$.

However, since L is not supposed to satisfy condition (42), we first replace it by its λ-regularization L^1_λ which satisfies all properties of Lemma 3.2. Therefore, there exists $u_{\mu,\lambda}\in\mathcal{X}_{p,q}$ satisfying

$$\dot u_{\mu,\lambda}(t)+\Lambda u_{\mu,\lambda}(t)+Ku_{\mu,\lambda}(t)+\mu\partial\psi\left(u_{\mu,\lambda}(t)\right)=-\bar\partial L^1_\lambda\left(t,u_{\mu,\lambda}(t)\right)\quad\text{on } [0,T]\tag{71}$$

and

$$\ell\left(u_{\mu,\lambda}(T)-u_{\mu,\lambda}(0),\frac{u_{\mu,\lambda}(T)+u_{\mu,\lambda}(0)}{2}\right)=\int_0^T\left\langle \dot u_{\mu,\lambda}(t),u_{\mu,\lambda}(t)\right\rangle dt.\tag{72}$$

We shall first find bounds for $u_{\mu,\lambda}$ in $\mathcal{X}_{p,q}$ that are independent of μ. Multiplying (71) by $u_{\mu,\lambda}$ and integrating, we obtain

$$\int_0^T\left\langle \dot u_{\mu,\lambda}(t)+\Lambda u_{\mu,\lambda}(t)+Ku_{\mu,\lambda}(t)+\mu\partial\psi\left(u_{\mu,\lambda}(t)\right),u_{\mu,\lambda}(t)\right\rangle dt=-\int_0^T\left\langle\bar\partial L^1_\lambda\left(t,u_{\mu,\lambda}(t)\right),u_{\mu,\lambda}(t)\right\rangle dt.\tag{73}$$

Since $\bar\partial L^1_\lambda(t,.)$ is a *maximal monotone* operator, we have $\int_0^T\langle\bar\partial L^1_\lambda(t,u_{\mu,\lambda}(t))-\bar\partial L^1_\lambda(t,0),u_{\mu,\lambda}(t)-0\rangle\,dt\geqslant 0$, and therefore

$$\int_0^T\left\langle\bar\partial L^1_\lambda\left(t,u_{\mu,\lambda}(t)\right),u_{\mu,\lambda}(t)\right\rangle dt\geqslant\int_0^T\left\langle\bar\partial L^1_\lambda(t,0),u_{\mu,\lambda}(t)\right\rangle dt.\tag{74}$$

Taking into account (72), (74) and the fact that $\int_0^T\partial\psi(u_\mu(t)),u_\mu(t)\rangle\geqslant 0$, it follows from (73) that

$$\ell\left(u_{\mu,\lambda}(0)-u_{\mu,\lambda}(T),\frac{u_{\mu,\lambda}(T)+u_{\mu,\lambda}(0)}{2}\right)+\int_0^T\left\langle \Lambda u_{\mu,\lambda}(t)+Ku_{\mu,\lambda}(t),u_{\mu,\lambda}(t)\right\rangle dt$$
$$\leqslant-\int_0^T\left\langle\bar\partial L^1_\lambda(t,0),u_{\mu,\lambda}(t)\right\rangle dt.$$

This implies $\{u_{\mu,\lambda}\}_\mu$ is bounded in L^p_X, and by the same argument as under condition (B), one can prove that $\{\dot u_{\mu,\lambda}\}_\mu$ is also bounded in $L^q_{X^*}$. Consider $u_\lambda\in\mathcal{X}_{p,q}$ such that $u_{\mu,\lambda}\rightharpoonup u_\lambda$ weakly in L^p_X and $\dot u_{\mu,\lambda}\rightharpoonup\dot u_\lambda$ in $L^q_{X^*}$. It follows just like in the proof under condition (B) that

$$\int_0^T\left[\left\langle \Lambda u_\lambda(t)+Ku_\lambda(t),u_\lambda(t)\right\rangle+L^1_\lambda\left(t,u_\lambda(t),\dot u_\lambda(t)+\Lambda u_\lambda(t)+Ku_\lambda(t)\right)\right]dt$$
$$+\ell\left(u_\lambda(0)-u_\lambda(T),\frac{u_\lambda(T)+u_\lambda(0)}{2}\right)=0,\tag{75}$$

and therefore

$$\dot u_\lambda(t) + \Lambda u_\lambda(t) + K u_\lambda(t) \in -\bar\partial L^1_\lambda\big(t, u_\lambda(t)\big) \quad \text{on } [0,T]. \tag{76}$$

Now we obtain estimates on u_λ in $\mathcal{X}_{p,q}$. Since ℓ and L^1_λ are bounded from below, it follows from (75) that $\int_0^T [\langle \Lambda u_\lambda(t) + K u_\lambda(t), u_\lambda(t)\rangle\, dt$ is bounded and therefore u_λ is bounded in L^p_X since

$$\begin{aligned}\int_0^T \big\langle \Lambda u_\lambda(t) + K u_\lambda(t), u_\lambda(t)\big\rangle\, dt &\geqslant -C\big(\|u\|_{L^p_X} + 1\big) - \int_0^T \big\langle \bar\partial L\big(t, u(t)\big), u(t)\big\rangle\, dt + \int_0^T \big\langle K u(t), u(t)\big\rangle\, dt \\ &\geqslant -C\big(\|u\|_{L^p_X} + 1\big) - w\big(\|u\|_{L^p_X}\big)\|u\|_{L^p_X} + w\big(\|u\|_{L^p_X}\big)\|u\|^2_{L^p_X} + \|u\|^{p+1}_{L^p_X}.\end{aligned}$$

Setting $v_\lambda(t) := \dot u_\lambda(t) + \Lambda u_\lambda(t) + K u_\lambda(t)$, we get from (76) that

$$-v_\lambda(t) = \bar\partial L^1_\lambda\big(t, u_\lambda(t)\big) = \bar\partial L\big(t, u_\lambda(t) + \lambda^{q-1}\big\|v_\lambda(t)\big\|_*^{q-2} J^{-1} v_\lambda(t)\big) \quad \text{on } [0,T].$$

This together with (75) implies that

$$\begin{aligned} 0 &= \int_0^T \big[\big\langle \Lambda u_\lambda(t) + K u_\lambda(t), u_\lambda(t)\big\rangle + \lambda\big\|v_\lambda(t)\big\|^q\big]\, dt \\ &\quad + \int_0^T L\big(t, u_\lambda(t) + \lambda\big\|v_\lambda(t)\big\|_*^{q-2} J^{-1} v_\lambda(t), \dot u_\lambda(t) + \Lambda u_\lambda(t) + K u_\lambda(t)\big)\, dt \\ &\quad + \ell\bigg(u_\lambda(0) - u_\lambda(T), \frac{u_\lambda(T) + u_\lambda(0)}{2}\bigg). \end{aligned} \tag{77}$$

It follows that $\int_0^T L(t, u_\lambda(t) + \lambda\|v_\lambda(t)\|_*^{q-2} J^{-1} v_\lambda(t), \dot u_\lambda(t) + \Lambda u_\lambda(t) + K u_\lambda(t))\, dt$ is bounded from above.

In view of (70), there exists then a constant $C > 0$ such that

$$\big\|\dot u_\lambda(t) + \Lambda u_\lambda(t) + K u_\lambda(t)\big\|_{L^q_{X^*}}\, dt \leqslant C. \tag{78}$$

It follows that

$$\|\dot u_\lambda\|_{L^q_{X^*}} \leqslant \|\Lambda u_\lambda\|_{L^q_{X^*}} + \|K u_\lambda\|_{L^q_{X^*}} + C \leqslant k\|\dot u_\lambda\|_{L^q_{X^*}} + w\big(\|u\|_{L^p_X}\big) + \|K u_\lambda\|_{L^q_{X^*}}$$

from which we obtain

$$(1-k)\|\dot u_\lambda\|_{L^q_{X^*}} \leqslant w\big(\|u_\lambda\|_{L^p_X}\big) + \|K u_\lambda\|_{L^q_{X^*}},$$

which means that $\|\dot u_\lambda\|_{L^q_{X^*}}$ is bounded. By letting λ go to zero in (77), we obtain

$$\ell\bigg(u(0) - u(T), \frac{u(T) + u(0)}{2}\bigg) + \int_0^T \big[\big\langle \Lambda u(t) + K u(t), u(t)\big\rangle + L\big(t, u(t), \dot u(t) + \Lambda u(t) + K u(t)\big)\big]\, dt = 0$$

where u is a weak limit of $(u_\lambda)_\lambda$ in $\mathcal{X}_{p,q}$. $\square$

Proof of Theorem 3.1. First we assume that ℓ satisfies condition (44), and we shall work towards eliminating the perturbation K. Let L^2_λ be the λ-regularization of L with respect to the second variable, in such a way that L^2_λ satisfies (59). Indeed

$$\begin{aligned}\int_0^T \big\langle \bar\partial L^2_\lambda\big(t, u(t)\big) + \Lambda u(t), u(t)\big\rangle\, dt &= \int_0^T \big\langle \bar\partial L\big(t, u(t)\big) + \Lambda u(t) + \lambda^{p-1}\|u\|^{p-2} J u(t), u(t)\big\rangle\, dt \\ &\geqslant \int_0^T \big\langle \bar\partial L\big(t, u(t)\big) + \Lambda u(t), u(t)\big\rangle\, dt \geqslant -C\|u\|_{L^p_X}. \end{aligned} \tag{79}$$

Moreover, we have in view of (56) that

$$\int_0^T L_\lambda^2(t,u,p)\,dt \geqslant -D + \frac{\lambda^{p-1}}{p}\|u\|_{L_X^p}^p. \tag{80}$$

From Lemma 3.3, we get for each $\epsilon > 0$, $u_{\epsilon,\lambda} \in \mathcal{X}_{p,q}$ such that

$$\int_0^T \big[\big\langle \Lambda u_{\epsilon,\lambda}(t) + \epsilon K u_{\epsilon,\lambda}(t), u_{\epsilon,\lambda}(t)\big\rangle + L_\lambda^2\big(t, u_{\epsilon,\lambda}(t), \dot u_{\epsilon,\lambda}(t) + \Lambda u_{\epsilon,\lambda}(t) + \epsilon K u_{\epsilon,\lambda}(t)\big)\big]\,dt$$
$$+ \ell\left(u_{\epsilon,\lambda}(0) - u_{\epsilon,\lambda}(T), \frac{u_{\epsilon,\lambda}(T) + u_{\epsilon,\lambda}(0)}{2}\right) = 0, \tag{81}$$

and

$$\dot u_{\epsilon,\lambda}(t) + \Lambda u_{\epsilon,\lambda}(t) + \epsilon K u_{\epsilon,\lambda}(t) \in -\bar\partial L_\lambda^2\big(t, u_{\epsilon,\lambda}(t)\big) \quad \text{on } [0,T]. \tag{82}$$

We shall first find bounds for $u_{\epsilon,\lambda}$ in $\mathcal{X}_{p,q}$ that are independent of ϵ. Multiplying (82) by $u_{\epsilon,\lambda}$ and integrating, we obtain

$$\int_0^T \big\langle \dot u_{\epsilon,\lambda}(t) + \Lambda u_{\epsilon,\lambda}(t) + \epsilon K u_{\epsilon,\lambda}(t), u_{\epsilon,\lambda}(t)\big\rangle\,dt = -\int_0^T \big\langle \bar\partial L_\lambda^2\big(t, u_{\epsilon,\lambda}(t)\big), u_{\epsilon,\lambda}(t)\big\rangle\,dt. \tag{83}$$

It follows from (79) and the above equality that

$$\int_0^T \big\langle \dot u_{\epsilon,\lambda}(t) + \epsilon K u_{\epsilon,\lambda}(t), u_{\epsilon,\lambda}(t)\big\rangle \leqslant C\|u_{\epsilon,\lambda}\|_{L_X^p}, \tag{84}$$

and therefore

$$\ell\left(u_{\epsilon,\lambda}(0) - u_{\epsilon,\lambda}(T), \frac{u_{\epsilon,\lambda}(T) + u_{\epsilon,\lambda}(0)}{2}\right) + \int_0^T \big\langle \epsilon K u_{\epsilon,\lambda}(t), u_{\epsilon,\lambda}(t)\big\rangle \leqslant C\|u_{\epsilon,\lambda}\|_{L_X^p},$$

which in view of (81) implies that

$$\left|\int_0^T L_\lambda^2\big(t, u_{\epsilon,\lambda}(t), \dot u_{\epsilon,\lambda}(t) + \Lambda u_{\epsilon,\lambda}(t) + \epsilon K u_{\epsilon,\lambda}(t)\big)\,dt\right| \leqslant C\|u_{\epsilon,\lambda}\|_{L_X^p}.$$

By (80), we deduce that $\{u_{\epsilon,\lambda}\}_\mu$ is bounded in L_X^p. The same reasoning as above then shows that $\{\dot u_{\epsilon,\lambda}\}_\mu$ is also bounded in $L_{X^*}^q$. Again, the regularity of Λ and the lower semi-continuity of L, yields the existence of $u_\lambda \in \mathcal{X}_{p,q}$ such that

$$\ell\left(u_\lambda(0) - u_\lambda(T), \frac{u_\lambda(T) + u_\lambda(0)}{2}\right) + \int_0^T \big[\big\langle \Lambda u_\lambda(t), u_\lambda(t)\big\rangle + L_\lambda^2\big(t, u_\lambda(t), \dot u_\lambda(t) + \Lambda u_\lambda(t)\big)\big]\,dt = 0. \tag{85}$$

In other words,

$$\int_0^T \big[\big\langle \Lambda u_\lambda(t), u_\lambda(t)\big\rangle + L\big(t, u_\lambda(t), \dot u_\lambda(t) + \Lambda u_\lambda(t)\big) + \lambda^{p-1}\big\|u_\lambda(t)\big\|^{p-2} J u_\lambda(t) + \lambda^{p-1}\big\|u_\lambda(t)\big\|^p\big]\,dt$$
$$+ \ell\left(u_\lambda(0) - u_\lambda(T), \frac{u_\lambda(T) + u_\lambda(0)}{2}\right) = 0. \tag{86}$$

Now since $I_{L,\ell,\Lambda}$ satisfies the selfdual Palais–Smale condition, we get that $(u_\lambda)_\lambda$ is bounded in $\mathcal{X}_{p,q}$. Suppose $u_\lambda \rightharpoonup \bar{u}$ in L^p_X and $\dot{u}_\lambda \rightharpoonup \dot{\bar{u}}$ in $L^q_{X^*}$. It follows from (57) that Λu_λ is bounded in $L^q_{X^*}$. Again, we deduce that

$$\ell\left(\bar{u}(T)-\bar{u}(0), \frac{\bar{u}(T)+\bar{u}(0)}{2}\right)+\int_0^T \left[\left\langle \Lambda\bar{u}(t), \bar{u}(t)\right\rangle + L\left(t, \bar{u}(t), \dot{\bar{u}}(t)+\Lambda\bar{u}(t)\right)\right] dt = 0.$$

Now, we show that we can do without assuming that ℓ satisfies (44), but that it is bounded below while $(0,0) \in \mathrm{Dom}(\ell)$. Indeed, let $\ell_\lambda := \ell_\lambda^{1,2}$ be the λ-regularization of the anti-selfdual Lagrangian ℓ in both variables. Then ℓ_λ satisfies (44) and therefore there exists $x_\lambda \in \mathcal{X}_{p,q}$ such that

$$\ell_\lambda\left(x_\lambda(T)-x_\lambda(0), \frac{x_\lambda(T)+x_\lambda(0)}{2}\right)+\int_0^T \left[\left\langle \Lambda x_\lambda(t), x_\lambda(t)\right\rangle + L\left(t, x_\lambda(t), \dot{x}_\lambda(t)+\Lambda x_\lambda(t)\right)\right] dt = 0. \tag{87}$$

Since ℓ is bounded from below, so is ℓ_λ. This together with (87) imply that the family $\int_0^T [\langle \Lambda x_\lambda(t), x_\lambda(t)\rangle + L(t, x_\lambda(t), \dot{x}_\lambda(t) + \Lambda x_\lambda(t))]\, dt$ is bounded above. Again, since $I_{L,\ell,\Lambda}$ is weakly coercive, we obtain that $(x_\lambda)_\lambda$ is bounded in $\mathcal{X}_{p,q}$. The continuity of the injection $\mathcal{X}_{p,q} \subseteq C([0,T]; H)$ also ensures the boundedness of $(x_\lambda(T))_\lambda$ and $(x_\lambda(0))_\lambda$ in H. Consider $\bar{x} \in \mathcal{X}_{p,q}$ such that $x_\lambda \rightharpoonup \bar{x}$ in L^p_X and $\dot{x}_\lambda \rightharpoonup \dot{\bar{x}}$ in $L^q_{X^*}$. It follows from the regularity of Λ and the lower semi-continuity of ℓ and L that

$$\ell\left(\bar{x}(T)-\bar{x}(0), \frac{\bar{x}(T)+\bar{x}(0)}{2}\right)+\int_0^T \left[\left\langle \Lambda\bar{x}(t), \bar{x}(t)\right\rangle + L\left(t, \bar{x}(t), \dot{\bar{x}}(t)+\Lambda\bar{x}(t)\right)\right] dt = 0,$$

and therefore $\bar{x}$ satisfies Eq. (61). $\square$

Remark 3.4. Note that the hypothesis that $I_{L,\ell,\Lambda}$ is weakly coercive, is only needed in the last part of the proof to deal with the case when ℓ is not assumed to satisfy (44). Otherwise, the hypothesis that $I_{L,\ell,\Lambda}$ satisfies the selfdual Palais–Smale condition would have been sufficient. This will be useful in the application to Schrödinger equations mentioned below.

3.1. Nonlinear evolutions involving a skew-adjoint operator

Suppose again that we have an evolution triple $X \subset H \subset X^*$, where X is reflexive, H is a Hilbert space and where each space is dense in the following one. Also assume that there exists a linear and symmetric duality map J between X and X^* in such a way that $\|x\|^2 = \langle x, Jx\rangle$. We can then consider X and X^* as Hilbert spaces with the following inner products,

$$\langle u, v\rangle_{X\times X} := \langle Ju, v\rangle \quad \text{and} \quad \langle u, v\rangle_{X^*\times X^*} := \left\langle J^{-1}u, v\right\rangle. \tag{88}$$

A typical example is the evolution triple $X = H^1_0(\Omega) \subset H := L^2(\Omega) \subset X^* = H^{-1}(\Omega)$ where the duality map is given by $J = -\Delta$.

If now $\bar{S}$ is an isometry on X^*, then $S = J^{-1}\bar{S}J$ is also an isometry on X, in such a way that

$$\langle u, p\rangle = \langle Su, \bar{S}p\rangle \quad \text{for all } u \in X \text{ and } p \in X^*. \tag{89}$$

Indeed, we have $\langle Su, \bar{S}p\rangle = \langle JSu, \bar{S}p\rangle_{X^*\times X^*} = \langle \bar{S}Ju, \bar{S}p\rangle_{X^*\times X^*} = \langle Ju, p\rangle_{X^*\times X^*} = \langle u, p\rangle$, from which we can deduce that

$$\|Su\|^2_X = \langle Su, Su\rangle_{X\times X} = \langle Su, JSu\rangle = \langle Su, \bar{S}Ju\rangle = \langle u, Ju\rangle = \|u\|^2_X.$$

Moreover, if L is an anti-selfdual Lagrangian on $X \times X^*$, then $L_S := L(Su, \bar{S}p)$ is also an anti-selfdual Lagrangian on $X \times X^*$, since

$$
\begin{aligned}
L_S^*(p,u) &= \sup\big\{\langle v,p\rangle + \langle u,q\rangle - L_S(v,q);\ (v,q)\in X\times X^*\big\}\\
&= \sup\big\{\langle Sv,\bar S p\rangle + \langle Su,\bar S q\rangle - L(Sv,\bar S q);\ (v,q)\in X\times X^*\big\}\\
&= L^*(\bar S p, Su) = L(-Su,-\bar S p) = L_S(-u,-p).
\end{aligned}
$$

We shall need the following facts about semi-groups of operators.

Definition 3.5. A C_0-group on H is a family of bounded operators $S=\{S_t\}_{t\in\mathbb{R}}$ satisfying

(i) $S_tS_s = S_{t+s}$ for each $t,s\in\mathbb{R}$.
(ii) $S(0)=I$.
(iii) The function $t\to S_tu\in C(\mathbb{R},H)$ for each $u\in H$.

We recall a celebrated result of Stone.

Proposition 3.1. *An operator $A: D(A)\subset H\to H$ on a Hilbert space H is skew-adjoint if and only if it is the infinitesimal generator of a C_0-group of unitary operators $(S_t)_{t\in\mathbb{R}}$ on H. In other words, we have $Ax=\lim_{t\downarrow 0}\frac{S_tx-x}{t}$ for every $x\in D(A)$.*

It follows from the above that if $(S_t)_t$ is such a group and if L is a time dependent anti-selfdual Lagrangian on $[0,T]\times H\times H$, then so is the Lagrangian $L_S(t,u,p):=L(t,S_tu,S_tp)$.

The same holds if $X\subset H\subset X^*$ is an evolution triple with a linear and symmetric duality map J. Indeed, let $(\bar S_t)_{t\in\mathbb{R}}$ be a C_0-unitary group of operators associated to a skew-adjoint operator A on the dual space X^* viewed as a Hilbert space (with scalar product $\langle J^{-1}p,q\rangle$). By defining the maps $(S_t)_{t\in\mathbb{R}}$ on X via the formula $S_t=J^{-1}\bar S_tJ$, we deduce from the above that if L is a time dependent anti-selfdual Lagrangian on $[0,T]\times X\times X^*$, then so is the Lagrangian $L_S(t,u,p):=L(t,S_tu,\bar S_tp)$.

These observations combined with Theorem 3.1 yield the following corollary.

Corollary 3.6. *Let $(\bar S_t)_{t\in\mathbb{R}}$ be a C_0-unitary group of operators associated to a skew-adjoint operator A on the Hilbert space X^*, and let $(S_t)_{t\in\mathbb{R}}$ be the corresponding group on X. For $p>1$ and $q=\frac{p}{p-1}$, assume that $\Lambda:\mathcal{X}_{p,q}\to L^q_{X^*}$ is a regular map such that for some nondecreasing continuous real function w, and $0\leqslant k<1$, it satisfies*

$$
\|\Lambda S_tx\|_{L^q_{X^*}} \leqslant k\|\dot x\|_{L^q_{X^*}} + w\big(\|x\|_{L^p_X}\big)\quad \text{for every } x\in\mathcal{X}_{p,q}, \tag{90}
$$

and

$$
\left|\int_0^T \big\langle \Lambda x(t),x(t)\big\rangle\,dt\right| \leqslant w\big(\|x\|_{L^p_X}\big)\quad \text{for every } x\in\mathcal{X}_{p,q}. \tag{91}
$$

Let ℓ be an anti-selfdual Lagrangian on $H\times H$ that is bounded below with $0\in\mathrm{Dom}(\ell)$, and let L be a time dependent anti-selfdual Lagrangian on $[0,T]\times X\times X^*$ such that for some $C>0$ and $r>1$, we have

$$
-C\leqslant \int_0^T L\big(t,u(t),0\big)\,dt\leqslant C\big(1+\|u\|^r_{L^p_X}\big)\quad \text{for every } u\in L^p_X. \tag{92}
$$

The functional

$$
I(u)=\int_0^T\big[L\big(t,S_tu(t),\bar S_t\dot u(t)+\Lambda S_tu(t)\big)+\big\langle \Lambda S_tu(t),S_tu(t)\big\rangle\big]\,dt+\ell\left(u(0)-u(T),\frac{u(T)+u(0)}{2}\right) \tag{93}
$$

is then selfdual on $\mathcal{X}_{p,q}$, and if in addition it is weakly coercive on $\mathcal{X}_{p,q}$, then it attains its minimum at $u\in\mathcal{X}_{p,q}$ in such a way that $I(u)=\inf_{w\in\mathcal{X}_{p,q}}I(w)=0$.

Moreover if $S_t = \bar{S}_t$ on X, then $v(t) = S_t u(t)$ is a "mild solution" of

$$\begin{cases} \Lambda v(t) + A v(t) + \dot{v}(t) \in -\bar{\partial} L(t, v(t)) \quad \text{on } [0,T], \\ \frac{v(0)+S_{(-T)}v(T)}{2} \in -\bar{\partial}\ell(v(0) - S_{(-T)}v(T)), \end{cases} \tag{94}$$

where being a mild solution means that for every $t \in [0,T]$,

$$v(t) = S_t v(0) - \int_0^t \bar{S}_{t-s}\big\{\bar{\partial} L\big(s, v(s)\big) - \Lambda v(s)\big\}\, ds. \tag{95}$$

Proof of Corollary 3.6. Define the nonlinear map $\Gamma : \mathcal{X}_{p,q} \to L^q_{X^*}$ by $\Gamma(u) = S_t^* \Lambda S_t(u)$. This map is also regular in view of the regularity of Λ. It follows from the previous observations that the anti-selfdual Lagrangian L_S satisfies (56). It remains to show that Γ satisfies conditions (57) and (A). Indeed for $x \in \mathcal{X}_{p,q}$, we have

$$\|\Gamma x\|_{L^q_{X^*}} = \big\|S_t^* \Lambda S_t x\big\|_{L^q_{X^*}} = \|\Lambda S_t x\|_{L^q_{X^*}} \leqslant k\|\dot{x}\|_{L^q_{X^*}} + w\big(\|x\|_{L^p_X}\big)$$

and

$$\left|\int_0^T \big\langle \Gamma x(t), x(t)\big\rangle\, dt\right| = \left|\int_0^T \big\langle \Lambda S_t x(t), S_t x(t)\big\rangle\, dt\right| \leqslant w\big(\|S_t x\|_{L^p_X}\big) = w\big(\|x\|_{L^p_X}\big).$$

Also it is easily seen that $I_{L_S,\ell,\Gamma}$ is weakly coercive, which means that all the hypothesis in Theorem 3.1 are satisfied. Hence there exists $x \in \mathcal{X}_{p,q}$ such that $I(x) = 0$. We now show that $v(t) = S_t x(t)$ is a mild solution of (94).

Indeed, we have

$$-\bar{S}_t \dot{x}(t) - \Lambda S_t x(t) \in \bar{\partial} L\big(t, S_t x(t)\big),$$

hence $-\dot{x}(t) - \bar{S}_{-t}\Lambda S_t x(t) \in \bar{S}_{-t}\bar{\partial} L(t, S_t x(t))$. By integrating between 0 and t, we get

$$x(t) = x(0) - \int_0^t \big\{\bar{S}_{-s}\bar{\partial} L\big(s, S_s x(s)\big) - \bar{S}_{-s}\Lambda S_s x(s)\big\}\, ds.$$

Substituting $v(t) = S_t x(t)$ in the above equation gives

$$S_{-t} v(t) = v(0) - \int_0^t \big\{\bar{S}_{-s}\bar{\partial} L\big(s, v(s)\big) - \bar{S}_{-s}\Lambda v(s)\big\}\, ds$$

and consequently

$$v(t) = S_t v(0) - S_t \int_0^t \big\{\bar{S}_{-s}\bar{\partial} L\big(s, v(s)\big) - \bar{S}_{-s}\Lambda v(s)\big\}\, ds = S_t v(0) - \int_0^t \bar{S}_{t-s}\big\{\bar{\partial} L\big(s, v(s)\big) - \Lambda v(s)\big\}\, ds,$$

which means that $v(t)$ is a mild solution for (94).

On the other hand, it is clear that the boundary condition $\frac{x(0)+x(T)}{2} \in -\bar{\partial}\ell(x(0) - x(T))$ translates after the change of variables into

$$\frac{v(0) + S_{(-T)}v(T)}{2} \in -\bar{\partial}\ell\big(v(0) - S_{(-T)}v(T)\big)$$

and we are done. □

4. Application to Navier–Stokes evolutions

The most basic time-dependent anti-selfdual Lagrangians are of the form $L(t,x,p)=\varphi(t,x)+\varphi^*(t,-p)$ where for each t, the function $x\to\varphi(t,x)$ is convex and lower semi-continuous on X. Let now $\psi: H\to\mathbb{R}\cup\{+\infty\}$ be another convex lower semi-continuous function which is bounded from below and such that $0\in \mathrm{Dom}(\psi)$, and set $\ell(a,b)=\psi(a)+\psi^*(-b)$. The above principle then yields that if for some $C_1, C_2>0$, we have

$$C_1\big(\|x\|^p_{L^p_X}-1\big)\leqslant \int_0^T \varphi\big(t,x(t)\big)\,dt\leqslant C_2\big(\|x\|^p_{L^p_X}+1\big)\quad \text{for all } x\in L^p_X,$$

then for every regular map Λ satisfying (57) and either one of conditions (A) or (B) in Theorem 3.1, the infimum of the functional

$$I(x)=\int_0^T\big[\varphi\big(t,x(t)\big)+\varphi^*\big(t,-\dot x(t)-\Lambda x(t)\big)+\big\langle \Lambda x(t),x(t)\big\rangle\big]\,dt+\psi\big(x(0)-x(T)\big)+\psi^*\bigg(-\frac{x(0)+x(T)}{2}\bigg)$$

on $\mathcal{X}_{p,q}$ is zero and is attained at a solution $x(t)$ of the following equation

$$\begin{cases} -\dot x(t)-\Lambda x(t)\in\partial\varphi(t,x(t)) & \text{for all } t\in[0,T],\\ -\frac{x(0)+x(T)}{2}\in\partial\psi(x(0)-x(T)). \end{cases}$$

As noted in the introduction, the boundary condition above is quite general and it includes as particular case the more traditional ones such as initial-value problems, periodic and anti-periodic orbits. It suffices to choose $\ell(a,b)=\psi(a)+\psi^*(-b)$ accordingly.

- For the initial boundary condition $x(0)=x_0$ for a given $x_0\in H$, we choose $\psi(x)=\frac14\|x\|^2_H-\langle x,x_0\rangle$.
- For periodic solutions $x(0)=x(T)$, ψ is chosen as:

$$\psi(x)=\begin{cases}0 & x=0,\\ +\infty & \text{elsewhere.}\end{cases}$$

- For anti-periodic solutions $x(0)=-x(T)$, it suffices to choose $\psi(x)=0$ for each $x\in H$.

As a consequence of the above theorem, we provide a variational resolution to evolution equations involving nonlinear operators such as the Navier–Stokes equation with various boundary conditions:

$$\begin{cases} \frac{\partial u}{\partial t}+(u\cdot\nabla)u+f=\nu\Delta u-\nabla p & \text{on } \Omega,\\ \operatorname{div} u=0 & \text{on } [0,T]\times\Omega,\\ u=0 & \text{on } [0,T]\times\partial\Omega, \end{cases}\tag{96}$$

where Ω is a smooth domain of $\mathbb{R}^n$, $f\in L^2_{X^*}([0,T])$, $\nu>0$.

Indeed, setting $X=\{u\in H^1_0(\Omega;\mathbf{R}^n);\ \operatorname{div} v=0\}$, and $H=L^2(\Omega)$, we write the above problem in the form

$$\begin{cases} \frac{\partial u}{\partial t}+\Lambda u\in-\partial\Phi(t,u) & \text{on } [0,T],\\ \frac{u(0)+u(T)}{2}\in-\partial\psi(u(0)-u(T)), \end{cases}\tag{97}$$

where ψ is any bounded below proper convex lower semi-continuous function on H, while the convex functional Φ and the nonlinear operator Λ are defined by:

$$\Phi(t,u)=\frac{\nu}{2}\int_\Omega\sum_{j,k=1}^3\bigg(\frac{\partial u_j}{\partial x_k}\bigg)^2\,dx+\int_\Omega\big\langle u(x),f(t,x)\big\rangle\,dx\quad\text{and}\quad \Lambda u:=(u\cdot\nabla)u.\tag{98}$$

Note that $\Lambda: X\to X^*$ is regular as long as the dimension $N\leqslant 4$. On the other hand, when Λ lifts to path space, we have the following.

Lemma 4.1.

(1) *When $N=2$, the operator $\Lambda:\mathcal{X}_{2,2}\to L^2_{X^*}$ is regular.*

(2) *When $N=3$, the operator Λ is regular from $\mathcal{X}_{4,\frac{4}{3}}\to L^{\frac{4}{3}}_{X^*}$ as well as from $\mathcal{X}_{2,\frac{4}{3}}\cap L^\infty(0,T;H)$ to $L^{\frac{4}{3}}_{X^*}$.*

Proof. First note that the three embeddings $\mathcal{X}_{2,2}\subseteq L^2_H$, $\mathcal{X}_{4,\frac{4}{3}}\subseteq L^2_H$, and $\mathcal{X}_{2,\frac{4}{3}}\subseteq L^2_H$, are compact.

Assuming that $N=3$, let $u^n\to u$ weakly in $\mathcal{X}_{4,\frac{4}{3}}$, and fix $v\in C^1([0,T]\times\Omega)$. We have that

$$\int_0^T\langle\Lambda u^n,v\rangle=\int_0^T\int_\Omega\sum_{j,k=1}^3 u^n_k\frac{\partial u^n_j}{\partial x_k}v_j\,dx\,dt=-\int_0^T\int_\Omega\sum_{j,k=1}^3 u^n_k\frac{\partial v_j}{\partial x_k}u^n_j\,dx.$$

Therefore

$$\begin{aligned}\left|\int_0^T\langle\Lambda u^n-\Lambda u,v\rangle\right|&=\left|\sum_{j,k=1}^3\int_0^T\int_\Omega\left(u^n_k\frac{\partial v_j}{\partial x_k}u^n_j-u_k\frac{\partial v_j}{\partial x_k}u_j\right)dx\,dt\right|\\&\leqslant\|v\|_{C^1([0,T]\times\Omega)}\sum_{j,k=1}^3\int_0^T\int_\Omega\left|u^n_ku^n_j-u_ku_j\right|dx\,dt.\end{aligned}\tag{99}$$

Also

$$\begin{aligned}\int_0^T\int_\Omega\left|u^n_ku^n_j-u_ku_j\right|dx\,dt&\leqslant\int_0^T\int_\Omega\left|u^n_ku^n_j-u_ku^n_j\right|dx\,dt+\int_0^T\int_\Omega\left|u_ku^n_j-u_ku_j\right|dx\,dt\\&\leqslant\left\|u^n_j\right\|_{L^2_H}\left\|u^n_k-u_k\right\|_{L^2_H}+\|u_k\|_{L^2_H}\left\|u^n_j-u_j\right\|_{L^2_H}\to 0.\end{aligned}\tag{100}$$

Moreover, for $N=3$ we have the following standard estimate [9]

$$\left\|\Lambda u^n\right\|_{X^*}\leqslant c\left|u^n\right|_H^{\frac{1}{2}}\left\|u^n\right\|_X^{\frac{3}{2}}.\tag{101}$$

Since $\mathcal{X}_{4,\frac{4}{3}}\subseteq C(0,T;H)$ is continuous, we obtain

$$\left\|\Lambda u^n\right\|_{L^{\frac{4}{3}}_{X^*}}\leqslant c\left|u^n\right|^{\frac{1}{2}}_{C(0,T;H)}\left\|u^n\right\|^{\frac{3}{4}}_{L^2_X}\leqslant c\left\|u^n\right\|^{\frac{1}{2}}_{\mathcal{X}_{4,\frac{4}{3}}}\left\|u^n\right\|^{\frac{3}{4}}_{L^2_X}\tag{102}$$

from which we conclude that Λu^n is a bounded sequence in $L^{\frac{4}{3}}_{X^*}$, and therefore the convergence of $\langle\Lambda u^n,v\rangle$ to $\langle\Lambda u,v\rangle$ holds for each $v\in L^4_X$.

Now, since $\mathcal{X}_{2,2}\subseteq C(0,T;H)$ is also continuous, the same argument works for $N=2$, the only difference being that we have the following estimate which is better that (101),

$$\left\|\Lambda u^n\right\|_{X^*}\leqslant c\left|u^n\right|_H\left\|u^n\right\|_X.\tag{103}$$

To consider the case $\Lambda:\mathcal{X}_{2,\frac{4}{3}}\cap L^\infty(0,T;H)\to L^{\frac{4}{3}}_{X^*}$, we note that relations (99) and (100) still hold if $u_n\to u$ weakly in $\mathcal{X}_{2,\frac{4}{3}}$. We also have estimate (101). However, unlike the above, one cannot deduce (102) since we do not have necessarily a continuous embedding from $\mathcal{X}_{2,\frac{4}{3}}\subseteq C(0,T;H)$. However, if (u_n) is also assumed to be bounded in $L^\infty(0,T;H)$, then we get the following estimate from (101),

$$\left\|\Lambda u^n\right\|_{L^{\frac{4}{3}}_{X^*}}\leqslant c\left|u^n\right|^{\frac{1}{2}}_{L^\infty(0,T;H)}\left\|u^n\right\|^{\frac{3}{4}}_{L^2_X}\tag{104}$$

which ensures the boundedness of Λu^n in $L^{\frac{4}{3}}_{X^*}$. $\square$

We now prove Corollaries 1.4 and 1.5 stated in the introduction.

Proof of Corollary 1.4. By the preceding lemma, one can verify that the operator $\Lambda : \mathcal{X}_{2,2} \to L^2_{X^*}$ satisfies conditions (23) and (24). Therefore the infimum of the functional

$$I(u) = \int_0^T \left[\Phi\big(t, u(t)\big) + \Phi^*\big(t, -\dot{u}(t) - (u \cdot \nabla)u(t)\big)\right] dt + \ell\left(u(0) - u(T), \frac{u(0) + u(T)}{2}\right)$$

on $\mathcal{X}_{2,2}$ is zero and is attained at a solution $u(t)$ of (96). $\square$

Proof of Corollary 1.5. We start by considering the following functional on the space $\mathcal{X}_{4,\frac{4}{3}}$.

$$I_\epsilon(u) := \int_0^T \left[\Phi_\epsilon\big(t, u(t)\big) + \Phi_\epsilon^*\big(t, -\dot{u}(t) - (u \cdot \nabla)u(t)\big)\right] dt + \ell\left(u(0) - u(T), \frac{u(0) + u(T)}{2}\right)$$

where $\Phi_\epsilon(t, u) = \Phi(t, u) + \frac{\epsilon}{4}\|u\|_X^4$. In view of the preceding lemma, the operator $\Lambda u := (u \cdot \nabla)u$ and Φ_ϵ satisfy all properties of Theorem 3.1. In particular, we have the estimate

$$\|\Lambda u\|_{X^*} \leqslant c|u|_H^{1/2}\|u\|_X^{3/2} \quad \text{for every } u \in X. \tag{105}$$

It follows from Theorem 3.1, that there exists $u_\epsilon \in \mathcal{X}_{4,\frac{4}{3}}$ with $I_\epsilon(u_\epsilon) = 0$. This implies that

$$\begin{cases} \frac{\partial u_\epsilon}{\partial t} + (u_\epsilon \cdot \nabla)u_\epsilon + f(t,x) = \nu\Delta u_\epsilon + \operatorname{div}(\epsilon\|u_\epsilon\|^2\nabla u_\epsilon) - \nabla p_\epsilon & \text{on } [0,T] \times \Omega, \\ \operatorname{div} u_\epsilon = 0 & \text{on } [0,T] \times \Omega, \\ u_\epsilon = 0 & \text{on } [0,T] \times \partial\Omega, \\ -\frac{u_\epsilon(0) + u_\epsilon(T)}{2} = \bar{\partial}\ell(u_\epsilon(0) - u_\epsilon(T)). \end{cases} \tag{106}$$

Now, we show that $(u_\epsilon)_\epsilon$ is bounded in $\mathcal{X}_{2,\frac{4}{3}}$. Indeed, multiply (106) by u_ϵ to get

$$\frac{d}{dt}\frac{|u_\epsilon(t)|^2}{2} + \nu\|u_\epsilon(t)\|_X^2 + \epsilon\|u_\epsilon(t)\|_X^4 = \langle f(t), u_\epsilon(t)\rangle \leqslant \frac{\nu}{2}\|u_\epsilon(t)\|_X^2 + \frac{2}{\nu}\|f(t)\|_{X^*}^2$$

so that

$$\frac{d}{dt}\frac{|u_\epsilon(t)|^2}{2} + \frac{\nu}{2}\|u_\epsilon(t)\|_X^2 + \epsilon\|u_\epsilon(t)\|_X^4 \leqslant \frac{2}{\nu}\|f(t)\|_{X^*}^2. \tag{107}$$

Integrating (107) over $[0,s]$ $(s < T)$, we obtain

$$\frac{|u_\epsilon(s)|^2}{2} - \frac{|u_\epsilon(0)|^2}{2} + \frac{\nu}{2}\int_0^s \|u_\epsilon(t)\|_X^2 + \epsilon\int_0^s \|u_\epsilon(t)\|_X^4 \leqslant \frac{2}{\nu}\int_0^s \|f(t)\|_{X^*}^2. \tag{108}$$

On the other hand, it follows from (106) that $\ell(u_\epsilon(0) - u_\epsilon(T), \frac{u_\epsilon(0)+u_\epsilon(T)}{2}) = \frac{|u_\epsilon(T)|^2}{2} - \frac{|u_\epsilon(0)|^2}{2}$. Considering this together with (108) with $s = T$, we get

$$\ell\left(u_\epsilon(0) - u_\epsilon(T), \frac{u_\epsilon(0) + u_\epsilon(T)}{2}\right) + \frac{\nu}{2}\int_0^T \|u_\epsilon(t)\|_X^2 + \epsilon\int_0^T \|u_\epsilon(t)\|_X^4 \leqslant \frac{2}{\nu}\int_0^T \|f(t)\|_{X^*}^2. \tag{109}$$

Since ℓ is bounded from below and is coercive in both variables, it follows from the above that $(u_\epsilon)_\epsilon$ is bounded in L^2_X, that $(u_\epsilon(T))_\epsilon$ and $(u_\epsilon(0))_\epsilon$ are bounded in H, and that $\epsilon\int_0^T \|u_\epsilon(t)\|^4$ is also bounded. It also follows from (108) coupled with the boundedness of $(u_\epsilon(0))_\epsilon$, that u_ϵ is bounded in $L^\infty(0,T;H)$. Estimate (105) combined with the boundedness of $(u_\epsilon)_\epsilon$ in $L^\infty(0,T;H) \cap L^2_X$ implies that $(\Lambda u_\epsilon)_\epsilon$ is bounded in $L^{4/3}_X$. We also have the estimate

$$\|\nu\Delta u_\epsilon + \operatorname{div}\big(\epsilon\|u_\epsilon\|^2\nabla u_\epsilon\big)\|_{X^*} \leqslant \nu\|u_\epsilon\|_X + \epsilon\|u_\epsilon\|_X^3$$

which implies that $\nu\Delta u_\epsilon + \operatorname{div}(\epsilon\|u_\epsilon\|^2\nabla u_\epsilon)$ is bounded in $L^{4/3}_{X^*}$.

It also follows from (106) that for each $v \in L^4_X$, we have

$$\int_0^T \left\langle \frac{\partial u_\epsilon}{\partial t}, v\right\rangle dt = \int_0^T \left\langle -(u_\epsilon \cdot \nabla)u_\epsilon - f(t,x) + \nu\Delta u_\epsilon + \operatorname{div}\left(\epsilon\|u_\epsilon\|^2\nabla u_\epsilon\right), v\right\rangle dt. \tag{110}$$

Since the right-hand side is uniformly bounded with respect to ϵ, so is the left-hand side, which implies that $\frac{\partial u_\epsilon}{\partial t}$ is bounded in $L^{4/3}_{X^*}$. Therefore, there exists $u \in \mathcal{X}_{2,4/3}$ such that

$$u_\epsilon \rightharpoonup u \quad \text{weakly in } L^2_X, \tag{111}$$

$$\frac{\partial u_\epsilon}{\partial t} \rightharpoonup \frac{\partial u_\epsilon}{\partial t} \quad \text{weakly in } L^{4/3}_{X^*}, \tag{112}$$

$$\operatorname{div}\left(\epsilon\|u_\epsilon\|^2\nabla u_\epsilon\right) \rightharpoonup 0 \quad \text{weakly in } L^{4/3}_{X^*}, \tag{113}$$

$$u_\epsilon(0) \rightharpoonup u(0) \quad \text{weakly in } H, \tag{114}$$

$$u_\epsilon(T) \rightharpoonup u(T) \quad \text{weakly in } H. \tag{115}$$

Letting ϵ approach to zero in (110), it follows from (111)–(115) that

$$\int_0^T \left\langle \frac{\partial u}{\partial t}, v\right\rangle dt = \int_0^T \left\langle -(u \cdot \nabla)u - f(t,x) + \nu\Delta u, v\right\rangle dt. \tag{116}$$

Also it follows from (114), (115) and (106) and the fact that $\bar\partial\ell$ is maximal monotone that

$$-\frac{u(0)+u(T)}{2} \in \bar\partial\ell\left(u(0)-u(T)\right). \tag{117}$$

(116) and (117) yield that u is a weak solution of

$$\begin{cases} \frac{\partial u}{\partial t} + (u\cdot\nabla)u + f(t,x) = \nu\Delta u - \nabla p & \text{on } [0,T]\times\Omega,\\ \operatorname{div} u = 0 & \text{on } [0,T]\times\Omega,\\ u = 0 & \text{on } [0,T]\times\partial\Omega,\\ -\frac{u(0)+u(T)}{2} \in \bar\partial\ell(u(0)-u(T)) & \text{on } \Omega. \end{cases} \tag{118}$$

Now we prove inequality (32). Since $I_\epsilon(u_\epsilon) = 0$, a standard argument (see the proof of Theorem 3.1) yields that $I(u) \leqslant \liminf_\epsilon I_\epsilon(u_\epsilon) = 0$, thereby giving that

$$I_\epsilon(u) := \int_0^T \left[\Phi\left(t,u(t)\right) + \Phi^*\left(t, -\dot u(t) - (u\cdot\nabla)u(t)\right)\right] dt + \ell\left(u(0)-u(T), \frac{u(0)+u(T)}{2}\right) \leqslant 0.$$

On the other hand it follows from (117) that $\ell(u(0)-u(T), -\frac{u(0)+u(T)}{2}) = \frac{|u(T)|^2}{2} - \frac{|u(0)|^2}{2}$. This together with the above inequality gives

$$\frac{|u(T)|^2}{2} + \int_0^T \left[\Phi\left(t,u(t)\right) + \Phi^*\left(t, -\dot u(t) - (u\cdot\nabla)u(t)\right)\right] dt \leqslant \frac{|u(0)|^2}{2}. \qquad \square$$

Corollary 4.2. *In dimension $N = 3$, there exists for any given α with $|\alpha| < 1$, a weak solution of the equation solutions*:

$$\begin{cases} \frac{\partial u}{\partial t} + (u\cdot\nabla)u + f(t,x) = \nu\Delta u - \nabla p & \textit{on } [0,T]\times\Omega,\\ \operatorname{div} u = 0 & \textit{on } [0,T]\times\Omega,\\ u = 0 & \textit{on } [0,T]\times\partial\Omega,\\ u(0) = \alpha u(T). \end{cases}$$

Proof. For each α with $|\alpha|<1$ there exists $\lambda>0$ such that $\alpha=\frac{\lambda-1}{\lambda+1}$. Now consider $\ell(a,b)=\psi_\lambda(a)+\psi_\lambda^*(-b)$ where $\psi_\lambda(a)=\frac{\lambda}{4}|a|^2$. □

Navier–Stokes evolutions driven by their boundary: We now consider the following evolution equation.

$$\begin{cases}\frac{\partial u}{\partial t}+(u\cdot\nabla)u+f=\nu\Delta u-\nabla p & \text{on } [0,T]\times\Omega,\\ \operatorname{div}u=0 & \text{on } [0,T]\times\Omega,\\ u(t,x)=u^0(x) & \text{on } [0,T]\times\partial\Omega,\\ u(0,x)=\alpha u(T,x) & \text{on } \Omega\end{cases} \tag{119}$$

where $\int_{\partial\Omega}u^0\cdot\mathbf{n}\,d\sigma=0$, $\nu>0$ and $f\in L^p_{X^*}$. Assuming that $u^0\in H^{3/2}(\partial\Omega)$ and that $\partial\Omega$ is connected, Hopf's extension theorem again yields the existence of $v^0\in H^2(\Omega)$ such that

$$v^0=u^0 \quad \text{on } \partial\Omega, \qquad \operatorname{div}v^0=0 \quad \text{and} \quad \int_\Omega\sum_{j,k=1}^n u_k\frac{\partial v_j^0}{\partial x_k}u_j\,dx\leqslant\epsilon\|u\|_X^2 \quad \text{for all } u\in X, \tag{120}$$

where $V=\{u\in H^1(\Omega;\mathbf{R}^n);\ \operatorname{div}u=0\}$. Setting $v=u+v^0$, then solving (119) reduces to finding a solution in the path space $\mathcal{X}_{2,2}$ corresponding to the Banach space $X=\{u\in H_0^1(\Omega;\mathbf{R}^n);\ \operatorname{div}v=0\}$ and the Hilbert space $H=L^2(\Omega)$ for

$$\begin{aligned}&\frac{\partial u}{\partial t}+(u\cdot\nabla)u+\big(v^0\cdot\nabla\big)u+(u\cdot\nabla)v^0\in-\partial\Phi(u),\\ &u(0)-\alpha u(T)=(\alpha-1)v^0,\end{aligned} \tag{121}$$

where $\Phi(t,u)=\frac{\nu}{2}\int_\Omega\sum_{j,k=1}^3(\frac{\partial u_j}{\partial x_k})^2\,dx+\langle g,u\rangle$, and where

$$g:=f-\nu\Delta v^0+\big(v^0\cdot\nabla\big)v^0\in L^p_{V^*}.$$

In other words, this is an equation of the form

$$\frac{\partial u}{\partial t}+\Lambda u\in-\partial\Phi(t,u) \tag{122}$$

where $\Lambda u:=(u\cdot\nabla)u+(v^0\cdot\nabla)u+(u\cdot\nabla)v^0$ is the nonlinear regular operator $N=2$ or $N=3$.

Now recalling the fact that the component $Bu:=(v^0\cdot\nabla)u$ is skew-symmetric, it follows from Hopf's estimate that

$$C\|u\|_V^2\geqslant\Phi(t,u)+\langle\Lambda u,u\rangle\geqslant(\nu-\epsilon)\|u\|^2+\langle g,u\rangle \quad \text{for all } u\in X.$$

As in Corollary 1.5 we have the following.

Corollary 4.3. *Assume $N=3$, and let ℓ be an anti-selfdual Lagrangian on $H\times H$ that is coercive in both variables. Then, there exists $u\in\mathcal{X}_{2,\frac{4}{3}}$ such that*

$$I(u)=\int_0^T\big[\Phi\big(t,u(t)\big)+\Phi^*\big(t,-\dot u(t)-\Lambda u(t)\big)+\big\langle u(t),\Lambda u(t)\big\rangle\big]\,dt+\ell\left(u(0)-u(T),\frac{u(0)+u(T)}{2}\right)\leqslant 0$$

and u is a weak solution of (119).

To obtain the boundary condition given in (121) that is $u(0)-\alpha u(T)=(\alpha-1)v^0$, consider $\ell(a,b)=\psi_\lambda(a)+\psi_\lambda^*(-b)$ where $\alpha=\frac{\lambda-1}{\lambda+1}$ and $\psi_\lambda(a)=\frac{\lambda}{4}|a|^2-4\langle a,v^0\rangle$.

5. Schrödinger and other nonlinear evolutions

5.1. Initial-value Schrödinger evolutions

Consider the following nonlinear Schrödinger equation

$$iu_t + \Delta u - |u|^{r-1}u = -i\bar{\partial}L(t,u), \quad (t,x) \in [0,T] \times \Omega, \tag{123}$$

where Ω is a bounded domain in $\mathbb{R}^N$, and L is a time dependent anti-selfdual Lagrangian on $[0,T] \times H_0^1(\Omega) \times H^{-1}(\Omega)$. Eq. (123) can be rewritten as

$$u_t + \Lambda u = -\bar{\partial}L(t,u), \quad (t,x) \in [0,T] \times \Omega,$$

where $\Lambda u = -i\Delta + i|u|^{r-1}u$. We can then deduce the following existence.

Theorem 5.1. *Suppose $1 \leqslant r \leqslant \frac{N}{N-2}$. Let $p = 2r$ and assume that L satisfies*

$$-C \leqslant \int_0^T L\big(t, u(t), 0\big)\,dt \leqslant C\big(1 + \|u\|^r_{L^p_{H_0^1}}\big) \quad \textit{for every } u \in L^p_{H_0^1}[0,T]. \tag{124}$$

$$\big\langle \bar{\partial}L(t,u), -\Delta u + |u|^{r-1}u\big\rangle \geqslant 0 \quad \textit{for each } u \in H^2(\Omega). \tag{125}$$

Let $u_0 \in H^2(\Omega)$ and $\ell(a,b) = \frac{1}{4}\|a\|_H^2 - \langle a, u_0\rangle + \|b - u_0\|_H^2$, then the following functional

$$I(u) = \int_0^T \big[L\big(u(t), \dot{u}(t) + \Lambda u(t)\big) + \big\langle \Lambda u(t), u(t)\big\rangle\big]\,dt + \ell\bigg(u(0) - u(T), \frac{u(T) + u(0)}{2}\bigg) \tag{126}$$

attains its minimum at $v \in \mathcal{X}_{p,q}$ in such a way that $I(v) = \inf_{u \in \mathcal{X}_{p,q}} I(u) = 0$ and

$$\begin{cases} \dot{v}(t) - i\Delta v(t) + i|v(t)|^{r-1}v(t) = -\bar{\partial}L(t, v(t)) & \textit{on } [0,T], \\ v(0) = u_0. \end{cases} \tag{127}$$

Proof. Let $X = H_0^1(\Omega)$ and $H = L^2(\Omega)$. Taking into account Theorem 1.3, we just need to verify (23), (24) and prove that I satisfies the *selfdual Palais–Smale condition* on $\mathcal{X}_{p,q}$. Note that (24) follows from the fact that $\langle \Lambda u, u\rangle = 0$. To prove (23), note that

$$\|\Lambda u\|_{H^{-1}} = \big\|-\Delta u + |u|^{r-1}u\big\|_{H^{-1}} \leqslant \|-\Delta u\|_{H^{-1}} + C\big\||u|^{r-1}u\big\|_{L^q(\Omega)} = \|u\|_{H_0^1} + C\|u\|^r_{L^{rq}}.$$

Since $p \geqslant 2$, we have $qr \leqslant 2r \leqslant \frac{2N}{N-2}$. It follows from the Sobolev inequality and the above that

$$\|\Lambda u\|_{H^{-1}} \leqslant \|u\|_{H_0^1} + C\|u\|^r_{H_0^1}$$

from which we obtain

$$\|\Lambda u\|_{L^q_{H^{-1}}} \leqslant \|u\|_{L^q_{H_0^1}} + C\|u\|^r_{L^{rq}_{H_0^1}} \leqslant C\big(\|u\|_{L^p_{H_0^1}} + \|u\|^r_{L^p_{H_0^1}}\big).$$

To show that I satisfies the *selfdual Palais–Smale condition*, we assume that $(u_n)_n \in \mathcal{X}_{p,q}$ is such that

$$\begin{cases} -\dot{u}_n(t) + i\Delta u_n(t) - i|u_n(t)|^{r-1}u_n(t) = -\frac{1}{n}\|u_n\|^{p-2}\Delta u_n + \bar{\partial}L(u_n(t)) & \text{on } [0,T], \\ u_n(0) = u_0. \end{cases} \tag{128}$$

Since $u_0 \in H^2(\Omega)$, it is standard that at least $u_n \in H^2(\Omega)$. Now multiply both sides of the above equation by $\Delta u_n(t) - |u_n(t)|^{r-1}u_n(t)$ and taking into account (125) we have

$$\big\langle \dot{u}_n(t), -\Delta u_n(t) + \big|u_n(t)\big|^{r-1}u_n(t)\big\rangle \leqslant 0$$

from which we obtain

$$\frac{1}{2}\big\|u_n(t)\big\|^2_{H_0^1} + \frac{1}{r+1}\big\|u_n(t)\big\|^{r+1} \leqslant \frac{1}{2}\big\|u(0)\big\|^2_{H_0^1} + \frac{1}{r+1}\big\|u(0)\big\|^{r+1}$$

which once combined with (128), gives the boundedness of $(u_n)_n$ in $\mathcal{X}_{p,q}$. $\square$

Here are two typical examples for anti-selfdual Lagrangians satisfying the assumptions of the above theorem

- $L(u,p)=\varphi(u)+\varphi^*(-p)$ where $\varphi=0$ which leads to a solution of:

$$\begin{cases} i\dot{v}(t)+\Delta v(t)-|v(t)|^{r-1}v(t)=0 & \text{on } [0,T],\\ v(0)=u_0. \end{cases}$$

- $L(u,p)=\varphi(u)+\varphi^*(\mathbf{a}.\nabla u-p)$ where $\varphi(u)=\frac{1}{2}\int_\Omega|\nabla u|^2\,dx$ and $\mathbf{a}$ is a vector field on Ω with compact support. In this case we have a solution for

$$\begin{cases} i\dot{v}(t)+\Delta v(t)-|v(t)|^{r-1}v(t)=-i\mathbf{a}.\nabla v+i\Delta v(t) & \text{on } [0,T],\\ v(0)=u_0. \end{cases}$$

5.2. Variational resolution for a Fluid driven by $-i\Delta^2$

Consider the problem of finding periodic type solutions for the following equation

$$\begin{cases} \frac{\partial u}{\partial t}+(u\cdot\nabla)u-i\Delta^2u+f=\nu\Delta u-\nabla p & \text{on } \Omega\subset\mathbb{R}^n,\\ \operatorname{div}u=0 & \text{on } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{129}$$

where $u=(u_1,u_2)$ and where the operator $i\Delta^2$ is defined in the following way:

$$i\Delta^2u=\left(\Delta^2u_2,-\Delta^2u_1\right)\quad\text{with } \operatorname{Dom}\left(i\Delta^2\right)=\left\{u\in H_0^1(\Omega);\ \Delta u\in H_0^1(\Omega)\text{ and } u=\Delta u=0\text{ on }\partial\Omega\right\}.$$

Theorem 5.2. *Let $(S_t)_{t\in\mathbb{R}}$ be the C_0-unitary group of operators associated to the skew-adjoint operator $i\Delta^2$. Assuming $N=2$, f in $L^2_{X^*}([0,T])$, and ℓ is an anti-selfdual Lagrangian on $H\times H$ that is bounded from below, then the infimum of the functional*

$$I(u)=\int_0^T\left[\Phi\left(t,S_tu(t)\right)+\Phi^*\left(t,-S_t\dot{u}(t)-\Lambda S_tu(t)\right)\right]dt+\ell\left(u(0)-u(T),\frac{u(0)+u(T)}{2}\right)$$

on $\mathcal{X}_{2,2}$ is zero and is attained at $u(t)$ in such a way that $v(t)=S_tu(t)$ is a solution of (129) *that satisfies the time-boundary condition*:

$$-\frac{v(0)+S_{(-T)}v(T)}{2}\in\bar{\partial}\ell\left(v(0)-S_{(-T)}v(T)\right). \tag{130}$$

Moreover, u verifies the following "energy identity":

$$\left\|u(t)\right\|_H^2+2\int_0^t\left[\Phi\left(t,S_tu(t)\right)+\Phi^*\left(t,-S_t\dot{u}(t)-\Lambda S_tu(t)\right)\right]dt=\left\|u(0)\right\|_H^2\quad\textit{for every } t\in[0,T]. \tag{131}$$

In particular, with appropriate choices for the boundary Lagrangian ℓ, the solution v can be chosen to verify either one of the following boundary conditions:

- *an initial value problem*: $v(0)=v_0$ *where v_0 is a given function in H*;
- *a periodic orbit*: $v(0)=S_{(-T)}v(T)$;
- *an anti-periodic orbit*: $v(0)=-S_{(-T)}v(T)$.

Proof. The duality map between X and X^* is $J=-\Delta$ and is therefore linear and symmetric. Also we have $S_t=e^{it\Delta^2}$ and therefore $S_tJ=JS_t$. The result follows from Corollary 3.6 and the remarks preceding it. □

6. A general nonlinear selfdual variational principle for weakly coercive functionals

In this section, we isolate the general variational principle behind the proofs of the last section. We shall actually extend the nonlinear selfdual variational principle mentioned in the introduction (Theorem 1.1) in two different ways. First, and as has already been noted in [5], the hypothesis of regularity on the operator Λ in Theorem 1.1 can be weakened (see Definition 6.1 below). More importantly, we shall also relax the coercivity condition (9) that proved prohibitive in the case of evolution equations.

We start with the following weaker notion for regularity.

Definition 6.1. A map $\Lambda : D(\Lambda) \subset X \to X^*$ is said to be *pseudo-regular* if whenever $(x_n)_n$ is a sequence in X such that $x_n \rightharpoonup x$ weakly in X and $\limsup_n \langle \Lambda x_n, x_n - x\rangle \leqslant 0$, then $\liminf_n \langle \Lambda x_n, x_n\rangle \geqslant \langle \Lambda x, x\rangle$ and $\Lambda x_n \rightharpoonup \Lambda x$ weakly in X^*.

It is clear that regular operators are necessarily pseudo-regular operators.

The following is an extension of Theorem 1.1.

Theorem 6.2. *Let L be an anti-selfdual Lagrangian on a reflexive Banach space X such that $0 \in \mathrm{Dom}(L)$. Let $\Lambda : D(\Lambda) \subset X \to X^*$ be a bounded pseudo-regular map such that $\overline{\mathrm{Dom}_1(L)} \subset D(\Lambda)$ and*

$$\big\langle \bar\partial L(x) + \Lambda x, x\big\rangle \geqslant -C\big(\|x\| + 1\big) \quad \text{for large } \|x\|. \tag{132}$$

Then for any $\lambda > 0$, the selfdual functional

$$I_\lambda(x) = L(x, \Lambda x + \lambda J x) + \langle \Lambda x + \lambda J x, x\rangle$$

attains its infimum at $x_\lambda \in X$ in such a way that $I_\lambda(x_\lambda) = \inf_{x\in X} I_\lambda(x) = 0$, and x_λ is a solution of the differential inclusion

$$0 \in \Lambda x_\lambda + \lambda J x_\lambda + \bar\partial L(x_\lambda). \tag{133}$$

In particular, if the functional $I_{L,\Lambda}$ satisfies the selfdual Palais–Smale condition, then there exists a solution for the equation

$$0 \in \Lambda x + \bar\partial L(x). \tag{134}$$

Remark 6.3. Theorem 6.2 is an extension of Theorem 1.1 which claims that the same conclusion holds under the following stronger coercivity assumption.

$$\lim_{\|x\|\to+\infty} H_L(0, -x) + \langle \Lambda x, x\rangle = +\infty. \tag{135}$$

Indeed, in order to show that condition (135) is stronger than both (132) and the selfdual Palais–Smale condition, note that for each $(x, p) \in X \times X^*$,

$$L(x, p) = \sup\big\{\langle y, p\rangle - H_L(x, y);\ y \in X\big\} \geqslant -H_L(x, 0) \geqslant H_L(0, -x),$$

in such a way that if $\|x_n\| \to +\infty$, then

$$\lim_{n\to+\infty} L\left(x_n, \Lambda x_n + \frac{1}{n} J x_n\right) + \langle x_n, \Lambda x_n\rangle + \frac{1}{n}\|x_n\|^2 \geqslant \lim_{n\to+\infty} H_L(0, -x_n) + \langle \Lambda x_n, x_n\rangle = +\infty.$$

Moreover, we have for large $\|x\|$,

$$\big\langle \bar\partial L(x) + \Lambda x, x\big\rangle = L\big(x, \bar\partial L(x)\big) + \langle \Lambda x, x\rangle \geqslant H_L(0, -x) + \langle \Lambda x, x\rangle \geqslant -C\big(\|x\| + 1\big).$$

For the proof of Theorem 6.2, we shall need the following lemma

Lemma 6.4. *Let L be an anti-selfdual Lagrangian on a reflexive Banach space X, let $\Lambda : D(\Lambda) \subseteq X \to X^*$ be a pseudo-regular map and let $F : D(F) \subseteq X \to X^*$ be a regular map. Assume $(x_n)_n$ is a sequence in $D(\Lambda) \cap D(F)$ such that $x_n \rightharpoonup x$ and $\Lambda x_n \rightharpoonup p$ for some $x \in X$ and $p \in X^*$. If $0 \in \Lambda x_n + F x_n + \bar\partial L(x_n)$ for each $n \in \mathbb{N}$, then necessarily $0 \in \Lambda x + F x + \bar\partial L(x)$.*

Proof. We have

$$\limsup_n \langle \Lambda x_n, x_n - x\rangle \leqslant \lim_{n\to\infty} \langle \Lambda x_n, -x\rangle + \limsup_n \bigl\{-L(x_n, \Lambda x_n + F x_n) - \langle F x_n, x_n\rangle\bigr\}$$
$$= \langle p, -x\rangle - \liminf_n \bigl\{L(x_n, \Lambda x_n + F x_n) + \langle F x_n, x_n\rangle\bigr\}. \tag{136}$$

Since L is weakly lower semi-continuous and F is regular, we have

$$L(x, p + Fx) + \langle Fx, x\rangle \leqslant \liminf_n \bigl\{L(x_n, \Lambda x_n + F x_n) + \langle F x_n, x_n\rangle\bigr\}$$

which together with (136) imply

$$\limsup_n \langle \Lambda x_n, x_n - x\rangle \leqslant \langle p, -x\rangle - L(x, p + Fx) - \langle Fx, x\rangle$$
$$= \langle p + Fx, -x\rangle - L(x, p + Fx).$$

L being an anti-selfdual Lagrangian, we have $L(x, p + Fx) \geqslant \langle p + Fx, -x\rangle$, and therefore

$$\limsup_n \langle \Lambda x_n, x_n - x\rangle \leqslant 0.$$

Now since Λ is pseudo-regular, we have $p = \Lambda x$ and $\liminf_n \langle \Lambda x_n, x_n\rangle \geqslant \langle \Lambda x, x\rangle$. It follows that

$$L(x, \Lambda x + Fx) + \langle \Lambda x + Fx, x\rangle \leqslant \liminf_n L(x_n, \Lambda x_n + F x_n) + \langle \Lambda x_n + F x_n, x_n\rangle = 0.$$

On the other hand, since L is an anti-selfdual Lagrangian, we have the reverse inequality $L(x, \Lambda x + Fx) + \langle \Lambda x + Fx, x\rangle \geqslant 0$ which implies that the latter is equal to zero. □

Proof of Theorem 6.2. Let $w(r) = \sup\{\|\Lambda u\|_* + 1;\ \|u\| \leqslant r\}$, set $Fu := w(\|u\|)Ju$, and consider L^2_λ to be the λ-regularization of L with respect to the second variable, i.e.

$$L^2_\lambda(x, p) := \inf\left\{L(x, q) + \frac{\|p - q\|^2_*}{2\lambda} + \frac{\lambda}{2}\|x\|^2;\ q \in X^*\right\}.$$

Since $0 \in \mathrm{Dom}(L)$, the Lagrangian L and consequently L^2_λ and therefore $H_{L^2_\lambda}(0, .)$ are bounded from below. Also we have

$$\lim_{\|x\|\to+\infty} H_{L^2_\lambda}(0, -x) + \langle \Lambda x + \epsilon F x, x\rangle = +\infty,$$

since $\langle \Lambda x + \epsilon F x, x\rangle \geqslant -w(\|x\|)\|x\| + \epsilon w(\|x\|)\|x\|^2$.

It follows from Theorem 1.1 that there exists $x_{\epsilon,\lambda}$ such that

$$L^2_\lambda(x_{\epsilon,\lambda}, \Lambda x_{\epsilon,\lambda} + \epsilon F x_{\epsilon,\lambda}) + \langle \Lambda x_{\epsilon,\lambda} + \epsilon F x_{\epsilon,\lambda}, x_{\epsilon,\lambda}\rangle = 0$$

which means that $\Lambda x_{\epsilon,\lambda} + \epsilon F x_{\epsilon,\lambda} \in -\bar\partial L^2_\lambda(x_{\epsilon,\lambda})$, and in other words, $\Lambda x_{\epsilon,\lambda} + \epsilon F x_{\epsilon,\lambda} + \lambda J x_{\epsilon,\lambda} \in -\bar\partial L(x_{\epsilon,\lambda})$. This together with (132), imply $\langle \epsilon F x_{\epsilon,\lambda} + \lambda J x_{\epsilon,\lambda}, x_{\epsilon,\lambda}\rangle \leqslant C\|x_{\epsilon,\lambda}\|$, thereby giving

$$\epsilon w\bigl(\|x_{\epsilon,\lambda}\|\bigr)\|x_{\epsilon,\lambda}\|^2 + \lambda\|x_{\epsilon,\lambda}\|^2 \leqslant C\|x_{\epsilon,\lambda}\|,$$

which in turn implies that $(F x_{\epsilon,\lambda})_\epsilon$ and $(x_{\epsilon,\lambda})_\epsilon$ are bounded. Since now Λ is a bounded operator, we get that $\Lambda x_{\epsilon,\lambda}$ is bounded in X^*. Suppose, up to a subsequence, $x_{\epsilon,\lambda} \rightharpoonup x_\lambda$ and $\Lambda x_{\epsilon,\lambda} \rightharpoonup p_\lambda$. It follows from Lemma 6.4 that for every $\lambda > 0$, we have

$$L(x_\lambda, \Lambda x_\lambda + \lambda J x_\lambda) + \langle \Lambda x_\lambda + \lambda J x_\lambda, x_\lambda\rangle = 0.$$

$(x_\lambda)_\lambda$ is therefore a selfdual Palais–Smale sequence, hence it is bounded in X and consequently it converges weakly – up to a subsequence – to $\bar x \in X$. Again, since Λ is a bounded operator, Λx_λ is also bounded in X^*, and again Lemma 6.4 yields $L(\bar x, \Lambda \bar x) + \langle \Lambda \bar x, \bar x\rangle = 0$, which means that $-\Lambda \bar x \in \bar\partial L(\bar x)$. □

Remark 6.5. Note that, we do not really need that Λ is a bounded operator, but a weaker condition of the form $\|\Lambda x\| \leqslant CH(0, x) + w(\|x\|)$ for some nondecreasing function w and some constant $C > 0$.

Let now $A : D(A) \subset X \to X^*$ be a closed linear operator on a reflexive Banach space X, and consider X_A to be the Banach space $D(A)$ equipped with the norm $\|x\|_A = \|x\|_X + \|Ax\|_{X^*}$. We have the following consequence.

Corollary 6.6. *Let $A : D(A) \subset X \to X^*$ be a closed linear operator on a reflexive Banach space X with a dense domain, and let Λ be a map from $D(A)$ into X^* that induces a pseudo-regular operator $\Lambda : X_A \to X_A^*$, and such that*

$$u \to \langle u, \Lambda u + Au\rangle \text{ is bounded from below.} \tag{137}$$

Suppose L is an anti-selfdual Lagrangian on $X \times X^*$ that satisfies the following conditions:

$$\text{For each } p \in \mathrm{Dom}_2(L), \text{ the functional } x \to L(x, p) \text{ is continuous on } X, \tag{138}$$

$$x \to L(x, 0) \text{ is bounded on the unit ball of } X. \tag{139}$$

Then for any $\lambda > 0$, there exists $u_\lambda \in X_A$ such that:

$$0 \in \Lambda u_\lambda + Au_\lambda + \lambda(J + A^*A)u_\lambda + \bar\partial L(u_\lambda). \tag{140}$$

Proof. Note first that $X_A \subseteq X \subseteq X^* \subseteq X_A^*$. We first show that the Lagrangian

$$\mathcal{M}(u, p) := \begin{cases} L(u, p), & p \in X^*, \\ +\infty & p \in X_A^* \setminus X^* \end{cases}$$

is an anti-selfdual Lagrangian on $X_A \times X_A^*$. Indeed, if $q \in X^*$, use the fact that X_A is dense in X and that the functional $x \to L(x, p)$ is continuous on X to write

$$\begin{aligned} \mathcal{M}^*(q, v) &= \sup\bigl\{\langle u, q\rangle + \langle v, p\rangle - \mathcal{M}(u, p);\ (u, p) \in X_A \times X_A^*\bigr\} \\ &= \sup\bigl\{\langle u, q\rangle + \langle v, p\rangle - L(u, p);\ (u, p) \in X_A \times X^*\bigr\} \\ &= L^*(q, v) = L(-v, -q) = \mathcal{M}(-v, -q). \end{aligned}$$

If now $q \in X_A^* \setminus X^*$, then there exists $\{x_n\}_n \subseteq X_A$ with $\|x_n\|_X \leqslant 1$ such that $\langle x_n, q\rangle \to +\infty$ as $n \to \infty$. Since $\{L(x_n, 0)\}_n$ is bounded, It follows that

$$\begin{aligned} \mathcal{M}^*(q, v) &= \sup\bigl\{\langle u, q\rangle + \langle v, p\rangle - \mathcal{M}(u, p);\ (u, p) \in X_A \times X^*\bigr\} \\ &\geqslant \sup\bigl\{\langle x_n, q\rangle - L(x_n, 0)\bigr\} \\ &= +\infty = \mathcal{M}(-v, -q). \end{aligned}$$

To verify condition (132) of Theorem 6.2, we note that

$$\bigl\langle \bar\partial L(u) + \Lambda u + Au, u\bigr\rangle \geqslant \bigl\langle \bar\partial L(0), u\bigr\rangle + \langle \Lambda u + Au, u\rangle \geqslant -C\bigl(1 + \|u\|_X\bigr) \geqslant -C\bigl(1 + \|u\|_{X_A}\bigr).$$

We have used the fact that $\bar\partial L$ is maximal monotone and that $\langle \Lambda u + Au, u\rangle$ is bounded from below. Now apply Theorem 6.2 to the Lagrangian $\mathcal{M}$, the pseudo-regular operator $\Lambda + A$ and the duality map $J + A^*A$ to conclude. □

Corollary 6.7. *Let the operators A, Λ and the space X_A be as in Corollary* 6.6 *and let φ be a proper convex lower semi-continuous function that is both coercive and bounded in X. Assume also the following conditions*:

$$\|\Lambda u\|_{X^*} \leqslant k\|Au\|_{X^*} + w\bigl(\|u\|_X\bigr) \quad \text{for some constant } 0 < k < 1 \text{ and a nondecreasing function } w. \tag{141}$$

Then there exists a solution $\bar x \in X_A$ to the equation

$$0 \in \Lambda x + Ax + \partial\varphi(x), \tag{142}$$

which can be obtained by minimizing the functional $I(x) = \varphi(x) + \varphi^*(-\Lambda x - Ax) + \langle x, \Lambda x + Ax\rangle$.

Proof. It is an immediate consequence of Corollary 6.6 applied to the Lagrangian $L(x, p) = \varphi(x) + \varphi^*(-p)$. We only need to prove that the functional I is weakly coercive on X_A. For that, suppose $\{x_n\}_n \subseteq X_A$ is such that $\|x_n\|_{X_A} \to \infty$, we show that

$$\varphi(x_n)+\varphi^*\left(-\Lambda x_n-Ax_n-\frac{1}{n}Jx_n\right)+\langle x_n,\Lambda x_n+Ax_n\rangle+\frac{1}{n}\|x_n\|_X\to\infty.$$

Indeed if not, and since $\langle x_n,\Lambda x_n+Ax_n\rangle+\frac{1}{n}\|x_n\|_X$ is bounded from below, we have $\varphi(x_n)+\varphi^*(-\Lambda x_n-Ax_n-\frac{1}{n}Jx_n)$ is bounded from above. The coercivity of φ on X ensures the boundedness of $\{\|x_n\|_X\}_n$. Now we show that $\{x_n\}$ is actually bounded in X_A. In fact, since φ is bounded on X we have that φ^* is coercive in X^* and in result

$$\left\|\Lambda x_n+Ax_n+\frac{1}{n}Jx_n\right\|_{X^*}\leqslant C$$

for some constant $C>0$. It follows from (141) and the above that

$$\begin{aligned}\|Ax_n\|_{X^*}&\leqslant\left\|\Lambda x_n+Ax_n+\frac{1}{n}Jx_n\right\|_{X^*}+\left\|\Lambda x_n+\frac{1}{n}Jx_n\right\|_{X^*}\\&\leqslant C+\|\Lambda x_n\|_{X^*}+\frac{1}{n}\|Jx_n\|_{X^*}\\&\leqslant C+k\|Ax_n\|_{X^*}+w\big(\|x_n\|_X\big)+\frac{1}{n}\|x_n\|_X.\end{aligned}$$

Hence $(1-k)\|Ax_n\|_{X^*}\leqslant C+w(\|x_n\|_X)+\frac{1}{n}\|x_n\|_X$, and therefore $\|Ax_n\|_{X^*}$ is bounded which results the boundedness of $\{x_n\}$ in X_A. □

We can also give a variational resolution for certain nonlinear systems.

Corollary 6.8. *Let φ be a bounded convex lower semi-continuous function on $X_1\times X_2$, let $A:X_1\to X_2^*$ be any bounded linear operator, let $B_1:X_1\to X_1^*$ (resp., $B_2:X_2\to X_2^*$) be two positive linear operators. Let $Y_i:=\{x\in X_i;\ B_ix\in X_i^*\}$, $i=1,2$. Assume $\Lambda:=(\Lambda_1,\Lambda_2):Y_1\times Y_2\to Y_1^*\times Y_2^*$ is a pseudo-regular operator such that*

$$\lim_{\|x\|_{X_1}+\|y\|_{X_2}\to\infty}\frac{\varphi(x,y)+\langle B_1x,x\rangle+\langle B_2y,y\rangle+\langle\Lambda(x,y),(x,y)\rangle}{\|x\|_{X_1}+\|y\|_{X_2}}=+\infty,$$

and

$$\big\|(\Lambda_1,\Lambda_2)(x,y)\big\|_{X_1^*\times X_2^*}\leqslant k\big\|(B_1,B_2)(x,y)\big\|_{X_1^*\times X_2^*}+w\big(\big\|(x,y)\big\|_{X_1\times X_2}\big)$$

for some continuous and nondecreasing function w, and some constant $0<k<1$. Then for any $(f,g)\in Y_1^*\times Y_2^*$, there exists $(\bar{x},\bar{y})\in Y_1\times Y_2$ which solves the following system

$$\begin{cases}-\Lambda_1(x,y)-A^*y-B_1x+f\in\partial_1\varphi(x,y),\\-\Lambda_2(x,y)+Ax-B_2y+g\in\partial_2\varphi(x,y).\end{cases}$$

The solution is obtained as a minimizer on $Y_1\times Y_2$ of the functional

$$\begin{aligned}I(x,y)={}&\psi(x,y)+\psi^*\big(-A^*y-B_1x-\Lambda_1(x,y),Ax-B_2y-\Lambda_2(x,y)\big)+\langle B_1x,x\rangle+\langle B_2y,y\rangle\\&+\big\langle\Lambda(x,y),(x,y)\big\rangle,\end{aligned}$$

where

$$\psi(x,y)=\varphi(x,y)-\langle f,x\rangle-\langle g,y\rangle.$$

Proof. Consider the following ASD Lagrangian (see [4])

$$L\big((x,y),(p,q)\big)=\psi(x,y)+\psi^*(-A^*y-p,Ax-q).$$

Setting $B:=(B_1,B_2)$, Corollary 6.7 yields that $I(x,y)=L((x,y),\Lambda(x,y)+B(x,y))+\langle\Lambda(x,y)+B(x,y),(x,y)\rangle$ attains its minimum at some point $(\bar{x},\bar{y})\in Y_1\times Y_2$ and that the minimum is 0. In other words,

$$
\begin{aligned}
0 &= I(\bar{x}, \bar{y}) \\
&= \psi(\bar{x}, \bar{y}) + \psi^*\big(-A^*\bar{y} - B_1\bar{x} - \Lambda_1(\bar{x}, \bar{y}), A\bar{x} - B_2\bar{y} - \Lambda_2(\bar{x}, \bar{y})\big) + \big\langle \Lambda(\bar{x}, \bar{y}) + B(\bar{x}, \bar{y}), (\bar{x}, \bar{y})\big\rangle \\
&= \psi(\bar{x}, \bar{y}) + \psi^*\big(-A^*\bar{y} - B_1\bar{x} - \Lambda_1(\bar{x}, \bar{y}), A\bar{x} - B_2\bar{y} - \Lambda_2(\bar{x}, \bar{y})\big) \\
&\quad + \big\langle\big(\Lambda_1(\bar{x}, \bar{y}) + B_1\bar{x} - A^*\bar{y}, \Lambda_2(\bar{x}, \bar{y}) + B_2\bar{y} + A\bar{x}\big), (\bar{x}, \bar{y})\big\rangle
\end{aligned}
$$

from which follows that

$$
\begin{cases} -A^*y - B_1x - \Lambda_1(x, y) \in \partial_1\varphi(x, y) - f, \\ Ax - B_2y - \Lambda_2(x, y) \in \partial_2\varphi(x, y) - g. \end{cases} \qquad \square
$$

6.1. A variational resolution for doubly nonlinear coupled equations

Let $\mathbf{b}_1 : \Omega \to \mathbf{R}^n$ and $\mathbf{b}_2 : \Omega \to \mathbf{R}^n$ be two compactly supported smooth vector fields on the neighborhood of a bounded domain Ω of $\mathbf{R}^n$. Consider the Dirichlet problem:

$$
\begin{cases} \Delta v + \mathbf{b}_1 \cdot \nabla u = |u|^{p-2}u + u^{m-1}v^m + f & \text{on } \Omega, \\ -\Delta u + \mathbf{b}_2 \cdot \nabla v = |v|^{p-2}v - u^m v^{m-1} + g & \text{on } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{143}
$$

We can use Corollary 6.7 to get

Theorem 6.9. *Assume f, g in L^p, $2 \leqslant p$, that $\operatorname{div}(\mathbf{b}_1) \geqslant 0$ and $\operatorname{div}(\mathbf{b}_2) \geqslant 0$ on Ω, and $1 \leqslant m < \frac{p-1}{2}$. Let $X = \{u \in H_0^1(\Omega);\ u \in L^p(\Omega) \text{ and } \Delta u \in L^q(\Omega)\}$ and consider on $X \times X$ the functional*

$$
\begin{aligned}
I(u, v) &= \Psi(u) + \Psi^*\big(\mathbf{b}_1 \cdot \nabla u + \Delta v - u^{m-1}v^m\big) + \Phi(v) + \Phi^*\big(\mathbf{b}_2 \cdot \nabla v - \Delta u + u^m v^{m-1}\big) \\
&\quad + \frac{1}{2}\int_\Omega \operatorname{div}(\mathbf{b}_1)|u|^2\,dx + \frac{1}{2}\int_\Omega \operatorname{div}(\mathbf{b}_1)|v|^2\,dx
\end{aligned}
$$

where

$$
\Psi(u) = \frac{1}{p}\int_\Omega |u|^p\,dx + \int_\Omega fu\,dx \quad \text{and} \quad \Phi(v) = \frac{1}{p}\int_\Omega |v|^p\,dx + \int_\Omega gv\,dx
$$

are defined on $L^p(\Omega)$ and Ψ^* and Φ^* are their Legendre transforms in $L^q(\Omega)$. Then there exists $(\bar{u}, \bar{v}) \in X \times X$ such that

$$
I(\bar{u}, \bar{v}) = \inf\big\{I(u, v);\ (u, v) \in X \times X\big\} = 0,
$$

and $(\bar{u}, \bar{v})$ is a solution of (143).

Proof. Let $A = \Delta$, $X_A = X$ and $X_1 = L^p(\Omega)$. Φ and Ψ are continuous and coercive on X_1. We need to verify condition (141) in Corollary 6.7. Indeed, by Hölder's inequality for $q = \frac{p}{p-1} \leqslant 2$ we obtain

$$
\big\|u^m v^{m-1}\big\|_{L^q(\Omega)} \leqslant \|u\|^m_{L^{2mq}(\Omega)}\|v\|^{(m-1)}_{L^{2(m-1)q}(\Omega)}
$$

and since $m < \frac{p-1}{2}$ we have $2mq < p$ and therefore

$$
\big\|u^m v^{m-1}\big\|_{L^q(\Omega)} \leqslant C\big(\|u\|^{2m}_{L^p(\Omega)} + \|v\|^{2(m-1)}_{L^p(\Omega)}\big). \tag{144}
$$

Also since $q \leqslant 2$,

$$
\begin{aligned}
\|\mathbf{b}_1 \cdot \nabla u\|_{L^q(\Omega)} &\leqslant C\|\mathbf{b}_1\|_{L^\infty(\Omega)}\|\nabla u\|_{L^2(\Omega)} \\
&\leqslant C\|\mathbf{b}_1\|_{L^\infty(\Omega)}\bigg(\int \langle -\Delta u, u\rangle\,dx\bigg)^{\frac{1}{2}} \leqslant C\|\mathbf{b}_1\|_{L^\infty(\Omega)}\|u\|^{\frac{1}{2}}_{L^p(\Omega)}\|\Delta u\|^{\frac{1}{2}}_{L^q(\Omega)} \\
&\leqslant k\|\Delta u\|_{L^q(\Omega)} + C(k)\|\mathbf{b}_1\|^2_{L^\infty(\Omega)}\|u\|_{L^p(\Omega)}
\end{aligned} \tag{145}
$$

for some $0 < k < 1$. Hence condition (141) follows from (144) and (145). $\square$

Also, it is also easy to verify that the nonlinear operator $\Lambda : X \times X \to L^q(\Omega) \times L^q(\Omega)$ defined by

$$\Lambda(u, v) = \left(-u^{m-1}v^m + \mathbf{b}_1 \cdot \nabla u, u^m v^{m-1} + \mathbf{b}_2 \cdot \nabla v\right)$$

is regular. It is worth noting that there is no restriction on the power p in the previous example, that is p can well be beyond the critical Sobolev exponent.

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