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Random homogenization of an obstacle problem ☆

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Abstract

We study the homogenization of an obstacle problem in a perforated domain, when the holes are periodically distributed and have random shape and size. The main assumption concerns the capacity of the holes which is assumed to be stationary ergodic.

Keywords: Homogenization; Obstacle problem; Free boundary problem

1. Introduction

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a given probability space. For every $\omega \in \Omega$ and every $\varepsilon > 0$, we consider a domain $D_\varepsilon(\omega)$ obtained by perforating holes from a bounded domain D of $\mathbb{R}^n$. We denote by $T_\varepsilon(\omega)$ the set of the holes (we then have $D_\varepsilon(\omega) = D \setminus T_\varepsilon(\omega)$). The goal of this paper is to study the asymptotic behavior as $\varepsilon \to 0$ of the solution of the following obstacle problem:

$$\min\left\{\int_D \frac{1}{2}|\nabla u|^2\,dx - \int_D f u\,dx;\, u \in H_0^1(D),\ u \geqslant 0 \text{ a.e. in } T_\varepsilon(\omega)\right\} \tag{1}$$

for some given function $f(x)$ in $L^2(D)$.

This is a classical homogenization problem and the asymptotic behavior of the solutions strongly depends on the properties of the set $T_\varepsilon(\omega)$. This type of problems was first studied by L. Carbone and F. Colombini [2] in periodic settings and then in more general frameworks by E. De Giorgi, G. Dal Maso and P. Longo [9], G. Dal Maso and P. Longo [7] and G. Dal Maso [6]. Our main reference will be the papers of D. Cioranescu and F. Murat [4,5], in which the particular case of a periodic repartition of holes is studied. More precisely, they take $T_\varepsilon(\omega) = T_\varepsilon$ (independent of ω) given by

$$T_\varepsilon = \bigcup_{k \in \mathbb{Z}^n} B_{a^\varepsilon}(\varepsilon k). \tag{2}$$

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It is then proved that there exists a critical radius $a^\varepsilon \ll \varepsilon$ such that the limiting problem is no longer an obstacle problem, but a simple elliptic boundary value problem with a new term that takes into account the effect of the holes. More precisely, in dimension $n \geqslant 3$, when $a^\varepsilon = r_0 \varepsilon^{\frac{n}{n-2}}$ then there exists a constant $\alpha_0 > 0$ such that $u = \lim_{\varepsilon\to 0} u^\varepsilon$ solves

$$\min\left\{\int_D \frac{1}{2}|\nabla u|^2 + \frac{1}{2}\alpha_0 u_-^2\, dx - \int_D f u\, dx;\ u \in H_0^1(D)\right\},$$

where $u_- = \max(0, -u)$.

Our goal is to generalize this result to the case where the holes are still located in small neighborhoods of lattice points $\varepsilon\mathbb{Z}^n$ but have random size and shape. More precisely, we assume that for any ε and ω, we can write

$$T_\varepsilon(\omega) = \left(\bigcup_{k\in\mathbb{Z}^n} S_\varepsilon(k,\omega)\right) \cap D,$$

where the holes $S_\varepsilon(k,\omega)$ satisfy, in particular,

$$S_\varepsilon(k,\omega) \subset B_{\varepsilon/2}(\varepsilon k) \quad \text{for all } k \in \mathbb{Z}^n.$$

Further assumptions concerning the sets $S_\varepsilon(k,\omega)$ will be made in the next section (we can already point out the fact that we will not exclude the case where $S_\varepsilon(k,\omega) = \emptyset$ for some k, thus allowing the fact that no holes may be present at some lattice points). The proof of the main theorem, which is detailed in Section 3, relies on the construction of an appropriate corrector. This construction is detailed in Sections 4 and 5, first in the case where the holes are balls of random radii in dimension $n \geqslant 2$, then when no assumptions are made on the shape of the holes (in dimension $n \geqslant 3$ only).

2. Assumptions and main result

Let us now make precise the assumptions on the holes $S_\varepsilon(k,\omega)$. The first assumption is mainly technical:

Assumption 1. There exists a (large) constant M such that for all $k \in \mathbb{Z}^n$ and a.e. $\omega \in \Omega$ we have

$$S_\varepsilon(k,\omega) \subset B_{M\varepsilon^{n/(n-2)}}(\varepsilon k) \quad \text{if } n \geqslant 3,$$
$$S_\varepsilon(k,\omega) \subset B_{\exp(-M\varepsilon^{-2})}(\varepsilon k) \quad \text{if } n = 2$$

for ε small.

As mentioned in the introduction, the asymptotic behavior of the u^ε strongly depends on the size of the holes. The critical size for which interesting phenomena is observed corresponds to finite, nontrivial capacity of the set T_ε. More precisely, we assume:

Assumption 2. For all $k \in \mathbb{Z}^n$ and a.e. $\omega \in \Omega$, there exists $\gamma(k,\omega)$ (independent of ε) such that

$$\operatorname{cap}\big(S_\varepsilon(k,\omega)\big) = \varepsilon^n \gamma(k,\omega),$$

where $\operatorname{cap}(A)$ denote the capacity of a subset A of $\mathbb{R}^n$ (as defined below). Moreover, we assume that there exists a constant $\bar\gamma > 0$:

$$\gamma(k,\omega) \leqslant \bar\gamma \quad \text{for all } k \in \mathbb{Z}^n \text{ and a.e. } \omega \in \Omega. \tag{3}$$

We take the following definitions for the capacity of a subset A of $\mathbb{R}^n$:

$$\operatorname{cap}(A) = \inf\left\{\int_{\mathbb{R}^n} |\nabla h|^2\, dx;\ h \in H^1(\mathbb{R}^n),\ h \geqslant 1 \text{ in } A,\ \lim_{|x|\to\infty} h(x) = 0\right\},$$

in dimension $n \geqslant 3$ and

$$\mathrm{cap}(A) = \inf\left\{ \int_{B_1} |\nabla h|^2\, dx;\, h \in H_0^1(B_1),\, h \geqslant 1 \text{ in } A \right\},$$

in dimension $n = 2$ and for sets $A \subset B_1$.

Finally, we need to make an assumption to ensure that some averaging process occurs as ε goes to zero:

Assumption 3. The process $\gamma : \mathbb{Z}^n \times \Omega \mapsto [0, \infty)$ is stationary ergodic: There exists a family of measure-preserving transformations $\tau_k : \Omega \to \Omega$ satisfying

$$\gamma(k + k', \omega) = \gamma(k, \tau_{k'}\omega) \quad \text{for all } k, k' \in \mathbb{Z}^n \text{ and } \omega \in \Omega,$$

and such that if $A \subset \Omega$ and $\tau_k A = A$ for all $k \in \mathbb{Z}^n$, then $P(A) = 0$ or $P(A) = 1$ (the only invariant set of positive measure is the whole set).

Let us make a few remarks concerning those assumptions: First of all, we stress out the fact that the shape of the holes S_ε is left unspecified and may change with ε; Only the rescaled capacity $\gamma(k, \omega)$ is independent of ε. The first assumption, which implies that the diameters of the holes decrease faster than ε, guarantees that the capacities of neighboring sets separate at the limit (i.e. that $\mathrm{cap}(\bigcup S_\varepsilon) \sim \sum \mathrm{cap}(S_\varepsilon)$). And the choice of scaling for the capacity guarantee that $\mathrm{cap}(T_\varepsilon)$ remains bounded as ε goes to zero (since $\#\{\mathbb{Z}^n \cap \varepsilon^{-1} D\} \leqslant C\varepsilon^{-n}$). When $n \geqslant 3$, we have $\mathrm{cap}(B_r) = c_n r^{n-2}$, and so Assumption 2 implies that if $S_\varepsilon(k, \omega)$ is a ball, then its radius is of the form $r(k, \omega)\varepsilon^{\frac{n}{n-2}}$. In particular, we recover Cioranescu–Murat's result in the periodic case. Finally, the hypothesis of stationarity is the most general extension of the notions of periodicity and almost periodicity for a function to have some self-averaging behavior.

Under those assumptions, we prove the following result:

Theorem 2.1. *Assume that $n \geqslant 3$ and that $T_\varepsilon(\omega) = \bigcup_{k \in \mathbb{Z}^n} S_\varepsilon(k, \omega)$ satisfies Assumptions 1, 2 and 3 listed above. Then there exists $\alpha_0 \geqslant 0$ such that when ε goes to zero, the solution $u^\varepsilon(x, \omega)$ of*

$$\min\left\{ \int_D \frac{1}{2}|\nabla u|^2\, dx - \int_D f u\, dx;\ u \in H_0^1(D),\ u \geqslant 0 \text{ a.e. in } T_\varepsilon(\omega) \right\}$$

converges weakly in $H^1(D)$ and almost surely $\omega \in \Omega$ to the solution $\bar{u}(x)$ of the following minimization problem:

$$\min\left\{ \int_D \frac{1}{2}|\nabla u|^2 + \frac{1}{2}\alpha_0 u_-^2 - f u\, dx;\ u \in H_0^1(D) \right\},$$

where $u_-(x) = \max(0, -u(x))$.

Moreover, if there exists $\underline{\gamma} > 0$ such that

$$\gamma(k, \omega) \geqslant \underline{\gamma} \quad \text{for all } k \in \mathbb{Z}^n \text{ and a.e. } \omega \in \Omega,$$

then $\alpha_0 > 0$.

The same result holds when $n = 2$ in the case where the sets $S_\varepsilon(k, \omega)$ are balls.

The general result (with holes of random shape) holds also in dimension $n = 2$. However, because the fundamental solution of Laplace's equation is different in that case, the proof is slightly different and more technical.

Note that the Euler–Lagrange equations for the minimization problems yield

$$\begin{cases} -\Delta u^\varepsilon = f & \text{in } D_\varepsilon, \\ u^\varepsilon(x) \geqslant 0 & \text{for } x \in T_\varepsilon, \\ u^\varepsilon(x) = 0 & \text{for } x \in \partial D \setminus T_\varepsilon \end{cases} \tag{4}$$

and

$$\begin{cases} -\Delta\bar{u} - \alpha_0\bar{u}_- = f & \text{in } D, \\ \bar{u}(x) = 0 & \text{for } x \in \partial D. \end{cases}$$

As in Cioranescu–Murat [4,5], the proof of this result relies on the construction of an appropriate corrector. More precisely, the key is the following result:

Proposition 2.2. *Assume that $n \geqslant 3$ and that $T_\varepsilon(\omega) = \bigcup_{k\in\mathbb{Z}^n} S_\varepsilon(k,\omega)$ satisfies Assumptions 1, 2 and 3 listed above. Then, there exists a nonnegative real number α_0 and a function $w^\varepsilon(x,\omega)$ such that*

$$\begin{cases} w^\varepsilon(x,\omega) = 1 & \text{for } x \in T_\varepsilon(\omega), \\ w^\varepsilon(x,\omega) = 0 & \text{for } x \in \partial D \setminus T_\varepsilon(\omega), \\ w^\varepsilon \text{ bounded in } L^\infty(D) \\ w^\varepsilon(\cdot,\omega) \to 0 \quad H^1(D)\text{-weak} \end{cases}$$

for almost all $\omega \in \Omega$, and

$$\begin{cases} \text{For all sequences } v^\varepsilon(x) \text{ satisfying:} \\ \quad \begin{cases} v^\varepsilon(x) \geqslant 0 & \text{for } x \in T_\varepsilon, \\ v^\varepsilon \text{ bounded in } L^\infty(D), \\ v^\varepsilon \to v & \text{in } H^1(D)\text{-weak} \end{cases} \\ \text{and for any } \phi \in \mathcal{D}(D) \text{ such that } \phi \geqslant 0, \text{ we have:} \\ \quad \lim_{\varepsilon\to 0} \int_D \phi \nabla w^\varepsilon \cdot \nabla v^\varepsilon \, dx \geqslant -\alpha_0 \int_D \phi v \, dx \\ \text{with equality if } v^\varepsilon(x) = 0 \text{ for } x \in T_\varepsilon. \end{cases} \tag{5}$$

The same result holds when $n = 2$ in the case where the sets $S_\varepsilon(k,\omega)$ are balls.

Condition (5) may seem rather complicated, but it is exactly the property that will be needed to prove Theorem 2.1. This condition is satisfied in particular if the Laplacian of w^ε is equal to α_0 outside the holes as in the following lemma:

Lemma 2.3. *Let $w^\varepsilon(x,\omega)$ be a function satisfying*

$$\begin{cases} \Delta w^\varepsilon = \alpha_0 & \text{in } D_\varepsilon(\omega), \\ w^\varepsilon(x,\omega) = 1 & \text{for } x \in T_\varepsilon(\omega), \\ w^\varepsilon(x,\omega) = 0 & \text{for } x \in \partial D \setminus T_\varepsilon(\omega) \end{cases} \tag{6}$$

for almost all $\omega \in \Omega$, and

$$w^\varepsilon(\cdot,\omega) \to 0 \quad H^1(D)\text{-weak a.s. } \omega \in \Omega.$$

Then w^ε satisfies (5).

The same conclusion holds if we only have

$$\int_{D_\varepsilon} |\Delta w^\varepsilon - \alpha_0| \, dx \to 0 \quad \text{a.s. } \omega \in \Omega.$$

We will strongly rely on this lemma in the sequel. In particular, in the case where the holes are all balls of random radii, we will construct w^ε by showing that for a critical α_0, the unique solution of (6) converges to 0 in H^1 weak for almost every ω.

Proof of Lemma 2.3. Since $\varphi = 0$ on ∂D, we have:

$$-\int_{D_\varepsilon} \Delta w^\varepsilon \varphi v^\varepsilon \, dx = \int_{D_\varepsilon} \varphi \nabla w^\varepsilon \cdot \nabla v^\varepsilon \, dx + \int_{D_\varepsilon} \nabla\varphi \cdot \nabla w^\varepsilon v^\varepsilon \, dx - \int_{\partial T^\varepsilon \cap D} w^\varepsilon_\nu \varphi v^\varepsilon \, d\sigma(x).$$

Since w^ε goes to zero in $H^1(D)$-weak and v^ε converges to v in $L^2(D)$-strong, it is readily seen that the second term in the right-hand side goes to zero as $\varepsilon \to 0$. Furthermore, it is readily seen that $w^\varepsilon \leqslant 1$ and $w^\varepsilon = 1$ on T_ε, and so $w^\varepsilon_\nu \geqslant 0$ on ∂T_ε. Since $v^\varepsilon \geqslant 0$ on T_ε and $\varphi \geqslant 0$, we deduce that

$$\lim_{\varepsilon\to 0} -\int_{D_\varepsilon} \Delta w^\varepsilon \varphi v^\varepsilon \, dx \leqslant \lim_{\varepsilon\to 0} \int_{D_\varepsilon} \varphi \nabla w^\varepsilon \cdot \nabla v^\varepsilon \, dx.$$

Finally, we have

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{D_\varepsilon} \Delta w^\varepsilon \varphi v^\varepsilon \, dx &= \lim_{\varepsilon\to 0} \int_{D_\varepsilon} \alpha_0 \varphi v^\varepsilon \, dx + \lim_{\varepsilon\to 0} \int_{D_\varepsilon} |\Delta w^\varepsilon - \alpha_0| \varphi v^\varepsilon \, dx \\ &\leqslant \lim_{\varepsilon\to 0} \int_{D} \alpha_0 \varphi v^\varepsilon \, dx\end{aligned}$$

using the fact that v^ε is bounded in L^∞. Hence we have

$$\lim_{\varepsilon\to 0} \int_{D_\varepsilon} \varphi \nabla w^\varepsilon \cdot \nabla v^\varepsilon \, dx \geqslant -\int_D \alpha_0 \varphi v \, dx.$$

It is now easy to check that all the inequalities becomes equalities whenever $v^\varepsilon = 0$ on T_ε. □

The proof of Proposition 2.2 will occupy most of this paper. It will be split into two parts: In Section 4, we consider the (simpler) case when the holes $S_\varepsilon(k,\omega)$ are all balls of random radius. In Section 5, we will use this first result to treat the general case (when the holes have unspecified shapes).

Before turning to this proof, we briefly give, in the next section, the proof of the main theorem.

3. Proof of Theorem 2.1

We introduce the initial and limit energies:

$$\mathcal{J}(v) = \int_D \frac{1}{2}|\nabla v|^2 - f v \, dx$$

and

$$\mathcal{J}_\alpha(v) = \int_D \frac{1}{2}|\nabla v|^2 + \frac{1}{2}\alpha_0 v_-^2 - f v \, dx.$$

With theses notations, we have that $u^\varepsilon(x,\omega)$ satisfies

$$\mathcal{J}(u^\varepsilon) = \inf_{v\in K_\varepsilon} \mathcal{J}(v)$$

with $K_\varepsilon = \{v \in H^1_0(D);\ v \geqslant 0 \text{ a.e. in } T_\varepsilon\}$ and standard estimates give the existence of a function $\bar u(x,\omega)$ such that

$$u^\varepsilon(\cdot,\omega) \to \bar u(\cdot,\omega) \quad \text{in } H^1_0(D)\text{-weak}.$$

We now have to show that for almost every ω, $\bar u(\cdot,\omega)$ satisfies:

$$\mathcal{J}_\alpha(\bar u) = \inf_{v\in H^1_0(D)} \mathcal{J}_\alpha(v).$$

This will follow from two lemmas:

Lemma 3.1. *For any test function φ in $\mathcal{D}(D)$ we have*

$$\lim_{\varepsilon\to 0}\int_D |\nabla w^\varepsilon|^2\varphi\,dx=\int_D \alpha_0\varphi\,dx.$$

Lemma 3.2. *Let u^ε be any bounded sequence in $H^1_0(D)$ such that $u^\varepsilon \geqslant 0$ in T_ε. If*

$$u^\varepsilon \rightharpoonup \bar u \quad \text{in } H^1(D)\text{-weak},$$

then

$$\liminf_{\varepsilon\to 0}\mathcal{J}(u^\varepsilon)\geqslant \mathcal{J}_\alpha(\bar u).$$

Proof of Theorem 2.1. For any $v\in\mathcal{D}(D)$, the function $v+v_-w^\varepsilon$ is nonnegative on $T_\varepsilon(\omega)$ and is thus admissible for the initial obstacle problem. In particular by definition of u^ε, we have

$$\mathcal{J}(u^\varepsilon)\leqslant \mathcal{J}(v+v_-w^\varepsilon).$$

Next, we can expand $\mathcal{J}(v+v_-w^\varepsilon)$ as follows:

$$\mathcal{J}(v+v_-w^\varepsilon)=\int \frac{1}{2}\big[|\nabla v|^2+|\nabla v_-|^2 w^{\varepsilon 2}+|v_-|^2|\nabla w^\varepsilon|^2\big]\,dx \\ +\int [v_-\nabla v_- w^\varepsilon\nabla w^\varepsilon+\nabla v\nabla v_- w^\varepsilon+\nabla v v_-\nabla w^\varepsilon]\,dx-\int [fv+fv_-w^\varepsilon]\,dx$$

and it is readily check that Lemma 3.1 and the weak convergence of w^ε to 0 in $H^1(D)$ implies

$$\lim_{\varepsilon\to 0}\mathcal{J}(v+v_-w^\varepsilon)=\mathcal{J}_\alpha(v),$$

as soon as $|v_-|^2\in\mathcal{D}(D)$. We deduce

$$\mathcal{J}_\alpha(v)\geqslant \limsup_{\varepsilon\to 0}\mathcal{J}(u^\varepsilon)$$

and so, using Lemma 3.2, we get:

$$\mathcal{J}_\alpha(v)\geqslant \mathcal{J}_\alpha(\bar u)$$

for all $v\in\mathcal{D}(D)$ such that $|v_-|^2\in\mathcal{D}(D)$.

Finally, by approximating separately the positive and the negative parts, one can show that every function $v\in H^1_0(D)$ is the limit of a sequence of functions $v_k\in\mathcal{D}(D)$ such that $(v_k)_-\in\mathcal{D}(D)$. Theorem 2.1 follows. □

Proof of Lemma 3.1. This is an immediate consequence of (5): Since $1-w^\varepsilon=0$ in T_ε we have

$$\lim_{\varepsilon\to 0}\int \varphi\nabla w^\varepsilon\cdot\nabla(1-w^\varepsilon)\,dx=\lim_{\varepsilon\to 0}\int \varphi_+\nabla w^\varepsilon\cdot\nabla(1-w^\varepsilon)\,dx-\lim_{\varepsilon\to 0}\int \varphi_-\nabla w^\varepsilon\cdot\nabla(1-w^\varepsilon)\,dx \\ =-\int \alpha_0\varphi_+\,dx+\int \alpha_0\varphi_-\,dx$$

and so

$$\lim_{\varepsilon\to 0}-\int \varphi\nabla w^\varepsilon\cdot\nabla w^\varepsilon\,dx=-\int \alpha_0\varphi\,dx. \qquad \square$$

Proof of Lemma 3.2. See Cioranescu and Murat [5], Proposition 3.1.

4. Proof of Proposition 2.2: Balls of random radius

Throughout this section, we assume that the sets $S_\varepsilon(k,\omega)$ are balls centered at εk. Since

$$\operatorname{cap}(B_r) = \begin{cases} n(n-2)\omega_n r^{n-2} & \text{if } n \geqslant 3, \\ -\dfrac{2\pi}{\log r} & \text{if } n = 2. \end{cases}$$

Assumption 2 becomes in this framework:

$$S_\varepsilon(k,\omega) = B_{a^\varepsilon(r(k,\omega))}(\varepsilon k) \quad \text{for all } k \in \mathbb{Z}^n$$

with

$$a^\varepsilon(r) = \begin{cases} r\varepsilon^{n/(n-2)} & \text{if } n \geqslant 3, \\ \exp(-r^{-1}\varepsilon^{-2}) & \text{if } n = 2, \end{cases}$$

and

$$r(k,\omega) = \begin{cases} \left(\dfrac{\gamma(k,\omega)}{n(n-2)\omega_n}\right)^{1/(n-2)} & \text{if } n \geqslant 3, \\ \gamma(k,\omega)/2\pi & \text{if } n = 2. \end{cases}$$

In particular that the process

$$r : \mathbb{Z}^n \times \Omega \mapsto [0,\infty)$$

is stationary ergodic and satisfies

$$r(k,\omega) \leqslant \bar r \quad \text{for all } k \in \mathbb{Z}^n \text{ and a.e. } \omega \in \Omega \tag{7}$$

for some constant $\bar r > 0$. Without loss of generality, we can always assume that $\bar r < 1/2$ (so that there is no overlapping of the holes for any $\varepsilon < 1$):

4.1. The auxiliary obstacle problem

After rescaling, we look for the corrector $w^\varepsilon(x,\omega)$ in the form

$$w^\varepsilon(x,\omega) = \varepsilon^2 v^\varepsilon(x/\varepsilon,\omega)$$

with $v^\varepsilon(y,\omega)$ solution to

$$\begin{cases} \Delta v = \alpha, & \text{in } \varepsilon^{-1}D_\varepsilon, \text{ a.e. } \omega \in \Omega, \\ v = \varepsilon^{-2} & \text{on } \bigcup_{k\in\mathbb{Z}^n} B_{\bar a^\varepsilon(k,\omega)}(k), \end{cases}$$

where

$$\bar a^\varepsilon(r) = \begin{cases} r\varepsilon^{2/(n-2)} & \text{if } n \geqslant 3, \\ \varepsilon^{-1}\exp(-r^{-1}\varepsilon^{-2}) & \text{if } n = 2, \end{cases}$$

and such that

$$\varepsilon^2 v^\varepsilon(x/\varepsilon) \to 0 \quad \text{in } H^1\text{-weak}.$$

One of the main tool in the proof is the fundamental solution of the Laplace equation, given by:

$$h(x) = \begin{cases} \dfrac{1}{n(n-2)\omega_n}\dfrac{1}{|x|^{n-2}} & \text{if } n \geqslant 3, \\ -\dfrac{1}{2\pi}\log|x| & \text{if } n = 2. \end{cases}$$

In particular, we note that

$$h|_{\partial B_{\bar{a}^\varepsilon(r(k,\omega))}(0)} = \begin{cases} \dfrac{1}{n(n-2)\omega_n r^{n-2}}\varepsilon^{-2} & \text{if } n \geqslant 3, \\ \dfrac{1}{2\pi}(\log(\varepsilon) + r^{-1}\varepsilon^{-2}) & \text{if } n = 2, \end{cases}$$

so we expect the rescaled corrector $v^\varepsilon(x,\omega)$ to behave, near the hole $B_{\bar{a}(k,\omega)}(k)$, like the function

$$h_k(x) := \begin{cases} \gamma(k,\omega)h(x-k) = \dfrac{r(k,\omega)^{n-2}}{|x-k|^{n-2}}, & \text{if } n \geqslant 3, \\ \gamma(k,\omega)h(x-k) = -r(k,\omega)\log|x-k| & \text{if } n = 2, \end{cases}$$

where

$$\gamma(k,\omega) = \begin{cases} \big(r(k,\omega)\big)^{n-2} n(n-2)\omega_n & \text{if } n \geqslant 3, \\ 2\pi r(k,\omega) & \text{if } n = 2. \end{cases}$$

Since h_k satisfies

$$\Delta h_k = -\gamma(k,\omega)\delta(x-k),$$

we will construct $v^\varepsilon(x,\omega)$ by solving

$$\begin{cases} \Delta v = \alpha - \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k,\omega)\delta(x-k) & \text{in } \varepsilon^{-1}D, \\ v = 0 & \text{on } \partial\varepsilon^{-1}D. \end{cases}$$

The main issue is thus to find the critical α for which the solution of the above equation has the appropriate behavior near $x = k$.

Following [3], this will be done by introducing the following obstacle problem, for every open set $A \subset \mathbb{R}^n$ and $\alpha \in \mathbb{R}$:

$$\bar{v}_{\alpha,A}(x,\omega) = \inf\left\{ v(x);\ \Delta v \leqslant \alpha - \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k,\omega)\delta(x-k),\ v \geqslant 0 \text{ in } A,\ v = 0 \text{ on } \partial A \right\}. \tag{8}$$

Clearly, the function $\bar{v}_{\alpha,A}$ is solution of

$$\Delta v = \alpha - \sum_{k\in\mathbb{Z}^n\cap A} \gamma(k,\omega)\delta(x-k) \tag{9}$$

whenever it is positive. Note that the function

$$h_{\alpha,k}(x) := \frac{\alpha}{2n}|x-k|^2 + h_k(x-k) \tag{10}$$

$$= \begin{cases} \dfrac{\alpha}{2n}|x-k|^2 + \dfrac{r(k,\omega)^{n-2}}{|x-k|^{n-2}}, & \text{if } n \geqslant 3, \\ \dfrac{\alpha}{2n}|x-k|^2 - r(k,\omega)\log|x-k| & \text{if } n = 2, \end{cases}$$

also satisfies

$$\Delta h_{\alpha,k}(x) = \alpha - \gamma(k,\omega)\delta(x-k).$$

It follows from (9) and the maximum principle that if $B_1(k) \subset A$, then, for all x in $B_1(k)$ and for almost every ω in Ω, we have

$$\bar{v}_{\alpha,A}(x,\omega) \geqslant \begin{cases} h_{\alpha,k}(x) - \dfrac{\alpha}{2n} - r^{n-2} & \text{if } n \geqslant 3, \\ h_{\alpha,k}(x) - \dfrac{\alpha}{2n} & \text{if } n = 2. \end{cases} \tag{11}$$

4.2. Critical α

The purpose of this section is to prove that for a critical α, $\bar{v}_{\alpha,A}$ behaves like $h_{\alpha,k}$ near $S_\varepsilon(k,\omega)$. For that purpose, we introduce the following quantity, which measures the size of the contact set:

$$\bar{m}_\alpha(A,\omega) = \left|\left\{x \in A;\, \bar{v}_{\alpha,A}(x,\omega) = 0\right\}\right|,$$

where $|A|$ denotes the Lebesgue measure of a set A.

The starting point of the proof is the following lemma:

Lemma 4.1. *The random variable $\bar{m}_\alpha$ is subadditive, and the process*

$$T_k m(A,\omega) = m(k + A,\omega)$$

has the same distribution for all $k \in \mathbb{Z}^n$.

Proof of Lemma 4.1. Assume that the finite family of sets $(A_i)_{i\in I}$ is such that

$$A_i \subset A \quad \text{for all } i \in I,$$
$$A_i \cap A_j = \emptyset \quad \text{for all } i \neq j,$$
$$\left|A - \bigcup_{i\in I} A_i\right| = 0$$

then $\bar{v}_{\alpha,A}$ is admissible for each A_i, and so $\bar{v}_{\alpha,A_i} \leqslant \bar{v}_{\alpha,A}$. It follows that

$$\{\bar{v}_{\alpha,A} = 0\} \cap A_i \subset \{\bar{v}_{\alpha,A_i} = 0\}$$

and so

$$\bar{m}_\alpha(A,\omega) = \sum_{i\in I}\left|\{\bar{v}_{\alpha,A} = 0\} \cap A_i\right| \leqslant \sum_{i\in I}\left|\{\bar{v}_{\alpha,A_i} = 0\}\right| = \sum_{i\in I}\bar{m}_\alpha(A_i,\omega),$$

which gives the subadditive property. Assumption 3 then yields

$$T_k m(A,\omega) = m(A,\tau_k\omega)$$

which gives the last assertion of the lemma. □

Since $\bar{m}_\alpha(A,\omega) \leqslant |A|$, and thanks to the ergodicity of the transformations τ_k, it follows from the subadditive ergodic theorem (see [8] and [1]) that for each α, there exists a constant $\bar{\ell}(\alpha)$ such that

$$\lim_{t\to\infty}\frac{\bar{m}_\alpha(B_t(0),\omega)}{|B_t(0)|} = \bar{\ell}(\alpha) \quad \text{a.s.},$$

where $B_t(0)$ denotes the ball centered at the origin with radius t. Note that the limit exists and is the same if instead of $B_t(0)$, we use cubes or balls centered at tx_0 for some x_0.

If we scale back and consider the function

$$\bar{w}_\alpha^\varepsilon(y,\omega) = \varepsilon^2\bar{v}_{\alpha,B_{\varepsilon^{-1}}(\varepsilon^{-1}x_0)}(y/\varepsilon,\omega) \quad \text{in } B_1(x_0),$$

we deduce

$$\lim_{\varepsilon\to 0}\frac{|\{y;\, \bar{w}_\alpha^\varepsilon(y,\omega) = 0\}|}{|B_1|} = \bar{\ell}(\alpha) \quad \text{a.s.}$$

The next lemma summarizes the properties of $\bar{\ell}(\alpha)$:

Lemma 4.2.

(i) *$\bar{\ell}(\alpha)$ is a nondecreasing functions of α.*

(ii) *If* $\alpha < 0$*, then* $\bar{\ell}(\alpha) = 0$*. Moreover, if the radii* $r(k,\omega)$ *are bounded from below, then* $\bar{\ell}(\alpha) = 0$ *for any* α *such that* $\alpha < n(n-2)\inf_{k\in\mathbb{Z}^n} r(k,\omega)^{n-2}$ *almost surely.*

(iii) *If* $\alpha \geqslant 2^n n(n-2)\sup_{k\in\mathbb{Z}^n} r(k,\omega)^{n-2}$ *(or* $\alpha \geqslant 8r$ *for* $n=2$*) almost surely, then* $\bar{\ell}(\alpha) > 0$.

Proof. (i) The proof follows immediately from the inequality

$$\bar{v}_{\alpha,A} \leqslant \bar{v}_{\alpha',A} \quad \text{for any } \alpha, \alpha' \text{ such that } \alpha' \leqslant \alpha.$$

(ii) If α is negative, then the function $\frac{\alpha}{2n}|x-x_0|^2 - \frac{\alpha}{2n}(tr)^2$, which is a sub-solution of (9), is positive in $tB_r(x_0)$ and vanishes along $\partial(tB_r(x_0))$ for any ball $B_r(x_0)$ and for any $t > 0$. We deduce:

$$\bar{v}_{\alpha,tB} > \frac{\alpha}{2n}|x-x_0|^2 - \frac{\alpha}{2n}(tr)^2 > 0 \quad \text{in } tB_r(x_0)$$

for all $t > 0$. Therefore $m_\alpha(tB,\omega) = 0$ for all $t > 0$, so $\bar{\ell}(\alpha) = 0$ for all $\alpha < 0$.

Furthermore, if $r(k,\omega)$ is bounded below:

$$r(k,\omega) \geqslant \underline{r} > 0 \quad \text{for all } k \in \mathbb{Z}^n, \text{ a.e. } \omega \in \Omega,$$

then, the function $\frac{\alpha}{2n}|x-k|^2 + \frac{\underline{r}^{n-2}}{|x-k|^{n-2}} - \frac{\alpha}{2n} - \underline{r}^{n-2}$ is a sub-solution of (9) in $B_1(k)$ which vanishes on $\partial B_1(k)$ and is strictly positive in $B_1(k)$ as long as $\alpha < n(n-2)\underline{r}^{n-2}$. As above, we deduce that $m_\alpha(tB,\omega) = 0$ for all $t > 0$ and for all $\alpha < n(n-2)\underline{r}^{n-2}$.

(iii) The function $h_{\alpha,k}(x) = \frac{\alpha}{2n}|x-k|^2 + \frac{r^{n-2}}{|x-k|^{n-2}}$ is radially symmetric and reaches its minimum when

$$|x-k| = R(\alpha,k) := \begin{cases} \left(\dfrac{n(n-2)r(k,\omega)^{n-2}}{\alpha}\right)^{1/n} & \text{when } n \geqslant 3, \\ \left(\dfrac{2r(k,\omega)}{\alpha}\right)^{1/2} & \text{when } n = 2. \end{cases} \tag{12}$$

In particular, for $\alpha > 2^n n(n-2)r(k,\omega)^{n-2}$ (or $n \geqslant 8r(k,\omega)$ when $n=2$), we have $R(\alpha,k) < 1/2$ and so the function

$$g_k(x) = \begin{cases} h_{\alpha,k}(x) - D_k & \text{in } B_{R(\alpha,k)}(k), \\ 0 & \text{in } \mathbb{R}^n \setminus B_{R(\alpha,k)}(k) \end{cases}$$

satisfies

$$\Delta g_k \leqslant \alpha - \gamma(k,\omega)\delta(x-k) \quad \text{in } C_1(k),$$

and

$$g_k = 0 \quad \text{in } C_1(k) \setminus B_{1/2}(k),$$

where $C_1(k)$ denotes the cube of size 1 centered at k, and the constant C_k is chosen in such a way that g_k and ∇g_k vanish along $\partial B_{R(\alpha,k)}$:

$$D(\alpha,k) := \begin{cases} \left(\dfrac{\alpha}{2n}\right)^{\frac{n-2}{n}} r^{\frac{2(n-2)}{n}} \left(\dfrac{n-2}{2}\right)^{\frac{2}{n}} \left(\dfrac{n}{n-2}\right) & \text{when } n \geqslant 3, \\ \dfrac{r}{2}\left(1 - \log\left(\dfrac{2r}{\alpha}\right)\right) & \text{when } n = 2. \end{cases} \tag{13}$$

By definition of $\bar{v}_{\alpha,tB}$, we deduce that

$$\bar{v}_{\alpha,tB}(x) \leqslant \sum_{k\in\mathbb{Z}^n\cap tB} g_k(x) \quad \text{in } tB \text{ a.s.}$$

In particular, this implies that $\bar{v}_{\alpha,tB}$ vanishes in $tB \setminus \bigcup_{k\in\mathbb{Z}^n} B_{1/2}(k)$, and so

$$\frac{\bar{m}_\alpha(tB,\omega)}{|tB|} \geqslant \frac{|C_1| - |B_{1/2}|}{|C_1|} = 1 - \frac{\omega_n}{2^n} \quad \text{a.s.}$$

We conclude

$$\bar{\ell}(\alpha) \geqslant 1 - \frac{\omega_n}{2^n} > 0. \qquad \square$$

Using Lemma 4.2, we can define

$$\alpha_0 = \sup\{\alpha; \bar{\ell}(\alpha) = 0\}.$$

Note that α_0 is finite under Assumption 3 (Lemma 4.2(iii)) and that $\alpha_0 \geqslant 0$ is strictly positive as soon as the $r(k,\omega)$ are bounded from below almost surely by a positive constant (Lemma 4.2(ii)).

Our goal, in the rest of this section, is to show that the function

$$w^\varepsilon(x,\omega) = \inf\left\{ w(x);\ \Delta w \leqslant \alpha_0 \text{ in } D \setminus T_\varepsilon,\ \begin{array}{l} w \geqslant 1 \text{ on } T_\varepsilon \cap D \\ w = 0 \text{ on } \partial D \setminus T_\varepsilon \end{array} \right\},$$

satisfies all the conditions of Proposition 2.2. For that purpose, we will introduce a series of intermediate functions.

4.3. Behavior of $\bar{v}_{\alpha,\varepsilon^{-1}A}$ near the holes

We fix a bounded subset A of $\mathbb{R}^n$ and we denote by

$$\bar{v}^\varepsilon_\alpha(x,\omega) = \bar{v}_{\alpha,\varepsilon^{-1}A}(x,\omega) \tag{14}$$

the solutions of (8) defined in $\varepsilon^{-1}A$. We also introduce the rescaled function

$$\bar{w}^\varepsilon_\alpha(y,\omega) = \varepsilon^2 \bar{v}^\varepsilon_\alpha(y/\varepsilon,\omega),$$

defined in A.

The key properties of $\bar{v}^\varepsilon_\alpha$ are given by the following lemma:

Lemma 4.3.

(i) *For every α and for every $k \in \mathbb{Z}^n$, we have*

$$\bar{v}^\varepsilon_\alpha(x) \geqslant \begin{cases} h_{\alpha,k}(x) - \dfrac{\alpha}{2n} - r^{n-2} & \text{if } n \geqslant 3, \\ h_{\alpha,k}(x) - \dfrac{\alpha}{2n} & \text{if } n = 2 \end{cases}$$

for all $x \in B_1(k)$ and almost everywhere $\omega \in \Omega$ (where $h_{\alpha,k}$ is defined by (10)*).*

(ii) *For every $\alpha > \alpha_0$, we have*

$$\bar{v}^\varepsilon_\alpha(x) \leqslant h_{\alpha,k}(x) + o(\varepsilon^{-2})$$

for all $x \in B_{1/2}(k)$ and almost everywhere $\omega \in \Omega$.

Since

$$h_{\alpha,k}|_{\partial B_{\bar{a}^\varepsilon(r(k,\omega))}(0)} = \begin{cases} \varepsilon^{-2} + \dfrac{\alpha_0}{2n}\left|\bar{a}^\varepsilon(r(k,\omega))\right|^2 & \text{if } n \geqslant 3, \\ \varepsilon^{-2} + \dfrac{\alpha_0}{4}\left|\bar{a}^\varepsilon(r(k,\omega))\right|^2 + r(k,\omega)\log\varepsilon & \text{if } n = 2, \end{cases}$$

we deduce the following corollary:

Corollary 4.4.

(i) *For every α and every $k \in \mathbb{Z}^n$ such that $r(k,\omega) > 0$, we have*

$$\bar{v}^\varepsilon_\alpha(x) \geqslant \varepsilon^{-2} + o(1) \quad \textit{on } \partial B_{\bar{a}^\varepsilon(r(k,\omega))}(k) \textit{ a.e. } \omega \in \Omega$$

and so

$$\bar{w}_\alpha^\varepsilon(x) \geqslant 1 + \mathrm{o}(\varepsilon^2) \quad \text{on } \partial T_\varepsilon(\omega) \text{ a.e. } \omega \in \Omega$$

for all α.

(ii) *For every* $\alpha > \alpha_0$ *and every* $k \in \mathbb{Z}^n$*, we have*

$$\bar{v}_\alpha^\varepsilon(x) \leqslant \varepsilon^{-2} + \mathrm{o}(\varepsilon^{-2}) \quad \text{on } \partial B_{\bar{a}^\varepsilon(r(k,\omega))}(k) \text{ a.e. } \omega \in \Omega$$

and so

$$\bar{w}_\alpha^\varepsilon(x) \leqslant 1 + \mathrm{o}(1) \quad \text{on } \partial T_\varepsilon(\omega) \text{ a.e. } \omega \in \Omega.$$

Proof of Lemma 4.3. (i) Immediate consequence of (11).

(ii) *Preliminary*: First of all since A is bounded, we have

$$A \subset B_R(x_0).$$

Without loss of generality, we can always assume that $B_R(x_0) = B_1(0)$. We then introduce

$$v_\alpha^\varepsilon(x,\omega) = \bar{v}_{\alpha,\varepsilon^{-1}B_1}(x,\omega),$$

the solutions of (8) in $B_{\varepsilon^{-1}}(0)$. It is readily seen that v_α^ε is admissible for (8) and thus

$$\bar{v}_\alpha^\varepsilon(x,\omega) \leqslant v_\alpha^\varepsilon(x,\omega) \quad \text{for all } x \in \varepsilon^{-1}A \text{ a.e. } \omega \in \Omega.$$

It is thus enough to prove (ii) for v_α^ε.

We will need the following consequence of Lemma 4.1 (see [3] for the proof):

Lemma 4.5. *For any ball* $B_r(x_0) \in B_1(0)$*, the following limit holds, a.s. in* ω

$$\lim_{\varepsilon\to 0} \frac{|\{v_\alpha^\varepsilon(x,\omega) = 0\} \cap B_{\varepsilon^{-1}r}(\varepsilon^{-1}x_1)|}{|B_{\varepsilon^{-1}r}|} = \bar{\ell}(\alpha).$$

Step 1: We can now start the proof: For any $\delta > 0$, we can cover $B_{\varepsilon^{-1}}$ by a finite number N ($\leqslant C\delta^{-n}$) of balls B_i with radius $\delta\varepsilon^{-1}$ and center $\varepsilon^{-1}x_i$. Since $\alpha > \alpha_0$, we have $\bar{\ell}(\alpha) > 0$. By Lemma 4.5, we deduce that for every i, there exists ε_i such that if $\varepsilon \leqslant \varepsilon_i$, then

$$\left|\left\{v_\alpha^\varepsilon(x,\omega) = 0\right\} \cap B_i\right| > 0 \quad \text{a.s. } \omega.$$

In particular, if $\varepsilon \leqslant \inf \varepsilon_i$, then $v_\alpha^\varepsilon(y_i) = 0$ for some y_i in B_i a.s. $\omega \in \Omega$. We now have to show that this implies that v_α^ε remains small in each B_i as long as we stay away from the lattice points $k \in \mathbb{Z}^n$. More precisely, we want to show that

$$\sup_{B_i \setminus \bigcup_{k\in\mathbb{Z}^n} B_{1/4}(k)} v_\alpha^\varepsilon \leqslant C\delta^2\varepsilon^{-2}.$$

Step 2: Let η be a nonnegative function such that $0 \leqslant \eta(x) \leqslant 1$ for all x, $\eta(x) = 1$ in $B_{1/8}$ and $\eta = 0$ in $\mathbb{R}^n \setminus B_{1/4}$. Then the function $u = v_\alpha^\varepsilon \star \eta$ is nonnegative on $2B_i$ and satisfies

$$-C \leqslant \Delta u \leqslant C,$$

where C is a universal constant depending only on n and $\bar{r}$. In particular, since B_i has radius $\delta\varepsilon^{-1}$, Harnack inequality yields:

$$\sup_{B_i} u \leqslant C \inf_{B_i} u + C\alpha(\delta\varepsilon^{-1})^2.$$

Step 3: We need the following lemma:

Lemma 4.6. *If* $\Delta v \leqslant \alpha$ *in* $B_r(y_0)$*, then*

$$\frac{1}{B_r} \int_{B_r(y_0)} v(x)\,dx \leqslant v(y_0) + \alpha C(n) r^2,$$

where $C(n)$ *is a universal constant.*

Proof. We note that the function $v(x) - \frac{\alpha}{2n}|x-y_0|^2$ is super-harmonic in $B_r(y_0)$. The lemma follows from the mean value formula. □

Now, we recall that $v_\alpha^\varepsilon(y_i) = 0$ and $\Delta v_\alpha^\varepsilon \leqslant \alpha$ in $B_{1/4}(y_i)$. So

$$\frac{1}{B_{1/4}} \int\limits_{B_{1/4}(y_i)} v_\alpha^\varepsilon(x)\,dx \leqslant v_\alpha^\varepsilon(y_i) + \alpha C(n).$$

In particular, we have

$$u(y_i) \leqslant \int\limits_{B_{1/4}(y_i)} v_\alpha^\varepsilon(x)\,dx \leqslant C(\alpha, n).$$

Step 4: Steps 2 and 3 yield

$$\sup_{B_i} u \leqslant C(\alpha, n)\big(1 + \alpha(\delta\varepsilon^{-1})^2\big)$$

and since $\Delta v_\alpha^\varepsilon \geqslant 0$ in $B_i \setminus \bigcap_{k\in\mathbb{Z}^n}\{k\}$, we have:

$$v_\alpha^\varepsilon(y) \leqslant \frac{1}{B_{1/8}} \int\limits_{B_{1/8}(y)} v_\alpha^\varepsilon(x)\,dx \leqslant Cu(y)$$

for all $y \in B_i \setminus \bigcap_{k\in\mathbb{Z}^n} B_{1/4}(k)$.

It follows that for every δ and for ε small enough, we have:

$$\sup_{B_{\varepsilon^{-1}} \setminus \bigcup_{k\in\mathbb{Z}^n} B_{1/4}(k)} v_\alpha^\varepsilon \leqslant C\delta^2\varepsilon^{-2}.$$

The definition of v_α^ε and the fact that $h_{\alpha,k} \geqslant 0$ on $\partial B_{1/2}$ implies that

$$v_\alpha^\varepsilon(x) \leqslant h_{\alpha,k}(x) + C\delta^2\varepsilon^{-2} \quad \text{in } B_{1/2}(k)$$

for all $k \in \mathbb{Z}^n$. □

4.4. Approximated corrector

We now want to use the function $\bar{v}_\alpha^\varepsilon$, solution of the obstacle problem (8), to study the properties of the free solution w_0^ε of

$$\begin{cases} \Delta w_0^\varepsilon = \alpha_0 - \sum_{k\in\mathbb{Z}^n\cap D} \gamma(k,\omega)\delta(x-\varepsilon k) & \text{in } D, \\ w_0^\varepsilon = 0 & \text{on } \partial D. \end{cases}$$

We define $h_{\alpha,k}^\varepsilon(x) = \varepsilon^2 h_{\alpha,k}(x/\varepsilon)$ for $n \geqslant 3$ and $h_{\alpha,k}^\varepsilon(x) = \varepsilon^2 h_{\alpha,k}(x/\varepsilon) + r\varepsilon^2 \log\varepsilon$ for $n = 2$. We have:

$$h_{\alpha,k}^\varepsilon(x) := \begin{cases} \dfrac{\alpha_0}{2n}|x-\varepsilon k|^2 + \dfrac{\varepsilon^n r(k,\omega)^{n-2}}{|x-\varepsilon k|^{n-2}} & \text{if } n \geqslant 3, \\ \dfrac{\alpha_0}{2n}|x-\varepsilon k|^2 - r(k,\omega)\varepsilon^2 \log|x-\varepsilon k| & \text{if } n = 2. \end{cases}$$

We then prove:

Lemma 4.7. *For every $k \in \mathbb{Z}^n$, w_0^ε satisfies*

$$h_{\alpha,k}^\varepsilon(x) - \mathrm{o}(1) \leqslant w_0^\varepsilon(x) \leqslant h_{\alpha,k}^\varepsilon(x) + \mathrm{o}(1) \quad \forall x \in B_{\varepsilon/2}(\varepsilon k) \cap D \text{ a.e. } \omega \in \Omega. \tag{15}$$

In particular:

$$w_0^\varepsilon(x) = 1 + \mathrm{o}(1) \quad \text{on } \partial T_\varepsilon \cap D. \tag{16}$$

Proof. For every α, we denote by $\bar{w}^\varepsilon_\alpha$ the function

$$\bar{w}^\varepsilon_\alpha(x) = \varepsilon^2 \bar{v}_{\alpha,\varepsilon^{-1}D}(x/\varepsilon),$$

defined in D and satisfying $\bar{w}^\varepsilon_\alpha = 0$ on ∂D.

1. For every $\alpha > \alpha_0$, we have

$$\Delta(w^\varepsilon_0 - w^\varepsilon_\alpha) \geqslant \alpha_0 - \alpha$$

and $w^\varepsilon_0 - w^\varepsilon_\alpha = 0$ on ∂D. This implies

$$w^\varepsilon_0(x_0) - w^\varepsilon_\alpha(x_0) \leqslant \int_D G(x_0, x)(\alpha_0 - \alpha)\,dx,$$

where $G(\cdot,\cdot)$ is the Green function on D ($\Delta G = \delta_{x_0}$ and $G = 0$ on ∂D). Note that we have

$$G(x_0, x) \geqslant -h(x - x_0) \quad \forall x, x_0 \in D,$$

and so

$$w^\varepsilon_0(x_0) - w^\varepsilon_\alpha(x_0) \leqslant (\alpha - \alpha_0) \int_D h(x - x_0)\,dx.$$

We deduce

$$\sup_D (w^\varepsilon_0 - w^\varepsilon_\alpha) \leqslant \begin{cases} C|D|^{1/(n-1)} \rho_D |\alpha - \alpha_0| & \text{if } n \geqslant 3, \\ C|D| \rho_D \log \rho_D |\alpha - \alpha_0| & \text{if } n = 2, \end{cases}$$

with

$$\rho_D = \inf\{\rho;\, D \subset B_\rho\}.$$

Hence we have

$$w^\varepsilon_0 \leqslant w^\varepsilon_\alpha + \mathrm{O}(\alpha - \alpha_0).$$

Using Lemma 4.3(ii) (since $\alpha > \alpha_0$), we deduce:

$$w^\varepsilon_0 \leqslant h^\varepsilon_{\alpha,k}(x) + \mathrm{O}(\alpha - \alpha_0) + \mathrm{o}(1) \quad \forall x \in B_{\varepsilon/2}(\varepsilon k) \text{ a.e. } \omega \in \Omega$$

which gives the second inequality in (15).

2. Similarly, we observe that for every $\alpha \leqslant \alpha_0$, we have

$$\Delta(w^\varepsilon_\alpha - w^\varepsilon_0) \geqslant \alpha - \alpha_0 - \alpha \mathbb{1}_{\{w^\varepsilon_\alpha = 0\}}.$$

Proceeding as before, we deduce that for $n \geqslant 3$,

$$\sup_D (w^\varepsilon_\alpha - w^\varepsilon_0) \leqslant C\rho_D\big[|D|^{1/(n-1)}(\alpha_0 - \alpha) + C\alpha\big|\{w^\varepsilon_\alpha = 0\}\big|^{1/(n-1)}\big]$$

and a similar inequality for $n = 2$. Using Lemma 4.3(i), we get

$$w^\varepsilon_0 \geqslant h^\varepsilon_{\alpha,k} - \mathrm{o}(\varepsilon^2) - \mathrm{O}(\alpha_0 - \alpha) - C\alpha\big|\{w^\varepsilon_\alpha = 0\}\big|^{1/(n-1)}.$$

Finally, since

$$\lim_{\varepsilon \to 0} \big|\{w^\varepsilon_\alpha = 0\}\big| = 0$$

for all $\alpha \leqslant \alpha_0$, and (15) follows. $\square$

4.5. Proof of Proposition 2.2

We are now in position to complete the proof of Proposition 2.2: We define

$$w^\varepsilon(x,\omega)=\inf\left\{w(x);\ \Delta w\leqslant\alpha_0 \text{ in } D\setminus T_\varepsilon,\ \begin{array}{l} w\geqslant 1 \text{ on } T_\varepsilon\cap D\\ w=0 \text{ on } \partial D\setminus T_\varepsilon\end{array}\right\},$$

it is readily seen that

$$\begin{cases} w^\varepsilon(x,\omega)=1 & \text{for } x\in\partial T_\varepsilon,\\ \Delta w^\varepsilon(x,\omega)=\alpha_0 & \text{for } x\in D\setminus T_\varepsilon,\\ w^\varepsilon(x,\omega)=0 & \text{for } x\in\partial D\setminus T_\varepsilon.\end{cases}$$

So in view of Lemma 2.3, we only have to show that $w^\varepsilon\to 0$ in $H^1(D)$-weak as ε goes to zero. More precisely, we will show that w^ε converges to zero in L^p strong and is bounded in H^1.

Strong convergence in L^p: First of all, (16) yields

$$w_0^\varepsilon(x)-\mathrm{o}(1)\leqslant w^\varepsilon(x,\omega)\leqslant w_0^\varepsilon(x)+\mathrm{o}(1)\quad \forall x\in D_\varepsilon \text{ a.e. } \omega\in\Omega,$$

which in turns imply (using Lemma 4.7 again):

$$h^\varepsilon_{\alpha,k}(x)-\mathrm{o}(1)\leqslant w^\varepsilon(x,\omega)\leqslant h^\varepsilon_{\alpha,k}(x)+\mathrm{o}(1)\quad \forall x\in B_{\varepsilon/2}(\varepsilon k) \text{ a.e. } \omega\in\Omega. \tag{17}$$

Next, a simple computation shows that

$$\int\limits_{B_\varepsilon\setminus B_{a^\varepsilon}} |h^\varepsilon_{\alpha,k}|^p\,dx\leqslant\begin{cases} C\varepsilon^n(\varepsilon^{\frac{2n}{n-2}}+\varepsilon^{2p}) & \text{if } n\geqslant 3,\\ C\varepsilon^2\varepsilon^{2p}(\log\varepsilon)^p & \text{if } n=2.\end{cases}$$

Since $\#\{\varepsilon\mathbb{Z}^n\cap D\}\leqslant C\varepsilon^n$ for all n, we deduce from (17) that

$$\|w^\varepsilon\|_{L^p}\leqslant\begin{cases} C(\varepsilon^{\frac{2n}{p(n-2)}}+\varepsilon^2) & \text{if } n\geqslant 3,\\ C\varepsilon^2(\log\varepsilon) & \text{if } n=2.\end{cases} \tag{18}$$

In particular

$$w^\varepsilon\to 0\quad \text{in } L^p\text{-strong, for all } p\in[1,\infty).$$

Bound in H^1: First of all, a simple integration by parts together with the fact that $w^\varepsilon=1$ on ∂T_ε yields

$$\int\limits_{D_\varepsilon}|\nabla w^\varepsilon|^2\,dx\leqslant\alpha_0|D|+\int\limits_{\partial T_\varepsilon}|\nabla w^\varepsilon|\,d\sigma(x),$$

where $\partial T_\varepsilon=\bigcup\partial S_\varepsilon(k,\omega)$. So we need an estimate in ∇w^ε along $\partial S_\varepsilon(k,\omega)=\partial B_{a^\varepsilon(r(k,\omega))}$.

We consider the function

$$z(x)=\begin{cases} w^\varepsilon(x)-h^\varepsilon_{\alpha,k}(x)+\frac{\alpha_0}{2n}r^2\varepsilon^{n/(n-2)} & \text{when } n\geqslant 3,\\ w^\varepsilon(x)-h^\varepsilon_{\alpha,k}(x)+\frac{\alpha_0}{2n}r^2e^{2\frac{\varepsilon^{-2}}{r}} & \text{when } n=2.\end{cases}$$

It satisfies

$$\begin{cases} \Delta z=0 & \text{in } B_{1/2}(\varepsilon k)\setminus B_{a^\varepsilon(r(k,\omega))}(\varepsilon k),\\ z(x)=\mathrm{o}(1) & \text{in } B_{1/2}(\varepsilon k)\setminus B_{a^\varepsilon(r(k,\omega))}(\varepsilon k),\\ z(x)=0 & \text{along } \partial B_{a^\varepsilon(r(k,\omega))}(\varepsilon k),\end{cases}$$

and so

$$\left|\nabla z(x)\right|\leqslant\begin{cases} \mathrm{o}(r^{n-2}\varepsilon^n\varepsilon^{-\frac{n(n-1)}{n-2}})=\mathrm{o}\big(\varepsilon^n a^\varepsilon(r)^{-(n-1)}\big) & \text{if } n\geqslant 3,\\ \mathrm{o}(\varepsilon^2e^{r^{-1}\varepsilon^{-2}})=\mathrm{o}\big(\varepsilon^n a^\varepsilon(r)^{-(n-1)}\big) & \text{if } n=2.\end{cases}$$

on $\partial B_{a^\varepsilon(r(k,\omega))}(\varepsilon k)$. It follows that

$$|\nabla w^\varepsilon| \leqslant \left|\nabla h^\varepsilon_{\alpha,k}(x)\right| + \left|\nabla z(x)\right| \leqslant C\varepsilon^n a^\varepsilon\big(r(k,\omega)\big)^{-(n-1)}$$

along $\partial B_{a^\varepsilon(r(k,\omega))}(\varepsilon k)$,

We deduce

$$\begin{aligned}\int_{D_\varepsilon} |\nabla w^\varepsilon|^2\,dx &\leqslant \alpha_0|D| + \int_{\partial T_\varepsilon} |\nabla w^\varepsilon|\,d\sigma(x)\\ &\leqslant \alpha_0|D| + \sum_{k\in\mathbb{Z}^n\cap\varepsilon^{-1}D}\ \int_{\partial B_{a^\varepsilon(r(k,\omega))}(\varepsilon k)} |\nabla w^\varepsilon|\,d\sigma(x)\\ &\leqslant \alpha_0|D| + C\varepsilon^{-n}a^\varepsilon(\bar r)^{n-1}\varepsilon^n a^\varepsilon(\bar r)^{-(n-1)}\\ &\leqslant C,\end{aligned}$$

and the proof is complete. □

5. Proof of Proposition 2.2: General case

In this section, we treat the case where the sets $S_\varepsilon(k,\omega)$ have unspecified shape, but satisfy Assumption 2:

$$\operatorname{cap}\big(S_\varepsilon(k,\omega)\big) = \varepsilon^n\gamma(k,\omega).$$

Throughout this section we assume $n \geqslant 3$.

The proof makes use of the result of the previous section, after noticing that away from εk, the hole $S_\varepsilon(k,\omega)$ is equivalent to a ball of radius $a^\varepsilon(r(k,\omega))$, where

$$a^\varepsilon(r) = r\varepsilon^{n/(n-2)}, \quad r(k,\omega) = \left(\frac{\gamma(k,\omega)}{n(n-2)\omega_n}\right)^{1/(n-2)}.$$

More precisely, we will rely on the following lemma:

Lemma 5.1. *For any $k\in\mathbb{Z}^n$ and $\omega\in\Omega$, let $\varphi^\varepsilon_k(x,\omega)$ be defined by*

$$\varphi^\varepsilon_k(x,\omega) = \inf\left\{v(x);\ \Delta v\leqslant 0,\ \begin{cases} v(x)\geqslant 1, & \forall x\in S_\varepsilon(k,\omega)\\ \lim_{|x|\to\infty} v(x) = 0 \end{cases}\right\}.$$

Then for any $\delta>0$, there exists R_δ such that

$$\left|\varphi^\varepsilon_k(x,\omega) - \varepsilon^n\gamma(k,\omega)h(x-\varepsilon k)\right| \leqslant \delta\varepsilon^n h(x-\varepsilon k)$$

for all x such that $|x-\varepsilon k|\geqslant a^\varepsilon(R_\delta)$ and for all $\varepsilon>0$.

Moreover, R_δ depends only on the constant M appearing in Assumption 1. *In particular, R_δ is independent of k and ω.*

1. For a given $\delta>0$, Lemma 5.1 implies that for every $k\in\mathbb{Z}^n$ and $\omega\in\Omega$ there exists a constant $R_\delta(k,\omega)$ such that

$$\left|\varphi^\varepsilon_k(x,\omega) - \frac{\varepsilon^n r(k,\omega)^{n-2}}{|x-\varepsilon k|^{n-2}}\right| \leqslant \delta\left(\frac{r}{R_\delta}\right)^{n-2} \quad \text{in } B_{2a^\varepsilon(R_\delta)}\setminus B_{a^\varepsilon(R_\delta)}(\varepsilon k) \tag{19}$$

for all $\varepsilon>0$. Moreover, it is readily seen that for any R there exists $\varepsilon_1(R)$ such that

$$a^\varepsilon(R)\leqslant\varepsilon^\sigma/4 \quad \text{for all } \varepsilon\leqslant\varepsilon_1 \tag{20}$$

for some $\sigma>1$. Finally, we note that by definition of φ^ε_k, we have

$$\int_{\mathbb{R}^n} |\nabla\varphi^\varepsilon_k|^2\,dx = \operatorname{cap}\big(S_\varepsilon(k)\big) = \varepsilon^n\gamma(k,\omega). \tag{21}$$

2. Next, let α_0 and w^ε be the coefficient and corresponding corrector constructed in the previous section, and associated with holes S_ε of radius $r(k,\omega)$. Lemma 4.7 implies that for δ and R given, there exists $\varepsilon_2(\delta,R) < \varepsilon_1(R)$ such that for all $\varepsilon \leqslant \varepsilon_2(\delta,R)$, we have
$$\left| w^\varepsilon(x) - \frac{\varepsilon^n r(k,\omega)^{n-2}}{|x-\varepsilon k|^{n-2}} \right| \leqslant \frac{\delta}{R^{n-2}} \quad \text{in } B_{\varepsilon/2}(\varepsilon k), \tag{22}$$
in dimension $n \geqslant 3$. Note that thanks to (20), Inequality (22) holds in particular in $B_{2a^\varepsilon(R)} \setminus B_{a^\varepsilon(R)}(\varepsilon k)$.

The corrector given by Proposition 2.2 will be constructed by gluing together the functions φ_k^ε (near the holes $S_\varepsilon(k)$) and the function w^ε (away from the holes). The gluing will have to be done in a very careful way so that the corrector satisfies all the properties listed in Proposition 2.2: For a given ε, we define δ_ε to be the smallest positive number such that (20) and (22) hold with $\delta = \delta_\varepsilon$ and $R = R_{\delta_\varepsilon}$. From the remarks above, we see that δ_ε is well defined as soon as ε is small enough (say smaller than $\varepsilon_2(1,R_1)$). Moreover, for any $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_2(\delta, R_\delta)$ such that
$$\delta_\varepsilon \leqslant \delta \quad \forall \varepsilon \leqslant \varepsilon_0.$$
In particular
$$\lim_{\varepsilon\to 0} \delta_\varepsilon = 0.$$
From now on, we write
$$R_\varepsilon = R_{\delta_\varepsilon}.$$

We are now ready to define the corrector $\bar{w}^\varepsilon$: Let $\eta_\varepsilon(x)$ be a function defined on D such that
$$\eta_\varepsilon(x) = 1 \quad \text{on } D \setminus \bigcup_{k\in\mathbb{Z}^n} B_{2a^\varepsilon(R_\varepsilon)}(\varepsilon k),$$
$$\eta_\varepsilon(x) = 0 \quad \text{on } \bigcup_{k\in\mathbb{Z}^n} B_{a^\varepsilon(R_\varepsilon)}(\varepsilon k)$$
and satisfying
$$|\nabla \eta_\varepsilon| \leqslant C a^\varepsilon(R_\varepsilon)^{-1} \quad \text{and} \quad |\Delta \eta_\varepsilon| \leqslant C a^\varepsilon(R_\varepsilon)^{-2}$$
in $B_{2a^\varepsilon(R_\varepsilon)} \setminus B_{a^\varepsilon(R_\varepsilon)}(k)$. We then define $\bar{w}^\varepsilon(x,\omega)$ in D by:
$$\bar{w}^\varepsilon(x,\omega) = \eta_\varepsilon(x) w^\varepsilon(x,\omega) + \big(1-\eta_\varepsilon(x)\big) \sum_{k\in\mathbb{Z}^n\cap D} \varphi_k^\varepsilon(x,\omega) 1_{B_{\varepsilon/2}(\varepsilon k)}(x).$$
It satisfies
$$\bar{w}^\varepsilon(x,\omega) = \begin{cases} \varphi_k^\varepsilon(x) & \text{in } B_{2a^\varepsilon(R_\varepsilon)}(k) \setminus S_\varepsilon(k)\ \forall k \in \mathbb{Z}^n, \\ w^\varepsilon(x) & \text{in } D \setminus \bigcup_{k\in\mathbb{Z}^n} B_{a^\varepsilon(R_\varepsilon)}. \end{cases}$$
To simplify the notations in the sequel, we denote
$$\varphi^\varepsilon(x) := \sum_{k\in\mathbb{Z}^n\cap D} \varphi_k^\varepsilon(x,\omega) 1_{B_{\varepsilon/2}(\varepsilon k)}(x).$$

The properties of $\bar{w}^\varepsilon$ are summarize in the following lemma, which implies Proposition 2.2 with (5) instead of the first equation:

Lemma 5.2. *The function $\bar{w}^\varepsilon$ satisfies*

(i) $\bar{w}^\varepsilon = 1$ *on S_ε for any $\varepsilon > 0$.*
(ii) $\bar{w}^\varepsilon$ *converges to zero as ε goes to zero in $L^p(D)$ strong for all $p \in [2,\infty)$ and*
$$\|\bar{w}^\varepsilon\|_{L^p} \leqslant C \varepsilon^{\frac{2n}{p(n-2)}} \quad \forall p \geqslant 2.$$

(iii) $\bar{w}^\varepsilon$ *is bounded in* $H^1(D)$.
(iv) $\bar{w}^\varepsilon$ *satisfies* (5).

Proof. (i) Immediate consequence of the definition of $\bar{w}^\varepsilon$ since $\varphi_k^\varepsilon = 1$ on $S_\varepsilon(k,\omega)$.

(ii) Assumption 1 yields

$$\varphi_k^\varepsilon(x,\omega) \leqslant C\varepsilon^n \gamma(k,\omega) h(x-\varepsilon k)$$

for all x such that $|x-\varepsilon k| \geqslant a^\varepsilon(M)$. Since $\varphi_k^\varepsilon \leqslant 1$ in $B_{a^\varepsilon(M)}(\varepsilon k)$, we deduce:

$$\begin{aligned}
\big\|(1-\eta_\varepsilon)\varphi^\varepsilon\big\|^p_{L^p(\mathbb{R}^n)} &\leqslant \sum_{k\in\mathbb{Z}^n\cap\varepsilon^{-1}D} \|\varphi_k^\varepsilon 1_{B_{\varepsilon/2}(\varepsilon k)}\|^p_{L^p(\bigcup B_{R(k)a(\varepsilon)})} \\
&\leqslant \sum_{k\in\mathbb{Z}^n\cap\varepsilon^{-1}D} \int_{B_{a^\varepsilon(M)(\varepsilon k)}} \big(\varphi_k^\varepsilon(x)\big)^p\,dx + C\sum_{k\in\mathbb{Z}^n\cap\varepsilon^{-1}D} \int_{B_{2a^\varepsilon(R_\varepsilon)(\varepsilon k)}} \big(\varepsilon^n\gamma(k)h(x-\varepsilon k)\big)^p\,dx \\
&\leqslant \sum_{k\in\mathbb{Z}^n\cap\varepsilon^{-1}D} a^\varepsilon(M)^n + C\bar{\gamma}\sum_{k\in\mathbb{Z}^n\cap\varepsilon^{-1}D} \varepsilon^{pn}\big(a^\varepsilon(R_\varepsilon)\big)^{n-p(n-2)}.
\end{aligned}$$

Using (20) and the definition of $a(\varepsilon)$, we deduce:

$$\begin{aligned}
\big\|(1-\eta_\varepsilon)\varphi^\varepsilon\big\|^p_{L^p(\mathbb{R}^n)} &\leqslant C\varepsilon^{-n}M^n\varepsilon^{\frac{n^2}{n-2}} + C\bar{\gamma}\sum_{k\in\mathbb{Z}^n\cap D}\varepsilon^{pn}\varepsilon^{n-p(n-2)} \\
&\leqslant CM^n\varepsilon^{\frac{2n}{n-2}} + C\bar{\gamma}\sum_{k\in\mathbb{Z}^n\cap D}\varepsilon^{n+2p} \\
&\leqslant CM^n\varepsilon^{\frac{2n}{n-2}} + C\bar{\gamma}\varepsilon^{2p},
\end{aligned}$$

where $2p \geqslant \frac{2n}{n-2}$ if $p\geqslant 2$ and $n\geqslant 3$.

Using (18), it follows that

$$\begin{aligned}
\|\bar{w}^\varepsilon\|_{L^p(D)} &\leqslant \|w^\varepsilon\|_{L^p(D)} + C\big(\varepsilon^{\frac{2n}{n-2}}\big)^{1/p} \\
&\leqslant C\varepsilon^{\frac{2n}{p(n-2)}}
\end{aligned}$$

for all $p\geqslant 2$.

(iii) Next, we want to show that $\bar{w}^\varepsilon$ is bounded in $H^1(D_\varepsilon)$. First, we note that in $B_{\varepsilon/2}(\varepsilon k)$, we have:

$$\nabla\bar{w}^\varepsilon = \nabla\eta_\varepsilon(w^\varepsilon - \varphi_k^\varepsilon) + \eta_\varepsilon\nabla\bar{w}_0^\varepsilon + (1-\eta_\varepsilon)\nabla\varphi_k^\varepsilon, \tag{23}$$

where the function $\nabla\eta_\varepsilon$ is supported in $B_{2a^\varepsilon(R_\varepsilon)}(\varepsilon k)\setminus B_{a^\varepsilon(R_\varepsilon)}(\varepsilon k)$ and satisfies

$$|\nabla\eta_\varepsilon| \leqslant C\big(a^\varepsilon(R)\big)^{-1}.$$

Since $|w^\varepsilon - \varphi_k^\varepsilon| \leqslant C\frac{\delta_\varepsilon}{R_\varepsilon^{n-2}}$ in $B_{2a^\varepsilon(R_\varepsilon)}(\varepsilon k)\setminus B_{a^\varepsilon(R_\varepsilon)}(\varepsilon k)$, we deduce

$$\begin{aligned}
\int_D \big|\nabla\eta_\varepsilon(w^\varepsilon - \varphi^\varepsilon)\big|^2\,dx &\leqslant \sum_{k\in\varepsilon\mathbb{Z}^n\cap D}\int_{B_{2a^\varepsilon(R_\varepsilon)}(\varepsilon k)} \big|\nabla\eta_\varepsilon(w^\varepsilon-\varphi_k^\varepsilon)\big|^2\,dx \\
&\leqslant \sum_{k\in\varepsilon\mathbb{Z}^n\cap D}\big(a^\varepsilon(R_\varepsilon)\big)^n\big(a^\varepsilon(R_\varepsilon)\big)^{-2}\frac{\delta_\varepsilon^2}{R_\varepsilon^{2(n-2)}} \\
&\leqslant \sum_{k\in\varepsilon\mathbb{Z}^n\cap D} R_\varepsilon^{-(n-2)}\varepsilon^n\delta_\varepsilon^2 \\
&\leqslant C\varepsilon^{-n}\varepsilon^n\delta_\varepsilon = C\delta_\varepsilon,
\end{aligned}$$

where we used the fact if ε is small enough, then $\delta_\varepsilon < 1$ and $R_\varepsilon \geqslant 1$. Finally, since w^ε and φ^ε are both bounded in H^1 (thanks to (21)), (23) implies

$$||\nabla\bar{w}^\varepsilon||_{L^2} \leqslant C.$$

(iv) Using Lemma 2.3, we only have to prove that

$$\lim_{\varepsilon\to 0}\int_{D_\varepsilon} |\Delta \bar w^\varepsilon - \alpha_0|\,dx = 0.$$

We have:

$$\Delta \bar w^\varepsilon = \alpha - (1-\eta_\varepsilon)\alpha + 2\nabla\eta_\varepsilon\cdot\nabla(w^\varepsilon - \varphi^\varepsilon) + \Delta\eta_\varepsilon(w^\varepsilon-\varphi^\varepsilon) \quad \text{in } D_\varepsilon.$$

Moreover, (22) and (19) yield

$$|w^\varepsilon - \varphi_k^\varepsilon| \leqslant \frac{\delta_\varepsilon}{R_\varepsilon^{n-2}} \quad \text{in } B_{2a^\varepsilon(R_\varepsilon)}\setminus B_{a^\varepsilon(R_\varepsilon)},$$

and by definition of w^ε and φ_k^ε, we have

$$\Delta\left(w^\varepsilon - \varphi_k^\varepsilon - \frac{\alpha_0}{2n}|x-\varepsilon k|^2\right) = 0 \quad \text{in } B_{4a^\varepsilon(R)}\setminus B_{a^\varepsilon(R_\varepsilon)/2}.$$

Interior gradient estimates thus imply

$$\left|\nabla(w^\varepsilon - \varphi_k^\varepsilon)\right| \leqslant \frac{\delta_\varepsilon}{R_\varepsilon^{n-2}} a^\varepsilon(R_\varepsilon)^{-1} + C a^\varepsilon(R_\varepsilon)$$

in $B_{2a^\varepsilon(R_\varepsilon)}\setminus B_{a^\varepsilon(R_\varepsilon)}$. We deduce (using (20)):

$$\begin{aligned}
\int_{D_\varepsilon} |\Delta\bar w^\varepsilon - \alpha|\,dx &\leqslant \int_{D_\varepsilon}(1-\eta_\varepsilon)\alpha\,dx + \int_{D_\varepsilon}|\nabla\eta_\varepsilon|\left|\nabla(w^\varepsilon-\varphi^\varepsilon)\right|dx + \int_{D_\varepsilon}|\Delta\eta_\varepsilon||w^\varepsilon - \varphi^\varepsilon|\,dx \\
&\leqslant \sum_{k\in\varepsilon\mathbb{Z}^n\cap\varepsilon^{-1}D} a^\varepsilon(R_\varepsilon)^n + \sum_{k\in\varepsilon\mathbb{Z}^n\cap\varepsilon^{-1}D} a^\varepsilon(R_\varepsilon)^{-1}\int_{B_{2a^\varepsilon(R_\varepsilon)}\setminus B_{a^\varepsilon(R)}} \left|\nabla(w^\varepsilon - \varphi_k^\varepsilon)\right|dx \\
&\quad + \sum_{k\in\varepsilon\mathbb{Z}^n\cap\varepsilon^{-1}D} a^\varepsilon(R_\varepsilon)^{-2}\int_{B_{2a^\varepsilon(R_\varepsilon)}\setminus B_{a^\varepsilon(R)}} |w_0^\varepsilon - \varphi_k^\varepsilon|\,dx \\
&\leqslant C\sum_{k\in\varepsilon\mathbb{Z}^n\cap\varepsilon^{-1}D} a^\varepsilon(R_\varepsilon)^n + C\sum_{k\in\varepsilon\mathbb{Z}^n\cap\varepsilon^{-1}D} \frac{\delta_\varepsilon}{R^{n-2}} a^\varepsilon(R_\varepsilon)^{-2}\left(a^\varepsilon(R)\right)^n \\
&\leqslant C\varepsilon^{-n}a^\varepsilon(R_\varepsilon)^n + C\delta_\varepsilon \sum_{k\in\varepsilon\mathbb{Z}^n\cap\varepsilon^{-1}D}\left(\frac{a^\varepsilon(R_\varepsilon)}{R_\varepsilon}\right)^{n-2} \\
&\leqslant C\varepsilon^{(\sigma-1)n} + C\delta_\varepsilon.
\end{aligned}$$

It follows that

$$\lim_{\varepsilon\to 0}\int_{D_\varepsilon}|\Delta\bar w^\varepsilon - \alpha_0|\,dx \leqslant C\delta$$

for any $\delta > 0$, which gives the result. □

Appendix A. Proof of Lemma 5.1

We recall that $n \geqslant 3$ in this section. For any $k\in\mathbb{Z}^n$, we define $\bar S_\varepsilon(k) = \varepsilon^{-\frac{n}{n-2}} S_\varepsilon(k)$. Then Assumption 2 yields:

$$\operatorname{cap}\left(\bar S_\varepsilon(k)\right) = \gamma(k) \leqslant \bar\gamma.$$

and Assumption 1 gives

$$\bar S_\varepsilon(k) \subset B_M(k). \tag{24}$$

For the sake of simplicity, we take $k=0$. We recall that h is defined by

$$h(x)=\frac{1}{n(n-2)\omega_n}\frac{1}{|x|^{n-2}}.$$

Lemma 5.1 will be a consequence of the following lemma:

Lemma 5.3. *Let φ be defined by*

$$\varphi(x)=\inf\left\{v(x);\ \Delta v\leqslant 0,\ \begin{cases} v(x)\geqslant 1, & \forall x\in \bar S_\varepsilon(k,\omega)\\ \lim_{|x|\to\infty} v(x)=0\end{cases}\right\}.$$

Then for any $\delta>0$, there exists R, depending only on δ and M such that

$$\left|\varphi(x,\omega)-\gamma h(x)\right|\leqslant \delta h(x)$$

for all x such that $|x|\geqslant R$.

Proof. We recall that φ solves

$$\begin{cases} \Delta\varphi(x)=0 & \text{for all } x\in\mathbb{R}^n\setminus S,\\ \varphi(x)=1 & \text{for all } x\in S,\\ \lim_{|x|\to\infty}\varphi(x)=0. \end{cases}$$

In particular, (24) and the maximum principle imply

$$\varphi(x)\leqslant M^{n-2}n(n-2)\omega_n h(x)=\frac{M^{n-2}}{|x|^{n-2}}\quad \text{in } \mathbb{R}^n\setminus B_M(0). \tag{25}$$

Next, we observe that

$$0=-\int_{\mathbb{R}^n\setminus S}\varphi\Delta\varphi\,dx=\int_{\mathbb{R}^n\setminus S}|\nabla\varphi|^2\,dx-\int_{\partial S}\varphi\varphi_\nu\,d\sigma(x)$$

and so

$$\int_{\mathbb{R}^n\setminus S}|\nabla\varphi|^2\,dx=\int_{\partial S}\varphi\varphi_\nu\,d\sigma(x)=\int_{\partial S}\varphi_\nu\,d\sigma(x).$$

Moreover, for any $R\geqslant M$, we have

$$0=\int_{B_R\setminus S}\Delta\varphi\,dx=\int_{\partial S}\varphi_\nu\,d\sigma(x)+\int_{\partial B_R}\varphi_\nu\,d\sigma(x).$$

We deduce:

$$\gamma=\int_{\mathbb{R}^n\setminus S}|\nabla\varphi|^2\,dx=-\int_{\partial B_R}\varphi_\nu\,d\sigma(x)\quad \text{for all } R\geqslant M. \tag{26}$$

We now introduce the function

$$\Theta(x)=h\left(\frac{x}{|x|^2}\right)^{-1}\varphi\left(\frac{x}{|x|^2}\right)=n(n-2)\omega_n\frac{1}{|x|^{n-2}}\varphi\left(\frac{x}{|x|^2}\right)$$

defined for $x\in B_{1/M}(0)$. A straightforward computation yields

$$\Delta\Theta=0\quad \text{in } B_{1/M}(0)$$

and (25) implies

$$\Theta(x)\leqslant M^{n-2}n(n-2)\omega_n\quad \text{in } B_{1/M}(0).$$

A more delicate computation, making use of the mean formula for harmonic functions, gives

$$\int_{\partial B_R} \varphi_\nu \, d\sigma(x) = -\Theta(0).$$

Hence (26) yields

$$\Theta(0) = \operatorname{cap}(\bar{S}_\varepsilon) = \gamma.$$

To conclude, we note that interior gradient estimates for harmonic functions imply the existence of a universal C (depending only on M) such that

$$\left|\Theta(x) - \gamma\right| \leqslant C|x| \quad \text{for all } |x| \leqslant 1/(2M).$$

Inverting back, we deduce

$$\left|\varphi(x) - \gamma h(x)\right| \leqslant \frac{C}{|x|} h(x) \quad \text{for all } |x| \geqslant 2M,$$

which yields the result. □

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