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Characteristic vector fields of generic distributions of corank 2

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Abstract

We study generic distributions $D \subset TM$ of corank 2 on manifolds M of dimension $n \geqslant 5$. We describe singular curves of such distributions, also called abnormal curves. For n even the singular directions (tangent to singular curves) are discrete lines in $D(x)$, while for n odd they form a Veronese curve in a projectivized subspace of $D(x)$, at generic $x \in M$. We show that singular curves of a generic distribution determine the distribution on the subset of M where they generate at least two different directions. In particular, this happens on the whole of M if n is odd. The distribution is determined by characteristic vector fields and their Lie brackets of appropriate order. We characterize pairs of vector fields which can appear as characteristic vector fields of a generic corank 2 distribution, when n is even.

Keywords: Tangent distributions; Generic distributions; Singular curves; Abnormal curves; Characteristic vector fields

1. Introduction

Let $M = M^n$ denote a smooth, paracompact differential manifold of dimension $n \geqslant 5$. Consider a smooth distribution D of rank $m = n - 2$ understood as a subbundle of TM of rank m spanned, locally, by m smooth linearly independent vector fields. Equivalently, we can write locally

$$D = \ker \omega_1 \cap \ker \omega_2,$$

where ω_1, ω_2 are C^∞ differential 1-forms on M, called later *cogenerators* of D.

Let $I = [0, 1] \subset \mathbb{R}$. Recall that horizontal curves of D on I are curves $\gamma : I \to M$ almost everywhere tangent to D. Let $\Omega(x_0)$ denote the set of absolutely continuous horizontal curves on I, with locally square integrable derivative, satisfying $\gamma(0) = x_0$. This set can be endowed with a structure of a Hilbert manifold. The endpoint map $\mathrm{End} : \Omega(x_0) \to M$ defined by

$$\mathrm{End}(\gamma) = \gamma(1)$$

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is differentiable (cf. [3], Ch. 1). Its singular points, i.e. curves $\gamma \in \Omega(x_0)$ such that the tangent map $D\,\mathrm{End}(\gamma)$ is not onto $T_{\gamma(1)}M$, are called *singular curves* of D, cf. [4,1,11]. Such curves can be rigid as described in [4,2]. Recent results of Chitour, Jean, and Trélat [5] show that for generic distributions such curves are of minimal order and of corank 1, but they cannot be rigid if the rank of D is larger then 2.

In sub-Riemannian geometry such curves, called there *abnormal*, attracted special attention after the discovery that they can be locally minimizing [10,9]. They are minimizing for generic distributions D of rank 2 [9], but they cannot be minimizing for generic D of rank larger then 2 [5].

In this paper we are interested in singular curves of corank 2 distributions and in establishing when such curves determine the distribution, cf. [11,8]. We say that *singular curves determine a distribution D on an open subset $U \subset M$* if for any other $\tilde{D}$ with the same singular curves on U we have $D|_U = \tilde{D}|_U$.

Given D, we denote by $\mathrm{Sing}_D(x)$ the set of smooth singular curves of D starting from x and $\mathrm{Sing}_D := \bigcup_{x\in M} \mathrm{Sing}_D(x)$. The *cone of singular vectors* $S(x) \subset D(x)$ is defined as

$$S(x) := \{ v \in T_xM\colon\ v = \dot{\gamma}(0),\ \gamma \in \mathrm{Sing}_D(x) \}.$$

The elements of the projectivization of $S(x)$ in the projective space $P(D(x))$ will be called *singular directions*.

Denote by $\mathcal{D}^m(M^n)$ the set of smooth distributions of rank m on M^n. Recall that a subset is called residual if it is a countable intersection of open and dense subsets. Our first main result says the following.

Theorem 1. *If $M = M^n$, $n \geqslant 5$, and M admits a distribution of corank 2 then there exists a subset $\mathcal{G} \subset \mathcal{D}^m(M^{m+2})$, which is residual (and therefore dense) in the Whitney C^∞-topology, such that for any $D \in \mathcal{G}$ the singular curves determine D in the region R of points $x \in M$ where the cone $S(x)$ has at least two different singular directions (if m is odd then $R = M$). Additionally, given a $D \in \mathcal{G}$ and a generic point $x \in M$, the set of singular directions consists of at most k points in $P(D(x))$, if $m = 2k$, and it is a Veronese curve in the projectivization of a subspace $D_{char}(x) \subset D(x)$ with* $\dim D_{char}(x) = k + 1$, *if $m = 2k + 1$.*

Remark 1. Theorem 1 follows from Theorems 2 and 3. The assumption that M admits a corank 2 distribution can be omitted if $\mathcal{D}^m(M^{m+2})$ is replaced by the class $\mathcal{D}^m_{sing}(M^{m+2})$ of singular distributions understood as sub-sheaves of $\Gamma(TM)$ generated locally by m smooth vector fields, linearly independent almost everywhere. The result also holds if, instead of such singular distributions, we consider singular co-distributions $D^* \subset T^*M$, locally generated by two 1-forms, linearly independent a.e. We recall that a Veronese curve in a projective space $P(V)$, $\dim V = r + 1$, is the curve obtained by projectivization of the image of the map $\mathbb{R}^2 \setminus \{0\} \ni (t_1, t_2) \mapsto \sum_{i=0}^r t_1^i t_2^{r-i} v_i$, where $v_0, \dots, v_r$ is a basis in V.

Early results in this direction concerned distributions of corank 1. Namely, the results in [17] implied that in many cases singular curves determine a corank 1 distribution D locally, up to a diffeomorphism. In [7] it was shown that for corank 1 distributions violating the Darboux condition on a nowhere dense subset of M the singular curves "almost always" determine the distribution (up to a diffeom.), if M is compact and distributions are close to each other. For distributions of corank $\geqslant 3$ a stronger property was proved by R. Montgomery [11] in the case of $\mathrm{rank}\, D$ odd: *the singular curves determine the distribution, if it is generic.* In [11] the same property was conjectured in the case of $\mathrm{rank}\, D$ even. This was proved in [8] in the case where $\mathrm{rank}\, D$ is not divisible by 4.

The case of $\mathrm{rank}\, D$ divisible by 4, $\mathrm{corank}\, D \geqslant 3$, is not settled yet. This case is degenerated as the class $\mathcal{D}^{4s}(M^n)$ may contain fat distributions [13] where the set of singular curves is empty. Note that the class of corank 1 distributions contains contact distributions ($\mathrm{rank}\, D$ even), where the set of singular curves is also empty. In this class, as well as for quasi-contact distributions ($\mathrm{rank}\, D$ odd), the Darboux theorem says that such distributions are all locally equivalent. This phenomenon can not hold in the case of distributions of corank D larger then 1. Namely, if $2 \leqslant \mathrm{rank}\, D \leqslant \dim M - 2$ and $(\mathrm{rank}\, D, \dim M) \neq (2, 4)$ then there must be infinite dimensional (functional) invariants, see [6], page 21, or [16]. For general results concerning singularities of corank 2 distributions see [12].

In the paper we introduce the notions of characteristic and horizontal characteristic vector fields X of D. We prove (Theorems 2 and 3) that the singular curves of D are integral curves of (horizontal) characteristic vector fields X and the (horizontal) characteristic vector fields, together with their Lie brackets, span the distribution in the generic case. This implies that (horizontal) characteristic vector fields contain all geometric information on D. Throughout the paper we assume that M admits a distribution of corank 2.

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2. Preliminaries

Consider a smooth distribution $D \subset TM$ of arbitrary rank. Let $D^\perp \subset T^*M$ denote the annihilator of D,

$$D^\perp(x) = \{p \in T_x^*M:\ pD(x) = 0\}.$$

By $D^\perp \setminus \{0\}$ we denote the annihilator of D with the zero section removed. We shall use

Fact 1. *If ω is a section of $D^\perp$ and X, Y are sections of D, then*

$$d\omega(X,Y) = -\omega([X,Y])$$

and both sides, evaluated at x, depend on the values of ω, X and Y at x, only.

This is a special case of the formula $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$. Below we use the usual notation $X \rfloor \eta = \eta(X,\ldots)$ for the interior product of a vector field X with a differential form η.

Proposition 1. *For any pair (ω, X), where ω is a section of $D^\perp \setminus \{0\}$ and X is a vector field in D such that $X \rfloor d\omega = 0$, the integral curves of X are singular curves of D.*

Proof. It follows from the invariant form of the Pontriagin Maximum Principle that singular curves $x(t)$ of D are projections to M of curves $\lambda(t) = (x(t), p(t))$ in $D^\perp \setminus \{0\}$ which satisfy the adjoint equation

$$\frac{d}{dt}\big(p(t)Y(x(t))\big) - p(t)\big([\dot{x}(t), Y]\big) = 0, \tag{AE}$$

for any smooth vector field Y on M, see e.g. Sussmann [14], page 540, Theorem 14.1, statement [II]. Above, given the vector field Y and $p \in D^\perp(x)$, $v \in D(x)$, the expression $p[Y, v]$ should be understood as the evaluation at x of the tensor field $\Gamma(D^\perp) \times \Gamma(D) \to C^\infty(M)$ defined by $(\omega, X) \mapsto \omega([X,Y]) = d\omega(Y,X) + X(\omega(Y))$, where $\omega([X,Y])(x) =: p[Y,v]$ depends on the values $\omega(x) = p$ and $X(x) = v$, only.

Consider smooth sections $\omega \in \Gamma(D^\perp \setminus \{0\})$ and $X \in \Gamma(D)$ such that $X \rfloor d\omega = 0$. Let γ, $t \mapsto x(t)$, be an integral curve of X, i.e. $\dot{x}(t) = X(x(t))$. Denote $p(t) = \omega(x(t))$. Then for arbitrary vector field Y on M we have

$$\begin{aligned} 0 &= d\omega(X,Y)\big(x(t)\big) \\ &= \big(X(\omega(Y)) - \omega([X,Y])\big)\big(x(t)\big) \\ &= \frac{d}{dt}\big(p(t)Y(x(t))\big) - p(t)\big([\dot{x}(t), Y]\big). \end{aligned} \tag{$\star$}$$

This means that γ satisfies (AE), thus γ is a singular curve. □

The converse statement is also true, for generic D of corank 2, around generic points (statement (i) in Theorems 2 and 3). If no genericity assumptions are made, some singular curves may not be integral curves of a vector field X as in Proposition 1 (see Example 1).

Fact 2. *If Ω is a local volume form and $\omega, \omega_1, \ldots, \omega_r$ are 1-forms on M^n, with $n - r - 1 = 2\ell$, then the vector field X given by*

$$X \rfloor \Omega = \omega_1 \wedge \cdots \wedge \omega_r \wedge (d\omega)^\ell \tag{1}$$

satisfies

$$X \rfloor \omega_i = 0, \quad i = 1, \ldots, r, \tag{2}$$

and

$$X \rfloor (d\omega|_{\ker\omega_1 \cap \cdots \cap \ker\omega_r}) = 0. \tag{3}$$

Vice versa, if (2) *and* (3) *hold at x and $(\omega_1 \wedge \cdots \wedge \omega_r \wedge (d\omega)^\ell)(x) \neq 0$, then* (1) *holds at x, up to a nonzero factor.*

Proof. Denote $\eta = \omega_1 \wedge \cdots \wedge \omega_r \wedge (d\omega)^\ell$ and assume $X(p) \neq 0$ (otherwise (2) and (3) are trivial). Then $\eta = \omega_i \wedge \hat{\eta}_i$, with some $\hat{\eta}_i$, and $\ker \eta(p) \subset \ker \omega_i(p)$. Thus (2) follows from $X \rfloor \eta = 0$, implied by (1). To prove the remaining part we fix a point $p \in M$. Then $\omega_j(p)$ are linearly independent and we can choose local coordinates $z^0, z^1, \ldots, z^r$, $x^1, \ldots, x^\ell, y^1, \ldots, y^\ell$ such that at the point p we have $\omega_1(p) = dz^1, \ldots, \omega_r(p) = dz^r$ and $d\omega|_D(p) = \sum_j dx^j \wedge dy^j$, where $D(p) = \bigcap_j \ker \omega_j(p)$ (use the Darboux algebraic lemma for $(d\omega|_D)(p)$). In such coordinates $X(p) = \partial/\partial z^0$, up to scalar factor, and checking Fact 2 is straightforward. □

3. Characteristic vector fields: rank $D = 2k + 1$

We begin our analysis with distributions $D \subset TM$ of odd rank $m = 2k + 1$. Consider local cogenerators ω_1, ω_2 of D which are linearly independent 1-forms, sections of $D^\perp$. Let ω be arbitrary smooth local section of $D^\perp$. We define a *characteristic vector field* $X = X_\omega$ of D, corresponding to ω, by the equality

$$X_\omega \rfloor \Omega = \omega_1 \wedge \omega_2 \wedge (d\omega)^k, \tag{CVF}$$

where Ω is a local volume form on $M = M^{2k+3}$ and on the right-hand side we have $(n-1)$-differential form. Basic properties of X_ω are listed in Proposition 2, in particular we have $X_\omega(x) \in D(x)$.

A nonvanishing section ω of $D^\perp$ is called *horizontal* if $X_\omega \rfloor d\omega = 0$. A characteristic vector field X_ω is called *horizontal* if it is defined by a horizontal section ω of $D^\perp$. Note that the horizontal characteristic pair (ω, X_ω) satisfies the assumption of Proposition 1, i.e., $X_\omega \rfloor d\omega = 0$. Thus the integral curves of X_ω are singular curves of D.

We denote by $C_{char}(x)$ (respectively, $C_{hor}(x)$) the set of characteristic vectors (respectively, horizontal characteristic vectors), which consists of vectors $X_\omega(x)$, where X_ω are characteristic (respectively, horizontal characteristic) vector fields of D.

Define local vector fields $Y_0, \ldots, Y_k$ via

$$Y_j \rfloor \Omega = \omega_1 \wedge \omega_2 \wedge (d\omega_1)^j \wedge (d\omega_2)^{k-j}. \tag{VF}$$

We will later show that $Y_j(x) \in D(x)$ (Proposition 2). Define a sub-distribution of D,

$$D_{char}(x) = \operatorname{span}\{Y_0(x), \ldots, Y_k(x)\}.$$

We recall that $\mathcal{D}^m(M^n)$ denotes the set of smooth distributions of rank m, on a smooth manifold M^n, and M^n is assumed to admit at least one such distribution.

Theorem 2. *There exists a subset $\mathcal{G} \subset \mathcal{D}^{2k+1}(M^{2k+3})$, which is residual (and therefore dense) in the Whitney C^∞-topology, such that for any $D \in \mathcal{G}$ the following conditions hold.*

(i) *At generic points in M the singular curves of D are exactly integral curves of horizontal characteristic vector fields X_ω, up to parametrization. At such points*

$$S(x) = C_{char}(x) = C_{hor}(x),$$

where $S(x)$ is the cone of singular vectors at x.

(ii) *At a generic point $x \in M$ the sub-distribution $D_{char}(x) \subset D(x)$ is of rank $k+1$, the cone $C_{char}(x) \subset D(x)$ of characteristic vectors linearly spans $D_{char}(x)$ and the projectivization of $C_{char}(x)$ is a Veronese curve in $P(D_{char}(x))$.*

(iii) *The characteristic vector fields of D, together with their first Lie brackets, span D at a generic point. The same holds for horizontal characteristic vector fields.*

(iv) *For any local cogenerators ω_1, ω_2 of D the following conditions hold at a generic point in M:*

$$Y_0(x), Y_1(x), \ldots, Y_k(x) \text{ are linearly independent}, \tag{G$'$}$$

$$\operatorname{span}_{r,s,t=0,\ldots,k}\{Y_r(x), [Y_s, Y_t](x)\} = D(x). \tag{G$''$}$$

Above and further on we say that a property holds at a generic point in M if it holds on an open, dense subset in M. $[X, Y]$ denotes the Lie bracket of vector fields X, Y.

Remark 2. (a) Statements (i), (ii), and (iii) imply Theorem 1, if $m = 2k + 1$.

(b) The sets of generic points in (i) and (ii) are the sets where (G′) holds, while the set of generic points in (iii) is given by both (G′) and (G″).

(c) The subset $\mathcal{G} \subset \mathcal{D}^{2k+1}(M^{2k+3})$ will be defined as the set of distributions whose 2-jet maps are transversal (in the sense of Thom transversality theorem) to the subset of 2-jets not satisfying the genericity conditions (G′) or (G″).

Remark 3. Statement (ii) follows from the relations between vector fields in (CVF) and (VF). Namely, let ω_1, ω_2 be local cogenerators of D. Consider an arbitrary section

$$\omega = a_1\omega_1 + a_2\omega_2$$

of $D^\perp$. Then $d\omega = a_1 d\omega_2 + a_2 d\omega_2 + da_1 \wedge \omega_1 + da_2 \wedge \omega_2$ and we see that the $(n-1)$-form

$$\omega_1 \wedge \omega_2 \wedge (d\omega)^k = \omega_1 \wedge \omega_2 \wedge (a_1 d\omega_1 + a_2 d\omega_2)^k$$

depends polynomially on the vector $a = (a_1, a_2)$. Thus (CVF) gives

$$X_\omega = \sum_{j=0}^{k} \binom{k}{j} a_1^j a_2^{k-j} Y_j, \tag{CVF′}$$

where Y_j were defined in (VF). Therefore the map $\omega \longmapsto X_\omega$ can be treated, at a fixed $x \in M$, as a homogeneous, degree k polynomial map $D^\perp(x) \to D(x)$. After [11], this map will be called *singular exp* and denoted

$$\mathrm{Sexp}_x(p) := X_\omega(x),$$

where ω is a local section of $D^\perp$ such that $\omega(x) = p$. This map defines the projectivized map $\mathrm{PSexp}_x : P(D^\perp(x)) \to P(D(x))$. When $Y_0(x), \ldots, Y_k(x)$ are linearly independent, the image of such a map is called a *Veronese curve*.

Note that $\mathrm{span}_{p\in D^\perp(x)} \mathrm{Sexp}_x(p) = D_{char}(x)$, which follows directly from the definitions.

Statement (i) of Theorem 2 implies that the singular cone $S(x)$ coincides with the cone C_{char} of characteristic vectors $X_\omega(x)$. This, together with Remark 3, implies

Corollary 1. *At points where* (G′) *is satisfied the singular cone $S(x)$ is an algebraic cone in $D_{char}(x)$ given by the Veronese curve $(a_1, a_2) \to X_\omega(x)$ in $P(D_{char}(x))$, defined by* (CVF′).

From the definitions (CVF) and (VF) it is easy to observe the following

Proposition 2. *Characteristic vector fields $X = X_\omega$ and the vector fields $Y_0, \ldots, Y_k$ defined via* (VF) *have the following properties in the domain of their definition*:

$$X(x) \in D(x), \tag{P1}$$

$$X_\omega(x) \in \ker d\omega|_{D(x)}, \tag{P2}$$

$$[X, Y](x) \in D(x), \tag{P3}$$

$$Y_j(x) \in D(x), \tag{P4}$$

$$[Y_i, Y_j](x) \in D(x), \tag{P5}$$

$$\mathrm{span}_{j=0,\ldots,k}\{Y_j(x)\} = \mathrm{span}_{p\in D^\perp(x)} \mathrm{Sexp}_x(p), \tag{P6}$$

where (P3) *holds for any other characteristic vector field Y and* (P4), (P5) *hold for all $i, j = 0, \ldots, k$.*

Proof. Let ω_1, ω_2 be cogenerators of D. Properties (P1) and (P4) follow from $X \rfloor \omega_i = 0$ and from $Y_j \rfloor \omega_i$, $i = 1, 2$, which are consequences of definitions (CVF) and (CV) and Fact 2. Similarly, (P2) follows from Fact 2.

(P3) is implied by (P1), (P2). Namely, for $X = X_\omega$, $Y = X_{\tilde\omega}$ and $i = 1, 2$ we have $0 = d\omega_i(X, Y) = -\omega_i([X, Y])$, thus $[X, Y] \in D$ (the first equality follows from (P2) and in the second we use Fact 1 and $\omega_i(X) = \omega_i(Y) = 0$, i.e. (P1)).

Property (P6) is a consequence of the formula (CVF′). Namely, from the definition $\mathrm{Sexp}_x(p) = X_\omega(x)$, where $\omega(x) = p$, and from the fact that the coefficients $a_1(x), a_2(x) \in \mathbb{R}$ in (CVF') are arbitrary we see that (P6) holds.

Finally, property (P5) can be shown using (P3) and (P6) in the following way. Note that $\mathrm{span}_{p\in D^\perp(x)} \mathrm{Sexp}_x(p) = D_{char}(x)$ is the "distribution" spanned by the characteristic vector fields (its rank may vary). Let $U \subset M$ be the open set where this rank is maximal. Then, locally on U, we have $Y_i = \sum_s \varphi_{is} X_s$, $Y_j = \sum_s \psi_{js} X_s$, where X_s are characteristic vectors fields that span D_{char} and φ_{is}, ψ_{js} are functions. It follows from (P3) and (P1) that $[Y_i, Y_j](x)$ is in $D(x)$ on U. If U is dense in M, we get (P5) on M, by continuity. Otherwise, we get (P5) on the closure $\mathrm{cl}\, U$, only. Consider the open set $V = M \setminus \mathrm{cl}\, U$ and the subset $U_1 \subset V$ where D_{char} is of maximal rank on V (smaller then the rank on U). Repeating the above argument gives that $[Y_i, Y_j](x)$ belongs to $D(x)$ in $\mathrm{cl}\, U_1$. After a finite number of such steps we get that $[Y_i, Y_j](x)$ is in $D(x)$ for any $x \in M$. □

Example 1. Let (x, y, z, u, w) be coordinates on $\mathbb{R}^5$. Consider

$$\omega_1 = dw - x\,du,$$
$$\omega_2 = dx + 2(y\,dz - z\,dy) + (y^2 + z^2)^2\,du.$$

Then $k = 1$ and the singular exp is a linear map. For $\Omega = dx \wedge dy \wedge dz \wedge du \wedge dw$ we compute

$$Y_0 = 2(y\partial_y + z\partial_z),$$
$$Y_1 = 4\big(\partial_u + x\partial_w + (y^2 + z^2)(z\partial_y - y\partial_z) + 2(y^2 + z^2)^2\partial_x\big).$$

We see that (G′) holds everywhere outside $S = \{y = z = 0\}$. Therefore, on the set $\mathbb{R}^5 \setminus S$, all characteristic vector fields can be written as $f_0Y_0 + f_1Y_1$ for certain functions f_0, f_1. By Theorem 2 the singular curves on $\mathbb{R}^5 \setminus S$ are exactly integral curves of characteristic horizontal vector fields. On the other hand, the curve $\gamma : t \mapsto (0, t, 0, 0, 0)$ is singular since it satisfies the adjoint equation (AE) in Section 2, with $p(t) = \omega_1(0, t, 0, 0, 0)$ (note that for $X = \partial_y$ and $\omega = \omega_1$ the calculation ($\star$) in the proof of Proposition 1 is valid). The tangent vector to γ at $0 \in \mathbb{R}^5$ is ∂_y. It cannot be extended to a field of the form $f_0Y_0 + f_1Y_1$. In particular there is no characteristic horizontal vector field such that γ is its integral curve.

For proving Theorem 2 we need

Lemma 1. *The subset $\mathcal{G}_2(x)$ of 2-jets at x of corank 2 distributions on M^{2k+3}, defined by the conditions* (G′) *and* (G″)*, is open and dense in the space of all 2-jets.*

Proof. It follows from the definition of $\mathcal{G}_2(x)$ that so defined set is open and its complement is a real algebraic subset in the space of 2-jets. If we show that $\mathcal{G}_2(x)$ is nonempty, it will follow that it is dense. We can take $M = \mathbb{R}^{2k+3}$ and $x = 0$.

Let $\mathbb{R}^{2k+3}$ be endowed with linear coordinates $p_1, \dots, p_{k+1}, q_1, \dots, q_k, z_1, z_2$. Consider two differential 1-forms on M

$$\omega_1 = dz_1 + \sum_1^k p_i\,dq_i,$$
$$\omega_2 = dz_2 + \sum_1^k q_i\,dp_{i+1} + \sum_1^k p_{k+1}p_i\,dp_{i+1},$$

and the corresponding distribution $D(x) = \ker\omega_1(x) \cap \ker\omega_2(x)$. Then

$$d\omega_1 = \sum_1^k dp_i \wedge dq_i,$$
$$d\omega_2 = \sum_1^k dq_i \wedge dp_{i+1} + \sum_1^k p_{k+1}\,dp_i \wedge dp_{i+1} - \sum_1^{k-1} p_i\,dp_{i+1} \wedge dp_{k+1}.$$

We then compute

$$(d\omega_1)^k = k!\prod_{i=1}^{k}(dp_i \wedge dq_i),$$

$$(d\omega_1)^j \wedge (d\omega_2)^{k-j} = a\prod_{i=1}^{j}(dp_i \wedge dq_i) \wedge \prod_{i=j+1}^{k}(dq_i \wedge dp_{i+1})$$
$$+ ap_{k+1}\prod_{i=1}^{j}(dp_i \wedge dq_i) \wedge dp_{j+1} \wedge dp_{j+2} \wedge \prod_{i=j+2}^{k}(dq_i \wedge dp_{i+1}) + \sum_{i=1}^{k-1} p_i \eta_i,$$

for $j = 0, \dots, k-1$, where $a = j!(k-j)!$ and η_i are some $(n-1)$-forms. Consider the volume form $\Omega = \prod_{i=1}^{k}(dp_i \wedge dq_i) \wedge dp_{k+1} \wedge dz_1 \wedge dz_2$ and the vector fields Y_j defined by

$$Y_j \rfloor \Omega = a\omega_1 \wedge \omega_2 \wedge (d\omega_1)^j \wedge (d\omega_2)^{k-j}.$$

Then

$$Y_k = \partial_{p_{k+1}},$$

$$Y_j = \partial_{p_{j+1}} + p_{k+1}\partial_{q_{j+1}} + \sum_{i=1}^{k-1} p_i Z_{ji},$$

for $j = 0, \dots, k-1$, where Z_{ji} are some vector fields. The Lie brackets at $0 \in \mathbb{R}^n$ of Y_k and Y_j are

$$[Y_k, Y_j](0) = \partial_{q_{j+1}},$$

for $j = 0, \dots, k-1$, and, together with $Y_0, \dots, Y_k$, span the distribution D at 0. The proof is complete. $\square$

Proof of Theorem 2. We define the exceptional subset $\mathcal{E} = \mathcal{E}' \cup \mathcal{E}''$ in the space of 2-jets on M of smooth distributions in $\mathcal{D}^{2k+1}(M^{2k+3})$, where $\mathcal{E}'$ consists of 2-jets which do not satisfy the genericity condition (G′) and $\mathcal{E}''$ consists of 2-jets that do not satisfy the condition (G″). Both these subsets are real algebraic subvarieties defined by a set of polynomial equations in the space of 2-jets, in a given coordinate system. These equations are expressed in terms of minors of the matrix of coefficients of the vector fields $Y_0, \dots, Y_k$, and $[Y_i, Y_j]$, $i, j = 0, \dots, k$. (Note that these vector fields are in D, by (P4) and (P5) in Proposition 2.) Moreover, these sets have empty interior as their complement contains the distribution germ constructed in Lemma 1. The space $\mathcal{G}$ of generic distributions is defined as consisting of those distributions which satisfy the Thom transversality theorem with respect to the stratified submanifolds $\mathcal{E}'$ and $\mathcal{E}''$. In particular, such distributions satisfy conditions (G′) and (G″) on open dense subsets in M. This statement implies condition (iv) of the theorem. Using (iv) we will deduce the other statements. Namely, statement (ii) follows from (G′) and property (P6) in Proposition 2 (note that (P6) can be written as $D_{char}(x) = \operatorname{span}\{Y_0(x), \dots, Y_k(x)\}$, where D_{char} is the distribution spanned by the characteristic vector fields). Condition (iii) follows from (G″) and (P6). Namely, using $D_{char} = \operatorname{span}\{Y_0, \dots, Y_k\}$ we see that (G″) implies $D_{char} + [D_{char}, D_{char}] = D$ at generic points.

In order to finish the proof we show that condition (i) is satisfied on the open subset U of M where condition (G′) holds. Assume that ω is a horizontal section of $D^\perp \setminus \{0\}$ and X_ω is the corresponding horizontal vector field. Then $X_\omega \rfloor d\omega = 0$ and we may use Proposition 1. Thus every integral curve of X_ω is a singular curve of D.

Vice versa, assume that γ, $t \mapsto x(t)$, is a singular curve. It follows from the Pontriagin Maximum Principle that for such a singular curve there exists a nonvanishing section $p(t)$ of $D^\perp$ along γ, which satisfies the adjoint equation (AE) in the proof of Proposition 1. Then (AE) implies $p(t)[\dot{x}(t), Y] = 0$, for any section Y of D. It follows that the vector $\dot{x}(t)$ lies in the kernel of the partial tensor field $\Gamma(D) \times \Gamma(D) \to C^\infty(M)$ defined by $(X, Y) \mapsto \omega([X, Y]) = d\omega(Y, X)$, where ω is any section of $D^\perp$ such that $\omega(x(t)) = p(t)$ (cf. Fact 1). This means $\dot{x}(t)\rfloor(d\omega|_{D(x(t))}) = 0$. It follows from Fact 2 that if $\operatorname{rank} d\omega|_{D(x)} = 2k = m-1$, the vectors v in $\ker d\omega(x)|_{D(x)}$ are exactly those which satisfy the equality $v \rfloor \Omega(x) = (\omega_1 \wedge \omega_2 \wedge d\omega^k)(x)$, or equivalently $v \rfloor \mu(x) = (d\omega|_{D(x)})^k$ (here $\mu(x)$ is a volume form on $D(x)$). The condition $\operatorname{rank} d\omega|_{D(x)} = 2k$ holds under the genericity assumption (G′). Namely, $d\omega|_{D(x)} = a_1\beta_1 + a_2\beta_2$, where $\beta_i = d\omega_i|_{D(x)}$. It follows from (G′) that $\operatorname{rank}(a_1\beta_1 + a_2\beta_2) = 2k$, for all $a = (a_1, a_2) \in \mathbb{R}^2$,

$a \neq 0$. (If $\operatorname{rank}(a_1\beta_1 + a_2\beta_2) < 2k$ then $(a_1\beta_1 + a_2\beta_2)^k = 0$ and, by the definition (VF) of $Y_0, \dots, Y_k$, we would have $\sum_0^k a_1^i a_2^{k-i} Y_i = 0$, which contradicts (G′).)

We conclude that for a singular curve γ, $t \mapsto x(t)$, the field of tangent vectors can be written as $v(t) = \dot{x}(t) = X_\omega(x(t))$, where X_ω is a characteristic vector field corresponding to a section ω of $D^\perp$, which is any extension to a neighborhood of the curve γ of the field of covectors $t \mapsto p(t)$ along γ. Therefore, any singular curve in U is an integral curve of a characteristic vector field.

It remains to show that the section ω can be taken horizontal. Take a point $x = x(t)$ on the curve γ. Since $\operatorname{rank} d\omega|_{D(x)} = 2k = m - 1$, the kernel of $d\omega|_{D(x)}$ is of dimension one and $v = \dot{x}(t)$ is in the kernel. This implies that there is a well defined, up to multiplicative factor, nonzero vector w tangent to $D^\perp$ at $p = \omega(x)$. Namely, let $v \in \ker d\omega|_{D(x)}$ and let X be any extension of v to a local section of D. It is not hard to verify that the Hamiltonian vector field $\vec{H}_X$ corresponding to the Hamiltonian $H_X : T^*M \to \mathbb{R}$, $H_X(\lambda) = \lambda(X)$, is tangent to $D^\perp$ at $\lambda = (x, p)$ and depends only on $v = X(x)$. (If $D = \operatorname{span}\{X_1, \dots, X_m\}$, take the unique u such that $v = \sum_i u_i X_i(x)$ and then take $w = \sum_i u_i \vec{H}_{X_i}(x, p)$.) A field of such vectors defines a bi-characteristic vector field $\vec{H}$ locally on $D^\perp$, which is nonvanishing over the region of x where (G′) holds. It is unique up to multiplicative factor (a function of $\lambda \in D^\perp$). Its trajectories $t \mapsto (x(t), p(t))$ satisfy (AE). The singular curves are exactly projections to M of the integral curves of $\vec{H}$. We can choose a submanifold $S \subset D^\perp$ which is a single cover of a neighborhood of the singular curve γ and coincides with the field of covectors $t \mapsto p(t)$ over the curve γ. For example, we can take a submanifold in $S_0 \subset D^\perp$ of dimension $n - 1$ which intersects the curve $t \mapsto \lambda(t) = (x(t), p(t))$ transversally at a single point and projects regularly on M. Then we define S as a submanifold of local integral curves of $\vec{H}$ passing through S_0. Since $\dot{x}(t) \neq 0$ the submanifold S has a nonsingular projection $\pi_S : S \to M$ on a neighborhood of γ, and π_S is a diffeomorphism onto $\pi_S(S)$. The foliation of trajectories of $\vec{H}$ projects on a foliation of singular curves of D, onto a neighborhood of γ. The section $\omega(x) := \pi_S^{-1}(x)$ is the desired section and X_ω has all trajectories being singular curves of D, including the curve γ. It follows from the second and the third equality in ($\star$) and from (AE) that $X_\omega \rfloor d\omega = 0$, i.e., the constructed section ω is horizontal.

We have shown that the sets of smooth singular curves and of integral curves of horizontal vector fields coincide on the set U where (G′) holds. Thus, $S(x) = C_{hor}(x)$. Clearly, $C_{hor}(x) \subset C_{char}(x)$. We have shown in the preceding paragraph that any vector $v \in C_{char}(x)$ belongs to $C_{hor}(x)$, thus $C_{hor}(x) = C_{char}(x)$. This finishes the proof of statement (i) and of Theorem 2. □

Remark 4. Notice that we do not need the distribution D to be jet-transversal to the exceptional sets $\mathcal{E}'$ and $\mathcal{E}''$. In order that conditions (i)–(iv) hold it is enough that it meets the sets $\mathcal{E}'$ and $\mathcal{E}''$ at a nowhere dense subset of M.

4. Characteristic vector fields: rank $D = 2k$

Consider smooth distribution D on M^{2k+2} of even rank $m = 2k$. Locally we can write $D = \ker \omega_1 \cap \ker \omega_2$, with smooth cogenerators ω_1, ω_2. A section ω of $D^\perp$ is called *characteristic* 1*-form* of D if it satisfies the *characteristic equation*

$$\omega_1 \wedge \omega_2 \wedge (d\omega)^k = 0. \tag{CE}$$

Given a local volume form Ω on M, the local vector field $X = X_\omega$ satisfying

$$X_\omega \rfloor \Omega = \omega \wedge (d\omega)^k \tag{CVF}$$

is called *characteristic vector field* of D, if ω is characteristic 1-form.

A characteristic 1-form ω is called *horizontal* if $X_\omega \rfloor d\omega = 0$ and the corresponding characteristic vector field X_ω is called *horizontal characteristic vector field* of D. We denote by $C_{char}(x) \subset T_xM$ (resp. $C_{hor}(x) \subset T_xM$) the cone of vectors at x generated by characteristic (resp. horizontal characteristic) vector fields.

Proposition 3. *Characteristic vector fields X_ω have properties* (P1), (P2), (P3) *from Proposition* 2*, i.e., $X_\omega(x) \in D(x)$, $X_\omega(x) \in \ker d\omega|_{D(x)}$, and $[X_\omega, Y](x) \in D(x)$, for any other characteristic vector field Y.*

Proof. Definition (CVF) and Fact 2 imply $X_\omega \rfloor \omega = 0$ and $X_\omega \in \ker d\omega|_{\ker\omega}$. Therefore we obtain (P2) as $D \subseteq \ker \omega$. Moreover $\bar{\omega} \wedge \omega \wedge (d\omega)^k = 0$ for any section $\bar{\omega}$ of $D^\perp$ since ω is characteristic. This gives $0 = X_\omega \rfloor (\bar{\omega} \wedge \omega \wedge (d\omega)^k) =$

$(X_\omega \rfloor \bar{\omega}) \wedge \omega \wedge (d\omega)^k$, since $X_\omega \rfloor (\omega \wedge (d\omega)^k) = X_\omega \rfloor (X_\omega \rfloor \Omega) = 0$. Therefore, $X_\omega \rfloor \bar{\omega} = 0$. This, together with $X_\omega \rfloor \omega = 0$, gives (P1). Property (P3) follows from (P1) and (P2), exactly as in the odd-rank case. □

Contrary to the case of odd rank, if $m = 2k$ the characteristic 1-forms and characteristic vector fields fill "discrete" subsets in $D^\perp$ and D. In order to explain this better we represent $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$ and re-introduce the *characteristic equation* of D,

$$\omega_1 \wedge \omega_2 \wedge (\lambda_1\, d\omega_1 + \lambda_2\, d\omega_2)^k = 0, \tag{CE}$$

with the unknown $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ depending on x. If Ω is a local volume form on M, we can write the characteristic equation in the equivalent form

$$P(\lambda_1, \lambda_2) := \sum_{j=0}^{k} a_j \lambda_1^{k-j} \lambda_2^j = 0, \tag{CE$'$}$$

where

$$a_j = \binom{k}{j} \frac{\omega_1 \wedge \omega_2 \wedge (d\omega_1)^{k-j} \wedge (d\omega_2)^j}{\Omega}$$

are locally defined functions on M. This is a homogeneous equation, thus its nonzero solutions at a given point x can be considered as points in the projective line.

Note that the *characteristic polynomial* P is defined by the distribution D uniquely, up to invertible factor, since the transformation $\omega_1 \to a_{11}\tilde{\omega}_1 + a_{12}\tilde{\omega}_2$ and $\omega_2 \to a_{21}\tilde{\omega}_1 + a_{22}\tilde{\omega}_2$ changes $\omega_1 \wedge \omega_2$ into $\det A \tilde{\omega}_1 \wedge \tilde{\omega}_2$, where $A = \{a_{ij}\}$. The solutions of (CE), when understood as elements $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$ of the annihilator $D^\perp$, depend only on D and not on the choice of ω_1 and ω_2.

We fix two cogenerators ω_1, ω_2 of D so that $a_0(x) \neq 0$. This can be done, locally, if $P \not\equiv 0$ and it means that $(1, 0)$ is not a root of $P = 0$. Let $Q = Q(a_0, \dots, a_k)$ denote the *discriminant* of the polynomial $\tilde{P}(t) := P(t, 1)$, which is a polynomial of the coefficients $a_0, \dots, a_k$. Denote $\mathrm{Discr}(x) = Q(a_0(x), \dots, a_k(x))$. We introduce the *genericity condition*

$$\mathrm{Discr}(x) \neq 0, \tag{G0}$$

equivalent to all roots of (CE$'$) being single. If a root $\lambda^0 = (t_0, 1)$ is single then $\tilde{P}(t_0) = 0$ and $\tilde{P}'(t_0) \neq 0$, thus the implicit function theorem is applicable. Consequently, the solution $(t_0, 1)$ has a locally unique continuation which depends smoothly on the coefficients of P.

We see that if $\operatorname{rank} D$ is even, the characteristic vector fields are locally defined and smooth in the region where $\mathrm{Discr} \neq 0$ (the condition $a_0(x) \neq 0$ is no more needed). They are unique up to order and multiplication by nonvanishing functions. There are at most k of them and they are given via the formula (CVF) where $\omega = \lambda_1^i\omega_1 + \lambda_2^i\omega_2$ is a solution of $\omega_1 \wedge \omega_2 \wedge (d\omega)^k = 0$, or equivalently, $\lambda_i = (\lambda_1^i, \lambda_2^i)$ is a solution of (CE) or (CE$'$).

In what follows we shall mostly work in the region R_2 *of points in* M*, where the characteristic equation* (CE) *has at least* 2 *single real roots* (*counted in the projective line*).

The region R_2 is an open subset in M, which follows from continuous dependence of (complex) solutions of polynomial equations with respect to the coefficients. The roots $\lambda^1 = (\lambda_1^1, \lambda_2^1)$ and $\lambda^2 = (\lambda_1^2, \lambda_2^2)$ being single, they depend analytically on the coefficients of the equation (treated as elements of the projective line). In particular, we can choose smooth sections $\lambda^i(x) = (\lambda_1^i(x), \lambda_2^i(x))$ so that each $\tilde{\omega}_i = \lambda_1^i\omega_1 + \lambda_2^i\omega_2$ is a smooth section of $D^\perp$ and $\tilde{\omega}^1, \tilde{\omega}^2$ are cogenerators of D. Such cogenerators satisfy the following equations (we omit tildes)

$$\big(\omega_1 \wedge \omega_2 \wedge (d\omega_1)^k\big)(x) = 0, \tag{C1}$$

$$\big(\omega_1 \wedge \omega_2 \wedge (d\omega_2)^k\big)(x) = 0, \tag{C2}$$

and will be called *characteristic cogenerators*. Thus we have proved

Proposition 4. *For any point $x \in R_2$ we can choose cogenerators ω_1 and ω_2 of D in a neighborhood of x so that* (C1) *and* (C2) *hold.*

Remark 5. Note that if $k > 2$ then the characteristic equation may have $r > 2$ real solutions. If they are all different and single at a given point, we may define in the same way r different characteristic 1-forms ω_i, $i = 1, \dots, r$ in a neighborhood of this point. In that case we can choose any two of them as characteristic cogenerators.

In order to state the main result consider two vector fields X_1, X_2 defined on an open subset $U \subset M^{2k+2}$. We define two distributions

$$\Gamma_1(x) := \operatorname{span}\{X_1, X_2, \operatorname{ad}^i_{X_1} X_2\colon\ i = 1, \dots, 2k-1\}(x), \tag{D1}$$

$$\Gamma_2(x) := \operatorname{span}\{X_1, X_2, \operatorname{ad}^i_{X_2} X_1\colon\ i = 2, \dots, 2k-1\}(x), \tag{D2}$$

where $\operatorname{ad}_X Y = [X, Y]$ denotes the Lie bracket of vector fields and we define inductively $\operatorname{ad}^i_X Y = \operatorname{ad}_X(\operatorname{ad}^{i-1}_X Y)$. We introduce *genericity conditions*, imposed on the pair (X_1, X_2),

$$\dim \Gamma_1(x) = 2k + 1, \tag{G1}$$

$$\dim \Gamma_2(x) = 2k + 1, \tag{G2}$$

equivalent to pointwise linear independence of the vector fields defining Γ_1 and Γ_2. Denote

$$Y_{-1} := X_1, \quad Y_0 := X_2, \quad Y_1 := \operatorname{ad}_{X_1} X_2, \quad \dots, \quad Y_{2k-1} := \operatorname{ad}^{2k-1}_{X_1} X_2.$$

Suppose (G1) be satisfied at $x \in M$ and define a nonvanishing 1-form ω_1 which satisfies

$$\omega_1(Y_i) = 0, \quad i = -1, 0, \dots, 2k-1. \tag{F1}$$

Then ω_1 is defined locally around x uniquely, up to a nonvanishing factor. Another useful *genericity conditions* is

$$\operatorname{rank}\bigl\{\omega_1\bigl([Y_i, Y_j]\bigr)(x)\bigr\}_{i,j=-1}^{2k-1} = 2k. \tag{G3}$$

The matrix in (G3) is antisymmetric, thus its rank is even, equal at most $2k$. Interchanging the role of X_1, X_2 and assuming (G2) we analogously define the vector fields

$$Z_{-1} := X_2, \quad Z_0 := X_1, \quad Z_1 := \operatorname{ad}_{X_2} X_1, \quad \dots, \quad Z_{2k-1} := \operatorname{ad}^{2k-1}_{X_2} X_1$$

and a nonvanishing 1-form ω_2 which satisfies

$$\omega_2(Z_i) = 0, \quad i = -1, 0, \dots, 2k-1. \tag{F2}$$

The next genericity condition is

$$\operatorname{rank}\bigl\{\omega_2\bigl([Z_i, Z_j]\bigr)(x)\bigr\}_{i,j=-1}^{2k-1} = 2k. \tag{G4}$$

We will also need

$$\dim\bigl(\Gamma_1(x) + \Gamma_2(x)\bigr) = 2k + 2 \tag{G5}$$

and

$$d\omega_1\bigl(X_2, [X_2, X_1]\bigr) \neq 0, \qquad d\omega_2\bigl(X_1, [X_1, X_2]\bigr) \neq 0. \tag{G6}$$

Theorem 3. *There exists a subset $\mathcal{G} \subset \mathcal{D}^{2k}(M^{2k+2})$, residual and therefore dense in Whitney C^∞-topology, such that for any distribution $D \in \mathcal{G}$ the following conditions hold.*

(i) *At generic points in M the singular curves of D are exactly integral curves of characteristic* (*equivalently, horizontal characteristic*) *vector fields, up to parametrization, and* (G0) *holds. At such points*

$$S(x) = C_{char}(x) = C_{hor}(x).$$

(ii) *Around generic points $x \in R_2$ there exist two characteristic vector fields X_1, X_2 of D which satisfy conditions* (G1)–(G6)*. Moreover, at such x we have*

$$D(x) = \Gamma_1(x) \cap \Gamma_2(x).$$

Remark 6. (a) Clearly, statements (i) and (ii) imply Theorem 1, if $m = 2k$.

(b) If k is odd then a generic distribution is not determined by characteristic vector fields in the region where the characteristic equation (CE$'$) has only one real single root as, by statement (i), all singular curves are orbits of a single vector field. If k is even then (CE$'$) has no or at least 2 real single roots at generic points.

(c) Any smooth distribution $D \subset TM$ can be modified on a contractible neighborhood of a point p so that its germ at p is an a priori given germ (this follows from connectedness of the Grassmannians). Thus, the region R_2 can always be made nonempty, by Lemma 2.

Remark 7. The open subset $U \subset M$ where condition (G0) holds is a disjoint union $U = U_0 \cup U_1 \cup \dots \cup U_k$, where U_r denotes the open subset on which the characteristic equation (CE$'$) has exactly r real roots. Characteristic vector fields $X_1, \dots, X_r$ corresponding to different real roots are linearly independent on U_r. (These vector fields span a local r-dimensional distribution D_{char} on U_r.) More precisely, at points in U_r there are r distinct local characteristic 1-forms $\omega_1, \dots, \omega_r$. The corresponding characteristic vector fields $X_1, \dots, X_r$ belong to the kernels $\ker d\omega_i|_D$, by property (P2) in Proposition 3. The kernels $\ker d\omega_i|_D$, $i = 1, \dots, r$, are 2-dimensional and span rank $2r$ subdistribution of D. This can be shown using e.g. Theorem 5 in [15] on the canonical form of two nondegenerate exterior 2-forms. Thus $X_1, \dots, X_r$ are linearly independent. It follows from Lemma 3 that the characteristic equation of a generic $2k$-distribution on M^{2k+2} may have k distinct real roots, thus U_k is nonempty for some generic distributions.

Statement (ii) admits the following converse. We introduce the *invariance condition*

$$[X_1, \Gamma_1] \subset \Gamma_1, \tag{I1}$$

$$[X_2, \Gamma_2] \subset \Gamma_2. \tag{I2}$$

(For brevity, we write $[X, \Delta] \subset \Delta$ instead of $[X, \Gamma^\infty(\Delta)] \subset \Gamma^\infty(\Delta)$, with $\Gamma^\infty(\Delta)$ denoting the set of local sections of Δ.) (I1) and (I2) mean invariance of Γ_i under the flow of X_i.

Theorem 4. *If X_1, X_2 are vector fields satisfying* (G1)–(G5)*, and* (I1), (I2)*, on an open subset $U \subset M$, then X_1, X_2 are characteristic vector fields of $D(x) = \Gamma_1(x) \cap \Gamma_2(x)$, corresponding to nonvanishing characteristic 1-forms ω_1, ω_2 given by* (F1), (F2).

Proof. Let ω_1, ω_2 be given by (F1), (F2). Then $\ker\omega_1(x)$ and $\ker\omega_2(x)$ are uniquely defined (which follows from (G1), (G2)) and $\Gamma_i = \ker\omega_i$, $i = 1, 2$. Take $D = \Gamma_1 \cap \Gamma_2$, then $\operatorname{codim} D(x) = 2$, by (G5). Conditions (I1), (I2), and Fact 1 give $X_i \in \ker d\omega_i|_{\Gamma_i}$. This and $X_i \in D$ imply $X_i \rfloor (\omega_1 \wedge \omega_2 \wedge (d\omega_i)^k) = 0$, therefore $\omega_1 \wedge \omega_2 \wedge (d\omega_i)^k = 0$ (if an n-form has nontrivial kernel then it vanishes). Thus ω_i, $i = 1, 2$, are characteristic cogenerators of D. From (G3) and (G4) it follows that the kernels of $d\omega_i|_{\Gamma_i}$ are of dimension 1, thus $\ker d\omega_i|_{\Gamma_i}(x) = \operatorname{span}\{X_i(x)\}$. This is equivalent to existence of nonvanishing f_1, f_2 such that $X_i \rfloor \Omega = f_i\omega_i \wedge (d\omega_i)^k$, which completes the proof. □

In the proof of Theorem 3 we will need the following

Lemma 2. *There exist polynomial vector fields X_1 and X_2 on $\mathbb{R}^{2\kappa+4}$, $\kappa \geqslant 1$, which satisfy* (I1), (I2), (G1), (G2), (G3), (G4), (G5) *and* (G6) *at generic points. Moreover, introducing the coordinates x_1, x_2, z_1, z_2, and $p_1, \dots, p_\kappa$, $q_1, \dots, q_\kappa$, we can take them in the form*

$$X_1 = \partial_{x_1} + \sum_1^\kappa \left(x_2^{i+1}\partial_{p_i} + x_2^{\kappa+i+1}\partial_{q_i}\right) + \sum_1^\kappa x_2^{i+1} q_i \partial_{z_1} + x_2\partial_{z_2},$$

$$X_2 = \partial_{x_2} + \sum_1^\kappa \left(x_1^{i+1}\partial_{q_i} + x_1^{\kappa+i+1}\partial_{p_i}\right) + \sum_1^\kappa x_1^{i+1} p_i \partial_{z_2} + x_1\partial_{z_1},$$

and then their germ at any point where $x_1 = x_2 = 0$ and $p_1 = q_1 = 1$ satisfies (I1), (I2) *and* (G1)–(G6)*, with $k = \kappa + 1$. In neighborhood of such points the characteristic equation of $D(x) = \Gamma_1(x) \cap \Gamma_2(x)$ has all roots real, ω_1, ω_2 defined by* (F1), (F2) *are characteristic cogenerators of D, and X_1, X_2 are characteristic vector fields corresponding to ω_1, ω_2.*

Proof. We will compute the Lie brackets defining Γ_1. Note that no coordinate function in X_1 and X_2 depends on z_1, z_2 (the corresponding terms in Lie brackets will vanish) and only some of them depend, linearly, on p_i or q_i. By direct computation we find

$$\begin{aligned}
Y_1 = [X_1, X_2] &= \sum_1^\kappa \bigl((i+1)x_1^i - (\kappa+i+1)x_2^{\kappa+i}\bigr)\partial_{q_i} \\
&\quad + \sum_1^\kappa \bigl((\kappa+i+1)x_1^{\kappa+i} - (i+1)x_2^i\bigr)\partial_{p_i} \\
&\quad + \sum_1^\kappa (i+1)x_1^i p_i \partial_{z_2} - \sum_1^\kappa (i+1)x_2^i q_i \partial_{z_1} \\
&\quad + \sum_1^\kappa \bigl(1 - x_1^{i+1}x_2^{i+1}\bigr)(\partial_{z_1} - \partial_{z_2}),
\end{aligned}$$

$$\begin{aligned}
Y_2 = \mathrm{ad}_{X_1}^2 X_2 &= \sum_1^\kappa (i+1)i x_1^{i-1}\partial_{q_i} + \sum_1^\kappa (\kappa+i+1)(\kappa+i)x_1^{\kappa+i-1}\partial_{p_i} \\
&\quad + \sum_1^\kappa (i+1)i x_1^{i-1} p_i \partial_{z_2} + \sum_1^\kappa \kappa x_2^{\kappa+2i+1}\partial_{z_1} \\
&\quad + 2\sum_1^\kappa (i+1)x_1^i x_2^{i+1}(\partial_{z_2} - \partial_{z_1}),
\end{aligned}$$

$$\begin{aligned}
Y_3 = \mathrm{ad}_{X_1}^3 X_2 &= \sum_2^\kappa (i+1)i(i-1)x_1^{i-2}\partial_{q_i} \\
&\quad + \sum_1^\kappa (\kappa+i+1)(\kappa+i)(\kappa+i-1)x_1^{\kappa+i-2}\partial_{p_i} \\
&\quad + \sum_2^\kappa (i+1)i(i-1)x_1^{i-2} p_i \partial_{z_2} \\
&\quad + 3\sum_1^\kappa (i+1)i x_1^{i-1}x_2^{i+1}(\partial_{z_2} - \partial_{z_1}),
\end{aligned}$$

$$\begin{aligned}
Y_4 = \mathrm{ad}_{X_1}^4 X_2 &= \sum_3^\kappa \frac{(i+1)!}{(i-3)!}x_1^{i-3}\partial_{q_i} + \sum_1^\kappa \frac{(\kappa+i+1)!}{(\kappa+i-3)!}x_1^{\kappa+i-3}\partial_{p_i} \\
&\quad + \sum_3^\kappa \frac{(i+1)!}{(i-3)!}x_1^{i-3} p_i \partial_{z_2} \\
&\quad + 4\sum_2^\kappa \frac{(i+1)!}{(i-2)!}x_1^{i-2}x_2^{i+1}(\partial_{z_2} - \partial_{z_1}).
\end{aligned}$$

To be more precise, if $\kappa = 1$ then the sum at ∂_{p_i} is empty, similarly as the other sums in the formula for Y_4, and thus $Y_4 = Y_{2\kappa+2} = 0$. In this case we stop our calculations here. If $\kappa \geqslant 2$ we continue the recursive procedure and get

$$\begin{aligned}
Y_{\kappa+1} = \mathrm{ad}_{X_1}^{\kappa+1} X_2 &= (\kappa+1)!\partial_{q_\kappa} + \sum_1^\kappa \frac{(\kappa+i+1)!}{i!}x_1^i\partial_{p_i} \\
&\quad + (\kappa+1)! p_\kappa \partial_{z_2}
\end{aligned}$$

$$+ (\kappa+1)\sum_{\kappa-1}^{\kappa}\frac{(i+1)!}{(i-\kappa+1)!}x_1^{i-\kappa+1}x_2^{i+1}(\partial_{z_2}-\partial_{z_1}),$$

$$Y_{\kappa+2} = \mathrm{ad}_{X_1}^{\kappa+2} X_2 = \sum_{1}^{\kappa}\frac{(\kappa+i+1)!}{(i-1)!}x_1^{i-1}\partial_{p_i} + (\kappa+2)!x_2^{\kappa+1}(\partial_{z_2}-\partial_{z_1}),$$

$$Y_{\kappa+3} = \mathrm{ad}_{X_1}^{\kappa+3} X_2 = \sum_{2}^{\kappa}\frac{(\kappa+i+1)!}{(i-2)!}x_1^{i-2}\partial_{p_i},$$

$$\vdots$$

$$Y_{2\kappa} = \mathrm{ad}_{X_1}^{2\kappa} X_2 = \sum_{\kappa-1}^{\kappa}\frac{(\kappa+i+1)!}{(i-\kappa+1)!}x_1^{i-\kappa+1}\partial_{p_i},$$

$$Y_{2\kappa+1} = \mathrm{ad}_{X_1}^{2\kappa+1} X_2 = (2\kappa+1)!\partial_{p_\kappa}.$$

We see that $[X_1, Y_{2\kappa+1}] = 0$ and $[X_1, Y_j] = Y_{j+1}$, for $j = 0, \dots, 2\kappa$, thus (I1) holds. By the symmetry of X_1 and X_2 condition (I2) also holds.

At the points where $x_1 = x_2 = 0$ the above vector fields take much simpler form

$$Y_{-1} = X_1 = \partial_{x_1},$$
$$Y_0 = X_2 = \partial_{x_2},$$
$$Y_1 = \mathrm{ad}_{X_1} X_2 = \partial_{z_1} - \partial_{z_2},$$
$$Y_2 = \mathrm{ad}_{X_1}^2 X_2 = 2!\partial_{q_1} + 2!p_1\partial_{z_2},$$
$$Y_3 = \mathrm{ad}_{X_1}^3 X_2 = 3!\partial_{q_2} + 3!p_2\partial_{z_2},$$
$$\vdots$$
$$Y_{\kappa+1} = \mathrm{ad}_{X_1}^{\kappa+1} X_2 = (\kappa+1)!\partial_{q_\kappa} + (\kappa+1)!p_\kappa\partial_{z_2},$$
$$Y_{\kappa+2} = \mathrm{ad}_{X_1}^{\kappa+2} X_2 = (\kappa+2)!\partial_{p_1},$$
$$\vdots$$
$$Y_{2\kappa} = \mathrm{ad}_{X_1}^{2\kappa} X_2 = (2\kappa)!\partial_{p_{\kappa-1}},$$
$$Y_{2\kappa+1} = \mathrm{ad}_{X_1}^{2\kappa+1} X_2 = (2\kappa+1)!\partial_{p_\kappa}$$

and we see that condition (G1) is satisfied at such points. By the symmetry we see that condition (G2) is also satisfied at these points.

From the definition of Y_j we get

(i) $[Y_{-1}, Y_j] = [X_1, Y_j] = Y_{j+1}$, for $0 \leqslant j < 2\kappa + 1$, and $[Y_{-1}, Y_{2\kappa+1}] = 0$.

Moreover, at the points where $x_1 = x_2 = 0$ we have

(ii) $[Y_0, Y_j] = [X_2, Y_j] = 0$, for all $1 < j \leqslant 2\kappa + 1$, and $[Y_0, Y_1] = -2\partial_{p_1} - 2q_1\partial_{z_1}$.

The Lie brackets of vector fields Y_r, Y_s which are tangent to the submanifold

$$S = \{x_1 = x_2 = 0\}$$

can be correctly computed at the points in S using the formulas for Y_r, Y_s restricted to S. All Y_j, $1 \leqslant j \leqslant 2\kappa + 1$, are tangent to S.

From the above formulas for Y_j restricted to S we easily find that:

(iii) Y_1 commutes on S with all Y_j, $1 < j \leqslant 2\kappa + 1$.
(iv) Y_r and Y_s commute on S if $2 \leqslant r, s \leqslant \kappa + 1$ or $\kappa + 2 \leqslant r, s \leqslant 2\kappa + 1$.
(v) Y_r and Y_s commute on S if $2 \leqslant r \leqslant \kappa + 1$, $\kappa + 2 \leqslant s \leqslant 2\kappa + 1$, $s - r \neq \kappa$.
(vi) $[Y_r, Y_{r+\kappa}] = -r!(\kappa + r)!\partial_{z_2}$ on S if $2 \leqslant r \leqslant \kappa + 1$.

We easily see that the 1-form ω_1 annihilating the vector fields $Y_{-1}, Y_0, Y_1, \ldots, Y_{2\kappa+1}$ restricted to S is

$$\omega_1 = dz_1 + dz_2 - \sum_1^{\kappa} p_i \, dq_i.$$

Thus the above calculations give the matrix $\omega_1([Y_i, Y_j])$ on the submanifold S. Namely, by (i)–(vi) we have on S

$$\omega_1\big([Y_r, Y_s]\big) = 0$$

for all $-1 \leqslant r, s \leqslant 2\kappa + 1$, with the following exceptions:

$$\omega_1\big([Y_0, Y_1]\big) = -\omega_1\big([Y_1, Y_0]\big)(0) = -2q_1,$$
$$\omega_1\big([Y_r, Y_{r+\kappa}]\big) = -\omega_1\big([Y_{r+\kappa}, Y_r]\big) = -r!(\kappa + r)!,$$

for $2 \leqslant r \leqslant \kappa + 1$. Therefore (G3) and the first part of (G6) hold at all points where $x_1 = x_2 = 0$ and $q_1 \neq 0$. By the symmetry between X_1 and X_2 we see that (G4) and the second part of (G6) hold, too, at the points where $x_1 = x_2 = 0$ and $p_1 \neq 0$.

Finally, at $S = \{x_1 = x_2 = 0\}$ the corresponding 1-form ω_2 annihilating Γ_2 is

$$\omega_2 = dz_1 + dz_2 - \sum_1^{\kappa} q_i \, dp_i.$$

We see that at points where $p = (p_1, \ldots, p_\kappa) \neq 0$ or $q = (q_1, \ldots, q_\kappa) \neq 0$ we have $\ker \omega_1(x) \neq \ker \omega_2(x)$, thus $\operatorname{corank} \Gamma_1(x) \cap \Gamma_2(x) = 2$ at such points and $\dim \Gamma_1(x) + \Gamma_2(x) = 2\kappa + 4$, i.e., (G5) holds.

The set of points where (I1), (I2), and (G1)–(G6) are satisfied is open and dense in $\mathbb{R}^n$, $n = 2\kappa + 4$. This follows from the fact that X_1 and X_2 have polynomial coefficients. Namely, negations of conditions (G1), (G2) and (G5) mean linear dependence of some Lie brackets of X_1 and X_2. Since the coefficients of $Y_{-1} = X_1$, $Y_0 = X_2$ and of the Lie brackets $Y_j = \mathrm{ad}_{X_1}^j X_2$ defining Γ_1 (respectively, Γ_2) depend polynomially on the coordinates, the negations of (G1), (G2) and (G5) can be expressed as polynomial equations. These equations are nontrivial as we have shown that they are not satisfied at some point in $\mathbb{R}^n$. Therefore, the set of their solutions is closed and nowhere dense in $\mathbb{R}^n$ and the set $U \subset \mathbb{R}^n$ of points where (G1), (G2) and (G5) are satisfied is open and dense in $\mathbb{R}^n$. The 1-form ω_1 is defined by the equations $\omega_1(Y_j) = 0$, $j = -1, 0, \ldots, 2\kappa + 1$, and can be taken with rational coefficients, the common denominator of which is nonzero on U. The negation of condition (G3) can then be expressed as a nontrivial polynomial equation on U. The same applies to (G4) and (G6). Therefore, negations of conditions (G3), (G4), (G5), and (G6) hold on closed, nowhere dense subsets of U. This means that (G1)–(G6) are satisfied on an open, dense subset V in $\mathbb{R}^n$. Finally, it follows from our proof that $[X_1, Y_j] = Y_{j+1}$, for $j = -1, 0, \ldots, 2\kappa + 1$, with $Y_{2\kappa+2} := 0$. Thus (I1) and (I2) hold on V.

To show that all characteristic roots are real note that $(1, 0)$ and $(0, 1)$ are such roots, at points in S, since ω_1, ω_2 are characteristic cogenerators. We will show that $\lambda = (1, 1)$ is the remaining root, which is of multiplicity $\kappa - 1 = k - 2$. In fact, consider the sub-distribution $N \subset D$ given by $D \cap TS$. Then $\operatorname{codim} N = 4$ for a generic point in S. Take $\omega = \omega_1 + \omega_2$, which corresponds to $\lambda = (1, 1)$. We see from the form of $\omega_i|_S$, $i = 1, 2$, that $d\omega_1|_N = -d\omega_2|_N$, thus $d\omega|_N = (d\omega_1 + d\omega_2)|_N = 0$. This means that $d\omega$ is of rank at most $\operatorname{codim} N = 4$ at a generic point in S. Therefore, $\lambda = (1, 1)$ is a solution of (CE) of multiplicity at least $k - 4/2 = k - 2$. On the other hand, it can not have higher multiplicity as there are two other real roots. The proof is complete. □

Lemma 3. *There exists a corank* 2 *distribution germ D at $0 \in \mathbb{R}^{2k+2}$, $k \geqslant 2$, which satisfies* (G0)*, all k roots of characteristic equation are real, and it has a pair of characteristic vector fields (X_1, X_2) that fulfill conditions* (G1)–(G6).

Proof. We shall first perturb the example in Lemma 2 so that condition (G0) is satisfied. (We were unable to find out, by hand calculations, whether the distribution in Lemma 2 satisfies (G0) at some points.) Consider a distribution D and its cogenerators ω_1, ω_2. Fix a local volume form Ω. The coefficients $a_0, \dots, a_k$ of the characteristic polynomial (CE′) depend polynomially on the first jet of (ω_1, ω_2). The map $j^1(\omega_1, \omega_2) \mapsto (a_0, \dots, a_k)$ is submersive at generic jets. (For polynomial maps it is enough to check submersivity at one point. It is easy to do it on the 1-jet $j^1\omega_1 = dz_1 + \sum_1^k x_i\, dy_i$, $j^1\omega_2 = dz_2 + \sum_1^k b_i x_i\, dy_i$, where the coefficients $a_0, \dots, a_k$ are symmetric functions $a_r = \sum_{j_1 \cdots j_r} b_{j_1} \cdots b_{j_r}$ of $b_1, \dots, b_k$.) The equation $\mathrm{Discr} = 0$ defines an algebraic subset of codimension 1 in the space of coefficients $a_0, \dots, a_k$. Thus there exist arbitrarily small perturbations of a given 1-jet $j^1(\omega_1, \omega_2)$ which give $\mathrm{Discr} \neq 0$.

Consider the distribution germ $D = D(\omega_1, \omega_2)$ defined in Lemma 2, with cogenerators (ω_1, ω_2) satisfying the genericity conditions (G1)–(G6) for a fixed pair of characteristic vector fields (X_1, X_2). Applying a shift, we may assume that this is a germ at $0 \in \mathbb{R}^{2k+2}$. Choose cogenerators $\tilde\omega_1$ and $\tilde\omega_2$ so that the coefficient a_0 of the characteristic equation (CE′), relative to these cogenerators, is nonzero (this is equivalent to $(1, 0)$ not being solution of (CE′)). By the above argument, a slight perturbation of the first jet of $(\tilde\omega_1, \tilde\omega_2)$ will satisfy (G0) at 0 and, by continuity, in its neighborhood. By surjectivity, we can choose the perturbation so that all roots of the characteristic equation will remain real. (The unperturbed polynomial equation has real roots, thus it is a product of linear terms.)

At the same time we can choose the perturbation so small that conditions (G1)–(G6) are still satisfied for the pair (X_1, X_2). This follows from the fact that the roots $\lambda^i = (\lambda_1^i, \lambda_2^i)$ of the characteristic equation (CE′) depend smoothly, as elements of the projective line, on the coefficients $a_0, \dots, a_k$ (the implicit function theorem is applicable here since they are single roots). Similarly, finite jets of $\lambda^i(x)$ depend smoothly on finite jets of $a_0, \dots, a_k$, thus they depend smoothly on finite jets of $(\tilde\omega_1, \tilde\omega_2)$. The lemma is proved. □

Proof of Theorem 3. Consider the space of homogeneous degree k real polynomials $\sum_{j=0}^k a_j \lambda_1^{k-j} \lambda_2^j$ and let $\mathcal{D}iscr$ denote the subset of such polynomials having $\mathrm{Discr} = 0$. As algebraic subset, it has a stratification into a finite number of submanifolds. The complement of this set is open and dense.

We first prove statement (ii). In the proof we will use the following condition.

(A) *The characteristic equation* (CE) *has at least* 2 *single real roots* (*in the projective line*).

Denote by $\mathcal{E}_0$ the subset, in the space of finite jets of distributions, which consists of D for which (G0) does not hold, that is $\mathrm{Discr}(x) = 0$ (this equation is independent of the choice of cogenerators). For $i = 1, 2$ denote by $\mathcal{E}_i$ the subset, in the space of finite jets of distributions, which consists of those distributions D for which the implication ((G0) and (A)) $\Longrightarrow$ (Gi) does not hold at a given x, for a pair of characteristic vector fields X_1, X_2. Similarly, let $\mathcal{E}_3$ be the subset consisting of those distributions D for which the implication ((G0), (A) and (G1)) $\Longrightarrow$ (G3) does not hold for a pair (X_1, X_2) and let $\mathcal{E}_4$ denote the subset of D for which the implication ((G0), (A) and (G2)) $\Longrightarrow$ (G4) does not hold for a pair (X_1, X_2). Finally, let $\mathcal{E}_6$ be the subset of D such that the implication ((G0), (A), (G1) and (G2)) $\Longrightarrow$ (G6) does not hold at a given x, for a pair of characteristic vector fields X_1, X_2. (In the definition of different subsets $\mathcal{E}_i$ the pair (X_1, X_2) is the same.) From the form of conditions (Gi), $i = 0, \dots, 4, 6$, it is easy to see that the subset $\mathcal{E}_i$, at a given x, is a real stratified submanifold and it has nonzero codimension as, by Lemma 3, there exist distribution jets which satisfy all (A) and (G0)–(G6). Note that ((G0), (A), (G1) and (G2)) $\Longrightarrow$ (G5) automatically holds, as (G0) means that the characteristic cogenerators defined by (F1), (F2) are linearly independent.

We define the set $\mathcal{G} \subset \mathcal{D}^{2k}(M^{2k+2})$ of generic distributions as those smooth distributions which have the 1-jet extensions transversal to all submanifolds in the stratified set $\mathcal{E}_0$ and, moreover, their finite jet extensions are transversal to the exceptional subsets $\mathcal{E}_i$, $i = 1, \dots, 4, 6$, in the space of appropriate jets. By the Thom transversality theorem, the set of such distributions is residual in the Whitney C^∞ topology.

Since all exceptional subsets $\mathcal{E}_0, \dots, \mathcal{E}_4, \mathcal{E}_6$ have nonzero codimension, and a distribution $D \in \mathcal{G}$ meets the subsets $\mathcal{E}_0, \dots, \mathcal{E}_4, \mathcal{E}_6$ at a closed, nowhere dense subset in M, it follows that D satisfies (G0) on an open, dense subset in M and it satisfies (G1)–(G6) on an open, nowhere dense subset in the region R_2 (note that (G5) is implied by (G0)).

Moreover, at generic points in R_2 there are characteristic vector fields X_1, X_2 of $D \in \mathcal{G}$ which fulfill (G1)–(G5), as well as (I1), (I2). They correspond to characteristic 1-forms ω_1, ω_2 satisfying (G0). Using Theorem 4 we conclude that $D = \Gamma_1 \cap \Gamma_2$. This proves statement (ii).

In order to prove statement (i) it is enough to show that if X_ω is a characteristic vector field, then there exists a function f on M such that $f\omega$ is horizontal. Then the proof will follow from equation (AE) as in the odd-rank case.

Note that if ω is characteristic, then $X_\omega \in \ker d\omega|_{\ker\omega}$ (Proposition 3). Consider a vector field Y satisfying $\omega(Y) = 1$. It is enough to find a function f such that $d(f\omega)(X_\omega, Y) = 0$. Since $\omega(X_\omega) = 0$, we have

$$d(f\omega)(X_\omega, Y) = df(X_\omega)\omega(Y) + f d\omega(X_\omega, Y) = X_\omega(f) + f d\omega(X_\omega, Y).$$

Each nontrivial solution of the ordinary differential equation $X_\omega(f) + f d\omega(X_\omega, Y) = 0$ gives a horizontal section $f\omega$. The proof is complete. □

Remark 8. It seems that statement (ii) in Theorem 3 and Theorem 4 can be modified so that similar results can be shown concerning a family of $r > 2$ characteristic vector fields, in the region where the characteristic equation has more then 2 real roots. However, proving an appropriate version of Lemma 2 is not easy. Complete understanding of this problem is left for further research.

Remark 9. If we admit complex solutions to the characteristic equation (CE) or (CE′), we obtain complex valued characteristic 1-forms and the corresponding complex vector fields, being sections of complexified cotangent and tangent bundles $T^{*\mathbb{C}}M$ and $T^{\mathbb{C}}M$ (the n-form Ω in (CVF) is still real). Denote by $C \subset M$ the subset where the characteristic equation has at least one nonreal root. It seems that arguments analogous to above, provided that an appropriate version of Lemma 2 is shown, should lead to a proof of the following analog of statement (ii) in Theorem 3.

Conjecture. *For any generic distribution $D = D^{2k}$ on M^{2k+2}, around generic points $x \in C$, there exists a pair of complex conjugated characteristic vector fields X_1, X_2 of D which satisfy conditions* (G1)–(G6) *in the complex sense and $D(x) = \mathrm{Re}\{\Gamma_1(x) \cap \Gamma_2(x)\}$.*

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