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Singular limits for a 4-dimensional semilinear elliptic problem with exponential nonlinearity

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Abstract

Using some nonlinear domain decomposition method, we prove the existence of branches of solutions having singular limits for some 4-dimensional semilinear elliptic problem with exponential nonlinearity.

Résumé

En utilisant une variante non linéaire de la méthode de décomposition de domaines, nous démontrons l'existence de branches de solutions ayant une limite singulière, pour une équation semilinéaire elliptique avec nonlinéarité exponentielle, en dimension 4.

Keywords: Singular perturbations; Concentration phenomena

1. Introduction and statement of the results

In the last decade important work has been devoted to the understanding of singularly perturbed problems, mostly in a variational framework. In general, a Liapunov–Schmidt type reduction argument is used to reduce the search of solutions of singularly perturbed nonlinear partial differential equations to the search of critical points of some function that is defined over some finite dimensional domain.

One of the purposes of the present paper is to present a rather efficient method to solve such singularly perturbed problems. This method has already been used successfully in geometric context (constant mean curvature surfaces, constant scalar curvature metrics, extremal Kähler metrics, manifolds with special holonomy, ...) but has never appeared in the framework of nonlinear partial differential equations. We felt that, given the interest in singular perturbation problems, it was worth illustrating this method on the following model problem:

Assume that $\Omega \subset \mathbb{R}^4$ is a regular bounded open domain in $\mathbb{R}^4$. We are interested in positive solutions of

$$\begin{cases} \Delta^2 u = \rho^4 \mathrm{e}^u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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when the parameter ρ tends to 0. Obviously, the application of the implicit function theorem yields, for ρ close to 0, the existence of a smooth one parameter family of solutions $(u_\rho)_\rho$ that converges uniformly to 0 as ρ tends to 0. This branch of solutions is usually referred to as the branch of *minimal solutions* and there is by now quite an important literature that is concerned with the understanding of this particular branch of solutions [12].

The problem we would like to consider is the existence of other branches of solutions as ρ tends to 0. To describe our result, let us denote by $G(x,\cdot)$ the solution of

$$\begin{cases} \Delta^2 G(x,\cdot) = 64\pi^2 \delta_x & \text{in } \Omega, \\ G(x,\cdot) = \Delta G(x,\cdot) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

It is easy to check that the function

$$R(x,y) := G(x,y) + 8\log|x-y| \tag{3}$$

is a smooth function. Finally, we define

$$W(x^1,\dots,x^m) := \sum_{j=1}^m R(x^j,x^j) + \sum_{j\neq\ell} G(x^j,x^\ell). \tag{4}$$

Our main result reads:

Theorem 1.1. *Assume that* $(x^1,\dots,x^m)$ *is a* nondegenerate *critical point of* W*, then there exist* $\rho_0>0$ *and* $(u_\rho)_{\rho\in(0,\rho_0)}$*, a one parameter family of solutions of* (1)*, such that*

$$\lim_{\rho\to 0} u_\rho = \sum_{j=1}^m G(x^j,\cdot)$$

in $C^{4,\alpha}_{\mathrm{loc}}(\Omega - \{x^1,\dots,x^m\})$.

This result is in agreement with the result of Lin and Wei [6] where sequences of solutions of (1) that blow up as ρ tends to 0 are studied. Indeed, in this paper, the authors show that blow up points can only occur at critical points of W.

Our result reduces the study of nontrivial branches of solutions of (1) to the search of critical points of the function W defined in (4). Observe that the assumption on the nondegeneracy of the critical point is a rather mild assumption since it is certainly fulfilled for generic choice of the regular bounded open domain Ω.

Semilinear equations involving fourth order elliptic operator and exponential nonlinearity appear naturally in conformal geometry and in particular in the prescription of the so called Q-curvature on 4-dimensional Riemannian manifolds [2,3]

$$Q_g = \frac{1}{12}\left(-\Delta_g S_g + S_g^2 - 3|\mathrm{Ric}_g|^2\right)$$

where Ric_g denotes the Ricci tensor and S_g is the scalar curvature of the metric g. Recall that the Q-curvature changes under a conformal change of metric

$$g_w = \mathrm{e}^{2w} g,$$

according to

$$P_g w + 2Q_g = 2Q_{g_w}\mathrm{e}^{4w} \tag{5}$$

where

$$P_g := \Delta_g^2 + \delta\left(\frac{2}{3}S_g I - 2\,\mathrm{Ric}_g\right)d \tag{6}$$

is the Paneitz operator, which is an elliptic 4th order partial differential operator [3] and which transforms according to

$$\mathrm{e}^{4w} P_{\mathrm{e}^{2w}g} = P_g, \tag{7}$$

under a conformal change of metric $g_w := e^{2w} g$. In the special case where the manifold is the Euclidean space, the corresponding Paneitz operator is simply given by

$$P_{g_{\mathrm{eucl}}} = \Delta^2$$

in that case (5) reduces to

$$\Delta^2 w = Q_{g_w} e^{4w}$$

the solutions of which give rise to conformal metric $g_w = e^{2w} g_{\mathrm{eucl}}$ whose Q-curvature is given by Q_{g_w}. There is by now an extensive literature about this problem and we refer to [3] and [9] for references and recent developments.

When $n = 2$, the analogue of the Q-curvature is nothing but the Gauss curvature and the corresponding problem has been studied for a long time. More relevant to the present paper is the study of nontrivial branches of solutions of

$$\begin{cases} -\Delta u = \rho^2 e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

that are defined on some domain of $\mathbb{R}^2$. The study of this equation goes back to 1853 when Liouville derived a representation formula for all solutions of (8) that are defined in $\mathbb{R}^2$, [7]. Beside the applications in geometry, elliptic equations with exponential nonlinearity also arise in the modeling of many physical phenomenon such as: thermionic emission, isothermal gas sphere, gas combustion, gauge theory [15], ...

When ρ tends to 0, the asymptotic behavior of nontrivial branches of solutions of (8) is well understood thanks to the pioneer work of Suzuki [14] that characterizes their possible limits. The existence of nontrivial branches of solutions was first proven by Weston [17] and then a general result has been obtained by Baraket and Pacard [1]. More recently these results were extended, with applications to the Chern–Simons vortex theory in mind, by Esposito [5] and Del Pino, Kowalczyk and Musso [4] to handle equations of the form

$$-\Delta u = \rho^2 V e^u$$

where V is a nonconstant (positive) function. We give in Section 9 some results concerning the fourth order analogue of this equation. Let us also mention that the construction of nontrivial branches of solutions of semilinear equations with exponential nonlinearities has allowed Wente to provide counterexamples to a conjecture of Hopf [16] concerning the existence of compact (immersed) constant mean curvature surfaces in Euclidean space.

We now describe the plan of the paper: In Section 2 we discuss rotationally symmetric solutions of (1). In Section 3 we study the linearized operator about the radially symmetric solution defined in the previous section. In Section 4, we discuss the analysis of the bi-Laplace operator in weighted spaces. Both sections strongly use the b-calculus that has been developed by Melrose [11] in the context of weighted Sobolev spaces and by Mazzeo [10] in the context of weighted Hölder spaces (see also [13]).

A first nonlinear problem is studied in Section 6 where the existence of an infinite dimensional family of solutions of (1) that are defined on large balls and that are close to the rotationally symmetric solution is proven. In Section 7, we prove the existence of an infinite dimensional family of solutions of (1) that are defined on Ω with small balls removed. Finally, in Section 8, we show how elements of these infinite dimensional families can be connected together to produce the solutions of (1) that are described in Theorem 1.1. This last section borrows ideas from applied mathematics were domain decomposition methods are of common use. Section 9 is devoted to some comments. In Section 10, we explain how the results of the previous analysis can be extended to handle equations of the form $\Delta^2 u = \rho^4 V e^u$.

Note added in proof: We should mention the recent preprint of M. Clapp, C. Muñoz and M. Musso, *Singular limits for the bi-Laplacian operator with exponential nonlinearity in* $\mathbb{R}^4$ where sufficient topological conditions are given that ensure the existence of critical points of the function W.

2. Rotationally symmetric solutions

We first describe the rotationally symmetric solutions of

$$\Delta^2 u - \rho^4 e^u = 0, \tag{9}$$

that will play a central rôle in our analysis. Given $\varepsilon > 0$, we define

$$u_\varepsilon(x) := 4 \log(1 + \varepsilon^2) - 4 \log(\varepsilon^2 + |x|^2)$$

that is clearly a solution of (9) when

$$\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4}. \tag{10}$$

Let us notice that Eq. (9) is invariant under some dilation in the following sense: If u is a solution of (9) and if $\tau > 0$, then $u(\tau\cdot) + 4\log\tau$ is also a solution of (9). With this observation in mind, we define, for all $\tau > 0$

$$u_{\varepsilon,\tau}(x) := 4\log\bigl(1+\varepsilon^2\bigr) + 4\log\tau - 4\log\bigl(\varepsilon^2 + \tau^2|x|^2\bigr). \tag{11}$$

3. A linear fourth order elliptic operator on $\mathbb{R}^4$

We define the linear fourth order elliptic operator

$$\mathbb{L} := \Delta^2 - \frac{384}{(1+|x|^2)^4} \tag{12}$$

which corresponds to the linearization of (9) about the solution u_1 $(= u_{\varepsilon=1})$ that has been defined in the previous section.

We are interested in the classification of bounded solutions of $\mathbb{L}w = 0$ in $\mathbb{R}^4$. Some solutions are easy to find. For example, we can define

$$\phi_0(x) := r\partial_r u_1(x) + 4 = 4\frac{1-r^2}{1+r^2},$$

where $r = |x|$. Clearly $\mathbb{L}\phi_0 = 0$ and this reflects the fact that (9) is invariant under the group of dilations $\tau \to u(\tau\cdot) + 4\log\tau$. We also define, for $i = 1, \ldots, 4$

$$\phi_i(x) := -\partial_{x_i} u_1(x) = \frac{8x_i}{1+|x|^2},$$

which are also solutions of $\mathbb{L}\phi_j = 0$ since these solutions correspond to the invariance of (9) under the group of translations $a \to u(\cdot + a)$.

The following result classifies all bounded solutions of $\mathbb{L}w = 0$ that are defined in $\mathbb{R}^4$.

Lemma 3.1. *Any bounded solution of $\mathbb{L}w = 0$ defined in $\mathbb{R}^4$ is a linear combination of ϕ_i for $i = 0, 1, \ldots, 4$.*

Proof. We consider on $\mathbb{R}^4$ the Euclidean metric $g_{\mathrm{eucl}} = \mathrm{d}x^2$ and the spherical metric

$$g_{S^4} = \frac{4}{(1+|x|^2)^2}\,\mathrm{d}x^2$$

induced by the inverse of the stereographic projection

$$\begin{aligned} \Pi : \mathbb{R}^4 &\longrightarrow S^4 \\ x &\longmapsto \left(\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}\right). \end{aligned}$$

According to [3] we have $P_{g_{S^4}} = \Delta_{S^4}^2 - 2\Delta_{S^4}$ and $P_{g_{\mathrm{eucl}}} = \Delta^2$. Therefore, we obtain from (7)

$$\left(\frac{4}{(1+|x|^2)^2}\right)^2 \bigl(\Delta_{S^4}^2 - 2\Delta_{S^4}\bigr) = \Delta^2.$$

In particular, if $w : (\mathbb{R}^4, g_{\mathrm{eucl}}) \to \mathbb{R}$ is a bounded solution of $\mathbb{L}w = 0$ then, $w : (\mathbb{R}^4, g_{S^4}) \to \mathbb{R}$ is a bounded solution of

$$\bigl(\Delta_{S^4}^2 - 2\Delta_{S^4} - 24\bigr)w = 0 \tag{13}$$

away from the north pôle $N \in S^4$ (with slight abuse of notation we identify w with $w \circ \Pi^{-1}$). It is easy to check that the isolated singularity at the north pôle is removable (since w is assumed to be bounded) and hence (13) holds on all S^4.

We now perform the eigenfunction decomposition of w in terms of the eigendata of the Laplacian on S^4. We decompose

$$w = \sum_{\ell \geqslant 0} w_\ell$$

where w_ℓ belongs to the ℓ-th eigenspace of $-\Delta_{S^4}$, namely, w_ℓ satisfies $\Delta_{S^4} w_\ell = -\lambda_\ell w_\ell$ with

$$\lambda_\ell := \ell(\ell+3).$$

We get from (13)

$$\big(\lambda_\ell^2 + 2\lambda_\ell - 24\big) w_\ell = 0.$$

Hence, $w_\ell = 0$ for all ℓ except eventually those for which $\lambda_\ell = 4$. This implies that $w : S^4 \longrightarrow \mathbb{R}$ is a combination of the eigenfunctions associated to $\ell = 1$ that are given by $\varphi_i(y) = y_i$ for $i = 1, \dots, 5$, where $y = (y_1, \dots, y_5) \in S^4$. The sphere being parameterized by the inverse of the stereographic projection we may write $y = \Pi(x)$. Then, the functions $4\varphi_i$ precisely correspond to the functions ϕ_i for $i = 1, \dots, 4$, while the function $4\varphi_5$ corresponds to the function ϕ_0. This completes the proof of the result. □

Let B_r denote the ball of radius r centered at the origin in $\mathbb{R}^4$.

Definition 3.1. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, we define the Hölder weighted space $\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4)$ as the space of functions $w \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}^4)$ for which the following norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4)} := \|w\|_{\mathcal{C}^{k,\alpha}(\bar{B}_1)} + \sup_{r \geqslant 1} \big(r^{-\mu} \| w(r\cdot) \|_{\mathcal{C}^{k,\alpha}(\bar{B}_1 - B_{1/2})} \big),$$

is finite.

More details about these spaces and their use in nonlinear problems can be found in [13]. Roughly speaking, functions in $\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4)$ are bounded by a constant times $(1+r^2)^{\mu/2}$ and have their ℓ-th partial derivatives that are bounded by $(1+r^2)^{(\mu-\ell)/2}$, for $\ell = 1, \dots, k+\alpha$.

As a consequence of the result of Lemma 3.1, we have the:

Proposition 3.1. *Assume that $\mu > 1$ and $\mu \notin \mathbb{N}$, then*

$$\begin{aligned} L_\mu : \mathcal{C}^{4,\alpha}_\mu\big(\mathbb{R}^4\big) &\longrightarrow \mathcal{C}^{0,\alpha}_{\mu-4}\big(\mathbb{R}^4\big) \\ w &\longmapsto \mathbb{L} w \end{aligned}$$

is surjective.

Proof. The mapping properties of L_μ are very sensitive to the choice of the weight μ. In particular, it is proved in [8], [11] and [10] (see also [13]) that L_μ has closed range and is Fredholm provided μ is not an indicial root of $\mathbb{L}$ at infinity. Recall that $\zeta \in \mathbb{R}$ is an indicial root of $\mathbb{L}$ at infinity if there exists a smooth function v on S^3 such that

$$\mathbb{L}\big(|x|^\zeta v\big) = \mathcal{O}\big(|x|^{\zeta-5}\big)$$

at infinity. It is easy to check that the indicial roots of $\mathbb{L}$ at infinity are all $\zeta \in \mathbb{Z}$. Indeed, let e be an eigenfunction of $-\Delta_{S^3}$ that is associated to the eigenvalue $\gamma(\gamma+2)$, where $\gamma \in \mathbb{N}$, hence

$$\Delta_{S^3} e = -\gamma(\gamma+2) e.$$

Then

$$\mathbb{L}\big(|x|^\zeta e\big) = (\zeta-\gamma)(\zeta-\gamma-2)(\zeta+2+\gamma)(\zeta+\gamma)|x|^{\zeta-4} e + \mathcal{O}\big(|x|^{\zeta-8}\big).$$

Therefore, we find that $-\gamma-2$, $-\gamma$, γ and $\gamma+2$ are indicial roots of $\mathbb{L}$ at infinity. Since the eigenfunctions of the Laplacian on the sphere constitute a Hilbert basis of $L^2(S^3)$, we have obtained all the indicial roots of $\mathbb{L}$ at infinity.

If $\mu \notin \mathbb{Z}$, some duality argument (in weighted Lebesgue spaces) shows that the operator L_μ is surjective if and only if the operator $L_{-\mu}$ is injective. And, still under this assumption

$$\dim \operatorname{Ker} L_\mu = \dim \operatorname{Coker} L_{-\mu}.$$

The result of Lemma 3.1 precisely states that the operator L_μ is injective when $\mu < -1$. Therefore, we conclude that L_μ is surjective when $\mu > 1$, $\mu \notin \mathbb{Z}$. This completes the proof of the result. □

4. Analysis of the bi-Laplace operator in weighted spaces

Given $x^1, \dots, x^m \in \Omega$ we define $X := (x^1, \dots, x^m)$,

$$\overline{\Omega}^*(X) := \overline{\Omega} - \{x^1, \dots, x^m\},$$

and we choose $r_0 > 0$ so that the balls $B_{2r_0}(x^i)$ of center x^i and radius r_0 are mutually disjoint and included in Ω. For all $r \in (0, r_0)$ we define

$$\overline{\Omega}_r(X) := \overline{\Omega} - \bigcup_{j=1}^m B_r(x^j).$$

With these notations, we have the:

Definition 4.1. Given $k \in \mathbb{R}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we introduce the Hölder weighted space $\mathcal{C}_\nu^{k,\alpha}(\overline{\Omega}^*(X))$ as the space of functions $w \in \mathcal{C}_{\mathrm{loc}}^{k,\alpha}(\overline{\Omega}^*(X))$ for which the following norm

$$\|w\|_{\mathcal{C}_\nu^{k,\alpha}(\overline{\Omega}^*(X))} := \|w\|_{\mathcal{C}^{k,\alpha}(\overline{\Omega}_{r_0/2}(X))} + \sum_{j=1}^m \sup_{r \in (0, r_0/2)} \left(r^{-\nu} \left\| w(x^j + r\cdot) \right\|_{\mathcal{C}^{k,\alpha}(\overline{B}_2 - B_1)}\right),$$

is finite.

Again, these spaces have already been used many times in nonlinear contexts and we refer to [13] for further details and references. Functions that belong to $\mathcal{C}_\nu^{k,\alpha}(\overline{\Omega}^*(X))$ are bounded by a constant times the distance to X to the power ν and have their ℓ-th partial derivatives that are bounded by a constant times the distance to X to the power $\nu - \ell$, for $\ell = 1, \dots, k + \alpha$.

When $k \geqslant 2$, we denote by $[\mathcal{C}_\nu^{k,\alpha}(\overline{\Omega}^*(X))]_0$ be the subspace of functions $w \in \mathcal{C}_\nu^{k,\alpha}(\overline{\Omega}^*(X))$ satisfying $w = \Delta w = 0$ on $\partial\Omega$.

We will use the following:

Proposition 4.1. *Assume that $\nu < 0$ and $\nu \notin \mathbb{Z}$, then*

$$\begin{aligned} \mathcal{L}_\nu : \left[\mathcal{C}_\nu^{4,\alpha}\left(\overline{\Omega}^*(X)\right)\right]_0 &\longrightarrow \mathcal{C}_{\nu-4}^{0,\alpha}\left(\overline{\Omega}^*(X)\right), \\ w &\longmapsto \Delta^2 w \end{aligned}$$

is surjective.

Proof. Again this result follows from the theory developed in [8], [11] and [10] (see also [13]). The mapping properties of $\mathcal{L}_\nu$ depend on the choice of the weight ν. The operator $\mathcal{L}_\nu$ has closed range and is Fredholm provided ν is not an indicial root of Δ^2 at the points x^j. This time, $\zeta \in \mathbb{R}$ is an indicial root of Δ^2 at x^j if there exists a smooth function v on S^3 such that

$$\Delta^2\left(|x - x^j|^\zeta v\right) = \mathcal{O}\left(|x - x^j|^{\zeta - 3}\right)$$

at x^j. As in Proposition 4.1, it is easy to check that the indicial roots of Δ^2 at x^j are all $\zeta \in \mathbb{Z}$.

If $\nu \notin \mathbb{Z}$, some duality argument (in weighted Lebesgue spaces) shows that the operator $\mathcal{L}_\nu$ is surjective if and only if the operator $\mathcal{L}_{-\nu}$ is injective. And, still under this assumption

$$\dim \operatorname{Ker} \mathcal{L}_\nu = \dim \operatorname{Coker} \mathcal{L}_{-\nu}.$$

We claim that the operator $\mathcal{L}_\nu$ is injective if $\nu > 0$. Indeed, isolated singularities of any solution $w \in \mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(X))$ of $\Delta^2 w = 0$ in Ω^* are removable if $\nu > 0$. Therefore, w is a bi-harmonic function in Ω with $w = \Delta w = 0$ on $\partial\Omega$. This implies that $w \equiv 0$ and hence $\mathcal{L}_\nu$ is injective when $\nu > 0$ as claimed.

We then conclude that $\mathcal{L}_\nu$ is surjective when $\nu < 0$, $\nu \notin \mathbb{Z}$. This completes the proof of the result. □

Given $y^1, \dots, y^m$ close enough to $x^1, \dots, x^m$, we set $Y := (y^1, \dots, y^m)$ and we define a family of diffeomorphisms $D(= D_{X,Y})$

$$D : \Omega \longrightarrow \Omega$$

depending smoothly on $y^1, \dots, y^m$ by

$$D(x) := x + \sum_{j=1}^m \chi_{r_0}\big(x - x^j\big)\big(x^j - y^j\big), \tag{14}$$

where χ_{r_0} is a cutoff function identically equal to 1 in $B_{r_0/2}$ and identically equal to 0 outside B_{r_0}. In particular, $D(y^j) = x^j$ for each j, provided $\|X - Y\| \leqslant r_0/2$.

The equation $\Delta^2 \tilde{w} = \tilde{f}$ where $\tilde{f} \in \mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}^*(Y))$ can be solved by writing $\tilde{w} = w \circ D$ and $\tilde{f} = f \circ D$ so that w is a solution of the problem

$$\Delta^2 w + \big(\Delta^2(w \circ D) - \big(\Delta^2 w\big) \circ D\big) \circ D^{-1} = f \tag{15}$$

where this time $f \in \mathcal{C}^{0,\alpha}_\nu(\overline{\Omega}^*(X))$. It should be clear that

$$\big\|\big(\Delta^2(w \circ D) - \big(\Delta^2 w\big) \circ D\big) \circ D^{-1}\big\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}^*(X))} \leqslant c\|Y - X\|\|w\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(X))} \tag{16}$$

provided $\|Y - X\| \leqslant r_0/2$.

We fix $\nu < 0$, $\nu \notin \mathbb{Z}$ and use the result of Proposition 4.1 to get a right inverse $\mathcal{G}_{\nu,X}$ for $\mathcal{L}_\nu : [\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(X))]_0 \to \mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}^*(X))$. The estimate (16) together with a perturbation argument shows that (15) is solvable provided Y is close enough to X. This provides a right inverse $\mathcal{G}_{\nu,Y}$ that depends continuously (and in fact smoothly) on the points $y^1, \dots, y^m$ in the sense that

$$f \in \mathcal{C}^{0,\alpha}_\nu\big(\overline{\Omega}^*(X)\big) \longmapsto \mathcal{G}_{\nu,Y}(f \circ D_{X,Y}) \circ (D_{X,Y})^{-1} \in \mathcal{C}^{4,\alpha}_\nu\big(\overline{\Omega}^*(X)\big)$$

depends smoothly on Y.

5. Bi-harmonic extensions

Given $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$ and $\psi \in \mathcal{C}^{2,\alpha}(S^3)$ we define $H^i (= H^i(\varphi, \psi; \cdot))$ to be the solution of

$$\begin{cases} \Delta^2 H^i = 0 & \text{in } B_1, \\ H^i = \varphi & \text{on } \partial B_1, \\ \Delta H^i = \psi & \text{on } \partial B_1, \end{cases} \tag{17}$$

where, as already mentioned, B_1 denotes the unit ball in $\mathbb{R}^4$.

We set $B_1^* = B_1 - \{0\}$. As in the previous section, we define:

Definition 5.1. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, we introduce the Hölder weighted space $\mathcal{C}^{k,\alpha}_\mu(\overline{B}_1^*)$ as the space of functions $w \in \mathcal{C}^{k,\alpha}_{\text{loc}}(\overline{B}_1^*)$ for which the following norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\mu(\overline{B}_1^*)} := \sup_{r \in (0,1/2)} \big(r^{-\mu}\big\|w(r\cdot)\big\|_{\mathcal{C}^{k,\alpha}(\overline{B}_2 - B_1)}\big),$$

is finite.

When $\Omega = B_1$, $m = 1$ and $x^1 = 0$, this agrees with the space and norm already defined in the previous section.

Let $e_1, \dots, e_4$ be the coordinate functions on S^3. We prove the:

Lemma 5.1. *Assume that*

$$\int_{S^3} (8\varphi - \psi)\,\mathrm{d}\,\mathrm{vol}_{S^3} = 0 \quad \textit{and also that} \quad \int_{S^3} (12\varphi - \psi) e_\ell\,\mathrm{d}\,\mathrm{vol}_{S^3} = 0 \tag{18}$$

for $\ell = 1, \ldots, 4$. *Then there exists* $c > 0$ *such that*

$$\left\| H^i(\varphi, \psi; \cdot)\right\|_{\mathcal{C}^{4,\alpha}_2(\bar{B}_1^*)} \leqslant c\big(\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)}\big).$$

Proof. There are many ways to proof this result. Here is a simple one that has the advantage to be quite flexible. We consider the eigenfunction decomposition of φ and ψ in terms of the eigenfunctions of $-\Delta_{S^3}$.

$$\varphi = \sum_{\ell \geqslant 0} \varphi_\ell \quad \text{and} \quad \psi = \sum_{\ell \geqslant 0} \psi_\ell, \tag{19}$$

where, for each $\ell \geqslant 0$, the functions φ_ℓ and ψ_ℓ belong to the ℓ-th eigenspace of $-\Delta_{S^3}$, namely

$$\Delta_{S^3}\varphi_\ell = -\ell(2+\ell)\varphi_\ell \quad \text{and} \quad \Delta_{S^3}\psi_\ell = -\ell(2+\ell)\psi_\ell.$$

Then the function H^i can be explicitly written as

$$H^i = \sum_{\ell \geqslant 0} r^\ell \left(\varphi_\ell - \frac{1}{4(\ell+2)}\psi_\ell\right) + \sum_{\ell \geqslant 0} \frac{1}{4(\ell+2)} r^{2+\ell}\psi_\ell. \tag{20}$$

Observe that, under the hypothesis (18), the coefficients of r^0 and r^1 vanish and hence, at least formally, the expansion of H only involves powers of r that are greater than or equal to 2.

We claim that

$$\|\varphi_\ell\|_{L^\infty} \leqslant c_\ell \|\varphi\|_{L^\infty}, \qquad \|\psi_\ell\|_{L^\infty} \leqslant c_\ell \|\psi\|_{L^\infty}$$

where the constant c_ℓ depends polynomially on ℓ. For example, we can write $\varphi_\ell = a_\ell e_\ell$ where $a_\ell \in \mathbb{R}$ and e_ℓ is an eigenvalue of $-\Delta_{S^3}$ that is normalized to have L^2 norm equal to 1. Then

$$|a_\ell| = \left| \int_{S^3} \varphi_\ell e_\ell \,\mathrm{d}v_{S^3} \right| \leqslant c\|\varphi\|_{L^2} \leqslant c\|\varphi\|_{L^\infty}.$$

Next, e_ℓ solves $\Delta_{S^3} e_\ell = -\ell(2+\ell)e_\ell$, we can use elliptic regularity theory to show that the $L^\infty(S^3)$ norm of e_ℓ depends polynomially on ℓ. The claim then follows at once.

This immediately yields the estimate

$$\sup_{r \leqslant 1/2} \big(r^{-2}\big|H^i\big| + \big|\Delta H^i\big|\big) \leqslant c\big(\|\varphi\|_{L^\infty} + \|\psi\|_{L^\infty}\big).$$

This estimate, together with the maximum principle and standard elliptic estimates yields

$$\sup_{r \leqslant 1}\big(r^{-2}\big|H^i\big| + \big|\Delta H^i\big|\big) \leqslant c\big(\|\varphi\|_{L^\infty} + \|\psi\|_{L^\infty}\big).$$

The estimate for the derivatives of H^i now follows at once from Schauder's estimates. □

Given $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$ and $\psi \in \mathcal{C}^{2,\alpha}(S^3)$ we define (when it exists!) $H^e (= H^e(\varphi, \psi; \cdot))$ to be the solution of

$$\begin{cases} \Delta^2 H^e = 0 & \text{in } \mathbb{R}^4 - B_1, \\ H^e = \varphi & \text{on } \partial B_1, \\ \Delta H^e = \psi & \text{on } \partial B_1, \end{cases} \tag{21}$$

that decays at infinity.

Definition 5.2. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we define the space $\mathcal{C}^{k,\alpha}_\nu(\mathbb{R}^4 - B_1)$ as the space of functions $w \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}^4 - B_1)$ for which the following norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\nu(\mathbb{R}^4 - B_1)} := \sup_{r \geqslant 1}\big(r^{-\nu}\big\|w(r\cdot)\big\|_{\mathcal{C}^{k,\alpha}_\nu(\bar{B}_2 - B_1)}\big),$$

is finite.

We prove the:

Lemma 5.2. *Assume that*

$$\int_{S^3} \psi \,\mathrm{d}\,\mathrm{vol}_{S^3} = 0. \tag{22}$$

Then there exists $c > 0$ such that

$$\big\| H^e(\varphi, \psi; \cdot)\big\|_{\mathcal{C}^{4,\alpha}_{-1}(\mathbb{R}^4 - B_1)} \leqslant c\big(\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)}\big).$$

Proof. We use the notations of the previous lemma. Now, the function H^e can be explicitly written as

$$H^e = r^{-2}\varphi_0 + \sum_{\ell \geqslant 1} r^{-2-\ell}\left(\varphi_\ell + \frac{1}{4\ell}\psi_\ell\right) - \sum_{\ell \geqslant 1} \frac{1}{4\ell} r^{-\ell}\psi_\ell. \tag{23}$$

Observe that (22) implies that the expansion of H^e only involves powers of r that are lower than or equal to -1. The proof is now identical to the proof of Lemma 5.1 and left to the reader. □

Under the hypothesis of Lemma 5.2, there is uniqueness of the bi-harmonic extension of the boundary data that decays at infinity.

If $F \subset L^2(S^3)$ is a space of functions defined on S^3, we define the space $F^\perp$ to be the subspace of functions of F that are $L^2(S^3)$-orthogonal to the functions $1, e_1, \ldots, e_4$. We will need the:

Lemma 5.3. *The mapping*

$$\begin{aligned} \mathcal{P}: \mathcal{C}^{4,\alpha}\big(S^3\big)^\perp \times \mathcal{C}^{2,\alpha}\big(S^3\big)^\perp &\longrightarrow \mathcal{C}^{3,\alpha}\big(S^3\big)^\perp \times \mathcal{C}^{1,\alpha}\big(S^3\big)^\perp, \\ (\varphi, \psi) &\longmapsto \big(\partial_r H^i - \partial_r H^e, \partial_r \Delta H^i - \partial_r \Delta H^e\big) \end{aligned}$$

where $H^i = H^i(\varphi, \psi; \cdot)$ and $H^e = H^e(\varphi, \psi; \cdot)$, is an isomorphism.

Proof. Granted the explicit formula given in the previous two lemmas, we have

$$\mathcal{P}(\varphi, \psi) = \left(\sum_{\ell \geqslant 2} (\ell + 1)\left(2\varphi_\ell + \frac{1}{\ell(\ell+2)}\psi_\ell\right), \sum_{\ell \geqslant 2} 2(\ell+1)\psi_\ell\right). \tag{24}$$

We denote by $W^{k,2}(S^3)$ the Sobolev space of functions on S^3 whose weak partial derivatives, up to order k are in $L^2(S^3)$. The norm in $W^{k,2}(S^3)$ can be chosen to be

$$\|\varphi\|_{W^{k,2}(S^3)} := \left(\sum_{\ell \geqslant 0} (1+\ell)^{2k} \|\varphi_\ell\|^2_{L^2(S^3)}\right)^{1/2}$$

when the function φ is decomposed over eigenspaces of Δ_{S^3}

$$\varphi = \sum_{\ell \geqslant 0} \varphi_\ell$$

where $\Delta_{S^3}\varphi_\ell = -\ell(\ell+2)\varphi_\ell$. It follows at once that

$$\mathcal{P}: W^{k+3,2}\big(S^3\big)^\perp \times W^{k+1,2}(S^3)^\perp \longrightarrow W^{k+2,2}\big(S^3\big)^\perp \times W^{k,2}\big(S^3\big)^\perp$$

is invertible. Elliptic regularity theory then implies that the corresponding map is also invertible when defined between the corresponding Hölder spaces. □

6. The first nonlinear Dirichlet problem

For all $\varepsilon, \tau > 0$, we set

$$R_\varepsilon := \tau/\sqrt{\varepsilon}.$$

Given $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$ and $\psi \in \mathcal{C}^{2,\alpha}(S^3)$ satisfying (18), we define

$$\mathbf{u} := u_1 + H^i\big(\varphi, \psi; (\cdot/R_\varepsilon)\big).$$

We would like to find a function u solution of

$$\Delta^2 u - 24 \mathrm{e}^u = 0 \tag{25}$$

which is defined in B_{R_ε} and is a perturbation of $\mathbf{u}$. Writing $u = \mathbf{u} + v$ and using the fact that H^i is bi-harmonic, we see that this amounts to solve the equation

$$\mathbb{L} v = \frac{384}{(1+r^2)^4}\big(\mathrm{e}^{H^i(\varphi,\psi;(\cdot/R_\varepsilon))+v} - 1 - v\big). \tag{26}$$

We will need the following:

Definition 6.1. Given $\bar r \geqslant 1$, $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, the weighted space $\mathcal{C}^{k,\alpha}_\mu(\overline{B}_{\bar r})$ is defined to be the space of functions $w \in \mathcal{C}^{k,\alpha}(\overline{B}_{\bar r})$ endowed with the norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\mu(\overline{B}_{\bar r})} := \|w\|_{\mathcal{C}^{k,\alpha}(B_1)} + \sup_{r\in[1,\bar r]} \big(r^{-\mu} \big\|w(r\cdot)\big\|_{\mathcal{C}^{k,\alpha}(\overline{B}_1 - B_{1/2})}\big).$$

For all $\sigma \geqslant 1$, we denote by

$$\mathcal{E}_\sigma : \mathcal{C}^{0,\alpha}_\mu(\overline{B}_\sigma) \longrightarrow \mathcal{C}^{0,\alpha}_\mu\big(\mathbb{R}^4\big)$$

the extension operator defined by $\mathcal{E}_\sigma = f$ in $\overline{B}_\sigma$ and

$$\mathcal{E}_\sigma(f)(x) = \chi\left(\frac{|x|}{\sigma}\right) f\left(\sigma \frac{x}{|x|}\right)$$

on $\mathbb{R}^4 - \overline{B}_\sigma$ where $t \mapsto \chi(t)$ is a smooth nonnegative cutoff function identically equal to 0 for $t \geqslant 2$ and identically equal to 1 for $t \leqslant 1$. It is easy to check that there exists a constant $c = c(\mu) > 0$, independent of $\sigma \geqslant 1$, such that

$$\big\|\mathcal{E}_\sigma(w)\big\|_{\mathcal{C}^{0,\alpha}_\mu(\mathbb{R}^4)} \leqslant c \|w\|_{\mathcal{C}^{0,\alpha}_\mu(\overline{B}_\sigma)}. \tag{27}$$

We fix

$$\mu \in (1,2)$$

and denote by $\mathcal{G}_\mu$ a right inverse for $\mathbb{L}$ provided by Proposition 3.1. To find a solution of (26), it is enough to find $v \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ solution of

$$v = N(\varepsilon, \tau, \varphi, \psi; v) \tag{28}$$

where we have defined

$$N(\varepsilon, \tau, \varphi, \psi; v) := \mathcal{G}_\mu \circ \mathcal{E}_{R_\varepsilon}\left(\frac{384}{(1+|\cdot|^2)^4}\big(\mathrm{e}^{H^i(\varphi,\psi;(\cdot/R_\varepsilon))+v} - 1 - v\big)\right).$$

Given $\kappa > 1$ (whose value will be fixed later on), we now further assume that the functions $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$, $\psi \in \mathcal{C}^{2,\alpha}(S^3)$ satisfying (18) and the constant $\tau > 0$ satisfy

$$\big|\log(\tau/\tau_*)\big| \leqslant 2\kappa\varepsilon \log 1/\varepsilon, \quad \|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} \leqslant \kappa\varepsilon \quad \text{and} \quad \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)} \leqslant \kappa\varepsilon, \tag{29}$$

where $\tau_* > 0$ is fixed.

We have the following technical:

Lemma 6.1. *Given $\kappa > 0$. There exist $\varepsilon_\kappa > 0$, $c_\kappa > 0$ and $\bar{c}_\kappa > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\kappa)$*

$$\left\| N(\varepsilon, \tau, \varphi, \psi; 0) \right\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant c_\kappa \varepsilon^2. \tag{30}$$

Moreover,

$$\left\| N(\varepsilon, \tau, \varphi, \psi; v_2) - N(\varepsilon, \tau, \varphi, \psi; v_1) \right\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant \bar{c}_\kappa \varepsilon^2 \| v_2 - v_1 \|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \tag{31}$$

and

$$\left\| N(\varepsilon, \tau, \varphi_2, \psi_2; v) - N(\varepsilon, \tau, \varphi_1, \psi_1; v) \right\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant \bar{c}_\kappa \varepsilon \left(\| \varphi_2 - \varphi_1 \|_{\mathcal{C}^{4,\alpha}(S^3)} + \| \psi_2 - \psi_1 \|_{\mathcal{C}^{2,\alpha}(S^3)} \right) \tag{32}$$

provided all $\tilde{v} \in \{v, v_1, v_2\} \subset \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ and all $\tilde{\varphi} \in \{\varphi, \varphi_1, \varphi_2\} \subset \mathcal{C}^{4,\alpha}(S^3)$, $\tilde{\psi} \in \{\psi, \psi_1, \psi_2\} \subset \mathcal{C}^{4,\alpha}(S^3)$ satisfying (18)*, also satisfy*

$$\| \tilde{v} \|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant 2 c_\kappa \varepsilon^2, \quad \| \tilde{\varphi} \|_{\mathcal{C}^{4,\alpha}(S^3)} \leqslant \kappa \varepsilon, \quad \| \tilde{\psi} \|_{\mathcal{C}^{2,\alpha}(S^3)} \leqslant \kappa \varepsilon,$$

and $|\log(\tau/\tau_*)| \leqslant 2\kappa \varepsilon \log 1/\varepsilon$.

Proof. The proof of these estimates follows from the result of Lemma 5.1 together with the assumption on the norms of φ and ψ. Let $c_\kappa^{(i)}$ denote constants that only depend on κ (provided ε is chosen small enough).

It follows from Lemma 5.1 that

$$\left\| H^i(\varphi, \psi; \cdot/R_\varepsilon) \right\|_{\mathcal{C}^{4,\alpha}_2(\bar{B}_{R_\varepsilon})} \leqslant c R_\varepsilon^{-2} \left(\| \varphi \|_{\mathcal{C}^{4,\alpha}(S^3)} + \| \psi \|_{\mathcal{C}^{2,\alpha}(S^3)} \right) \leqslant c_\kappa^{(1)} \varepsilon^2.$$

Therefore, we get

$$\left\| \left(1 + |\cdot|^2\right)^{-4} \left(\mathrm{e}^{H^i(\varphi, \psi; \cdot/R_\varepsilon)} - 1 \right) \right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_\varepsilon})} \leqslant c_\kappa^{(2)} \varepsilon^2.$$

Making use of Proposition 3.1 together with (27) we conclude that

$$\left\| N(\varepsilon, \tau, \varphi, \psi; 0) \right\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant c_\kappa \varepsilon^2.$$

To derive the second estimate, we use the fact that

$$\left\| \left(1 + |\cdot|^2\right)^{-4} \mathrm{e}^{H^i(\varphi, \psi; \cdot/R_\varepsilon)} \left(\mathrm{e}^{v_2} - \mathrm{e}^{v_1} - v_2 + v_1 \right) \right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_\varepsilon})} \leqslant c_\kappa^{(3)} \varepsilon^2 \| v_2 - v_1 \|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}$$

and

$$\left\| \left(1 + |\cdot|^2\right)^{-4} \left(\mathrm{e}^{H^i(\varphi, \psi; \cdot/R_\varepsilon)} - 1 \right) \left(v_2 - v_1 \right) \right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_\varepsilon})} \leqslant c_\kappa^{(4)} \varepsilon^2 \| v_2 - v_1 \|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)},$$

provided $v_1, v_2 \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ satisfy $\| v_i \|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant 2 c_\kappa \varepsilon^2$.

Finally, in order to derive the third estimate, we use

$$\begin{aligned} &\left\| \left(1 + |\cdot|^2\right)^{-4} \left(\mathrm{e}^{H^i(\varphi_2, \psi_2; \cdot/R_\varepsilon)} - \mathrm{e}^{H^i(\varphi_1, \psi_1; \cdot/R_\varepsilon)} \right) \mathrm{e}^{v} \right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar{B}_{R_\varepsilon})} \\ &\quad \leqslant c_\kappa^{(5)} \varepsilon \left\| H^i(\varphi_2 - \varphi_1, \psi_2 - \psi_1; \cdot/R_\varepsilon) \right\|_{\mathcal{C}^{4,\alpha}_2(\bar{B}_{R_\varepsilon})}, \end{aligned}$$

provided $v \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ satisfies $\| v \|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant 2 c_\kappa \varepsilon^2$. The second and third estimates again follow from Proposition 3.1 and (27). $\square$

Reducing $\varepsilon_\kappa > 0$ if necessary, we can assume that,

$$\bar{c}_\kappa \varepsilon^2 \leqslant \frac{1}{2} \tag{33}$$

for all $\varepsilon \in (0, \varepsilon_\kappa)$. Then, (30) and (31) in Lemma 6.1 are enough to show that

$$v \longmapsto N(\varepsilon, \tau, \varphi, \psi; v)$$

is a contraction from

$$\left\{v \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4):\ \|v\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant 2c_\kappa \varepsilon^2\right\}$$

into itself and hence has a unique fixed point $v(\varepsilon,\tau,\varphi,\psi;\cdot)$ in this set. This fixed point is a solution of (26) in $\overline{B}_{R_\varepsilon}$.

We summarize this in the:

Proposition 6.1. *Given $\kappa > 1$, there exist $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ (only depending on κ) such that given $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$, $\psi \in \mathcal{C}^{2,\alpha}(S^3)$ and $\tau > 0$ satisfying* (18) *and*

$$\left|\log(\tau/\tau_*)\right| \leqslant 2\kappa\varepsilon\log 1/\varepsilon, \quad \|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} \leqslant \kappa\varepsilon \quad \text{and} \quad \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)} \leqslant \kappa\varepsilon, \tag{34}$$

the function

$$u(\varepsilon,\tau,\varphi,\psi;\cdot) := u_1 + H^i(\varphi,\psi;\cdot/R_\varepsilon) + v(\varepsilon,\tau,\varphi,\psi;\cdot),$$

solves (25) *in $\overline{B}_{R_\varepsilon}$. In addition*

$$\left\|v(\varepsilon,\tau,\varphi,\psi;\cdot)\right\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant 2c_\kappa\varepsilon^2 \tag{35}$$

and

$$\left\|v(\varepsilon,\tau,\varphi_2,\psi_2;\cdot) - v(\varepsilon,\tau,\varphi_1,\psi_1;\cdot)\right\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant 2\tilde{c}_\kappa\varepsilon\left(\|\varphi_2-\varphi_1\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi_2-\psi_1\|_{\mathcal{C}^{2,\alpha}(S^3)}\right). \tag{36}$$

The last estimate easily follows from (31) and (32) in Lemma 6.1. Observe that the function $v(\varepsilon,\tau,\varphi,\psi;\cdot)$ being obtained as a fixed point for contraction mappings, it depends continuously on the parameter τ.

7. The second nonlinear Dirichlet problem

For all $\varepsilon \in (0, r_0^2)$, we set

$$r_\varepsilon = \sqrt{\varepsilon}.$$

Recall that $G(x,\cdot)$ denotes the unique solution of

$$\Delta^2 G(x,\cdot) = 64\pi^2\delta_x$$

in Ω, with $G(x,\cdot) = \Delta G(x,\cdot) = 0$ on $\partial\Omega$. In addition, the following decomposition holds

$$G(x,y) = -8\log|x-y| + R(x,y)$$

where $y \longmapsto R(x,y)$ is a smooth function.

Given $x^1,\dots,x^m \in \Omega$. The data we will need are the following:

(i) Points $Y := (y^1,\dots,y^m) \in \Omega^m$ close enough to $X := (x^1,\dots,x^m)$.
(ii) Parameters $\Lambda := (\lambda^1,\dots,\lambda^m) \in \mathbb{R}^m$ close to 0.
(iii) Boundary data $\Phi := (\varphi^1,\dots,\varphi^m) \in (\mathcal{C}^{4,\alpha}(S^3))^m$ and $\Psi := (\psi^1,\dots,\psi^m) \in (\mathcal{C}^{2,\alpha}(S^3))^m$ each of which satisfies (22).

With all these data, we define

$$\tilde{\mathbf{u}} := \sum_{j=1}^m \left(1+\lambda^j\right)G\left(y^j,\cdot\right) + \sum_{j=1}^m \chi_{r_0}\left(\cdot - y^j\right)H^e\left(\varphi^j,\psi^j;\left(\cdot - y^j\right)/r_\varepsilon\right) \tag{37}$$

where χ_{r_0} is a cutoff function identically equal to 1 in $B_{r_0/2}$ and identically equal to 0 outside B_{r_0}.

We define $\rho > 0$ by

$$\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4}.$$

We would like to find a solution of the equation

$$\Delta^2 u - \rho^4 e^u = 0, \tag{38}$$

which is defined in $\overline{\Omega}_{r_\varepsilon}(Y)$ and is a perturbation of $\tilde{\mathbf{u}}$. Writing $u = \tilde{\mathbf{u}} + \tilde{v}$, this amounts to solve

$$\Delta^2 \tilde{v} = \rho^4 e^{\tilde{\mathbf{u}}+\tilde{v}} - \Delta^2 \tilde{\mathbf{u}}. \tag{39}$$

We need to define an auxiliary weighed space:

Definition 7.1. Given $\bar{r} \in (0, r_0/2)$, $k \in \mathbb{R}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we define the Hölder weighted space $\mathcal{C}^{k,\alpha}_\nu(\overline{\Omega}_{\bar{r}}(X))$ as the space of functions $w \in \mathcal{C}^{k,\alpha}(\overline{\Omega}_{\bar{r}}(X))$ that is endowed with the norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\nu(\overline{\Omega}_{\bar{r}}(X))} := \|w\|_{\mathcal{C}^{k,\alpha}(\overline{\Omega}_{r_0/2}(X))} + \sum_{j=1}^m \sup_{r\in[\bar{r}, r_0/2)} \bigl(r^{-\nu} \bigl\|w\bigl(x^j + r\cdot\bigr)\bigr\|_{\mathcal{C}^{k,\alpha}(\overline{B}_2 - B_1)}\bigr).$$

For all $\sigma \in (0, r_0/2)$ and all $Y \in \Omega^m$ such that $\|X - Y\| \leqslant r_0/2$, we denote by

$$\tilde{\mathcal{E}}_{\sigma,Y}\colon \mathcal{C}^{0,\alpha}_\nu\bigl(\overline{\Omega}_\sigma(Y)\bigr) \longrightarrow \mathcal{C}^{0,\alpha}_\nu\bigl(\overline{\Omega}^*(Y)\bigr),$$

the extension operator defined by $\tilde{\mathcal{E}}_{\sigma,Y}(f) = f$ in $\overline{\Omega}_\sigma(Y)$,

$$\tilde{\mathcal{E}}_{\sigma,Y}(f)(y^j + x) = \tilde{\chi}\left(\frac{|x|}{\sigma}\right) f\left(y^j + \sigma \frac{x}{|x|}\right)$$

in $B_\sigma(y^j) - B_{\sigma/2}(y^j)$, for each $j = 1, \dots, m$ and $\tilde{\mathcal{E}}_{\sigma,Y}(f) = 0$ in each $B_{\sigma/2}(y^j)$, where $t \mapsto \tilde{\chi}(t)$ is a cutoff function identically equal to 1 for $t \geqslant 1$ and identically equal to 0 for $t \leqslant 1/2$. It is easy to check that there exists a constant $c = c(\nu) > 0$ only depending on ν such that

$$\bigl\|\tilde{\mathcal{E}}_{\sigma,Y}(w)\bigr\|_{\mathcal{C}^{0,\alpha}_\nu(\overline{\Omega}^*(Y))} \leqslant c\|w\|_{\mathcal{C}^{0,\alpha}_\nu(\overline{\Omega}_\sigma(Y))}. \tag{40}$$

We fix

$$\nu \in (-1, 0),$$

and denote by $\mathcal{G}_{\nu,Y}$ a right inverse for Δ^2 provided by Proposition 4.1. Clearly, it is enough to find $\tilde{v} \in \mathcal{C}^{4,\alpha}_\nu(\Omega^*(Y))$ solution of

$$\tilde{v} = \widetilde{N}(\varepsilon, \Lambda, Y, \Phi, \Psi; \tilde{v}) \tag{41}$$

where we have defined

$$\widetilde{N}(\varepsilon, \Lambda, Y, \Phi, \Psi; \tilde{v}) := \mathcal{G}_{\nu,Y} \circ \tilde{\mathcal{E}}_{r_\varepsilon,Y}\bigl(\rho^4 e^{\tilde{\mathbf{u}}+\tilde{v}} - \Delta^2 \tilde{\mathbf{u}}\bigr).$$

Given $\kappa > 0$ (whose value will be fixed later on), we further assume that Φ and Ψ satisfy

$$\|\Phi\|_{(\mathcal{C}^{4,\alpha}(S^3))^m} \leqslant \kappa\varepsilon, \quad \text{and} \quad \|\Psi\|_{(\mathcal{C}^{2,\alpha}(S^3))^m} \leqslant \kappa\varepsilon. \tag{42}$$

Moreover, we assume that the parameters Λ and the points Y are chosen to satisfy

$$|\Lambda| \leqslant \kappa\varepsilon, \quad \text{and} \quad \|Y - X\| \leqslant \kappa\sqrt{\varepsilon}. \tag{43}$$

Then, the following result holds:

Lemma 7.1. *Given $\kappa > 1$. There exist $\varepsilon_\kappa > 0$, $c_\kappa > 0$ and $\bar{c}_\kappa > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\kappa)$*

$$\bigl\|\widetilde{N}(\varepsilon, \Lambda, Y, \Phi, \Psi; 0)\bigr\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(Y))} \leqslant c_\kappa \varepsilon^{3/2}. \tag{44}$$

Moreover,

$$\bigl\|\widetilde{N}(\varepsilon, \Lambda, Y, \Phi, \Psi; \tilde{v}_2) - \widetilde{N}(\varepsilon, \Lambda, Y, \Phi, \Psi; \tilde{v}_1)\bigr\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(Y))} \leqslant \bar{c}_\kappa \varepsilon^2 \|\tilde{v}_2 - \tilde{v}_1\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(Y))} \tag{45}$$

and

$$\begin{aligned}&\big\|\widetilde{N}(\varepsilon,\Lambda,Y,\Phi_2,\Psi_2;\tilde{v})-\widetilde{N}(\varepsilon,\Lambda,Y,\Phi_1,\Psi_1;\tilde{v})\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(Y))}\\&\quad\leqslant \bar{c}_\kappa\varepsilon^{1/2}\big(\|\Phi_2-\Phi_1\|_{(\mathcal{C}^{4,\alpha}(S^3))^m}+\|\Psi_2-\Psi_1\|_{(\mathcal{C}^{2,\alpha}(S^3))^m}\big)\end{aligned}\tag{46}$$

provided $v\in\{\tilde{v},\tilde{v}_1,\tilde{v}_2\}\subset\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(Y))$, $\tilde{\Phi}\in\{\Phi,\Phi_1,\Phi_2\}\subset(\mathcal{C}^{4,\alpha}(S^3))^m$, $\tilde{\Psi}\in\{\Psi,\Psi_1,\Psi_2\}\subset(\mathcal{C}^{2,\alpha}(S^3))^m$ *satisfying* (22)*, also satisfy*

$$\|v\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(Y))}\leqslant 2c_\kappa\varepsilon^{3/2},\quad \|\tilde{\Phi}\|_{(\mathcal{C}^{4,\alpha}(S^3))^m}\leqslant\kappa\varepsilon,\quad \|\tilde{\Psi}\|_{(\mathcal{C}^{2,\alpha}(S^3))^m}\leqslant\kappa\varepsilon,$$

and $|\Lambda|\leqslant\kappa\varepsilon$*,* $\|Y-X\|\leqslant\kappa\sqrt{\varepsilon}$*.*

Proof. The proof of the first estimate follows from the result of Lemma 5.2 together with (42), (43). More precisely, we have

$$\big\|\rho^4 e^{\tilde{\mathbf{u}}}\big\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}_{r_\varepsilon}(Y))}\leqslant c_\kappa\varepsilon^{(4-\nu)/2}\quad\text{and}\quad\big\|\Delta^2\tilde{\mathbf{u}}\big\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}_{r_\varepsilon}(Y))}\leqslant c_\kappa\varepsilon^{3/2}.$$

The proof of the first estimate follows from (43) and Proposition 4.1.

The proof of the second estimate follows from

$$\big\|\rho^4\big(e^{\tilde{\mathbf{u}}+v_2}-e^{\tilde{\mathbf{u}}+v_1}\big)\big\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}_{r_\varepsilon}(Y))}\leqslant c_\kappa\varepsilon^2\|v_2-v_1\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(Y))}$$

and the third estimate follows from

$$\big\|\rho^4\big(e^{\tilde{\mathbf{u}}_2+v}-e^{\tilde{\mathbf{u}}_1+v}\big)\big\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}_{r_\varepsilon}(Y))}\leqslant c_\kappa\varepsilon^{(4-\nu)/2}\big(\|\Phi_2-\Phi_1\|_{(\mathcal{C}^{4,\alpha}(S^3))^m}+\|\Psi_2-\Psi_1\|_{(\mathcal{C}^{2,\alpha}(S^3))^m}\big)$$

and

$$\big\|\Delta^2(\tilde{\mathbf{u}}_2-\tilde{\mathbf{u}}_1)\big\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}_{r_\varepsilon}(Y))}\leqslant c_\kappa\varepsilon^{1/2}\big(\|\Phi_2-\Phi_1\|_{(\mathcal{C}^{4,\alpha}(S^3))^m}+\|\Psi_2-\Psi_1\|_{(\mathcal{C}^{2,\alpha}(S^3))^m}\big)$$

(where $\tilde{\mathbf{u}}_j$ corresponds to $\tilde{\mathbf{u}}$ when $\Phi=\Phi_j$ and $\Psi=\Psi_j$) together with (43) and Proposition 4.1. □

Reducing ε_κ is necessary, we can assume that

$$\bar{c}_\kappa\varepsilon^2\leqslant\frac{1}{2}$$

for all $\varepsilon\in(0,\varepsilon_\kappa)$. Then, (44) and (45) are enough to show that

$$\tilde{v}\longmapsto\widetilde{N}(\varepsilon,\Lambda,Y,\Phi,\Psi;\tilde{v})$$

is a contraction from

$$\big\{\tilde{v}\in\mathcal{C}^{4,\alpha}_\nu\big(\overline{\Omega}^*(Y)\big):\ \|\tilde{v}\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(Y))}\leqslant 2c_\kappa\varepsilon^{3/2}\big\}$$

into itself and hence has a unique fixed point $\tilde{v}(\varepsilon,\Lambda,Y,\Phi,\Psi;\cdot)$ in this set. This fixed point is a solution of (39).

We summarize this in the:

Proposition 7.1. *Given* $\kappa>0$*, there exists* $\varepsilon_\kappa>0$ *and* $c_\kappa>0$ (*only depending on* κ) *such that for all* $\varepsilon\in(0,\varepsilon_\kappa)$*, for all set of parameters* Λ*, points* Y *satisfying*

$$|\Lambda|\leqslant\kappa\varepsilon,\quad\textit{and}\quad\|Y-X\|\leqslant\kappa\sqrt{\varepsilon}$$

and boundary functions Φ *and* Ψ *satisfying* (22) *and*

$$\|\Phi\|_{(\mathcal{C}^{4,\alpha}(S^3))^m}\leqslant\kappa\varepsilon,\quad\textit{and}\quad\|\Psi\|_{(\mathcal{C}^{2,\alpha}(S^3))^m}\leqslant\kappa\varepsilon.$$

The function

$$\tilde{u}(\varepsilon,\Lambda,Y,\Phi,\Psi;\cdot):=\sum_{j=1}^m\big(1+\lambda^j\big)G_{y^j}+\sum_{j=1}^m\chi_{r_0}\big(\cdot-y^j\big)H^e\big(\varphi^j,\psi^j;(\cdot-y^j)/r_\varepsilon\big)+\tilde{v}(\varepsilon,\Lambda,Y,\Phi,\Psi;\cdot),$$

solves (38) *in* $\overline{\Omega}_{r_\varepsilon}(Y)$. *In addition*

$$\left\|\tilde{v}(\varepsilon,\Lambda,Y,\Phi,\Psi;\cdot)\right\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)} \leqslant 2c_\kappa\varepsilon^{3/2} \tag{47}$$

and

$$\begin{aligned}&\left\|\tilde{v}(\varepsilon,\Lambda,Y,\Phi_2,\Psi_2;\cdot)-\tilde{v}(\varepsilon,\Lambda,Y,\Phi_1,\Psi_1;\cdot)\right\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)}\\&\quad\leqslant 2\bar{c}_\kappa\varepsilon^{1/2}\big(\|\Phi_2-\Phi_1\|_{(\mathcal{C}^{4,\alpha}(S^3))^m}+\|\Psi_2-\Psi_1\|_{(\mathcal{C}^{2,\alpha}(S^3))^m}\big).\end{aligned} \tag{48}$$

Again the last estimate follows from (45) and (46) in Lemma 7.1.

Observe that the function $\tilde{v}_{\varepsilon,\Lambda,Y,\Phi,\Psi}$ being obtained as a fixed point for contraction mappings, it depends continuously on the parameters Λ and the points Y.

8. The nonlinear Cauchy-data matching

Keeping the notations of the previous sections, we gather the results of Propositions 6.1 and 7.1. From now on $\kappa > 1$ is fixed large enough (we will shortly see how) and $\varepsilon \in (0,\varepsilon_\kappa)$.

Assume that $X = (x^1,\dots,x^m) \in \Omega^m$ is a nondegenerate critical point of the function W defined in the introduction. For all $j = 1,\dots,m$, we define $\tau_*^j > 0$ by

$$-4\log\tau_*^j = R\big(x^j,x^j\big)+\sum_{\ell\neq j}G\big(x^\ell,x^j\big). \tag{49}$$

We assume that we are given:

(i) Points $Y := (y^1,\dots,y^m) \in \Omega^m$ close to $X := (x^1,\dots,x^m)$ satisfying (43).
(ii) Parameters $\Lambda := (\lambda^1,\dots,\lambda^m) \in \mathbb{R}^m$ satisfying (43).
(iii) Parameters $T := (\tau^1,\dots,\tau^m) \in (0,\infty)^m$ satisfying (29) (where, for each $j = 1,\dots,m$, τ_* is replaced by τ_*^j).

We set

$$R_\varepsilon^j := \tau^j/\sqrt{\varepsilon}.$$

First, we consider some set of boundary data

$$\Phi := \big(\varphi^1,\dots,\varphi^m\big) \in \big(\mathcal{C}^{4,\alpha}\big(S^3\big)\big)^m \quad \text{and} \quad \Psi := \big(\psi^1,\dots,\psi^m\big) \in \big(\mathcal{C}^{2,\alpha}\big(S^3\big)\big)^m$$

satisfying (18) and (29).

Recall that

$$\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4}.$$

Thanks to the result of Proposition 6.1, we can find u_{int} a solution of

$$\Delta^2 u - \rho^4 \mathrm{e}^u = 0$$

in each $B_{r_\varepsilon}(y^j)$, that can be decomposed as

$$\begin{aligned}u_{\mathrm{int}}(\varepsilon,T,Y,\Phi,\Psi;x) &:= u\big(\varepsilon,\tau^j,\varphi^j,\psi^j;R_\varepsilon^j\big(x-y^j\big)/r_\varepsilon\big)+4\log\bigg(\frac{\tau^j(1+\varepsilon^2)}{2\varepsilon^2}\bigg)\\&= u_{\varepsilon,\tau^j}\big(x-y^j\big)+H^i\big(\varphi^j,\psi^j;(x-y^j)/r_\varepsilon\big)+v\big(\varepsilon,\tau^j,\varphi^j,\psi^j;R_\varepsilon^j\big(x-y^j\big)/r_\varepsilon\big)\end{aligned}$$

in $B_{r_\varepsilon}(y^j)$.

Similarly, given some boundary data

$$\tilde{\Phi} := \big(\tilde{\varphi}^1,\dots,\tilde{\varphi}^m\big) \in \big(\mathcal{C}^{4,\alpha}\big(S^3\big)\big)^m \quad \text{and} \quad \tilde{\Psi} := \big(\tilde{\psi}^1,\dots,\tilde{\psi}^m\big) \in \big(\mathcal{C}^{2,\alpha}\big(S^3\big)\big)^m$$

satisfying (22) and (42), we use the result of Proposition 7.1, to find u_{ext} a solution of

$$\Delta^2 u - \rho^4 \mathrm{e}^u = 0$$

in $\overline{\Omega}_{r_\varepsilon}(Y)$, that can be decomposed as

$$u_{\mathrm{ext}}(\varepsilon, \Lambda, Y, \tilde{\Phi}, \tilde{\Psi}; x) = \sum_{j=1}^{m} \left(1 + \lambda^j\right) G\left(y^j, x\right) + \sum_{j=1}^{m} \chi_{r_0}\left(x - y^j\right) H^e\left(\tilde{\varphi}^j, \tilde{\psi}^j; \left(x - y^j\right)/r_\varepsilon\right) + \tilde{v}(\varepsilon, \Lambda, Y, \tilde{\Phi}, \tilde{\Psi}; x).$$

It remains to determine the parameters and the boundary functions in such a way that the function that is equal to u_{int} in $\bigcup_j \overline{B}_{r_\varepsilon}(y^j)$ and that is equal to u_{ext} in $\overline{\Omega}_{r_\varepsilon}(Y)$ is $\mathcal{C}^3$ function. This amounts to find the boundary data and the parameters so that, for each $j = 1, \ldots, m$

$$u_{\mathrm{int}} = u_{\mathrm{ext}}, \quad \partial_r u_{\mathrm{int}} = \partial_r u_{\mathrm{ext}}, \quad \Delta u_{\mathrm{int}} = \Delta u_{\mathrm{ext}}, \quad \partial_r \Delta u_{\mathrm{int}} = \partial_r \Delta u_{\mathrm{ext}}, \tag{50}$$

on $\partial B_{r_\varepsilon}(y^j)$. Assuming we have already done so, this provides for each ε small enough a function $u_\varepsilon \in \mathcal{C}^3(\overline{\Omega})$ (which is obtained by patching together the function u_{int} and the function u_{ext}) weak solution of $\Delta^2 u - \rho^4 \mathrm{e}^u = 0$ and elliptic regularity theory implies that this solution is in fact smooth. This will complete the proof of our result since, as ε tends to 0, the sequence of solutions we have obtained satisfies the required properties, namely, away from the points x^j the sequence u_ε converges to $\sum_j G(x^j, \cdot)$.

Before we proceed, some remarks are due. First it will be convenient to observe that the functions u_{ε,τ^j} can be expanded as

$$u_{\varepsilon,\tau^j}(x) = -8 \log|x| - 4 \log \tau^j + \mathcal{O}(\varepsilon) \tag{51}$$

near $\partial B_{r_\varepsilon}$. Also, the function

$$\sum_{j=1}^{m} \left(1 + \lambda^j\right) G\left(y^j, x\right)$$

that appears in the expression of u_{ext} can be expanded as

$$\sum_{\ell=1}^{m} \left(1 + \lambda^\ell\right) G\left(y^\ell, y^j + x\right) = -8\left(1 + \lambda^j\right) \log|x| + \left(1 + \lambda^j\right) R\left(y^j, x\right) + \sum_{\ell \neq j} \left(1 + \lambda^\ell\right) G\left(y^\ell, x\right) + \nabla_x \left(R\left(y^j, \cdot\right) + \sum_{\ell \neq j} G\left(y^\ell, \cdot\right) \right)\Big|_{y^j} \cdot x + \mathcal{O}\left(|x|^2\right) \tag{52}$$

near $\partial B_{r_\varepsilon}(y^j)$.

In (50), all functions are defined on $\partial B_{r_\varepsilon}(y^j)$, nevertheless, it will be convenient to solve, instead of (50) the following set of equations

$$\begin{aligned} &(u_{\mathrm{int}} - u_{\mathrm{ext}})\left(y^j + r_\varepsilon \cdot\right) = 0, \qquad (\partial_r u_{\mathrm{int}} - \partial_r u_{\mathrm{ext}})\left(y^j + r_\varepsilon \cdot\right) = 0, \\ &(\Delta u_{\mathrm{int}} - \Delta u_{\mathrm{ext}})\left(y^j + r_\varepsilon \cdot\right) = 0, \qquad (\partial_r \Delta u_{\mathrm{int}} - \partial_r \Delta u_{\mathrm{ext}})\left(y^j + r_\varepsilon \cdot\right) = 0, \end{aligned} \tag{53}$$

on S^3. Here all functions are considered as functions of $z \in S^3$ and we have simply used the change of variables $x = y^j + r_\varepsilon z$ to parameterize $\partial B_{r_\varepsilon}(y^j)$.

Since the boundary data we have chosen satisfy (18) and (22), we can decompose

$$\Phi = \Phi_0 + \Phi_1 + \Phi^\perp, \qquad \Psi = 8\Phi_0 + 12\Phi_1 + \Psi^\perp$$

and

$$\tilde{\Phi} = \tilde{\Phi}_0 + \tilde{\Phi}_1 + \tilde{\Phi}^\perp, \qquad \tilde{\Psi} = \tilde{\Psi}_1 + \tilde{\Psi}^\perp$$

where the components of Φ_0, $\tilde{\Phi}_0$ are constant functions on S^3, the components of Φ_1, $\tilde{\Phi}_1$, $\tilde{\Psi}_1$ belong to

$$\mathrm{Ker}(\Delta_{S^3} + 3) = \mathrm{Span}\{e_1, \ldots, e_4\}$$

and where the components of $\Phi^\perp, \Psi^\perp, \tilde\Phi^\perp, \tilde\Psi^\perp$ are $L^2(S^3)$ orthogonal to the constant function and the functions $e_1, \dots, e_4$. Observe that the components of Ψ over the constant functions or functions in $\mathrm{Ker}(\Delta_{S^3}+3)$ are determined by the corresponding components of Φ. Moreover, $\tilde\Psi$ has no component over constant functions.

We first consider the $L^2(S^3)$-orthogonal projection of (53) onto the space of functions that are orthogonal to the constant function and the functions $e_1, \dots, e_4$. This yields the system

$$\begin{cases} \varphi^{j,\perp} - \tilde\varphi^{j,\perp} = M_0^{(j)}(\varepsilon, \Lambda, T, Y, \Phi, \tilde\Phi, \Psi, \tilde\Psi), \\ \partial_r H^i\big(\varphi^{j,\perp}, \psi^{j,\perp}; \cdot\big) - \partial_r H^e\big(\tilde\varphi^{j,\perp}, \tilde\psi^{j,\perp}; \cdot\big) = M_1^{(j)}(\varepsilon, \Lambda, T, Y, \Phi, \tilde\Phi, \Psi, \tilde\Psi), \\ \psi^{j,\perp} - \tilde\psi^{j,\perp} = M_2^{(j)}(\varepsilon, \Lambda, T, Y, \Phi, \tilde\Phi, \Psi, \tilde\Psi), \\ \partial_r \Delta H^i\big(\varphi^{j,\perp}, \psi^{j,\perp}; \cdot\big) - \partial_r \Delta H^e\big(\tilde\varphi^{j,\perp}, \tilde\psi^{j,\perp}; \cdot\big) = M_3^{(j)}(\varepsilon, \Lambda, T, Y, \Phi, \tilde\Phi, \Psi, \tilde\Psi) \end{cases} \tag{54}$$

where the functions $M_k^{(j)}$ are nonlinear functions of the parameters ε, Λ, Y, T and the boundary data Φ, $\tilde\Phi$, Ψ and $\tilde\Psi$. Moreover, using (51) and (52) and also (35) (keeping in mind that $\mu \in (1,2)$) and (47) (keeping in mind that $\nu \in (-1,0)$), we conclude that, for each $j = 1, \dots, m$ and $k = 0, 1, 2, 3$

$$\big\| M_k^{(j)} \big\|_{\mathcal{C}^{4-k,\alpha}(S^3)} \leqslant c\varepsilon \tag{55}$$

for some constant $c > 0$ independent of κ (provided $\varepsilon \in (0, \varepsilon_\kappa)$).

Thanks to the result of Lemma 5.3 and (55), this last system can be re-written as

$$\big(\Phi^\perp, \tilde\Phi^\perp, \Psi^\perp, \tilde\Psi^\perp\big) = M(\varepsilon, \Lambda, T, Y, \Phi, \tilde\Phi, \Psi, \tilde\Psi)$$

where

$$\|M\|_{(\mathcal{C}^{4,\alpha}(S^3))^{2m} \times (\mathcal{C}^{2,\alpha}(S^3))^{2m}} \leqslant c\varepsilon$$

for some constant $c > 0$ independent of κ (provided $\varepsilon \in (0, \varepsilon_\kappa)$). Moreover, (36) and (48) imply (reducing ε_κ if necessary) that the mapping M is a contraction from the ball of radius $\kappa\varepsilon$ in $(\mathcal{C}^{4,\alpha}(S^3))^{2m} \times (\mathcal{C}^{2,\alpha}(S^3))^{2m}$ into itself and as such has a unique fixed point in this set. Observe that this fixed point depends continuously on ε, Λ, T, Y and also on Φ_0, $\tilde\Phi_0$, Φ_1, $\tilde\Phi_1$ and $\tilde\Psi_1$.

We insert this fixed point in (53) and now project the corresponding system over the set of functions spanned by $e_1, \dots, e_4$ and finally over the set of constant functions. We define, for all $j = 1, \dots, n$

$$E_j(Y; \cdot) := R\big(y^j, \cdot\big) + \sum_{\ell \neq j} G\big(y^\ell, \cdot\big).$$

The first projection, over the set of functions spanned by $e_1, \dots, e_4$, yields the system of equations

$$\begin{cases} \Phi_1 = \overline{M}_1(\varepsilon, \Lambda, T, Y, \Phi_0, \tilde\Phi_0, \Phi_1, \tilde\Phi_1, \tilde\Psi_1), \\ \tilde\Phi_1 = \overline{M}_2(\varepsilon, \Lambda, T, Y, \Phi_0, \tilde\Phi_0, \Phi_1, \tilde\Phi_1, \tilde\Psi_1), \\ \Psi_1 = \overline{M}_3(\varepsilon, \Lambda, T, Y, \Phi_0, \tilde\Phi_0, \Phi_1, \tilde\Phi_1, \tilde\Psi_1), \\ \sqrt{\varepsilon}\nabla E_j\big(Y; y^j\big) = \overline{M}_4^{(j)}(\varepsilon, \Lambda, T, Y, \Phi_0, \tilde\Phi_0, \Phi_1, \tilde\Phi_1, \tilde\Psi_1) \end{cases} \tag{56}$$

where the functions $\overline{M}_k$ (and also $\overline{M}_4^{(j)}$) are nonlinear functions depending continuously on the parameters ε, Λ, T, Y and the components of the boundary data Φ_0, $\tilde\Phi_0$, Φ_1, $\tilde\Phi_1$ and $\tilde\Psi_1$. Moreover,

$$|\overline{M}_k| \leqslant c\varepsilon \quad \text{and} \quad \big|\overline{M}_4^{(j)}\big| \leqslant c\varepsilon$$

for some constant $c > 0$ independent of κ (provided $\varepsilon \in (0, \varepsilon_\kappa)$).

Let us comment briefly on how these equations are obtained. These equations simply come from (50) when expansions (51) and (52) are taken into account, together with the expression of $H^i(\varphi^j, \psi^j; \cdot)$ and $H^e(\tilde\varphi^j, \tilde\psi^j; \cdot)$ given in Lemmas 5.1 and 5.2, and also the estimates (35) and (47). Observe that the projection of the term $x \to \nabla E_j(Y; y^j) \cdot x$ that arises in (52), as well as the projection of its partial derivative with respect to r, over the set of constant function is equal to 0. Moreover, this term projects identically over the set of functions spanned by $e_1, \dots, e_4$ as well as its derivative with respect to r. Finally, its Laplacian vanishes identically.

Recall that we have defined in the introduction the function

$$W(Y) := \sum_{j=1}^m R\big(y^j, y^j\big) + \sum_{j_1 \neq j_2} G\big(y^{j_1}, y^{j_2}\big).$$

Using the symmetries of the functions G and R, namely the fact that

$$G(x,y)=G(y,x) \quad \text{and} \quad R(x,y)=R(y,x)$$

we get

$$\nabla W|_Y = 2\big(\nabla E_1\big(Y;y^1\big),\dots,\nabla E_m\big(Y;y^m\big)\big).$$

Now, we have assumed that the point $X=(x^1,\dots,x^m)$ is a nondegenerate critical point of the functional W and hence

$$\nabla W|_X = 0,$$

and

$$\big(\mathbb{R}^4\big)^m \ni Z \longmapsto D(\nabla W)|_X(Z) \in \big(\mathbb{R}^4\big)^m$$

is invertible. Therefore, the last equation can be rewritten as

$$\sqrt{\varepsilon}(Y-X) = \overline{M}_5(\varepsilon,\Lambda,T,Y,\Phi_0,\tilde{\Phi}_0,\Phi_1,\tilde{\Phi}_1,\tilde{\Psi}_1).$$

If we define the parameters $U := (u^1,\dots,u^m)$ by

$$u^j = -4\log\tau^j + 8\lambda^j \log r_\varepsilon - \big(1+\lambda^j\big)R\big(y^j,y^j\big) - \sum_{\ell\neq j}\big(1+\lambda^\ell\big)G\big(y^\ell,y^j\big),$$

the projection of (53) over the constant function leads to the system

$$\begin{cases} U = \overline{M}_6(\varepsilon,\Lambda,T,Y,\Phi_0,\tilde{\Phi}_0,\Phi_1,\tilde{\Phi}_1,\tilde{\Psi}_1),\\ \tilde{\Phi}_0 = \overline{M}_7(\varepsilon,\Lambda,T,Y,\Phi_0,\tilde{\Phi}_0,\Phi_1,\tilde{\Phi}_1,\tilde{\Psi}_1),\\ \Phi_0 = \overline{M}_8(\varepsilon,\Lambda,T,Y,\Phi_0,\tilde{\Phi}_0,\Phi_1,\tilde{\Phi}_1,\tilde{\Psi}_1),\\ \Lambda = \overline{M}_9(\varepsilon,\Lambda,T,Y,\Phi_0,\tilde{\Phi}_0,\Phi_1,\tilde{\Phi}_1,\tilde{\Psi}_1) \end{cases} \tag{57}$$

where the function $\overline{M}_k$ satisfies the usual properties. We set

$$Z = \sqrt{\varepsilon}(Y-X)$$

so that the system we have to solve reads

$$(U,\Lambda,Z,\Phi_0,\tilde{\Phi}_0,\Phi_1,\tilde{\Phi}_1,\tilde{\Psi}_1) = \overline{M}(\varepsilon,U,\Lambda,Z,\Phi_0,\tilde{\Phi}_0,\Phi_1,\tilde{\Phi}_1,\tilde{\Psi}_1). \tag{58}$$

where as usual, the nonlinear function $\overline{M}$ depends continuously on the parameters T,Λ,Z and the functions $\Phi_0,\tilde{\Phi}_0,\Phi_1,\tilde{\Psi}_1$ and is bounded (in the appropriate norm) by a constant (independent of ε and κ) time ε, provided $\varepsilon\in(0,\varepsilon_\kappa)$. Observe that

$$U,\Lambda\in\mathbb{R}^m, \quad Z\in\big(\mathbb{R}^4\big)^m, \quad \Phi_0,\tilde{\Phi}_0\in\mathbb{R}^m, \quad \Phi_1,\tilde{\Phi}_1,\tilde{\Psi}_1\in\big(\mathrm{Ker}(\Delta_{S^3}+3)\big)^m.$$

In addition, reducing ε_κ if necessary, this nonlinear mapping sends the ball of radius $\kappa\varepsilon$ (for the natural product norm) into itself, provided κ is fixed large enough and $\varepsilon\in(0,\varepsilon_\kappa)$. Applying Schauder's fixed point theorem in the ball of radius $\kappa\varepsilon$ in the product space where the entries live yields the existence of a solution of (58) and this completes the proof of Theorem 1.1.

9. Comments

Let us comment on how the condition "$(x^1,\dots,x^m)$ is a *nondegenerate* critical point of W" enters in our analysis since, we confess, it is somehow very well hidden in the technicalities of the proof.

The condition "$(x^1,\dots,x^m)$ is a critical point of W" enters in the estimate (52) when $Y=X$ and $\Lambda=0$, since, in this case we have

$$\sum_{\ell=1}^m G\big(x^\ell,x^j+x\big) = -8\log|x| + E_j\big(X;x^j\big) + \mathcal{O}\big(|x|^2\big)$$

while, if $(x^1, \dots, x^m)$ were not a critical point of W, then $\nabla E_j(X; x^j) \neq 0$ and we would only have

$$\sum_{\ell=1}^{m} G\big(x^\ell, x^j + x\big) = -8 \log |x| + E_j\big(X; x^j\big) + \mathcal{O}\big(|x|\big)$$

and this would not be enough: roughly speaking this says that the approximate solution we have constructed is not close to any solution of the problem. Given the result of Lin and Wei [6], the condition on "$(x^1, \dots, x^m)$ being a critical point of W" is certainly a necessary condition for the result of Theorem 1.1 to hold.

The origin of the "nondegeneracy" assumption takes its roots in the result of Lemma 3.1 that classifies all the solutions of the linearized equation about the rotationally symmetric solution. The existence of elements ϕ_i, for $i = 1, \dots, 4$ in the kernel of $\mathbb{L}$ has forced us in Proposition 3.1 to work with weights $\mu > 1$ to obtain the surjectivity of the operator L_μ. This choice has one important consequence: In Lemma 5.1, we had to restrict our attention to boundary data that satisfy the constraints (18) and (22) (even though only the second constraint in (18) is important to understand where the nondegeneracy condition comes from) to obtain bi-harmonic extensions in the unit ball that vanish at the origin at least quadratically. A second reading will convince the reader that this property was crucial in the estimate of Lemma 6.1. Indeed, the main estimate in this lemma arises from the fact that

$$\big|H^i(\varphi, \psi; \cdot/R_\varepsilon)\big| \leqslant c_\kappa^{(1)} \varepsilon^2 |x|^2.$$

Without the second hypothesis in (18) we would only have

$$\big|H^i(\varphi, \psi; \cdot/R_\varepsilon)\big| \leqslant c_\kappa^{(1)} \varepsilon^{3/2} |x|$$

that would have led in Lemma 6.1 to the estimate

$$\big\|N(\varepsilon, \tau, \varphi, \psi; 0)\big\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R})^4} \leqslant c_\kappa \varepsilon^{3/2}.$$

But since $\mu \in (1, 2)$ this implies that, on the boundary $\partial B_{R_\varepsilon}$ the function $v(\varepsilon, \tau, \varphi, \psi; \cdot)$ is bounded by a constant times $\varepsilon^{(3-\mu)/2}$ and since

$$\varepsilon^{(3-\mu)/2} \gg \varepsilon$$

the function v would be much larger than the functions $H^i(\varphi, \psi; \cdot/R_\varepsilon)$ on this boundary and hence could not be considered as a small perturbation anymore. Given the fact that, in the construction of H^i and H^e we could not prescribe all possible boundary data, we had to "find" new degrees of freedom to compensate the constraints imposed by (18) and (22). The introduction of the parameters τ^j and λ^j enter at this point to compensate the first condition imposed by (18) and also the condition imposed by (22). The points y^j close to x^j are introduced to compensate the second condition imposed by (18) and this is precisely where the nondegeneracy of the critical points of W comes into play.

Let us point out that the *nondegeneracy* condition strictly speaking can be weakened as this has been done for example in [4] and [5] in the case of Eq. (8). The idea being that the nondegeneracy is essentially used to solve the last equation in (56) by some disguised version of the Implicit Function Theorem. But, remembering that the problem we want to solve is a variational problem, this last equation can be rephrased essentially as the gradient of a function W_ε that is defined on Ω^m and that converges (in a sense to be made precise) to the function W as ε tends to 0. Some extra work is needed, but in any case, we could have used some variational techniques to find critical points of this functional. Since nondegeneracy of critical points is a generic condition and in order not to make the exposition of this "nonlinear domain decomposition technique" as clear as possible, we have chosen not to follow this route.

10. Further results

Modifying very little the previous analysis, it is possible to extend the result of Theorem 1.1 to handle more general equations. We will illustrate this on one example.

As usual, let us assume that $\Omega \subset \mathbb{R}^4$ is a regular bounded open subset and let us choose $z^1, \dots, z^p \in \Omega$ and $\alpha^1, \dots, \alpha^p \in (0, +\infty)$. We would like to extend the result of Theorem 1.1 to the equation

$$\begin{cases} \Delta^2 u = \rho^4 \mathrm{e}^u - 64\pi^2 \sum_{i=1}^{p} \alpha^i \delta_{z^i} & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{59}$$

Namely, we are still looking for solutions that concentrate at some points $x^1, \dots, x^m \in \Omega$, as the parameter $\rho \to 0$ and, in order to keep the amount of technicalities as low as possible, we will assume that the set of concentration points x^j and the set of singularities z^i are disjoint. This problem is very much in the spirit of the work of [5] and [4] even though we do not know any applications in physics. On the other hand solutions of this problem might be of interest to understand constant Q-curvature metrics with singularities.

Setting

$$v := u + \sum_{i=1}^{p} \alpha^i G\big(z^i, \cdot\big)$$

we can rephrase the equation satisfied by u as an equation satisfied by v, namely

$$\begin{cases} \Delta^2 v = \rho^4 \mathrm{e}^{-\sum \alpha^i G(z^i,\cdot)} \mathrm{e}^v & \text{in } \Omega, \\ v = \Delta v = 0 & \text{on } \partial\Omega. \end{cases} \tag{60}$$

This equation is a particular case of the more general problem

$$\begin{cases} \Delta^2 u = \rho^4 V \mathrm{e}^u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{61}$$

where $V : \Omega \to [0, +\infty)$ is a smooth function. We are still looking for solutions of this last equation that concentrate at some points $x^1, \dots, x^m$, as the parameter $\rho \to 0$. In order to keep the technicalities as low as possible, we will assume that the set of concentration points x^j and the set of zeros of V are disjoint.

As in the introduction, we introduce the functional

$$W\big(x^1, \dots, x^m\big) := \sum_{j=1}^{m} R\big(x^j, x^j\big) + \sum_{j \neq \ell} G\big(x^j, x^\ell\big) + 2\sum_{j=1}^{m} \log V\big(x^j\big). \tag{62}$$

It is easy to check that the result of Theorem 1.1 holds when (1) is replaced by (61) and (62) replaces (4). We briefly describe the main modifications that are needed to prove this modified result.

Only Sections 6, 7 and 8 have to be slightly modified. In Section 6, (25) has to be replaced by

$$\Delta^2 u = 24\mathrm{e}^u + \varepsilon^4 g$$

where g is a bounded function (in fact bounded in $\mathcal{C}^{0,\alpha}(B_{R_\varepsilon})$ by some constant independent of ε). It is easy to check that the analysis goes through. The presence of the term $\varepsilon^4 g$ does not alter the estimates in Lemma 6.1 and in fact, keeping the notations of introduced in the proof of Lemma 6.1, we have

$$\big\|\varepsilon^4 g\big\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(B_{R_\varepsilon})} \leqslant c\varepsilon^{2+\mu/2}.$$

The result of Proposition 6.1 remains unchanged. Section 7 applies *vertabim* and Proposition 7.1 is unchanged.

In Section 8, the main modification due is in the definition of u_{int}. Indeed, for each $j = 1, \dots, m$ we apply the result of the modified version of Section 6 with

$$g = \frac{1}{\tau_j^4}\big(\Delta^2 \log V\big)\big(y^j + \varepsilon \cdot /\tau_j\big).$$

This induces in each $B_{r_\varepsilon}(y^j)$ a solution of

$$\Delta^2 u = \rho^4 V \mathrm{e}^u$$

that can be decomposed as

$$u_{\mathrm{int}}(x) = u_{\varepsilon,\tau^j}\big(x - y^j\big) - \log V(x) + H^i\big(\varphi^j, \psi^j; \big(x - y^j\big)/r_\varepsilon\big) + v\big(\varepsilon, \tau^j, \varphi^j, \psi^j; R^j_\varepsilon\big(x - y^j\big)/r_\varepsilon\big).$$

The remaining of the analysis of Section 8 remains essentially unchanged once the definition of E_j is modified into

$$E_j(Y; \cdot) := R\big(y^j, \cdot\big) + \sum_{\ell \neq j} G\big(y^\ell, \cdot\big) + \log V\big(y^j\big).$$

We leave the details to the reader.

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