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Local exact Lagrangian controllability of the Burgers viscous equation

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Abstract

We study and give the definition of the exact Lagrangian controllability of the viscous Burgers equation and prove a local result. We give similar results for the heat equation in dimension 1.

Résumé

On définit et étudie la contrôlabilité exacte lagrangienne de l'équation de Burgers dissipative, et prouve un résultat local. On donne des résultats similaires en dimension 1 pour l'équation de la chaleur.

Keywords: Controllability; Boundary control; Lagrangian description; Burgers equation; Evolution equation

1. Introduction

We consider here the problem of the motion of a fluid in dimension 1, modeled by the viscous Burgers equation on a bounded interval. We assume that the velocity of the fluid is prescribed at one extremity of the interval; this quantity will be referred to as "the control". The controllability of the motion of a given set of fluid particles is considered here by means of the aforementioned control.

2. Formulation of the problem

Let us give a precise framework of our problem.

Let $I = (0, 1)$ be the interval in which there is a fluid whose velocity u is assumed to satisfy the classical viscous Burgers equation on an interval of time $[0, T]$ where $T > 0$:

$$u_t + \left(\frac{u^2}{2}\right)_x - u_{xx} = 0, \quad \forall (t, x) \in (0, T) \times I, \tag{2.1}$$

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$$u(t, x=0) = 0, \quad \forall t \in (0, T), \tag{2.2}$$
$$u(t=0, x) = 0, \quad \forall x \in (0, 1). \tag{2.3}$$

As we have said in Section 1 the control $t \in (0, T) \mapsto u(t, x=1)$ defines the last boundary condition required to study Eq. (2.1) with boundary Dirichlet conditions, thus we will also impose that for some function $h : [0, T] \to \mathbb{R}$ (the control)

$$u(t, x=1) = h(t), \quad \forall t \in (0, T). \tag{2.4}$$

Definition 2.1. For any $h : [0, T] \to \mathbb{R}$, we will say that u is a solution of P_h if it satisfies (2.1)–(2.4).

Notice that for the sake of simplicity we have made the choice of the rest for the initial state of the fluid, which seems to us more physically relevant for our problem, but we will indicate some extensions later (see Section 5 and the following ones). Let us also note that we have to define in which space we consider the solution to the problem P_h. This will be considered later.

Now let us (presumably) consider and define the flow of u by

$$\begin{aligned} &\phi(h) : [0, T] \times \mathbb{R} \to \mathbb{R}, \\ &\partial_t \phi(h)(t, x) = z\big(t, \phi(h)(t, x)\big), \\ &\phi(h)(0, x) = x, \end{aligned} \tag{2.5}$$

where, for $t \geqslant 0$

$$\begin{aligned} &z(t, x) = 0, \quad x \leqslant 0, \\ &z(t, x) = u(t, x), \quad x \in (0, 1), \\ &z(t, x) = h(t), \quad x \geqslant 1, \end{aligned}$$

where u is solution of P_h. Of course one has to take care of the fact that $\phi(h)$ might not exist, but if u is proven to be regular up to the boundary then (2.5) defines the usual flow of the extension z of u to $[0, T) \times \mathbb{R}$. Otherwise one has to use the theory of DiPerna and Lions [12]. In this paper all boundary controls h that are considered will be shown to be regular enough so that the preceding flow is well-defined, and in particular in Theorem 3.1 further, the regularity of h is in fact stronger than the one announced.

The question addressed here is the following one:

Question 2.1. Given two closed intervals I_1, I_2, both included in $(0, 1)$, does it exist a function $h : [0, T] \to \mathbb{R}$ such that if u the solution of P_h, $\phi(h)$ given by (2.5) satisfies: for any $x \in I_1$, $\phi(h)(T, x) \in I_2$ and $\phi(h)(T, \cdot) : I_1 \to I_2$ is a homeomorphism?

Definition 2.2. If the answer to Question 2.1 is true for any couple (I_1, I_2) of closed intervals included in $(0, 1)$ we will say the Lagrangian controllability holds for the viscous Burgers equation.

What is thus meant by the Lagrangian controllability is the controllability in Lagrangian coordinates, that is to say, roughly speaking, that one follows the particles of fluid and tries to modify the final position of some given set of particles by acting on one end of the channel containing the fluid.

3. Statement of the main result

The main result of this paper is the following:

Let us denote $I_1 = [\alpha_1, \alpha_2]$, $I_2 = [\beta_1, \beta_2]$.

Then we prove the local Lagrangian controllability of the viscous Burgers equation:

Theorem 3.1. *There exists $\varepsilon > 0$ such that if for $i = 1, 2$ $|\alpha_i - \beta_i| < \varepsilon$, there exists $h \in H^1(0, T)$ for which $\phi(h)$ defined by* (2.5) *satisfies $\phi(h)(T, \alpha_1) = \beta_1$ $\phi(h)(T, \alpha_2) = \beta_2$. Moreover one can impose*

$$\Big\| \max_{i=1,2} \big|\phi(h)(\cdot\,, \alpha_i) - \alpha_i\big| \Big\|_{L^\infty(0,T)} \to 0$$

when $\varepsilon \to 0$.

Let us mention that this theorem is also true in the dimension 1 on a bounded interval for the heat equation, which can be seen as a simplification of the viscous Burgers equation where the nonlinear transport term is neglected, and the proof is similar. The same result also holds true globally for the heat equation (Theorem 3.2 below) and has been proven in [23]. We will thus point out in Section 3.1 what makes the difference in the present work with the work [23] for the heat equation.

Remark 1. Due to the regularity ($u \in W^{1,1}(0, T; H^1(0, 1))$) that will appear in the proof of Theorem 3.1 (namely Theorem 4.1 and the following comments) the flow of u is well-defined and is at time T a diffeomorphism; it thus gives a positive local answer to Question 2.1.

3.1. Previous known result: The case of the heat equation

In this section we discuss the interest of the present main theorem (Theorem 3.1) in comparison with previous known results on the Lagrangian controllability.

Let us recall the following theorem proven in [23]:

Theorem 3.2. *Let $w : [0, T] \times I \to \mathbb{R}$ satisfying*

$$\begin{aligned}
&w_t - w_{xx} = 0 \quad \textit{in } (0, T) \times (0, 1),\\
&w(t = 0, x) = 0, \quad x \in (0, 1),\\
&w(t, x = 0) = 0, \quad t \in (0, T),\\
&t \in (0, T) \mapsto w(t, x = 1) \textit{ is the control,}
\end{aligned}$$

then for any I_1 and I_2 closed intervals included in $(0, 1)$, one can choose $t \to w(t, x = 1)$ in $L^2(0, T)$ such that the corresponding flow of w, Ψ satisfies: $\Psi(T, \cdot)$ is a diffeomorphism from I_1 onto I_2.

Of course this theorem does not require that I_1 and I_2 should be close to each other, but, in the present work we were not able to prove a corresponding global result as Theorem 3.2 for the viscous Burgers equation which is the model with which we deal here.

As we said it, the proof of Theorem 3.1 given in the present paper also applies to the case of the heat equation provided that I_1 and I_2 are close enough to each other:

- Therefore, at a primary sight, localizing the result of Theorem 3.2 might appear as a weakening of the ability of deforming intervals in the case of the heat equation.
  First to go in this direction let us mention that in fact, the proof given in [23] of Theorem 3.2 showed that, again for the flow of the heat equation, one can prescribe any preserving order homeomorphism between I_1 and I_2.
  To assert this one just has to note that, at it is explained below, one drives every particle located initially in I_1 into the region where the control acts. In this region one can modify the position of the particles according to any preserving order homeomorphism, and then perform the method given in [23].
- But second, however, it is important to mention that to obtain Theorem 3.2, we used in [23] what is related to, in the literature of controllability, as an “implicit control”, meaning that you will not give explicitly what will be your device to steer the system. The method in [23] also deeply uses the exact zero controllability of the heat equation (see [15,18,27]).
  For the model (the viscous Burgers equation) treated in this paper, such results are not true (see [16,20,18]) thus we have introduced another method which only works for the moment being for the local Lagrangian controllability in the case of the viscous Burgers equation and also in the case of the heat equation.

Explicit controls can be constructed for other (Eulerian) control systems by minimizing procedures, see e.g. [14,26,31], but to our knowledge, neither in the framework of [23], nor in the present context.
Moreover, the minimizing procedure is a way to prove some controllability results for linear or almost linear partial differential equations (see e.g. [14]), whereas the type of controllability (i.e. the Lagrangian controllability) that we consider in Theorem 3.1 is highly nonlinear and for the moment being, suffers a lack of minimizing methods allowing to prove the existence of a control leading to the desired result.
- Besides, the argument presented further in Section 4 to obtain Theorem 3.1 relies on a local inverse property which can be, at least theoretically, approximated by a fixed point method and thus may be the threshold to numerical computations.
- Local controllability can also be particularly interesting regarding applications when one wants to treat the dispersion of particles whose interactions with the fluid are neglected (for example some particular types of pollution where the particles of pollutant are moved by the stream of the fluid), and you do not want to spread the particles too far from the region where they are initially located. Being able to give a local result is then interesting since the conclusion of Theorem 3.1 asserts that the interval I_1 will not be moved too far away contrary to the argument in the proof of Theorem 3.2 in [23] where you "flush" I_1 in the control zone.

Concerning the more specific fluid-structure interaction controllability problems, we refer to [2] and [13] as well as (e.g.) [5] and [11] for the existence of the model in dimension 2 or 3, and to [28] and [33] for the existence of the model in dimension 1.

We have to mention that before studying the Lagrangian controllability (i.e. Question 2.1 addressed in Section 2) of fluid models, the Eulerian controllability (i.e. controlling the speed of fluids) seems to be the first goal since what one might expect is that knowing the velocity of a fluid allows you theoretically to know the position of the fluid itself.

For fluid models in dimension 2 or 3, the controllability of the velocity has been, if not totally understood, nowadays, quite deeply studied.

In [7,8] the (Eulerian) exact and approximate controllability of the Euler and Navier–Stokes equations have been derived with different type of boundary conditions. By means of Carleman estimates, the papers [17,24,25] deal also with the local exact zero controllability of the Navier–Stokes equations.

For the Burgers equation it has been proven in [20] that it is not controllable even in large time, as well as for the non-viscous Burgers equation in [1] and [22].

Thus the result presented in this work may suggest that the relationship between the Eulerian controllability and the Lagrangian one is not clear.

One may also mention, to emphasize the last remarks, that a direct Lagrangian approach has been recently proven to be more efficient than the Eulerian one for computations of inverse problems in oceanography in [30].

4. Proof of Theorem 3.1

The previous section has enlightened the independence between the Eulerian and the Lagrangian controllability, but the proof of Theorem 3.1 may rely on some approximate controllability properties of the linearized Burgers equation around the zero trajectory (namely the heat equation) as it will be shown below, which is usually a very difficult argument used in the Eulerian controllability for higher dimensional fluid models.

In order to prove this theorem, we will proceed as follows: we will move locally around the null solution of the Burgers equation in two directions which will be correctly chosen in order to steer α_1 and α_2 by the flow to any points sufficiently closed to them.

Let us therefore consider two elements h_1 and h_2 of $H^1(0,T)$ and let us consider u_1 and u_2 be respectively the solutions of P_{h_1} and P_{h_2} whose existence is now quite straightforward. More precisely with less regularity one can prove (see for instance [20])

Theorem 4.1. *There exists a continuous function $K \geqslant 0$ with $K(0) = 0$ such that for any $h \in H^{1/2}(0,T)$ there exists a unique $u \in X := H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1))$ solution of P_h, moreover one has*

$$\|u\|_X \leqslant K\left(\|h\|_{H^{1/2}(0,T)}\right).$$

For other general controllability questions on the viscous Burgers equation including existence and regularity results we also refer to [18].

For λ and μ two real numbers we consider $h(t) = \lambda h_1(t) + \mu h_2(t)$ where h_1 and h_2 are $W^{3,\infty}(0,T)$ with $h_i^{(k)}(0) = 0$ with $k = 0, 1, 2$. Let $u \in X$ be the solution of P_h, it is clear that $u \in W^{1,1}(0,T; H^1(0,1))$.

Let us remark that according to the regularity of u ($u \in W^{1,1}(0,T; H^1(0,1))$), $\phi(h)(T,\cdot)$ will be a diffeomorphism from $[\alpha_1, \alpha_2]$ onto $[\phi(h)(T,\alpha_1), \phi(h)(T,\alpha_2)]$. Since $\phi(0) = \mathrm{Id}$ and according to the regularity of ϕ that we will use further, the second part of Theorem 3.1 will then be true.

Let us define the following C^1 map.

$$\Theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}, \tag{4.1}$$

$$(\lambda, \mu) \mapsto \big(\phi(\lambda h_1 + \mu h_2)(T, \alpha_1), \phi(\lambda h_1 + \mu h_2)(T, \alpha_2)\big). \tag{4.2}$$

Of course the fact that it is C^1 is in general not clear, but here it suffices to see that due to the regularizing property of the dissipative term in (2.1), u will be as regular in space and time as h_i and then will depend regularly of h_i (and thus λ and μ); this last regularity is in general false (see [6,9,29]) particularly for hyperbolic systems, for which one might lose regularity properties. In our case one thus has only to use the regularity of solutions of ordinary differential equations with respect to the parameters, see [4]. Then one gets after straightforward computations

$$\frac{\partial \Theta}{\partial \lambda}(0,0) = \left(\int_0^T v_1(s, \alpha_1)\,\mathrm{d}s, \int_0^T v_1(s, \alpha_2)\,\mathrm{d}s \right), \tag{4.3}$$

$$\frac{\partial \Theta}{\partial \mu}(0,0) = \left(\int_0^T v_2(s, \alpha_1)\,\mathrm{d}s, \int_0^T v_2(s, \alpha_2)\,\mathrm{d}s \right) \tag{4.4}$$

where for $i = 1, 2$ v_i satisfies

$$v_{it} - v_{ixx} = 0 \quad \text{in } (0,T) \times (0,1), \tag{4.5}$$

$$v_i(t, x = 0) = 0, \quad \forall t \in (0,T), \tag{4.6}$$

$$v_i(t, x = 1) = h_i, \quad \forall t \in (0,T), \tag{4.7}$$

$$v_i(t = 0, x) = 0, \quad \forall x \in (0,1). \tag{4.8}$$

In order to briefly justify (4.3) (or (4.4)) it suffices to write

$$\Phi(\lambda h_1 + \mu h_2)(T, \alpha_1) = \alpha_1 + \int_0^T u_{\lambda h_1 + \mu h_2}\big(s, \Phi(\lambda h_1 + \mu h_2)(s, \alpha_1)\big)\,\mathrm{d}s$$

and differentiate this identity with respect to λ and μ at $(0,0)$ (here $u_{\lambda h_1 + \mu h_2}$ denotes the solution of $P_{\lambda h_1 + \mu h_2}$).

Let A be the matrix whose rows are given by the right-hand sides of (4.3) and (4.4).

Proving that Θ is a local diffeomorphism will ensure our result. It might seem clear that A is an invertible linear matrix for many choices of h_1 and h_2, but we are going to present a remark relying on the approximate controllability of the heat equation that we believe to be interesting and that asserts partially the proof.

For that, one considers for $g \in L^2(0,1)$ the solution $\xi \in H^1(0,T; L^2(0,1)) \cap C(0,T; H_0^1(0,1))$ of the following backward heat equation

$$\xi_t + \xi_{xx} = g \quad \text{in } (0,T) \times (0,1), \tag{4.9}$$

$$\xi(t, x = 0) = \xi(t, x = 1) = 0, \quad \forall t \in (0,T), \tag{4.10}$$

$$\xi(t = T, x) = 0, \quad \forall x \in (0,1). \tag{4.11}$$

Now let us make the following computations:

$$\int_0^1\int_0^T v_1(s,y)\,\mathrm{d}s\,g(y)\,\mathrm{d}y = \int_0^T\int_0^1 v_1(s,y)g(y)\,\mathrm{d}y\,\mathrm{d}s \tag{4.12}$$
$$= \int_0^T\int_0^1 v_1(s,y)(\xi_t+\xi_{xx})(s,y)\,\mathrm{d}y\,\mathrm{d}s$$
$$= \int_0^T\int_0^1 \big(-v_{1t}(s,y)+v_{1xx}(s,y)\big)\xi(s,y)\,\mathrm{d}y\,\mathrm{d}s + \int_0^T h_1(s)\xi_x(s,1)\,\mathrm{d}s$$
$$= \int_0^T h_1(s)\xi_x(s,1)\,\mathrm{d}s. \tag{4.13}$$

Now assume that the left-hand side of (4.12) is zero for every h_1, then due to (4.13) one has

$$\xi_x(s,1)=0, \quad \text{for a.e. } s \tag{4.14}$$

(let us recall that due to (4.9) $s\mapsto \xi_x(s,1)$ makes sense in $L^2(0,T)$).

Then ξ satisfies (4.9), (4.10), (4.14), thus one can apply Holmgren's unique continuation theorem (see [21]) to ξ_t (which eliminates g which does not depend on t) to conclude that $\xi_t=0$. According to (4.11) one gets $\xi=0$.

Thus the maps

$$\begin{aligned} H^1(0,T) &\to L^2(0,1),\\ h_1 &\mapsto \left(y\mapsto \textstyle\int_0^T v_1(s,y)\,\mathrm{d}s\right) \end{aligned} \tag{4.15}$$

has a dense image in $L^2(0,1)$. The same argument shows that

$$\begin{aligned} &H^1(0,T)\to L^2(0,1),\\ &h_2\mapsto \left(y\mapsto \int_0^T v_2(s,y)\,\mathrm{d}s\right) \end{aligned}$$

has a dense image in $L^2(0,1)$. Thus one can choose h_1 and h_2 (as small as one wants) such that for almost every α_1 and almost every α_2 the rank of the matrix A is 2.

According to the local inverse mapping theorem, one concludes that Θ is a local diffeomorphism from a neighborhood of $(0,0)$ in $\mathbb{R}^2$ to a neighborhood of (α_1,α_2). Thus we have proven Theorem 3.1 for a generic set of pairs (α_1,α_2).

To prove the theorem in the general case let us proceed as follows: take for $z\in(0,1)$, and let us consider $G(\cdot,\cdot,z)$ be the unique element in $H^1(0,T;H^{1/2-\nu}(0,1))\cap C(0,T;H^{3/2-\nu}(0,1)\cap H_0^1(0,1))$ (for all $\nu>0$) satisfying

$$G_t(\cdot,\cdot,z)+G_{xx}(\cdot,\cdot,z)=\delta_z(\cdot), \quad \forall (t,x)\in(0,T)\times(0,1), \tag{4.16}$$
$$G(t=T,x,z)=0, \quad \forall x\in(0,1), \tag{4.17}$$

where δ_z is the Dirac mass at x.

Let us remark that $G(\cdot,\cdot,z)$ will have the same spatial regularity of Green's function at $z\in(0,1)$ of the Laplace equation in $(0,1)$ and thus will be regular on $[0,T]\times(0,1)\setminus\{z\}$.

Then, according to the preceding remark and the *a priori* regularity $\forall\nu>0$,

$$G(\cdot,\cdot,z)\in H^1\big(0,T;H^{1/2-\nu}(0,1)\big)\cap C\big(0,T;H^{3/2-\nu}(0,1)\cap H_0^1(0,1)\big),$$

one has using (4.16), (4.17)

$$\int_0^T v_i(s,\alpha_j)\,\mathrm{d}s = \int_0^T h_i(s)G_x(s,1,\alpha_j)\,\mathrm{d}s, \quad \forall (i,j)\in\{1,2\}. \tag{4.18}$$

Since $\alpha_1 \neq \alpha_2$ the functions $G_x(\cdot\,,1,\alpha_1)$ and $G_x(\cdot\,,1,\alpha_2)$ are not proportional (and are at least in $C^1([0,T])$). Let us forget for a moment that h_1 and h_2 are assumed to be vanish at $t=0$, then we take $h_i = G_x(\cdot\,,1,\alpha_i)$ and have

$$\det(A) = \int_0^T G_x(s,1,\alpha_1)^2\,\mathrm{d}s \int_0^T G_x(s,1,\alpha_2)^2\,\mathrm{d}s - \left(\int_0^T G_x(s,1,\alpha_1)G_x(s,1,\alpha_2)\,\mathrm{d}s\right)^2$$

which is positive according to the Cauchy–Schwarz inequality in $L^2([0,T])$.

Now for $\alpha > 0$ we take a nonnegative function ρ_α in $C^\infty([0,T])$ such that $\rho_\alpha^{(i)}(0) = 0$ for $i = 0,1,2$ and $\rho_\alpha(x) = 1$ for $x \geqslant \alpha$ and $\|\rho_\alpha\|_\infty \leqslant 1$. When $\alpha \to 0$ $\rho_\alpha G_x(\cdot\,,1,\alpha_i) \to G_x(\cdot\,,1,\alpha_i)$ in $L^\infty(0,T)$.

Taking $h_i = \rho_\alpha G_x(\cdot\,,1,\alpha_i)$ will give again, for $\alpha > 0$ small enough, $\det(A) > 0$.

A h_i in the appropriate space for which $\det(A)$ is not zero.

Let us point out that once one couple (h_1,h_2) is found to ensure that $\det(A) \neq 0$ one may (by bilinearity of the determinant) choose h_1 and h_2 as small as one may want.

5. The case of initial moving fluid

We have assumed in Theorem 3.1 that u satisfies the initial condition (2.3). This is not essential in the sense that if $u(t=0)$ is not zero one can prove, along the same lines of the proof of Theorem 3.1, the following result:

Theorem 5.1. *Let $u_0 \in C^2(0,1)$ satisfy $u_0(0) = 0$ and $u_0(1) = 0$. For $h \in H^1(0,T)$ let v denotes the solution of*

$$v_t + \left(\frac{v^2}{2}\right)_x - v_{xx} = 0 \quad \text{in } (0,T)\times(0,1), \tag{5.1}$$

$$v(t,x=0) = 0, \quad \forall t \in (0,T), \tag{5.2}$$

$$v(t,x=1) = h(t), \quad \forall t \in (0,T), \tag{5.3}$$

$$v(t=0,x) = u_0(x), \quad \forall x \in (0,1). \tag{5.4}$$

Let $\Phi(h)$ denotes the flow of v and Ψ denotes the flow of the solution u of (2.1), (2.2), *$u(t,1) = 0$, $\forall t \in (0,T)$, $u(0,x) = u_0(x)$, $\forall x \in (0,1)$.*

Then there exists $\varepsilon > 0$ such that for any $\beta_1 \in (0,1)$ and $\beta_2 \in (0,1)$ such that

$$\bigl|\beta_1 - \Psi(T,\alpha_1)\bigr| < \varepsilon, \qquad \bigl|\beta_2 - \Psi(T,\alpha_2)\bigr| < \varepsilon$$

there exists $h \in H^1(0,T)$ such that $\Phi(h)(T,\alpha_1) = \beta_1$ and $\Phi(h)(T,\alpha_2) = \beta_2$.

Let us sketch the proof: as we already said it, it is essentially the same as the one of Theorem 3.1, but one has to take care of the fact that due to u_0, when $h = 0$ the flow of v is not trivial and thus there is the a "drift" (comparable to finite dimensional control systems with drifts – see e.g. [6,32] for full descriptions of such finite dimensional control systems) in the Lagrangian controllability question. This explains that $[\beta_1,\beta_2]$ has to be close to the image of $[\alpha_1,\alpha_2]$ by $\Phi(0)(T,\cdot) = \Psi(T,\cdot)$ in the formulation of Theorem 5.1.

For $h \in W^{3,\infty}(0,T)$ the flow $\Phi(h)$ of v satisfies

$$\forall t \in [0,T], \quad \forall x \in (0,1), \quad \Phi(h)(t,x) = x + \int_0^t v\bigl(s,\Phi(h)(s,x)\bigr)\,\mathrm{d}s. \tag{5.5}$$

If one differentiates Eq. (5.5) with respect to h at $h = 0$ in the direction $\mathfrak{h} \in W^{3,\infty}(0,T)$ one gets that $\mathfrak{g}(\mathfrak{h}) := \frac{\partial\Phi}{\partial h}(0)\cdot\mathfrak{h}$ satisfies

$$\mathfrak{g}(\mathfrak{h})(0,x) = 0, \tag{5.6}$$

$$\frac{\partial \mathfrak{g}(\mathfrak{h})}{\partial t}(t,x) = \frac{\partial v}{\partial h}(0)\cdot\mathfrak{h}\bigl(s,\Phi(0)(t,x)\bigr) + \frac{\partial v}{\partial x}\bigg|_{h=0}\bigl(t,\Phi(0)(t,x)\bigr)\mathfrak{g}(\mathfrak{h})(t,x). \tag{5.7}$$

Let us point out that we have $\Phi(0)=\Psi$. Therefore according to (5.6) and (5.7), one gets

$$\mathfrak{g}(\mathfrak{h})(T,x)=\int_0^T g(\mathfrak{h})\big(s,\Psi(s,x)\big)\mathrm{e}^{-\int_T^s u_x(\rho,\Psi(\rho,x))\,\mathrm{d}\rho}\,\mathrm{d}s, \tag{5.8}$$

where $g(\mathfrak{h})$ satisfies

$$g(\mathfrak{h})_t-g(\mathfrak{h})_{xx}+u_x g(\mathfrak{h})+g(\mathfrak{h})_x u=0 \quad \text{in } (0,T)\times(0,1), \tag{5.9}$$

$$g(\mathfrak{h})(t,x=0)=0, \quad \forall t\in(0,T), \tag{5.10}$$

$$g(\mathfrak{h})(t,x=1)=\mathfrak{h}(t), \quad \forall t\in(0,T), \tag{5.11}$$

$$g(\mathfrak{h})(t=0,x)=0, \quad \forall x\in(0,1). \tag{5.12}$$

If we can prove that for two different $\mathfrak{h}_1$ and $\mathfrak{h}_2$ the matrix

$$\begin{pmatrix} \mathfrak{g}(\mathfrak{h}_1)(T,\alpha_1) & \mathfrak{g}(\mathfrak{h}_1)(T,\alpha_2) \\ \mathfrak{g}(\mathfrak{h}_2)(T,\alpha_1) & \mathfrak{g}(\mathfrak{h}_2)(T,\alpha_2) \end{pmatrix}$$

is invertible we can conclude as in the proof of Theorem 3.1.

Instead of (4.16) one considers $G(\cdot,\cdot,z)$ the solution of (4.17) and

$$\begin{aligned} &G_t(t,x,z)+G_{xx}(t,x,z)+uG_x(t,x,z)=\delta_z(t,x)\mathrm{e}^{-\int_T^t u_x(\rho,\Psi(\rho,z))\,\mathrm{d}\rho}\,\mathrm{d}s, \quad \forall (t,x)\in(0,T)\times(0,1),\\ &G(\cdot,x,z)=0, \quad x\in\{0,1\}. \end{aligned} \tag{5.13}$$

and, similarly to (4.18), one gets

$$\mathfrak{g}(\mathfrak{h}_i)(T,\alpha_j)=\int_0^T \mathfrak{h}_i(s)G_x(s,1,\alpha_j)\,\mathrm{d}s. \tag{5.14}$$

One can therefore conclude as in the proof of the theorem by the following argument:

Let $\mu:\mathbb{R}\to\mathbb{R}$ be a C^∞ map with support in $[-1,1]$ such that $0\leqslant\mu\leqslant 1$ and $\int_{[-1,1]}\mu\,\mathrm{d}x=1$, and let us define

$$\mu_\alpha(x):=\mu(\alpha x)/\alpha, \qquad G_x^\alpha(t,x,z):=\int_0^T \mu(t-s)G_x(s,x,z)\,\mathrm{d}s.$$

Again by considering $\mathfrak{h}_i=\rho_\alpha G_x^\alpha(\cdot,1,\alpha_i)$ for $i=1,2$ for $\alpha>0$ small enough (ρ_α has been defined in the proof of Theorem 3.1) which tends to $G_x(\cdot,1,\alpha_i)$ in $L^\infty(0,T)$ will give

$$\det\begin{pmatrix} \mathfrak{g}(\mathfrak{h}_1)(T,\alpha_1) & \mathfrak{g}(\mathfrak{h}_1)(T,\alpha_2) \\ \mathfrak{g}(\mathfrak{h}_2)(T,\alpha_1) & \mathfrak{g}(\mathfrak{h}_2)(T,\alpha_2) \end{pmatrix}$$

for α small enough since as in the proof of Theorem 3.1 for $\mathfrak{h}_i=G_x(s,1,\alpha_j)$ the above determinant is positive.

6. Prescribing the homeomorphism

As it is recalled in Section 3.1, it has been proven in [23] that one can prescribe any preserving order homeomorphism between I_1 and I_2 in the framework of the heat equation.

In our present model, it would mean that if $\mathcal{H}:I_1\to I_2$ is a homeomorphism which preserves order then there exists h such that $\phi(h)(\cdot,T)=\mathcal{H}$.

In fact, we are not able to prove this property here, nevertheless, it is straightforward that the following proposition holds:

Proposition 6.1. *Let us consider for $k\in\mathbb{N}$ a finite number of points $(a_i)_{i=1,\dots,k}$, with $\alpha_1=a_1<a_2<\cdots<a_k=\alpha_2$. There exists $\varepsilon>0$ such that if k points $(b_i)_{i=1,\dots,k}$ satisfy $\max_{i=1,\dots,k}|a_i-b_i|\leqslant\varepsilon$ then there exists $h\in H^1(0,T)$ such that $\phi(h)(T,a_i)=b_i$ for $i=1,\dots,k$.*

Proof. It suffices to adapt the proof of Theorem 3.1 by considering the following map

$$(\lambda_i)_{i=1,\dots,k} \in \mathbb{R}^k \mapsto \left(\phi\left(\sum_{j=1}^{k} \lambda_j h_j\right)(T, a_i)\right)_{i=1,\dots,k} \in \mathbb{R}^k, \tag{6.1}$$

and then by choosing h_j, $j = 1, \dots, k$, so that the map (6.1) defines a local diffeomorphism in a neighborhood of 0, which is possible through an analogous argument as the one of Theorem 3.1. □

Remark 2. If one tries to make numerical simulations on how to prescribe any given homeomorphism, in the context of Theorem 3.1, then one has certainly to use Proposition 6, since numerical homeomorphisms may be described by discrete maps. So far as we know, these numerical tests have not yet been performed successfully, the main difficulty being the way one computes the control, even in the case of the heat equation.

7. The semilinear heat equation

One can also deal with semilinear heat equations for which almost the same proof as the one of Theorem 3.1 gives:

Theorem 7.1. *Assume that u satisfies*

$$u_x - u_{xx} + \varepsilon u|u|^{p-1} = 0 \quad \text{in } (0,T) \times (0,1) \text{ with } \varepsilon = 1 \text{ or } \varepsilon = -1 \tag{7.1}$$

instead of (2.1)*, together with* (2.2)–(2.4)*, then there exists $\varepsilon > 0$ such that if for $i = 1, 2$ $|\alpha_i - \beta_i| < \varepsilon$, there exists $h \in H^1(0,T)$ for which the solution u of* (7.1), (2.2)–(2.4) *is well-defined on $t \in (0,T)$ and the flow $\phi(h)$ of u defined by* (2.5) *satisfies $\phi(h)(T, \alpha_1) = \beta_1$ $\phi(h)(T, \alpha_2) = \beta_2$. Moreover one can impose*

$$\left\| \max_{i=1,2} \left|\phi(h)(\cdot, \alpha_i) - \alpha_i\right| \right\|_{L^\infty(0,T)} \to 0$$

when $\varepsilon \to 0$.

Sketch of the proof. Let us remark that due to some blow-up phenomena that might occur when $\varepsilon = -1$ in (7.1) it might be strange to get such a result, but the proof of Theorem 3.1 shows that the desired locality implies that one has to take h_1 and h_2 very small and thus one can then avoid the blow-up for Eq. (7.1) on $[0, T]$ by taking h_1 and h_2 small enough in $W^{3,\infty}(0,T)$ (see [3]).

The remaining part of the proof is identical to the one of Theorem (3.1).

We refer to [10] where such a phenomenon (preventing the blow-up meanwhile – and by – controlling) is described in the classical framework of controllability using stabilization methods and the connectivity of the set of the steady states of (7.1). □

Let us also mention that in the work [23] partial global results were obtained for the Lagrangian controllability provided that T is large enough, using again implicit controls and the results in [10]. In the context of Theorem 7.1, the localization allows to give again an almost explicit control.

8. Some other models: an example and a counter-example

8.1. Other viscous conservation laws equations

The choice of the nonlinearity in the quasilinear equation (2.1) is not essential, and one can easily adapt the proof to viscous approximations of more general conservation laws. Thus one gets

Theorem 8.1. *Assume that u satisfies instead of* (2.1) *the following equations*:

$$u_t - u_{xx} + \big(f(u)\big)_x = 0 \quad \text{in } (0,T) \times (0,1), \tag{8.1}$$

together with (2.2)–(2.4)*, where f is C^3, convex, $f(0) = f'(0) = 0$, then the result of Theorem* 3.1 *remains true.*

Remark 3. The assumptions $f(0) = f'(0) = 0$ ensure that for the Θ corresponding to (8.1), one stills gets (4.3) and (4.4).

8.2. The case of the non-viscous Burgers equation

In the case of the non-viscous Burgers equation, namely when (2.1) is replaced by

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{8.2}$$

for which the boundary controllability has been studied in [1,22], the situation is in some sense clearer: since the good notion of solution is the one of viscosity solutions, both points α_1 and α_2 are only possibly moved leftward through any control $h(t)$ at $x = 1$ (when $h \geqslant 0$ the solution of (8.2) with (2.2), (2.4) is zero).

Moreover if one replaces (2.2) by another control, i.e.

$$u(t, x = 0) = \tilde{h}(t), \quad \forall t \in (0, T) \tag{8.3}$$

for some function $\tilde{h}$, we have then a control system (8.2), (2.3), (8.3), (2.4), for which, due to the jump condition at shocks (see [19]), the result of Theorem 3.1 is not true: basically if α_1 is moved leftward, so has to be α_2.

9. The case of higher dimensions

In higher dimensions it has been proven in [23] the following

Theorem 9.1. *Let Ω be a bounded convex open subset of $\mathbb{R}^N$, with $N \geqslant 1$. Let F_1 and F_2 be two closed C^3 isotopic subsets of $\mathbb{R}^N$ included in Ω. Let also ω an open subset of $\mathbb{R}^N$ whose closure is in $\Omega \setminus F_1 \cup F_2$. Then there exist $T > 0$ and $h \in L^2((0, T) \times \omega)$ such that if u denotes the weak solution of*

$$u_t - \Delta u = h \quad \text{in } (0, T) \times \Omega, \tag{9.1}$$

$$u = 0 \quad \text{on } \big((0, T) \times \partial\Omega\big) \cup (0 \times \Omega), \tag{9.2}$$

then for every $x_0 \in F_1$ the solution $\nu(\cdot, x_0)$ of

$$\frac{\partial \nu}{\partial t}(t, x_0) = \nabla u\big(t, \nu(t, x_0)\big), \quad t \in (0, T),$$
$$\nu(0, x_0) = x_0$$

is well defined for $t \in [0, T]$ and satisfies $\nu(T, x_0) \in F_2$, moreover

$$F_1 \to F_2,$$
$$x_0 \to \nu(T, x_0)$$

is into and onto.

Let us recall that F_1 and F_2 will be said to be C^3 isotopic if there exists a C^3 map $H : [0, 1] \times \Omega \to \Omega$ such that $\forall t \in [0, 1]$ $H(t, \cdot)$ is a diffeomorphism and $H(1, F_1) = F_2$, $H(0, \cdot) = \mathrm{Id}$.

One may wonder if this or an equivalent result is true for some classical models of fluids in dimension 2 or 3. Of course the result described in Theorem 9.1 can be related to the motion of the dust in a fluid (the air) but it should be interesting and more realistic to get such a result for the Stokes or the Navier–Stokes equations.

We are not able yet to prove such a result for the Stokes and Navier–Stokes equations. Let us only mention that if one considers a formal flow of the solution of the controlled Burgers (vector) equation on a bounded domain Ω of $\mathbb{R}^3$, namely

$$v_t - \Delta v + (v \cdot \nabla)v = 0, \tag{9.3}$$

$$v(t, \sigma) = h(t, \sigma), \quad \forall \sigma \in \Sigma_0, \tag{9.4}$$

$$v(t, \sigma) = 0, \quad \forall \sigma \in \Sigma \setminus \Sigma_0, \tag{9.5}$$

$$v(0, x) = 0, \quad x \in \Omega, \tag{9.6}$$

where h is the control acting on a part Σ_0 of $\Sigma = \partial\Omega$, for which we want to study the effect of the control on the flow of v (i.e. the solution of $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = v(t, x(t)), x(0)$ given), then the linearized flow around zero with respect to the control leads to consider, at least formally in the case when $\partial\Omega$ is not regular for example, the map

$$\begin{aligned} L^2\big(0, T, L^2(\Sigma_0)^3\big) &\to L^2(\Omega)^3, \\ h &\mapsto \left(y \mapsto \int_0^T u(s, y)\,\mathrm{d}s\right) \end{aligned} \tag{9.7}$$

where the vector valued function u satisfies

$$u_t - \Delta u = 0 \quad \text{in } (0, T) \times \Omega, \tag{9.8}$$

$$u(t, \sigma) = h(t, \sigma), \quad \forall\sigma \in \Sigma_0,\ \forall t \in (0, T), \tag{9.9}$$

$$u(t, \sigma) = 0, \quad \forall\sigma \in \Sigma \setminus \Sigma_0,\ \forall t \in (0, T), \tag{9.10}$$

$$u(0, \cdot) = 0 \quad \text{in } \Omega. \tag{9.11}$$

To derive formally 9.7, one simply use the fact that the flow Ψ of v satisfies formally

$$\Psi(s, x) = x + \int_0^s v\big(\rho, \Psi(s, x)\big)\,\mathrm{d}\rho$$

and one derives this expression with respect to h at $h = 0$. The same argument as in the proof of Theorem 3.1 gives the following

Proposition 9.2. *The map*

$$\begin{aligned} L^2\big(0, T, L^2(\Sigma_0)^3\big) &\to L^2(\Omega)^3, \\ h &\mapsto \left(y \mapsto \int_0^T u(s, y)\,\mathrm{d}s\right) \end{aligned}$$

where u satisfies (9.8)–(9.11) *has a dense image in* $L^2(\Omega)^3$.

Of course, what seems to us the main difficulties to conclude for the nonlinear problem from Proposition 9.2 is that, in general, the approximate controllability of the linearized operator is not enough to conclude for the initial control problems and that we also might lose some topological or geometrical properties of the subset that is being moved by the flow unless some regularity of v is prescribed or asserted set of $\mathbb{R}^2$ and γ a closed C^1 Jordan curve bounding an open subset of $\mathbb{R}^2$ compactly included in Ω. Let $\Sigma := \partial\Omega$such that if γ' is a closed C^1 Jordan curve "close enough" to γ, then for any $x_0 \in \gamma$, the solution of $x_0 \to x(T)$ being into and onto.

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