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Metrics of constant curvature on a Riemann surface with two corners on the boundary

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Abstract

In this paper we classify all solutions of

$$\begin{cases} -\Delta u = |x|^{2\alpha} e^u, & \text{in } \mathbb{R}^2_+, \\ \frac{\partial u}{\partial t} = c_1 |x|^\alpha e^{\frac{u}{2}}, & \text{on } \partial\mathbb{R}^2_+ \cap \{s > 0\}, \\ \frac{\partial u}{\partial t} = c_2 |x|^\alpha e^{\frac{u}{2}}, & \text{on } \partial\mathbb{R}^2_+ \cap \{s < 0\} \end{cases}$$

with the finite energy condition

$$\int_{\mathbb{R}^2_+} |x|^{2\alpha} e^u \, dx < \infty, \qquad \int_{\partial\mathbb{R}^2_+} |x|^\alpha e^{\frac{u}{2}} \, ds < \infty.$$

Here c_1, c_2 are constants and $\alpha > -1$.

Keywords: Liouville equation; Gauss curvature; Geodesic curvature; Conner

1. Introduction

On a Riemann surface, one of the interesting geometric problems is to determine which functions can be realized as the Gaussian curvature of some pointwise conformal metric. The classical uniformization theorem tell us that every smooth Riemannian metric on a two-dimensional surface is pointwise conformal to one with constant curvature. This question is by now well understood from many different perspectives, and successfully approached by many different methods.

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On this basis, research can move on to surfaces with singularities. This, however, is by no means a straightforward generalization of the smooth case. Results for smooth surfaces might not be true for surfaces with singularities. For instance, there exist many surfaces with conical singularities that do not admit a conformal metric of constant Gauss curvature. In fact, a closed surface with two conical singularities admits a conformal metric of constant Gauss curvature if and only if its singularities have the same angle and are in antipodal positions – thus, such a surface necessarily has the shape of an American football; this was proved by Troyanov [18]. Therefore a surface with exactly one singularity (the teardrop) does not carry a conformal metric of constant Gauss curvature.

This result was obtained by methods from complex analysis. It is known, however, that the existence question for conformal metrics is intimately linked to the Liouville equation. In recent years, very powerful PDE methods have been developed to precisely determine the asymptotic behavior of solutions of this equation near singularities.

The purpose of the present paper then is to bring to bear the full force of those methods on the existence problem for conformal metrics with prescribed singularities. In fact, we shall investigate the more general situation of surfaces with boundary. When we have a boundary, the natural curvature condition there, the analogue of the constant Gauss curvature condition in the interior, is the one of constant geodesic curvature.

To continue the discussion about surfaces with singularities, let us first recall their definition, following [18]. A conformal metric ds^2 on a Riemannian surface Σ (possibly with boundary) has a conical singularity of order α (a real number with $\alpha > -1$) at a point $p \in \Sigma \cup \partial\Sigma$ if in some neighborhood of p

$$ds^2 = e^{2u}\left|z - z(p)\right|^{2\alpha}|dz|^2$$

where z is a coordinate of Σ defined in this neighborhood and u is smooth away from p and continuous at p. The point p is then said to be a conical singularity of *angle* $\theta = 2\pi(\alpha+1)$ if $p \notin \partial\Sigma$ and a *corner* of angle $\theta = \pi(\alpha+1)$ if $p \in \partial\Sigma$. For example, a football has two singularities of equal angle, while a teardrop has only one singularity. Both these examples correspond to the case $-1 < \alpha < 0$; in case $\alpha > 0$, the angle is larger than 2π, leading to a different geometric picture. Such singularities also appear in orbifolds and branched coverings. They can also describe the ends of complete Riemannian surfaces with finite total curvature. If (Σ, ds^2) has conical singularities of order $\alpha_1, \alpha_2, \ldots, \alpha_n$ at $p_1, p_2, \ldots, p_n$, then ds^2 is said to represent the divisor $\mathbf{A} := \sum_{i=1}^n \alpha_i p_i$.

For a closed surface with more than two conical singularities, the existence problem of constant Gauss curvature already becomes subtle. When all singularities have order $\alpha \in (-1, 0)$, Luo and Tian [14] gave a necessary and sufficient condition. For the case of general α, a necessary and sufficient condition was given by [20] recently for a closed surface with 3 conical singularities. See also [8] for a simpler proof.

As already mentioned, the objective of this paper is to consider surfaces (with boundary) with corners on their boundary and to study the existence problem of conformal metrics with constant Gauss curvature and constant geodesic curvature on their boundary. Our first result shows that a disk with two corners admits a conformal metric with constant Gauss curvature and constant geodesic curvature on its boundary if and only if the two corners have the same angle. This is analogous to the result of [19]. The disk is conformally equivalent to $\mathbb{R}^2_+ \cup \{\infty\}$. Note that the case of a metric with zero geodesic curvature on its boundary can be reduced to Troyanov's result.

Theorem 1.1. *It is possible to construct a metric g with constant Gauss curvature on the unit disk D and constant geodesic curvature on $\Gamma_\pm := \partial D \cap \{(x, y) \in \mathbb{R}^2 \mid \pm y > 0\}$ admitting two corners $p_1 = (1, 0)$ with order $\alpha_1 > -1$ and $p_2 = (-1, 0)$ with order $\alpha_2 > -1$ if and only if*

$$\alpha_1 = \alpha_2.$$

In Theorem 1.1, the constant geodesic curvatures on Γ_+ and Γ_- may be different. All solutions can be explicitly written down, see Theorem 1.2. Theorem 1.1 is not difficult to prove. But it is a good starting point for our research.

What we do in fact is more general than this generalization of Troyanov's result. Let us denote $\mathbb{R}^2_+ = \{(s, t) \mid t > 0\}$. We consider

$$\begin{cases} -\Delta u = |x|^{2\alpha} e^u, & \text{in } \mathbb{R}^2_+, \\ \frac{\partial u}{\partial t} = c_1 |x|^\alpha e^{\frac{u}{2}}, & \text{on } \partial\mathbb{R}^2_+ \cap \{s > 0\}, \\ \frac{\partial u}{\partial t} = c_2 |x|^\alpha e^{\frac{u}{2}}, & \text{on } \partial\mathbb{R}^2_+ \cap \{s < 0\} \end{cases} \tag{1}$$

with the energy conditions

$$\int_{\mathbb{R}^2_+} |x|^{2\alpha} e^u \, dx < \infty, \qquad \int_{\partial\mathbb{R}^2_+} |x|^{\alpha} e^{\frac{u}{2}} \, ds < \infty. \tag{2}$$

Here c_1, c_2 are constants and $\alpha > -1$.

We call $u \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ a weak solution of (1)–(2) if it satisfies

$$\int_{\mathbb{R}^2_+} \nabla u \cdot \nabla \varphi \, dx + c_1 \int_{\partial\mathbb{R}^2_+ \cap \{s>0\}} |x|^{\alpha} e^{\frac{u}{2}} \varphi \, ds + c_2 \int_{\partial\mathbb{R}^2_+ \cap \{s<0\}} |x|^{\alpha} e^{\frac{u}{2}} \varphi \, ds = \int_{\mathbb{R}^2_+} |x|^{2\alpha} e^u \varphi \, dx$$

for any smooth function $\varphi(x)$ on $\overline{\mathbb{R}^2_+}$ with compact support. Since $u \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ implies $e^u \in L^p_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ for all $p > 1$, by standard elliptic regularity we conclude that any weak solution u of (1) is a classical solution when $\alpha \geqslant 0$ while u is smooth away from the origin and $u \in W^{2,q}$ near the origin for $1 < q < -\frac{1}{\alpha}$ when $-1 < \alpha < 0$. In particular, u is continuous at the origin in any case. In the sequel, we assume that a solution u of (1)–(2) always satisfies $u \in C^2(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+} \setminus \{0\})$ and that u is continuous at the origin.

Geometrically, a solution u of (1)–(2) determines a metric $ds^2 = |z|^{2\alpha} e^u |dz|^2$ with constant scalar curvature 1 on $\mathbb{R}^2_+$ and with geodesic curvature $-c_1$ on $\partial\mathbb{R}^2_+ \cap \{s > 0\}$ and geodesic curvature $-c_2$ on $\partial\mathbb{R}^2_+ \cap \{s < 0\}$. Moreover $ds^2 = |z|^{2\alpha} e^u |dz|^2$ has a conical singularity at $z = 0$. Let 1 and -1 be two points on the boundary of the unit disk D. We take a conformal transformation ϕ mapping D to $\mathbb{R}^2_+$ and ∂D to $\partial\mathbb{R}^2_+$ with $\phi(1) = 0$ and $\phi(-1) = \infty$. With such a conformal transformation, the metrics studied in Theorem 1.1 are solutions of (1)–(2). Our main result in this paper is to show the converse, namely, any solution of (1)–(2) is in fact obtained from a metric in Theorem 1.1.

Theorem 1.2. *Let u be a solution of* (1)–(2). *Then $ds^2 = e^u |z|^{2\alpha} |dz|^2$ comes from a conformal metric as in Theorem* 1.1. *More precisely, there exists $\lambda > 0$ such that*:

(1) *When $\alpha = 2k$, $k = 0, 1, 2, \ldots,$ then $c_1 = c_2$. And when $\alpha = 2k + 1$, $k = 0, 1, 2, \ldots,$ then $c_1 = -c_2$. In this case the metric is*

$$ds^2 = \frac{8(\alpha+1)^2 \lambda^{2\alpha+2} |z|^{2\alpha} |dz|^2}{(\lambda^{2\alpha+2} + |z^{\alpha+1} - z_0|^2)^2}$$

for some $z_0 = (s_0, t_0)$ with $s_0 \in \mathbb{R}$ and $t_0 = \frac{c_1 \lambda^{\alpha+1}}{\sqrt{2}}$.

(2) *When $\alpha \neq k$, $k = 0, 1, 2, \ldots,$ then for any c_1 and c_2, the metric is*

$$ds^2 = \frac{8(\alpha+1)^2 \lambda^{2\alpha+2} |z|^{2\alpha} |dz|^2}{(\lambda^{2\alpha+2} + |z^{\alpha+1} - z_0|^2)^2}$$

for some $z_0 = (s_0, t_0)$ with $s_0 = \frac{\lambda^{\alpha+1}(c_1 \cos(\pi\alpha) - c_2)}{\sqrt{2}\sin(\pi\alpha)}$ and $t_0 = \frac{c_1 \lambda^{\alpha+1}}{\sqrt{2}}$.

This result is a natural generalization of the classification result of Chen and Li [4] for the Liouville equation

$$-\Delta u = e^u \quad \text{in } \mathbb{R}^2 \tag{3}$$

with finite area $\int_{\mathbb{R}^2} e^u < \infty$ and the classification result of Li and Zhu [13] for solutions of

$$\begin{cases} -\Delta u = e^u & \text{in } \mathbb{R}^2_+, \\ \frac{\partial u}{\partial t} = c e^{\frac{u}{2}} & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{4}$$

Geometrically, the result of Chen–Li covers the case of the standard sphere. In fact, their classification result tells us that any solution of the Liouville equation (3) with finite area can be compactified as a metric on the standard sphere with constant curvature. Similarly, the result of Li–Zhu deals with a portion of the standard sphere cut by a 2-plane. Namely, from their result we know that any solution of (4) can be compactified as a metric on such a portion of the standard sphere with constant Gauss curvature and constant geodesic curvature on the boundary. In this spirit, our result (for $-1 < \alpha < 1$) then deals with a portion of the standard sphere cut by two 2-planes with angle $\pi(\alpha + 1)$.

It would then be interesting to consider portions of the standard sphere cut by 3 or more 2-planes. This is related to the result of Umehara and Yamada [20], see also [8]. We will return to this issue later. In another direction, our result is a generalization of Prajapat and Tarantello [17], who classify solutions of the Liouville equation with one singularity. For the case $c_1 = c_2 = 0$, Theorem 1.2 can be reduced to their result. For other classication results, or different proofs, see [3,5–7,10,9,12,16], and [22].

Our method to deal with (1)–(2) can be viewed as a combination of the methods developed for those previous results. We shall make particular use of [11] and [13]. The main issue is the determination of

$$d := -\lim_{|x|\to\infty} \frac{u(x)}{\ln|x|}.$$

Note that Eqs. (1) are no longer translation invariant and a solution of (1)–(2) will no longer be radially symmetric if one of $c_i \neq 0$ for $i = 1, 2$. The methods used in [13] and [17] can therefore not be directly utilized to prove Theorem 1.2. However, after we have shown that the metric $ds^2 = e^{\tilde{u}}|dz|^2 = |z|^{2\alpha} e^u |dz|^2$ has two conical singularities at $z = 0$ and $z = \infty$, we can define

$$\eta(z) = \left(\frac{\partial^2 \tilde{u}}{\partial z^2} - \frac{1}{2}\left(\frac{\partial \tilde{u}}{\partial z}\right)^2\right)|dz|^2, \quad z \text{ in } \mathbb{R}^2_+,$$

which can be extended to a projective connection on $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ as defined in [19]. Then the problem is reduced to a linear partial differential system, see (31) and (32). Finally we solve this boundary problem and demonstrate Theorem 1.2.

2. Projective connections

In this section, we will state the definition and the properties of the projective connection discussed in the papers of Troyanov [19] and Mandelbaum [15]. In the last section, we will demonstrate our main result in the sense of a projective connection on $\mathbb{C} \cup \{\infty\}$.

Assume that Σ is a Riemann surface. Let η be a quadratic differential. If

(1) $\eta(z) = \phi(z)|dz|^2$ is a meromorphic quadratic differential in each local coordinate (U, z) on Σ,
(2) $\eta(w) = \eta(z) + \{z, w\}|dw|^2$ in the overlap of two local coordinates (U, z), (V, w),

then η is called a *projective connection* on Σ. Here $\{\,,\}$ denotes the Schwarzian derivative:

$$\{z, w\} = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2$$

a function z of w.

A point $p \in \Sigma$ is called a regular point of the projective connection η if η is holomorphic at this point. We say that η has a *regular singularity* of weight ρ at p if

$$\eta(z) = \left(\frac{\rho}{z^2} + \frac{\sigma}{z} + \phi_1(z)\right)|dz|^2$$

where $\phi_1(z)$ is holomorphic, and z is a local coordinate at p with $z(p) = 0$.

The projective connection is said to be compatible with the divisor $\mathbf{A} := \sum_{i=1}^n \alpha_i p_i$ if it is regular in $\Sigma - \{p_1, \ldots, p_n\}$ and has, for each i, a regular singularity of weight $\rho_i = -\frac{1}{2}\alpha_i(\alpha_i + 2)$ at p_i. The next two lemmas are examples about some results for the projective connection from [19].

Lemma 2.1. *The definition of the weight for a singular point is independent of the choice of local coordinate.*

Lemma 2.2. *If $ds^2 = e^u|dz|^2$ is a conformal metric of constant curvature on Σ representing the divisor* $\mathbf{A}$ *then*

$$\eta(z) = \left(\frac{\partial^2 u}{\partial z^2} - \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2\right)|dz|^2$$

defines a projective connection compatible with the divisor $\mathbf{A}$.

3. Asymptotic behavior

We will first rewrite Eq. (1). Set $\tilde{u} = u + 2\alpha \ln|x|$. Then $\tilde{u}$ satisfies

$$\begin{cases} -\Delta\tilde{u} = e^{\tilde{u}}, & \text{in } \mathbb{R}^2_+, \\ \frac{\partial\tilde{u}}{\partial t} = c_1 e^{\frac{\tilde{u}}{2}}, & \text{on } \partial\mathbb{R}^2_+ \cap \{s > 0\}, \\ \frac{\partial\tilde{u}}{\partial t} = c_2 e^{\frac{\tilde{u}}{2}}, & \text{on } \partial\mathbb{R}^2_+ \cap \{s < 0\} \end{cases} \tag{5}$$

with the energy conditions

$$\int_{\mathbb{R}^2_+} e^{\tilde{u}}\, dx < \infty, \tag{6}$$

$$\int_{\partial\mathbb{R}^2_+} e^{\frac{\tilde{u}}{2}}\, ds < \infty. \tag{7}$$

Proposition 3.1. *Any solution $\tilde{u}$ of* (5) *with* (6) *and* (7) *is bounded from above in the region $\overline{\mathbb{R}^2_+ \setminus B^+_\varepsilon(0)}$, for each $\varepsilon > 0$.*

To prove Proposition 3.1, we need the following lemma.

Lemma 3.2. *Assume that u is a solution of*

$$\begin{cases} -\Delta u = 0, & \text{in } B^+_R, \\ \frac{\partial u}{\partial t} = f(x), & \text{on } \{t = 0\} \cap \partial B^+_R, \\ u = 0, & \text{on } \partial B^+_R \cap \overline{B}^+_R \end{cases}$$

with $f \in L^1(\{t = 0\} \cap \partial B^+_R)$ for any $R > 0$. Then for every $\delta_1 \in (0, 4\pi)$ we have

$$\int_{B^+_R} \exp\left\{\frac{(4\pi - \delta_1)|u(x)|}{\|f\|_1}\right\} dx \leqslant \frac{16\pi^2 R^2}{\delta_1}$$

and for every $\delta_2 \in (0, 2\pi)$

$$\int_{\partial B^+_R \cap \{t=0\}} \exp\left\{\frac{(2\pi - \delta_2)|u(x)|}{\|f\|_1}\right\} ds \leqslant \frac{4\pi R}{\delta_2}$$

where $\|f\|_1 = \int_{\{t=0\}\cap\partial B^+_R} |f|\, ds$.

Proof. Set

$$\Gamma_1 = \{t = 0\} \cap \overline{B}^+_R, \qquad \Gamma_2 = \{t > 0\} \cap \partial B^+_R.$$

Let

$$\phi(y) = \frac{1}{2\pi} \int_{\Gamma_1} \left(\log\frac{2R}{|x - y|} + \log\frac{2R}{|x - \overline{y}|}\right) \big|f(x)\big|\, dx$$

where $\overline{y}$ is the reflection point of y about $\{t = 0\}$.

A direct computation yields

$$\begin{cases} -\Delta\phi = 0, & \text{in } B^+_R, \\ \frac{\partial\phi}{\partial t} = -|f|, & \text{on } \Gamma_1. \end{cases}$$

Note that $\phi \geqslant 0$ for $x \in B_R^+$ since $\frac{2R}{|x-y|} \geqslant 1$ for any $x, y \in B_R^+$. We have

$$\begin{cases} -\Delta(u-\phi) = 0, & \text{in } B_R^+, \\ \frac{\partial(u-\phi)}{\partial t} = f + |f|, & \text{on } \Gamma_1, \\ u - \phi \leqslant 0, & \text{on } \Gamma_2. \end{cases}$$

It follows from the maximum principle and the Hopf lemma that $u \leqslant \phi$ in $\overline{B}_R^+$.

By a similar argument we also have

$$\begin{cases} -\Delta(u+\phi) = 0, & \text{in } B_R^+, \\ \frac{\partial(u+\phi)}{\partial t} = f - |f|, & \text{on } \Gamma_1, \\ u + \phi \geqslant 0, & \text{on } \Gamma_2 \end{cases}$$

which implies that $u \geqslant -\phi$ in $\overline{B}_R^+$. Therefore we have $|u| \leqslant \phi$ in $\overline{B}_R^+$ and thus we have

$$\int_{B_R^+} \exp\left\{\frac{(4\pi - \delta_1)|u(x)|}{\|f\|_1}\right\} dx \leqslant \int_{B_R^+} \exp\left\{\frac{(4\pi - \delta_1)\phi}{\|f\|_1}\right\} dx,$$

and

$$\int_{\Gamma_1} \exp\left\{\frac{(2\pi - \delta_2)|u(x)|}{\|f\|_1}\right\} ds \leqslant \int_{\Gamma_1} \exp\left\{\frac{(2\pi - \delta_2)\phi}{\|f\|_1}\right\} ds.$$

At this point, using Jensen's inequality, we can follow the argument of [1], proof of Theorem 1, to conclude the result. □

Proof of Proposition 3.1. We first fix $\varepsilon > 0$, and assume that $\tilde{u}$ is a solution of (5) with (6) and (7). From Theorem 2 of [1] it suffices to show that, for any $x_0 \in \partial\mathbb{R}_+^2 \setminus \overline{B}_\varepsilon^+(0)$, $\tilde{u}$ is bounded from above on $\overline{B}_R^+(x_0)$ for some small number $R > 0$, with a bound that is independent of the point x_0. In the following, we denote by C various constants independent of x_0.

Write $g = e^{\tilde{u}}$, $f = c(x)e^{\frac{\tilde{u}}{2}}$ where $c(x)$ is a function on $\partial\mathbb{R}_+^2 \setminus \{0\}$ with $c(x) = c_1$ when $s > 0$ and $c(x) = c_2$ when $s < 0$, where we write $x = (s,t)$. Then $\tilde{u}$ satisfies

$$\begin{cases} -\Delta\tilde{u} = g, & \text{in } B_R^+(x_0), \\ \frac{\partial\tilde{u}}{\partial t} = f, & \text{on } \Gamma_1. \end{cases}$$

It is clear that $f \in L^1(\partial\mathbb{R}_+^2)$. Set $f = f_1 + f_2$ with $\|f_1\|_{L^1(\partial\mathbb{R}_+^2)} \leqslant \pi$ and $f_2 \in L^\infty(\partial\mathbb{R}_+^2)$. Let Γ_1 and Γ_2 be as Lemma 3.2. Define $\tilde{u}_1$, $\tilde{u}_2$ and $\tilde{u}_3$ by

$$\begin{cases} -\Delta\tilde{u}_1 = e^{\tilde{u}}, & \text{in } B_R^+(x_0), \\ \frac{\partial\tilde{u}_1}{\partial t} = 0, & \text{on } \Gamma_1, \\ \tilde{u}_1 = 0, & \text{on } \Gamma_2. \end{cases}$$

$$\begin{cases} -\Delta\tilde{u}_2 = 0, & \text{in } B_R^+(x_0), \\ \frac{\partial\tilde{u}_2}{\partial t} = f_1, & \text{on } \Gamma_1, \\ \tilde{u}_2 = 0, & \text{on } \Gamma_2. \end{cases}$$

$$\begin{cases} -\Delta\tilde{u}_3 = 0, & \text{in } B_R^+(x_0), \\ \frac{\partial\tilde{u}_3}{\partial t} = f_2, & \text{on } \Gamma_1, \\ \tilde{u}_3 = 0, & \text{on } \Gamma_2. \end{cases}$$

Extending $\tilde{u}_1$ evenly we have

$$\begin{cases} -\Delta\tilde{u}_1 = e^{\tilde{u}}, & \text{in } B_R(x_0), \\ \tilde{u}_1 = 0, & \text{on } \partial B_R(x_0). \end{cases}$$

By using Theorem 2 in [1] and (6) we have

$$\|\tilde{u}_1\|_{L^\infty(\overline{B}^+_R(x_0))} \leqslant C.$$

For $\tilde{u}_2$, by Lemma 3.2, we have

$$\int_{B^+_R(x_0)} \exp\bigl(2|\tilde{u}_2|\bigr)\,dx \leqslant C, \qquad \int_{\Gamma_1} \exp\bigl(|\tilde{u}_2|\bigr)\,ds \leqslant C$$

and in particular $\|\tilde{u}_2\|_{L^q(B^+_R(x_0))} \leqslant C$ and $\|\tilde{u}_2\|_{L^q(\Gamma_1)} \leqslant C$ for any $q > 1$.

For $\tilde{u}_3$, it is obvious that

$$\|\tilde{u}_3\|_{L^\infty(\overline{B}^+_{\frac{R}{2}}(x_0))} \leqslant C.$$

Let $\tilde{u}_4 = \tilde{u} - \tilde{u}_1 - \widetilde{u}_2 - \tilde{u}_3$. Then we have

$$\begin{cases} -\Delta\tilde{u}_4 = 0, & \text{in } B^+_R(x_0),\\ \frac{\partial\tilde{u}_4}{\partial t} = 0, & \text{on } \Gamma_1. \end{cases}$$

Extending $\tilde{u}_4$ evenly, $\tilde{u}_4$ becomes a harmonic function on $B_R(x_0)$. Then the mean value theorem for harmonic functions implies that

$$\bigl\|\tilde{u}^+_4\bigr\|_{L^\infty(\overline{B}^+_{\frac{R}{2}}(x_0))} \leqslant C\bigl\|\tilde{u}^+_4\bigr\|_{L^1(B^+_R(x_0))}.$$

Notice that

$$\tilde{u}^+_4 \leqslant \tilde{u}^+ + |\tilde{u}_1| + |\tilde{u}_2| + |\tilde{u}_3|,$$

and

$$\int_{\mathbb{R}^2_+} \tilde{u}^+\,dx \leqslant \int_{\mathbb{R}^2_+} e^{\tilde{u}^+}\,dx < \infty.$$

We get

$$\bigl\|\tilde{u}^+_4\bigr\|_{L^\infty(\overline{B}^+_{\frac{R}{2}}(x_0))} \leqslant C.$$

Finally, we write

$$\begin{cases} -\Delta\tilde{u} = e^{\tilde{u}} = g, & \text{in } B^+_R(x_0),\\ \frac{\partial\tilde{u}}{\partial t} = c(x)e^{\frac{\tilde{u}}{2}} = f, & \text{on } \Gamma_1. \end{cases}$$

The standard elliptic estimates imply that

$$\|\tilde{u}^+\|_{L^\infty(\overline{B}^+_{\frac{R}{4}}(x_0))} \leqslant C,$$

since $\|g\|_{L^q(B^+_{\frac{R}{2}}(x_0))} \leqslant C$ and $\|f\|_{L^q(\partial B^+_{\frac{R}{2}}(x_0)\cap\{t=0\})} \leqslant C$ for any $q > 1$. □

As in the proof of Proposition 3.1, in the sequel we always let $c(x)$ be a function on $\partial\mathbb{R}^2_+ \setminus \{0\}$ with $c(x) = c_1$ when $s > 0$ and $c(x) = c_2$ when $s < 0$, where $x = (s,t)$. In virtue of Proposition 3.1, we obtain the asymptotic behavior of the solution of (1)–(2). More precisely, we have the following

Proposition 3.3. *Let u be a solution of* (1)–(2). *Let*

$$d = \frac{1}{\pi}\int_{\mathbb{R}^2_+} |x|^{2\alpha} e^u\,dx - \frac{1}{\pi}\int_{\partial\mathbb{R}^2_+} c(x)|x|^\alpha e^{\frac{u}{2}}\,ds.$$

Then we have

$$\lim_{|x|\to\infty}\frac{u(x)}{\ln|x|}=-d.$$

Proof. Let

$$w(x)=\frac{1}{2\pi}\int_{\mathbb{R}^2_+}\big(\log|x-y|+\log|\bar{x}-y|-2\log|y|\big)|y|^{2\alpha}e^{u(y)}\,dy$$
$$-\frac{1}{2\pi}\int_{\partial\mathbb{R}^2_+}\big(\log|x-y|+\log|\bar{x}-y|-2\log|y|\big)c(y)|y|^{\alpha}e^{\frac{u(y)}{2}}\,dy$$

where $\bar{x}$ is the reflection point of y about $\{t=0\}$. It is easy to check that $w(x)$ satisfies

$$\begin{cases}\Delta w=|x|^{2\alpha}e^{u}, & \text{in } \mathbb{R}^2_+,\\ \frac{\partial w}{\partial t}=-c(x)|x|^{\alpha}e^{\frac{u}{2}}, & \text{on } \partial\mathbb{R}^2_+\setminus\{0\}\end{cases}$$

and

$$\lim_{|x|\to\infty}\frac{w(x)}{\ln|x|}=d.$$

Consider $v(x)=u+w$. Then $v(x)$ satisfies

$$\begin{cases}\Delta v=0, & \text{in } \mathbb{R}^2_+,\\ \frac{\partial v}{\partial t}=0, & \text{on } \partial\mathbb{R}^2_+\setminus\{0\}.\end{cases}$$

We extend $v(x)$ to $\mathbb{R}^2$ by even reflection such that $v(x)$ is harmonic in $\mathbb{R}^2$. From Proposition 3.1 we know $v(x)\leqslant C(1+\ln(|x|+1))$ for some positive constant C. Thus $v(x)$ is a constant. This completes the proof. □

Remark 3.4. From (2), it is easy to check that $d\geqslant 2+2\alpha$.

4. The exact value of *d*

In this section, we want to compute the value of d. We need to distinguish two cases. When $c_1\leqslant 0$ and $c_2\leqslant 0$, we will employ a similar argument as in [11] when they proved $\gamma_i<2$ in Proposition 7.1 to show that $d>2+2\alpha$. Here $c_1\leqslant 0$ and $c_2\leqslant 0$ are crucial such that $w(x)<0$ in D^+, see Proposition 4.1. Once we have proved that $d>2+2\alpha$, we can obtain an extension of $u(x)$ near ∞, see (11). Then we can use the Pohozaev identity of (1) to prove $d=4+4\alpha$. When $c_i>0$ for $i=1$ or $i=2$, this method will not work well. We will use the moving sphere method, which was used in [13], to show $d>2(1+\alpha)$. Let us start with the negative case.

Proposition 4.1. *If $c_1\leqslant 0$ and $c_2\leqslant 0$ in* (1)–(2)*, we have $d>2+2\alpha$.*

Proof. Assume by contradiction that $d=2+2\alpha$. Let v be the Kelvin transformation of u, i.e. $v(x)=u(\frac{x}{|x|^2})-(4\alpha+4)\ln|x|$. Then v satisfies

$$\begin{cases}-\Delta v=|x|^{2\alpha}e^{v}, & \text{in } \mathbb{R}^2_+,\\ \frac{\partial v}{\partial t}=c(x)|x|^{\alpha}e^{\frac{v}{2}}, & \text{on } \partial\mathbb{R}^2_+\setminus\{0\}\end{cases}$$

with the energy conditions

$$\int_{\mathbb{R}^2_+}|x|^{2\alpha}e^{v}\,dx<\infty$$

and

$$\int_{\partial\mathbb{R}^2_+} |x|^\alpha e^{\frac{v}{2}}\,dt < \infty.$$

Here $c(x)$ is a function as in the above section.

Let D^+ be a small half disk centered at zero. Define $w(x)$ by

$$w(x) = \frac{1}{2\pi}\int_{D^+}\big(\log|x-y| + \log|\bar{x}-y|\big)|y|^{2\alpha}e^{v(y)}\,dy \\ - \frac{1}{2\pi}\int_{\partial D^+\cap\{t=0\}}\big(\log|x-y| + \log|\bar{x}-y|\big)c(y)|y|^\alpha e^{\frac{v(y)}{2}}\,dy$$

and define $g(x) = v(x) + w(x)$. It is clear that

$$\begin{cases} \Delta g = 0, & \text{in } D^+, \\ \frac{\partial g}{\partial t} = 0, & \text{on } \{\partial D^+ \cap \{t=0\}\} \setminus \{0\}. \end{cases}$$

Therefore by extending $g(x)$ to $D \setminus \{0\}$ evenly we obtain a harmonic $g(x)$ in $D \setminus \{0\}$.

On the other hand, we can check that

$$\lim_{|x|\to 0}\frac{w}{-\log|x|} = 0$$

which implies

$$\lim_{|x|\to 0}\frac{g(x)}{-\log|x|} = \lim_{|x|\to 0}\frac{v(x)+w(x)}{-\log|x|} = 2\alpha + 2.$$

Since $g(x)$ is harmonic in $D\backslash\{0\}$, we have $g(x) = -(2\alpha+2)\log|x| + g_0(x)$ with a smooth harmonic function g_0 in D. By the definition, we have $w(x) < 0$ since $c(x)$ is negative. Thus, we have

$$\int_{D^+} |x|^{2\alpha}e^v\,dx = \int_{D^+} |x|^{2\alpha}e^{g-w}\,dx \geqslant \int_{D^+} |x|^{2\alpha}|x|^{-2\alpha-2}e^{g_0}\,dx = \infty,$$

which is a contradiction with $\int_{\mathbb{R}^2_+} |x|^{2\alpha}e^v dx < \infty$. Hence we have shown that $d > 2\alpha + 2$. □

From $d > 2\alpha + 2$ we can improve the estimates for e^u to

$$e^u \leqslant C|x|^{-2-2\alpha-\varepsilon}, \quad \text{for } |x| \text{ near } \infty. \tag{8}$$

Then by using potential analysis, we obtain

$$-d\ln|x| - C \leqslant u(x) \leqslant -d\ln|x| + C$$

for some constant $C > 0$ and $\varepsilon > 0$, see [4]. Furthermore following the idea for the derivation of gradient estimates in [2] and [21], we get

$$\big|\langle x, \nabla u\rangle + d\big| \leqslant C|x|^{-\varepsilon} \quad \text{for } |x| \text{ near } \infty,$$

consequently we have

$$\left|u_r + \frac{d}{r}\right| \leqslant C|x|^{-1-\varepsilon} \quad \text{for } |x| \text{ near } \infty. \tag{9}$$

In a similar way, we can also get

$$|u_\theta| \leqslant C|x|^{-\varepsilon} \quad \text{for } |x| \text{ near } \infty. \tag{10}$$

From (9) and (10) we can also get by standard potential analysis that

$$u(x) = -d\ln|x| + C + \mathrm{O}\big(|x|^{-1}\big) \quad \text{for } |x| \text{ near } \infty. \tag{11}$$

Here (r, θ) is the polar coordinate system on $\mathbb{R}^2$, and C, ε are some positive constants.

Proposition 4.2. *If $d > 2+2\alpha$, then we have $d = 4+4\alpha$.*

Proof. Firstly we establish the Pohozaev identity of (1)–(2). Multiply equation (1) by $x\cdot\nabla u$ and integrate over B_R^+ to obtain

$$-\int_{B_R^+}(x\cdot\nabla u)\Delta u\,dx = \int_{B_R^+}|x|^{2\alpha}e^u x\cdot\nabla u\,dx.$$

Since

$$(x\cdot\nabla u)\Delta u = \operatorname{div}\big((x\cdot\nabla u)\nabla u\big) - \operatorname{div}\left(\frac{x|\nabla u|^2}{2}\right),$$
$$|x|^{2\alpha}e^u x\cdot\nabla u = \operatorname{div}\big(x|x|^{2\alpha}e^u\big) - \operatorname{div}(x)|x|^{2\alpha}e^u - e^u x\cdot\nabla|x|^{2\alpha},$$

and

$$x\cdot\nabla|x|^{2\alpha} = 2\alpha|x|^{2\alpha},$$

we obtain

$$\int_{\partial B_R^+\cap\{t>0\}} x\cdot\nu\frac{|\nabla u|^2}{2} - (\nu\cdot\nabla u)(x\cdot\nabla u)\,ds + \int_{\partial B_R^+\cap\{t=0\}} x\cdot\nu\frac{|\nabla u|^2}{2} - (\nu\cdot\nabla u)(x\cdot\nabla u)\,ds$$
$$= \int_{\partial B_R^+\cap\{t>0\}} x\cdot\nu|x|^{2\alpha}e^u\,ds + \int_{\partial B_R^+\cap\{t=0\}} x\cdot\nu|x|^{2\alpha}e^u\,ds - (2+2\alpha)\int_{B_R^+}|x|^{2\alpha}e^u\,dx,$$

where ν is the outward unit normal vector to ∂B_R^+. Hence we have

$$R\int_{\partial B_R^+\cap\{t>0\}}\frac{|\nabla u|^2}{2} - \left|\frac{\partial u}{\partial r}\right|^2 ds + \int_{\partial B_R^+\cap\{t=0\}}\frac{\partial u}{\partial t}(x\cdot\nabla u)\,ds = R\int_{\partial B_R^+\cap\{t>0\}}|x|^{2\alpha}e^u\,ds - (2+2\alpha)\int_{B_R^+}|x|^{2\alpha}e^u\,dx.$$

Since

$$\int_{\partial B_R^+\cap\{t=0\}}\frac{\partial u}{\partial t}(x\cdot\nabla u)\,ds = \int_{-R}^{R}c(x)|x|^\alpha e^{\frac{u}{2}}s\partial_s u\,ds$$
$$= 2\int_{-R}^{R}c(x)|x|^\alpha s\partial_s e^{\frac{u}{2}}\,ds$$
$$= 2c(x)|s|^\alpha s e^{\frac{u}{2}}\big|_{-R}^{R} - (2+2\alpha)\int_{-R}^{R}c(x)|x|^\alpha e^{\frac{u}{2}}\,ds,$$

we get the Pohozaev identity

$$R\int_{\partial B_R^+\cap\{t>0\}}\frac{|u_\theta|^2}{2R^2} - \frac{|u_r|^2}{2}\,ds = R\int_{\partial B_R^+\cap\{t>0\}}|x|^{2\alpha}e^u\,ds - (2+2\alpha)\int_{B_R^+}|x|^{2\alpha}e^u\,dx$$
$$- 2c(x)|s|^\alpha s e^{\frac{u}{2}}\big|_{-R}^{R} + (2+2\alpha)\int_{-R}^{R}c(x)|x|^\alpha e^{\frac{u}{2}}\,ds.$$

In virtue of (8), (9) and (10), we let $R\to\infty$ in the Pohozaev identity and get

$$d = 4+4\alpha. \qquad \square$$

Next let us consider the case $c_i > 0$ for $i = 1$ or $i = 2$.

Proposition 4.3. *If $c_i > 0$ for $i = 1$ or $i = 2$, then $d \geqslant 4 + 4\alpha$ and consequently $d = 4 + 4\alpha$.*

Proof. Without loss of generality, we assume that $c_1 > 0$. First we have $d \geqslant 2(1+\alpha)$ from Remark 3.4. To prove $d \geqslant 4(\alpha+1)$, we will derive a contradiction from assuming $d < 4(1+\alpha)$.

Case 1. $c_1 > 0$ and $c_2 \geqslant 0$.

In this case $c(x) \geqslant 0$, where $c(x)$ is a function defined as in the proof of Proposition 3.1. We assume $2(1+\alpha) \leqslant d < 4(1+\alpha)$ by contradiction. For any $\lambda > 0$, set $E_\lambda = \{x \in \mathbb{R}^2_+ \colon |x| > \frac{1}{\sqrt{\lambda}}\}$ and $u_\lambda(x) = u(\lambda x) + 2(1+\alpha)\ln\lambda$. Then $u_\lambda(x)$ satisfies

$$\begin{cases} -\Delta u_\lambda(x) = |x|^{2\alpha} e^{u_\lambda}, & \text{in } E_\lambda, \\ \frac{\partial u_\lambda}{\partial t} = c(x)|x|^\alpha e^{\frac{u_\lambda}{2}}, & \text{on } \partial E_\lambda \cap \partial\mathbb{R}^2_+. \end{cases} \tag{12}$$

Set

$$\begin{aligned} v_\lambda(x) &= v(\lambda x) + 2(1+\alpha)\ln\lambda \\ &= u\left(\frac{x}{\lambda|x|^2}\right) + 2(\alpha+1)\ln\frac{1}{\lambda|x|^2} \end{aligned}$$

where $v(x)$ is the Kelvin transformation of $u(x)$, i.e. $v(x) = u(\frac{x}{|x|^2}) - 4(\alpha+1)\ln|x|$. So, $v_\lambda(x)$ is also a solution of (12).

Set $w_\lambda = u - v_\lambda$. Since E_λ does not contain the point $x = 0$, w_λ is smooth in E_λ, and w_λ satisfies

$$\begin{cases} -\Delta w_\lambda(x) = c_1(x)|x|^{2\alpha} w_\lambda, & \text{in } E_\lambda, \\ \frac{\partial w_\lambda}{\partial t} = c(x)c_2(x)|x|^\alpha w_\lambda, & \text{on } \partial E_\lambda \cap \partial\mathbb{R}^2_+, \\ w_\lambda = 0, & \text{on } \partial E_\lambda \cap \{t > 0\}, \end{cases} \tag{13}$$

where $c_1(x) = e^{\xi_1(x)}$ and $c_2(x) = \frac{1}{2}e^{\frac{\xi_2(x)}{2}}$, ξ_i $(i = 1, 2)$ are two functions between u and v_λ.

Claim 1. *For λ large enough, $w_\lambda(x) \geqslant 0$ for all $x \in E_\lambda$.*

Step 1. $\exists R_0$ such that for all $x \in \{x \in \mathbb{R}^2_+, \frac{2}{\sqrt{\lambda}} \leqslant |x| \leqslant R_0\}$, we have $w_\lambda \geqslant 0$.

For $x \in \{x \in \mathbb{R}^2_+, \frac{2}{\sqrt{\lambda}} \leqslant |x| \leqslant R_0\}$ with R_0 small enough, we have

$$\begin{aligned} w_\lambda(x) &= u(x) - u\left(\frac{x}{\lambda|x|^2}\right) + 2(\alpha+1)\ln\bigl(\lambda|x|^2\bigr) \\ &\geqslant o(1) + 2(\alpha+1)\ln 4 > 0. \end{aligned}$$

Step 2. $\exists R_1 \leqslant R_0$ such that for all $x \in \{x \in \mathbb{R}^2_+, \frac{1}{\sqrt{\lambda}} \leqslant |x| \leqslant \frac{2}{\sqrt{\lambda}} \leqslant R_1\}$, we have $w_\lambda \geqslant 0$.

Set $A_\lambda = \{x \in \mathbb{R}^2_+, \frac{1}{\sqrt{\lambda}} \leqslant |x| \leqslant \frac{2}{\sqrt{\lambda}} \leqslant R_1\}$ and $g(x) = 1 - |x|^{\alpha+1}$ and let $\overline{w}_\lambda(x) = \frac{w_\lambda(x)}{g(x)}$. Then, by Step 1 and (13), $\overline{w}_\lambda(x)$ satisfies

$$\begin{cases} \Delta\overline{w}_\lambda(x) + \frac{2}{g}\nabla g\cdot\nabla\overline{w}_\lambda(x) + (c_1(x)|x|^{2\alpha} + \frac{\Delta g}{g})\overline{w}_\lambda(x) = 0, & \text{in } A_\lambda, \\ \frac{\partial\overline{w}_\lambda(x)}{\partial t} = c(x)c_2(x)|x|^\alpha\overline{w}_\lambda(x), & \text{on } \partial A_\lambda \cap \{t = 0\}, \\ \overline{w}_\lambda \geqslant 0, & \text{on } \partial A_\lambda \cap \{t > 0\}. \end{cases} \tag{14}$$

Since $v_\lambda \leqslant \max_{\mathbb{R}^2_+} u$ in $\overline{E}_\lambda$, there exists some positive constant C_0 such that $c_1(x) \leqslant C_0$. By a direct computation,

$$\begin{aligned} c_1(x)|x|^{2\alpha} + \frac{\Delta g}{g} &\leqslant g^{-1}\bigl(-(\alpha+1)^2|x|^{\alpha-1} + C_0|x|^{2\alpha}\bigl(1 - |x|^{\alpha+1}\bigr)\bigr) \\ &\leqslant g^{-1}|x|^{\alpha-1}\bigl(-(\alpha+1)^2 + C_0|x|^{\alpha+1}\bigr) < 0, \end{aligned}$$

if $|x| < \{\frac{(\alpha+1)^2}{C_0}\}^{\frac{1}{\alpha+1}}$. Therefore, we choose $R_1 < \min\{\{\frac{(\alpha+1)^2}{C_0}\}^{\frac{1}{\alpha+1}}, 1\}$ small enough. Then, from (14), it follows from the maximum principle and the Hopf Lemma that $w_\lambda \geqslant 0$ in A_λ. Here we have used the fact that $c(x) \geqslant 0$.

Step 3. $\exists R_2 \leqslant R_1$ such that for $\sqrt{\lambda} \geqslant \frac{1}{R_2}$, we have $w_\lambda \geqslant 0$ for all $x \in \{x \in \mathbb{R}^2_+, |x| > R_0\}$.

For $x \in \{x \in R^2_+, |x| > R_0\}$ and $d < 4\alpha + 4$, as $|x| \to \infty$ we have

$$\lim_{|x|\to\infty} \frac{u(x) + 4(\alpha+1)\ln|x|}{\ln|x|} = -d + 4(\alpha+1) > 0.$$

Then there exists some constant $C > 0$ such that

$$u(x) + 4(\alpha+1)\ln|x| > -C, \quad \text{for } |x| > R_0.$$

Therefore, for λ large enough we have

$$\begin{aligned} w_\lambda(x) &= u(x) + 4(\alpha+1)\ln|x| - u\left(\frac{x}{\lambda|x|^2}\right) + 2(\alpha+1)\ln\lambda \\ &\geqslant -C - \max_{\mathbb{R}^2_+} u + 2(\alpha+1)\ln\lambda \geqslant 0. \end{aligned}$$

Thus we finish the proof of Claim 1.

Now we define

$$\lambda_0 = \inf\{\lambda > 0 \mid w_\mu(x) \geqslant 0 \quad \text{in } E_\mu \text{ for all } \mu \geqslant \lambda\}.$$

Claim 2. $\lambda_0 > 0$.

Assume by contradiction that $\lambda_0 = 0$, that is, for all $\lambda > 0$, we have $w_\lambda(x) \geqslant 0$ in E_λ. Then, we have for all $x \in \mathbb{R}^2_+$

$$\begin{cases} w_{\frac{1}{|x|^2}}(x) = 0, \\ w_{\frac{1}{|x|^2}}(rx) \geqslant 0, \quad \forall 0 < r < 1. \end{cases}$$

Since

$$w_\lambda(x) = u(x) - u\left(\frac{x}{\lambda|x|^2}\right) + 2(\alpha+1)\ln(\lambda|x|^2),$$

by a direct computation, we have

$$w_{\frac{1}{|x|^2}}(rx) = u(rx) - u\left(\frac{x}{r}\right) + 4(\alpha+1)\ln r. \tag{15}$$

In (15), taking firstly $|x| = r$ and then let $r \to 0^+$, we get $w_{\frac{1}{|x|^2}}(rx) \to -\infty$. Thus we get a contradiction with $w_{\frac{1}{|x|^2}}(rx) \geqslant 0$ for all $0 < r < 1$ and all $x \in \mathbb{R}^2_+$.

Claim 3. $w_{\lambda_0}(x) = 0$, $\forall x \in \mathbb{R}^2_+$.

Assume by contradiction $w_{\lambda_0} \geqslant 0$ for all $x \in \mathbb{R}^2_+$. Then from (13) we obtain firstly

$$\begin{cases} \Delta w_{\lambda_0}(x) \leqslant 0, & \text{in } E_{\lambda_0}, \\ \frac{\partial w_{\lambda_0}}{\partial t} \geqslant 0, & \text{on } \partial E_{\lambda_0} \cap \partial\mathbb{R}^2_+, \\ w_{\lambda_0} = 0, & \text{on } \partial E_{\lambda_0} \cap \{t > 0\}. \end{cases} \tag{16}$$

Then we use the strong maximum principle and the Hopf Lemma to obtain

$$\begin{cases} w_{\lambda_0}(x) > 0, & \text{in } E_{\lambda_0}, \\ \frac{\partial w_{\lambda_0}}{\partial \nu} > 0, & \text{on } \partial E_{\lambda_0} \cap \{t > 0\} \end{cases} \tag{17}$$

where ν denotes the outward unit normal of the surface $\partial B_{\sqrt{1/\lambda_0}}(0) \cap \{t > 0\}$.

Next note that by the definition of λ_0, we can assume that there exists a sequence $\lambda_k \to \lambda_0$ with $\lambda_k < \lambda_0$ such that

$$\inf_{E_{\lambda_k}} w_{\lambda_k} < 0.$$

If we can prove that

$$w_{\lambda_0}(x) \geqslant C \quad \text{for } x \in \overline{E}_{\frac{\lambda_0}{2}} \tag{18}$$

for some constant $C = C(\lambda_0) > 0$, then from the continuity of u at $x = 0$ we get

$$w_{\lambda_k} \geqslant \frac{C}{2}, \quad \forall x \in \overline{E}_{\frac{\lambda_0}{2}}.$$

for k large enough. It follows that there exists $x_k = (s_k, t_k) \in \overline{E}_{\lambda_k} \setminus E_{\lambda_0/2}$ such that

$$w_{\lambda_k}(x_k) = \inf_{E_{\lambda_k}} w_{\lambda_k} < 0.$$

It is clear that $\sqrt{\frac{1}{\lambda_k}} < |x_k| < \sqrt{\frac{2}{\lambda_0}}$ and, due to the boundary condition, $t_k > 0$. Hence $\nabla w_{\lambda_k}(x_k) = 0$. After passing to a subsequence (still denoted as x_k) $x_k \to x_0 = (s_0, t_0)$, it follows that

$$w_{\lambda_0}(x_0) = 0, \qquad \nabla w_{\lambda_0}(x_0) = 0. \tag{19}$$

By (17) we have $t_0 = 0$ and $|s_0| = \sqrt{\frac{1}{\lambda_0}}$.

However, we would like to show

$$\frac{\partial w_{\lambda_0}(x_0)}{\partial \nu} > 0, \quad \text{for } x_0 = (s_0, 0),\ |s_0| = \sqrt{\frac{1}{\lambda_0}} \tag{20}$$

if $w_{\lambda_0}(x)$ satisfies (16). Here ν denotes the outward unit normal of the surface $\partial B_{\sqrt{1/\lambda_0}}(0) \cap \{t \geqslant 0\}$.

Therefore from (19) and (20) we get a contradiction. Thus to prove Claim 3, it suffices to show (18) and (20).

Proof of (18). First, for $x \in \overline{E}_{\frac{\lambda_0}{2}}$, we have

$$\begin{aligned} v_{\lambda_0} &= u\left(\frac{x}{\lambda_0 |x|^2}\right) + 2(\alpha+1)\ln\frac{1}{\lambda_0|x|^2} \\ &\leqslant \max_{\mathbb{R}^2_+} u + 2(\alpha+1)\ln 2 \leqslant C. \end{aligned}$$

Notice that $\min_{\partial E_{\frac{\lambda_0}{2}} \cap \{t>0\}} w_{\lambda_0} \geqslant \varepsilon$ for some $0 < \varepsilon < 1$. Without loss of generality, we assume $\lambda_0 = 2$. For $0 < r < 1$, we introduce an auxiliary function

$$\varphi(x) = \frac{\varepsilon\mu}{2(c+1)} - \frac{\log|x|}{\log\sqrt{1/r}} \cdot \varepsilon + \frac{\varepsilon(1-\mu)(t - 1/\sqrt{r})}{2}\left(\sqrt{\frac{1}{r}}\right)^{\alpha}$$

when $\alpha \geqslant 0$. Here $c = \max\{c_1, c_2\}$; $0 < \mu < 1$ will be chosen later. When $-1 < \alpha < 0$, we use instead the auxiliary function

$$\varphi(x) = \frac{\varepsilon\mu}{2(c+1)} - \frac{\log|x|}{\log\sqrt{1/r}} \cdot \varepsilon + \frac{\varepsilon(1-\mu)t}{2}.$$

We shall only present the details for the case $\alpha \geqslant 0$ as the case $-1 < \alpha < 0$ can be treated in a similar way. Let $P(x) = w_{\lambda_0}(x) - \varphi(x)$. Then we get

$$\begin{cases} \Delta P(x) = \Delta w_{\lambda_0}(x) \leqslant 0, & \text{in } E_1 \setminus E_r, \\ \frac{\partial P(x)}{\partial t} = c(x)c_2(x)|x|^{\alpha} w_{\lambda_0} - \frac{\varepsilon(1-\mu)}{2}\left(\sqrt{\frac{1}{r}}\right)^{\alpha}, & \text{on } \partial(E_1 \setminus E_r) \cap \{t = 0\}. \end{cases}$$

We will show

$$P(x) \geqslant 0, \quad x \in E_1 \setminus E_r. \tag{21}$$

We prove it by contradiction. If (21) does not hold, there exists some $x_0 = (s_0, t_0)$ such that

$$P(x_0) = \min_{\overline{E}_1 \setminus E_r} P(x) < 0.$$

Since we have $P(x) \geqslant 0$ on $\partial E_1 \cap \{t > 0\}$ and $P(x) > w_{\lambda_0}(x) \geqslant 0$ on $\partial E_r \cap \{t > 0\}$, then it follows from the maximum principle that $t_0 = 0$ and $1 < |s_0| < \sqrt{\frac{1}{r}}$ and $\frac{\partial P(x)}{\partial t}|_{x_0} \geqslant 0$.

In virtue of $P(x_0) < 0$ and $v_{\lambda_0}(x_0) < C_1$, we have $c_2(x_0) < C_0$ for some constant $C_0 > 0$ and moreover

$$w_{\lambda_0}(x_0) < \varphi(x_0) < \frac{\varepsilon\mu}{2(c+1)}. \tag{22}$$

On the other hand, in virtue of $\frac{\partial P(x)}{\partial t}|_{x_0} \geqslant 0$ we have

$$\begin{aligned} 0 &\leqslant (c+1)c_2(x_0)|x_0|^\alpha w_{\lambda_0}(x_0) - \frac{\varepsilon(1-\mu)}{2}\left(\sqrt{\frac{1}{r}}\right)^\alpha \\ &\leqslant \left\{\sqrt{\frac{1}{r}}\right\}^\alpha \left(C_0(c+1)w_{\lambda_0}(x_0) - \frac{\varepsilon(1-\mu)}{2}\right). \end{aligned}$$

Hence

$$w_{\lambda_0}(x_0) \geqslant \frac{\varepsilon(1-\mu)}{2C_0(c+1)}. \tag{23}$$

From (22) and (23), we have

$$\mu > \frac{1}{1+C_0}.$$

If we choose μ such that $0 < \mu < \frac{1}{1+C_0}$ from the beginning we reach a contradiction.

Since $P(x) \geqslant 0$, we then let $r \to 0$ and have proved (18) with $C = \frac{\varepsilon}{2(1+c)(1+C_0)}$.

Proof of (20). Without loss of generality, we assume $\lambda_0 = 1$ and $s_0 = 1$. Set $\Omega = \{x = (s,t) \mid 1 < s^2 + t^2 < 4,\ s > 0,\ 0 < t < \frac{1}{4}\}$. Let

$$h(x) = \varepsilon(s-1)(t+\mu),$$

and

$$g(x) = h(x) - h\left(\frac{x}{|x|^2}\right)$$

where $0 < \varepsilon, \mu < 1$ are chosen later. A direct computation yields $\Delta g(x) = 0$ for $x \in \Omega$. Now consider

$$f(x) = w_{\lambda_0}(x) - g(x).$$

Then we have

$$\begin{cases} \Delta f(x) \geqslant 0, & \text{in } \Omega, \\ \frac{\partial f(x)}{\partial t} = c(x)c_2(x)|x|^\alpha w_{\lambda_0}(x) - \frac{\partial g(x)}{\partial t}, & \text{on } \partial\Omega \cap \{t = 0\}. \end{cases}$$

Next we want to show

$$f(x) \geqslant 0, \quad \forall x \in \Omega,$$

for suitably chosen ε and μ.

In fact, we argue by the contradiction and assume that there exists some $x_1 = (s_1, t_1) \in \overline{\Omega}$ such that

$$f(x_1) = \min_{\overline{\Omega}} f(x) < 0.$$

Since $f(x) = 0$ on $\partial\Omega \cap \partial E_1$ and $f(x) \geqslant 0$ on $\partial\Omega \cap \{\partial E_{\frac{1}{4}} \cup \{t = \frac{1}{4}\}\}$, we can use the maximum principle to obtain $t_1 = 0$, $1 < s_1 < 2$ and $\frac{\partial f(x_1)}{\partial t} \geqslant 0$ on $\partial\Omega \cap \{t = 0\}$.

A simple calculation yields

$$\frac{\partial g(x_1)}{\partial t} = \varepsilon(s_1 - 1)\big(1 + s_1^{-3}\big).$$

In virtue of $\frac{\partial f(x_1)}{\partial t} \geqslant 0$ on $\partial\Omega \cap \{t = 0\}$, we obtain

$$cc_2(x_1)|s_1|^\alpha w_{\lambda_0}(x_1) \geqslant \varepsilon(s_1 - 1)\big(1 + s_1^{-3}\big).$$

Hence, we get

$$2^\alpha cc_2(x_1)w_{\lambda_0}(x_1) \geqslant \varepsilon(s_1 - 1)\big(1 + s_1^{-3}\big)$$

for $\alpha \geqslant 0$. And

$$cc_2(x_1)w_{\lambda_0}(x_1) \geqslant \varepsilon(s_1 - 1)\big(1 + s_1^{-3}\big)$$

for $-1 < \alpha < 0$. Here $c = \max\{c_1, c_2\}$. On the other hand, we have

$$w_{\lambda_0}(x_1) < f(x_1) = \varepsilon\mu(s_1 - 1)\left(1 + \frac{1}{s_1}\right).$$

Therefore, if $\alpha \geqslant 0$ we have

$$2^\alpha\big(1 + cc_2(x_1)\big)\mu \geqslant \frac{1 + s_1^{-3}}{1 + s_1^{-1}} > \frac{3}{4},$$

and if $-1 < \alpha < 0$, we have

$$\big(1 + cc_2(x_1)\big)\mu \geqslant \frac{1 + s_1^{-3}}{1 + s_1^{-1}} > \frac{3}{4}.$$

If we choose μ such that $0 < \mu < \frac{3}{2^{a+2}(1+c\sup_{\mathbb{R}^2_+} c_2(x))}$ for $a = \max\{\alpha, 0\}$ from the beginning we reach a contradiction. Thus we have proved that $f(x) \geqslant 0$ for $x \in \Omega$. Since $f(x_0) = 0$, i.e. x_0 is minimum point of $f(x)$ in $\overline{\Omega}$, it follows from the Hopf Lemma that

$$\frac{\partial f(x_0)}{\partial \nu} \geqslant 0.$$

A direct calculation shows that

$$\frac{\partial w_{\lambda_0}(x_0)}{\partial \nu} = \frac{\partial f(x_0)}{\partial \nu} + \frac{\partial g(x_0)}{\partial \nu} \geqslant \frac{\partial g(x_0)}{\partial \nu} = 2\varepsilon\mu > 0.$$

We finish the proof of (20).

In Claim 3, $w_{\lambda_0}(x) = 0$ implies that

$$u(x) = u\left(\frac{x}{\lambda_0|x|^2}\right) + 2(\alpha + 1)\ln\frac{1}{\lambda_0|x|^2}. \tag{24}$$

Hence it follows from (24) that $d = 4(1 + \alpha)$. This contradicts our assumption $d < 4(1 + \alpha)$. Thus we proved $d \geqslant 4(1 + \alpha)$. From Proposition 4.2 we know $d = 4(1 + \alpha)$.

Case 2. $c_1 > 0$ and $c_2 < 0$.

In this case, we will follow the argument of Case 1. The main difference between the case $c_2 \geqslant 0$ and $c_2 < 0$, in view of the maximum principle and the Hopf Lemma, is to show Step 2 in the proof of Claim 1. Actually we can prove this step in the case $c_2 < 0$ by using a suitable test function. This will become evident from the rest of the argument.

Step 2 of Claim 1. $\exists R_1 \leqslant R_0$ such that for all $x \in A_\lambda = \{x \in \mathbb{R}^2_+, \frac{1}{\sqrt{\lambda}} \leqslant |x| \leqslant \frac{2}{\sqrt{\lambda}} \leqslant R_1\}$, we have $w_\lambda \geqslant 0$.

Let $x=(s,t)$ and $z=x+(0,+\frac{\mu}{\sqrt{\lambda}})$, where μ is a positive number that will be determined later. Set $g(x)=1-|z|^{\alpha+1}$ and $\overline{w}_\lambda(x)=\frac{w_\lambda(x)}{g(x)}$. Then, by Step 1 and (13), $\overline{w}_\lambda(x)$ satisfies

$$\begin{cases} \Delta\overline{w}_\lambda(x)+\frac{2}{g}\nabla g\cdot\nabla\overline{w}_\lambda(x)+(c_1(x)|x|^{2\alpha}+\frac{\Delta g}{g})\overline{w}_\lambda(x)=0, & \text{in } A_\lambda,\\ \frac{\partial\overline{w}_\lambda(x)}{\partial t}=(c_1c_2(x)|x|^\alpha-\frac{1}{g}\frac{\partial g}{\partial t})\overline{w}_\lambda(x), & \text{on } \partial A_\lambda\cap\{t=0\}\cap\{s>0\},\\ \frac{\partial\overline{w}_\lambda(x)}{\partial t}=(c_2c_2(x)|x|^\alpha-\frac{1}{g}\frac{\partial g}{\partial t})\overline{w}_\lambda(x), & \text{on } \partial A_\lambda\cap\{t=0\}\cap\{s<0\},\\ \overline{w}_\lambda\geqslant 0, & \text{on } \partial A_\lambda\cap\{t>0\}. \end{cases} \tag{25}$$

Since $v_\lambda\leqslant\max_{\mathbb{R}^2_+}u$ in $\overline{E}_\lambda$, then there exists some positive constant C_0 such that $0<c_1(x),c_2(x)\leqslant C_0$. Since $x\in A_\lambda=\{x\in\mathbb{R}^2_+,\ \frac{1}{\sqrt{\lambda}}\leqslant|x|\leqslant\frac{2}{\sqrt{\lambda}}\leqslant R_1\}$, we have $|x|\sim|z|\sim|\frac{1}{\sqrt{\lambda}}|$. Then by a direct computation, we obtain

$$c_1(x)|x|^{2\alpha}+\frac{\Delta g}{g}\leqslant g^{-1}\big(-(\alpha+1)^2|z|^{\alpha-1}+C_0|x|^{2\alpha}\big(1-|z|^{\alpha+1}\big)\big)<0,$$

if λ is large enough. Similarly, we have

$$\begin{aligned} c_2c_2(x)|x|^{2\alpha}-\frac{1}{g}\frac{\partial g}{\partial t}&\geqslant g^{-1}\bigg((\alpha+1)|z|^{\alpha-1}\frac{\mu}{\sqrt{\lambda}}+c_2C_0|x|^\alpha\big(1-|z|^{\alpha+1}\big)\bigg)\\ &\geqslant g^{-1}\bigg((\alpha+1)C\mu\bigg|\frac{1}{\sqrt{\lambda}}\bigg|^\alpha+c_2C_0\bigg|\frac{1}{\sqrt{\lambda}}\bigg|^\alpha\bigg)>0, \end{aligned}$$

on $\partial A_\lambda\cap\{t=0\}\cap\{s<0\}$ for sufficiently large μ. It is obvious that $c_1c_2(x)|x|^{2\alpha}-\frac{1}{g}\frac{\partial g}{\partial t}>0$ on $\partial A_\lambda\cap\{t=0\}\cap\{s>0\}$ since $c_1>0$. Then, from (25), we can again use the maximum principle and the Hopf Lemma to obtain $w_\lambda\geqslant 0$ in A_λ.

The proof of Claim 3 requires some simple modifications when we use the maximum principle and the Hopf Lemma. But these can be carried out just by changing test functions as in the previous argument. Here we omit the details. Thus we complete the proof of the theorem. □

Remark 4.4. Actually the spherical symmetry (24) is inherited by the solution of (1)–(2). From the proof of Proposition 4.3, it is sufficient to establish Step 3 when $d=4(1+\alpha)$. But this can be done with the help of the asymptotic estimate (11).

5. Proof of main theorems

In this section we prove our main theorems. Theorem 1.1 can be obtained directly from Proposition 4.3, since we can show that the solution u to (1)–(2) has a removable singularity at $z=\infty$ by using the Kelvin transformation as in many conformal problems. To prove Theorem 1.2, we follow closely the argument in [19]. The crucial step is to construct a projective connection on S^2 by using the conformal metric on $\overline{\mathbb{R}^2_+}\cup\{\infty\}$ with constant curvature 1 and constant geodesic curvature $c(x)$ on the boundary.

First, we prove Theorem 1.1:

Proof of Theorem 1.1. To prove Theorem 1.1, it suffices to show that any solution of (1)–(2) determines a metric as in Theorem 1.1. For this point, we first prove that the metric $ds^2=|x|^{2\alpha}e^{u(x)}|dz|^2$, u being a solution of (1)–(2), has two conical singularities at 0 and ∞ with the same order. The existence of this metric is shown in Theorem 1.2.

Let v be the Kelvin transformation of u. If u is a solution of (1)–(2), then $v\in C^2(\mathbb{R}^2_+)\cap C^1(\overline{\mathbb{R}^2_+}\setminus\{0\})$ and satisfies

$$\begin{cases} -\Delta v=|x|^{2\alpha}e^v, & \text{in } \mathbb{R}^2_+,\\ \frac{\partial v}{\partial t}=c(x)|x|^\alpha e^{\frac{v}{2}}, & \text{on } \partial\mathbb{R}^2_+\setminus\{0\}. \end{cases} \tag{26}$$

To prove the result, we first show that v is continuous at $x=0$, that is the singularity $z=0$ of v is removable. Applying the asymptotic estimate (11) we have

$$\begin{aligned} v(x)&=u\bigg(\frac{x}{|x|^2}\bigg)-4(\alpha+1)\ln|x|\\ &=\big(d-4(\alpha+1)\big)\ln|x|+\mathrm{O}(1)\quad\text{for } |x| \text{ near } 0. \end{aligned}$$

Since $d = 4(1+\alpha)$, we get that v is bounded near 0. Thus, by standard elliptic regularity, we conclude that v is a $C^2(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+})$ solution of (1) when $\alpha \geqslant 0$. While, for $\alpha \in (-1, 0)$, v is smooth away from the origin and $v \in W^{2,p}$ for $1 < p < -\frac{1}{\alpha}$ near the origin. In particular, in any case, v is continuous at the origin.

Next note that $ds^2 = e^{\tilde{u}}\, dx^2$ for $\tilde{u} = u(x) + 2\alpha \log|x|$, where u is a solution of (1)–(2). So the metric ds^2 has a conical singularity at $z = 0$ with order α. Let $\widetilde{v}(x) = \tilde{u}(\frac{x}{|x|^2}) - 4\log|x|$ be Kelvin transformation of $\tilde{u}$. Then we obtain near $z = 0$

$$\begin{aligned}\widetilde{v}(x) &= u\left(\frac{x}{|x|^2}\right) - 2\alpha \log|x| - 4\log|x| \\ &= 2\alpha \log|x| + v(x)\end{aligned}$$

since $v(x)$ is continuous function at $z = 0$. By the definition of a conical singularity, we get that the metric $ds^2 = e^{\tilde{u}}\, dx^2$ has a conical singularity at $z = \infty$ with the same order as at $z = 0$. □

Lemma 5.1. *Let u be a solution of* (1)–(2)*, and $ds^2 = e^{\tilde{u}}|dz|^2$, where $\tilde{u} = u + 2\alpha \ln|z|$. Define*

$$\eta(z) = \left(\frac{\partial^2 \tilde{u}}{\partial z^2} - \frac{1}{2}\left(\frac{\partial \tilde{u}}{\partial z}\right)^2\right)|dz|^2.$$

Then $\eta(z)$ can be extended to a projective connection on $\mathbb{S}^2 = \mathbb{C} \cup \infty$, still denoted by $\eta(z)$, that is compatible with the divisor $\mathbf{A} = \alpha \cdot 0 + \alpha \cdot \infty$.

Proof. First, from the assumption, we know that $\tilde{u}$ satisfies

$$\begin{cases} -\Delta \tilde{u} = e^{\tilde{u}}, & \text{in } \mathbb{R}^2_+, \\ \frac{\partial \tilde{u}}{\partial t} = c(x) e^{\frac{\tilde{u}}{2}}, & \text{on } \partial\mathbb{R}^2_+ \setminus \{0\}, \end{cases} \tag{27}$$

with the energy conditions

$$\int_{\mathbb{R}^2_+} e^{\tilde{u}}\, dx < \infty, \tag{28}$$

$$\int_{\partial\mathbb{R}^2_+} e^{\frac{\tilde{u}}{2}}\, dt < \infty. \tag{29}$$

Let $f(z) = \frac{\partial^2 \tilde{u}}{\partial z^2} - \frac{1}{2}(\frac{\partial \tilde{u}}{\partial z})^2$, then from (27), $f(z)$ is holomorphic on $\mathbb{R}^2_+$ and $\operatorname{Im} f = \frac{1}{2}(\frac{1}{2}\frac{\partial \tilde{u}}{\partial s}\frac{\partial \tilde{u}}{\partial t} - \frac{\partial^2 \tilde{u}}{\partial s \partial t})$. On the other hand, since on $\partial\mathbb{R}^2_+ \setminus \{0\}$, $\frac{\partial \tilde{u}}{\partial t} = c(z) e^{\frac{\tilde{u}}{2}}$, we have $\frac{\partial^2 \tilde{u}}{\partial s \partial t} = \frac{c(z)}{2} e^{\frac{\tilde{u}}{2}} \frac{\partial \tilde{u}}{\partial s} = \frac{1}{2}\frac{\partial \tilde{u}}{\partial s}\frac{\partial u}{\partial t}$. This implies $f(z)$ is real on $\partial\mathbb{R}^2_+ \setminus \{0\}$, and we may extend $f(z)$ to a holomorphic function on $\mathbb{C} \setminus \{0\}$ by $f(z) = \overline{f(\bar{z})}$ for $z \in \mathbb{R}^2_-$. Thus we extend η to $\mathbb{C}$ such that η is holomorphic on $\mathbb{C} - \{0\}$.

Next we show that $\eta(z)$ is a projective connection on $\mathbb{C} \cup \infty$. Let (V, w) and (U, z) be coordinate charts with $U \cap V \neq \emptyset$. If $U \cap V \subset \overline{\mathbb{R}^2_+} \cup \{\infty\}$, then by following the argument in [19] and by using the fact that $ds^2 = e^{\tilde{u}}|dz|^2$ is a conformal metric on $\mathbb{R}^2_+ \cup \infty$, we have $ds^2 = e^{\tilde{u}}|dz|^2 = e^{v}|dw|^2$ with $v = \tilde{u} + \frac{1}{2}\log|\frac{dz}{dw}|$, and consequently we get

$$\begin{aligned}\eta(w) &= \left(\frac{\partial^2(\tilde{u} + \frac{1}{2}\log|\frac{dz}{dw}|)}{\partial w^2} - \frac{1}{2}\left(\frac{\partial(\tilde{u} + \frac{1}{2}\log|\frac{dz}{dw}|)}{\partial w}\right)^2\right)|dw|^2 \\ &= \eta(z) + \{z, w\}|dw|^2.\end{aligned} \tag{30}$$

If $U \cap V \subset \mathbb{R}^2_-$, since $\overline{\bar{z}_{\bar{w}}} = z_w$, we get from (30)

$$\begin{aligned}\eta(w) &= \overline{\eta(\bar{w})} = \overline{\eta(\bar{z}) + \{\bar{z}, \bar{w}\}\, d\bar{w}^2} \\ &= \eta(z) + \{z, w\}|dw|^2.\end{aligned}$$

So, in any case, $\eta(w) = \eta(z) + \{z, w\}\,dw^2$ when $U \cap V \neq \emptyset$. This means that η is a projective connection on $S^2 = \mathbb{C} \cup \infty$.

Next, we want to show that η has a regular singularity at 0 and at ∞ of weight $\rho = -\frac{1}{2}\alpha(\alpha+2)$. We prove this statement only at the singular point 0, since the same argument applies at ∞ by using the Kelvin transformation. Since 0 is a conical singular point of the metric $ds^2 = e^{\tilde u}\,dz^2$ on $\mathbb{R}^2_+ \cup \{\infty\}$, we set $\tilde u = u(x) + 2\alpha \log|x|$ in $B_r(0) \cap \overline{\mathbb{R}^2_+}$, where $u(x)$ is a continuous solution of (1)–(2).

First, we consider the case $\alpha \geqslant 0$. In this case, since $u(x)$ is a continuous solution of (1)–(2), u is of class C^2 in $\overline{\mathbb{R}^2_+}$ by classical elliptic regularity theory. Then we have

$$\frac{\partial^2 \tilde u}{\partial z^2} - \frac{1}{2}\left(\frac{\partial \tilde u}{\partial z}\right)^2 = \frac{\partial^2 u}{\partial z^2} - \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2 - \frac{\alpha}{z}\frac{\partial u}{\partial z} - \frac{\alpha(\alpha+2)}{2z^2}$$

in $\overline{\mathbb{R}^2_+} \setminus \{0\}$. Hence we obtain

$$\eta(z) = \left(-\frac{\alpha(\alpha+2)}{2z^2} - \frac{\alpha}{z}\frac{\partial u(z)}{\partial z} + \phi(z)\right)dz^2, \quad \text{for } z \in \overline{\mathbb{R}^2_+} \setminus \{0\},$$
$$\eta(z) = \left(-\frac{\alpha(\alpha+2)}{2z^2} - \frac{\alpha}{z}\frac{\overline{\partial u(\bar z)}}{\partial \bar z} + \overline{\phi(\bar z)}\right)dz^2, \quad \text{for } z \in \mathbb{R}^2_-,$$

where $\phi(z) = \frac{\partial^2 u}{\partial z^2} - \frac{1}{2}(\frac{\partial u}{\partial z})^2$ for $z \in \overline{\mathbb{R}^2_+} \setminus \{0\}$. This proves that $\eta(z)$ has a regular singularity of weight $\rho = -\frac{1}{2}\alpha(\alpha+2)$ at $z = 0$ in this case.

When $-1 < \alpha < 0$, u might not to be C^2 and the computation above might not work. However, we may take a method used in [19] to lift the metric to a local branched cover: We set $z = w^m$ $(m \in \mathbb{N})$, then the metric can be lifted in the w-plane: $ds'^2 = e^{u'}\,dw^2$ with $u' = \tilde u + 2\log|\frac{dz}{dw}| = u + 2(m(\alpha+1)-1)\log|w| + 2\log m$, when z is in the upper half plane. Therefore ds'^2 has a conical singularity at $w = 0$ of order $\alpha' = m(\alpha+1) - 1$. Since Eq. (27) is invariant under conformal transformations, u' satisfies (27) in terms of w. Now choosing m large enough, we have $\alpha' > 0$. Then we can use the same argument as in [19] and the extension technique above to get

$$\eta(z) = \left(-\frac{\alpha(\alpha+2)}{2z^2} + \frac{\sigma}{z} + \phi(z)\right)dz^2$$

where $\phi(z)$ is holomorphic function. □

Proof of Theorem 1.2. From Lemma 5.1, we know that $\eta(z)$ is a projective connection on $S^2 = \mathbb{C} \cup \{\infty\}$ with regular singularities at $z = 0$ and $z = \infty$. It follows from Proposition 2 in [19] that

$$\eta(z) = -\frac{\alpha(\alpha+2)}{2} \cdot \frac{dz^2}{z^2}$$

in the standard coordinate z.

Setting $h = e^{-\frac{\tilde u}{2}}$, then we have

$$\frac{\partial^2 h}{\partial z^2} = \frac{\alpha(\alpha+2)}{4} \cdot \frac{h}{z^2}, \quad \text{for any } z \in \mathbb{R}^2_+, \tag{31}$$

and the boundary condition is

$$\frac{\partial h}{\partial \bar z} - \frac{\partial h}{\partial z} = -\frac{ic(x)}{2}, \quad \text{on } \partial\mathbb{R}^2_+ \setminus \{0\}. \tag{32}$$

All solutions of (31) are of the form

$$h(z, \bar z) = f(\bar z) z^{-\frac{\alpha}{2}} + g(\bar z) z^{1+\frac{\alpha}{2}},$$

for any $z \in \mathbb{R}^2_+$. Since h is real and analytic, we have

$$h(z, \bar z) = a(\bar z z)^{-\frac{\alpha}{2}} + p z^{1+\frac{\alpha}{2}} \bar z^{-\frac{\alpha}{2}} + \bar p \bar z^{1+\frac{\alpha}{2}} z^{-\frac{\alpha}{2}} + b(z\bar z)^{1+\frac{\alpha}{2}}, \quad \text{for any } z \in \mathbb{R}^2_+.$$

Here, $a, b \in \mathbb{R}$ and $p \in \mathbb{C}$. Since $\tilde{u} = u + 2\alpha \ln|x|$ near 0 for some continuous function u, it is clear that $a \neq 0$. Then rewriting $h(z, \bar{z})$, we have

$$h = a \cdot \left(\frac{|1 + \bar{\mu}\bar{z}^{\alpha+1}|^2 + \nu|z|^{2\alpha+2}}{|z|^{\alpha}} \right),$$

for some parameters $\mu = \frac{p}{a} \in \mathbb{C}$ and $\nu = \frac{ab - p\bar{p}}{a^2} \in \mathbb{R}$. Therefore, a conformal metric should be

$$ds^2 = \frac{|dz|^2}{h^2} = \frac{1}{a^2} \cdot \frac{|z|^{2\alpha}|dz^2|}{(|1 + \bar{\mu}\bar{z}^{\alpha+1}|^2 + \nu|z|^{2\alpha+2})^2}.$$

Setting $w = \frac{1}{z}$, we have

$$ds^2 = \frac{1}{a^2} \cdot \frac{|w|^{2\alpha}|dw^2|}{(|\bar{\mu} + w^{\alpha+1}|^2 + \nu)^2}.$$

On the other hand, if we assume (r, θ) is the polar coordinate system in $\mathbb{R}^2$, then we have

$$h(r, \theta) = ar^{-\alpha} + pre^{i\theta(1+\alpha)} + \bar{p}re^{-i\theta(1+\alpha)} + br^{2+\alpha}.$$

And its boundary condition (32) can be rewritten as

$$-\frac{\partial h}{\partial \theta}\left(e^{i\theta} + e^{-i\theta}\right) + ir\frac{\partial h}{\partial r}\left(e^{i\theta} - e^{-i\theta}\right) = rc(r, \theta),$$

for $\theta = 0$ and $\theta = \pi$. Here $c(r, \theta) = c_1$ if $\theta = 0$ and $c(r, \theta) = c_2$ if $\theta = \pi$. Therefore we obtain by using the partial derivative $\frac{\partial h}{\partial \theta}$ at $\theta = 0$ and $\theta = \pi$ respectively

$$2(\alpha + 1)(\bar{p} - p) = -ic_1,$$

and

$$2(\alpha + 1)\left(\bar{p}e^{-i\alpha\pi} - pe^{i\alpha\pi}\right) = -ic_2.$$

Then there are two cases.

In the first case, α is an integer: When $\alpha = 2k$, $k = 0, 1, 2, \ldots$, then $c_1 = c_2$. And when $\alpha = 2k + 1$, $k = 0, 1, 2, \ldots$, then $c_1 = -c_2$. In this case one can only determine $\mathrm{Im}\{p\}$, namely $\mathrm{Im}\{p\} = \frac{c_1}{4(\alpha+1)}$. Now we set $\frac{\mathrm{Im}\{p\}}{a} = \frac{c_1\lambda^{\alpha+1}}{\sqrt{2}}$. Then we have

$$a = \frac{\sqrt{2}}{4(\alpha + 1)\lambda^{\alpha+1}},$$

and consequently

$$ds^2 = \frac{8(\alpha + 1)^2\lambda^{2(\alpha+1)}|w|^{2\alpha}|dw^2|}{(|w^{\alpha+1} - w_0|^2 + \nu)^2},$$

where $w_0 = (x_0, t_0)$ for some real number x_0 and $t_0 = \frac{c_1\lambda^{\alpha+1}}{\sqrt{2}}$. Set

$$u = \log \frac{8(\alpha + 1)^2\lambda^{2(\alpha+1)}}{(|w^{\alpha+1} - w_0|^2 + \nu)^2}.$$

Then it follows from the definition of the conformal metric that u is a solution of (1)–(2). Hence we have $\nu = \lambda^{2\alpha+2}$. This implies

$$ds^2 = \frac{8(\alpha + 1)^2\lambda^{2(\alpha+1)}|w|^{2\alpha}|dw^2|}{(|w^{\alpha+1} - w_0|^2 + \lambda^{2\alpha+2})^2}.$$

In the second case, $\alpha \neq k$, $k = 0, 1, 2, \ldots$. For any c_1 and c_2, one can then find a unique complex number p. In this case, we also set $\frac{\mathrm{Im}\{p\}}{a} = \frac{c_1\lambda^{\alpha+1}}{\sqrt{2}}$. Then we have

$$a = \frac{\sqrt{2}}{4(\alpha + 1)\lambda^{\alpha+1}},$$

and consequently we have

$$ds^2 = \frac{8(\alpha+1)^2\lambda^{2(\alpha+1)}|w|^{2\alpha}|dw^2|}{(|w^{\alpha+1}-w_0|^2+\nu)^2},$$

where $w_0 = (x_0, t_0)$ is a fixed point for

$$x_0 = \frac{\lambda^{\alpha+1}(c_1\cos(\pi\alpha)-c_2)}{\sqrt{2}\sin(\pi\alpha)} \quad \text{and} \quad t_0 = \frac{c_1\lambda^{\alpha+1}}{\sqrt{2}}. \tag{33}$$

Then as in the first case, we can get

$$ds^2 = \frac{8(\alpha+1)^2\lambda^{2(\alpha+1)}|w|^{2\alpha}|dw^2|}{(|w^{\alpha+1}-w_0|^2+\lambda^{2\alpha+2})^2}.$$

We complete the proof. □

Since the domain $\overline{\mathbb{R}^2_+}\setminus\{0\}$ is simply connected, in this paper we consider $z^{1+\alpha}$ as a well-defined function, even if for non-integer α. In polar coordinates, we have

$$e^u = \frac{8(\alpha+1)^2\lambda^{2(\alpha+1)}}{((r^{1+\alpha}\cos(1+\alpha)\theta - x_0)^2 + (r^{1+\alpha}\sin(1+\alpha)\theta - t_0)^2 + \lambda^{2\alpha+2})^2},$$

where x_0 and t_0 are given by (33).

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