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Sign changing bubble tower solutions in a slightly subcritical semilinear Dirichlet problem

Solutions changeant de signe du type « tour de bulles » du problème de Dirichlet pour l'équation $-\Delta u = |u|^{\frac{4}{N-2}-\varepsilon}u$

Angela Pistoia [a,*,1], Tobias Weth [b]

[a] *Dipartimento di Metodi e Modelli Matematici, Università di Roma "La Sapienza", via A. Scarpa 16, 00161 Roma, Italy*
[b] *Mathematisches Institut, Universität Giessen, Arndtstr. 2, 35392 Giessen, Germany*

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Abstract

We consider the problem $-\Delta u = |u|^{p-1-\varepsilon}u$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded smooth domain in $\mathbb{R}^N$ symmetric with respect to $x_1, \ldots, x_N$, which contains the origin, $N \geqslant 3$, $p = \frac{N+2}{N-2}$ and ε is a positive parameter. As ε goes to zero, we construct sign changing solutions with multiple blow up at the origin. These solutions have, as ε goes to zero, more and more annular-shaped nodal domains.
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Résumé

Nous considérons le problème $-\Delta u = |u|^{p-1-\varepsilon}u$ dans Ω, $u = 0$ sur $\partial\Omega$, où Ω est un domaine borné de $\mathbb{R}^N$ symétrique par rapport à $x_1, \ldots, x_N$ qui contient l'origine, $N \geqslant 3$, $p = \frac{N+2}{N-2}$ et ε est un paramètre positif. Quand $\varepsilon \to 0$, nous construisons des solutions changeant de signe qui ressemblent à une superposition de transitoires centrées dans l'origine. Ces solutions admettent, quand $\varepsilon \to 0$, de plus en plus de domaines nodeaux ressemblant à un anneau.
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* Corresponding author.
E-mail addresses: pistoia@dmmm.uniroma1.it (A. Pistoia), tobias.weth@math.uni-giessen.de (T. Weth).
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1. Introduction

We consider the problem

$$\begin{cases} -\Delta u = |u|^{p-1-\varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 3$, $p = \frac{N+2}{N-2}$ and ε is a positive parameter.

First, let us consider the critical case, i.e. $\varepsilon = 0$. Pohozaev proved in [39] that problem (1.1) has no solutions if Ω is starshaped. On the other hand, Kazdan and Warner observed in [32] that (1.1) has a positive radial solution when Ω is an annulus. In [5] Bahri and Coron proved that (1.1) has a positive solution, provided that Ω has nontrivial topology. To our knowledge there are only few results about existence of sign changing solutions of problem (1.1) in the critical case. In [30] the authors provide existence and multiplicity of sign changing solutions in specific cases, like for instance in the case of tori. In [19] the authors obtain existence and multiplicity results for sign changing solutions in domains with small holes and in some contractible domains with an involution symmetry. We also recall the classical result of Ding [25] who showed that (1.1) has infinitely many sign changing solutions in the whole space $\Omega = \mathbb{R}^N$. Concerning sign changing solutions for different problems with critical growth, we refer to the papers [1,2, 15,17,18,27,31,34].

In this paper, we deal with the slightly subcritical case, i.e. $\varepsilon > 0$. In order to state old and new results, it is useful to recall some well known definitions. We denote by G the Green's function of the Laplacian with Dirichlet boundary condition on $\partial\Omega$ and by H its regular part, i.e. $G(x,y) = C_N|x-y|^{2-N} - H(x,y)$, $x, y \in \Omega$, where $C_N = 1/(N-2)\omega_N$ and ω_N denotes the surface area of the unit sphere in $\mathbb{R}^N$. Observe that $\Delta_x H = 0$ in $\Omega \times \Omega$ and $G = 0$ on $\partial(\Omega \times \Omega)$. The leading term $H(x,x)$ of the regular part of the Green's function is called the Robin's function of Ω at x.

It is well known that problem (1.1) has always a positive least energy solution u_ε which is obtained by solving the variational problem

$$\inf\left\{ \|u\|^2 = \int_\Omega |\nabla u|^2\colon\ u \in \mathrm{H}_0^1(\Omega),\ \int_\Omega |u|^{p+1-\varepsilon} = 1 \right\}.$$

In [13,26,29,42,41] it was proved that, as ε goes to zero, u_ε blows up and concentrates at a single point ξ in Ω, which is a critical point (a minimum point) of the Robin's function. Conversely, it was shown in [41,37] that, if ξ is a "stable" critical point of the Robin's function, then problem (1.1) has for ε small a solution which blows up at ξ as ε goes to zero. In general, problem (1.1) can have positive solutions which concentrate simultaneously at different points $\xi_1, \ldots, \xi_k$ of Ω, $k \geqslant 2$, as ε goes to zero. This was analysed in [40,6,37]: the condition which ensures existence and multiplicity of solutions which blows up at more than one point involve both the Green's function and Robin's function. As far as the existence of positive solutions to (1.1) is concerned, we want to point out the fact that if u_ε solves (1.1) and blows up at some points $\xi_1, \ldots, \xi_k$ of Ω, then necessarily each ξ_i is a simple blow up point (see [33]). More precisely, the profile of the solution u_ε near each blow up point ξ_i can be approximated as

$$u_\varepsilon(x) \sim \frac{\lambda_i\sqrt{\varepsilon}}{(\lambda_i^2\varepsilon + |x-\xi_i|^2)^{(N-2)/2}},$$

with $\alpha_N = [N(N-2)]^{(N-2)/4}$, for some constant λ_i which depends only on N and k. Roughly speaking, we can also say that u_ε has k *simple positive bubbles*.

The existence of one sign changing solution to (1.1) for $\varepsilon \in (0, p-1)$ was first proved in [16] and [8]. Later, multiple sign changing solutions and their nodal properties were studied in [7,9]. In all these papers, the authors consider a larger class of nonlinearities with superlinear and subcritical growth. In particular, in [7,9] it is shown that, for fixed $\varepsilon \in (0, p-1)$, problem (1.1) has a sequence of sign solutions $\pm u_\varepsilon^n$, with $\|u_\varepsilon^n\| \to \infty$ as n goes to ∞, and such that u_ε^n has at most $n+1$ nodal domains. Recently in [10], the authors proved the existence of N pairs $\pm u_\varepsilon^{(j)}$, $j = 1, \ldots, N$, of sign changing solutions such that each $u_\varepsilon^{(j)}$ blows up positively at a point $\xi_1^{(j)} \in \Omega$ and negatively at a point $\xi_2^{(j)} \in \Omega$, with $\xi_1^{(j)} \neq \xi_2^{(j)}$, as ε goes to 0. We point out that each of these solutions considered in [10] has *one simple positive bubble and one simple negative bubble.* More precisely the profile of $u_\varepsilon^{(j)}$ near each blow up point $\xi_i^{(j)}$, $i = 1, 2$, can be approximated when ε goes to 0 as

$$u_\varepsilon^{(j)}(x) \sim \alpha_N (-1)^i \frac{\lambda_i \sqrt{\varepsilon}}{(\lambda_i^2 \varepsilon + |x - \xi_i^{(j)}|^2)^{(N-2)/2}},$$

with $\alpha_N = [N(N-2)]^{(N-2)/4}$, for some constant λ_i which depends only on N.

In this paper, we focus on a new phenomenon: we observe the presence of *sign changing bubble towers* constituted by superposition of positive bubbles and negative bubbles of different blow up orders. This stands in strong contrast to the fact that positive solutions to (1.1) can only have simple bubbles in the subcritical case (see [33]). We assume that Ω is a bounded smooth domain in $\mathbb{R}^N$ symmetric with respect to $x_1, \dots, x_N$, which contains the origin. Our main result reads:

Theorem 1.1. *For any integer $k \geqslant 1$, there exists $\varepsilon_k > 0$ such that for any $\varepsilon \in (0, \varepsilon_k)$ there exists a pair of solutions u_ε and $-u_\varepsilon$ to problem* (1.1) *such that*

$$u_\varepsilon(y) = \alpha_N \sum_{i=1}^{k} (-1)^i \left(\frac{M_i \varepsilon^{\frac{2i-1}{N-2}}}{M_i^2 \varepsilon^{2\frac{2i-1}{N-2}} + |y|^2} \right)^{\frac{N-2}{2}} \big(1 + \mathrm{o}(1)\big),$$

where $\alpha_N := [N(N-2)]^{\frac{N-2}{4}}$, $M_1, \dots, M_k$ are positive constants depending only on N and k and $\mathrm{o}(1) \to 0$ *uniformly on compact subsets of Ω, as $\varepsilon \to 0$. Moreover, u_ε is even with respect to the variables $x_1, \dots, x_N$.*

It seems that this is the first result dealing with sign changing bubble tower solutions for superlinear boundary value problems close to the critical exponent. The asymptotic expansion and some energy estimates derived in the course of the proof allow to draw interesting consequences concerning the number and shape of the nodal domains of the solution u_ε. More precisely, we have

Theorem 1.2. *For $\varepsilon > 0$ sufficiently small, the solution u_ε constructed in Theorem* 1.1 *has precisely k nodal domains $\Omega_\varepsilon^1, \dots, \Omega_\varepsilon^k$ such that Ω_ε^j contains the sphere $S_\varepsilon^j = \{y \in \mathbb{R}^N\colon\ |y| = \varepsilon^{\frac{2j-1}{N-2}}\}$, and $(-1)^j u_\varepsilon > 0$ on Ω_ε^j for all j. Consequently, $0 \in \Omega_\varepsilon^k$, and Ω_ε^1 is the only nodal domain of u which touches the boundary $\partial\Omega$.*

In particular, we obtain solutions with arbitrarily many annular shaped nodal domains as ε goes to 0.

The proof of Theorem 1.1 relies on a form of Lyapunov–Schmidt procedure (see [4]), which reduces the construction of the searched solutions to a finite dimensional variational problem, in a general scheme already followed in the study of bubble towers in [20,21,24].

Let us point out that the situation turns out to be very different in the slightly supercritical case, i.e. $\varepsilon < 0$. We recall that, if the domain Ω has a small hole, problem (1.1) has *positive* solutions blowing up at two or three points, see [22] and [38]. Moreover, if Ω has some symmetries, problem (1.1) has solutions blowing up at an arbitrary number of points (see [23,35,38]). An interesting nonexistence result obtained in [11] states that, for any domain Ω, there are no positive solutions to (1.1) blowing up at a single point as ε goes to zero. A natural question then arises: is it possible to construct a sign changing bubble tower solution to (1.1) (as in Theorem 1.1) when ε is negative and small enough? We conjecture that the answer is negative. Indeed this is suggested by the estimates obtained in the present paper. More precisely, in order to detect sign changing bubble tower solutions in the slightly supercritical case, we could reduce the problem in the same way to a finite dimensional one, but for small negative ε the reduced functional (the function $\tilde{I}_\varepsilon$ defined in Lemma 4.1 below) does not have any critical points (as follows from Lemma 4.2 and estimate (4.24)).

It is also worth to compare Theorem 1.1 with recent results on positive bubble tower solutions for the *supercritical* Dirichlet problem

$$\begin{cases} -\Delta u = u^{p+\varepsilon} + \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 4$, ε and λ are positive parameters. In [20] the authors considered the case when Ω is a ball and they proved the existence of radial solutions to (1.2), which have a multiple bubble at the origin, provided ε and λ are small enough. Recently, the result was extended in [28], where the authors constructed solutions to problem (1.2), which have multiple blow up at finitely many points which are the critical points of a function whose definition involves the Green's function. Successively, existence of bubble tower solutions was established for the Neumann supercritical problem

$$\begin{cases} -\Delta u + u = u^{p+\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

when Ω is even with respect to $N-1$ variables, $N \geqslant 3$ and 0 is a point in $\partial\Omega$ with positive mean curvature. In [24] the authors proved the existence of solutions to (1.3) which resemble the form of a superposition of bubbles centered at 0.

The proof of Theorem 1.1 relies on a form of Lyapunov–Schmidt procedure (see [4]), which reduces the construction of the searched solutions to a finite-dimensional variational problem, in a general scheme already followed in the study of bubble towers in [20,21,24]. The paper is organized as follows. In Section 2 we collect basic tools, and we introduce a change of coordinates. In Section 3 we apply a finite-dimensional reduction method to the transformed problem. We like to warn the reader that in this section we did not repeat the proofs of the estimates required for the reduction procedure, referring the reader to [20,21,24]. In Section 4 we derive an asymptotic expansion for the reduced energy functional. Here we decided to include the details, since the expansion shows crucial differences in comparison with [20,21,24]. Finally, in Section 5 we complete the proof of Theorem 1.1 and we prove Theorem 1.2.

2. Preliminaries

It is well known (see [3,14,43]) that the functions

$$w_\mu(y) = \alpha_N \frac{\mu^{\frac{N-2}{2}}}{(\mu^2 + |y|^2)^{\frac{N-2}{2}}}, \quad \mu > 0,$$

with $\alpha_N := [N(N-2)]^{\frac{N-2}{4}}$, are the only radial solutions of the equation

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^N.$$

We define π_μ to be the unique solution to the problem

$$\begin{cases} \Delta \pi_\mu = 0 & \text{in } \Omega, \\ \pi_\mu = -w_\mu & \text{on } \partial\Omega. \end{cases}$$

We remark that the function $Pw_\mu := w_\mu + \pi_\mu$ is the projection onto $\mathrm{H}^1_0(\Omega)$ of the function w_μ, i.e.

$$\begin{cases} -\Delta P w_\mu = w_\mu^p & \text{in } \Omega, \\ P w_\mu = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known that the following expansion holds

$$\pi_\mu(y) = -\alpha_N \mu^{\frac{N-2}{2}} H(0, y) + \mathrm{o}\big(\mu^{\frac{N-2}{2}}\big) \text{ uniformly in } \Omega. \tag{2.1}$$

Let us consider parameters $\mu_1 > \mu_2 > \cdots > \mu_k$. We look for a solution to (1.1) of the form

$$u(y) = \sum_{i=1}^{k} (-1)^i \big(w_{\mu_i}(y) + \pi_{\mu_i}(y)\big) + \psi(y), \tag{2.2}$$

where the rest term ψ is a small function which is even with respect to the variables $y_1, \ldots, y_N$.

As in [20,21,24], we rewrite this problem in different variables. We consider spherical coordinates $y = y(\rho, \Theta)$ centered at the origin given by $\rho = |y|$ and $\Theta = \frac{y}{|y|}$. We define the transformation

$$v(x, \Theta) = \mathcal{T}(u)(x, \Theta) := \left(\frac{p-1}{2}\right)^{\frac{2}{p-1}} \mathrm{e}^{-x} u\big(\mathrm{e}^{-\frac{p-1}{2}x}\Theta\big).$$

We denote by D the subset of $S=\mathbb{R}\times S^{N-1}$ where the variables (x,Θ) vary. After these changes of variables, problem (1.1) becomes

$$\begin{cases} L_0(v)=c_\varepsilon \mathrm{e}^{-\varepsilon x}|v|^{p-1-\varepsilon}v & \text{in } D,\\ v=0 & \text{on } \partial D,\end{cases} \tag{2.3}$$

where

$$c_\varepsilon=\left(\frac{p-1}{2}\right)^{\frac{2\varepsilon}{p-1}}$$

and

$$L_0(v)=-\left(\frac{p-1}{2}\right)^2\Delta_{S^{N-1}}v-v''+v. \tag{2.4}$$

L_0 is the transformed operator associated to $-\Delta$. Here and in what follows, $'=\frac{\partial}{\partial x}$ and $\Delta_{S^{N-1}}$ denotes the Laplace–Beltrami operator on S^{N-1}.

We observe then that

$$\mathcal{T}(w_\mu)(x,\Theta)=W_\xi(x):=W(x-\xi),$$

where

$$W(x):=\left(\frac{4N}{N-2}\right)^{\frac{N-2}{4}}\mathrm{e}^{-x}\left(1+\mathrm{e}^{-\frac{4}{N-2}x}\right)^{-\frac{N-2}{2}},\quad \text{with } \mu=\mathrm{e}^{-\frac{p-1}{2}\xi}. \tag{2.5}$$

W is the unique solution of the problem

$$\begin{cases} W''-W+W^p=0 & \text{in } \mathbb{R},\\ W'(0)=0,\quad W>0, & \\ W(x)\to 0 & \text{as } x\to\pm\infty.\end{cases} \tag{2.6}$$

We see also that setting

$$\Pi_\xi=\mathcal{T}(\pi_\mu),\quad \text{with } \mu=\mathrm{e}^{-\frac{p-1}{2}\xi}, \tag{2.7}$$

then Π_ξ solves the boundary problem

$$\begin{cases} L_0(\Pi_\xi)=0 & \text{in } D,\\ \Pi_\xi=-W_\xi & \text{on } \partial D.\end{cases} \tag{2.8}$$

We note the useful fact that this transformation leaves the associated energies invariant (up to a constant). Indeed, the energy functional associated to problem (2.3) is

$$I_\varepsilon(v):=\frac{1}{2}\left(\frac{p-1}{2}\right)^2\int_D|\nabla_\Theta v|^2\,\mathrm{d}x\,\mathrm{d}\Theta+\frac{1}{2}\int_D\left(|v'|^2+|v|^2\right)\mathrm{d}x\,\mathrm{d}\Theta-\frac{c_\varepsilon}{p+1-\varepsilon}\int_D \mathrm{e}^{-\varepsilon x}|v|^{p+1-\varepsilon}\,\mathrm{d}x\,\mathrm{d}\Theta \tag{2.9}$$

and the energy functional associated to problem (1.1) is

$$J_\varepsilon(u):=\frac{1}{2}\int_\Omega|\nabla u|^2\,\mathrm{d}y-\frac{1}{p+1-\varepsilon}\int_\Omega|u|^{p+1-\varepsilon}\,\mathrm{d}y. \tag{2.10}$$

Then we have the identity

$$I_\varepsilon(v)=\left(\frac{2}{N-2}\right)^{N-1}J_\varepsilon(u),\quad v=\mathcal{T}(u). \tag{2.11}$$

Let us consider points $0<\xi_1<\xi_2<\cdots<\xi_k$. We look for a solution to (2.3) of the form

$$v(x,\Theta)=\sum_{i=1}^k(-1)^i\left(W(x-\xi_i)+\Pi_{\xi_i}(x,\Theta)\right)+\phi(x,\Theta), \tag{2.12}$$

where the rest term ϕ is a small function which is symmetric with respect to the variables $\Theta_1, \dots, \Theta_N$.

A crucial remark is that $v(x) \sim \sum_{i=1}^k (-1)^i W(x - \xi_i)$ solves (2.3) if and only if (going back in the change of variables)

$$u(y) \sim \alpha_N \sum_{i=1}^k (-1)^i \left(\frac{\mathrm{e}^{-\frac{2\xi_i}{N-2}}}{\mathrm{e}^{-\frac{4\xi_i}{N-2}} + |y|^2} \right)^{\frac{N-2}{2}}$$

solves (1.1). Therefore, the ansatz given for v provides (for large values of the ξ_i's) a sign changing bubble-tower solution for (1.1).

Let us write

$$W_i(x) := W(x - \xi_i), \quad \Pi_i := \Pi_{\xi_i}, \quad V_i = W_i + \Pi_i, \quad V := \sum_{i=1}^k (-1)^i V_i. \tag{2.13}$$

We consider the ansatz $v = V + \phi$. In terms of ϕ, problem (2.3) becomes

$$\begin{cases} L(\phi) = N(\phi) + R & \text{in } D, \\ \phi = 0 & \text{on } \partial D, \end{cases} \tag{2.14}$$

where

$$L(\phi) := L_0(\phi) - c_\varepsilon \mathrm{e}^{-\varepsilon x} f_\varepsilon'(V)\phi, \tag{2.15}$$

$$N(\phi) := c_\varepsilon \mathrm{e}^{-\varepsilon x}\big[f_\varepsilon(V + \phi) - f_\varepsilon(V) - f_\varepsilon'(V)\phi\big], \tag{2.16}$$

$$R := c_\varepsilon \mathrm{e}^{-\varepsilon x} f_\varepsilon(V) - \sum_{i=1}^k (-1)^i W_i^p. \tag{2.17}$$

Here, we set $f_\varepsilon(s) := |s|^{p-1-\varepsilon} s$.

3. The reduction method

Rather than solving (2.14) directly, we consider first the following intermediate problem: given points $\xi := (\xi_1, \dots, \xi_k) \in \mathbb{R}^k$ find a function ϕ symmetric with respect to the variables $\Theta_1, \dots, \Theta_N$ such that for certain constants c_i

$$\begin{cases} L(\phi) = N(\phi) + R + \sum_{i=1}^k c_i Z_i & \text{in } D, \\ \phi = 0 & \text{on } \partial D, \\ \int_D Z_i \phi \,\mathrm{d}x\,\mathrm{d}\Theta = 0 & \text{if } i = 1, \dots, k \end{cases} \tag{3.1}$$

where the Z_i's are defined as follows. Let

$$z_i(y) = \mu_i \frac{\partial}{\partial \mu_i} w_{\mu_i}(y) \quad \text{for } i = 1, \dots, k, \text{ with } \mu_i = \mathrm{e}^{-\frac{2}{N-2}\xi_i}.$$

Each z_i solves the linearized problem (see [12])

$$-\Delta z = p w_{\mu_i}^{p-1} z \quad \text{in } \mathbb{R}^N.$$

Let $P z_i$ be the projections onto $\mathrm{H}_0^1(\Omega)$ of the function z_i, i.e. $\Delta P z_i = \Delta z_i$ in Ω, $P z_i = 0$ on $\partial\Omega$. Let $Z_i(x, \Theta) := \mathcal{T}(P z_i)(x, \Theta)$. Then Z_i solves

$$\begin{cases} L_0(Z_i) = p W_i^{p-1} W_i' & \text{in } D, \\ Z_i = 0 & \text{on } \partial D. \end{cases}$$

In order to solve problem (3.1), it is necessary to understand first its linear part. Given a function h, we consider the problem of finding ϕ such that for certain real numbers c_i the following is satisfied

$$\begin{cases} L(\phi) = h + \sum_{i=1}^k c_i Z_i & \text{in } D, \\ \phi = 0 & \text{on } \partial D, \\ \int_D Z_i \phi \,\mathrm{d}x\,\mathrm{d}\Theta = 0 & \text{if } i = 1, \dots, k \end{cases} \tag{3.2}$$

where the linear operator L is defined in (2.15). We need uniformly bounded solvability in proper functional spaces for problem (3.2), for a proper range of the ξ_i's. To this end, it is convenient to introduce the following norm. Given a small but fixed number $0<\sigma<1$, we define:

$$\|g\|_* := \sup_{(x,\Theta)\in D}\left(\sum_{i=1}^{k} \mathrm{e}^{-(1-\sigma)|x-\xi_i|}\right)^{-1}\big|g(x,\Theta)\big|.$$

Although this norm depends on σ and the numbers $0<\xi_1<\cdots<\xi_k$, we do not indicate this dependence in our notation. In fact, different choices of σ and $\xi=(\xi_1,\dots,\xi_k)$ lead to equivalent norms. Let $\mathcal{C}_*$ be the Banach space of all continuous functions $g: D\to\mathbb{R}$ which are symmetric with respect to the variables $\Theta_1,\dots,\Theta_N$ and for which $\|g\|_*<+\infty$.

Arguing exactly as in Propositions 1 and 2 in [20] and in Propositions 5.1 and 5.2 in [21], we obtain the following result.

Proposition 3.1. *There exist $\varepsilon_0>0$, $R_0, R_1>0$ and $C>0$ such that if $\varepsilon\in(0,\varepsilon_0)$ and if $\xi=(\xi_1,\dots,\xi_k)$ satisfies*

$$R_0<\xi_1,\quad R_0<\min_{1\leqslant i<k}(\xi_{i+1}-\xi_i),\quad \xi_k<\frac{R_1}{\varepsilon},\tag{3.3}$$

then for any $h\in\mathcal{C}_*$ problem (3.2) *admits a unique solution $T_\varepsilon(\xi,h)\in\mathcal{C}_*$, with*

$$\big\|T_\varepsilon(\xi,h)\big\|_*\leqslant C\|h\|_* \quad \textit{and} \quad |c_i|\leqslant C\|h\|_*.$$

Moreover, the map $\xi\to T_\varepsilon(\xi,h)$, with values in $\mathcal{L}(\mathcal{C}_*)$, is of class C^1 and

$$\big\|D_\xi T_\varepsilon(\xi,h)\big\|_{\mathcal{L}(\mathcal{C}_*)}\leqslant C$$

uniformly in ξ satisfying conditions (3.3).

Now, we are ready to solve problem (3.1). We shall do this after restricting conveniently the range of the parameters ξ_i. Let us consider for a number M large but fixed, the following conditions:

$$\xi_1>\frac{1}{2}\log\left(\frac{1}{M\varepsilon}\right),\quad \min_{1\leqslant i<k}(\xi_{i+1}-\xi_i)>\log\left(\frac{1}{M\varepsilon}\right),\quad \xi_k<k\log\left(\frac{1}{M\varepsilon}\right).\tag{3.4}$$

Arguing exactly as in Proposition 3 in [20] and in Lemma 6.1 in [21], we prove the following result.

Proposition 3.2. *There exists $\varepsilon_0>0$ and $C>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$ and for any $\xi=(\xi_1,\dots,\xi_k)$ which satisfies* (3.4) *there exists a unique solution $\phi=\phi(\xi)$, $c=(c_1(\xi),\dots,c_k(\xi))$ to problem* (3.1) *which satisfies $\|\phi\|_*\leqslant C\varepsilon$. Moreover, the map $\xi\to\phi(\xi)$ is of class C^1 for the $\|\cdot\|_*$ norm and $\|D_\xi\phi\|_*\leqslant C\varepsilon$.*

4. Estimates for the reduced functional

In this section, we fix a large number M and assume that conditions (3.4) hold true for $\xi=(\xi_1,\dots,\xi_k)$. According to the results of the previous section, our problem has been reduced to that of finding points ξ_i so that the constants c_i which appear in (3.1), for the solution ϕ given by Proposition 3.2, are all equal to zero. Thus, we need to solve the system of equations

$$c_i(\xi)=0\quad\text{for any } i=1,\dots,k.\tag{4.1}$$

If (4.1) holds, then $v=V+\phi$ will be a solution to (2.14) or equivalently to (2.3). This system turns out to be equivalent to a variational problem, related to the functional (2.9) associated to problem (2.3). Indeed, by the same (standard) arguments as given on p. 301 in [20], the following result is proved.

Lemma 4.1. *The function $V+\phi$ is a solution to* (2.3) *if ξ is a critical point of the function*

$$\xi\mapsto\tilde{I}_\varepsilon(\xi):=I_\varepsilon(V+\phi),$$

where $V=V(\xi)$ is given by (2.13), *$\phi=\phi(\xi)$ is given by Proposition* 3.2, *and I_ε is defined in* (2.9).

The following estimate is crucial for finding critical points of $\tilde{I}_\varepsilon$. It can be proved exactly as Lemma 4 in [20] and Lemma 6.2 in [24].

Lemma 4.2. *The following expansion holds*:

$$\tilde{I}_\varepsilon(\xi) = I_\varepsilon(V) + \mathrm{o}(\varepsilon),$$

where the term $\mathrm{o}(\varepsilon)$ *is uniform over all points satisfying constraints* (3.4), *for some given* $M > 0$.

We make the following choices for the points ξ_i:

$$\xi_1 = -\frac{1}{2}\log\varepsilon + \log\Lambda_1, \tag{4.2}$$

$$\xi_{i+1} - \xi_i = -\log\varepsilon - \log\Lambda_{i+1}, \quad i = 1, \ldots, k-1, \tag{4.3}$$

where the Λ_i's are positive parameters. For notational convenience, we also set $\Lambda := (\Lambda_1, \ldots, \Lambda_k)$.

The advantage of the above choice is the validity of the expansion of the functional (2.9) given in the following lemma.

Proposition 4.3. *For any* $\delta > 0$, *there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *the following expansion holds*

$$I_\varepsilon(V) = ka_4 + \varepsilon\Psi_k(\Lambda) - \frac{k^2 - 2k + 2}{2}a_2\varepsilon\log\varepsilon + ka_1\varepsilon + \varepsilon R_\varepsilon(\Lambda)$$

where

$$\Psi_k(\Lambda) := \frac{a_5 H(0,0)}{\Lambda_1^2} + ka_2\log\Lambda_1 + \sum_{i=2}^{k}\bigl[a_3\Lambda_i - (k-i+1)a_2\log\Lambda_i\bigr]$$

and as $\varepsilon \to 0$ *the term* R_ε *converges to* 0 *uniformly on the set of* Λ_i*'s with* $\delta < \Lambda_i < \delta^{-1}$, $i = 1, \ldots, k$. *Here* a_i, $i = 0, \ldots, 5$, *are positive constants depending only on* N.

The proof of this expansion relies on arguments inspired by [20,21,24]. For the convenience of the reader, we present the details here. As a first step, we collect some asymptotic estimates in the following lemma.

Lemma 4.4. *For fixed* $\delta > 0$ *and* $\delta < \Lambda_i < \delta^{-1}$, $i = 1, \ldots, k$, *the following estimates hold as* ε *goes to zero*:

$$\int_D |V|^{p+1} = k\omega_{N-1}\int_{\mathbb{R}} W^{p+1}(x)\,\mathrm{d}x + \mathrm{o}(1), \tag{4.4}$$

$$\int_D \bigl(|V|^{p+1} - |V|^{p+1-\varepsilon}\bigr) = k\omega_{N-1}\int_{\mathbb{R}} W^{p+1}(x)\log W(x)\,\mathrm{d}x + \mathrm{o}(1), \tag{4.5}$$

$$\int_D x|V|^{p+1} = \left(\sum_{j=1}^{k}\xi_j\right)\omega_{N-1}\int_{\mathbb{R}} W^{p+1}(x)\,\mathrm{d}x + \mathrm{o}(1). \tag{4.6}$$

Here ω_{N-1} *is the surface area of* S^{N-1}. *Moreover, considering the numbers* $\chi_1 = 0$, $\chi_l = \frac{\xi_{l-1}+\xi_l}{2}$, $l = 2, \ldots, k$, $\chi_{k+1} = +\infty$ *and setting* $D_l = \{(x,\Theta) \in D\colon\ \chi_l \leqslant x < \chi_{l+1}\}$, *we have for* $i, j, l = 1, \ldots, k$

$$\int_{D_l} W_i^p W_j = \mathrm{o}(\varepsilon) \quad \textit{if } i \neq l, \tag{4.7}$$

$$\int_{D_l} W_l^p W_j = a_3\mathrm{e}^{-|\xi_j - \xi_l|} + \mathrm{o}(\varepsilon) \quad \textit{if } j \neq l, \tag{4.8}$$

$$\int_{D_l}\left[V_l^{p+1}-|V|^{p+1}+(p+1)V_l^p\sum_{j\neq l}(-1)^{l+j}V_j\right]=\mathrm{o}(\varepsilon), \tag{4.9}$$

$$\int_{D_l}\left(W_i^p-V_i^p\right)V_j=\mathrm{o}(\varepsilon)\quad \textit{if } i\neq j. \tag{4.10}$$

Here we have set $a_3:=(\frac{4N}{N-2})^{\frac{N-2}{4}}\omega_{N-1}\int_{\mathbb{R}}\mathrm{e}^x W^p(x)\,\mathrm{d}x$.

Proof. Throughout this proof, C stands for a generic constant depending only on N and k whose value may change in every step of the calculation. From (2.5) we directly deduce the estimate

$$\left|W(x)\right|\leqslant C\mathrm{e}^{-|x|}\quad \text{in } D \tag{4.11}$$

which will be frequently used in the following. Combining the assumption $\delta<\Lambda_i<\delta^{-1}$ with (4.2) and (4.3), we get

$$\xi_1>-\log\left(\frac{\sqrt{\varepsilon}}{\delta}\right)\quad\text{and}\quad \xi_{i+1}-\xi_i>-\log\left(\frac{\varepsilon}{\delta}\right)\quad \text{for } i=1,\dots,k-1. \tag{4.12}$$

We start by verifying (4.7), first for $i\neq l$ and $j\neq l$. In this case (4.11) implies

$$\begin{aligned}\int_{D_l}W_i^pW_j&\leqslant C\int_{\chi_l}^{\chi_{l+1}}\mathrm{e}^{-p|x-\xi_i|}\mathrm{e}^{-|x-\xi_j|}\\&\leqslant C(\chi_{l+1}-\chi_l)\max\left\{\mathrm{e}^{-p|\chi_l-\xi_i|},\mathrm{e}^{-p|\chi_{l+1}-\xi_i|}\right\}\max\left\{\mathrm{e}^{-|\chi_l-\xi_j|},\mathrm{e}^{-|\chi_{l+1}-\xi_j|}\right\}\\&\leqslant C\frac{\xi_{l+1}-\xi_{l-1}}{2}\mathrm{e}^{-\frac{p|\xi_l-\xi_i|}{2}}\mathrm{e}^{-\frac{|\xi_l-\xi_j|}{2}}\leqslant C(-\log\varepsilon)\varepsilon^{(p+1)/2}=\mathrm{o}(\varepsilon).\end{aligned}$$

Next we consider $j=l$. By (4.11) we get for $i<l$

$$\begin{aligned}\int_{D_l}W_i^pW_l&\leqslant C\int_{\chi_l}^{\chi_{l+1}}\mathrm{e}^{-p(x-\xi_i)}\mathrm{e}^{x-\xi_l}\,\mathrm{d}x=C\mathrm{e}^{-p(\xi_l-\xi_i)}\int_{\frac{\xi_{l-1}-\xi_l}{2}}^{\frac{\xi_{l+1}-\xi_l}{2}}\mathrm{e}^{-(p-1)x}\,\mathrm{d}x\\&\leqslant C\mathrm{e}^{-p(\xi_l-\xi_i)}\mathrm{e}^{(p-1)\frac{\xi_l-\xi_{l-1}}{2}}\leqslant C\mathrm{e}^{-\frac{p+1}{2}(\xi_l-\xi_{l-1})}\leqslant C\varepsilon^{\frac{p+1}{2}}=\mathrm{o}(\varepsilon).\end{aligned}$$

For $i>l$, we find similarly

$$\begin{aligned}\int_{D_l}W_i^pW_l&\leqslant C\int_{\chi_l}^{\chi_{l+1}}\mathrm{e}^{p(x-\xi_i)}\mathrm{e}^{-(x-\xi_l)}\,\mathrm{d}x=C\mathrm{e}^{p(\xi_l-\xi_i)}\int_{\frac{\xi_{l-1}-\xi_l}{2}}^{\frac{\xi_{l+1}-\xi_l}{2}}\mathrm{e}^{(p-1)x}\,\mathrm{d}x\\&\leqslant C\mathrm{e}^{p(\xi_l-\xi_i)}\mathrm{e}^{(p-1)\frac{\xi_{l+1}-\xi_l}{2}}\leqslant C\mathrm{e}^{-\frac{p+1}{2}(\xi_{l+1}-\xi_l)}=\mathrm{o}(\varepsilon),\end{aligned}$$

and thus (4.7) is proved in all cases. Next we derive (4.8) for $j<l$, using the definition of W given in (2.5):

$$\begin{aligned}\int_{D_l}W_l^pW_j&=\omega_{N-1}\int_{\frac{\xi_{l-1}-\xi_l}{2}}^{\frac{\xi_{l+1}-\xi_l}{2}}W^p(x)W(x+\xi_l-\xi_j)\,\mathrm{d}x+\mathrm{o}(\varepsilon)\\&=\omega_{N-1}\left(\frac{4N}{N-2}\right)^{\frac{N-2}{4}}\mathrm{e}^{-(\xi_l-\xi_j)}\int_{\frac{\xi_{l-1}-\xi_l}{2}}^{\frac{\xi_{l+1}-\xi_l}{2}}W^p(x)\mathrm{e}^{-x}\left(1+\mathrm{e}^{-\frac{4}{N-2}(x+\xi_l-\xi_j)}\right)^{-\frac{N-2}{2}}\,\mathrm{d}x+\mathrm{o}(\varepsilon)\end{aligned}$$

$$= \omega_{N-1}\left(\frac{4N}{N-2}\right)^{\frac{N-2}{4}} \mathrm{e}^{-(\xi_l-\xi_j)}\bigl(1+\mathrm{o}(1)\bigr)\int\limits_{\mathbb{R}} W^p(x)\mathrm{e}^{-x}\,\mathrm{d}x + \mathrm{o}(\varepsilon)$$
$$= a_3\mathrm{e}^{-|\xi_j-\xi_l|} + \mathrm{o}(\varepsilon).$$

The proof for $j > l$ is similar, since $W(-x) = W(x)$ for all $x \in \mathbb{R}$. In particular, (4.7) and (4.8) yield

$$\int\limits_{D_l} W_i^p W_j = \mathrm{O}(\varepsilon) \quad \text{if } i \neq l \text{ or } j \neq l. \tag{4.13}$$

Next we show (4.4), and we set $\widetilde{W} = \sum_{i=1}^k (-1)^i W_i$ and $\widetilde{\Pi} = \sum_{i=1}^k (-1)^i \Pi_i$, so that $\widetilde{W} = V - \widetilde{\Pi}$. From (2.1), (2.7) and (4.12) we infer

$$0 \leqslant W_i - V_i = -\Pi_i \leqslant C\mathrm{e}^{-\xi_i} \leqslant C\sqrt{\varepsilon} \quad \text{in } D \text{ for all } i. \tag{4.14}$$

Hence the mean value theorem implies

$$\left|\int\limits_D \bigl|\widetilde{W}\bigr|^{p+1} - |V|^{p+1}\right| \leqslant (p+1)\int\limits_D \left(\sum_{i=1}^k W_i\right)^p \bigl|\widetilde{\Pi}\bigr| = \mathrm{o}(1), \tag{4.15}$$

and, by (4.13),

$$\begin{aligned}\int\limits_D \left(\sum_{l=1}^k W_l^{p+1} - \bigl|\widetilde{W}\bigr|^{p+1}\right) &= \sum_{l=1}^k \int\limits_{D_l} \left(W_l^{p+1} - \left|W_l + \sum_{j\neq l}(-1)^{l+j}W_j\right|^{p+1}\right) + \mathrm{o}(1)\\ &\leqslant (p+1)\sum_{l=1}^k \int\limits_{D_l} \left(\sum_{i=1}^k W_i\right)^p \left(\sum_{j\neq l} W_j\right) + \mathrm{o}(1)\\ &\leqslant C\sum_{l,i=1}^k \int\limits_{D_l} W_i^p\left(\sum_{j\neq l} W_j\right) + \mathrm{o}(1) = \mathrm{o}(1).\end{aligned} \tag{4.16}$$

Combining (4.15) and (4.16) we get

$$\int\limits_D |V|^{p+1} = \sum_{l=1}^k \int\limits_D W_l^{p+1} + \mathrm{o}(1) = k\omega_{N-1}\int\limits_{\mathbb{R}} W^{p+1}(x)\,\mathrm{d}x + \mathrm{o}(1),$$

which shows (4.4). Since the proof of (4.5) is similar, we omit it. To show (4.6), we use again (4.13) and (4.14) to estimate

$$\begin{aligned}\int\limits_D x|V|^{p+1} &= \int\limits_D x\bigl|\widetilde{W}\bigr|^{p+1} + \mathrm{o}(1) = \sum_{l=1}^k \int\limits_{D_l} xW_l^{p+1} + \mathrm{o}(1)\\ &= \omega_{N-1}\sum_{l=1}^k \int\limits_{\frac{\xi_{l-1}-\xi_l}{2}}^{\frac{\xi_{l+1}-\xi_l}{2}} (x+\xi_l)W^{p+1}(x)\,\mathrm{d}x + \mathrm{o}(1)\\ &= \omega_{N-1}\left(\sum_{l=1}^k \xi_l\right)\int\limits_{\mathbb{R}} W^{p+1}(x)\,\mathrm{d}x + k\int\limits_{\mathbb{R}} xW^{p+1}(x)\,\mathrm{d}x + \mathrm{o}(1)\\ &= \omega_{N-1}\left(\sum_{l=1}^k \xi_l\right)\int\limits_{\mathbb{R}} W^{p+1}(x)\,\mathrm{d}x + \mathrm{o}(1).\end{aligned}$$

Here we used that $W(-x)=W(x)$ for all $x\in\mathbb{R}$. It remains to show (4.9) and (4.10). Via a Taylor expansion, we get

$$\begin{aligned}
&\int_{D_l}\Big(V_l^{p+1}-|V|^{p+1}+(p+1)V_l^p\sum_{j\neq l}(-1)^{l+j}V_j\Big)\\
&\quad=\int_{D_l}\Big(V_l^{p+1}-\Big|V_l+\sum_{j\neq l}(-1)^{l+j}V_j\Big|^{p+1}+(p+1)V_l^p\sum_{j\neq l}(-1)^{l+j}V_j\Big)\\
&\quad\leqslant\frac{p(p+1)}{2}\int_{D_l}\Big(\sum_{j=1}^k V_j\Big)^{p-1}\Big(\sum_{j\neq l}V_j\Big)^2\leqslant C\max_{\substack{i,j\\ j\neq l}}\int_{D_l}V_i^{p-1}V_j^2\\
&\quad\leqslant C\max_{\substack{i,j\\ j\neq l}}\Big(\int_{D_l}V_i^pV_j\Big)^{\frac{p-1}{p}}\Big(\int_{D_l}V_j^{p+1}\Big)^{\frac1p}\\
&\quad\leqslant C\max_{\substack{i,j\\ j\neq l}}\Big(\int_{D_l}W_i^pW_j\Big)^{\frac{p-1}{p}}\Big(\int_{D_l}W_j^{p+1}\Big)^{\frac1p}=\mathrm{O}\big(\varepsilon^{\frac{p-1}{p}}\big)\mathrm{o}\big(\varepsilon^{\frac1p}\big)=\mathrm{o}(\varepsilon),
\end{aligned}$$

where (4.7) and (4.13) are used in the last line. Hence (4.9) is proved. Finally, by (4.13), (4.14) and the mean value theorem, we get for $i\neq j$

$$\begin{aligned}
0&\leqslant\int_{D_l}\big(W_i^p-V_i^p\big)V_j\leqslant p\int_{D_l}W_i^{p-1}(W_i-V_i)V_j\\
&\leqslant p\Big(\int_{D_l}W_i^pV_j\Big)^{\frac{p-1}{p}}\Big(\int_{D_l}(W_i-V_i)^pV_j\Big)^{1/p}=\mathrm{O}\big(\varepsilon^{\frac{p-1}{p}}\big)\mathrm{O}\big(\sqrt{\varepsilon}\big)=\mathrm{o}(\varepsilon),
\end{aligned}$$

so that (4.10) holds. □

Proof of Proposition 4.3 (completed). Using (4.4) and (4.5), we find

$$\begin{aligned}
I_\varepsilon(V)&=I_0(V)+\frac{1}{p+1}\int_D|V|^{p+1}-\frac{c_\varepsilon}{p+1-\varepsilon}\int_D\mathrm{e}^{-\varepsilon x}|V|^{p+1-\varepsilon}\\
&=I_0(V)-\frac{1}{p+1}\int_D\big(\mathrm{e}^{-\varepsilon x}-1\big)|V|^{p+1}+\Big(\frac{1}{p+1}-\frac{1}{p+1-\varepsilon}\Big)\int_D\mathrm{e}^{-\varepsilon x}|V|^{p+1-\varepsilon}\\
&\quad+\frac{1-c_\varepsilon}{p+1-\varepsilon}\int_D\mathrm{e}^{-\varepsilon x}|V|^{p+1-\varepsilon}+\frac{1}{p+1}\int_D\mathrm{e}^{-\varepsilon x}\big(|V|^{p+1}-|V|^{p+1-\varepsilon}\big)\\
&=I_0(V)-\frac{1}{p+1}\int_D\big(\mathrm{e}^{-\varepsilon x}-1\big)|V|^{p+1}+k\varepsilon a_1+\mathrm{o}(\varepsilon),
\end{aligned}\tag{4.17}$$

where

$$a_1:=\frac{\omega_{N-1}}{p+1}\Big[\Big(\frac{1}{p+1}+\frac{2}{p-1}\log\frac{p-1}{2}\Big)\int_{\mathbb{R}}W^{p+1}(x)\,\mathrm{d}x+\int_{\mathbb{R}}W^{p+1}(x)\log W(x)\,\mathrm{d}x\Big],$$

and from (4.6) we deduce

$$-\frac{1}{p+1}\int_D\big(\mathrm{e}^{-\varepsilon x}-1\big)|V|^{p+1}=\varepsilon\frac{1}{p+1}\int_D x|V|^{p+1}+\mathrm{o}(\varepsilon)=\varepsilon a_2\sum_{j=1}^k\xi_j+\mathrm{o}(\varepsilon),\tag{4.18}$$

where

$$a_2 := \omega_{N-1}\frac{1}{p+1}\int_{\mathbb{R}} W^{p+1}(x)\,\mathrm{d}x.$$

Next we note that

$$\begin{aligned}
&I_0(V) - \sum_{i=1}^{k} I_0(V_i) - \frac{1}{p+1}\int_D \left[\sum_{i=1}^{k} V_i^{p+1} - |V|^{p+1}\right] \\
&\quad = \frac{1}{2}\left(\frac{p-1}{2}\right)^2 \int_D \left[|\nabla_\Theta V|^2 + \frac{1}{2}\left|V'\right|^2 + |V|^2 - \sum_{i=1}^{k}\left(|\nabla_\Theta V_i|^2 + \frac{1}{2}|V_i'|^2 + |V_i|^2\right)\right] \\
&\quad = \sum_{\substack{i,j=1\\ i>j}}^{k} (-1)^{i+j}\int_D \left(\frac{p-1}{2}\right)^2 \nabla_\Theta V_i \nabla_\Theta V_j + V_i' V_j' + V_i V_j \\
&\quad = \sum_{\substack{i,j=1\\ i>j}}^{k} (-1)^{i+j}\int_D \left[-\left(\frac{p-1}{2}\right)^2 \Delta_{S^{N-1}} V_i - V_i'' + V_i\right] V_j \\
&\quad = \sum_{\substack{i,j=1\\ i>j}}^{k} (-1)^{i+j}\int_D W_i^p V_j,
\end{aligned}$$

so that

$$I_0(V) - \sum_{i=1}^{k} I_0(V_i) = \frac{1}{p+1}\int_D \left[\sum_{i=1}^{k} V_i^{p+1} - |V|^{p+1}\right] + \sum_{\substack{i,j=1\\ i>j}}^{k} (-1)^{i+j}\int_D W_i^p V_j.$$

Let χ_l and D_l be defined as in Lemma 4.4. Since $0 \leqslant V_i \leqslant W_i$ for all i, we can replace the letter W by V once or twice in the estimate (4.7), and thus we obtain

$$\begin{aligned}
I_0(V) - \sum_{i=1}^{k} I_0(V_i) &= \frac{1}{p+1}\sum_{l=1}^{k}\int_{D_l}\left[V_l^{p+1} - |V|^{p+1} + (p+1)\sum_{l>j}(-1)^{l+j} W_l^p V_j\right] + \mathrm{o}(\varepsilon) \\
&= \frac{1}{p+1}\sum_{l=1}^{k}\int_{D_l}\left[V_l^{p+1} - |V|^{p+1} + (p+1)\sum_{j\neq l}(-1)^{l+j} V_l^p V_j\right] \\
&\quad + \sum_{l=1}^{k}\sum_{j\neq l}(-1)^{l+j}\int_{D_l}\left(W_l^p - V_l^p\right)V_j - \sum_{l=1}^{k}\sum_{j>l}(-1)^{l+j}\int_{D_l} W_l^p V_j + \mathrm{o}(\varepsilon),
\end{aligned}$$

so that by (4.9) and (4.10) we have

$$I_0(V) - \sum_{i=1}^{k} I_0(V_i) = -\sum_{l=1}^{k}\sum_{j>l}(-1)^{l+j}\int_{D_l} W_l^p V_j + \mathrm{o}(\varepsilon).$$

Since moreover $\Pi_j = \mathrm{O}(\mathrm{e}^{-\xi_i}) = \mathrm{O}(\varepsilon^{3/2})$ for $j \geqslant 2$ uniformly in D, we have

$$0 \leqslant \int_{D_l} W_l^p (W_j - V_j) = \int_{D_l} W_l^p \Pi_j = \mathrm{o}(\varepsilon) \quad \text{for } j > l,$$

and we can use (4.7) and (4.8) to get

$$I_0(V) - \sum_{i=1}^{k} I_0(V_i) = -\sum_{l=1}^{k}\sum_{j>l}(-1)^{l+j}\int_{D_l} W_l^p W_j + \mathrm{o}(\varepsilon)$$

$$= \sum_{l=1}^{k-1}\int_{D_l} W_l^p W_{l+1} + \mathrm{o}(\varepsilon) = a_3\sum_{l=1}^{k-1}\mathrm{e}^{-|\xi_{l+1}-\xi_l|} + \mathrm{o}(\varepsilon). \tag{4.19}$$

Finally, by (2.11) we have

$$I_0(V_i) = \left(\frac{2}{N-2}\right)^{N-1} J_0(\omega_{\mu_i} + \pi_{\mu_i})$$

and it is well known that

$$J_0(\omega_{\mu_i} + \pi_{\mu_i}) = \frac{1}{N}\int_{\mathbb{R}^N}\omega_1^{p+1} + \frac{\alpha_N}{2}H(0,0)\int_{\mathbb{R}^N}\omega_1^p\big(\mu_i^{N-2}\big) + \mathrm{o}\big(\mu_i^{N-2}\big). \tag{4.20}$$

Therefore, since $\mu_1^{N-2} = \mathrm{e}^{-2\xi_1} = \mathrm{O}(\varepsilon)$ and $\mu_i^{N-2} = \mathrm{e}^{-2\xi_i} = \mathrm{o}(\varepsilon)$ for $i \geqslant 2$, we obtain

$$\sum_{i=1}^{k} I_0(V_i) = ka_4 + a_5 H(0,0)\mathrm{e}^{-2\xi_1} + \mathrm{o}(\varepsilon), \tag{4.21}$$

where

$$a_4 := \left(\frac{2}{N-2}\right)^{N-1}\frac{1}{N}\int_{\mathbb{R}^N}\omega_1^{p+1}(y)\,\mathrm{d}y \tag{4.22}$$

and

$$a_5 := \left(\frac{2}{N-2}\right)^{N-1}\frac{\alpha_N}{2}\int_{\mathbb{R}^N}\omega_1^{p}(y)\,\mathrm{d}y. \tag{4.23}$$

Combining (4.17)–(4.19) and (4.21) we deduce that

$$I_\varepsilon(V) = k\varepsilon a_1 + \varepsilon a_2\sum_{i=1}^{k}\xi_i + a_3\sum_{i=1}^{k-1}\mathrm{e}^{-|\xi_{i+1}-\xi_i|} + ka_4 + a_5 H(0,0)\mathrm{e}^{-2\xi_1} + \mathrm{o}(\varepsilon). \tag{4.24}$$

We note that, by (4.2) and (4.3), we get

$$\sum_{i=1}^{k}\xi_i = -\frac{k^2-2k+2}{2}\log\varepsilon + k\log\Lambda_1 - \sum_{i=2}^{k}(k-i+1)\log\Lambda_i, \tag{4.25}$$

$$\sum_{i=1}^{k-1}\mathrm{e}^{-|\xi_{i+1}-\xi_i|} = \varepsilon\sum_{i=2}^{k}\Lambda_i, \tag{4.26}$$

$$\mathrm{e}^{-2\xi_1} = \frac{\varepsilon}{\Lambda_1^2}. \tag{4.27}$$

The claim now follows from (4.24)–(4.27). □

5. Proof of main results

Let us complete the proof of the existence of sign changing-bubble tower solutions to problem (1.1).

Proof of Theorem 1.1. In virtue of Lemma 4.1, we need to find a critical point of the function $\tilde{I}_\varepsilon$. Performing the change of variables $\xi = \xi(\varepsilon, \Lambda)$ given in (4.2) and (4.3), it is sufficient to find a critical point of the function $\mathcal{I}_\varepsilon(\Lambda) = \varepsilon^{-1}(\tilde{I}_\varepsilon(\xi(\varepsilon,\Lambda)) - ka_4)$.

From Lemma 4.2 and Proposition 4.3, we get

$$\mathcal{I}_\varepsilon(\Lambda) = \psi_k(\Lambda) + c_1 + c_2 \log\varepsilon + \mathrm{o}(1),$$

where the term o(1) is uniform and c_1, c_2 are constants depending only on N and k.

We observe that the function ψ_k has a minimum point $\Lambda^* = (\Lambda_1^*, \dots, \Lambda_k^*)$ where Λ_1^* is the minimum on $(0, +\infty)$ of the function

$$\Lambda_1 \to a_5 \frac{H(0,0)}{\Lambda_1^2} + k a_2 \log \Lambda_1$$

and, for $i = 2, \dots, k$, Λ_i^* is the minimum on $(0, +\infty)$ of the function

$$\Lambda_i \to a_3 \Lambda_i - (k - i + 1) a_2 \log \Lambda_i.$$

Since Λ^* is stable with respect to uniform perturbation, for ε small enough there exists $\Lambda^\varepsilon = (\Lambda_1^\varepsilon, \dots, \Lambda_k^\varepsilon)$ critical point of $\mathcal{I}_\varepsilon(\Lambda)$, such that $\Lambda_i^\varepsilon \to \Lambda_i^*$ as $\varepsilon \to 0$, for $i = 1, \dots, k$.

Therefore, the point $\xi^\varepsilon = (\xi_1^\varepsilon, \dots, \xi_k^\varepsilon)$, where

$$\xi_1^\varepsilon = \log \frac{\Lambda_1^\varepsilon}{\varepsilon^{1/2}}, \qquad \xi_i^\varepsilon = \log \frac{\Lambda_1^\varepsilon}{\Lambda_2^\varepsilon \cdots \Lambda_i^\varepsilon \varepsilon^{(2i-1)/2}}, \quad i = 2, \dots, k,$$

is a critical point of $\tilde{I}_\varepsilon$ and the function $V + \phi(\xi^\varepsilon)$ is a solution to (2.3).

The claim follows, since $\mu_i = \mathrm{e}^{-\frac{2}{N-2}\xi_i} = M_i \varepsilon^{\frac{2i-1}{N-2}}$. □

Finally, let us prove the nodal properties of solutions found in Theorem 1.1.

Proof of Theorem 1.2. Let u_ε be a solution as in Theorem 1.1. We put

$$\begin{aligned}
\tilde{u}_\varepsilon(y) &= \sum_{i=1}^k (-1)^i \left(\frac{M_i \varepsilon^{\frac{2i-1}{N-2}}}{M_i^2 \varepsilon^{2\frac{2i-1}{N-2}} + |y|^2} \right)^{\frac{N-2}{2}} \\
&= \sum_{i=1}^k (-1)^i \left(\frac{1}{M_i \varepsilon^{\frac{2i-1}{N-2}} + M_i^{-1} \varepsilon^{-\frac{2i-1}{N-2}} |y|^2} \right)^{\frac{N-2}{2}}.
\end{aligned}$$

Then

$$u_\varepsilon(y) = \alpha_N \tilde{u}_\varepsilon(y)\big(1 + \mathrm{o}(1)\big), \quad y \in \Omega, \tag{5.1}$$

by Theorem 1.1, with $\mathrm{o}(1) \to 0$ uniformly on compact subsets of Ω. We consider the spheres $S_\varepsilon^j = \{y \in \mathbb{R}^N \colon |y| = \varepsilon^{\frac{2j-1}{N-2}}\}$, $j = 1, \dots, k$. We may fix a compact subset $K \subset \Omega$ such that $S_\varepsilon^j \subset K$ for $j = 1, \dots, k$ and $\varepsilon > 0$ sufficiently small. For $y \in S_\varepsilon^j$ we estimate

$$\begin{aligned}
\tilde{u}_\varepsilon(y) &= \sum_{i=1}^k (-1)^i \left(\frac{1}{M_i \varepsilon^{\frac{2i-1}{N-2}} + M_i^{-1} \varepsilon^{\frac{4j-2i-1}{N-2}}} \right)^{\frac{N-2}{2}} \\
&= \varepsilon^{-\frac{2j-1}{2}} \sum_{i=1}^k (-1)^i \left(\frac{1}{M_i \varepsilon^{\frac{2(i-j)}{N-2}} + M_i^{-1} \varepsilon^{\frac{2(j-i)}{N-2}}} \right)^{\frac{N-2}{2}} \\
&= (-1)^j \varepsilon^{-\frac{2j-1}{2}} \left(\left(\frac{1}{M_j + M_j^{-1}} \right)^{\frac{N-2}{2}} + \mathrm{o}(1) \right) \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Hence $(-1)^j \tilde{u}_\varepsilon > 0$ on S_ε^j for $j = 1, \dots, k$ and $\varepsilon > 0$ small, and together with (5.1) this implies that also $(-1)^j u_\varepsilon > 0$ on S_ε^j for ε small. Thus u_ε has at least k nodal domains $\Omega_1, \dots, \Omega_k$ such that Ω_j contains the sphere S_ε^j.

Next we show that u_ε has not more than k nodal domains for ε small. For this we recall that, by (2.11), Lemma 4.2, Proposition 4.3 and (4.22), we have

$$J_\varepsilon(u_\varepsilon) \to \frac{k}{N}\int\limits_{\mathbb{R}^N} \omega_1^{p+1}(y)\,\mathrm{d}y = \frac{k}{N}S^{N/2} \quad \text{as } \varepsilon \to 0, \tag{5.2}$$

where J_ε is defined in (2.10) and S is the best Sobolev constant for the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, i.e.,

$$S = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,\mathrm{d}y}{(\int_\Omega |u|^{p+1}\,\mathrm{d}y)^{2/(p+1)}}.$$

We put $c_\varepsilon := \inf_{u \in N_\varepsilon} J_\varepsilon$ for $\varepsilon \geqslant 0$, where N_ε is the Nehari manifold given by

$$N_\varepsilon = \left\{u \in H_0^1(\Omega) \setminus \{0\}\colon \int\limits_\Omega |\nabla u|^2 = \int\limits_\Omega |u|^{p+1-\varepsilon}\right\}.$$

It is well known and easy to see that $c_\varepsilon \to c_0 = \frac{1}{N}S^{N/2}$ as $\varepsilon \to 0$, and therefore

$$J_\varepsilon(u_\varepsilon) < (k+1)c_\varepsilon \quad \text{for } \varepsilon > 0 \text{ small} \tag{5.3}$$

by (5.2). We now suppose by contradiction that u_ε has at least $k+1$ pairwise different nodal domains $\Omega_1, \dots, \Omega_{k+1}$. We let χ_i be the characteristic function corresponding to the set Ω_i. Then $u_\varepsilon \chi_i \in H_0^1(\Omega)$ for $i = 1, \dots, k$ by [36, Lemma 1]. Moreover,

$$\begin{aligned}\int\limits_\Omega \left|\nabla(u_\varepsilon \chi_i)\right|^2 \mathrm{d}y &= \int\limits_\Omega \nabla u_\varepsilon \nabla(u_\varepsilon \chi_i)\,\mathrm{d}y = -\int\limits_\Omega \Delta u_\varepsilon(u_\varepsilon \chi_i)\,\mathrm{d}y \\ &= \int\limits_\Omega |u_\varepsilon|^{p-1-\varepsilon} u_\varepsilon \chi_i\,\mathrm{d}y = \int\limits_\Omega |u_\varepsilon \chi_i|^{p+1-\varepsilon}\,\mathrm{d}y,\end{aligned}$$

so that $u_\varepsilon \chi_i \in N_\varepsilon$. Since also $u_\varepsilon \in N_\varepsilon$, we obtain

$$\begin{aligned}J_\varepsilon(u_\varepsilon) &= \left(\frac{1}{2} - \frac{1}{p+1-\varepsilon}\right)\int\limits_\Omega |u_\varepsilon|^{p+1-\varepsilon}\,\mathrm{d}y \geqslant \left(\frac{1}{2} - \frac{1}{p+1-\varepsilon}\right)\sum_{i=1}^{k+1}\int\limits_\Omega |u_\varepsilon \chi_i|^{p+1-\varepsilon}\,\mathrm{d}y \\ &= \sum_{i=1}^{k+1} J_\varepsilon(u_\varepsilon \chi_i) \geqslant (k+1)c_\varepsilon,\end{aligned}$$

contrary to (5.3). The contradiction shows that u_ε has at most k nodal domains for ε small enough. This completes the proof of Theorem 1.2. □

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