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Ann. I. H. Poincaré – AN 24 (2007) 279–310

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Nearly time optimal stabilizing patchy feedbacks

Feedback de type patchy stabilisant en temps presque optimal

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Received 24 November 2005; received in revised form 2 February 2006; accepted 2 March 2006

Available online 13 October 2006

Abstract

We consider the time optimal stabilization problem for a nonlinear control system $\dot{x} = f(x, u)$. Let $T(y)$ be the minimum time needed to steer the system from the state $y \in \mathbb{R}^n$ to the origin, and call $\mathcal{A}(\tau)$ the set of initial states that can be steered to the origin in time $T(y) \leqslant \tau$. Given any $\varepsilon > 0$, in this paper we construct a patchy feedback $u = U(x)$ such that every solution of $\dot{x} = f(x, U(x))$, $x(0) = y \in \mathcal{A}(\tau)$ reaches an ε-neighborhood of the origin within time $T(y) + \varepsilon$.

Résumé

On considère un problème de stabilization en temps optimal pour un système de commandé non-linéaire du type $\dot{x} = f(x, u)$. Notons par $T(y)$ le temps minimal nécessaire pour aller de $y \in \mathbb{R}^n$ à l'origine, et par $\mathcal{A}(\tau)$ l'ensemble des $y \in \mathbb{R}^n$ tels que $T(y) \leqslant \tau$. Pour chaque $\varepsilon > 0$, on construit un feedback $u = U(x)$ de type patchy tel que toutes les solutions de $\dot{x} = f(x, U(x))$, $x(0) = y \in \mathcal{A}(\tau)$ atteignent un ε-voisinage de l'origine en temps au plus $T(y) + \varepsilon$.

MSC: 34A; 34D; 49E; 93D

Keywords: Time optimal stabilization; Discontinuous feedback control; Robustness

1. Introduction

Consider an optimization problem for a nonlinear control system of the form

$$\dot{x} = f(x, u), \quad u(t) \in \mathbf{U}, \tag{1.1}$$

where $x \in \mathbb{R}^n$ describes the state of the system, the upper dot denotes a derivative w.r.t. time, while $\mathbf{U} \subset \mathbb{R}^m$ is the set of admissible control values. A central issue in the theory of optimal control is the existence of a feedback control $u = U(x)$ such that *all* trajectories of

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2006.03.010

$$\dot{x} = f\big(x, U(x)\big) \tag{1.2}$$

are optimal, for a given performance criterion. In most cases, the optimal feedback law $u = U(x)$ is not continuous. As shown in Example 1.1 in [26] or Example 2 in [10], even near-optimal feedback laws can usually be found only within a class of discontinuous functions.

Therefore, it is essential to provide suitable definitions of "generalized solutions" for discontinuous ODE's. In particular, we recall the concept of "sample-and-hold" solutions and Euler solutions [18] (limits of sample-and-hold solutions), which were successfully implemented both within the context of stabilization problems [16,30,32,33] and of near-optimal feedbacks [17,19,26] (see also [25] for a discussion of further definitions of generalized solutions relevant for optimization problems). We point out that a major difficulty which in general arises in the construction of discontinuous feedbacks is that, as illustrated by Examples 5.3 and 5.4 in [29], *arbitrary* discontinuous feedback can generate too many trajectories, some of which fail to be optimal. In fact, Example 5.3 in [29] shows that the set of Carathéodory solutions of the optimal closed-loop equation (1.2) contains, in addition to all optimal trajectories, some other arcs that are not optimal. Moreover, Example 5.4 in [29] exhibits an optimal control problem in which the optimal trajectories are Euler solutions, but the closed-loop equation (1.2) has many other Euler solutions which are not optimal. To avoid such a type of behavior, it is thus necessary to implement a careful construction of the feedback law so to achieve the desired optimality of all sample-and-hold trajectories as in [17,19,26], or of all Carathéodory solutions as it is shown in the present paper.

A different strategy, proposed by Piccoli [27] and Sussmann [34], takes as primary object of investigation an optimal "synthesis" which is just a collection of optimal trajectories not necessarily arising from a feedback control. A general notion of regular synthesis is discussed in [29] where a sufficiency theorem for optimal synthesis is proved. For other definitions of regular synthesis we refer to [8,12]. The existence and the structure of an optimal synthesis has been the subject of a large body of literature on nonlinear control. At present, detailed results are known for time optimal planar systems of the form

$$\dot{x} = f(x) + g(x)u, \quad u \in [-1, 1],\ x \in \mathbb{R}^2,$$

see [9] and the references therein. For more general classes of optimal control problems, or in higher space dimensions, the construction of an optimal synthesis faces severe difficulties. On one hand, the optimal synthesis can have an extremely complicated structure, and only few regularity results are presently known (see [23]). Already for systems in two space dimensions, an accurate description of all generic singularities of a time optimal synthesis involves the classification of eighteen topological equivalence classes of singular points [11,27,28]. In higher dimensions, an even larger number of different singularities arises, and the optimal synthesis can exhibit pathological behavior such as the famous "Fuller phenomenon" (see [24,35]), where every optimal control has an infinite number of switchings. On the other hand, even in cases where a regular synthesis exists, the performance achieved by the optimal synthesis may not be robust. In other words, small perturbations can greatly affect the behavior of the synthesis (e.g. see Example 5.3 in [29]).

Because of the difficulties faced in the construction of an optimal syntheses, it seems natural to slightly relax our requirements, and look for nearly-optimal feedbacks instead. This is indeed the main purpose of the present paper. Within this wider class, one can hope to find a feedback law whose discontinuities are sufficiently "tame", providing the existence of trajectories in the usual Carathéodory sense, all of which are "almost optimal". Moreover, the new feedback laws will have a simpler structure and better robustness properties than a regular synthesis.

For sake of definiteness, we shall study the problem of steering the system (1.1) from any initial state $y \in \mathbb{R}^n$ to the origin in minimum time, under the basic assumptions

(H) The set $\mathbf{U} \subset \mathbb{R}^m$ of admissible control values is bounded. Moreover, the function $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is twice continuously differentiable and has sublinear growth:

$$\big|f(x,u)\big| \leqslant c\big(1 + |x|\big) \quad \text{for all } u \in \mathbf{U}. \tag{1.3}$$

For $y \in \mathbb{R}^n$, call $T(y)$ the minimum time needed to steer the system from the state $y \in \mathbb{R}^n$ to the origin, i.e. set

$$T(y) \doteq \inf\big\{t \geqslant 0;\ \text{there exists some trajectory } x(\cdot) \text{ of (1.1) that satisfies } x(0) = y,\ x(t) = 0\big\}. \tag{1.4}$$

Roughly speaking, our main theorem states the following. If we relax a bit the optimality requirements, asking that every initial state y be steered inside an ε-neighborhood of the origin within time $T(y) + \varepsilon$, then this can be accomplished by a *patchy feedback*, for any fixed $\varepsilon > 0$. Other relevant references for the problem of optimal (discontinuous) feedback construction are [7,13,15,20–22,31].

Patchy feedback controls were first introduced in [1] in order to study asymptotic stabilization problems. They have a particularly simple structure, being piecewise constant in the state space $\mathbb{R}^n$. Moreover, the Carathéodory solutions of the corresponding Cauchy problems (1.2) enjoy important robustness properties [2–4], which are particularly relevant in many practical situations. Indeed, one of the main reasons for using a state feedback is precisely the fact that open loop controls are usually very sensitive to disturbances. In particular, we have shown in [2] that a patchy feedback is "fully robust" with respect to perturbation of the external dynamics, and to measurement errors having sufficiently small total variation so to avoid the chattering behavior that may arise at discontinuity points.

We recall here the main definitions (see [1]):

Definition 1.1. By a *patch* we mean a pair (Ω, g) where $\Omega \subset \mathbb{R}^n$ is an open domain with smooth boundary $\partial\Omega$, and g is a smooth vector field defined on a neighborhood of the closure $\overline{\Omega}$ of Ω, which points strictly inward at each boundary point $x \in \partial\Omega$.

Calling $\mathbf{n}(x)$ the outer normal at the boundary point x, we thus require

$$\big\langle g(x), \mathbf{n}(x)\big\rangle < 0 \quad \text{for all } x \in \partial\Omega. \tag{1.5}$$

Definition 1.2. We say that $g : \Omega \mapsto \mathbb{R}^n$ is a *patchy vector field* on the open domain Ω if there exists a family of patches $\{(\Omega_\alpha, g_\alpha);\ \alpha \in \mathcal{A}\}$ such that

- $\mathcal{A}$ is a totally ordered set of indices,
- the open sets Ω_α form a locally finite covering of Ω,
- the vector field g can be written in the form

$$g(x) = g_\alpha(x) \quad \text{if } x \in \Omega_\alpha \Big\backslash \bigcup_{\beta>\alpha} \Omega_\beta. \tag{1.6}$$

We shall occasionally adopt the longer notation $(\Omega, g, (\Omega_\alpha, g_\alpha)_{\alpha\in\mathcal{A}})$ to indicate a patchy vector field, specifying both the domain and the single patches.

By setting

$$\alpha^*(x) \doteq \max\{\alpha \in \mathcal{A};\ x \in \Omega_\alpha\}, \tag{1.7}$$

we can write (1.6) in the equivalent form

$$g(x) = g_{\alpha^*(x)}(x) \quad \text{for all } x \in \Omega. \tag{1.8}$$

Remark 1.1. Notice that the patches $(\Omega_\alpha, g_\alpha)$ are not uniquely determined by a patchy vector field (Ω, g). Indeed, whenever $\alpha < \beta$, by (1.6) the values of g_α on the set $\Omega_\alpha \cap \Omega_\beta$ are irrelevant. Therefore, if the open sets Ω_α form a locally finite covering of Ω and we assume that, for each $\alpha \in \mathcal{A}$, the vector field g_α satisfies (1.5) at every point $x \in \partial\Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta$, then the vector field g defined according with (1.6) is again a patchy vector field. To see this, it suffices to construct vector fields $\tilde{g}_\alpha$ (defined on a neighborhood of $\overline{\Omega}_\alpha$ as g_α) which satisfy the inward pointing property (1.5) at every point $x \in \partial\Omega_\alpha$ and such that $\tilde{g}_\alpha = g_\alpha$ on $\Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta$ (cf. [1, Remark 2.1]). In fact, with the same arguments one deduces that, to guarantee that a vector field g defined on an open domain Ω according with (1.6) be a patchy vector field, it is sufficient to require that each vector field g_α satisfy (1.5) at every point $x \in \partial\Omega_\alpha \setminus ((\bigcup_{\beta>\alpha} \Omega_\beta) \cup \partial\Omega)$.

If g is a patchy vector field, the differential equation

$$\dot{x} = g(x) \tag{1.9}$$

has several useful properties. In particular, in [1] it was proved that the set of Carathéodory solutions of (1.8) is closed (in the topology of uniform convergence) but possibly not connected. Moreover, given an initial condition

$$x(t_0) = x_0, \tag{1.10}$$

the Cauchy problem (1.9), (1.10) has at least one forward solution, and at most one backward solution, in the Carathéodory sense. For every Carathéodory solution $x = x(t)$ of (1.9), the map $t \mapsto \alpha^*(x(t))$ is left continuous and non-decreasing.

Remark 1.2. In some situations it is useful to adopt a more general definition of patchy vector field than the one formulated above. Indeed, one can consider patches $(\Omega_\alpha, g_\alpha)$ where the domain Ω_α has a piecewise smooth boundary (see [3]). In this case, the inward-pointing condition (1.5) can be expressed requiring that

$$g(x) \in \mathring{T}_\Omega(x) \tag{1.11}$$

where $\mathring{T}_\Omega(x)$ denotes the interior of the tangent cone to Ω at the point x, defined by

$$T_\Omega(x) \doteq \left\{ v \in \mathbb{R}^n \colon \liminf_{t\downarrow 0} \frac{d(x+tv, \Omega)}{t} = 0 \right\}. \tag{1.12}$$

Clearly, at any regular point $x \in \partial\Omega$, the interior of the tangent cone $T_\Omega(x)$ is precisely the set of all vectors $v \in \mathbb{R}^n$ that satisfy $\langle v, \mathbf{n}(x)\rangle < 0$ and hence (1.11) coincides with the inward-pointing condition (1.5). One can easily see that all the results concerning patchy vector fields established in [1,2] remain true within this more general formulation.

Definition 1.3. Let $(\Omega, g, (\Omega_\alpha, g_\alpha)_{\alpha\in\mathcal{A}})$ be a patchy vector field. Assume that there exist control values $v_\alpha \in \mathbf{U}$ such that, for each $\alpha \in \mathcal{A}$, there holds

$$g_\alpha(x) = f(x, v_\alpha) \quad \forall x \in D_\alpha \doteq \Omega_\alpha \Big\backslash \bigcup_{\beta>\alpha} \Omega_\beta. \tag{1.13}$$

Then, the piecewise constant map

$$U(x) \doteq v_\alpha \quad \text{if } x \in D_\alpha \tag{1.14}$$

is called a *patchy feedback* control on Ω, and referred to as $(\Omega, U, (\Omega_\alpha, v_\alpha)_{\alpha\in\mathcal{A}})$.

Remark 1.3. By Definitions 1.2 and 1.3, the vector field

$$g(x) = f\big(x, U(x)\big)$$

defined in connection with a given patchy feedback $(\Omega, U, (\Omega_\alpha, v_\alpha)_{\alpha\in\mathcal{A}})$ is precisely the patchy vector field $(\Omega, g, (\Omega_\alpha, g_\alpha)_{\alpha\in\mathcal{A}})$ associated with a family of fields $\{g_\alpha \colon \alpha \in \mathcal{A}\}$ satisfying (1.5). Notice that, recalling the notation (1.7), for all $x \in \Omega$ we have

$$U(x) = v_{\alpha^*(x)}. \tag{1.15}$$

As observed in Remark 1.1, the values of the vector fields $f(x, v_\alpha)$ on the set $\Omega_\alpha \cap \Omega_\beta$ are irrelevant whenever $\alpha < \beta$, and it is not necessary that $f(x, v_\alpha)$ satisfy the inward-pointing condition (1.5) at the points of $\partial\Omega_\alpha \cap (\bigcup_{\beta>\alpha} \Omega_\beta)$. Moreover, all the properties of a patchy feedback continue to hold even in the case where we assume that the inward-pointing condition (1.5) fails to be satisfied at the points of $(\partial\Omega_\alpha \cap \Sigma) \setminus \bigcup_{\beta>\alpha} \Omega_\beta$, for some region Σ of the boundary $\partial\Omega$. Clearly, in this case every Carathéodory trajectory of the patchy vector field g can eventually reach the boundary $\partial\Omega$ only crossing points of the region Σ.

To state our main results, we first need to relax the minimum time problem. Call $\mathcal{U}$ the family of admissible control functions, i.e. all measurable functions $t \mapsto u(t)$, $t \geqslant 0$, with $u(t) \in \mathbf{U}$ almost everywhere. For $y \in \mathbb{R}^n$ and $u \in \mathcal{U}$, we denote by $t \mapsto x(t; y, u)$ the solution of the Cauchy problem

$$\dot{x}(t) = f\big(x(t), u(t)\big), \quad x(0) = y. \tag{1.16}$$

The global existence and the uniqueness of this solution are guaranteed by the assumptions (H). Now fix $\varepsilon > 0$ arbitrarily small and define the penalization function

$$\varphi_\varepsilon(x) \doteq \begin{cases} \dfrac{|x|^2}{\varepsilon^2 - |x|^2} & \text{if } |x| < \varepsilon, \\ \infty & \text{if } |x| \geqslant \varepsilon. \end{cases} \tag{1.17}$$

Consider the following ε-approximate minimization problem:

$$\inf_{t\geqslant 0;\ u\in\mathcal{U}} \big\{t+\varphi_\varepsilon\big(x(t;y,u)\big)\big\}. \tag{1.18}$$

We denote this infimum by $V(y)$, for every $y\in\mathbb{R}^n$, and refer to $y\mapsto V(y)\in[0,\infty]$ as the *value function* for (1.18). Observe that $V(y)\leqslant T(y)$. Hence, for a fixed time $T>0$, the set of points that can be steered to the origin within time T is contained in the sub-level set

$$\Lambda_T \doteq \big\{y\in\mathbb{R}^n;\ V(y)\leqslant T\big\}. \tag{1.19}$$

With the above notations, our main result can be stated as follows.

Theorem 1. *Let the assumptions* (H) *hold and, given $\varepsilon>0$, $T>0$, let Λ_T be the sub-level set defined in* (1.19) *in connection with the value function V for* (1.18). *Then, there exists a patchy feedback control $u=U(x)$, defined on*

$$\Lambda_{T,\varepsilon} \doteq \big\{y\in\mathbb{R}^n;\ V(y)\leqslant T;\ |y|\geqslant\varepsilon\big\}, \tag{1.20}$$

such that, for each $y\in\Lambda_{T,\varepsilon}$, every Carathéodory solution of

$$\dot{x}=f\big(x,U(x)\big),\quad x(0)=y \tag{1.21}$$

reaches the ball

$$B_\varepsilon \doteq \big\{x\in\mathbb{R}^n;\ |x|\leqslant\varepsilon\big\}$$

within time $V(y)+\varepsilon$.

The assumptions (H) are very general. They do not even imply the existence of optimal controls, even for the relaxed problem (1.18). We recall that the standard existence theory requires the additional assumptions

(H′) The set $\mathbf{U}\subset\mathbb{R}^m$ of admissible control values is compact. For every $x\in\mathbb{R}^n$, the set of velocities $\{f(x,u);\ u\in\mathbf{U}\}$ is convex.

If both (H) and (H′) hold, then the infimum in (1.4) and in (1.18) are actually attained (e.g. cf. [14]). Moreover, the minimum time function $T:\mathbb{R}^n\mapsto[0,\infty]$ is lower semicontinuous. This fact is a well known consequence of the closure property of the graph of the set valued map $S:[0,\infty)\times\mathbb{R}^n\rightsquigarrow\mathbb{R}^n$ defined by $S(t,y)\doteq\{x(t;y,u);\ u\in\mathcal{U}\}$. Because of the lower semicontinuity of the minimum time function, and by (1.3), it follows that, for every $\tau\geqslant 0$, the attainable set

$$A(\tau) \doteq \big\{y\in\mathbb{R}^n;\ T(y)\leqslant\tau\big\} \tag{1.22}$$

is compact. Since $V(y)\leqslant T(y)$ for all $y\in\mathbb{R}^n$, from Theorem 1 one thus obtains

Corollary. *Let the assumptions* (H) *and* (H′) *hold, and let $\varepsilon>0$, $\tau>0$ be given. Then there exists a patchy feedback control $u=U(x)$, defined on the set*

$$A_\varepsilon(\tau) \doteq \big\{y\in\mathbb{R}^n;\ T(y)\leqslant\tau;\ |y|\geqslant\varepsilon\big\}, \tag{1.23}$$

such that, for each $y\in A_\varepsilon(\tau)$, every Carathéodory solution of (1.21) *reaches the ball B_ε within time $T(y)+\varepsilon$.*

In all previous papers [1–3] the construction of a stabilizing patchy feedback did not make any use of a control-Lyapunov function for (1.1). Instead, the feedback law was obtained by patching together a finite number of open-loop controls. We remark that a straightforward adaptation of this strategy would not work here. Indeed, let $\varepsilon>0$ be given. As in [1], we can then cover the set $A_\varepsilon(\tau)$ with finitely many tubes $\Omega_1,\dots,\Omega_N$ and construct a patchy feedback $u=U_\alpha(x)$ steering each point $y\in\Omega_\alpha$ inside the ball B_ε within time $T(y)+\varepsilon$. However, we cannot guarantee that the patchy feedback

$$u(x)=U_{\alpha^*(x)},\quad \alpha^*(x)\doteq\max\{\alpha;\ x\in\Omega_\alpha\} \tag{1.24}$$

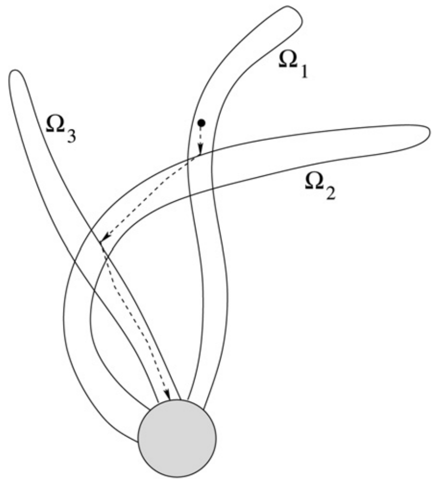

Fig. 1.

is nearly-optimal (see Fig. 1). Indeed, call $T_\alpha(y)$ the time taken by the control U_α to steer the point $y \in \Omega_\alpha$ inside B_ε. Let $t \mapsto x(t)$ be a trajectory of the patchy feedback (1.24), with $x(0) = y$, $x(\tau) \in B_\varepsilon$. Assume $\alpha^*(t) = \alpha$ for $t \in]t_{\alpha-1}, t_\alpha]$. The near-optimality of each feedback implies $T_\alpha(x) \leqslant T(x) + \varepsilon$ for every x. Moreover

$$T_\alpha\big(x(t_{\alpha-1})\big) - T_\alpha\big(x(t_\alpha)\big) = (t_\alpha - t_{\alpha-1}).$$

Unfortunately, from the above inequalities one can only deduce

$$T\big(x(t_{\alpha-1})\big) - T\big(x(t_\alpha)\big) \geqslant (t_\alpha - t_{\alpha-1}) - \varepsilon$$

and hence $\tau \leqslant T(y) + N\varepsilon$. This is a useless information, because the number N of tubes may well approach infinity as $\varepsilon \to 0$.

To overcome this problem, in the present paper we perform an entirely different construction of the patchy feedback. As starting point, instead of open-loop controls, we use the value function V for the problem (1.18), together with a piecewise quadratic approximation $\widetilde{V}$. This has the form

$$\widetilde{V}(x) = \min_j V_j(x), \qquad V_j(x) = a_j + b_j \cdot x + c|x|^2$$

and satisfies $\widetilde{V}(x) \leqslant V(x) + \varepsilon$ for each point x. The result will be achieved by constructing a patchy feedback such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{V}\big(x(t)\big) = \nabla\widetilde{V}\big(x(t)\big) \cdot f\big(x(t), u(x(t)\big) \leqslant \varepsilon$$

at a.e. time t.

2. Preliminary results

Throughout the paper, by $B(x,r)$ we denote the closed ball centered at x with radius r, and set $B_r \doteq B(0,r)$. The closure, the interior and the boundary of a set Ω are written as $\overline{\Omega}$, $\overset{\circ}{\Omega}$ and $\partial\Omega$, respectively, while $\mathrm{diam}(\Omega)$ denotes the diameter of a bounded set Ω. The distance of a point x from a set Ω is denoted by $d_\Omega(x)$, while $d_\Omega(E) \doteq \inf_{x\in E} d_\Omega(x)$ denotes the distance between two sets Ω, E. The number of elements of a finite set $\mathcal{J}$ is denoted by $|\mathcal{J}|$.

We begin by observing that the infimum in (1.18) provides an upper bound for the time needed to steer the system (1.1) from y to the ball B_ε. Hence, for every $T \geqslant 0$, the sub-level set Λ_T of the value function V for (1.18) is contained in the set of points that can be steered to the ball B_ε within time T. On the other hand, notice that the scalar Cauchy problem

$$\dot{z} = c(1+z), \qquad z(0) = \varepsilon, \tag{2.1}$$

has solution

$$z(t) = (1+\varepsilon)\,e^{ct} - 1. \tag{2.2}$$

Therefore, because of (1.3), a comparison argument yields

$$\Lambda_T \subseteq B_{z(T)}, \tag{2.3}$$

for every $T \geqslant 0$.

In connection with the relaxed minimization problem (1.18), we now show that the value function V is Lipschitz continuous on Λ_T and locally semiconcave, that is, for any x_0, there exists a constant $c_0 > 0$ such that there holds

$$V(x_1) + V(x_2) - 2V\left(\frac{x_1+x_2}{2}\right) \leqslant c_0|x_1 - x_2|^2 \tag{2.4}$$

for all x_1, x_2 in a neighborhood of x_0. We refer to [14] for the definition and properties of semiconcave functions.

Lemma 1. *With the assumptions* (H)*, for any fixed $\varepsilon, T > 0$ the restriction of the value function V for* (1.18) *to the sublevel set Λ_T is Lipschitz continuous and locally semiconcave. Indeed, there exists a positive constant λ such that, for every point $y_0 \in \Lambda_T$ where V is differentiable, there holds*

$$V(y) \leqslant V(y_0) + \big\langle \nabla V(y_0), y - y_0 \big\rangle + \lambda |y - y_0|^2 \quad \forall y \in \Lambda_T. \tag{2.5}$$

Proof.

1. First observe that, since we are only proving something about the value function V for (1.18), it is not restrictive to assume that the additional hypotheses (H′) hold. Indeed, allowing the set of controls to range in the closure of $\mathbf{U}$ does not affect the value function. Moreover, if the sets of velocities $\{f(x,u);\ u \in \mathbf{U}\}$ are not convex, we can replace the original system (1.1) by a chattering one (see [6]), such that the problem (1.18) yields exactly the same value function. This in particular implies that the value function V is lower semicontinuous and that the sub-level set (1.19) is compact.

2. Next, observe that, since the function f is twice continuously differentiable and the sets Λ_T, $\mathbf{U}$ are compact, by standard differentiability properties of the trajectories of a control system (1.1), there holds

$$\sup_{\substack{t\in[0,T],\ y\in\Lambda_T\\ u\in\mathcal{U}}} \left|\frac{\partial}{\partial y} x(t; y, u)\right| \leqslant \exp\big\{T\|D_x f\|_{\mathbf{L}^\infty(\Lambda_T)}\big\} \doteq M_1, \tag{2.6}$$

$$\sup_{\substack{t\in[0,T],\ y\in\Lambda_T\\ u\in\mathcal{U}}} \left|\frac{\partial^2}{\partial y^2} x(t; y, u)\right| \leqslant \big[M_1\big(1 + T M_1^3 \|D_x^2 f\|_{\mathbf{L}^\infty(\Lambda_T)}\big)\big] \doteq M_2, \tag{2.7}$$

where

$$\begin{aligned} \|D_x f\|_{\mathbf{L}^\infty(\Lambda_T)} &\doteq \sup_{x\in\Lambda_T,\ u\in\mathbf{U}} \left|\frac{\partial}{\partial x} f(x,u)\right| < \infty, \\ \|D_x^2 f\|_{\mathbf{L}^\infty(\Lambda_T)} &\doteq \sup_{x\in\Lambda_T,\ u\in\mathbf{U}} \left|\frac{\partial^2}{\partial x^2} f(x,u)\right| < \infty, \end{aligned} \tag{2.8}$$

provide a bound on the first and second partial derivatives of f w.r.t. the x-variable, over the set Λ_T. Then, because of (2.6), there exists a constant c_1 such that

$$\big|x(t; y_2, u) - x(t; y_1, u)\big| \leqslant c_1|y_2 - y_1| \quad \forall t \in [0,T],\ y_1, y_2 \in \Lambda_T,\ u \in \mathcal{U}. \tag{2.9}$$

3. Given $y_0 \in \Lambda_T$, by the previous assumptions at point **1** there exists an optimal control $u_0 \in \mathcal{U}$, and a time t_0, such that

$$t_0 + \varphi_\varepsilon\big(x(t_0; y_0, u_0)\big) = V(y_0) \leqslant T. \tag{2.10}$$

This, by definition (1.17) of φ_ε, of course implies

$$t_0 \leqslant T, \quad \big|x(t_0; y_0, u_0)\big| \leqslant \varepsilon\sqrt{\frac{T}{1+T}}. \tag{2.11}$$

Hence, using (2.11) together with (2.9), we find that there exists some constant $\delta > 0$, depending only on ε, T, and on c_1, but not on the point $y_0 \in \Lambda_T$, such that

$$\bigl|x(t_0; y, u_0)\bigr| \leqslant \varepsilon\sqrt{\frac{2T+1}{2(1+T)}} \quad \forall y \in B(y_0, \delta) \cap \Lambda_T. \tag{2.12}$$

Observe now that, since $V(y)$ is the infimum in (1.18), there holds

$$V(y) \leqslant V^0(y) \doteq t_0 + \varphi_\varepsilon\bigl(x(t_0; y, u_0)\bigr) \quad \forall y. \tag{2.13}$$

Because of (2.12), the map $y \mapsto V^0(y)$ defined in (2.13) is twice continuously differentiable at every point of $B(y_0, \delta) \cap \Lambda_T$. Hence, since (2.10) implies $V^0(y_0) = V(y_0)$, there holds

$$V^0(y) \leqslant V(y_0) + \bigl\langle \nabla V^0(y_0), y - y_0 \bigr\rangle + \lambda_0 |y - y_0|^2 \quad \forall y \in B(y_0, \delta) \cap \Lambda_T, \tag{2.14}$$

with

$$\lambda_0 \doteq \sup_{\eta \in B(y_0, \delta) \cap \Lambda_T} \bigl|D^2 V^0(\eta)\bigr|.$$

The gradient of the function V^0 is computed by

$$\nabla V^0(y) = \nabla\varphi_\varepsilon\bigl(x(t_0; y, u_0)\bigr) \cdot \frac{\partial}{\partial y} x(t_0; y, u_0).$$

Thus, relying on (2.6), (2.12), and setting

$$M_0 \doteq \sup_{\substack{t \in [0,T],\ y \in \Lambda_T \\ u \in \mathcal{U}}} \bigl|x(t; y, u)\bigr|, \tag{2.15}$$

we obtain

$$\bigl|\nabla V^0(y)\bigr| \leqslant \frac{2\varepsilon^2 |x(t_0; y, u_0)|}{(\varepsilon^2 - |x(t_0; y, u_0)|^2)^2} \cdot M_1 \leqslant \frac{8(1+T)^2 M_0 M_1}{\varepsilon^2} \doteq c_2 \quad \forall y \in B(y_0, \delta) \cap \Lambda_T. \tag{2.16}$$

With similar computations, using (2.7), (2.12), we find that a bound on the second derivative of V^0 is provided by

$$\begin{aligned} \bigl|D^2 V^0(y)\bigr| &\leqslant \bigl|D^2\varphi_\varepsilon\bigl(x(t_0; y, u_0)\bigr)\bigr| \left|\frac{\partial}{\partial y} x(t; y, u)\right| + \bigl|\nabla\varphi_\varepsilon\bigl(x(t_0; y, u_0)\bigr)\bigr| \left|\frac{\partial^2}{\partial y^2} x(t; y, u)\right| \\ &\leqslant \left[\frac{8(1+T)^2 M_1}{\varepsilon^2} + \frac{64(1+T)^3 M_0^2 M_1}{\varepsilon^4} + \frac{8(1+T)^2 M_0 M_2}{\varepsilon^2}\right] \doteq c_3 \quad \forall y \in B(y_0, \delta) \cap \Lambda_T. \end{aligned} \tag{2.17}$$

Notice that the constants c_2, c_3 depend only on ε, T, and on the function f, but not on the point $y_0 \in \Lambda_T$. Then, (2.13), (2.14), together with (2.16), (2.17) yield

$$V(y) \leqslant V(y_0) + (c_2 + \delta c_3)|y - y_0| \quad \forall y \in B(y_0, \delta) \cap \Lambda_T,\ \forall y_0 \in \Lambda_T, \tag{2.18}$$

which, in turn, implies

$$y_1, y_2 \in \Lambda_T,\ |y_1 - y_2| < \delta \quad \Longrightarrow \quad \bigl|V(y_1) - V(y_2)\bigr| \leqslant (c_2 + \delta c_3)|y_1 - y_2|. \tag{2.19}$$

Since the set Λ_T is compact, we deduce from (2.19) that the map V is (globally) Lipschitz continuous on Λ_T.

4. Given $y_0 \in \Lambda_T$, in connection with the constants λ_0, δ, c_2 introduced at point **3** choose

$$\lambda > \max\left\{\lambda_0, \frac{\mathrm{Lip}(V) + c_2}{\delta}\right\},$$

and observe that, because of (2.16), there holds

$$\begin{aligned} \bigl\langle \nabla V^0(y_0), y - y_0 \bigr\rangle + \lambda|y - y_0|^2 &\geqslant (-c_2 + \lambda\delta)|y - y_0| \\ &\geqslant \mathrm{Lip}(V) \cdot |y - y_0| \end{aligned} \quad \forall y \in \Lambda_T \setminus B(y_0, \delta). \tag{2.20}$$

Thus, (2.13), (2.14), together with (2.20), yield

$$V(y) \leqslant V(y_0) + \bigl\langle \nabla V^0(y_0), y - y_0 \bigr\rangle + \lambda\bigl|y - y_0\bigr|^2 \quad \forall y \in \Lambda_T. \tag{2.21}$$

5. Fix $\rho > 0$. By the above arguments there exist a positive constant $\lambda = \lambda_\rho$ so that, for every fixed $y_0 \in \Lambda_{T+\rho}$, the estimate (2.21) holds for all $y \in \Lambda_{T+\rho}$. Next, given $x_0 \in \Lambda_T$, choose δ_0 so that $B(x_0, \delta_0) \subset \Lambda_{T+\rho}$. Then, for any $y_1, y_2 \in B(x_0, \delta_0) \cap \Lambda_T$, one has $\frac{y_1+y_2}{2} \in \Lambda_{T+\rho}$. Hence, applying (2.21) for $y = y_i$, $i = 1, 2$, and $y_0 = \frac{y_1+y_2}{2}$, and summing up the corresponding inequalities, since $y_1 - y_0 = y_0 - y_2$ we obtain

$$V(y_1) + V(y_2) \leqslant 2V\left(\frac{y_1+y_2}{2}\right) + \lambda\left[|y_1 - y_0|^2 + |y_2 - y_0|^2\right] \leqslant 2V\left(\frac{y_1+y_2}{2}\right) + \lambda|y_1 - y_2|^2,$$

which shows that the estimate (2.4) is verified, with $c_0 = \lambda$, for all $y_1, y_2 \in \Lambda_T$ in the ball $B(x_0, \delta_0)$. Therefore, the map V is locally semiconcave on Λ_T.

6. To conclude the proof of the lemma, consider a point $y_0 \in \Lambda_T$ where V is differentiable, and observe that, by (2.21), one has

$$\left\langle \nabla V(y_0) - \nabla V^0(y_0), \frac{y - y_0}{|y - y_0|} \right\rangle \leqslant \lambda\left[|y - y_0| + \frac{\mathrm{o}(|y - y_0|)}{|y - y_0|}\right] \tag{2.22}$$

for all $y \in \Lambda_T$. Thus, taking $y_\sigma \doteq y_0 + \sigma(\nabla V(y_0) - \nabla V^0(y_0))/(|\nabla V(y_0) - \nabla V^0(y_0)|)$, $\sigma > 0$ from (2.22) we deduce

$$\left|\nabla V(y_0) - \nabla V^0(y_0)\right| \leqslant \lambda\left[\sigma + \frac{\mathrm{o}(\sigma)}{\sigma}\right] \quad \forall \sigma > 0. \tag{2.23}$$

By letting $\sigma \to 0$ in (2.23) we obtain $\nabla V^0(y_0) = \nabla V(y_0)$ which, together with (2.21), yields (2.5), completing the proof of the lemma. □

We next show that the value function V enjoys an infinitesimal decrease property at every point where it is differentiable, which is expressed in terms of an Hamilton–Jacobi inequality.

Lemma 2. *With the assumptions* (H)*, given $\varepsilon, T > 0$, let V be the value function for* (1.18)*. Then, there exists $0 < \varepsilon_0 < \varepsilon$ such that, letting $\Lambda_{T,\varepsilon_0}$ be the set defined in* (1.20)*, for each $y \in \Lambda_{T,\varepsilon_0}$ at which V is differentiable there holds*

$$\inf_{v \in \mathbf{U}} \left\{\left\langle \nabla V(y), f(y, v)\right\rangle\right\} + 1 \leqslant 0. \tag{2.24}$$

Proof. Given $\varepsilon, T > 0$, set

$$\varepsilon_0 \doteq \varepsilon\sqrt{\frac{4T+1}{2(1+2T)}}, \qquad \varepsilon_0' \doteq \varepsilon\sqrt{\frac{2T}{1+2T}}, \tag{2.25}$$

$$\tau_0 \doteq c^{-1}\ln\left(\frac{\varepsilon_0 + 1}{\varepsilon_0' + 1}\right), \tag{2.26}$$

where c denotes the constant in (1.3), and observe that, by definition (1.17) of φ_ε, one has

$$\varphi_\varepsilon(x) \geqslant 2T \quad \text{whenever} \quad |x| \geqslant \varepsilon_0'. \tag{2.27}$$

Then, recalling that (2.2) provides the solution to the scalar Cauchy problem (2.1), by a comparison argument, and because of (1.3), we deduce that

$$|y| \geqslant \varepsilon_0 \quad \Longrightarrow \quad \left|x(t; y, u)\right| \geqslant \varepsilon_0' \quad \forall t \in [0, \tau_0],\ u \in \mathcal{U}. \tag{2.28}$$

Hence, (2.27) together with (2.28), yields

$$t + \varphi_\varepsilon\big(x(t; y, u)\big) \geqslant 2T \quad \forall t \in [0, \tau_0],\ |y| \geqslant \varepsilon_0,\ u \in \mathcal{U}. \tag{2.29}$$

From (2.29) we deduce that, for every $y \in \Lambda_{T,\varepsilon_0}$, the value function for (1.18) satisfies

$$V(y) = \inf_{t > \tau_0;\ u \in \mathcal{U}} \left\{t + \varphi_\varepsilon\big(x(t; y, u)\big)\right\} > \tau_0. \tag{2.30}$$

Thus, we reach the conclusion of the lemma observing that by standard arguments in control theory (e.g. see [5,14]) one can show that the value function for (2.30) satisfies the Hamilton–Jacoby inequality (2.24) at every point where V is differentiable. □

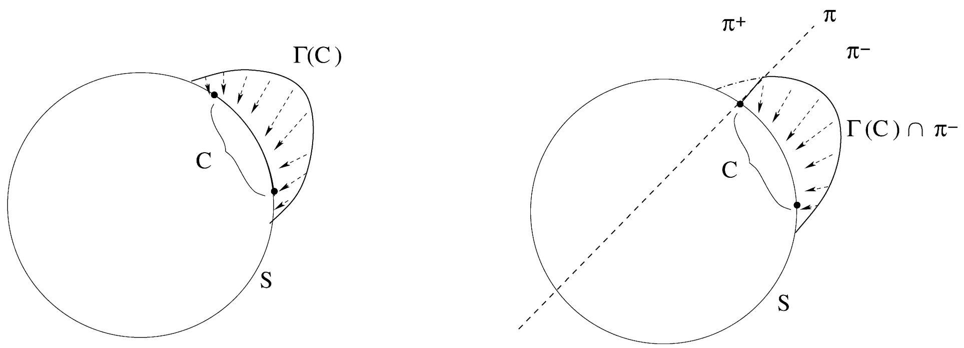

Fig. 2.

Remark 2.1. Notice that, in the proof of Theorem 1, we shall only need to have at a disposal a value function V satisfying the conclusions of Lemmas 1 and 2.

We state now two technical results which will be useful later in the construction of an almost time optimal patchy feedback. We shall provide a proof of them in Appendix A at the end of the paper. Throughout the following, for any given subset C of a sphere S, we let $\partial_S C$ denote the boundary of C relative to the topology of S.

Lemma 3. *Given $r_0 > 0$, let S be a sphere with radius $r \geqslant r_0$, and let g be a bounded, Lipschitz continuous vector field on $\mathbb{R}^n$ which points strictly inward at the points of a closed set $C \subset S$ that has a piecewise smooth relative boundary $\partial_S C$. More precisely, letting $\mathbf{n}_S(y)$ denote the unit outer normal to S at the point y, assume that*

$$\big\langle \mathbf{n}_S(y), g(y)\big\rangle \leqslant -\bar{c} \quad \forall y \in C, \tag{2.31}$$

for some constant $\bar{c} > 0$. Denote by $t \mapsto x(t, y)$ the solution of the Cauchy problem $\dot{x} = g(x)$, $x(0) = y$. Then there exists $\bar{\varepsilon} > 0$, depending only on r_0, $\bar{c}$, $\|g\|_{\mathbf{L}^\infty}$, and on the Lipschitz constant $\mathrm{Lip}(g)$ of g, such that the following holds. Define

$$\Gamma_{\bar{\varepsilon}}(C) \doteq \big\{x(\tau, y);\ y \in B(C, \bar{\varepsilon}) \cap S,\ d_C{}^2(y) - \bar{\varepsilon}^2 < \tau \leqslant 0\big\}. \tag{2.32}$$

Then the vector field g is transversal to the boundary of $\Gamma_{\bar{\varepsilon}} \doteq \Gamma_{\bar{\varepsilon}}(C)$. Indeed, it points strictly inward on the set

$$\partial^- \Gamma_{\bar{\varepsilon}} \doteq \big\{x\big(d_C{}^2(y) - \bar{\varepsilon}^2, y\big); y \in \mathring{B}(C, \bar{\varepsilon}) \cap S\big\} \tag{2.33}$$

and strictly outward on the set

$$\partial^+ \Gamma_{\bar{\varepsilon}} \doteq \partial \Gamma_{\bar{\varepsilon}} \cap S. \tag{2.34}$$

The lens-shaped domain (2.32) provides the basic building block for the construction of the patchy feedback produced in the next section. In some situations it will be necessary to restrict such domains cutting them along hyperplanes in order to preserve the (almost) time-optimality property of the feedback law (cf. Fig. 2). The next lemma provides an a-priori lower bound on the distance between the upper boundary of a collection of such domains and the union of spheres around which the domains are constructed.

Lemma 4. *Given $0 < r_0 < r_0'$, let $B_1, \dots, B_\nu$ be a finite collection of balls with surfaces $S_1, \dots, S_\nu$, having radii $r_1, \dots, r_\nu \in [r_0, r_0']$, and satisfying*

$$\left.\begin{aligned} &S_i \cap \bigcup_{j=1}^{\nu} \mathring{B}_j \neq \emptyset, \\ &S_i \Big\backslash \bigcup_{j=1}^{\nu} \mathring{B}_j \neq \emptyset, \end{aligned}\right\} \quad \forall i = 1, \dots, \nu. \tag{2.35}$$

Consider the sets

$$C_i \doteq S_i \Big\backslash \left(\bigcup_{j=1}^{\nu} \mathring{B}_j \right) \quad \forall i = 1, \dots, \nu, \tag{2.36}$$

and let $g_1, \dots, g_\nu$ be bounded, Lipschitz continuous vector fields which point strictly inward (*towards the interior of $S_1, \dots, S_\nu$*) *on $C_1, \dots, C_\nu$, respectively. Then, there exist constants $\bar{\varepsilon}', c_4 > 0$, depending only on ν, r_0, r'_0, and on $\|g_i\|_{\mathbf{L}^\infty}$, $\mathrm{Lip}(g_i)$, $i = 1, \dots, \nu$, such that the following holds. Let*

$$\begin{aligned} &\Pi = \bigcup_{k=1}^{\nu} \Pi_k, \quad \Pi_k \doteq \{\pi_{k,i};\ i \in \mathcal{J}_k\}, \\ &\mathcal{J}_k \subset \{1, \dots, \nu\} \setminus \{k\} \quad \forall k, \end{aligned} \tag{2.37}$$

be a (*possibly empty*) *collection of hyperplanes enjoying the properties*:

- $\pi_{k,i} = \pi_{i,k}$ *for all* $i \in \mathcal{J}_k$, $k \in \mathcal{J}_i$;
- $\pi_{k,i} \in \Pi_h$ *if and only if either* $h = k$, $i \in \mathcal{J}_h$, *or* $h = i$, $k \in \mathcal{J}_h$;
- *if* $\mathring{B}_k \cap \mathring{B}_i \neq \emptyset$, $i \in \mathcal{J}_k$, *then* $\pi_{k,i}$ *is the hyperplane passing through* $S_k \cap S_i$ (*cf. Fig.* 3);
- *if* $\mathring{B}_k \cap \mathring{B}_i = \emptyset$, $i \in \mathcal{J}_k$, *then* $\pi_{k,i}$ *is an hyperplane separating* S_k *and* S_i, *i.e. s.t.* S_k, S_i *are entirely contained in the opposite closed half spaces determined by* $\pi_{k,i}$ (*cf. Fig.* 4).

For every $\mathcal{J}_k \neq \emptyset$, and for any $i \in \mathcal{J}_k$, call $\pi^-_{k,i}$ the open half space determined by $\pi_{k,i}$ that contains $C_k \setminus \partial_{S_k} C_k$. Then, setting (*cf. Fig.* 5)

$$\Gamma^{\mathcal{J}_k}_{\bar{\varepsilon}'} \doteq \Gamma^{\mathcal{J}_k}_{\bar{\varepsilon}'}(C_k) \doteq \begin{cases} \Gamma_{\bar{\varepsilon}'}(C_k) \cap \bigcap_{i \in \mathcal{J}_k} \pi^-_{k,i} & \text{if } \mathcal{J}_k \neq \emptyset, \\ \Gamma_{\bar{\varepsilon}'}(C_k) & \text{otherwise,} \end{cases} \qquad k = 1, \dots, \nu, \tag{2.38}$$

$$C \doteq \bigcup_{k=1}^{\nu} C_k, \qquad \mathcal{G} \doteq \bigcup_{k=1}^{\nu} \overline{\Gamma^{\mathcal{J}_k}_{\bar{\varepsilon}'}}, \tag{2.39}$$

$$\partial^- \mathcal{G} \doteq \partial \mathcal{G} \Big\backslash \bigcup_{k=1}^{\nu} B_k, \tag{2.40}$$

one has

$$d_C(\partial^- \mathcal{G}) \geqslant c_4. \tag{2.41}$$

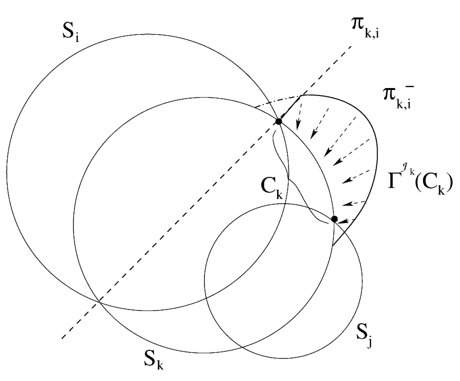

Fig. 3.

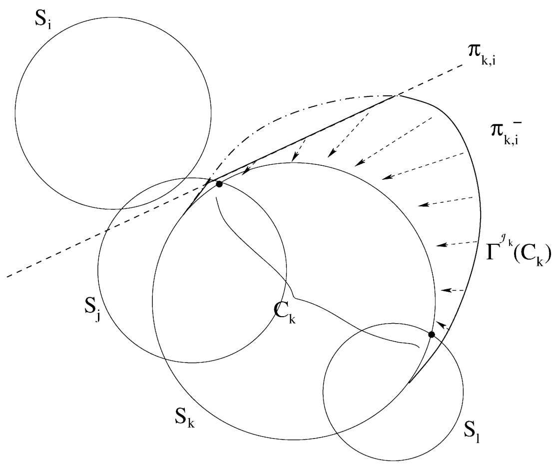

Fig. 4.

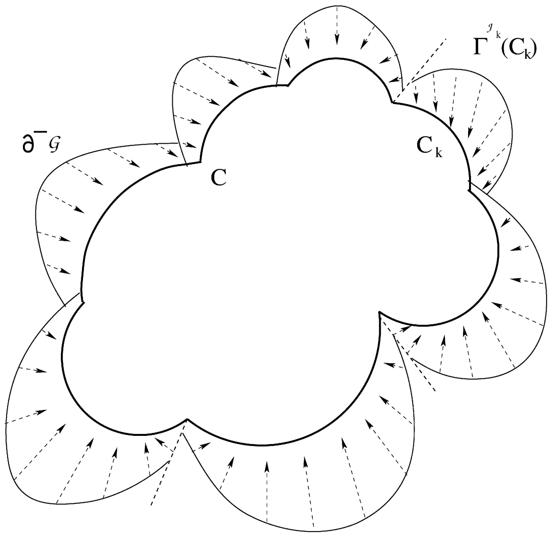

Fig. 5.

3. Proof of the theorem

The proof will be given in several steps.

1. Given $\varepsilon, T > 0$ ($\varepsilon < \min\{1, T\}$), fix some constant $T' > T + 1$, and observe that by Lemma 1 the value function V for (1.18) is Lipschitz continuous on $\Lambda_{T'} \doteq \{y \in \mathbb{R}^n;\ V(y) \leqslant T'\}$. Hence, by Rademacher's theorem V is differentiable a.e. in $\Lambda_{T'}$. Then, letting $\lambda > 0$ be the constants provided by Lemma 1 in connection with the set $\Lambda_{T'}$, for each $y \in \Lambda_{T'}$ at which V is differentiable define a quadratic function V^y setting

$$V^y(x) \doteq V(y) + \nabla V(y) \cdot (x - y) + (1 + \lambda)|x - y|^2. \tag{3.1}$$

Notice that, because of (2.5), there holds

$$V(x) + |x - y|^2 \leqslant V^y(x) \quad \forall x \in \Lambda_{T'}. \tag{3.2}$$

Moreover, according with Lemma 2, there exists some constant $\varepsilon_0 > 0$ so that, given a constant

$$0 < \varepsilon_1 < \frac{\varepsilon}{4T'}, \tag{3.3}$$

for every $y \in \Lambda_{T',\varepsilon_0} \doteq \{y \in \Lambda_{T'};\ |y| \geqslant \varepsilon_0\}$ where V is differentiable we can choose a control value $v^y \in \mathbf{U}$ such that

$$\big\langle \nabla V(y), f\big(y, v^y\big)\big\rangle < -1 + \varepsilon_1. \tag{3.4}$$

Choose the constant ε_0 so that, setting

$$\begin{aligned} &\varepsilon_0' \doteq \varepsilon\sqrt{\frac{4T'+1}{2(1+2T')}}, \qquad \varepsilon_0'' \doteq \varepsilon\sqrt{\frac{2T'}{1+2T'}}, \\ &\tau_0 \doteq c^{-1}\ln\left(\frac{\varepsilon_0'+1}{\varepsilon_0''+1}\right), \end{aligned} \tag{3.5}$$

where c denotes the constant in (1.3), there holds

$$\varepsilon_0 < \min\left\{\frac{\varepsilon}{4}, \varepsilon_0'\right\}, \qquad \frac{\varepsilon_0^2}{\varepsilon^2 - \varepsilon_0^2} < \frac{\tau_0}{2}. \tag{3.6}$$

Notice that, by definition (1.17) of φ_ε, and because of (3.6), the value function V for (1.18) satisfies

$$V(x) \leqslant \frac{\varepsilon_0^2}{\varepsilon^2 - \varepsilon_0^2} < \frac{\tau_0}{2} \quad \forall x \in B_{\varepsilon_0}. \tag{3.7}$$

Next, choose some other constant

$$L' > L \doteq \operatorname{diam}(\Lambda_{T'}) + \sqrt{n}\,\operatorname{Lip}(V) + \sqrt{n\big(\operatorname{Lip}(V)\big)^2 + 4T'(1+\lambda)}, \tag{3.8}$$

where $\operatorname{Lip}(V)$ denotes the Lipschitz constant of V on $\Lambda_{T'}$. Hence, since the assumptions (H) imply the Lipschitz continuity in x of the function $f(x,u)$ on the compact set $B_{L'} \times \mathbf{U}$, uniformly for $u \in \mathbf{U}$, and because also ∇V^y is Lipschitz continuous with a Lipschitz constant independent on $y \in \Lambda_{T'}$, there will be some constant $c_5 > 0$ (depending only on L') such that

$$\left.\begin{array}{l} \big|\langle \nabla V^y(x_1), f(x_1,u)\rangle - \langle \nabla V^y(x_2), f(x_2,u)\rangle\big| \leqslant c_5|x_1 - x_2| \\ \big|f(x_1,u) - f(x_2,u)\big| \leqslant c_5|x_1-x_2| \end{array}\right. \quad \forall x_1, x_2 \in B_{L'},\ u \in \mathbf{U}. \tag{3.9}$$

Then, setting

$$c_6 \doteq \sqrt{n}\,\operatorname{Lip}(V) + 4(1+\lambda)\operatorname{diam}(\Lambda_{T'}), \tag{3.10}$$

and choosing $\varepsilon_2 > 0$ so that

$$\varepsilon_2 < \min\left\{\sqrt{\frac{\varepsilon}{3}}, \frac{\varepsilon_1}{8c_5c_6}, \sqrt{\frac{\tau_0}{2}}, \frac{T'-T}{1+\operatorname{Lip}(V)}, L'-L\right\}, \tag{3.11}$$

we deduce from (3.4), (3.9) that, for every $y \in \Lambda_{T',\varepsilon_0}$ where V is differentiable there holds

$$\big\langle \nabla V^y(x), f\big(x, v^y\big)\big\rangle < -1 + 2\varepsilon_1 \quad \forall x \in B(y, 2\varepsilon_2) \cap B_{L'}. \tag{3.12}$$

2. By the Lipschitz continuity of V on the set $\Lambda_{T'}$ it follows that, for each $y \in \Lambda_{T',\varepsilon_0}$ at which V is differentiable, there holds

$$\big|V^y(x) - V(x)\big| \leqslant c_7|x-y| \quad \forall x \in \Lambda_{T'},$$

for some positive constant c_7. Hence, since the set $\Lambda_{T',\varepsilon_0}$ is compact (cf. point **1** of the proof of Lemma 1), we can cover it with finitely many balls (of sufficiently small radius), centered at points of $\Lambda_{T',\varepsilon_0}$ where V is differentiable, say $y_1, \dots, y_N$, so that, setting

$$\left.\begin{array}{l} V_i(x) \doteq V^{y_i}(x), \quad 1 \leqslant i \leqslant N, \\ \widetilde{V}(x) \doteq \min\limits_i V_i(x) \end{array}\right. \quad \forall x \in \mathbb{R}^n, \tag{3.13}$$

there holds

$$V(x) \leqslant \widetilde{V}(x) \leqslant V(x) + \varepsilon_2^2 \quad \forall x \in \Lambda_{T',\varepsilon_0}. \tag{3.14}$$

Next, observing that (3.2) implies

$$V_i(x) > V(x) + \varepsilon_2^2 \quad \forall x \in \Lambda_{T'} \setminus B(y_i, \varepsilon_2),$$

we deduce from (3.14) that

$$\widetilde{V}(x) < V_i(x) \quad \forall x \in \Lambda_{T',\varepsilon_0} \setminus B(y_i, \varepsilon_2). \tag{3.15}$$

Relying on (3.15), letting

$$\mathcal{P}_i \doteq \big\{x \in \mathbb{R}^n;\ V_i(x) = \widetilde{V}(x)\big\}, \tag{3.16}$$

we find that

$$\mathcal{P}_i \cap \Lambda_{T',\varepsilon_0} \subset B(y_i, \varepsilon_2), \quad 1 \leqslant i \leqslant N. \tag{3.17}$$

Hence, by (3.12), (3.17) we have

$$\big\langle \nabla V_i(x), f\big(x, v^i\big)\big\rangle < -1 + 2\varepsilon_1 \quad \forall x \in B(\mathcal{P}_i, \varepsilon_2) \cap \Lambda_{T',\varepsilon_0} \cap B_{L'},\ 1 \leqslant i \leqslant N, \tag{3.18}$$

where we have set $v^i \doteq v^{y_i}$ $(1 \leqslant i \leqslant N)$, while (3.2) yields

$$\widetilde{V}(x) \geqslant V(x) + |x - y_i|^2 \quad \forall x \in \mathcal{P}_i \cap \Lambda_{T'},\ 1 \leqslant i \leqslant N. \tag{3.19}$$

3. The patchy feedback $u = U(x)$ will be constructed looking at the level sets of the function $\widetilde{V}$ defined in (3.13). To this end, observe first that, because of (3.14), and by the choice (3.11) of ε_2, there holds

$$\widetilde{V}(x) < T' - \mathrm{Lip}(V)\cdot\varepsilon_2 \quad \forall x \in \Lambda_T. \tag{3.20}$$

Moreover, notice that, relying on the definitions (3.1) of V^{y_i} and (3.8) of the constant L, one finds

$$\begin{aligned} 1 \leqslant i \leqslant N, \quad V_i(x) < T' \quad \Longrightarrow \quad |x| &\leqslant |x - y_i| + \mathrm{diam}(\Lambda_{T'}) \\ &\leqslant \bigl|\nabla V(y_i)\bigr| + \sqrt{\bigl|\nabla V(y_i)\bigr|^2 + 4T'(1+\lambda)} + \mathrm{diam}(\Lambda_{T'}) \\ &\leqslant \sqrt{n}\,\mathrm{Lip}(V) + \sqrt{n\bigl(\mathrm{Lip}(V)\bigr)^2 + 4T'(1+\lambda)} + \mathrm{diam}(\Lambda_{T'}) \\ &\leqslant L, \end{aligned} \tag{3.21}$$

which, in turn, by the definition (3.13) of $\widetilde{V}$, and because of (3.14), yields

$$\bigl\{x \in \mathbb{R}^n;\ \widetilde{V}(x) < T'\bigr\} \subset \Lambda_{T'} \cap B_L. \tag{3.22}$$

On the other hand, observe that all level sets of each quadratic function V_i are spheres. Therefore, every level set

$$\Sigma_\tau \doteq \bigl\{x \in \mathbb{R}^n;\ \widetilde{V}(x) = \tau\bigr\} \quad \tau \geqslant \tau_0, \tag{3.23}$$

is contained in a finite union of spheres, and each upper level set $\{x \in \mathbb{R}^n;\ \widetilde{V}(x) \geqslant \tau\}$, $\tau \geqslant \tau_0$, is connected. Moreover, notice that by (3.7), (3.11), (3.14), we derive

$$\max_{|x|=\varepsilon_0} \widetilde{V}(x) < \frac{\tau_0}{2} + \varepsilon_2^2 \leqslant \tau_0, \tag{3.24}$$

and hence we find that

$$\bigl\{x \in \mathbb{R}^n;\ \widetilde{V}(x) \geqslant \tau_0\bigr\} \cap B_{\varepsilon_0} = \emptyset. \tag{3.25}$$

Thus, setting

$$T'' \doteq T' - \mathrm{Lip}(V)\cdot\varepsilon_2, \quad \mathcal{D} \doteq \bigl\{x \in \mathbb{R}^n;\ \tau_0 < \widetilde{V}(x) < T''\bigr\}, \tag{3.26}$$

thanks to (3.11), (3.14), (3.20), (3.22), (3.25), we deduce that

$$\Lambda_{T,\varepsilon} \subset B(\mathcal{D}, \varepsilon_2) \subset \Lambda_{T',\varepsilon_0} \cap B_L. \tag{3.27}$$

We will establish the theorem by constructing the patchy feedback $u = U(x)$ on the domain $\mathcal{D}$. Notice that, with the same arguments used in the proof of Lemma 2, by the choice of the constants ε_0', τ_0 in (3.5) we find that

$$V(x) > \tau_0 \quad \forall x \in \Lambda_{T'},\ |x| \geqslant \varepsilon_0'. \tag{3.28}$$

Hence, since the definition (3.13) of $\widetilde{V}$ implies

$$V(x) \leqslant \tau_0 \quad \forall x \in \Sigma_{\tau_0},$$

we deduce from (3.6), (3.28) that

$$\Sigma_{\tau_0} \subset B_{\varepsilon_0'} \subset B_\varepsilon. \tag{3.29}$$

Next, observe that, since all functions V_i, $1 \leqslant i \leqslant N$, have the same coefficient of the quadratic term, it follows that, for each couple of indices $k \neq i$, the set

$$\pi_{k,i} \doteq \pi_{i,k} \doteq \bigl\{x \in \mathbb{R}^n;\ V_k(x) = V_i(x)\bigr\} \tag{3.30}$$

is an hyperplane, and the difference of the gradients $\nabla V_i(x) - \nabla V_k(x)$ is a constant vector on $\pi_{k,i}$. Then, letting $\mathbf{n}_{k,i}$ denote the unit normal to $\pi_{k,i}$, pointing towards the half space

$$\pi_{k,i}^+ \doteq \pi_{i,k}^- \doteq \bigl\{x \in \mathbb{R}^n;\ V_k(x) > V_i(x)\bigr\}, \tag{3.31}$$

one has

$$\nabla V_i(x) - \nabla V_k(x) = -c\mathbf{n}_{k,i} \quad \forall x \in \pi_{k,i}, \tag{3.32}$$

for some constant $c = c_{k,i} \geqslant 0$. Denote as $\pi^-_{k,i}$ the other half space determined by $\pi_{k,i}$, i.e. set

$$\pi^-_{k,i} \doteq \pi^+_{i,k} \doteq \{x \in \mathbb{R}^n;\ V_k(x) < V_i(x)\}. \tag{3.33}$$

4. The basic step in the construction of $U(x)$ is the following. We shall fix a suitably small time size Δt and, in connection with an increasing sequence of times $\{\tau_m\}_{m\geqslant 0}$ with the property

$$\exists p \quad \text{s.t.} \quad \tau_{m+p} > \tau_m + \Delta_t \quad \forall m,$$

we will construct, for every $m \geqslant 0$, a patchy feedback whose domain contains the region

$$\mathcal{D}_m \doteq \{x \in \mathbb{R}^n;\ \tau_m < \widetilde{V}(x) \leqslant \tau_{m+1}\},$$

so that all the trajectories $x(t)$ of the corresponding closed-loop system (1.2) satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{V}\big(x(t)\big) \leqslant -1 + 3\varepsilon_1 \quad \text{for a.e. } t,$$

and eventually enter the set where $\widetilde{V} < \tau_m$. To this end, fix any $\tau \in [\tau_0, T'[$ and consider the level set Σ_τ of $\widetilde{V}$. By construction, Σ_τ is contained in the union of finitely many spheres, say $S_{i_1}, \ldots, S_{i_{\nu_\tau}}$. Here we denote as $S_{i_\ell} \doteq \{x \in \mathbb{R}^n;\ V_{i_\ell}(x) = \tau\}$ the surface of the ball $B_{i_\ell} \doteq \{x \in \mathbb{R}^n;\ V_{i_\ell}(x) \leqslant \tau\}$. Notice that, since the definition (3.13) of $\widetilde{V}$ implies $\widetilde{V}(x) < \tau$ for all $x \in \mathring{B}_{i_\ell}$, by definition (3.23) it follows that

$$\Sigma_\tau = \bigcup_{\ell=1}^{\nu_\tau} \Sigma_{\tau,i_\ell}, \qquad \Sigma_{\tau,i_\ell} \doteq S_{i_\ell} \Big\backslash \bigcup_{q=1}^{\nu_\tau} \mathring{B}_{i_q}, \tag{3.34}$$

$$\{x \in \mathbb{R}^n;\ \widetilde{V}(x) < \tau\} = \bigcup_{\ell=1}^{\nu_\tau} \mathring{B}_{i_\ell}. \tag{3.35}$$

We can assume that the set of indices $\mathcal{I}_\tau \doteq \{i_1, \ldots, i_{\nu_\tau}\}$ includes only those indices $i \in \{1, \ldots, N\}$ for which there exists some point $\bar{x} \in \mathcal{D}$ satisfying

$$\tau = V_i(\bar{x}) < V_j(\bar{x}) \quad \forall j \neq i.$$

This means that

$$\mathcal{I}_\tau = \left\{ i \in \{1, \ldots, N\};\ \left(\mathcal{P}_i \Big\backslash \bigcup_{j\neq i} \mathcal{P}_j\right) \cap \Sigma_\tau \neq \emptyset \right\}, \tag{3.36}$$

and, in particular, implies that

$$S_i \Big\backslash \bigcup_{j\in\mathcal{I}_\tau} \mathring{B}_j \neq \emptyset \quad \forall i \in \mathcal{I}_\tau. \tag{3.37}$$

Moreover, we may write Σ_τ as the union of η_τ connected components $\Sigma^1_\tau, \ldots, \Sigma^{\eta_\tau}_\tau$, so that setting

$$\mathcal{I}^h_\tau \doteq \left\{ i \in \mathcal{I}_\tau;\ \left(\mathcal{P}_i \Big\backslash \bigcup_{j\neq i} \mathcal{P}_j\right) \cap \Sigma^h_\tau \neq \emptyset \right\}, \tag{3.38}$$

there holds

$$S_i \cap \bigcup_{j\in\mathcal{I}^h_\tau} \mathring{B}_j \neq \emptyset \quad \forall i \in \mathcal{I}^h_\tau,\ h = 1, \ldots, \eta_\tau. \tag{3.39}$$

Notice also that, by (3.13), (3.23), (3.34), (3.37), every set $\Sigma_{\tau,i}$, $i \in \mathcal{I}_\tau$, is nonempty and one has

$$\Sigma_{\tau,i} = \{x \in \mathbb{R}^n;\ V_i(x) = \widetilde{V}(x) = \tau\} \quad \forall i \in \mathcal{I}_\tau, \tag{3.40}$$

while the definitions (3.30), (3.31), (3.33) imply

$$\begin{matrix} \pi_{k,i} \cap S_k = \pi_{k,i} \cap S_i = S_k \cap S_i, & \\ \Sigma_{\tau,k} \subset \pi_{k,i} \cup \pi^-_{k,i}, \qquad \Sigma_{\tau,i} \subset \pi_{k,i} \cup \pi^+_{k,i} \end{matrix} \quad \forall k, i \in \mathcal{I}_\tau. \tag{3.41}$$

Therefore, relying on (3.41) we deduce that, for every pair of indices $k, i \in \mathcal{I}_\tau$, $k \neq i$, one of the following two cases occurs:

- if $\mathring{B}_k \cap \mathring{B}_i \neq \emptyset$, then $\pi_{k,i}$ is the hyperplane passing through $S_k \cap S_i$;
- if $\mathring{B}_k \cap \mathring{B}_i = \emptyset$, then $S_k \subset \pi_{k,i} \cup \pi^-_{k,i}$ and $S_i \subset \pi_{k,i} \cup \pi^+_{k,i}$, i.e. $\pi_{k,i}$ is an hyperplane separating S_k and S_i.

5. By the above construction, and relying on (3.11), (3.22), we deduce that

$$\begin{matrix} \Sigma_{\tau,i} \subset \mathcal{P}_i \cap B_L, \\ B(\Sigma_{\tau,i}, \varepsilon_2) \subset B_{L'}, \end{matrix} \qquad \forall \tau \in [\tau_0, T''[,\ i \in \mathcal{I}_\tau. \tag{3.42}$$

Hence, thanks to (3.18), (3.27), (3.42), we find

$$\big\langle \nabla V_i(x), f\big(x, v^i\big)\big\rangle < -1 + 2\varepsilon_1 \quad \forall x \in B\big(\Sigma_{\tau,i}, \varepsilon_2\big),\ \tau \in [\tau_0, T''[,\ i \in \mathcal{I}_\tau. \tag{3.43}$$

Relying on (3.43), we shall construct around each set $\Sigma_{\tau,i}$, $i \in \mathcal{I}_\tau$, a lens-shaped domain $\Gamma_{\tau,i}$ of the form (2.32) as in Lemma 3, so that the boundary of $\Gamma_{\tau,i}$ is transversal to the flow of the vector field $g_i(x) \doteq f(x, v^i)$. Namely, letting $x(t; y, v^i)$ denote the solution of the Cauchy problem $\dot{x} = g_i(x)$, $x(0) = y$, we will prove the following

Claim 1. *There exists a positive constants ε_3 so that, for every given $\tau \in [\tau_0, T''[$, $k \in \mathcal{I}_\tau$, the vector field $g_{\tau,k}(x) \doteq f(x, v^k)$ is transversal to the boundary of the domain*

$$\Gamma_{\tau,k} \doteq \big\{x\big(s; y, v^k\big);\ y \in B(\Sigma_{\tau,k}, \varepsilon_3) \cap S_k, d_{\Sigma_{\tau,k}}{}^2(y) - \varepsilon_3^2 < s \leqslant 0\big\}. \tag{3.44}$$

Namely, it points strictly inward on the upper boundary

$$\partial^- \Gamma_{\tau,k} \doteq \partial \Gamma_{\tau,k} \Big\backslash \bigcup_{j \in \mathcal{I}_\tau} B_j$$

and strictly outward on the lower boundary

$$\partial^+ \Gamma_{\tau,k} \doteq \partial \Gamma_{\tau,k} \cap S_k.$$

Moreover, there holds

$$\big|g_{\tau,k}(x)\big| \geqslant c_8 \quad \forall x \in \overline{\Gamma_{\tau,k}}, \tag{3.45}$$

$$\overline{\Gamma_{\tau,k}} \subset B(\Sigma_{\tau,k}, \varepsilon_2) \subset B(y_k, 2\varepsilon_2), \tag{3.46}$$

for some constant $c_8 > 0$ independent on $\tau \in [\tau_0, T''[$, $k \in \mathcal{I}_\tau$.

6. Proof of Claim 1. In order to establish the claim, we shall first derive an upper and lower uniform bound for the radii of the spheres

$$S_i \doteq \big\{x \in \mathbb{R}^n;\ V_i(x) = \tau\big\} \quad i \in \mathcal{I}_\tau,\ \tau \in [\tau_0, T''[, \tag{3.47}$$

and we will prove an a priori estimate for $\langle \mathbf{n}_i, f(x, v^i)\rangle$, $x \in \Sigma_{\tau,i}$, independent of $\tau \in [\tau_0, T''[$, and $i \in \mathcal{I}_\tau$ ($\mathbf{n}_i$ denoting the unit outer normal to S_i). To this end observe that by (1.3) one has

$$\big|g_{\tau,i}(x)\big| = \big|f(x, v^i)\big| \leqslant c_9 \doteq c(1 + L') \quad \forall x \in B_{L'}. \tag{3.48}$$

Then, for every fixed $i \in \mathcal{I}_\tau$, $\tau \in [\tau_0, T''[$, writing

$$V_i(x) = (1 + \lambda)|x - \omega_i|^2 + b_i \quad \forall x \in S_i,$$

for some point $\omega_i \in \mathbb{R}^n$ and some constant b_i, and using (3.3), (3.42), (3.43), (3.48), we derive the estimate

$$2(1 + \lambda)|x - \omega_i| = \big|\nabla V_i(x)\big| \geqslant \frac{|\langle \nabla V_i(x), f(x, v^i)\rangle|}{|f(x, v^i)|} \geqslant \frac{1}{2c_9} \quad \forall x \in \Sigma_{\tau,i}. \tag{3.49}$$

On the other hand, from the definition (3.1) of $V_i = V^{y_i}$, recalling that $y_i \in \Lambda_{T'}$, and relying on (3.10), (3.27), one deduces the a priori bound

$$\big|\nabla V_i(x)\big| \leqslant \big|\nabla V(x)\big| + 4(1 + \lambda)\,\mathrm{diam}(\Lambda_{T'}) \leqslant c_6 \quad \forall x \in \mathcal{D}. \tag{3.50}$$

Hence, thanks to (3.49), (3.50), we find that the radius $r_i = |x - \omega_i|$, $x \in \Sigma_{\tau,i}$, of the sphere S_i satisfies

$$\frac{1}{4c_9(1+\lambda)} \leqslant r_i \leqslant \frac{c_6}{2(1+\lambda)} \quad \forall i \in \mathcal{I}_\tau,\ \tau \in \left[\tau_0, T''\right[, \tag{3.51}$$

while (3.3), (3.43), together with (3.50), yield

$$\left\langle \mathbf{n}_i, f\left(x, v^i\right)\right\rangle = \frac{\langle \nabla V_i(x), f(x, v^i)\rangle}{|\nabla V_i(x)|} \leqslant -\frac{1}{2c_6} \quad \forall x \in \Sigma_{\tau,i},\ i \in \mathcal{I}_\tau,\ \tau \in \left[\tau_0, T''\right[. \tag{3.52}$$

Therefore, because of (3.51), (3.52), we can apply Lemma 3 to every set $\Sigma_{\tau,k}$, $\tau \in [\tau_0, T''[$, $k \in \mathcal{I}_\tau$, in connection with the vector field $g_{\tau,k}$. Thus we deduce the existence of some constant $\varepsilon_3 > 0$, so that the field $g_{\tau,k}(x) = f(x, v^k)$ is transversal to the boundary of the domain $\Gamma_{\tau,k}$ defined in (3.44). Concerning (3.46), observe that choosing ε_3 such that $\varepsilon_3(c_9\varepsilon_3 + 1) < \varepsilon_2$, thanks to (3.42), (3.48) we obtain

$$\begin{aligned} d_{\Sigma_{\tau,k}}\left(x\left(s; y, v^k\right)\right) + d_{\Sigma_{\tau,k}}(y) &\leqslant \left|x\left(s; y, v^k\right) - y\right| \\ &\leqslant |s|\, \|g_{\tau,k}\|_{\mathbf{L}^\infty(B_{L'})} + d_{\Sigma_{\tau,k}}(y) \\ &\leqslant |s|c_9 + \varepsilon_3 < \varepsilon_2 \quad \forall y \in B(\Sigma_{\tau,k}, \varepsilon_3) \cap S_k,\ -\varepsilon_3^2 < s \leqslant 0. \end{aligned} \tag{3.53}$$

Hence, because of (3.42), (3.17), (3.27), relying on (3.53) we find

$$\overline{\Gamma_{\tau,k}} \subset B(\Sigma_{\tau,k}, \varepsilon_2) \subset B(\mathcal{P}_k, \varepsilon_2) \subset B(y_k, 2\varepsilon_2), \tag{3.54}$$

which proves (3.46). Finally, observe that, for every given $k \in \mathcal{I}_\tau$, $\tau \in [\tau_0, T''[$, fixing some point $\bar{x} \in \Sigma_{\tau,k}$, thanks to (3.46), and because of (3.3), (3.9)–(3.11), (3.52), we derive

$$\begin{aligned} \left|f\left(x, v^k\right)\right| &\geqslant \left|f\left(\bar{x}, v^k\right)\right| - \left|f\left(x, v^k\right) - f\left(\bar{x}, v^k\right)\right| \\ &\geqslant \left|\left\langle \mathbf{n}_k, f\left(\bar{x}, v^k\right)\right\rangle\right| - c_5 \cdot |x - \bar{x}| \\ &\geqslant \frac{1}{2c_6} - 4c_5\varepsilon_2 \\ &\geqslant \frac{1}{4c_6} \quad \forall x \in \overline{\Gamma_{\tau,k}}, \end{aligned} \tag{3.55}$$

which yields (3.45), thus completing the proof of our claim.

7. Given $\tau \in [\tau_0, T''[$, $k \in \mathcal{I}_\tau$, consider now the domain $\Gamma_{\tau,k}$ defined in (3.44), and observe that, because of (3.16), (3.43), (3.46), every trajectory $x(t)$ of $\dot{x} = g_{\tau,k}(x)$, passing through points of $\Gamma_{\tau,k} \cap \mathcal{P}_k$, satisfies

$$\begin{aligned} \widetilde{V}\left(x(t)\right) &= V_k\left(x(t)\right) \\ &= V_k\left(x(s)\right) + \int_s^t \left\langle \nabla V_k\left(x(\sigma)\right), f\left(x(\sigma), v^k\right)\right\rangle \mathrm{d}\sigma \\ &\leqslant V_k\left(x(s)\right) + (-1 + 2\varepsilon_1)(t - s) \\ &= \widetilde{V}\left(x(s)\right) + (-1 + 2\varepsilon_1)(t - s) \quad \forall t > s. \end{aligned} \tag{3.56}$$

However, there may well be points $x(t) \in \Gamma_{\tau,k}$ where $V_k(x(t)) > \widetilde{V}(x(t))$. Near these points there is no guarantee that (3.56) should hold. To address this difficulty, we will consider the set of all indices $i \neq k$ such that $V_i(\bar{x}) < V_k(\bar{x})$ for some $\bar{x} \in \Gamma_{\tau,k}$, and such that

$$\min_{x \in \overline{\Gamma_{\tau,k}}} \left\langle \nabla V_k(x) - \nabla V_i(x), f\left(x, v^k\right)\right\rangle < 0. \tag{3.57}$$

In this case, we shall replace $\Gamma_{\tau,k}$ with the smaller domain

$$\Gamma_{\tau,k} \cap \left\{x \in \mathbb{R}^n;\ V_k(x) < V_i(x)\right\}.$$

Then, setting

$$\mathcal{J}_{\tau,k} \doteq \left\{i \in \{1, \ldots, N\} \setminus \{k\};\ \mathcal{P}_i \cap \Gamma_{\tau,k} \neq \emptyset,\ \min_{x \in \overline{\Gamma_{\tau,k}}} \left\langle \nabla V_k(x) - \nabla V_i(x), f\left(x, v^k\right)\right\rangle < 0\right\}, \tag{3.58}$$

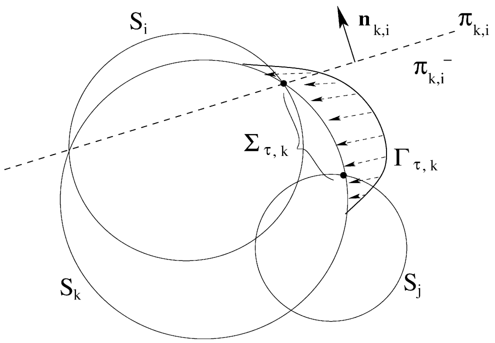

Fig. 6. ($i \notin \mathcal{J}_{\tau,k}$).

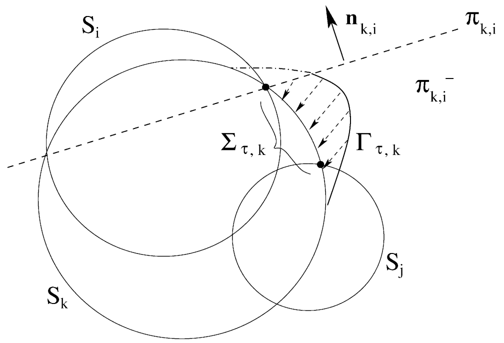

Fig. 7. ($i \in \mathcal{J}_{\tau,k}$).

consider the domain (see Fig. 7)

$$\widetilde{\Gamma}_{\tau,k} \doteq \begin{cases} \Gamma_{\tau,k} \cap \bigcap_{i \in \mathcal{J}_{\tau,k}} \pi_{k,i}^- & \text{if } \mathcal{J}_{\tau,k} \neq \emptyset, \\ \Gamma_{\tau,k} & \text{otherwise,} \end{cases} \tag{3.59}$$

which, according with the definitions (2.32), (2.38), is precisely equal to $\Gamma_{\varepsilon_3}^{\mathcal{J}_{\tau,k}}(\Sigma_{\tau,k})$.

Notice that, because of (3.37), (3.39), (3.51), (3.52), and by the observations at point **4**, for every fixed $h = 1, \dots, \eta_\tau$, the spheres S_i, $i \in \mathcal{I}_\tau^h$, and the collection of hyperplanes

$$\Pi_\tau^h \doteq \{\pi_{k,i};\ k, i \in \mathcal{I}_\tau^h\}$$

satisfy the assumptions of Lemma 4. Hence, in the case where

$$\bigcup_{k \in \mathcal{I}_\tau^h} \mathcal{J}_{\tau,k} \subset \mathcal{I}_\tau^h,$$

we are in the position to apply the conclusion of Lemma 4 in connection with the collection of hyperplanes Π_τ^h and of sets (see Fig. 8)

$$\{\Sigma_{\tau,i};\ i \in \mathcal{I}_\tau^h\}$$

defined in (3.34), in order to derive a uniform estimate of the distance of the (upper) boundary of

$$\mathcal{G}_\tau^h \doteq \bigcup_{k \in \mathcal{I}_\tau^h} \overline{\widetilde{\Gamma}_{\tau,k}} \tag{3.60}$$

from the set

$$\Sigma_\tau^h = \bigcup_{i \in \mathcal{I}_\tau^h} \Sigma_{\tau,i}.$$

As a consequence, we obtain an estimate of the decrease of $\widetilde{V}$ along trajectories of $g_{\tau,k}$ passing through $\widetilde{\Gamma}_{\tau,k}$, $k \in \mathcal{I}_\tau^h$. More precisely, setting

$$\begin{aligned} \mathcal{I}_1^* &\doteq \Big\{\tau \in [\tau_0, T''[;\ \bigcup_{k \in \mathcal{I}_\tau^h} \mathcal{J}_{\tau,k} \subset \mathcal{I}_\tau^h\ \forall h = 1, \dots, \eta_\tau\Big\}, \\ \mathcal{G}_\tau &\doteq \bigcup_{k \in \mathcal{I}_\tau} \overline{\widetilde{\Gamma}_{\tau,k}} = \bigcup_{h=1}^{\eta_\tau} \mathcal{G}_\tau^h, \end{aligned} \tag{3.61}$$

we will prove the following

Claim 2. *The domains $\widetilde{\Gamma}_{\tau,k}$, $k \in \mathcal{I}_\tau$, $\tau \in [\tau_0, T''[$, defined in* (3.59) *enjoy the following properties.*

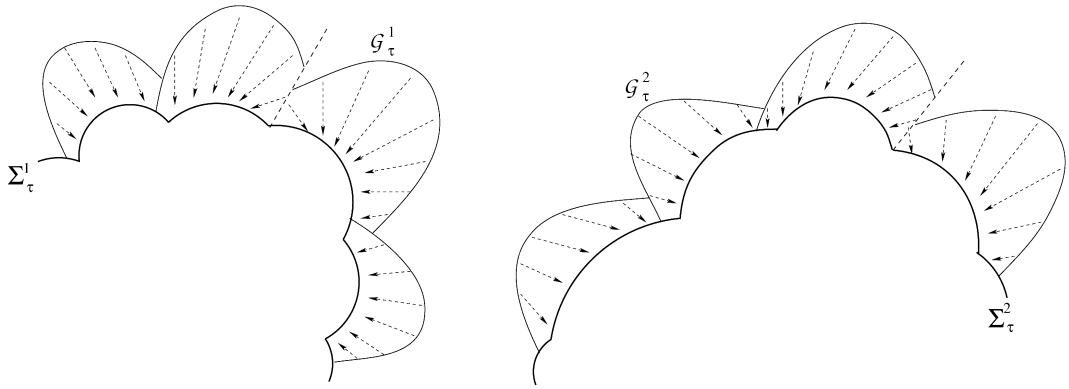

Fig. 8.

(i) *For any* $k \in \mathcal{I}_\tau$, $\tau \in [\tau_0, T''[$, *the vector field* $g_{\tau,k}(x) \doteq f(x, v^k)$ *points strictly inward at every point of the upper boundary*

$$\partial^- \widetilde{\Gamma}_{\tau,k} \doteq \partial \widetilde{\Gamma}_{\tau,k} \Big\backslash \bigcup_{j \in \mathcal{I}_\tau} B_j. \tag{3.62}$$

(ii) *For any* $y \in \overline{\widetilde{\Gamma}_{\tau,k}} \setminus \bigcup_{j \in \mathcal{I}_\tau} B_j$, $k \in \mathcal{I}_\tau$, $\tau \in [\tau_0, T''[$, *there exists a time* $\mathcal{T}_{\tau,k}(y) > 0$ *so that one has*

$$x\big(\mathcal{T}_{\tau,k}(y);\ y, v^k\big) \in \Sigma_\tau, \tag{3.63}$$

$$x\big(t; y, v^k\big) \in \widetilde{\Gamma}_{\tau,k} \quad \forall t \in \big]0, \mathcal{T}_{\tau,k}(y)\big], \tag{3.64}$$

and there holds

$$\widetilde{V}\big(x\big(t; y, v^k\big)\big) \leqslant \widetilde{V}\big(x\big(s; y, v^k\big)\big) + (-1 + 3\varepsilon_1)(t - s) \quad \forall 0 \leqslant s < t \leqslant \mathcal{T}_{\tau,k}(y), \tag{3.65}$$

where ε_1 *is the constant satisfying* (3.3).

(iii) *For any* $\tau \in [\tau_0, T''[$, *one has*

$$\tau^\sharp \doteq \sup\left\{t \in [\tau, T''[;\ \Sigma_s \subset \mathring{\mathcal{G}}_\tau \Big\backslash \bigcup_{j \in \mathcal{I}_\tau} B_j\ \forall s \in]\tau, t]\right\} > \tau. \tag{3.66}$$

Moreover, there exists a positive constant ε_4 *so that there holds*

$$\tau^\sharp > \tau + \varepsilon_4 \quad \forall \tau \in \mathcal{I}_1^*. \tag{3.67}$$

8. Proof of Claim 2. By Claim 1 we know that, for every $k \in \mathcal{I}_\tau$, the vector field $g_{\tau,k}$ is inward-pointing on the region $\partial^- \widetilde{\Gamma}_{\tau,k} \cap \partial^- \Gamma_{\tau,k}$. On the other hand, recalling (3.32), the inequality (3.57) guarantees that $g_{\tau,k}$ enjoys the inward-pointing condition also at the boundary points $x \in \partial^- \widetilde{\Gamma}_{\tau,k} \cap \mathring{\Gamma}_{\tau,k} \cap \pi_{k,i}$, $i \in \mathcal{J}_{\tau,k}$. Then, observing that

$$\partial^- \widetilde{\Gamma}_{\tau,k} \setminus \partial^- \Gamma_{\tau,k} = \partial^- \widetilde{\Gamma}_{\tau,k} \cap \mathring{\Gamma}_{\tau,k} \cap \bigcup_{i \in \mathcal{J}_{\tau,k}} \pi_{k,i},$$

by continuity it follows that $g_{\tau,k}(x) \in \mathring{T}_{\widetilde{\Gamma}_{\tau,k}}(x)$ at every point $x \in \partial^- \widetilde{\Gamma}_{\tau,k}$ ($\mathring{T}_{\widetilde{\Gamma}_{\tau,k}}$ denoting the interior of the tangent cone to $\widetilde{\Gamma}_{\tau,k}$ defined as in (1.12)), which proves the property (i) of Claim 2. Concerning the property (ii), observe first that by property (i) a trajectory $\gamma_y(\cdot)$ of $g_{\tau,k}$ starting at a point $y \in Q_{\tau,k} \doteq \overline{\widetilde{\Gamma}_{\tau,k}} \setminus \bigcup_{j \in \mathcal{I}_\tau} B_j$ cannot escape from $Q_{\tau,k}$ through a point of $\partial^- \widetilde{\Gamma}_{\tau,k}$. Thus, since (3.45) shows that $|g_{\tau,k}|$ is bounded away from zero, and because by (3.34) one has $\partial Q_{\tau,k} \setminus \partial^- \widetilde{\Gamma}_{\tau,k} \subset \bigcup_{j \in \mathcal{I}_\tau} S_j = \Sigma_\tau$, it follows that $\gamma_y(\cdot)$ must cross the level set Σ_τ in finite time $\mathcal{T}_{\tau,k}(y) > 0$, and hence (3.63), (3.64) are verified. In fact, with the same arguments above one can show that every trajectory $\gamma_y(\cdot)$ starting at a point of

$$Q_{\tau,k}^h \doteq \overline{\widetilde{\Gamma}_{\tau,k}} \Big\backslash \bigcup_{j \in \mathcal{I}_\tau^h} B_j, \quad 1 \leqslant h \leqslant \eta_\tau, \tag{3.68}$$

crosses the set $\Sigma_\tau^h \subset \Sigma_\tau$ in finite time $\mathcal{T}_{\tau,k}^h(y) \geqslant \mathcal{T}_{\tau,k}(y)$. Next, observe that setting

$$I_{\tau,k} \doteq \{i \in \{1,\dots,N\};\ \mathcal{P}_i \cap \widetilde{\Gamma}_{\tau,k} \neq \emptyset\}, \tag{3.69}$$

by definition (3.58) for every $i \in I_{\tau,k} \setminus (\mathcal{J}_{\tau,k} \cup \{k\})$ there will be some point $\bar{x}_i \in \overline{\Gamma_{\tau,k}}$ such that (see Fig. 6)

$$\langle \nabla V_k(\bar{x}_i) - \nabla V_i(\bar{x}_i), f(\bar{x}_i, v^k)\rangle \geqslant 0. \tag{3.70}$$

Thus, relying on (3.43), (3.46), (3.70), we derive

$$\begin{aligned}\langle \nabla V_i(\bar{x}_i), f(\bar{x}_i, v^k)\rangle &= \langle \nabla V_k(\bar{x}_i), f(\bar{x}_i, v^k)\rangle - \langle \nabla V_k(\bar{x}_i) - \nabla V_i(\bar{x}_i), f(\bar{x}_i, v^k)\rangle \\ &< -1 + 2\varepsilon_1.\end{aligned} \tag{3.71}$$

Then, since (3.27), (3.46) imply $\overline{\widetilde{\Gamma}_{\tau,k}} \subset \overline{\Gamma_{\tau,k}} \subset B(y_k, 2\varepsilon_2) \cap B_L$, using (3.9), (3.11), (3.71), we find

$$\begin{aligned}\langle \nabla V_i(x), f(x, v^k)\rangle &\leqslant \langle \nabla V_i(\bar{x}_i), f(\bar{x}_i, v^k)\rangle + \left|\langle \nabla V_i(x), f(x, v^k)\rangle - \langle \nabla V_i(\bar{x}_i), f(\bar{x}_i, v^k)\rangle\right| \\ &< -1 + 2\varepsilon_1 + c_5 4\varepsilon_2 \\ &< -1 + 3\varepsilon_1 \quad \forall x \in \overline{\widetilde{\Gamma}_{\tau,k}},\ i \in I_{\tau,k}.\end{aligned} \tag{3.72}$$

Hence, setting

$$x(t) \doteq x(t; y, v^k), \quad y \in Q_{\tau,k}^h,\ 1 \leqslant h \leqslant \eta_\tau, \tag{3.73}$$

and observing that, for every fixed $0 \leqslant s < t \leqslant \mathcal{T}_{\tau,k}^h(y)$, by (3.16), (3.69) there will be some index $i(s) \in I_{\tau,k}$ such that $\widetilde{V}(x(s)) = V_{i(s)}(x(s))$, relying on (3.43), (3.46), (3.64), (3.72), we derive

$$\begin{aligned}\widetilde{V}(x(t)) &\leqslant V_{i(s)}(x(t)) \\ &= V_{i(s)}(x(s)) + \int_s^t \langle \nabla V_{i(s)}(x(\sigma)), f(x(\sigma), v^k)\rangle \,\mathrm{d}\sigma \\ &\leqslant V_{i(s)}(x(s)) + (-1 + 3\varepsilon_1)(t - s) \\ &= \widetilde{V}(x(s)) + (-1 + 3\varepsilon_1)(t - s),\end{aligned} \tag{3.74}$$

which yields (3.65) since $\mathcal{T}_{\tau,k}^h(y) \geqslant \mathcal{T}_{\tau,k}(y)$. Observe now that, by the observations at point **7**, we can apply Lemma 4 for every collection of sets $\{\Sigma_{\tau,k};\ k \in \mathcal{I}_\tau^h\}$, and hyperplanes $\{\pi_{k,i};\ k, i \in \mathcal{I}_\tau^h\}$, $h = 1,\dots,\eta_\tau$, $\tau \in \mathcal{I}_1^*$. Thus we deduce that there exists some constant $c_{10} > 0$ such that

$$d_{\Sigma_\tau^h}(\partial^- \mathcal{G}_\tau^h) \geqslant c_{10} \quad \forall h = 1,\dots,\eta_\tau,\ \tau \in \mathcal{I}_1^*. \tag{3.75}$$

Since $x(\mathcal{T}_{\tau,k}^h(y)) \in \Sigma_\tau^h$, relying on (3.64), (3.75) we find that, for every fixed $\tau \in \mathcal{I}_1^*$, $1 \leqslant h \leqslant \eta_\tau$, $k \in \mathcal{I}_\tau^h$, using the same notation in (3.73) one has

$$\left|y - x(\mathcal{T}_{\tau,k}^h(y))\right| \geqslant d_{\Sigma_\tau^h}(\partial^- \mathcal{G}_\tau^h) \geqslant c_{10} \quad \forall y \in \partial^- \mathcal{G}_\tau^h \cap Q_{\tau,k}^h. \tag{3.76}$$

On the other hand, by (3.27), (3.48), (3.64), we derive

$$\left|y - x(\mathcal{T}_{\tau,k}^h(y))\right| \leqslant c_9 \mathcal{T}_{\tau,k}^h(y) \quad \forall y \in \partial^- \mathcal{G}_\tau^h \cap Q_{\tau,k}^h, \tag{3.77}$$

which, together with (3.76), yields

$$\mathcal{T}_{\tau,k}^h(y) \geqslant \frac{c_{10}}{c_9} \quad \forall y \in \partial^- \mathcal{G}_\tau^h \cap Q_{\tau,k}^h. \tag{3.78}$$

Therefore, observing that by (3.23) one has

$$\widetilde{V}(x(\mathcal{T}_{\tau,k}^h(y))) = \tau \quad \forall y \in \partial^- \mathcal{G}_\tau^h \cap Q_{\tau,k}^h,$$

thanks to (3.78), and relying on (3.3), (3.74), we deduce that, for every fixed $\tau \in \mathcal{I}_1^*$, $1 \leqslant h \leqslant \eta_\tau$, $k \in \mathcal{I}_\tau^h$, there holds

$$\begin{aligned}\widetilde V(y) &\geqslant \widetilde V\big(x\big(\mathcal{T}^h_{\tau,k}(y)\big)\big) + (1-3\varepsilon_1)\mathcal{T}^h_{\tau,k}(y)\\ &\geqslant \tau + \frac{\mathcal{T}^h_{\tau,k}(y)}{4}\\ &\geqslant \tau + \frac{c_{10}}{4c_9} \quad \forall y\in\partial^-\mathcal{G}^h_\tau\cap Q^h_{\tau,k}.\end{aligned} \tag{3.79}$$

Hence, since by definitions (2.40), (3.60), (3.68), one has

$$\partial^-\mathcal{G}^h_\tau = \bigcup_{k\in\mathcal{I}^h_\tau} \partial^-\mathcal{G}^h_\tau\cap Q^h_{\tau,k},$$

it follows from (3.79) that

$$\widetilde V(y) > \tau + \varepsilon_4 \quad \forall y\in\partial^-\mathcal{G}^h_\tau,\ 1\leqslant h\leqslant \eta_\tau,\ \tau\in\mathcal{I}^*_1, \tag{3.80}$$

where $\varepsilon_4 \doteq c_{10}/(8c_9)$. Moreover, with the same computations in (3.79) we derive also the estimates

$$\widetilde V(y) > \tau \quad \forall y\in\mathcal{G}^h_\tau \Big\backslash \bigcup_{j\in\mathcal{I}^h_\tau} B_j, \quad 1\leqslant h\leqslant\eta_\tau,\ \tau\in\big[\tau_0, T''\big[, \tag{3.81}$$

$$\widetilde V(y) > \tau + \frac{\min_h d_{\Sigma^h_\tau}(\partial^-\mathcal{G}^h_\tau)}{8c_9} \quad \forall y\in\partial^-\mathcal{G}^h_\tau,\ 1\leqslant h\leqslant\eta_\tau,\ \tau\in\big[\tau_0, T''\big[. \tag{3.82}$$

Notice that (3.81), in particular, implies $\partial^-\mathcal{G}^h_\tau\cap\Sigma^h_\tau=\emptyset$, for all $1\leqslant h\leqslant\eta_\tau$, and hence one has

$$\chi_\tau \doteq \tau + \frac{\min_h d_{\Sigma^h_\tau}\big(\partial^-\mathcal{G}^h_\tau\big)}{8c_9} > \tau. \tag{3.83}$$

To conclude, observe that by construction, for every given $\tau\in[\tau_0, T''[$, $1\leqslant h\leqslant\eta_\tau$, the set

$$\big\{x\in\mathbb{R}^n;\ \widetilde V(x)\geqslant\tau\big\}\setminus\partial^-\mathcal{G}^h_\tau \tag{3.84}$$

consists of two connected components, one of which, say $\mathcal{O}^h$, contains Σ^h_τ. Thus, since (3.61) implies $\mathring{\mathcal{G}}_\tau = \bigcup_{h=1}^{\eta_\tau}\mathring{\mathcal{G}}^h_\tau$, and because of (3.81), there holds

$$\bigcup_{h=1}^{\eta_\tau}\mathcal{O}_h = \Sigma_\tau \cup\Big(\mathring{\mathcal{G}}_\tau \Big\backslash \bigcup_{j\in\mathcal{I}_\tau} B_j\Big). \tag{3.85}$$

On the other hand, (3.80), (3.82), imply

$$\begin{aligned}&\big\{x\in\mathbb{R}^n;\ \tau\leqslant\widetilde V(x)\leqslant\chi_\tau\big\}\subset\bigcup_{h=1}^{\eta_\tau}\mathcal{O}_h \quad \forall\tau\in\big[\tau_0,T''\big[,\\ &\big\{x\in\mathbb{R}^n;\ \tau\leqslant\widetilde V(x)\leqslant\tau+\varepsilon_4\big\}\subset\bigcup_{h=1}^{\eta_\tau}\mathcal{O}_h \quad \forall\tau\in\mathcal{I}^*_1,\end{aligned} \tag{3.86}$$

and hence (3.85), (3.86) together yield

$$\big\{x\in\mathbb{R}^n;\ \tau\leqslant\widetilde V(x)\leqslant\chi_\tau\big\}\subset\Sigma_\tau\cup\Big(\mathring{\mathcal{G}}_\tau\Big\backslash\bigcup_{j\in\mathcal{I}_\tau}B_j\Big) \quad \forall\tau\in\big[\tau_0,T''\big[, \tag{3.87}$$

$$\big\{x\in\mathbb{R}^n;\ \tau\leqslant\widetilde V(x)\leqslant\tau+\varepsilon_4\big\}\subset\Sigma_\tau\cup\Big(\mathring{\mathcal{G}}_\tau\Big\backslash\bigcup_{j\in\mathcal{I}_\tau}B_j\Big) \quad \forall\tau\in\mathcal{I}^*_1. \tag{3.88}$$

Recalling the definition (3.23) of Σ_τ, we recover from (3.87), (3.88) the inclusions

$$\mathring{\mathcal{G}}_\tau \Big\backslash \bigcup_{j\in\mathcal{I}_\tau} B_j \supset \big\{x\in\mathbb{R}^n;\ \tau<\widetilde{V}(x)\leqslant\chi_\tau\big\}\quad \forall\tau\in\big[\tau_0,T''\big[, \tag{3.89}$$

$$\mathring{\mathcal{G}}_\tau \Big\backslash \bigcup_{j\in\mathcal{I}_\tau} B_j \supset \big\{x\in\mathbb{R}^n;\ \tau<\widetilde{V}(x)\leqslant\tau+\varepsilon_4\big\}\quad \forall\tau\in\mathcal{I}_1^* \tag{3.90}$$

which, in turn, together with (3.83), yield (3.66), (3.67), and thus we complete the proof of the claim.

Notice that, by definitions (3.38), (3.58), (3.59), and from the above proof of Claim 2 it follows that the inclusion in (3.90) is verified also for all time τ in the set

$$\mathcal{I}_2^* \doteq \big\{\tau\in\big[\tau_0,T''\big[\setminus\mathcal{I}_1^*;\ \eta_t=\eta_\tau,\ I_t^h\subset\mathcal{I}_\tau^h\ \forall t\in\big[\tau,\tau^\sharp\big[,\ h=1,\dots,\eta_\tau\big\}. \tag{3.91}$$

Hence, we derive

$$\tau^\sharp>\tau+\varepsilon_4\quad\forall\tau\in\mathcal{I}_2^*. \tag{3.92}$$

9. Relying on the properties (i), (ii) stated in Claim 2, for every fixed $\tau\in[\tau_0,T''[$ we shall construct now a patchy feedback on the open region

$$\Omega_\tau \doteq \mathring{\mathcal{G}}_\tau \Big\backslash \bigcup_{j\in\mathcal{I}_\tau} B_j. \tag{3.93}$$

To this end we first need to slightly enlarge some of the domains defined in (3.59). Namely, for every $k\in\mathcal{I}_\tau$, consider the set

$$\widehat{\mathcal{J}}_{\tau,k} \doteq \big\{i\in\mathcal{J}_{\tau,k}\cap\mathcal{I}_\tau;\ i>k,\ k\in\mathcal{J}_{\tau,i}\big\}, \tag{3.94}$$

fix some positive constant $\rho\ll\varepsilon_3$, denote by $\pi_{k,i}^\rho$ the hyperplane parallel to $\pi_{k,i}$ that lies in the half space $\pi_{k,i}^+=\{x\in\mathbb{R}^n;\ V_k(x)>V_i(x)\}$ at a distance ρ from $\pi_{k,i}$, and call $\pi_{k,i}^{\rho,-}$ the half space determined by $\pi_{k,i}^\rho$ that contains $\pi_{k,i}$. Then, set

$$\widehat{\Gamma}_{\tau,k} \doteq \begin{cases} \Gamma_{\tau,k}\cap\displaystyle\bigcap_{i\in\mathcal{J}_{\tau,k}\setminus\widehat{\mathcal{J}}_{\tau,k}}\pi_{k,i}^-\cap\bigcap_{i\in\widehat{\mathcal{J}}_{\tau,k}}\big(\pi_{k,i}^{\rho,-}\cap\Gamma_{\tau,i}\big) & \text{if } \mathcal{J}_{\tau,k}\neq\widehat{\mathcal{J}}_{\tau,k},\ \widehat{\mathcal{J}}_{\tau,k}\neq\emptyset,\\ \Gamma_{\tau,k}\cap\displaystyle\bigcap_{i\in\widehat{\mathcal{J}}_{\tau,k}}\big(\pi_{k,i}^{\rho,-}\cap\Gamma_{\tau,i}\big) & \text{if } \mathcal{J}_{\tau,k}=\widehat{\mathcal{J}}_{\tau,k}\neq\emptyset,\\ \widetilde{\Gamma}_{\tau,k} & \text{if } \widehat{\mathcal{J}}_{\tau,k}=\emptyset, \end{cases} \tag{3.95}$$

$$\Omega_{\tau,k} \doteq \widehat{\Gamma}_{\tau,k}\Big\backslash\bigcup_{j\in\mathcal{I}_\tau}B_j, \tag{3.96}$$

and observe that, by definitions (3.59), (3.62), (3.94), (3.95), (3.96), one has

$$\partial\widehat{\Gamma}_{\tau,k}\Big\backslash\bigcup_{\substack{h\in\mathcal{I}_\tau\\ h>k}}\widehat{\Gamma}_{\tau,h}\subset\partial\widetilde{\Gamma}_{\tau,k},$$

$$\partial\Omega_{\tau,k}\Big\backslash\Big(\Sigma_{\tau,k}\cup\bigcup_{\substack{h\in\mathcal{I}_\tau\\ h>k}}\Omega_{\tau,h}\Big)\subset\partial^-\widetilde{\Gamma}_{\tau,k}.$$

Thus, by property (i) of Claim 2 it follows that the vector field $g_{\tau,k}(x)=f(x,v^k)$ satisfies the inward-pointing condition (1.5) at every point $x\in\partial\Omega_{\tau,k}\setminus(\Sigma_\tau\cup\bigcup_{\substack{h\in\mathcal{I}_\tau\\ h>k}}\Omega_{\tau,h})$. Then, letting g_τ denote the vector field on Ω_τ defined by

$$g_\tau(x)\doteq g_{\tau,k}(x)\quad\text{if } x\in\Delta_{\tau,k}\doteq\Omega_{\tau,k}\Big\backslash\bigcup_{\substack{h\in\mathcal{I}_\tau\\ h>k}}\Omega_{\tau,h}, \tag{3.97}$$

and considering the map $U_\tau:\Omega_\tau\to\mathbf{U}$ defined by

$$U_\tau(x)\doteq v^k\quad\text{if } x\in\Delta_{\tau,k}, \tag{3.98}$$

in view of Remark 1.3 we deduce that the triple $(\Omega_\tau, g_\tau, (\Omega_{\tau,k}, g_{\tau,k})_{k\in\mathcal{I}_\tau})$ is a patchy vector field on Ω_τ associated to the patchy feedback $(\Omega_\tau, U_\tau, (\Omega_{\tau,k}, v^k)_{k\in\mathcal{I}_\tau})$. Notice that, by definitions (3.59), (3.62), (3.94)–(3.97), one has

$$\Delta_{\tau,k} \subset \overline{\widetilde{\Gamma}_{\tau,k}} \Big\backslash \bigcup_{j\in\mathcal{I}_\tau} B_j \quad \forall k \in \mathcal{I}_\tau,$$

and hence we may apply the property (ii) of Claim 2 to a trajectory of g_τ passing through the domain $\Delta_{\tau,k}$.

Claim 3. *The patchy vector field g_τ on the domain Ω_τ, $\tau \in [\tau_0, T''[$, defined in* (3.97) *enjoys the following properties.*

(i) *For any $y \in \Omega_\tau$, and for every Carathéodory trajectory $\gamma_y(\cdot)$ of*

$$\dot{x} = g_\tau(x) \tag{3.99}$$

starting at y, there exists a time $\mathcal{T}_\tau(y, \gamma_y) > 0$ so that one has

$$\gamma_y\big(\mathcal{T}_\tau(y, \gamma_y)\big) \in \Sigma_\tau, \tag{3.100}$$

and there holds

$$t + \widetilde{V}\big(\gamma_y(t)\big) \leqslant \widetilde{V}(y) + 3\varepsilon_1 t \quad \forall 0 \leqslant t \leqslant \mathcal{T}_\tau(y, \gamma_y). \tag{3.101}$$

(ii) *For any $\tau \in [\tau_0, T''[$, one has*

$$\tau^\sharp \doteq \sup\big\{t \in \big[\tau, T''\big[;\ \Sigma_s \subset \Omega_\tau\ \forall s \in\,]\tau, t]\big\} > \tau. \tag{3.102}$$

Moreover, there exists a positive constant ε_4 so that there holds

$$\tau^\sharp > \tau + \varepsilon_4 \quad \forall \tau \in \mathcal{I}_1^* \cup \mathcal{I}_2^*. \tag{3.103}$$

10. Proof of Claim 3. Given $y \in \Omega_\tau$, let γ_y be a trajectory of (3.99) starting at y, and set

$$t_{\max}(\gamma_y) \doteq \sup\big\{t > 0;\ \gamma_y \text{ is defined on } [0, t]\big\}. \tag{3.104}$$

By the properties of the patchy vector fields recalled in Section 1 and relying on Claim 2 one can recursively construct two increasing sequences of times $0 = t_0 < t_1 < \cdots < t_{\bar\nu} \leqslant t_{\max}$, and of indices $i_1 < i_2 < \cdots < i_{\bar\nu} \in \mathcal{I}_\tau$ with the following properties:

(a) γ_y is a solution of $\dot{x} = g_{\tau,i_\nu}(x)$ taking values in Δ_{τ,i_ν} for all $t \in\,]t_{\nu-1}, t_\nu]$, $1 \leqslant \nu \leqslant \bar\nu$;
(b) $\gamma_y(t_\nu) \in \partial\Omega_{\tau,i_{\nu+1}}$ for all $1 \leqslant \nu < \bar\nu$, and $\gamma_y(t_{\bar\nu}) \in \Sigma_\tau \cup \bigcup_{\substack{i\in\mathcal{I}_\tau\\ i>i_{\bar\nu}}} \partial\Omega_{\tau,i}$;
(c) $t_\nu - t_{\nu-1} < \mathcal{T}_{\tau,i_\nu}(\gamma_y(t_{\nu-1}))$ for all $1 \leqslant \nu < \bar\nu$, and $t_{\bar\nu} - t_{\bar\nu-1} \leqslant \mathcal{T}_{\tau,i_{\bar\nu}}(\gamma_y(t_{\bar\nu-1}))$.

Notice that, since $\{i_\nu\}_\nu$ is strictly increasing, and because $\bar\nu \leqslant |\mathcal{I}_\tau| \leqslant N$ (N being the number of quadratic function V_i that appear in the definition (1.13) of the map $\widetilde{V}$), we can produce a sequence of times t_ν, and of indices $i_\nu \in I_\tau$, $1 \leqslant \nu \leqslant \hat\nu$, of such type so that $t_{\hat\nu} = t_{\max}$. Hence, since $\gamma_y(t_{\hat\nu}) \in \bigcup_{\substack{i\in\mathcal{I}_\tau\\ i>i_{\hat\nu}}} \partial\Omega_{\tau,i}$ would imply that the trajectory γ_y could be prolonged after time $t_{\hat\nu}$, which is in contrast with the maximality of $t_{\hat\nu}$, by property b) it follows that $\gamma_y(t_{\hat\nu}) \in \Sigma_\tau$, proving (3.100). Next, applying repeatedly the estimate (3.65) of Claim 2, and recalling that $\gamma_y(0) = y$, we derive

$$\begin{aligned}
\widetilde{V}\big(\gamma_y(t)\big) &\leqslant \widetilde{V}\big(\gamma_y(t_\nu)\big) + (-1 + 3\varepsilon_1)(t - t_\nu) \\
&\leqslant \widetilde{V}(y) + (-1 + 3\varepsilon_1)t \quad \forall t \in\,]t_{\nu-1}, t_\nu],\ 0 < \nu \leqslant \hat\nu,
\end{aligned}$$

which yields (3.101). To conclude the proof of the claim, we only need to observe that, by definition (3.93), the estimates (3.102), (3.103) are precisely the same as the estimates (3.66), (3.67), (3.92) established at point **8**.

11. Relying on Claim 3, we shall construct now a patchy feedback on the region $\mathcal{D}$ defined in (3.26). To this end, proceeding by induction on $m \geqslant 0$, we introduce a sequence of times τ_m defined as follows. Observe that, by definition (3.91), for every $\tau \in [\tau_0, T''[\backslash(\mathcal{I}_1^* \cup \mathcal{I}_2^*)$ one has

$$\Theta_\tau \doteq \big\{t \in\,]\tau, \tau^\sharp[;\ \text{either } \eta_t < \eta_\tau,\ \text{or } \big|\mathcal{I}_t^h\big| > \big|\mathcal{I}_\tau^h\big| \text{ for some } 1 \leqslant h \leqslant \eta_\tau\big\} \neq \emptyset.$$

Then, letting τ_0 be the constant defined in (3.5), for every $m > 0$, set

$$\tau_m \doteq \begin{cases} \tau_{m-1} + \varepsilon_4 & \text{if } \tau_{m-1} \in \mathcal{I}_1^* \cup \mathcal{I}_2^*, \\ \inf \Theta_{\tau_{m-1}} & \text{otherwise.} \end{cases} \tag{3.105}$$

By construction, and because of (3.102), (3.103), there holds

$$\Omega_{\tau_m} \supset \left\{x \in \mathbb{R}^n;\ \tau_m < \widetilde{V}(x) \leqslant \tau_{m+1}\right\} \quad \forall m \geqslant 0. \tag{3.106}$$

Moreover, observing that $t \mapsto \eta_t$ is a decreasing map and that $\eta_t \leqslant N$, $|\mathcal{I}_t^h| \leqslant N$, for all t and h, it follows that $\{\tau_m\}_{m \geqslant 0}$ is a strictly increasing sequence enjoying the property

$$\tau_m \notin \mathcal{I}_1^* \cup \mathcal{I}_2^* \quad \Longrightarrow \quad \exists p > m, \quad p < m + N^2 \quad \text{s.t.} \quad \tau_p \in \mathcal{I}_1^* \cup \mathcal{I}_2^*. \tag{3.107}$$

In turn, (3.105), (3.107) imply that for every m there exists some $p > m$, $p < m + N^2$, such that $\tau_p > \tau_m + \varepsilon_4$. Thus, we deduce that there will be some integer μ such that $\tau_\mu \leqslant T'' < \tau_{\mu+1}$, and hence, by (3.26), (3.106) one has

$$\mathcal{D} \subset \Omega \doteq \bigcup_{m=0}^{\mu} \Omega_{\tau_m}. \tag{3.108}$$

Let's introduce the total ordering

$$(m,k) \prec (p,h) \quad \text{if either} \quad m > p \quad \text{or else} \quad m = p,\ k < h, \tag{3.109}$$

on the index set

$$\mathcal{A} = \left\{(m,k):\ m = 0, \ldots, \overline{m},\ k \in \mathcal{I}_{\tau_m}\right\}.$$

Then, if we define the vector field g on Ω by setting

$$g(x) \doteq g_{\tau_m,k}(x) \quad \text{if} \quad x \in D_{m,k} \doteq \Omega_{\tau_m,k} \Big\backslash \bigcup_{(m,k) \prec (p,h)} \Omega_{\tau_p,h}, \tag{3.110}$$

and consider the map $U : \Omega \to \mathbf{U}$ defined by

$$U(x) \doteq v^k \quad \text{if} \quad x \in D_{m,k}, \tag{3.111}$$

in view of the observations at point **9** we deduce that the triple $(\Omega, g, (\Omega_{\tau_m,k}, g_{\tau_m,k})_{(m.k) \in \mathcal{A}})$ is a patchy vector field on Ω associated to the patchy feedback $(\Omega, U, (\Omega_{\tau_m,k}, v^k)_{(m.k) \in \mathcal{A}})$, so that one has

$$g(x) = f\big(x, U(x)\big) \quad \forall x \in \Omega. \tag{3.112}$$

Given $y \in \Omega$, let γ_y be a Carathéodory trajectory of (1.9) starting at y, and define $t_{\max}(\gamma_y)$ as in (3.104). By the properties of the patchy vector fields and relying on Claim 3 one can recursively construct an increasing sequences of times $0 = t_0 < t_1 < \cdots < t_{\bar\nu} \leqslant t_{\max}$, and a decreasing sequence of indices $m_1 > m_2 > \cdots > m_{\bar\nu}$, so that, setting $\gamma_\nu \doteq \gamma \restriction_{]t_{\nu-1}, t_\nu]}$, $1 \leqslant \nu \leqslant \bar\nu$, there holds:

(a) γ_y is a solution of $\dot{x} = g_{\tau_{m_\nu}}(x)$ taking values in Ω_{m_ν} for all $t \in\,]t_{\nu-1}, t_\nu]$, $1 \leqslant \nu \leqslant \bar\nu$;
(b) $\gamma_y(t_{\nu-1}) \in \partial\Omega_{\tau_{m_\nu}}$ for all $1 < \nu \leqslant \bar\nu$, and $\gamma_y(t_{\bar\nu}) \in \Sigma_{\tau_0} \cup \bigcup_{m_{\bar\nu} < p} \partial\Omega_{\tau_p}$;
(c) $t_\nu - t_{\nu-1} < \mathcal{T}_{\tau_{m_\nu}}(\gamma_y(t_{\nu-1}), \gamma_\nu)$ for all $1 \leqslant \nu < \bar\nu$, and $t_{\bar\nu} - t_{\bar\nu-1} \leqslant \mathcal{T}_{\tau_{m_{\bar\nu}}}(\gamma_y(t_{\bar\nu-1}), \gamma_{\bar\nu})$.

Notice that, since $\{m_\nu\}_\nu$ is strictly decreasing, and because $\bar\nu \leqslant \mu$, we can produce a sequence of times t_ν, and of indices m_ν, $1 \leqslant \nu \leqslant \hat\nu$, of such type so that $t_{\hat\nu} = t_{\max}$. Thus, since $\gamma_y(t_{\hat\nu}) \in \bigcup_{m_{\bar\nu} < p} \partial\Omega_{\tau_p}$ would imply that the trajectory γ_y could be prolonged after time $t_{\hat\nu}$, which is in contrast with the maximality of $t_{\hat\nu}$, by property (b) it follows that $\gamma_y(t_{\hat\nu}) \in \Sigma_{\tau_0}$, and hence, by (3.29), one has $\gamma_y(t_{\hat\nu}) \in B_\varepsilon$. Next, given $y \in \mathcal{D}$, applying repeatedly the estimate (3.101) of Claim 3, we derive

$$\begin{aligned} \widetilde{V}\big(\gamma_y(t_\nu)\big) &\leqslant \widetilde{V}\big(\gamma_y(t_{\nu-1})\big) + (-1 + 3\varepsilon_1)\big(t_\nu - t_{\nu-1}\big) \\ &\leqslant \widetilde{V}(y) + (-1 + 3\varepsilon_1) \cdot t_\nu \quad \forall 0 < \nu \leqslant \hat\nu. \end{aligned} \tag{3.113}$$

Relying on the estimate (3.113) in the case $\nu = \hat\nu$, and thanks to (3.3), (3.11), (3.14), (3.26), (3.27), we find

$$\begin{aligned} t_{\hat{\mathfrak{v}}} &\leqslant \frac{\widetilde{V}(y)}{1-3\varepsilon_1} \\ &\leqslant \big(1+2\varepsilon_1\big)\big(V(y)+\varepsilon_2^2\big) \\ &\leqslant V(y)+2\varepsilon_1 T' + (1+2\varepsilon_1)\varepsilon_2^2 \\ &< V(y)+\varepsilon, \end{aligned} \tag{3.114}$$

which establish the conclusion of the theorem observing that γ_y reaches the ball B_ε within a time $\leqslant t_{\hat{\mathfrak{v}}}$ since $\gamma_y(t_{\hat{\mathfrak{v}}}) \in B_\varepsilon$. □

Appendix A

We provide here a proof of the two technical lemmas stated in Section 2, concerning the properties of lens-shaped domains of the form (2.32) constructed around a collection of spheres with uniformly bounded (from above and from below) radii.

Proof of Lemma 3. Fix $r_0 > 0$, and observe that the unit normal to a sphere S with radius $r \geqslant r_0$ is Lipschitz continuous with Lipschitz constant $1/r_0$:

$$\big|\mathbf{n}_S(y_1) - \mathbf{n}_S(y_2)\big| = \frac{|y_1 - y_2|}{r} \leqslant \frac{|y_1 - y_2|}{r_0} \quad \forall y_1, y_2 \in S.$$

Hence, by (2.31), and thanks to the Lipschitz continuity of the field g and of the unit normal $\mathbf{n}_S$, we deduce that there exist $\bar{\varepsilon} > 0$ sufficiently small, and $\bar{c}' > 0$, depending only on r_0, c_0, and on $\mathrm{Lip}(g)$, so that

$$\big\langle \mathbf{n}_S(x), g(x)\big\rangle \leqslant -\bar{c}', \quad \forall x \in \partial^+ \Gamma_{\bar{\varepsilon}}, \tag{A.1}$$

proving the transversality property of the vector field g to the boundary $\partial^+\Gamma_{\bar{\varepsilon}}$. Next, observe that the set $\partial^-\Gamma_{\bar{\varepsilon}}$ in (2.33) is a piecewise smooth hypersurface parametrized by

$$y \mapsto \Phi(y) \doteq x\big({d_C}^2(y) - \bar{\varepsilon}^2, y\big), \quad y \in B(C, \bar{\varepsilon}) \cap S.$$

Hence, the tangent space to $\partial^-\Gamma_{\bar{\varepsilon}}$ at every regular point $x = \Phi(y)$ of $\partial^-\Gamma_{\bar{\varepsilon}}$ is the image of the tangent space to S at y under the differential of Φ, i.e. there holds

$$T_{\partial^-\Gamma_{\bar{\varepsilon}}}\big(\Phi(y)\big) = d\Phi(y) \cdot T_S(y). \tag{A.2}$$

By standard differentiability properties of the trajectories of $\dot{x} = g(x)$, one finds that at the points in which ${d_C}^2(y)$ is differentiable there holds

$$d\Phi(y) = \big\langle \nabla {d_C}^2(y), \cdot\big\rangle g\big(\Phi(y)\big) + X\big(\big(d_C(y)\big)^2 - \bar{\varepsilon}^2\big)$$

where $X(t)$ denotes the fundamental matrix solution of the linear problem $\dot{v} = Dg(x(t, y)) \cdot v$, that coincides with the identity matrix Id at time $t = 0$. Thus, observing that at the points where ${d_C}^2(y)$ is differentiable one has $|\nabla {d_C}^2(y)| \leqslant 2d_C(y)$, we obtain

$$\begin{aligned} \big|d\Phi(y) - \mathrm{Id}\big| &\leqslant 2d_C(y)\|g\|_{\mathbf{L}^\infty} + \big(\big(\bar{\varepsilon}^2 - {d_C}^2(y)\big)\,\mathrm{Lip}(g)\big)\,\mathrm{e}^{((\bar{\varepsilon}^2 - {d_C}^2(y))\,\mathrm{Lip}(g))} \\ &\leqslant 2\bar{\varepsilon}\|g\|_{\mathbf{L}^\infty} + \big(\bar{\varepsilon}^2 \cdot \mathrm{Lip}(g)\big)\,\mathrm{e}^{(\bar{\varepsilon}^2\,\mathrm{Lip}(g))}. \end{aligned} \tag{A.3}$$

In turn, (A.3) together with (A.2) implies

$$\big|\mathbf{n}_{\partial^-\Gamma_{\bar{\varepsilon}}}(x) - \mathbf{n}_S\big(\Phi^{-1}(x)\big)\big| \leqslant c_{11}\bar{\varepsilon} \tag{A.4}$$

($\mathbf{n}_{\partial^-\Gamma_{\bar{\varepsilon}}}(x)$ denoting the unit normal to $\partial^-\Gamma_{\bar{\varepsilon}}$), for some constant $c_{11} > 0$ depending only on $\|g\|_{\mathbf{L}^\infty}, \mathrm{Lip}(g)$. Then, by the Lipschitz continuity of g and of the unit normal $\mathbf{n}_S$, we deduce from (2.31), (A.4) that, choosing $\bar{\varepsilon} > 0$ sufficiently small, there exist some constant $\bar{c}'' > 0$, depending only on r_0, c_0, and on $\|g\|_{\mathbf{L}^\infty}, \mathrm{Lip}(g)$, so that at every regular point $x = \Phi(y)$ of $\partial^-\Gamma_{\bar{\varepsilon}}$ there holds

$$\big\langle \mathbf{n}_{\partial^-\Gamma_{\bar{\varepsilon}}}(x), g(x)\big\rangle \leqslant -\bar{c}''.$$

Clearly, by continuity this implies that $g(x) \in \mathring{T}_{\Gamma_{\bar\varepsilon}}(x)$ at every irregular point of $\partial^- \Gamma_{\bar\varepsilon}$ ($\mathring{T}_{\Gamma_{\bar\varepsilon}}$ denoting the interior of the tangent cone to $\Gamma_{\bar\varepsilon}$ defined as in (1.12)), thus showing that the vector field g is inward-pointing on the boundary $\partial^- \Gamma_{\bar\varepsilon}$, which completes the proof of the lemma. □

Remark 4.1. Relying on the proof of Lemma 3 one can show that there exists some constant $c_{12} > 0$ (depending only on $r_0, \bar c$, $\|g\|_{\mathbf{L}^\infty}$, and on $\mathrm{Lip}(g)$), so that there holds

$$d_S\big(x\big(d_C{}^2(y) - \bar\varepsilon^2, y\big)\big) > c_{12}\bar\varepsilon^2 \quad \forall y \in B(C, \bar\varepsilon/2) \cap S. \tag{A.5}$$

Indeed, notice that thanks to the Lipschitz continuity of the field g we may choose the constants $\bar c', \bar\varepsilon$ so that the estimate in (A.1) holds for all points $x \in \Gamma_{\bar\varepsilon}$, i.e. such that

$$\big|\big\langle \mathbf{n}_S(y), g(x)\big\rangle\big| \geqslant \bar c' \quad \forall x \in \Gamma_{\bar\varepsilon},\ \forall y \in \partial^+ \Gamma_{\bar\varepsilon}. \tag{A.6}$$

Relying on (A.6) we then deduce that

$$\begin{aligned} d_S\big(\Phi(y)\big) &\geqslant \big|\big\langle \Phi(y) - y, \mathbf{n}_S(y)\big\rangle\big| \\ &\geqslant \bar c'\big(\bar\varepsilon^2 - d_C{}^2(y)\big) \\ &> \frac{\bar c' \bar\varepsilon^2}{2} \quad \forall y \in B(C, \bar\varepsilon/2) \cap S, \end{aligned} \tag{A.7}$$

which proves (A.5), with $c_{12} \doteq \bar c'/2$,

Proof of Lemma 4.

1. We will provide a proof of a more general result than the one stated in the lemma. Namely, we will show that there exist constants $\bar\varepsilon', c_4 > 0$, so that, for every given set of indices $\mathcal{I} \subset \{1, \dots, \nu\}$, if we consider the sets

$$C^{\mathcal{I}} \doteq \bigcup_{k \in \mathcal{I}} C_k, \qquad \mathcal{G}^{\mathcal{I}} \doteq \bigcup_{k \in \mathcal{I}} \overline{\Gamma_{\bar\varepsilon'}^{\mathcal{J}_k}}, \tag{A.8}$$

$$\partial^- \mathcal{G}^{\mathcal{I}} \doteq \partial \mathcal{G}^{\mathcal{I}} \Big\backslash \bigcup_{k=1}^{\nu} B_k, \tag{A.9}$$

one has

$$d_{C^{\mathcal{I}}}\big(\partial^- \mathcal{G}^{\mathcal{I}}\big) \geqslant c_4. \tag{A.10}$$

Clearly, in the particular case where $\mathcal{I} = \{1, \dots, \nu\}$, we have

$$C^{\mathcal{I}} = C, \quad \mathcal{G}^{\mathcal{I}} = \mathcal{G}, \quad \partial^- \mathcal{G}^{\mathcal{I}} = \partial^- \mathcal{G},$$

and hence we recover the estimate (2.41) from (A.10). The proof of (A.10), for an arbitrary set $\mathcal{I} \subset \{1, \dots, \nu\}$, will be obtained proceeding by induction on the number $|\Pi|$ of hyperplanes contained in the set Π considered in (2.37). Notice that, setting

$$\partial^{-\mathcal{I}} \Gamma_{\bar\varepsilon'}^{\mathcal{J}_k} \doteq \partial \Gamma_{\bar\varepsilon'}^{\mathcal{J}_k} \Big\backslash \Bigg(\bigcup_{j=1}^{\nu} B_j \cup \bigcup_{\substack{j \in \mathcal{I} \\ j \neq k}} \overline{\Gamma_{\bar\varepsilon'}^{\mathcal{J}_j}} \Bigg) \quad \forall k = 1, \dots, \nu, \tag{A.11}$$

by definitions (A.8), (A.9), one has

$$\partial^- \mathcal{G}^{\mathcal{I}} = \bigcup_{k \in \mathcal{I}} \overline{\partial^{-\mathcal{I}} \Gamma_{\bar\varepsilon'}^{\mathcal{J}_k}},$$

and hence there holds

$$d_{C^{\mathcal{I}}}\big(\partial^- \mathcal{G}^{\mathcal{I}}\big) \geqslant \min_{k \in \mathcal{I}} d_{C^{\mathcal{I}}}\big(\overline{\partial^{-\mathcal{I}} \Gamma_{\bar\varepsilon'}^{\mathcal{J}_k}}\big). \tag{A.12}$$

Thus, in order to establish (A.10), it will be sufficient to prove by induction on $|\Pi|$ that there exist some constants $\bar\varepsilon', c_4 > 0$, so that there holds

$$d_{C^{\mathcal{I}}}\big(\partial^{-\mathcal{I}}\Gamma_{\bar\varepsilon'}^{\mathcal{J}_k}\big) > c_4 \quad \forall k \in \mathcal{I}. \tag{A.13}$$

2. Consider first the case where $\Pi = \emptyset$, i.e. assume that $\Pi_k = \emptyset$ for all k, and fix some set of indices $\mathcal{I} \subset \{1, \dots, \nu\}$. Then, recalling the definitions (2.32), (2.33), and observing that

$$\partial^- \Gamma_{\bar\varepsilon}(C_k) = \partial \Gamma_{\bar\varepsilon}(C_k) \setminus B_k \quad \forall k = 1, \dots, \nu, \tag{A.14}$$

by (2.38), (A.11), we have

$$\begin{aligned} &\Gamma_{\bar\varepsilon}^{\mathcal{J}_k} = \Gamma_{\bar\varepsilon}(C_k),\\ &\partial^{-\mathcal{I}}\Gamma_{\bar\varepsilon'}^{\mathcal{J}_k} = \partial^- \Gamma_{\bar\varepsilon}(C_k) \Big\backslash \Bigg(\bigcup_{j\neq k} B_j \cup \bigcup_{\substack{j\in\mathcal{I}\\ j\neq k}} \overline{\Gamma_{\bar\varepsilon}(C_j)}\Bigg), \quad \forall k = 1, \dots, \nu. \end{aligned} \tag{A.15}$$

Let $\bar\varepsilon, c_{12}$ be the constants (depending only on r_0 and $g_1, \dots, g_\nu$) provided by Lemma 3 and Remark 4.1 for all sets $C_1, \dots, C_\nu$, and observe that, choosing $\bar\varepsilon$ sufficiently small so that

$$\|g_k\|_{\mathbf{L}^\infty} \bar\varepsilon^2 < \bar\varepsilon/4 \quad \forall k = 1, \dots, \nu, \tag{A.16}$$

and setting

$$R_{\bar\varepsilon} \doteq \big\{ y \in S_k;\ \bar\varepsilon/2 \leqslant d_{C_k}(y) \leqslant \bar\varepsilon \big\},$$

by the definition (2.36) of C_k there holds

$$B\big(R_{\bar\varepsilon}, \|g_k\|_{\mathbf{L}^\infty}\bar\varepsilon^2\big) \subset \bigcup_{j=1}^{\nu} \mathring{B}_j \quad \forall k = 1, \dots, \nu. \tag{A.17}$$

Moreover, since the solution $\tau \mapsto x(\tau, y)$ of the Cauchy problem $\dot x = g_k(x)$, $x(0) = y$, satisfies

$$\big| x(\tau, y) - y \big| \leqslant \|g_k\|_{\mathbf{L}^\infty} \tau \quad \forall \tau,$$

we deduce from (A.17) that

$$\begin{aligned} &\overline{\Gamma_{\bar\varepsilon}(C_k)} \Big\backslash \bigcup_{j=1}^{\nu} \mathring{B}_j \subset \big\{ x(\tau, y);\ y \in B(C_k, \bar\varepsilon/2) \cap S_k, d_{C_k}{}^2(y) - \bar\varepsilon^2 \leqslant \tau \leqslant 0 \big\} \cap B\big(C_k, 2\bar\varepsilon\big)\\ &\quad \forall k = 1, \dots, \nu. \end{aligned} \tag{A.18}$$

On the other hand, by (A.14), (A.15) one has

$$\partial^{-\mathcal{I}}\Gamma_{\bar\varepsilon}^{\mathcal{J}_k} \subset \overline{\Gamma_{\bar\varepsilon}(C_k)} \Big\backslash \bigcup_{j=1}^{\nu} \mathring{B}_j \quad \forall k = 1, \dots, \nu. \tag{A.19}$$

Thus, relying on (A.18), (A.19) we find

$$\partial^{-\mathcal{I}}\Gamma_{\bar\varepsilon}^{\mathcal{J}_k} \subset \big\{ x\big(d_{C_k}{}^2(y) - \bar\varepsilon^2, y\big), y \in B(C_k, \bar\varepsilon/2) \cap S_k \big\} \cap \Bigg(B(C_k, 2\bar\varepsilon) \Big\backslash \bigcup_{j=1}^{\nu} \mathring{B}_j \Bigg), \tag{A.20}$$

which, in turn, applying (A.5), yields

$$d_{S_k}\big(\partial^{-\mathcal{I}}\Gamma_{\bar\varepsilon}^{\mathcal{J}_k}\big) > c_{12}\bar\varepsilon^2 \quad \forall k = 1, \dots, \nu. \tag{A.21}$$

Observe now that, since the radii of S_i, $i = 1, \dots, \nu$, are uniformly bounded by r_0', and because the definitions (2.36), (2.39) imply

$$C = \partial \Bigg(\bigcup_{j=1}^{\nu} B_j \Bigg),$$

it follows that there will be some constants $c_{13}, c_{14} > 0$, depending only on r_0', such that there holds

$$d_C(y) \geqslant c_{13} {d_{S_k}}^2(y) \quad \forall y \in B(C_k, c_{14}) \Big\backslash \left(\bigcup_{j=1}^{\nu} \mathring{B}_j \right), \quad \forall k = 1, \ldots, \nu. \tag{A.22}$$

Hence, thanks to (A.20), (A.22), choosing $\bar{\varepsilon}$ sufficiently small so that

$$\bar{\varepsilon} < \frac{c_{14}}{2}, \tag{A.23}$$

and observing that

$$d_{C^{\mathcal{I}}}(y) \geqslant d_C(y) \quad \forall y \in \mathbb{R}^n, \tag{A.24}$$

we recover from (A.21) the estimates (A.13), with $c_4 = c_{12}^2 \cdot c_{13}\bar{\varepsilon}^2$, and $\bar{\varepsilon}' = \bar{\varepsilon}$ satisfying (A.16), (A.23).
3. Given $p \geqslant 1$, suppose now that there exists some constants $\bar{c}_p > 0$ so that, letting $\bar{\varepsilon}$ be the constant provided by Lemma 3 and satisfying (A.16), (A.23), when $|\Pi| < p$ for every set of indices $\mathcal{I} \subset \{1, \ldots, \nu\}$ there holds

$$d_{C^{\mathcal{I}}}\big(\partial^- \mathcal{G}^{\mathcal{I}}\big) \geqslant \bar{c}_p, \tag{A.25}$$

$$d_{C^{\mathcal{I}}}\big(\partial^{-\mathcal{I}} \Gamma_{\bar{\varepsilon}}^{\mathcal{J}_k}\big) > \bar{c}_p \quad \forall k \in \mathcal{I}. \tag{A.26}$$

Then, consider the case where $|\Pi| = p$. Fix $\mathcal{I} \subset \{1, \ldots, \nu\}$, $k \in \mathcal{I}$. Our goal is to show that there exists some constant $\bar{c}_{p+1} > 0$ so that the estimate in (A.26) is verified with $\bar{c}_{p+1}$ in place of $\bar{c}_p$. Clearly, if $\Pi_k = \emptyset$ we recover the estimates in (A.26) from the proof derived at point **2**. Hence, we need to consider only the case where $|\Pi_k| = |\mathcal{J}_k| > 0$. Then, recalling the definitions (2.32), (2.33), by (2.38), (A.11), (A.14), one has

$$\Gamma_{\bar{\varepsilon}}^{\mathcal{J}_k} = \Gamma_{\bar{\varepsilon}}(C_k) \cap \bigcap_{i \in \mathcal{J}_k} \pi_{k,i}^-, \tag{A.27}$$

$$\partial^{-\mathcal{I}} \Gamma_{\bar{\varepsilon}}^{\mathcal{J}_k} = E_1^{\mathcal{I}} \cup E_2^{\mathcal{I}},$$

$$E_1^{\mathcal{I}} \doteq \left(\partial^- \Gamma_{\bar{\varepsilon}}(C_k) \Big\backslash \left(\bigcup_{j \neq k} B_j \cup \bigcup_{\substack{j \in \mathcal{I} \\ j \neq k}} \overline{\Gamma_{\bar{\varepsilon}}^{\mathcal{J}_j}} \right) \right) \cap \bigcap_{i \in \mathcal{J}_k} \pi_{k,i}^-,$$

$$E_2^{\mathcal{I}} \doteq \bigcup_{i \in \mathcal{J}_k} E_{2,i}^{\mathcal{I}}, \quad E_{2,i}^{\mathcal{I}} \doteq \left(\left(\Gamma_{\bar{\varepsilon}}(C_k) \cap \bigcap_{j \in \mathcal{J}_k} (\pi_{k,j}^- \cup \pi_{k,j}) \right) \Big\backslash \left(\bigcup_{j=1}^{\nu} B_j \cup \bigcup_{\substack{j \in \mathcal{I} \\ j \neq k}} \overline{\Gamma_{\bar{\varepsilon}}^{\mathcal{J}_j}} \right) \right) \cap \pi_{k,i}. \tag{A.28}$$

Observe first that, letting $\bar{\varepsilon}$ be the constant provided by Lemma 3 and satisfying (A.16), (A.23), by the proof established at point **2** one immediately deduces the inequality

$$d_{S_k}\big(E_1^{\mathcal{I}}\big) \geqslant d_{S_k}\left(\partial^- \Gamma_{\bar{\varepsilon}}(C_k) \Big\backslash \bigcup_{j \neq k} B_j \right) > c_{12}\bar{\varepsilon}^2, \tag{A.29}$$

which, together with (A.22), (A.24), yields

$$d_{C^{\mathcal{I}}}\big(E_1^{\mathcal{I}}\big) > c_{12}^2 c_{13} \bar{\varepsilon}^4. \tag{A.30}$$

Hence, if $E_2^{\mathcal{I}} = \emptyset$ we recover from (A.30) the estimates in (A.26) with $\bar{c}_{p+1} \doteq c_{12}^2 c_{13}\bar{\varepsilon}^4$ in place of $\bar{c}_p$. On the other hand, observe that if we let S_j^p, $j \in \mathcal{I}$, denote the surfaces of the balls

$$B_j^p \doteq B(B_j, \bar{c}_p/2), \quad j \in \mathcal{I}, \tag{A.31}$$

and we consider the set

$$C^{p,\mathcal{I}} \doteq \bigcup_{k \in \mathcal{I}} C_k^p, \quad C_k^p \doteq S_k^p \Big\backslash \left(\bigcup_{j \notin \mathcal{I}} \mathring{B}_j \cup \bigcup_{j \in \mathcal{I}} \mathring{B}_j^p \right),$$

by construction one has

$$\partial\Bigl(\bigcup_{j\notin\mathcal{I}} B_j \cup \bigcup_{j\in\mathcal{I}} B_j^p\Bigr) = \Bigl(C \Bigm\backslash \bigcup_{j\in\mathcal{I}} B_j^p\Bigr) \cup C^{p,\mathcal{I}}, \tag{A.32}$$

$$\begin{aligned} d_{C^{\mathcal{I}}}\Bigl(C \Bigm\backslash \bigcup_{j\in\mathcal{I}} B_j^p\Bigr) &\geqslant d_{C^{\mathcal{I}}}\bigl(C \cap C^{p,\mathcal{I}}\bigr) \\ &\geqslant d_{C^{\mathcal{I}}}\bigl(C^{p,\mathcal{I}}\bigr) \\ &\geqslant \frac{\bar{c}_p}{2}. \end{aligned} \tag{A.33}$$

Thus, for every $i \in \mathcal{J}_k$ for which there holds

$$\pi_{k,i} \cap \Bigl(\bigcup_{j\in\mathcal{I}} B_j^p \Bigm\backslash \bigcup_{j=1}^{\nu} \mathring{B}_j\Bigr) = \emptyset, \tag{A.34}$$

since the definition (A.28) implies

$$E_{2,i}^{\mathcal{I}} \subset \Bigl(\mathbb{R}^n \Bigm\backslash \bigcup_{j=1}^{\nu} B_j\Bigr) \cap \pi_{k,i},$$

it follows that

$$E_{2,i}^{\mathcal{I}} \subset \mathbb{R}^n \Bigm\backslash \Bigl(\bigcup_{j\notin\mathcal{I}} B_j \cup \bigcup_{j\in\mathcal{I}} B_j^p\Bigr). \tag{A.35}$$

Then, relying on (A.32), (A.33), (A.35), we deduce that

$$\begin{aligned} d_{C^{\mathcal{I}}}\bigl(E_{2,i}^{\mathcal{I}}\bigr) &\geqslant \min\Bigl\{d_{C^{\mathcal{I}}}\Bigl(C \Bigm\backslash \bigcup_{j\in\mathcal{I}} B_j^p\Bigr),\ d_{C^{\mathcal{I}}}\bigl(C^{p,\mathcal{I}}\bigr)\Bigr\} \\ &\geqslant \frac{\bar{c}_p}{2}, \end{aligned} \tag{A.36}$$

for all $i \in \mathcal{J}_k$ that satisfy (A.34). Hence, in the case where (A.34) holds for all $i \in \mathcal{J}_k$, we obtain from (A.30), (A.36) the estimate in (A.26) with $\bar{c}_{p+1} \doteq \min\{c_{12}^2 c_{13} \bar{\varepsilon}^4, c_p/2\}$ in place of $\bar{c}_p$. Therefore, to complete the proof of the lemma it remains to derive an estimate of $d_{C^{\mathcal{I}}}(E_2^{\mathcal{I}})$ when $E_2^{\mathcal{I}} \neq \emptyset$ and (A.34) does not hold for some $i \in \mathcal{J}_k$.

4. With the same definitions and notations introduced at point **3**, consider a set of indices $\mathcal{I} \subset \{1, \dots, \nu\}$ for which $E_2^{\mathcal{I}} \neq \emptyset$, and such that (A.34) is not satisfied for some $i \in \mathcal{J}_k$. Set

$$\widetilde{\mathcal{J}}_k \doteq \Bigl\{i \in \mathcal{J}_k;\ \pi_{k,i} \cap \Bigl(\bigcup_{j\in\mathcal{I}} B_j^p \Bigm\backslash \bigcup_{j=1}^{\nu} \mathring{B}_j\Bigr) \neq \emptyset\Bigr\}, \tag{A.37}$$

and observe that, by the proof derived at point **3**, there holds

$$d_{C^{\mathcal{I}}}\bigl(E_{2,i}^{\mathcal{I}}\bigr) \geqslant \frac{\bar{c}_p}{2} \quad \forall i \in \mathcal{J}_k \setminus \widetilde{\mathcal{J}}_k. \tag{A.38}$$

On the other hand, for every fixed $i \in \widetilde{\mathcal{J}}_k$, by definition (A.31) there will be some constant $\rho \in [0, (c_p/2)]$ such that, letting $\widetilde{S}_j$, $j \in \mathcal{I}$, denote the surfaces of the balls

$$\widetilde{B}_j \doteq B\bigl(B_j, \rho\bigr), \quad j \in \mathcal{I}, \tag{A.39}$$

and considering the set

$$\widetilde{C}^{\mathcal{I}} \doteq \bigcup_{h\in\mathcal{I}} \widetilde{C}_h, \quad \widetilde{C}_h \doteq \widetilde{S}_h \Bigm\backslash \Bigl(\bigcup_{j\notin\mathcal{I}} \mathring{B}_j \cup \bigcup_{j\in\mathcal{I}} \mathring{\widetilde{B}}_j\Bigr), \tag{A.40}$$

there holds

$$\pi_{k,i} \cap \widetilde{C}^{\mathcal{I}} \neq \emptyset.$$

Then, as a first step towards an estimate of $d_{C^{\mathcal{I}}}(E^{\mathcal{I}}_{2,i})$ we will show that, setting

$$\Upsilon_i \doteq \widetilde{C}^{\mathcal{I}} \cap \pi_{k,i}, \tag{A.41}$$

there holds

$$d_{\Upsilon_i}\big(E^{\mathcal{I}}_{2,i}\big) \geqslant \frac{\bar{c}_p}{2}, \quad \forall i \in \widetilde{\mathcal{J}}_k. \tag{A.42}$$

Recalling that $\pi_{k,i} = \pi_{i,k}$, set

$$\Pi^* \doteq \Pi \setminus \{\pi_{k,i}\}, \quad \Pi^*_k \doteq \Pi_k \setminus \{\pi_{k,i}\}, \quad \Pi^*_i \doteq \Pi_i \setminus \{\pi_{k,i}\}, \tag{A.43}$$

and observe that, by the properties of $\pi_{k,i}$, one has $\pi_{k,i} \notin \Pi_j$ for all $j \neq k, i$, and hence there holds

$$\Pi^* = \bigcup_{j \neq k,i} \Pi_j \cup \Pi^*_k \cup \Pi^*_i.$$

Moreover, because of (A.43), one has $|\Pi^*| < |\Pi| = p$. Therefore, setting

$$\mathcal{J}^*_i \doteq \mathcal{J}_i \setminus \{k\}, \quad \Gamma^{\mathcal{J}_i,*}_{\bar{\varepsilon}} \doteq \Gamma_{\bar{\varepsilon}}(C_i) \cap \bigcap_{j \in \mathcal{J}^*_i} \pi^-_{i,j}, \quad \mathcal{I}^* \doteq \mathcal{I} \setminus \{k\}, \tag{A.44}$$

and defining

$$\begin{gathered}\mathcal{G}^{*,\mathcal{I}^*} \doteq \begin{cases} \overline{\Gamma^{\mathcal{J}_i,*}_{\bar{\varepsilon}}} \cup \bigcup_{j \in \mathcal{I}^* \setminus \{i\}} \overline{\Gamma^{\mathcal{J}_j}_{\bar{\varepsilon}}} & \text{if } i \in \mathcal{I}, \\ \mathcal{G}^{\mathcal{I}^*} & \text{if } i \notin \mathcal{I}, \end{cases} \\ \partial^- \mathcal{G}^{*,\mathcal{I}^*} \doteq \partial \mathcal{G}^{*,\mathcal{I}^*} \Big\backslash \bigcup_{j=1}^{\nu} B_j,\end{gathered} \tag{A.45}$$

by the inductive hypothesis we can apply the inequality (A.25) in connection with the set of hyperplanes Π^* and hence, in particular, for the set of indices $\mathcal{I}^*$ there holds

$$d_{C^{\mathcal{I}^*}}\big(\partial^- \mathcal{G}^{*,\mathcal{I}^*}\big) \geqslant \bar{c}_p. \tag{A.46}$$

Relying on (A.46), and observing that by (A.44), one has

$$\overline{\Gamma^{\mathcal{J}_i}_{\bar{\varepsilon}'}} \cap \pi_{i,k} = \overline{\big(\Gamma^{\mathcal{J}_i,*}_{\bar{\varepsilon}'} \cap \pi^-_{i,k}\big)} \cap \pi_{i,k} = \overline{\Gamma^{\mathcal{J}_i,*}_{\bar{\varepsilon}'}} \cap \pi_{i,k},$$

since $\pi_{i,k} = \pi_{k,i}$ we find that

$$\begin{aligned} B\big(C^{\mathcal{I}^*}, \bar{c}_p\big) \cap \pi_{k,i} &\subset \left(\bigcup_{j=1}^{\nu} B_j \cup \overline{\Gamma^{\mathcal{J}_i,*}_{\bar{\varepsilon}}} \cup \bigcup_{j \in \mathcal{I}^*} \overline{\Gamma^{\mathcal{J}_j}_{\bar{\varepsilon}}} \right) \cap \pi_{k,i} \\ &\subset \left(\bigcup_{j=1}^{\nu} B_j \cup \bigcup_{j \in \mathcal{I}^*} \overline{\Gamma^{\mathcal{J}_j}_{\bar{\varepsilon}}} \right) \cap \pi_{k,i}. \end{aligned} \tag{A.47}$$

Moreover observe that, letting

$$\widetilde{C}^{\mathcal{I}^*} \doteq \bigcup_{h \in \mathcal{I}^*} \widetilde{C}_h,$$

by definitions (A.39), (A.40) one has

$$\widetilde{C}^{\mathcal{I}^*} \subset B\big(C^{\mathcal{I}^*}, \rho\big), \quad \rho < c_p/2,$$

and hence there holds

$$B\big(\widetilde{C}^{\mathcal{I}^*}, c_p/2\big) \subset B\big(C^{\mathcal{I}^*}, c_p\big). \tag{A.48}$$

Thus, (A.47), (A.48) together, yield

$$B\big(\widetilde{C}^{\mathcal{I}^*}, c_p/2\big) \cap \pi_{k,i} \subset \left(\bigcup_{j=1}^{\nu} B_j \cup \bigcup_{j \in \mathcal{I}^*} \overline{\Gamma_{\bar{\varepsilon}}^{\mathcal{J}_j}} \right) \cap \pi_{k,i}. \tag{A.49}$$

On the other hand, observe that, by the properties of $\pi_{k,i}$, we have

$$\widetilde{S}_k \cap \widetilde{S}_i = \widetilde{S}_k \cap \pi_{k,i} = \widetilde{S}_i \cap \pi_{k,i},$$

which, in turn, by definition (A.40) implies

$$\widetilde{C}_k \cap \pi_{k,i} = \widetilde{C}_i \cap \pi_{k,i}. \tag{A.50}$$

Thanks to (A.50), it follows from (A.49) that

$$B(\Upsilon_i, c_p/2) \cap \pi_{k,i} \subset \left(\bigcup_{j=1}^{\nu} B_j \cup \bigcup_{j \in \mathcal{I}^*} \overline{\Gamma_{\bar{\varepsilon}}^{\mathcal{J}_j}} \right) \cap \pi_{k,i}. \tag{A.51}$$

Then, recalling the definition (A.28) of $E_{2,i}^{\mathcal{I}}$, we deduce from (A.51) that

$$E_{2,i}^{\mathcal{I}} \subset \mathbb{R}^n \setminus B(\Upsilon_i, c_p/2),$$

which clearly implies (A.42). Finally, in order to obtain an estimate of $d_C(E_{2,i}^{\mathcal{I}})$, notice that, since the radii of S_i, $i = 1, \dots, \nu$, are uniformly bounded by r_0', and thanks to the properties of the hyperplanes in Π, there will be some constants $c_{15}, c_{16} > 0$, depending only on r_0', such that there holds

$$d_{S_k}(y) > c_{15} d_{\Upsilon_i}^2(y) \quad \forall y \in \big(B\big(S_k, c_{16}\big) \cap \pi_{k,i}\big) \Big\backslash \left(\bigcup_{j=1}^{\nu} B_j \right), \ \forall i \in \mathcal{J}_k, \ k = 1, \dots, \nu. \tag{A.52}$$

Therefore, choosing $\bar{\varepsilon}$ so that

$$\bar{\varepsilon} < \frac{c_{16}}{2}, \tag{A.53}$$

and observing that by the same computations at point **2** one has

$$E_{2,i}^{\mathcal{I}} \subset \partial^{-\mathcal{I}} \Gamma_{\bar{\varepsilon}}^{\mathcal{J}_k} \subset B(S_k, 2\bar{\varepsilon}) \Big\backslash \left(\bigcup_{j=1}^{\nu} B_j \right),$$

we deduce from (A.42), (A.52) that

$$d_{S_k}\big(E_{2,i}^{\mathcal{I}}\big) \geqslant c_{15} \frac{\bar{c}_p^2}{4} \quad \forall i \in \widetilde{\mathcal{J}}_k. \tag{A.54}$$

Hence, letting $\bar{\varepsilon}$ be the constant provided by Lemma 3 and satisfying (A.16), (A.23), (A.53), relying on (A.22), (A.24) we recover from (A.54) the estimates

$$d_{C^{\mathcal{I}}}\big(E_{2,i}^{\mathcal{I}}\big) \geqslant c_{13} c_{15}^2 \frac{\bar{c}_p^4}{16} \quad \forall i \in \widetilde{\mathcal{J}}_k, \tag{A.55}$$

which, together with (A.30), (A.38), yield

$$\begin{aligned} d_{C^{\mathcal{I}}}\big(\partial^{-\mathcal{I}} \Gamma_{\bar{\varepsilon}}^{\mathcal{J}_k}\big) &\geqslant \min\Big\{ d_{C^{\mathcal{I}}}\big(E_1\big), \min_{i \in \mathcal{J}_k} d_{C^{\mathcal{I}}}\big(E_{2,i}^{\mathcal{I}}\big) \Big\} \\ &\geqslant \bar{c}_{p+1}, \end{aligned} \tag{A.56}$$

where $\bar{c}_{p+1} \doteq \min\{c_{12}^2 c_{13} \bar{\varepsilon}^4, \bar{c}_p/2, \ c_{13} \cdot c_{15}^2 \cdot \bar{c}_p^4/16\}$. This establishes the estimate in (A.26) in the case where $E_2^{\mathcal{I}} \neq \emptyset$, and (A.34) does not hold for some $i \in \mathcal{J}_k$, and the proof of the lemma is completed. □

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