Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 26 (2009) 139–158

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Morse theory for indefinite nonlinear elliptic problems ☆

Kung-Ching Chang, Mei-Yue Jiang *

LMAM, School of Mathematical Sciences, Peking University, Beijing, 100871, PR China

Received 22 February 2007; accepted 22 August 2007

Available online 3 December 2007

Abstract

Using the heat flow as a deformation, a Morse theory for the solutions of the nonlinear elliptic equation:

$$-\Delta u - \lambda u = a_+(x)|u|^{q-1}u - a_-(x)|u|^{p-1}u + h(x,u)$$

in a bounded domain $\Omega \subset \mathbf{R}^N$ with the Dirichlet boundary condition is established, where $a_\pm \geqslant 0$, $\operatorname{supp}(a_-) \cap \operatorname{supp}(a_+) = \emptyset$, $\operatorname{supp}(a_+) \neq \emptyset$, $1 < q < 2^* - 1$ and $p > 1$. Various existence and multiplicity results of solutions are presented.

1. Introduction

We study the nonlinear elliptic equations with indefinite nonlinearities. Arising from differential geometry and biology, the problem has been received much attention in recent years, see [1–5,8,9,16–18,21,22,24,26].

One of the modelling problems can be stated as follows: Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary, we study the existence and multiplicity of positive, negative and sign-changing solutions of the following elliptic boundary value problem:

$$\begin{aligned} &-\Delta u = \lambda u + a_+(x)|u|^{q-1}u - a_-(x)|u|^{p-1}u + h(x,u) \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $a_\pm : \overline{\Omega} \to \mathbf{R}$ are continuous functions and $h : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ is a C^1 function, and λ is a real parameter. We assume

(A1) $a_\pm \geqslant 0$, $\overline{\Omega}_+ \cap \overline{\Omega}_- = \emptyset$ and $\Omega_+ \neq \emptyset$, where $\overline{\Omega}_\pm = \operatorname{supp}(a_\pm)$,
(A2) $1 < q < 2^* - 1 = \frac{N+2}{N-2}$, $p > 1$,
(A3) there exists a constant $C > 0$ such that

$$\bigl|h(x,\xi)\bigr| \leqslant C\bigl(1 + |\xi|\bigr), \quad \xi \in \mathbf{R}.$$

The case $1 < p \leqslant q < 2^* - 1$ has been studied by many of the previous papers, while the case $1 < q < p$ and $q < 2^* - 1$ by [5], but only for positive solutions.

☆ This work was supported by NSFC, RFDP of the Ministry of Education of China.
* Corresponding author.
E-mail address: mjiang@math.pku.edu.cn (M.-Y. Jiang).

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2007.08.004

The paper is a continuation of our previous paper [16]. The simple decomposition lemma in [16], on which the computation of the critical groups at infinity of the associated functional I (see below) relies, plays an important role in dealing with this kind of indefinite nonlinearities.

In [14], the first author observed that there are several advantages if we use the heat flow for Eq. (1.1) to establish the Morse theory for isolated critical points, i.e., instead of the gradient flow:

$$\partial_t v(t) = v(t) - (-\Delta)^{-1}\big(\lambda v + a_+(x)|v|^{q-1}v - a_-(x)|v|^{p-1}v + h(x,v)\big)$$

for the associated functional of (1.1):

$$I(u) = \int_\Omega \left[\frac{1}{2}\big(|\nabla u|^2 - \lambda u^2\big) - \frac{a_+(x)}{q+1}|u|^{q+1} + \frac{a_-(x)}{p+1}|u|^{p+1} - H(x,u)\right] dx,$$

where $H(x,u) = \int_0^u h(x,s)\,ds$, in $\mathbf{R} \times H_0^1(\Omega)$, the following heat semi-flow:

$$\partial_t v(t,x) = (\Delta + \lambda)v + a_+(x)|v|^{q-1}v - a_-(x)|v|^{p-1}v + h(x,v), \quad (t,x) \in \mathbf{R}^+ \times \overline{\Omega}$$

in $\mathbf{R}^+ \times C_0^1(\overline{\Omega})$ is considered. The heat semi-flow is the L^2 gradient flow of I and can be used as deformation of the level sets of I as the gradient flow.

The disadvantage of this method, to our knowledge, is that a more restrictive exponent on the nonlinear term $|u|^{q-1}u$ is needed, i.e., $q < p_1^* < 2^* - 1$, where p_1^* is defined in Section 3. But one of the advantages of the heat semi-flow is the positive invariance of the cones $\pm\tilde{P}$, where $\tilde{P} = \{u \in C_0^1(\overline{\Omega}) \mid u(x) \geqslant 0,\ x \in \overline{\Omega}\}$. This was observed in [1] and plays an important role in the existence of sign-changing solutions of (1.1).

Recall the case of definite nonlinearity, i.e., $a_- = 0$, $\Omega_+ = \Omega$, if there is a positive invariant set $\mathcal{D}$ for the gradient flow, then one can estimate the number of solutions inside and outside of $\mathcal{D}$, separately. Few abstract critical point theorems can be applied in the study of multiple solutions, see [13]. In this paper, we shall extend these results to fit the indefinite nonlinearity by the above two ingredients. Our main purpose is to develop the above tools in dealing with the multiple solution problems for indefinite nonlinearities. The main results are stated in Section 5, among other things, the following theorem will be proved:

Theorem. *Under the assumption* (A1), (A2′), (A3′), (A4) *and* (A6)*, to Eq.* $(0.1)_{\lambda,\gamma}$ *there exists* $\gamma^* > 0$ *such that* $\forall \gamma > \gamma^*$, $\exists -\infty < \lambda_-(\gamma) < \lambda_1 < \lambda_+(\gamma)$ *such that*:

(1) *For* $0 < \lambda < \lambda_-(\gamma)$*, there exist at least one positive, one negative and one sign-changing solutions.*
(2) *For* $\lambda_-(\gamma) < \lambda < \lambda_1$*, there exist at least three positive, three negative and one sign-changing solutions.*
(3) *For* $\lambda_1 < \lambda < \lambda_+(\gamma)$*, there exist at least two positive, two negative and one sign-changing solutions.*
(4) *For* $\lambda_2 < \lambda < \lambda_+(\gamma)$*, there exist at least two positive, two negative and three sign-changing solutions.*

Results in [1,4,5,16,22] are extended.

2. A decomposition lemma and the critical groups at infinity

We extend the decomposition lemma in [16] to the problem (1.1). The main difference is that the exponent p may be greater than $2^* - 1$. For Eq. (1.1), the associated functional is defined on the Banach space $E = H_0^1(\Omega) \cap L_{a_-}^{p+1}(\Omega)$, where

$$L_{a_-}^{p+1}(\Omega) = \left\{u \in \mathcal{D}'(\Omega) \,\Big|\, \int_\Omega a_-(x)\big|u(x)\big|^{p+1}\,dx < \infty\right\}.$$

Thus $E \neq H_0^1(\Omega)$ if $\Omega_- \neq \emptyset$ and $p > 2^* - 1$, and $E = H_0^1(\Omega)$ if $p \leqslant 2^* - 1$ by the embedding theorem. The norm on E is defined by $\|u\| = (\|\nabla u\|_2^2 + \|u\|_{L_{a_-}^{p+1}}^2)^2$.

The space is the closed subspace

$$E = \big\{(u_1, u_2) \in H_0^1(\Omega) \times L_{a_-}^{p+1}(\Omega) \mid u_1 = u_2\big\}$$

of $H_0^1(\Omega) \times L_{a_-}^{p+1}(\Omega)$, and both $H_0^1(\Omega)$ and $L_{a_-}^{p+1}(\Omega)$ are reflexive, according to Pettis Theorem, we have

Lemma 2.1. *The Banach space E is reflexive.*

The following decomposition lemma was proved in [16] if $p \leqslant 2^* - 1$. Let

$$E_1 = H_0^1(\overline{\Omega_0 \cup \Omega_-}) \cap L_{a_-}^{p+1}(\overline{\Omega_0 \cup \Omega_-})$$

and

$$E_2 = \{u \in H_0^1(\overline{\Omega_0 \cup \Omega_+}) \mid \Delta u(x) = 0\ \forall x \in \Omega_0\}.$$

Theorem 2.2.

$$E = E_1 \oplus E_2.$$

Proof. (1) $\forall u \in E$, let

$$v(x) = \begin{cases} u(x), & x \in \Omega_-, \\ u(x) - w_0(x), & x \in \Omega_0 \end{cases}$$

and

$$w(x) = \begin{cases} w_0(x), & x \in \Omega_0, \\ u(x), & x \in \Omega_+, \end{cases}$$

where $w_0 \in H^1(\Omega_0)$ is given by

$$\begin{cases} \Delta w_0 = 0, & x \in \Omega_0, \\ w_0(x) = 0, & x \in \partial\Omega_0 \cap \partial\Omega_-, \\ w_0(x) = u(x), & x \in \partial\Omega_0 \cap \partial\Omega_+. \end{cases}$$

Then $v \in E_1$ and $w \in E_2$, and $u = v + w$.

(2) The decomposition is unique, i.e., if $u = v + w = 0$ for $v \in E_1$ and $w \in E_2$, then $v = w = 0$.

Indeed we have $v(x) = u(x) = 0\ \forall x \in \Omega_-$ and $w(x) = u(x) = 0\ \forall x \in \Omega_+$, then $w_0|_{\partial\Omega_0} = 0$, hence $w_0 = 0$ by the maximum principle, and $v = w = 0$.

(3) Define the mapping $\pi : u \to (v, w)$ from E to $E_1 \oplus E_2$. It is linear and bounded. Moreover, it is also surjective. Indeed, for $(v, w) \in E_1 \oplus E_2$, let

$$u(x) = \begin{cases} v(x), & x \in \Omega_-, \\ v(x) + w(x), & x \in \Omega_0, \\ w(x), & x \in \Omega_+, \end{cases}$$

then $u \in E$ and $\pi(u) = (v, w)$. Therefore E is isomorphic to $E_1 \oplus E_2$ by Banach Theorem. □

Again, the spaces E_1 and E_2 are decomposable. Indeed, let

$$E_3 = \left\{v_3 \in E_1 \,\Big|\, \int_\Omega \nabla v_3 \cdot \nabla\phi\, dx = 0,\ \forall \phi \in H_0^1(\Omega_0) \cup H_0^1(\Omega_-)\right\}$$

and

$$E_4 = \left\{v_4 \in E_2 \,\Big|\, \int_\Omega \nabla v_4 \cdot \nabla\phi\, dx = 0,\ \forall \phi \in H_0^1(\Omega_+)\right\}.$$

It is easy to verify that

$$E_1 = \big(H_0^1(\Omega_-) \cap L_{a_-}^{p+1}(\Omega_-)\big) \oplus H_0^1(\Omega_0) \oplus E_3$$

and

$$E_2 = H_0^1(\Omega_+) \oplus E_4.$$

These decompositions were used in [16] in the computation of the critical groups of I at infinity. In order to compute these groups, we follow the method in Section 4 of [16], by introducing a family of functionals I_s, $s \in [0,1]$, as follows:

$$I_s(u) = \frac{1}{2}\int_\Omega \left[|\nabla v|^2 + 2s\nabla v\cdot\nabla w + |\nabla w|^2 - \lambda\left(v^2 + 2sv\cdot w + w^2\right)\right]dx$$
$$\quad - \int_\Omega \left[\frac{a_+(x)}{q+1}|w|^{q+1} - \frac{a_-(x)}{p+1}|v|^{p+1} + sH(x, v+w)\right]dx$$

for $(v,w) \in E_1 \times E_2$.

We note that $I_1(v,w) = I(v+w) = I(u)$ and that $I_0(v,w) = J_-(v) + J_+(w)$ is of separable variables, where

$$J_-(v) = \int_\Omega \left[\frac{1}{2}\left(|\nabla v|^2 - \lambda v^2\right) + \frac{a_-(x)}{p+1}|v|^{p+1}\right]dx$$

and

$$J_+(w) = \int_\Omega \left[\frac{1}{2}\left(|\nabla w|^2 - \lambda w^2\right) - \frac{a_+(x)}{q+1}|w|^{q+1}\right]dx.$$

We shall compute the critical groups of I at infinity via those of I_0. One can easily figure out the critical groups for $J_\pm$ and so does for I_0. Thus it remains to show that the critical groups for I_s are invariant along $s \in [0,1]$.

Definition 2.3. Let C be a constant. A sequence $\{u_k\} = \{(v_k, w_k)\} \subset E_1 \oplus E_2$ is said a weak Palais–Smale sequence for I_s, if

$$I_s(v_k, w_k) \leqslant C \quad \text{and} \quad \left\|I_s'(v_k,w_k)\right\|_{E^*} = \mathrm{o}\left(\|u_k\|_E\right) = \mathrm{o}\left(\|v_k\|_{E_1} + \|w_k\|_{E_2}\right).$$

Obviously, a Palais–Smale sequence is a weak Palais–Smale sequence.

Lemma 2.4. *Assume* (A1), (A2),

(A3′) $h(x,\xi) = \mathrm{o}(|\xi|)$ *as* $|\xi| \to +\infty$ *uniformly in* x,
(A4) $\lambda \notin \sigma(\Omega_0)$.

Then any weak Palais–Smale sequence for I_s is bounded in E.

Proof. From the definition

$$I_s(v_k, w_k) \leqslant C \quad \text{and} \quad \left\|I_s'(v_k,w_k)\right\|_{E^*} = \mathrm{o}\left(\|v_k\|_{E_1} + \|w_k\|_{E_2}\right),$$

we have

$$I_s(v_k,w_k) = \frac{1}{2}\int_\Omega \left(|\nabla v_k|^2 + 2s\nabla v_k\cdot\nabla w_k + |\nabla w_k|^2 - \lambda\left(v_k^2 + 2sv_k\cdot w_k + w_k^2\right)\right)dx$$
$$\quad - \int_\Omega \left(\frac{a_+(x)}{q+1}|w_k|^{q+1} - \frac{a_-(x)}{p+1}|v_k|^{p+1} + sH(x, v_k+w_k)\right)dx \leqslant C$$

and

$$\begin{aligned}\left\langle I_s'(v_k,w_k),(v_k,w_k)\right\rangle &= \int_\Omega \left(|\nabla v_k|^2 + 2s\nabla v_k\cdot\nabla w_k + |\nabla w_k|^2 - \lambda\left(v_k^2 + 2sv_k\cdot w_k + w_k^2\right)\right)dx \\ &\quad - \int_\Omega \left(a_+(x)|w_k|^{q+1} - a_-(x)|v_k|^{p+1} + sh(x, v_k+w_k)(v_k+w_k)\right)dx \\ &= \mathrm{o}\left(\|v_k\|_{E_1}^2 + \|w_k\|_{E_2}^2\right).\end{aligned}$$

These imply

$$\left(\frac{1}{2}-\frac{1}{q+1}\right)\int_\Omega a_+(x)|w_k|^{q+1}\,dx-\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega a_-(x)|v_k|^{p+1}\,dx\leqslant C+\mathrm{o}\big(\|v_k\|_{E_1}^2+\|w_k\|_{E_2}^2\big). \tag{2.1}$$

Now let $\eta\in C^\infty(\Omega)$ satisfy

$$\eta(x)=\begin{cases}1, & x\in\Omega_-,\\ 0, & x\in\Omega_+,\end{cases}$$

again, we have

$$\begin{aligned}\big\langle I_s(v_k,w_k),(\eta v_k,\eta w_k)\big\rangle&=\int_\Omega\big[\nabla v_k\cdot\nabla(\eta v_k)+s\nabla(\eta v_k)\cdot\nabla w_k+s\nabla(\eta w_k)\cdot\nabla v_k+\nabla w_k\cdot\nabla(\eta w_k)\big]\,dx\\&\quad+\int_\Omega a_-(x)|v_k|^{p+1}\,dx+\mathrm{O}\big(\|u_k\|_2^2\big)\\&=\mathrm{o}\big(\|u_k\|_E\big)\|\eta u_k\|_E=\mathrm{o}\big(\|u_k\|_E^2\big).\end{aligned} \tag{2.2}$$

The first integral on the right-hand side of (2.2) equals

$$\begin{aligned}&\int_\Omega\left[\frac{1}{2}\nabla v_k^2\cdot\nabla\eta+\eta|\nabla v_k|^2+s\eta\nabla v_k\cdot\nabla w_k\right]dx\\&\qquad+\int_\Omega\left[s\eta\nabla v_k\cdot\nabla w_k+s\nabla\eta\cdot\nabla(v_kw_k)+\frac{1}{2}\nabla w_k^2\cdot\nabla\eta+\eta|\nabla w_k|^2\right]dx\\&\quad=\int_\Omega\big[(1-s)\big(|\nabla v_k|^2+|\nabla w_k|^2\big)+s\big|\nabla(v_k+w_k)\big|^2\big]\eta\,dx-\frac{1}{2}\int_\Omega\big(|v_k|^2+|w_k|^2+2sv_kw_k\big)\Delta\eta\,dx.\end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2) it follows

$$\int_\Omega a_-(x)|v_k|^{p+1}\,dx\leqslant\mathrm{O}\big(\|u_k\|_2^2\big)+\mathrm{o}\big(\|u_k\|_E^2\big). \tag{2.4}$$

Combining (2.4) with (2.1) we get

$$\int_\Omega a_+(x)|w_k|^{q+1}\,dx\leqslant C+\mathrm{O}\big(\|u_k\|_2^2\big)+\mathrm{o}\big(\|u_k\|_E^2\big). \tag{2.5}$$

The assumption $I_s(v_k,w_k)\leqslant C$ and (2.5) imply

$$\frac{1}{2}\int_\Omega\big(|\nabla v_k|^2+2s\nabla v_k\cdot\nabla w_k+|\nabla w_k|^2\big)\,dx+\int_\Omega a_-(x)|v_k|^{p+1}\,dx\leqslant C+\mathrm{O}\big(\|u_k\|_2^2\big)+\mathrm{o}\big(\|u_k\|_E^2\big). \tag{2.6}$$

However, there is a constant C_1 such that

$$\begin{aligned}\int_\Omega\big(|\nabla v|^2+2s\nabla v\cdot\nabla w+|\nabla w|^2\big)\,dx&\geqslant(1-s)\int_\Omega\big(|\nabla v|^2+|\nabla w|^2\big)\,dx+s\int_\Omega|\nabla u|^2\,dx\\&\geqslant C_1\int_\Omega\big(|\nabla v|^2+|\nabla w|^2\big)\,dx.\end{aligned} \tag{2.7}$$

Inserting (2.7) into (2.6) we have

$$\int_\Omega\big(|\nabla v_k|^2+|\nabla w_k|^2\big)\,dx+\int_\Omega a_-(x)|v_k|^{p+1}\,dx\leqslant C+\mathrm{O}\big(\|u_k\|_2^2\big)+\mathrm{o}\big(\|u_k\|_E^2\big). \tag{2.8}$$

Therefore

$$\|u_k\|_E^2 \leqslant C\big(1+\|u_k\|_2^2\big) \tag{2.9}$$

for some constant C.

In view of (2.9), it remains to prove that $\|v_k\|_2+\|w_k\|_2$ is bounded. This is proved by contradiction. Suppose not, we have a weak Palais–Smale sequence $\{(v_k,w_k)\}$ satisfying $\|v_k\|_2+\|w_k\|_2\to+\infty$ as $k\to\infty$. Define

$$\tilde u_k=\frac{u_k}{\|v_k\|_2+\|w_k\|_2},\qquad \tilde v_k=\frac{v_k}{\|v_k\|_2+\|w_k\|_2},\qquad \tilde w_k=\frac{w_k}{\|v_k\|_2+\|w_k\|_2},$$

(2.8) implies that $\{\tilde v_k\}$, $\{\tilde w_k\}$, and $\{\tilde u_k\}$ are bounded. By Lemma 2.1, E is reflexive, so we may assume, after a subsequence that

$$\tilde v_k\rightharpoonup v_0,\quad \tilde w_k\rightharpoonup w_0,\quad \tilde u_k\rightharpoonup u_0\quad \text{weakly in } H_0^1(\Omega)$$

and

$$\tilde v_k\to v_0,\quad \tilde w_k\to w_0,\quad \tilde u_k\to u_0\quad \text{strongly in } L^2(\Omega)$$

with $u_0=v_0+w_0$. Then $u_0\neq 0$ as $\|v_0\|_2+\|w_0\|_2=1$. Setting $z_k=\partial_v I_s(v_k,w_k)$ and $\phi=\frac{v_k}{(\|v_k\|_2+\|w_k\|_2)^2}$ we have

$$\begin{aligned}\langle z_k,\phi\rangle &= \int_\Omega \big(|\nabla\tilde v_k|^2-\lambda\tilde v_k^2\big)\,dx+s\int_\Omega(\nabla\tilde v_k\cdot\nabla\tilde w_k-s\lambda\tilde v_k\cdot\tilde w_k)\,dx\\ &\quad+\big(\|v_k\|_2+\|w_k\|_2\big)^{p-1}\int_\Omega a_-(x)|\tilde v_k|^{p+1}\,dx+\mathrm{o}(1).\end{aligned}$$

Therefore

$$\big(\|v_k\|_2+\|w_k\|_2\big)^{p-1}\int_\Omega a_-(x)|\tilde v_k|^{p+1}\,dx\leqslant C$$

and

$$\int_\Omega a_-(x)|\tilde v_k|^{p+1}\,dx\to 0$$

by $p>1$ and $\|v_k\|_2+\|w_k\|_2\to+\infty$. It follows from $\tilde v_k\rightharpoonup v_0$ that

$$\int_\Omega a_-(x)|v_0|^{p+1}\,dx\leqslant \varliminf_{k\to\infty}\int_\Omega a_-(x)|\tilde v_k|^{p+1}=0$$

and then $\operatorname{supp}(v_0)\subset\overline{\Omega}_0$.

Similarly, we have

$$\int_\Omega a_+(x)|w_0|^{q+1}\,dx=0$$

and $\operatorname{supp}(w_0)\subset\overline{\Omega}_0$ by computing $\langle\partial_w I_s(v_k,w_k),\frac{w_k}{(\|v_k\|_2+\|w_k\|_2)^2}\rangle$. Hence $w_0=0$ provided by $\Delta w_0=0$ in Ω_0 and then $\operatorname{supp}(u_0)\subset\overline{\Omega}_0$.

Let us choose $\phi\in H_0^1(\Omega_0)$ as an element in E, then we have

$$\begin{aligned}\frac{1}{\|v_k\|_2+\|w_k\|_2}\big\langle I_s'(v_k,w_k),\phi\big\rangle &= \int_\Omega(\nabla\tilde v_k\cdot\nabla\phi-\lambda\tilde v_k\phi)\,dx+s\int_\Omega(\nabla\tilde w_k\cdot\nabla\phi-\lambda\tilde w_k\phi)\,dx\\ &=\int_\Omega(\nabla\tilde u_k\cdot\nabla\phi-\lambda\tilde u_k\phi)\,dx+s\int_\Omega(\nabla\tilde w_k\cdot\nabla\phi-\lambda\tilde w_k\phi)\,dx=\mathrm{o}(1).\end{aligned}$$

Thus

$$\int_\Omega (\nabla u_0 \cdot \nabla \phi - \lambda u_0 \phi)\, dx = 0 \quad \forall \phi \in H_0^1(\Omega_0)$$

by $\tilde{w}_k \rightharpoonup 0$. According to the assumption $\lambda \notin \sigma(\Omega_0)$, $u_0 = 0$ in Ω_0 and $u_0 = 0$ in E. This is a contradiction. □

Lemma 2.5. *Under the assumption of Lemma* 2.4*, every weak Palais–Smale sequence of I_s contains a convergent subsequence in E.*

Proof. (1) Let $\{u_k\} = \{(v_k, w_k)\}$ be a weak Palais–Smale sequence of I_s for some $s \in [0, 1]$. According to Lemma 2.4, $\{u_k\}$ is bounded in E.

(2) Applying Lemma 2.1, there is a $u^* \in E$ such that $u_k \rightharpoonup u^*$ in E. After a subsequence we have

$$u_k(x) \to u^*(x) \quad \text{a.e. in } \Omega$$

and

$$u_k \to u^* \quad \text{in } L^2(\Omega) \text{ and in } L^{q+1}(\Omega).$$

(3) By the definition of weak Palais–Smale sequence, we have

$$\big\langle I_s'(u_k) - I_s'(u^*), u_k - u^*\big\rangle = \mathrm{o}\big(\|u_k\|_E\big)\|u_k - u^*\|_E + \mathrm{o}(1) = \mathrm{o}(1) \tag{2.10}$$

provided by the boundedness of $\{u_k\}$ in E. On the other hand, there holds

$$\big\langle I_s'(u_k) - I_s'(u^*), u_k - u^*\big\rangle = \|u_k - u^*\|_s^2 + \int_\Omega a_-\big(|u_k|^{p-1}u_k - |u^*|^{p-1}u^*\big)(u_k - u^*)\, dx + \mathrm{o}(1), \tag{2.11}$$

where

$$\|u\|_s^2 = \int_\Omega \big(|\nabla v|^2 + 2s\nabla v \cdot \nabla w + |\nabla w|^2\big)\, dx$$

is an equivalent norm of $\|u\|_{H_0^1} = (\int_\Omega |\nabla u|^2\, dx)^{1/2}$ by (2.7). From an elementary inequality, there is a constant C such that

$$C\int_\Omega a_-|u_k - u^*|^{p+1}\, dx \leqslant \int_\Omega a_-\big(|u_k|^{p-1}u_k - |u^*|^{p-1}u^*\big)(u_k - u^*)\, dx. \tag{2.12}$$

Combining (2.10)–(2.12) we get $u_k \to u^*$ in E. □

As a consequence of Lemmas 2.4 and 2.5 we have

Corollary 2.6. *Under the assumptions of Lemma* 2.5*, the functional I satisfies the Palais–Smale condition on E.*

The proofs of Lemmas 2.4 and 2.5 also yield (see Proposition 3.3 in [16]).

Theorem 2.7. *Under the assumptions of* (A1), (A2), (A3′) *and* (A4)*, there are constants A and $\delta > 0$ such that* $\forall s \in [0, 1]$,

$$\big\|I_s'(u)\big\|_{E^*} \geqslant \delta\|u\|_E \quad \text{if } I_s(u) \leqslant A.$$

Using the deformation I_s we have the following theorem on the critical groups at infinity, its proof is referred to Theorem 4.1 in [16].

Theorem 2.8. *Under the assumptions of* (A1), (A2), (A3′) *and* (A4)*, all critical groups of I at infinity are trivial, i.e.,*

$$C_*(I, \infty) := H_*\big(E, I^a\big) = \{0\}, \quad * = 0, 1, 2, \ldots$$

for $a \leqslant A$, where A is the constant in Theorem 2.7 *and $I^a = \{u \in E \mid I(u) \leqslant a\}$.*

Also one defines

$$I_\pm(u)=\int\limits_\Omega\left[\frac{1}{2}\left(|\nabla u|^2-\lambda u_\pm^2\right)-\frac{a_+(x)}{q+1}|u_\pm|^{q+1}+\frac{a_-(x)}{p+1}|u_\pm|^{p+1}-H_\pm(x,u)\right]dx,$$

where $u_\pm=\max\{\pm u,0\}$, $H_\pm(x,\xi)$ is the primitive of $h_\pm(x,\xi)$ and

$$h_\pm(x,\xi)=\begin{cases}h(x,\xi), & \pm\xi\geqslant 0,\\ 0, & \text{otherwise.}\end{cases}$$

We have the following facts for $I_\pm$:

(1) $I_\pm(u)=I(u)$ if $\pm u\geqslant 0$;

(2) Let K and $K_\pm$ be the critical sets of I and $I_\pm$, respectively, then $K_\pm=K\cap(\pm P)$, where $P=\{u\in H_0^1(\Omega)\mid u(x)\geqslant 0 \text{ a.e. } x\in\Omega\}$.

As we proved in [16], the following theorem holds.

Theorem 2.9. *Under the assumptions of* (A1), (A2), (A3′) *and* (A4)*, the critical groups of $I_\pm$ at infinity are well defined and trivial, i.e.,*

$$C_*(I_\pm,\infty)=\{0\},\quad *=0,1,2,\ldots.$$

Remark 2.10. In the following sections we shall consider the functional on the space $C_0^1(\overline{\Omega})$ rather than the space E. Let $\tilde I=I|_{C_0^1(\overline{\Omega})}$ and $\tilde I_\pm=I_\pm|_{C_0^1(\overline{\Omega})}$. It is well known [25] that

$$H_*\big(C_0^1(\overline{\Omega}),\tilde I^a\big)=H_*\big(E,I^a\big)=\{0\},\quad *=0,1,2,\ldots,$$

and

$$H_*\big(C_0^1(\overline{\Omega}),\tilde I_\pm^a\big)=H_*\big(E,I_\pm^a\big)=\{0\},\quad *=0,1,2,\ldots.$$

This means that the critical groups at infinity for both I and $\tilde I$ are trivial.

Remark 2.11. The results in this section hold if the term $|u|^{q-1}u$ is replaced by $g(u)$ satisfying:

(g1) $|g(u)|\leqslant C(1+|u|^q)$ $u\in\mathbf{R}$,

(g2) there are constants $\theta>2$ and $R>0$ such that

$$g(u)u\geqslant\theta G(u)>0,\quad |u|\geqslant R$$

where $G(u)=\int_0^u g(s)\,ds$.

3. The heat flow

Let $T>0$ and $\Omega^T=(0,T)\times\Omega$ and $\phi\in C_0^1(\overline{\Omega})$, in this section we study the L^∞ a priori estimate for the solution $v\in C^1(\overline{\Omega}^T)$ of the nonlinear heat equation:

$$\begin{aligned}&\partial_t v(t,x)=(\Delta+\lambda)v+a_+(x)|v|^{q-1}v-a_-(x)|v|^{p-1}v+h(x,v),\quad (t,x)\in\Omega^T,\\ &v(t,x)=0,\quad (t,x)\in[0,T)\times\partial\Omega,\\ &v(0,x)=\phi(x),\quad x\in\overline{\Omega},\end{aligned}\tag{3.1}$$

and explain how the heat flow can be used in Morse theory for the associated functional. It is well known that the solution v may blow up at finite time. However, if one adds a finite energy condition, the blow up phenomena can be ruled out, see Ackermann, Bartsch, Kaplicky and Quittner [1], Cazenave and Lions [11], Chang [14], Giga [19] and Quittner [27]. For technical reasons we assume

$$\text{(A2}'\text{)}\quad q<p_1^*:=\begin{cases}+\infty, & n=1,\\ 7, & n=2,\\ \frac{18}{5}, & n=3,\\ \frac{9n^2-4n+16\sqrt{n(n-1)}}{(3n-4)^2}, & n>3\end{cases}$$

and $p > 1$. We shall prove

Theorem 3.1. *Assume* (A1), (A2′) *and* (A3). *If v is a solution of* (3.1) *which blows up at a finite time T, then $I(v(t,\cdot)) \to -\infty$ as $t \to T - 0$.*

Theorem 3.2. *Assume* (A1), (A2′) *and*

(A3′) $h(x,\xi) = \mathrm{o}(|\xi|)$ *as* $|\xi| \to \infty$ *uniformly in* x,
(A4) $\lambda \notin \sigma(\Omega_0)$,

where $\Omega_0 = \Omega \setminus \overline{(\Omega_+ \cup \Omega_-)}$ and $\sigma(\Omega_0)$ is the spectrum of $-\Delta$ on Ω_0 with Dirichlet boundary condition. If v is a global solution of (3.1) *satisfying $I(v(t,\cdot)) \geqslant -C_0$ for some constant C_0, then the ω-limit set $\omega(\phi) \neq \emptyset$ and contains critical points of I.*

Combining Theorems 3.1 and 3.2 we have

Theorem 3.3. *Assume* (A1), (A2′), (A3′) *and* (A4)*, the Morse theory for isolated critical points of the functional I holds in the Banach space $C_0^1(\overline{\Omega})$. The Morse theory is related to the order preserving parabolic semi-flow.*

In the remaining of this section we give details of the proofs.

Lemma 3.4. *If $v \in C^1(\overline{\Omega}^T)$ is a solution of* (3.1) *and if $I(v(t,\cdot)) \geqslant -C_0$ $\forall t \in [0,T]$ for some constant C_0, then there is a constant $C_T(\phi, C_0)$ depending only on ϕ and C_0 such that*

$$\|v(t,\cdot)\|_2 \leqslant C_T(\phi, C_0).$$

Proof. It follows from the Hölder inequality that

$$\begin{aligned}\|v(t,\cdot) - \phi\|_2 &= \left\| \int_0^t \partial_s v(s,\cdot)\,ds \right\|_2 \\ &\leqslant T^{1/2}\left(\int_0^T \|\partial_s v(s,\cdot)\|_2^2\,ds\right)^{1/2} \\ &\leqslant \big[T\big(I(\phi) - I\big(v(T,\cdot)\big)\big)\big]^{1/2}\end{aligned}$$

provided by

$$\int_0^T \|\partial_s v(s,\cdot)\|_2^2\,ds = -\int_0^T \frac{d}{dt} I\big(v(t,\cdot)\big)\,dt = I(\phi) - I\big(v(T,\cdot)\big).$$

Now the conclusion follows easily. □

Lemma 3.5. *Under the assumptions of* (A1), (A2), (A3′) *and* (A4)*, if $v \in C^1(\Omega^T)$ is a global solution of* (3.1) *and if $I(v(t,\cdot)) \geqslant -C_0$ $\forall t \in [0,+\infty)$ for some constant C_0, then there is a constant C_1 such that*

$$\|v(t,\cdot)\|_2 \leqslant C_1, \quad t \in [0,\infty).$$

Proof. We prove it by contradiction. Suppose not, there exists a sequence $\{t_k\}$ such that $\|v(t_k,\cdot)\|_2 \geqslant k$. According to Lemma 3.4, $t_k \to +\infty$ and we may assume $t_{k+1} - t_k > 1$. By Lemma 2.4, a contradiction follows if we can construct a weak Palais–Smale sequence of I close to $\{v(t_k,\cdot)\}$.

(1) Claim: $\exists\eta > 0$ such that

$$\|v(t,\cdot)\|_2 \geqslant \frac{k}{2}, \quad \forall t \in [t_k - \eta, t_k].$$

Indeed, $\forall s < t_k$

$$\begin{aligned}
\big|\|v(s,\cdot)\|_2^2 - \|v(t_k,\cdot)\|_2^2\big| &= 2\left|\int_s^{t_k}\int_\Omega v\partial_t v(t,x)\,dx\,dt\right| \\
&\leqslant 2\left(\int_s^{t_k}\|v(t,\cdot)\|_2^2\,dt \int_s^{t_k}\|\partial_t v(t,\cdot)\|_2^2\,dt\right)^{1/2} \\
&\leqslant \big(I(\phi) + C_0\big) + \int_s^{t_k}\|v(t,\cdot)\|_2^2\,dt.
\end{aligned}$$

From the Gronwall inequality

$$\|v(t_k,\cdot)\|_2^2 \leqslant \big(\|v(s,\cdot)\|_2^2 + C_1\big)e^{(t_k - s)}, \quad s \leqslant t_k,$$

where $C_1 = I(\phi) + C_0$. Thus we can find an $\eta > 0$ satisfying

$$\|v(t,\cdot)\|_2 \geqslant \frac{k}{2}, \quad t \in [t_k - \eta, t_k] \text{ and } k \gg 1.$$

(2) Claim: $\exists s_k \in [t_k - \eta, t_k]$ such that

$$\int_\Omega |\partial_t v(s_k,x)|^2\,dx \leqslant \frac{I(\phi) + C_0}{\eta}.$$

This is due to the fact:

$$\int_0^{+\infty}\int_\Omega |\partial_t v(t,x)|^2\,dx\,dt \leqslant I(\phi) + C_0.$$

(3) Now let $u_k = v(s_k,\cdot)$, then

$$I(u_k) = I\big(v(s_k,\cdot)\big) \leqslant I(\phi)$$

and $\forall\psi \in E$,

$$\begin{aligned}
\big|\langle I'(u_k), \psi\rangle\big| &= \left|\int_\Omega \partial_t v(s_k,x)\psi\,dx\right| \\
&\leqslant \left(\int_\Omega |\partial_t v(s_k,x)|^2\,dx\right)^{1/2}\|\psi\|_2 \\
&\leqslant \eta^{-1/2}\big(I(\phi) + C_0\big)^{1/2}\|\psi\|_E.
\end{aligned}$$

That is,

$$\|I'(u_k)\|_{E^*} \leqslant \eta^{-1/2}\big(I(\phi) + C_0\big)^{1/2} = \mathrm{o}\big(\|u_k\|_2\big) = \mathrm{o}\big(\|u_k\|_E\big).$$

Therefore, $\{u_k\}$ is a weak Palais–Smale sequence and is bounded by Lemma 2.4. This is a contradiction. $\square$

Having Lemmas 3.4 and 3.5, now we can prove the main estimate.

Lemma 3.6. *Let* $J=[0,T]$*. Assume* (A1), (A2′) *and* (A3)*, if* $v\in C^1(\overline{\Omega^T})$ *is a solution of* (3.1) *satisfying*

$$\|v(t,\cdot)\|_2\leqslant C_J,\quad t\in[0,T],$$

where T *is either finite or infinite, then* $\|v(t,\cdot)\|_\infty$ *is bounded on* J*.*

Proof. We estimate $\|v(t,\cdot)\|_\infty$ in various subdomains of Ω separately.

(1) For $\epsilon>0$, let $\Omega_{+,\epsilon}$ be the ϵ-neighborhood of Ω_+ in Ω. Since $\overline{\Omega}_+\cap\overline{\Omega}_-=\emptyset$, there exists $\epsilon>0$ such that $\overline{\Omega}_{+,\epsilon}\cap\overline{\Omega}_-=\emptyset$, then $\partial\Omega_{+,\epsilon}\setminus\partial\Omega\subset\overline{\Omega}_0$ and $(\partial\Omega_{+,\epsilon}\setminus\partial\Omega)\cap\overline{\Omega}_+=\emptyset$. We fix ϵ from now on and $\forall(t_0,x_0)\in\Omega^T$ and $\forall R>0$, denote $Q_R(t_0,x_0)=(t_0-R^2,t_0+R_0^2)\times B_R(x_0)$. Now v satisfies

$$\partial_t v=\Delta v+\lambda v+h(x,v),\quad (t,x)\in[0,T]\times\Omega_0.$$

In case $Q_R(t_0,x_0)\subset\Omega_0^T$, according to Moser's iteration on the local boundedness of the weak solution v on Ω_0, see [23], there exist R_0 and $C>0$ such that $\forall R\in(0,R_0]$, there holds:

$$\sup_{Q_{\frac{R}{2}}(t_0,x_0)}\left|v(t,x)\right|\leqslant\left(\frac{C}{R^{n+2}}\int\limits_{Q_R(t_0,x_0)}\left|v(t,x)\right|^2dx\,dt\right)^{1/2}+C.$$

In case $x_0\in\partial\Omega\cap\Omega_0$ and $t_0>0$ we also have

$$\sup_{Q_{\frac{R}{2}}(t_0,x_0)\cap\Omega^T}\left|v(t,x)\right|\leqslant\left(\frac{C}{R^{n+2}}\int\limits_{Q_R(t_0,x_0)\cap\Omega^T}\left|v(t,x)\right|^2dx\,dt\right)^{1/2}+C.$$

If T is finite, one fixes $R>0$, then $[\frac{R^2}{4},T]\times(\partial\Omega_{+,\epsilon}\setminus\partial\Omega)$ is covered by finitely many, say M, cylinders in the family $\{Q_{\frac{R}{2}}(t_0,x_0)\mid(t_0,x_0)\in[\frac{R^2}{4},T]\times(\partial\Omega_{+,\epsilon}\setminus\partial\Omega)\}$. Thus

$$\sup_{[\frac{R^2}{4},T]\times(\partial\Omega_{+,\epsilon}\setminus\partial\Omega)}\left|v(t,x)\right|\leqslant M\left[\left(\frac{C}{R^{n+2}}\int\limits_{\Omega^T}\left|v(t,x)\right|^2dx\,dt\right)^{1/2}+C\right].\tag{3.2}$$

If $T=+\infty$, applying the same arguments to the domains $[k-1,k+1]\times\Omega_0$, $k=2,3,\ldots,$ we get

$$\sup_{[k-1,k+1]\times(\partial\Omega_{+,\epsilon}\setminus\partial\Omega)}\left|v(t,x)\right|\leqslant M\left[\left(\frac{C}{R^{n+2}}\int\limits_{Q_R(t_0,x_0)\cap\Omega^T}\left|v(t,x)\right|^2dx\,dt\right)^{1/2}+C\right].$$

The number M and all constants are independent of k, so we have

$$\sup_{[\frac{R^2}{4},+\infty)\times(\partial\Omega_{+,\epsilon}\setminus\partial\Omega)}\left|v(t,x)\right|\leqslant M\left[\left(\frac{C}{R^{n+2}}\int\limits_{Q_R(t_0,x_0)\cap\Omega^T}\left|v(t,x)\right|^2dx\,dt\right)^{1/2}+C\right].\tag{3.3}$$

According to the assumption:

$$\|v(t,\cdot)\|_2\leqslant C_J,\quad t\in[0,T],$$

the right-hand sides of (3.2) and (3.3) are bounded by $M[C+(\frac{C}{R^{n+2}})^{1/2}C_J]$, and then v is bounded on $[\frac{R^2}{4},T]\times(\partial\Omega_{+,\epsilon}\setminus\partial\Omega)$.

(2) By a standard argument of the variation of constant formula, we have

$$\sup_{[0,\frac{R^2}{2}]\times\overline{\Omega}}\left|v(t,x)\right|\leqslant C_1\sup_{\Omega}\left|\phi(x)\right|.\tag{3.4}$$

Combining (3.4) with the estimates in the last step we obtain

$$\left|v(t,x)\right|\leqslant C_2,\quad (t,x)\in J\times(\partial\Omega_{+,\epsilon}\setminus\partial\Omega),\tag{3.5}$$

where C_2 is a constant depending on C_J.

(3) Let us consider Eq. (3.1) on the subdomain $\Omega\setminus\Omega_{+,\epsilon}$:

$$\partial_t v_1(t,x) = (\Delta+\lambda)v_1 - a_-(x)|v_1|^{p-1}v_1 + h(x,v_1), \quad \forall (t,x) \in J \times (\Omega \setminus \Omega_{+,\epsilon}),$$
$$v_1(t,x) = v(t,x), \quad \forall (t,x) \in J \times (\partial\Omega_{+,\epsilon} \setminus \partial\Omega),$$
$$v_1(t,x) = 0, \quad \forall (t,x) \in J \times \big(\partial\Omega \cap (\overline{\Omega \setminus \Omega_{+,\epsilon}})\big),$$
$$v_1(0,x) = \phi(x), \quad \forall x \in \Omega \setminus \Omega_{+,\epsilon}.$$

By the uniqueness,

$$v(t,x) = v_1(t,x), \quad \forall (t,x) \in J \times (\Omega \setminus \Omega_{+,\epsilon}).$$

We apply the weak maximum principle due to De Giorgi's iteration [23], it follows

$$\|v\|_{L^\infty(J\times(\Omega\setminus\Omega_{+,\epsilon}))} = \|v_1\|_{L^\infty(J\times(\Omega\setminus\Omega_{+,\epsilon}))} \leqslant C_3 \tag{3.6}$$

for a constant depending on C_2, ϕ, λ, h and C_J.

(4) Finally, we consider Eq. (3.1) on the subdomain $\Omega_{+,\epsilon}$:

$$\partial_t v_2(t,x) = (\Delta+\lambda)v_2 + a_+(x)|v_2|^{q-1}v_2 + h(x,v_2), \quad \forall (t,x) \in J \times \Omega_{+,\epsilon},$$
$$v_2(t,x) = v(t,x), \quad \forall (t,x) \in J \times (\partial\Omega_{+,\epsilon} \setminus \partial\Omega),$$
$$v_2(t,x) = 0, \quad \forall (t,x) \in J \times (\partial\Omega \cap \Omega_{+,\epsilon}),$$
$$v_2(0,x) = \phi(x), \quad \forall x \in \Omega_{+,\epsilon}.$$

Since $q < p_1^*$ is assumed, after the iteration estimate due to Quittner, see [1], and (3.6), we have

$$\|v\|_{L^\infty(J\times\Omega_{+,\epsilon})} = \|v_2\|_{L^\infty(J\times\Omega_{+,\epsilon})} \leqslant C_4. \tag{3.7}$$

In summary we have proved the boundedness of $\sup_{t\in J}\|v(t,\cdot)\|_\infty$. □

Proof of Theorem 3.1. This follows from Lemmas 3.4 and 3.6. In fact, if $I(v(t,\cdot)) \geqslant -C_0$ for some constant $C_0 > 0$, then $\|v(t,\cdot)\|_2 \leqslant C_T$. By Lemma 3.6, $\|v(t,\cdot)\|_\infty$ is bounded on $[0,T]$. This contradicts with the assumption that T is the blow up time. □

Proof of Theorem 3.2. Let v be the global solution of (3.1). After Lemma 3.6 we have $\|v(t,\cdot)\|_\infty \leqslant C_2$ $\forall t$. Then by a standard argument, see [10], Theorem 9.4.2, $\omega(\phi) \neq \emptyset$ and $\forall u \in \omega(\phi)$, it is a critical point of I with $I(u) = \lim_{t\to+\infty} I(v(t,\cdot)) \geqslant -C_0$. According to the regularity theory, the topology can be taken on $C_0^1(\overline{\Omega})$. □

Proof of Theorem 3.3. Assume now that the functional I has only isolated critical points. If the global orbit $O(\phi) = \{v(t,\cdot) \mid t \in \mathbf{R}^+\}$ exists and satisfies $I(v(t,\cdot)) \geqslant -C_0$, again by Lemma 3.6 and Theorem 9.4.2 in [10], the limit set $\omega(\phi)$ must be a singleton. Moreover, the limit exists in $C_0^1(\overline{\Omega})$ topology. □

The heat flow $v(t,\phi)$ with initial value ϕ is used to replace the pseudo-gradient flow. In order to establish the Morse theory for the isolated critical points, it is sufficient to prove the following deformation lemma via the heat flow: Let K be the critical set of $\tilde I$, $a < d$, if $K \cap \tilde I^{-1}(a,d] = \emptyset$ and if $K_a = K \cap \tilde I^{-1}(a)$ is isolated, then $\tilde I^a$ is a strong deformation retract of $\tilde I^b$, where $\tilde I = I|_{C_0^1(\overline{\Omega})}$.

Indeed, $\forall \phi \in \tilde I^{-1}(a,d]$, let $T_\phi > 0$ be the maximal existence time of $v(t,\phi)$, and let $O(\phi) = \{v(t,\phi) \mid t \in [0,T_\phi)\}$ be its orbit. Let t_ϕ be the arriving time of the orbit $O(\phi)$ at the level $\tilde I^{-1}(a)$. According to Theorems 3.1 and 3.2, $t_\phi > 0$ is either finite, $t_\phi < T_\phi$ if $\tilde I(\omega(\phi)) < a$, or $t_\phi = +\infty$ and $\omega(\phi) \in \tilde I^{-1}(a)$. In both cases, we can rescale the time variable as follows: Let

$$\tau = \tau(t) = \frac{\tilde I(\phi) - \tilde I(v(t,\phi))}{\tilde I(\phi) - a}, \quad \forall \phi \in \tilde I^1(a,d].$$

Then we have

$$\begin{cases} \tau(0) = 0, \\ \tau(+\infty) = 1 & \text{if } \omega(\phi) \in \tilde I^{-1}(a), \\ \tau(+\infty) > 1 & \text{if } \tilde I(\omega(\phi)) < a. \end{cases}$$

Let $t(\tau)$ be the inverse function of $\tau(t)$ and

$$\eta(\tau,\phi)=\begin{cases} v(t(\tau),\phi) & \text{if } (\tau,x)\in[0,1]\times\tilde I^{-1}(a,d],\\ \phi & \text{if } (\tau,\phi)\in[0,1]\times\tilde I_a.\end{cases}$$

Observing the relation

$$\frac{d}{dt}\tilde I\big(v(t,\phi)\big)=-\int_\Omega\big|\partial_t v(t,\phi)\big|^2\,dx<0,\quad \forall t<T_\phi,$$

the flow $v(t,\phi)$ is transversal to each level set $\tilde I^{-1}(c)$ for all $c\in(a,d]$ and $\phi\in\tilde I^{-1}(a,d]$.

A standard argument in [11] and [14] can be applied to verify that η is the desired strong deformation retract. Since η is a rescaling of the heat semi-flow which is order preserving, so is η.

Remark 3.7. Without the requirements of the strong order preserving and of the $C_0^1(\overline{\Omega})$ topology, the Morse theory for the isolated critical points of the functional I holds on the space $E=H_0^1(\Omega)\cap L_{a_-}^{p+1}(\Omega)$ for $q<2^*-1$.

This is due to the Palais–Smale condition on E, see Corollary 2.6.

Remark 3.8. Theorems 3.1–3.2 hold for the solutions of the following heat flow:

$$\begin{aligned}&\partial_t v(t,x)=(\Delta+\lambda)v+a_+(x)|v_\pm|^{q-1}v_\pm-a_-(x)|v_\pm|^{p-1}v_\pm+h_\pm(x,v),\quad (t,x)\in\Omega^T,\\ &v(t,x)=0,\quad (t,x)\in[0,T)\times\partial\Omega,\\ &v(0,x)=\phi(x),\quad x\in\overline{\Omega}.\end{aligned}\tag{3.8}$$

These flows preserve the cones $\tilde P$ and $-\tilde P$, respectively.

4. Critical points theorems

In this section, for $\underline{u},\bar u\in C_0^1(\overline{\Omega})$ with $\underline{u}\leqslant\bar u$, the order interval $[\underline{u},\bar u]$ is defined to be the set $\{u\in C_0^1(\overline{\Omega})\mid \underline{u}(x)\leqslant u(x)\leqslant\bar u(x)\ \forall x\in\overline{\Omega}\}$.

Theorem 4.1. *Let the assumptions* (A1), (A2), (A3′) *and* (A4) *be satisfied. Suppose that Eq.* (1.1) *has two pairs of sub- and super-solutions* $\underline{v}_i<\bar v_i$, $i=1,2$, *satisfying* $\underline{v}_1<\bar v_2$ *and* $[\underline{v}_1,\bar v_1]\cap[\underline{v}_2,\bar v_2]=\emptyset$. *Then* (1.1) *has at least a solution* $u\in[\underline{v}_1,\bar v_2]\setminus([\underline{v}_1,\bar v_1]\cup[\underline{v}_2,\bar v_2])$, *which is a mountain pass point of* $\tilde I$ *if there are only finite many critical points of* $\tilde I$ *on* $[\underline{v}_1,\bar v_2]\setminus([\underline{v}_1,\bar v_1]\cup[\underline{v}_2,\bar v_2])$.

Proof. We only work on the ordered interval $[\underline{v}_1,\bar v_2]$, which is a bounded set in the L^∞-norm, so we can use the modified gradient flow of I:

$$\dot\eta=\eta-(-\Delta+kI)^{-1}\big(\lambda\eta+a_+|\eta|^{q-1}\eta-a_-|\eta|^{p-1}\eta+h(x,\eta)+k\eta\big),\quad k>0\text{ large}$$

as a deformation for the functional $\tilde I$. All intervals $[\underline{v}_1,\bar v_2]$, $[\underline{v}_1,\bar v_1]$ and $[\underline{v}_2,\bar v_2]$ are positively invariant w.r.t. the modified gradient flow. This can be proved as in the definite nonlinearity case. Let S_i be the critical set of $\tilde I$ located in the intervals $[\underline{v}_i,\bar v_i]$, $i=1,2$, and $S=S_1\cup S_2$, then $[\underline{v}_i,\bar v_i]$ and $[\underline{v}_1,\bar v_2]$ are isolated neighborhoods of S_i and S, $([\underline{v}_i,\bar v_i],\emptyset)$, and $([\underline{v}_1,\bar v_2],\emptyset)$ are the index pairs of S_i and S, respectively, $i=1,2$. All intervals $[\underline{v}_1,\bar v_2]$, $[\underline{v}_2,\bar v_2]$, $[\underline{v}_1,\bar v_2]$ are contractible, hence we have

$$C_*(\tilde I,S_i)=C_*(\tilde I,S)=\delta_{*,0}G,\quad *=0,1,2,\dots,$$

where $C_*(\tilde I,S)$, $*=0,1,2,\dots$, are the critical groups for isolated critical set S, G is the coefficient group of homology, see [15]. Following the Morse relation, there must be a critical point $u\in[\underline{v}_1,\bar v_2]\setminus([\underline{v}_1,\bar v_1]\cup[\underline{v}_2,\bar v_2])$ with $C_1(\tilde I,u)\neq 0$ provided by the finiteness of the critical points. Thus u is a mountain pass point of $\tilde I$. □

Remark 4.2. Without assuming the finiteness of critical points, a minimax proof can be found in [22] (proof of Theorem 1.3).

In the study of sign-changing solutions of definitely nonlinear problem, i.e, $a_- = 0$, we have combined the information of critical groups at infinity and the positive invariance of the positive and negative cones under the flow to obtain an abstract critical point theorem in [13], see also [7]. Now we are going to use the heat flow, which preserves the positive and negative cones in $C_0^1(\overline{\Omega})$, and the results in Section 2 on the critical groups at infinity to get a similar result with indefinite nonlinearities.

With the aid of the functionals $I_\pm$ and the heat flow for the functionals $\tilde{I}_\pm$, we can prove the following theorem. Let $\tilde{P} = P \cap C_0^1(\overline{\Omega})$.

Theorem 4.3. *Under the assumptions* (A1), (A2′), (A3′) *and* (A4)*, if* $\underline{u} < 0 < \bar{u}$ *is a pair of sub and super-solutions of* (1.1)*, then there exist at least* 3 *distinct solutions* $u_\pm, u_0$ *of* (1.1) *such that* $u_\pm \in \pm\tilde{P} \setminus D$ *and* $u_0 \notin \tilde{P} \cup (-\tilde{P}) \cup D$*, where* $D = [\underline{u}, \bar{u}]$.

Proof. (1) Let K be the critical set of $\tilde{I}$ and $K^\pm$ be that of $\tilde{I}^\pm$, then $K^\pm = \pm\tilde{P} \cap K$. We know by the excision that

$$H_*\big(\pm\tilde{P} \cup \tilde{I}_\pm^a, \tilde{I}_\pm^a\big) \approx H_*\big(\pm\tilde{P}, \pm\tilde{P} \cap \tilde{I}_\pm^a\big), \quad * = 0, 1, 2, \ldots. \tag{4.1}$$

In the following we choose $a < A$, which is defined in Theorem 2.7 and $\min\{\tilde{I}(u) \mid u \in D\} > a$. We claim if $1 < q < p_1^*$, then

$$H_*\big(C_0^1(\overline{\Omega}), \tilde{I}_\pm^a\big) \approx H_*\big(\pm\tilde{P} \cup \tilde{I}_\pm^a, \tilde{I}_\pm^a\big) = 0, \quad * = 0, 1, 2, \ldots. \tag{4.2}$$

We follow the argument in [6] (proof of Theorem 3.2) to prove (4.2). We only consider the functional $\tilde{I}_+$. Let $v(t, \phi)$ be the solution of (3.8). By the strong maximum principle for sub- and super-solutions we know that 0 is in the interior of D (see Section 3.2 in [20]). According to the strong maximum principle of the heat equation and Remark 3.8, using the fact that 0 is in the interior of D, $\forall \phi \in C_0^1(\overline{\Omega})$, $\exists t(\phi) \geqslant 0$ such that $v(t(\phi), \phi) \in \operatorname{int}(\tilde{P} \cup D \cup \tilde{I}_+^a)$. Thus there is a neighborhood U_ϕ of ϕ in $C_0^1(\overline{\Omega})$ such that $v(t(\phi), \psi) \in \operatorname{int}(\tilde{P} \cup D \cup \tilde{I}_+^a)$ for all $\psi \in U_\phi$. Consider the covering $\{U_\phi \mid \phi \in C_0^1(\overline{\Omega})\}$ of $C_0^1(\overline{\Omega})$, we take a locally finite partition of unity $(\pi_j)_{j \in J}$ subordinating to $\{U_\phi\}$, then $\operatorname{supp}(\pi_j) \subset U_{\phi_j}$ for some ϕ_j. Let

$$T : C_0^1(\overline{\Omega}) \to [0, +\infty), \quad T(\phi) = \sum_j \pi_j(\phi) t(\phi_j)$$

and

$$\eta : [0, 1] \times C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega}), \quad \eta(s, \phi) = v\big(sT(\phi), \phi\big).$$

Since $\tilde{P} \cup D \cup \tilde{I}_+^a$ is positively invariant w.r.t. the heat flow, $\eta(s, \tilde{P} \cup D \cup \tilde{I}_+^a) \subset \tilde{P} \cup D \cup \tilde{I}_+^a$ for $s \in [0, 1]$. Moreover, $\eta(1, C_0^1(\overline{\Omega})) \subset \tilde{P} \cup D \cup \tilde{I}_+^a$. Now, let $\eta_1 = \eta|_{[0,1] \times (\tilde{P} \cup D \cup \tilde{I}_+^a)}$, then

$$\eta_1 : [0, 1] \times \big(\tilde{P} \cup D \cup \tilde{I}_+^a\big) \to \tilde{P} \cup D \cup \tilde{I}_+^a$$

is a homotopy between the identity and $\eta_1(1, \cdot)$. Hence

$$\eta_1(1, \cdot)_* : H_*\big(\tilde{P} \cup D \cup \tilde{I}_+^a\big) \to H_*\big(\tilde{P} \cup D \cup \tilde{I}_+^a\big)$$

is an isomorphism. This implies that

$$i_* : H_*\big(\tilde{P} \cup D \cup \tilde{I}_+^a\big) \to H_*\big(C_0^1(\overline{\Omega})\big)$$

is injective, and that

$$\eta_1(1, \cdot)_* : H_*\big(C_0^1(\overline{\Omega})\big) \to H_*\big(\tilde{P} \cup D \cup \tilde{I}_+^a\big)$$

is surjective, where $i : \tilde{P} \cup D \cup \tilde{I}_+^a \to C_0^1(\overline{\Omega})$ is the inclusion map. Therefore

$$H_*\big(\tilde{P} \cup D \cup \tilde{I}_+^a\big) = H_*\big(C_0^1(\overline{\Omega})\big) = 0, \quad * = 1, 2, \dots,$$
$$H_0\big(\tilde{P} \cup D \cup \tilde{I}_+^a\big) = G$$

since $H_0(\tilde{P} \cup D \cup \tilde{I}_+^a) \neq 0$ and $H_0(C_0^1(\overline{\Omega})) = G$. This proves that i_* is an isomorphism and

$$H_*\big(\tilde{P} \cup D \cup \tilde{I}_+^a\big) = H_*\big(C_0^1(\overline{\Omega})\big) = \delta_{*,0} G, \quad * = 1, 2, \dots. \tag{4.3}$$

Since both $\tilde{P}$ and D are closed and convex, $\tilde{P} \cap D$ is a strong deformation retract of D, i.e., there is a continuous map $r : [0, 1] \times D \to \tilde{P} \cap D$ such that

$$r(s, u) = u, \quad u \in \tilde{P} \cap D,$$
$$r(0, u) = u, \quad u \in D,$$
$$r(1, u) \in \tilde{P} \cap D, \quad u \in D.$$

It follows from $\min\{\tilde{I}(u) | u \in D\} > a$ that the map $\tilde{r} : [0, 1] \times (\tilde{P} \cup D \cup \tilde{I}_+^a) \to \tilde{P} \cup D \cup \tilde{I}_+^a$ given by

$$\tilde{r}(s, u) = \begin{cases} r(s, u) & \text{if } u \in D, \\ u & \text{if } u \notin D \end{cases}$$

is a strong deformation retract, which induces an isomorphism

$$i_* : H_*\big(\tilde{P} \cup \tilde{I}_+^a\big) \cong H_*\big(\tilde{P} \cup D \cup \tilde{I}_+^a\big).$$

By (4.3) we have

$$H_*\big(\tilde{P} \cup \tilde{I}_+^a\big) = \delta_{*,0} G, \quad * = 0, 1, 2, \dots, \tag{4.4}$$

and that $i_* : H_*(\tilde{P} \cup \tilde{I}_+^a) \to H_*(C_0^1(\overline{\Omega}))$ is also an isomorphism.

Consider the following commutative diagram:

$$\begin{array}{ccccccccccc}
H_l(\tilde{I}_+^a) & \longrightarrow & H_l(\tilde{P} \cup \tilde{I}_+^a) & \longrightarrow & H_l(\tilde{P} \cup \tilde{I}_+^a, \tilde{I}_+^a) & \longrightarrow & H_{l-1}(\tilde{I}_+^a) & \longrightarrow & H_{l-1}(\tilde{P} \cup \tilde{I}_+^a) & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H_l(\tilde{I}_+^a) & \longrightarrow & H_l\big(C_0^1(\overline{\Omega})\big) & \longrightarrow & H_l\big(C_0^1(\overline{\Omega}), \tilde{I}_+^a\big) & \longrightarrow & H_{l-1}(\tilde{I}_+^a) & \longrightarrow & H_{l-1}\big(C_0^1(\overline{\Omega})\big) & \longrightarrow & ,
\end{array}$$

all vertical maps $i_* : H_*(\tilde{I}_+^a) \to H_*(\tilde{I}_+^a)$ and $i_* : H_*(\tilde{P} \cup \tilde{I}_+^a) \to H_*(C_0^1(\overline{\Omega}))$, $* = 0, 1, \dots$, are induced by the inclusion, and are isomorphisms, the rows are exact, so by the Five Lemma, Theorem 2.9 and Remark 2.10, (4.2) holds.

(2) There exists $u_0 \in K$ such that $u_0 \notin \tilde{P} \cup (-\tilde{P}) \cup D$.

If $K \subset \tilde{P} \cup (-\tilde{P}) \cup D$, then by the same argument as above we have

$$H_*\big(C_0^1(\overline{\Omega}), \tilde{I}^a\big) \approx H_*\big(\tilde{P} \cup (-\tilde{P}) \cup D \cup \tilde{I}^a, \tilde{I}^a\big) = 0. \tag{4.5}$$

On the other hand, by $\pm\tilde{P} \cap \tilde{I}_\pm^a = \pm\tilde{P} \cap \tilde{I}_a$ and $\min\{\tilde{I}(u) | u \in D\} > a$, after excision we obtain

$$\begin{aligned} H_*\big(\tilde{P} \cup (-\tilde{P}) \cup D \cup \tilde{I}^a, \tilde{I}^a\big) &\approx H_*\big(\tilde{P} \cup (-\tilde{P}) \cup D, \tilde{I}^a \cap \big(\tilde{P} \cup (-\tilde{P})\big)\big) \\ &\approx H_*\big(\tilde{P} \cup (-\tilde{P}) \cup D, \big(\tilde{I}_+^a \cap \tilde{P}\big) \cup \big(-\tilde{P} \cap \tilde{I}_-^a\big)\big). \end{aligned} \tag{4.6}$$

We see from (4.1), (4.2) and the exact sequence of homology for the pairs $(\pm\tilde{P}, \pm\tilde{P} \cap \tilde{I}_\pm^a)$ that

$$H_*\big(\pm\tilde{P} \cap \tilde{I}_\pm^a\big) = \delta_{*,0} G, \quad * = 0, 1, 2, \dots, \tag{4.7}$$

since $\pm\tilde{P}$ is contractible. The set $\tilde{P} \cup (-\tilde{P}) \cup D$ is also contractible provided by the convexity of $\pm\tilde{P}$, D and $0 \in \tilde{P} \cap (-\tilde{P}) \cap D$. Applying the exact sequence to the pair $(\tilde{P} \cup (-\tilde{P}) \cup D, (\tilde{I}_+^a \cap \tilde{P}) \cup (-\tilde{P} \cap \tilde{I}_-^a))$ and (4.7) we get

$$\begin{aligned} H_*\big(\tilde{P} \cup (-\tilde{P}) \cup D \cup \tilde{I}^a, \tilde{I}^a\big) &\approx H_*\big(\tilde{P} \cup (-\tilde{P}) \cup D, \big(\tilde{I}_+^a \cap \tilde{P}\big) \cup \big(-\tilde{P} \cap \tilde{I}_-^a\big)\big) \\ &\approx \delta_{*,1} G, \quad * = 0, 1, 2, \dots, \end{aligned}$$

which contradicts with (4.5). The existence of $u_0 \in K$ such that $u_0 \notin \tilde{P} \cup (-\tilde{P}) \cup D$ follows.

(3) There exist $u_\pm \in \pm\tilde{P} \setminus D$.

It follows from the contractibility of $\tilde{P} \cup D$ and $\tilde{P}$, the exact sequence and (4.7) that

$$H_*\big(\tilde{P} \cup D, \tilde{P} \cap \tilde{I}_+^a\big) \cong H_*\big(\tilde{P}, \tilde{P} \cap \tilde{I}_+^a\big) = 0, \quad * = 0, 1, 2, \ldots. \tag{4.8}$$

Both $\tilde{P} \cup D$ and D are contractible and positively invariant sets of the heat flow, we have

$$C_*(\tilde{I}, K \cap D) = \delta_{*,0}G, \quad * = 0, 1, 2, \ldots. \tag{4.9}$$

Therefore, $K \cap (\tilde{P} \setminus D) \neq \emptyset$ by (4.8), (4.9) and the Morse relation, i.e., $\exists u_+ \in K \cap (\tilde{P} \setminus D)$. Similarly, $\exists u_- \in K \cap (-\tilde{P} \setminus D)$. □

Remark 4.4. One can show that both u_+ and u_- are mountain pass points of $\tilde{I}$ if the number of positive and negative solutions of (1.1) is finite.

Indeed, we have

$$\begin{aligned} H_*\big(\tilde{P} \cup (-\tilde{P}) \cup D, \big(\tilde{P} \cap \tilde{I}^a\big) \cup \big(-\tilde{P} \cap \tilde{I}^a\big)\big) &\cong H_*\big(\tilde{P} \cup (-\tilde{P}), \big(\tilde{P} \cup (-\tilde{P})\big) \setminus \{0\}\big) \\ &\cong \delta_{*,1}G, \quad * = 0, 1, 2, \ldots. \end{aligned} \tag{4.10}$$

From (4.10) and the Morse relation on the invariant set $\tilde{P} \cup (-\tilde{P}) \cup D$, we have

$$m_1 = \sum_{\pm} \operatorname{rank} C_1(\tilde{I}, u_\pm) = 2$$

and at least one of $\operatorname{rank} C_1(\tilde{I}, u_\pm) > 0$. Therefore

$$\operatorname{rank} C_1(\tilde{I}, u_\pm) = 1$$

and $u_\pm$ are mountain pass points of $\tilde{I}$, see [12].

This conclusion holds even without the finiteness assumption. It can be proved by the minimax argument in the invariant set $\tilde{P} \cup (-\tilde{P}) \cup D$, see the proof of Theorem 1.3 in [22].

5. Multiple solutions

Applying the previous results, now we turn to study the multiplicity of solutions of Eq. (1.1).

Combining Theorems 2.8 and 2.9 we have

Theorem 5.1. *Under the assumptions* (A1), (A2), (A3′), (A4) *and* (A5) $\lambda \notin \sigma(\Omega)$, *Eq.* (1.1) *has three nonzero solutions, among them, one is positive, one is negative if* $\lambda < \lambda_1$, *and has a nonzero solution if* $\lambda > \lambda_1$. *Moreover, if the number of positive and negative solutions is finite, then there is a sign-changing solution of* (1.1).

Proof. The proof has been given in [16] (Proof of Theorem 5.1) we repeat it here for the convenience.

(1) First we note from (A5) for all cases, $u = 0$ is an isolated critical point of $I(u)$ with

$$C_*(I, 0) = \delta_{*,i_0}G$$

where $i_0 = \sum_{\lambda_i < \lambda} \dim(-\Delta - \lambda_i)$. According to Theorem 2.8, $H_*(E, I^a) \cong \{0\}$ $* = 0, 1, 2, \ldots$, hence by the Morse relation, see [12], there must be a nonzero solution of (1.1). In case $\lambda < \lambda_1$, $i_0 = 0$ and 0 is a local minimizer. The existence of a positive (and a negative) solution follows from the mountain pass theorem for I_+ (and I_-, respectively). And the existence of the third solutions follows from the Morse relation and Theorem 2.8.

(2) Next, with the finiteness assumption of the positive and negative solutions, let $\{u_i^+\}_1^l$ and $\{u_j^-\}_1^m$ be the sets of positive and negative solutions, respectively. We assume that there is no sign-changing solution. Let

$$\chi_\pm\big(u^\pm\big) = \sum (-1)^k \operatorname{rank} C_k(I_\pm, u^\pm)$$

and

$$\chi\big(u^\pm\big) = \sum (-1)^k \operatorname{rank} C_k(I, u^\pm).$$

It is known that $\chi(u^{\pm}) = \chi_{\pm}(u^{\pm})$. According to the Morse relations for I_+, I_-, I and Theorems 2.8–2.9, we have

$$\chi_+(0) + \sum_1^l \chi_+(u_i^+) = 0, \tag{5.1}$$

$$\chi_-(0) + \sum_1^m \chi_-(u_j^-) = 0 \tag{5.2}$$

and

$$\chi(0) + \sum_1^l \chi(u_i^+) + \sum_1^m \chi(u_j^-) = 0. \tag{5.3}$$

It was proved in [16] that $\chi_+(0) = \chi_-(0)$. Thus from (5.1)–(5.3) we get

$$\chi(0) = \chi_-(0) + \chi_+(0) = 2\chi_-(0).$$

It is known from the previous paragraph (1) that $\chi(0)$ is odd, while the right-hand side of the above equality is even. This contradiction concludes the existence of a sign-changing solution of (1.1). □

In order to emphasize the role of the nonlinear terms $a_+|u|^{q-1}u$ and $a_-|u|^{p-1}u$ in the modelling equation, we assume further

$$\text{(A6)} \quad h(x,u) = \begin{cases} \mathrm{o}(a_{\pm}(x)|u|^{\alpha}) & \text{as } |u| \to 0 \quad \text{uniformly in } x \in \Omega_{\pm}, \\ 0, & x \in \Omega_0, \end{cases}$$

where $\alpha = \max\{p, q\}$. We study two cases:

(I) $p < q$ or $p = q$ with $\int_{\Omega} (a_+ - a_-)\phi_1^{p+1} dx < 0$, where ϕ_1 is the first eigenfunction $-\Delta$.
(II) $p > q$.

Case I. According to [4], $\exists \lambda_+ > \lambda_1$ such that (1.1) admits a positive solution for $\lambda \in (\lambda_1, \lambda_+)$, but no positive solutions for $\lambda > \lambda_+$. By the same reason, $\exists \lambda_- > \lambda_1$ such that (1.1) admits a negative solution for $\lambda \in (\lambda_1, \lambda_-)$, but no negative solutions for $\lambda > \lambda_-$.

Theorem 5.2. *Under the assumptions* (A1), (A2′), (A3′), (A4) *and* (A6) *in Case* (I).

(1) *If $\lambda_2 < \lambda < \lambda_{\pm}$ and $\lambda \notin \sigma(\Omega)$, then Eq.* (1.1) *has at least two positive, two negative and three sign-changing solutions.*
(2) *If $\lambda_1 < \lambda < \lambda_{\pm} < \lambda_2$, then Eq.* (1.1) *has at least two positive, two negative and one sign-changing solutions.*

Proof. We only prove the first statement. The second one can be proved similarly.

(i) By the assumption $\lambda_2 < \lambda < \lambda_{\pm}$, there are two pairs of sub- and super-solutions: $\underline{v}_1 < \bar{v}_1 < 0 < \underline{v}_2 < \bar{v}_2$, where $\bar{v}_1 = -\epsilon\phi_1$ and $\underline{v}_2 = \epsilon\phi_1$ for small ϵ. Applying Theorem 4.1, for small ϵ, we have a positive solution $u_+^{\epsilon} \in [\underline{v}_2, \bar{v}_2]$, a negative solution $u_-^{\epsilon} \in [\underline{v}_1, \bar{v}_1]$ and $u_0^{\epsilon} \in [\underline{v}_1, \bar{v}_2] \setminus ([\underline{v}_1, \bar{v}_1] \cup [\underline{v}_2, \bar{v}_2])$. Since $\epsilon > 0$ can be chosen arbitrary small, all solutions $0 \neq u \in [\underline{v}_1, \bar{v}_2] \setminus ([\underline{v}_1, \bar{v}_1] \cup [\underline{v}_2, \bar{v}_2])$ must be sign-changing. If the number of solutions of (1.1) in $[\underline{v}_1, \bar{v}_2]$ is finite, then we can assume $C_*(I, u_0^{\epsilon}) = \delta_{*,1}G$, by Theorem 4.1. But we know $C_1(I, 0) = 0$, hence $u_0^{\epsilon} \neq 0$, i.e., u_0^{ϵ} is sign-changing.

Moreover, by the assumption $\lambda \notin \sigma(\Omega)$, $\mathrm{ind}(I', 0) = \pm 1$ and we conclude that there exists one more sign-changing solution u_1 in $[\underline{v}_1, \bar{v}_2]$ by computing the degree on the invariant set $[\underline{v}_1, \bar{v}_2]$. Indeed, if u_0^{ϵ} is the only sign-changing solution of (1.1), then

$$\deg\bigl(I', [\underline{v}_1, \bar{v}_2], 0\bigr) = \deg\bigl(I', [\underline{v}_1, \bar{v}_1], 0\bigr) + \deg\bigl(I', [\underline{v}_2, \bar{v}_2], 0\bigr) + \mathrm{ind}(I', 0) + \mathrm{ind}\bigl(I', u_0^{\epsilon}\bigr).$$

This is impossible because

$$\deg\bigl(I', [\underline{v}_1, \bar{v}_2], 0\bigr) = \deg\bigl(I', [\underline{v}_1, \bar{v}_1], 0\bigr) = \deg\bigl(I', [\underline{v}_2, \bar{v}_2], 0\bigr) = 1$$

provided by the contractibility of $[\underline{v}_1, \bar{v}_2]$, $[\underline{v}_i, \bar{v}_i]$, $i = 1, 2$, and $\mathrm{ind}(I', u_0^\epsilon) = -1$.

In summary there are one positive, one negative and two sign-changing solutions in the interval $D = [\underline{v}_1, \bar{v}_2]$.

(ii) Now applying Theorem 4.3 to the set $[\underline{v}_1, \bar{v}_2]$, we conclude that, outside D, (1.1) has a positive, a negative and a sign-changing solutions.

Combing (i) and (ii), the proof is completed. $\square$

Remark 5.3. One can analyze various cases in more details and count the number of solutions by the previous method. The results extend those in [16], where only the case $\lambda < \lambda_1$ was studied, and in [1], where $p = q$ was assumed.

Case II. We add a parameter $\gamma > 0$ to Eq. (1.1) as follows:

$$-\Delta u = \lambda u + a_+(x)|u|^{q-1}u - \gamma a_-(x)|u|^{p-1}u + h(x,u) \tag*{$(1.1)_{\lambda,\gamma}$}$$

and let $I_{\lambda,\gamma}$ be the associated functional. We assume (A6). Following the argument in [5], $(1.1)_{\lambda,\gamma}$ possesses a positive super-solution $\bar{u}_\gamma$ as $\gamma > 0$ large and a positive sub-solution $\underline{u}_\gamma$, which can be arbitrarily small for $\lambda > \lambda_1 - \epsilon$, $\epsilon > 0$ small. Indeed, let w_γ be the unique solution of

$$\begin{cases} -\Delta w = \bar{\lambda} w - \gamma a_- w^p, & x \in \Omega, \\ w(x) > 0, & x \in \Omega, \\ w(x) = 0, & x \in \partial\Omega \end{cases}$$

where $\bar{\lambda} \in (\lambda, \lambda_1(\overline{\Omega_0 \cup \Omega_+}))$, then $\exists \gamma_1 = \gamma_1(\lambda)$ such that for $\gamma \geqslant \gamma_1$, $\bar{u}_\gamma = w_{\gamma_1}$ is a positive super-solution. And $\underline{u}_\gamma$ is the positive solution bifurcating from $(0, \lambda_1)$. By the assumption (A6) and $q < p$, the branch of solutions of $(\lambda(\alpha), u_\alpha)$ with $\lambda(0) = \lambda_1$ and $u_\alpha = \alpha\phi_1 + o(\alpha)$ is on the left side of $(0, \lambda_1)$ locally, i.e., $\lambda(\alpha) < \lambda_1$ as $\alpha > 0$ small. Moreover, by simple computation, see [5],

$$I_{\lambda,\gamma}(\underline{u}_\gamma) \leqslant I_{\lambda_1,\gamma}(\underline{u}_\gamma) < 0 \quad \text{as } \lambda \geqslant \lambda_1,$$

and

$$I_{\lambda,\gamma}(\underline{u}_\gamma) \leqslant I_{\lambda_1,\gamma}(\underline{u}_\gamma) + \epsilon\|\underline{u}_\gamma\|^2 \quad \text{as } \lambda \in [\lambda_1 - \epsilon, \lambda_1]$$

for $\epsilon > 0$ small. Therefore,

$$I_{\lambda,\gamma}(\underline{u}_\gamma) < 0 \quad \text{as } \epsilon > 0 \text{ small and } \lambda \geqslant \lambda_1 - \epsilon.$$

Define

$$\gamma_0(\lambda) = \inf\bigl\{\gamma > 0 \mid (1.1)_{\lambda,\gamma} \text{ admits a solution } u > 0 \text{ and } I_{\lambda,\gamma}(u) < 0\bigr\}.$$

Following the proof of [5] Lemma 5.4, we have $\gamma_0(\lambda) > 0$ such that for $\lambda \in [\lambda_1, \lambda_1(\overline{\Omega_0 \cup \Omega_+}))$, and that $(1.1)_{\lambda,\gamma_0(\lambda)}$ admits a positive solution.

Let

$$\gamma^* = \inf\bigl\{\gamma_0(\lambda) \mid \lambda > \lambda_1\bigr\} \geqslant \gamma_0(\lambda_1) > 0.$$

In [5], Alama and Tarantello defined, for $\gamma > \gamma^*$,

$$\lambda_+(\gamma) = \sup\bigl\{\lambda > 0 \mid (1.1)_{\lambda,\gamma} \text{ admits a positive solution}\bigr\}$$

and

$$\lambda_-(\gamma) = \inf\bigl\{\lambda \mid (1.1)_{\lambda,\gamma} \text{ admits a solution } u \text{ with } 0 < u < u_+(x) \text{ a.e.}\bigr\},$$

where u_+ is a positive solution of $(1.1)_{\lambda_+(\gamma),\gamma}$, and proved that: $-\infty < \lambda_-(\gamma) < \lambda_1 < \lambda_+(\gamma) < \lambda_1(\overline{\Omega_0 \cup \Omega_+})$ in Lemma 5.9. Moreover, there exists a pairs of positive sub- and super-solutions $(\underline{u}_\gamma, \bar{u}_\gamma)$ for $\lambda \in (\lambda_-(\gamma), \lambda_+(\gamma))$.

Now we use these facts to prove

Theorem 5.4. *Under the assumption* (A1), (A2′), (A3′), (A4) *and* (A6)*, to Eq.* $(1.1)_{\lambda,\gamma}$*, there exist* $\gamma^* > 0$ *and* $-\infty < \lambda_-(\gamma) < \lambda_1 < \lambda_+(\gamma)$ *such that* $\forall \gamma > \gamma^*$,

(1) *For* $0 < \lambda < \lambda_-(\gamma)$*, there exist at least one positive, one negative and one sign-changing solutions.*
(2) *For* $\lambda_-(\gamma) < \lambda < \lambda_1$*, there exist at least three positive, three negative and one sign-changing solutions.*
(3) *For* $\lambda_1 < \lambda < \lambda_+(\gamma)$*, there exist at least two positive, two negative and one sign-changing solutions.*
(4) *For* $\lambda_2 < \lambda < \lambda_+(\gamma)$*, there exist at least two positive, two negative and three sign-changing solutions.*

Proof. Case (1) follows directly from Theorem 5.1.

Case (2): Let $(\underline{u}_\gamma, \bar{u}_\gamma)$ (and $(\underline{v}_\gamma, \bar{v}_\gamma)$) be pairs of positive (negative) sub- and super-solutions of $(1.1)_{\lambda,\gamma}$, respectively. Besides, $(-\epsilon\phi_1, \epsilon\phi_1)$ is also a pair of sub- and super-solutions. According to Theorem 4.1, there exist positive solutions $u_\gamma^1 \in [\underline{u}_\gamma, \bar{u}_\gamma]$ and $u_\gamma^2 \in [0, \bar{u}_\gamma] \setminus [\underline{u}_\gamma, \bar{u}_\gamma]$, negative solutions $v_\gamma^1 \in [\underline{v}_\gamma, \bar{v}_\gamma]$ and $v_\gamma^2 \in [\underline{v}_\gamma, 0] \setminus [\underline{v}_\gamma, \bar{v}_\gamma]$. Moreover, we apply Theorem 4.3 to $D := [\underline{v}_\gamma, \bar{u}_\gamma]$, there exist one positive solution u_γ^3, one negative solution v_γ^3 and one sign-changing solution v, all are outside D.

Case (3): It is similar to Case (2), but there exist only one positive and one negative solutions inside D.

Case (4): Similar to the first paragraph of the proof of Theorem 5.2, there exist one positive, one negative and two sign-changing solutions in D. Again, there are one positive, one negative and one sign-changing solutions outside D. □

Remark 5.5. The remaining two cases are as follows:

Case (5): $\lambda = \lambda_-(\gamma)$, there are at least two positive, two negative and one sign-changing solutions, see [5], Theorems 4.1 and 4.2.

Case (6): $\lambda = \lambda_+(\gamma)$, there are at least one positive and one negative solutions, see [5].

Remark 5.6. We also have some information on the type of solutions from the point of view of critical points.

Case (1): Both positive and negative solutions are mountain pass points.

Case (2): Among the three positive (negative, respectively) solutions, one is a local minimizer, and two are mountain pass points.

Case (3): Among the two positive (negative, respectively) solutions, one is a local minimizer, the other is a mountain pass point.

Case (4): The same situation occurs as in Case (3). Moreover, at least one of the sign-changing solutions is a mountain pass point.

Remark 5.7. The positive solutions for Eq. $(1.1)_{\lambda,\gamma}$ have been obtained in [5].

Remark 5.8. As in [16], under the assumptions of theorems in this section, if we assume further that h is odd in u, then there is a sequence of solutions $\{u_k\}$ of (1.1) such that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$.

References

[1] N. Ackermann, T. Bartsch, P. Kaplicky, P. Quittner, A priori bounds, nodal equilibria and connecting orbits in indefinite superlinear parabolic problems, Trans. Amer. Math. Soc., in press.
[2] H. Amann, J. Lépez-Gómez, A priori bounds and multiple solutions for superlinear indefinite nonlinear elliptic equations, J. Differential Equations 146 (1998) 336–374.
[3] S. Alama, M. Del Pino, Solutions of elliptic equations with indefinite nonlinearities via Morse theory and linking, Ann. Inst. H. Poincaré Anal. Nonlinéaire 13 (1996) 95–115.
[4] S. Alama, G. Tarantello, On semilinear elliptic equations with indefinite nonlinearities, Calc. Var. 1 (1993) 439–475.
[5] S. Alama, G. Tarantello, Elliptic problems with nonlinearities indefinite in sign, J. Funct. Anal. 141 (1996) 159–215.
[6] T. Bartsch, Critical point theory on partially ordered Hilbert spaces, J. Funct. Anal. 186 (2001) 117–152.
[7] T. Bartsch, Z.-Q. Wang, On the existence of sign changing solutions for semilinear Dirichlet problem, Topol. Methods Nonlinear Anal. 7 (1996) 115–131.
[8] H. Berestycki, I. Capuzzo-Dolcetta, L. Nirenberg, Superlinear indefinite elliptic problem and nonlinear Liouville theorems, Topol. Methods Nonlinear Anal. 4 (1994) 59–78.
[9] H. Berestycki, I. Capuzzo-Dolcetta, L. Nirenberg, Variational methods for indefinite superlinear homogeneous elliptic problems, NoDEA 2 (1995) 533–572.
[10] T. Cazenave, A. Haraux, An Introduction to Semilinear Evolution Equations, Clarendon Press, Oxford, 1998.
[11] T. Cazenave, P.L. Lions, Solutions globales d'équations de la chaleur semi linéaires, Comm. Partial Differential Equations 9 (1984) 955–978.
[12] K.C. Chang, Infinite Dimensional Morse Theory and Multiple Solution Problems, Birkhäuser, Boston, 1993.

[13] K.C. Chang, Morse theory in nonlinear analysis, in: A. Ambrosetti, K.C. Chang, I. Ekeland (Eds.), Nonlinear Functional Analysis and Applications to Differential Equations, World Scientific, 1998, pp. 60–101.
[14] K.C. Chang, Heat methods in nonlinear elliptic equations, in: H. Brézis, K.C. Chang, S.J. Li, P.H. Rabinowitz (Eds.), Topological and Variational Methods and Their Applications, World Scientific, 2003, pp. 65–76.
[15] K.C. Chang, N. Ghoussoub, The Conley index and the critical groups via an extension of Gromoll–Meyer theory, Topol. Methods Nonlinear Anal. 7 (1996) 77–93.
[16] K.C. Chang, M.Y. Jiang, Dirichlet problems with indefinite nonlinearities, Cacl. Var. 20 (2004) 257–282.
[17] W. Chen, C. Li, Indefinite elliptic problems in a domain, Discrete Contin. Dynam. Systems 3 (1997) 333–340.
[18] Y.H. Du, S.J. Li, Nonlinear Liouville theorems and a priori estimates for indefinite superlinear elliptic equations, Adv. Differential Equations 10 (2005) 841–860.
[19] Y. Giga, A bound for global solutions of semilinear heat equations, Comm. Math. Phys. 103 (1986) 415–421.
[20] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, 2001.
[21] M. Grossi, P. Magrone, M. Matzeau, Linking type solutions for elliptic equations with indefinite nonlinearities, Discrete and Contin. Dynam. Systems 7 (2001) 703–718.
[22] S.-J. Li, Z.-Q. Wang, Mountain pass theorem in order interval and multiple solutions for semilinear elliptic Dirichlet problems, J. Anal. Math. 81 (2000) 373–396.
[23] G.M. Lieberman, Second Order Parabolic Differential Equations, World Scientific, 1998.
[24] P. Magrone, On a class of semilinear elliptic equations with potential changing sign, Dynam. Systems Appl. 9 (2000) 459–467.
[25] R. Palais, Homotopy theory of infinite dimensional manifolds, Topology 5 (1966) 115–132.
[26] M. Ramos, S. Terracini, C. Troestler, Superlinear indefinite elliptic problems and Pohozaev type identities, J. Funct. Anal. 159 (1998) 596–628.
[27] P. Quittner, A priori bounds for global solutions of a semilinear parabolic problem, Acta Math. Univ. Comenian (N.S.) 68 (1999) 195–203.