Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 24 (2007) 171–205
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Anti-self-dual Lagrangians: Variational resolutions of non-self-adjoint equations and dissipative evolutions

Lagrangiens anti-autoduaux : Résolution variationnelle d'équations non-autoadjointes et de systèmes d'évolution dissipatifs

Nassif Ghoussoub [1]

Department of Mathematics, University of British Columbia, Vancouver BC, Canada V6T 1Z2

Received 21 January 2005; received in revised form 15 January 2006; accepted 15 February 2006

Available online 29 September 2006

Abstract

We develop the concept and the calculus of *anti-self-dual* (*ASD*) *Lagrangians* and their derived vector fields which seem inherent to many partial differential equations and evolutionary systems. They are natural extensions of gradients of convex functions – hence of self-adjoint positive operators – which usually drive dissipative systems, but also provide representations for the superposition of such gradients with skew-symmetric operators which normally generate unitary flows. They yield variational formulations and resolutions for large classes of non-potential boundary value problems and initial-value parabolic equations. Solutions are minima of newly devised energy functionals, however, and just like the self (and anti-self) dual equations of quantum field theory (e.g. Yang–Mills) the equations associated to such minima are not derived from the fact they are critical points of the functional I, but because they are also zeroes of suitably derived Lagrangians. The approach has many advantages: it solves variationally many equations and systems that cannot be obtained as Euler–Lagrange equations of action functionals, since they can involve non-self-adjoint or other non-potential operators; it also associates variational principles to variational inequalities, and to various dissipative initial-value first order parabolic problems. These equations can therefore be analyzed with the full range of methods – computational or not – that are available for variational settings. Most remarkable are the permanence properties that ASD Lagrangians possess making their calculus relatively manageable and their domain of applications quite broad.

Résumé

On introduit et développe la notion de *Lagrangien anti-autodual* qui apparait dans plusieurs problèmes de géométrie et de physique théorique. Cette classe inclut les champs de gradient de fonctions convexes qui sont à la base de systèmes dissipatifs, mais aussi la superposition de ces derniers avec les opérateurs anti-symétriques qui, par contre, engendrent des flots conservatifs. Comme pour les équations autoduales de Yang–Mills, ces Lagrangiens permettent la résolution variationnelle de plusieurs équations différentielles du premier ordre qui ne rentrent donc pas dans le cadre de la théorie de Euler–Lagrange. Les solutions

E-mail address: nassif@math.ubc.ca (N. Ghoussoub).

[1] Research partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada. The author gratefully acknowledges the hospitality and support of the Centre de Recherches Mathématiques in Montréal where this work was initiated and the Université de Nice where it was developed.

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2006.02.002

proviennent de minima de certaines (nouvelles) fonctionelles d'énergie, mais les équations ne sont pas derivées du fait qu'elles sont des points critiques, mais du fait qu'elles sont des racines de Lagrangiens positifs associés obtenus par une extension d'une astuce de Bogomolnyi. Cette nouvelle approche variationelle a plusieurs avantages, surtout qu'elle est applicable dans plusieurs situations, puisque la classe des Lagrangiens anti-autoadjoints est assez riche, étant stable – entre autres – par les opérations du calcul fonctionnel de l'analyse convexe, ainsi que celui des opérateurs anti-symétriques.

MSC: 35F10; 35J65; 47N10; 58E30

Keywords: Selfdual Lagrangians; Variational principles; Legendre–Fenchel duality; Anti-selfdual Lagrangians; Skew symmetry; Variational resolution

1. Introduction

Non-self-adjoint problems such as the *transport equation*:

$$\begin{cases} -\sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + a_0 u = |u|^{p-1}u + f & \text{on } \Omega \subset \mathbb{R}^n, \\ u(x) = 0 & \text{on } \Sigma_-, \end{cases} \tag{1}$$

where $\mathbf{a} = (a_i)_i : \overline{\Omega} \to \mathbb{R}^n$ is a smooth vector field, $p > 1$, $f \in L^2(\Omega)$, and $\Sigma_- = \{x \in \partial\Omega;\ \mathbf{a}(x) \cdot \mathbf{n}(x) < 0\}$, $\mathbf{n}$ being the outer normal on $\partial\Omega$, are not of Euler–Lagrange type and their solutions are not normally obtained as critical points of functionals of the form $\int_\Omega F(x, u(x), \nabla u(x))\,\mathrm{d}x$. Similarly, dissipative initial value problems such as the heat equation or those describing *porous media*:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u^m + f & \text{on } [0,T] \times \Omega, \\ u(t,x) = 0 & \text{on } [0,T] \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{on } \Omega, \end{cases} \tag{2}$$

cannot be solved by the standard methods of the calculus of variations since they do not correspond to Euler–Lagrange equations of action functionals of the form $\int_0^T L(t, x(t), \dot{x}(t))\,\mathrm{d}t$.

However, physicists have managed to formulate – if not solve – variationally many of the basic first order equations of quantum field theory by minimizing their associated action functionals. These are the celebrated self (anti-self) dual equations of Yang–Mills, Seiberg–Witten and Ginzburg–Landau which are not derived from the fact they are critical points (i.e., from the corresponding Euler–Lagrange equations) but from the fact that they are zeros of certain derived Lagrangian obtained by Bogomolnyi's trick of completing squares. But this is the case as long as the action functional attains a natural and a priori known minimum (see for example [19]).

From a totally different perspective, Brezis and Ekeland [7,8] formulated about 30 years ago an intriguing minimization principle which can be associated to the heat equation and other gradient flows of convex energy functionals. Again the applicability of their principle was conditional on identifying the minimum value of the functional. Later, Auchmuty [1,2] proposed a framework in which he formalizes and generalizes the Brezis–Ekeland procedure in order to apply it to operator equations of non-potential type. However, the applicability of this variational principle remained conditional on evaluating the minimum value and in most cases could not be used to establish existence and uniqueness of solutions. In this paper, we develop a variational framework where the infimum can be automatically identified, making such an approach applicable for the resolution of a wide array of partial differential equations, and evolutionary systems not normally covered by the standard Euler–Lagrange theory.

The basic idea is simple and is an elaboration on our work in [16] where we gave complete variational proofs of the existence and uniqueness of gradient flows of convex energy functionals, and the one in [9] where we give a variational proof for the existence and uniqueness of solutions of certain non-linear transport equations. Starting with an equation of the form

$$-Au \in \partial\varphi(u), \tag{3}$$

it is well known that it can be formulated – and sometimes solved – variationally whenever $A : X \to X^*$ is a self-adjoint bounded linear operator and φ is a differentiable or convex functional on X. Indeed, in this case it can be reduced to the equation $0 \in \partial\psi(u)$, where ψ is the functional $\psi(u) = \varphi(u) + \frac{1}{2}\langle Au, u\rangle$. A solution can then be obtained for example by minimization whenever φ is convex and A is positive.

But this variational procedure fails when A is not self-adjoint, or when A is a non-potential operator (i.e., when A is not a gradient vector field), and definitely when A is not linear. In this case, the Brezis–Ekeland procedure – as formalized by Auchmuty – consists of simply minimizing the functional

$$I(u) = \varphi(u) + \varphi^*(-Au) + \langle u, Au\rangle, \tag{4}$$

where φ^* is the Fenchel–Legendre dual of φ defined on X^* by $\varphi^*(p) = \sup\{\langle x, p\rangle - \varphi(x);\ x \in X\}$. The basic Legendre inequality yields that $\alpha := \inf_{u\in X} I(u) \geqslant 0$, and the following simple observation was made by several authors: if the infimum $\alpha = 0$ and if it is attained at $\bar{u} \in X$, then we are in the limiting case of the Fenchel–Legendre duality, $\varphi(\bar{u}) + \varphi^*(-A\bar{u}) = \langle \bar{u}, -A\bar{u}\rangle$ and therefore $-A\bar{u} \in \partial\varphi(\bar{u})$.

Note that the procedure does not require any assumption on A, and very general coercivity assumptions on φ often ensure the existence of a minimum. However, the difficulty here is different from standard minimization problems in that besides the problem of existence of a minimum, one has to insure that the infimum is actually zero. This is obviously not the case for general operators A, though one can always write (and many authors did) the variational principle (4) for the operator equation (3).

In this paper, we tackle the real difficulty of when the infimum α is actually zero and we identify a class of vector fields F for which the equation $F(u) = 0$ and the initial-value problem $\dot{u}(t) = F(u(t))$ can be formulated and solved variationally. Our method is based on the concept of *anti-self-dual Lagrangians* (ASD) which are simply lower semi-continuous convex functions (in both variables) L on the state space $X \times X^*$ that satisfy the following self-duality property:

$$L^*(p, x) = L(-x, -p) \quad \text{for all } (x, p) \in X \times X^*. \tag{5}$$

From such a Lagrangian, we derive an *anti-self-dual vector field* – denoted $\bar{\partial}L$ but not to be confused with the subdifferential ∂L of the convex function L – in such a way that stationary equations and initial-value problems of the form

$$\Lambda u \in \bar{\partial}L(u) \quad \text{and} \quad \begin{cases} \dot{u}(t) + \Lambda u(t) \in \bar{\partial}L(u(t)), \\ u(0) = u_0 \end{cases} \tag{6}$$

will be solved variationally by simply minimizing functionals of the form:

$$I(u) = L(u, \Lambda u) + \langle \Lambda u, u\rangle \quad \text{and} \quad I(u) = \int_0^T \bigl\{L\bigl(u(t), \dot{u}(t)\bigr) + \bigl\langle \Lambda u(t), u(t)\bigr\rangle\bigr\}\,\mathrm{d}t + \ell_{u_0}\bigl(u(0), u(T)\bigr) \tag{7}$$

for an appropriate boundary Lagrangian ℓ_{u_0}, the key point being that the infimum in both cases is actually zero. What is remarkable here is that the notion of ASD vector fields covers quite a large class of boundary value problems and evolution equations. A typical example is the vector field

$$F(u) = Bu + \partial\varphi(u) + f \tag{8}$$

where $B : X \to X^*$ is a skew-adjoint operator and $f \in X^*$. F is clearly not a potential field in the standard differential sense, yet we shall be able to write

$$F(u) = \bar{\partial}L(u), \tag{9}$$

where L is the ASD Lagrangian $L(x, p) = \varphi(x) + \langle f, x\rangle + \varphi^*(-Bx - p - f)$ and therefore resolve the equation $0 \in F(u)$ variationally. It is worth noting that ASD vector fields form a subset of the class of maximal monotone operators for which there is already an extensive theory [3,6]. The interesting point here however, is the fact that this subclass corresponds to what one might call "integrable maximal monotone operators", i.e., those to which one can associate a Lagrangian which allows for a variational resolution. The advantages of this class are numerous:

- It possesses remarkable permanence properties that maximal monotone operators either do not satisfy or do so via substantially more elaborate methods. It contains in particular subdifferentials of convex lower semi-continuous functions and their superposition with skew-adjoint operators, but more importantly, it enjoys all variational aspects of convex analysis, and is stable under similar type of operations making the calculus of ASD Lagrangians as manageable, yet much more encompassing.

- It is stable under the addition of appropriate boundary Lagrangians to an "interior" ASD Lagrangian allowing for the resolution of problems with various linear and non-linear boundary constraints that are not amenable to the standard variational theory.
- It allows for the lifting of ASD Lagrangians defined on state spaces to ASD Lagrangians on path spaces leading to a unified approach for stationary and dynamic equations. More precisely, ASD flows of the form $\dot{u}(t) \in \bar{\partial} L(u(t))$ with a variety of time-boundary conditions can be reformulated and resolved as $0 \in \bar{\partial}\mathcal{L}(u)$ where $\mathcal{L}$ is a corresponding ASD Lagrangian on path space, a phenomenon that leads to natural and quite interesting iterations (see [18]).
- The class of R-self-dual Lagrangians corresponding to a general automorphism R and their R-self-dual vector fields go beyond the theory of maximal monotone operators, as they include extensions of Hamiltonian systems and other twisted differential operators. The corresponding class of PDEs and their variational principles will be studied in [11].

In this paper, we establish the algebraic properties of ASD Lagrangians, emphasizing issues on how to build and identify complex ASD Lagrangians from the more basic ones. To keep the key ideas transparent, we chose to deal with cases when operators are linear and mostly bounded, leaving the more analytically involved cases of unbounded and non-linear operators to forthcoming papers. This case already has many interesting features and cover several basic partial differential equations/systems and evolutions. The theory however is evolving in many directions which are currently being developed in a series of papers:

- In [17], we give a thorough analysis of cases where the operators are not bounded, and more importantly, we establish the existence of "ASD flows" $\dot{u}(t) \in \bar{\partial} L(u(t))$ under minimal hypothesis on L. This setting includes all known results on gradient flows of convex functions but also more general parabolic equations involving first order operators such as transport, as well as certain Schrödinger equations.
- The dual notion of anti-symetric Hamiltonians is developed in [10], where we also deal with the case when the operator Λ – in (6) above – is non-linear providing applications to Navier–Stokes, and Choquard–Pekar equations as well as complex Ginsburg–Landau evolutions.
- Self-dual variational principles provide a natural approach to problems about Lagrangian intersections for Hamiltonian systems [13,18], about periodic and anti-periodic orbits for such systems [14], as well as quasi-periodic solutions of certain Schrödinger equations [15].
- The general theory of self-dual equations and their variational principles is developed in [11]. The full scope of R-selfduality where R is any automorphism of the space is exploited to cover yet a larger set of equations with old and new boundary conditions including Hamiltonian systems and non-linear Cauchy–Riemann equations.

In this introductory paper, we start by presenting – in Section 2 – the special variational properties of R-self-dual Lagrangians. This should already give an idea of their relevance in the existence theory of certain PDEs, and will hopefully motivate the study of their permanence properties. Beyond this first section, we shall only deal with the anti-symmetric case, i.e., when $R(x) = -x$, in which case R-self-dual Lagrangians will be called *anti-self-dual Lagrangians* (*ASD*). This class of Lagrangians already covers a great deal of applications which warranted that this paper as well as [10] and [17] be solely devoted to this case. In Section 3, we establish the basic permanence properties of anti-self-dual Lagrangians and in Section 4 we present their first special variational features while focusing on homogeneous stationary equations and systems. In Section 5 we deal with boundary value problems where compatible boundary Lagrangians are appropriately added to the "interior Lagrangian", in order to solve problems with prescribed boundary terms. In Section 6, we show how ASD Lagrangians "lift" to path spaces allowing us to solve with the same variational approach several parabolic equations – including gradient flows. In Section 7, we associate to each ASD vector field, a semi-group of contractions which emphasizes again how this class of vector filed is a natural extension of the superposition of dissipative subgradients of convex functions with conservative skew-adjoint operators.

This paper was meant to be mostly an introduction to the basics of the theory of ASD Lagrangians, so we stuck with the simplest of examples leaving more involved applications to forthcoming papers. At this stage, I would like to express my gratitude to Yann Brenier, Ivar Ekeland, Eric Séré, Abbas Moameni and Leo Tzou for the many extremely fruitful discussions and their extremely valuable input into this project.

2. A variational principle for R-self-dual Lagrangians

We consider the class $\mathcal{L}(X)$ of *convex Lagrangians* L on a reflexive Banach space X: these are all functions $L : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ which are convex and lower semi-continuous (in both variables) and which are not identically $+\infty$. The Legendre–Fenchel dual (in both variables) of L is defined at any $(q, y) \in X^* \times X$ by

$$L^*(q, y) = \sup\bigl\{\langle q, x\rangle + \langle y, p\rangle - L(x, p);\ x \in X, p \in X^*\bigr\}.$$

Definition 2.1. Given a bounded linear operator $R : X \to X$, say that:

(1) L is a *R-self-dual Lagrangian* on $X \times X^*$, if

$$L^*(p, x) = L(Rx, R^* p) \quad \text{for all } (p, x) \in X^* \times X. \tag{10}$$

(2) L is *partially R-self dual*, if

$$L^*(0, x) = L(Rx, 0) \quad \text{for all } x \in X. \tag{11}$$

(3) L is *anti-self dual on the graph of Λ*, the latter being a map from $D(\Lambda) \subset X$ into X^*, if

$$L^*(\Lambda x, x) = L(Rx, R^* \circ \Lambda x) \quad \text{for all } x \in D(\Lambda). \tag{12}$$

A typical example of an R^{-1}-self-dual Lagrangian is $L(x, p) = \varphi(Rx) + \varphi^*(p)$ and $M(x, p) = \varphi(x) + \varphi^*(R^* p)$ where φ is a convex lower semi-continuous function and R is an invertible operator on X. More generally, $L(x, p) = \varphi(Rx) + \varphi^*(S^* p)$ is an $(S \circ R)^{-1}$-self-dual Lagrangian. Moreover, if $\Lambda : X \to X^*$ is such that $\Lambda \circ (S \circ R)^{-1}$ is skew-adjoint, then

$$L(x, p) = \varphi(Rx) + \varphi^*(\Lambda x + S^* p)$$

is also an $(S \circ R)^{-1}$-self-dual Lagrangian.

Our basic premise in this paper is that many boundary value problems can be solved by minimizing functionals of the form $I(x) = L(x, \Lambda x)$ where L is a R-self-dual Lagrangian and provided $\Lambda \circ R$ is a skew-adjoint operator. However, their main relevance to our study stems from the fact that – generically – the infimum is actually equal to 0. It is this latter property that allows for novel variational formulations and resolutions of several basic PDEs and evolution equations, which – often because of lack of self-adjointness – do not normally fit the Euler–Lagrange framework.

As mentioned above, if L is a R-self-dual Lagrangian and if $\Lambda : X \to X^*$ is an operator such that $\Lambda \circ R$ is skew adjoint, then the Lagrangian $L_\Lambda(x, p) = L(x, \Lambda x + p)$ is again R-self-dual. In other words, Minimizing $L(x, \Lambda x)$ amounts to minimizing $L_\Lambda(x, 0)$ which is covered by the following very simple – yet far reaching – proposition. Again, its relevance comes from the evaluation of the minimum and not from the – more standard – question about its attainability.

We start by noticing that for a R-self-dual Lagrangian, we readily have:

$$L(Rx, R^* p) \geqslant \langle Rx, p\rangle \quad \text{for every } (x, p) \in X \times X^*, \tag{13}$$

and if L is partially anti-self-dual, then

$$I(x) = L(Rx, 0) \geqslant 0 \quad \text{for every } x \in X, \tag{14}$$

So, we are looking into an interesting variational situation, where the minima can also be zeros of the functionals. Here are some necessary conditions for the existence of such minima.

Proposition 2.1. *Let L be a convex lower-semi continuous functional on a reflexive Banach space $X \times X^*$. Assume that L is a partially R-self-dual Lagrangian for some automorphism R of X, and that for some $x_0 \in X$, the function $p \to L(x_0, p)$ is bounded above on a neighborhood of the origin in X^*. Then there exists $\bar{x} \in X$ such that*:

$$\begin{cases} L(R\bar{x}, 0) = \inf_{x \in X} L(x, 0) = 0, \\ (0, \bar{x}) \in \partial L(R\bar{x}, 0). \end{cases} \tag{15}$$

Proof. This follows from the basic duality theory in convex optimization. Indeed, if $(\mathcal{P}_p)$ is the primal minimization problem $h(p) = \inf_{x\in X} L(x, p)$ in such a way that $(\mathcal{P}_0)$ is the initial problem $h(0) = \inf_{x\in X} L(x, 0)$, then the dual problem $(\mathcal{P}^*)$ is $\sup_{y\in X} -L^*(0, y)$, and we have the weak duality formula

$$\inf \mathcal{P}_0 := \inf_{x\in X} L(x,0) \geqslant \sup_{y\in X} -L^*(0,y) := \sup \mathcal{P}^*.$$

The "partial R-selfduality" of L gives that

$$\inf_{x\in X} L(x,0) \geqslant \sup_{y\in X} -L^*(0,y) = \sup_{y\in X} -L(Ry,0). \tag{16}$$

Note that h is convex on X^* and that its Legendre conjugate satisfies $h^*(y) = L^*(0, y) = L(Ry, 0)$ on X. If now h is subdifferentiable at 0 (i.e., if the problem $(\mathcal{P}_0)$ is stable), then for any $\bar{x} \in \partial h(0)$, we have $h(0) + h^*(\bar{x}) = 0$, which means that

$$-\inf_{x\in X} L(x,0) = -h(0) = h^*(\bar{x}) = L^*(0,\bar{x}) = L(R\bar{x},0) \geqslant \inf_{x\in X} L(x,0).$$

It follows that $\inf_{x\in X} L(x, 0) = L(R\bar{x}, 0) = 0$ in view of (14), and that the infimum of $(\mathcal{P})$ is attained at $R\bar{x}$, while the supremum of $(\mathcal{P}^*)$ is attained at $\bar{x}$. In this case we can write

$$L(R\bar{x},0) + L^*(0,\bar{x}) = 0$$

which yields in view of the limiting case of Legendre duality, that $(0, \bar{x}) \in \partial L(R\bar{x}, 0)$.

If now for some $x_0 \in X$, the function $p \to L(x_0, p)$ is bounded above on a neighborhood of the origin in X^*, then $h(p) \leqslant \inf_{x\in X} L(x, p) \leqslant L(x_0, p)$ and therefore h is subdifferentiable at 0 and we are done. □

Remark 2.2.

(i) The above holds under the condition that $x \to L(Rx, 0)$ is coercive in the following sense:

$$\lim_{\|x\|\to\infty} \frac{L(Rx,0)}{\|x\|} = +\infty. \tag{17}$$

Indeed since $h^*(y) = L^*(0, y) = L(Ry, 0)$ on X, we get that h^* is coercive on X, which means that h is bounded above on neighborhoods of zero in X^*.

(ii) The proof above requires only that L is a Lagrangian satisfying

$$L^*(0,x) \geqslant L(Rx,0) \geqslant 0 \quad \text{for all } x \in X. \tag{18}$$

Now we can deduce the following

Theorem 2.3. *Let $R : X \to X$ be an automorphism on a reflexive Banach space X and let $\Lambda : X \to X^*$ be an operator such that $\Lambda \circ R$ is skew adjoint. Let L be a Lagrangian on $X \times X^*$ that is R-self-dual on the graph of $-\Lambda^*$, and assume that* $\lim_{\|x\|\to\infty} \frac{L(Rx,\Lambda Rx)}{\|x\|} = +\infty$. *Then there exists $\bar{x} \in X$, such that*:

$$\begin{cases} L(R\bar{x}, \Lambda R\bar{x}) = \inf_{x\in X} L(x, \Lambda x) = 0, \\ (-\Lambda^*\bar{x}, \bar{x}) \in \partial L(R\bar{x}, \Lambda R\bar{x}). \end{cases} \tag{19}$$

Proof. We first prove that the Lagrangian defined as $M(x, p) = L(x, \Lambda x + p)$ is partially R-self-dual. Indeed fix $(q, y) \in X^* \times X$, set $r = \Lambda x + p$ and write:

$$\begin{aligned} M^*(q,y) &= \sup\bigl\{\langle q,x\rangle + \langle y,p\rangle - L(x,\Lambda x+p);\ (x,p) \in X\times X^*\bigr\} \\ &= \sup\bigl\{\langle q,x\rangle + \langle y,r-\Lambda x\rangle - L(x,r);\ (x,r) \in X\times X^*\bigr\} \\ &= \sup\bigl\{\langle q-\Lambda^* y,x\rangle + \langle y,r\rangle - L(x,r);\ (x,r) \in X\times X^*\bigr\} \\ &= L^*(q-\Lambda^* y, y). \end{aligned}$$

If $q=0$, then $M^*(0,y)=L^*(-\Lambda^*y,y)=L(Ry,-R^*\Lambda^*y)=L(Ry,\Lambda Ry)=M(Ry,0)$, and M is therefore partially R-self-dual. It follows from the previous proposition applied to M, that there exists $\bar{x}\in X$ such that:

$$L(R\bar{x},\Lambda R\bar{x})=M(R\bar{x},0)=\inf_{x\in X}M(x,0)=\inf_{x\in X}L(x,\Lambda x)=0.$$

Now note that

$$L(R\bar{x},\Lambda R\bar{x})=L(R\bar{x},-R^*\Lambda^*\bar{x})=L^*(-\Lambda^*\bar{x},\bar{x}),$$

hence

$$L(R\bar{x},\Lambda R\bar{x})+L^*(-\Lambda^*\bar{x},\bar{x})=0=\big\langle(R\bar{x},\Lambda R\bar{x}),(-\Lambda^*\bar{x},\bar{x})\big\rangle.$$

It follows from the limiting case of Legendre duality that $(-\Lambda^*\bar{x},\bar{x})\in\partial L(R\bar{x},\Lambda R\bar{x})$. □

3. Basic properties of anti-self-dual Lagrangians

The concept of R-selfduality for a general automorphism R is relevant to many equations such as Hamiltonian systems and Cauchy–Riemann systems, and will be pursued in full generality in a forthcoming paper [11]. We shall however concentrate in the sequel on the class of *anti-self-dual Lagrangians* (*ASD*), meaning those R-self-dual Lagrangians corresponding to the inversion operator $R(x)=-x$. In other words,

(1) L is said to be an *anti-self-dual Lagrangian* on $X\times X^*$, if

$$L^*(p,x)=L(-x,-p)\quad\text{for all }(p,x)\in X^*\times X. \tag{20}$$

(2) L is *partially anti-self dual*, if

$$L^*(0,x)=L(-x,0)\quad\text{for all }x\in X. \tag{21}$$

(3) L is *anti-self dual on the graph of* Λ, the latter being a map from $D(\Lambda)\subset X$ into X^*, if

$$L^*(\Lambda x,x)=L(-x,-\Lambda x)\quad\text{for all }x\in D(\Lambda). \tag{22}$$

(4) More generally, if $Y\times Z$ is any subset of $X\times X^*$, we shall say that L is *anti-self dual on the elements of* $Y\times Z$ if $L^*(p,x)=L(-x,-p)$ for all $(p,x)\in Y\times Z$.

Denote by $\mathcal{L}_{\mathrm{AD}}(X)$ the class of anti-self-dual (ASD) Lagrangians on a given Banach space X. We shall see that this is already a very interesting and natural class of Lagrangians as they appear in several basic PDEs and evolution equations. The basic example of an anti-self-dual Lagrangian is given by a function L on $X\times X^*$, of the form

$$L(x,p)=\varphi(x)+\varphi^*(-p) \tag{23}$$

where φ is a convex and lower semi-continuous function on X and φ^* is its Legendre conjugate on X^*. We shall call them the *Basic ASD-Lagrangians*. A key element of this theory is that the family of ASD Lagrangians is much richer and goes well beyond convex functions and their conjugates, since they are naturally compatible with skew-symmetric operators. Indeed if $\Lambda:X\to X^*$ is skew-symmetric (i.e., $\Lambda^*=-\Lambda$), the Lagrangian

$$M(x,p)=\varphi(x)+\varphi^*(-\Lambda x-p) \tag{24}$$

is also anti-self dual, and if in addition Λ is invertible then the same holds true for

$$N(x,p)=\varphi\big(x+\Lambda^{-1}p\big)+\varphi^*(\Lambda x). \tag{25}$$

3.1. Permanence properties of ASD Lagrangians

The class $\mathcal{L}_{\mathrm{AD}}(X)$ enjoys a remarkable number of permanence properties. Indeed, we define on the class of Lagrangians $\mathcal{L}(X)$ the following operations:

Scalar multiplication: If $\lambda > 0$ and $L \in \mathcal{L}(X)$, define the Lagrangian $\lambda \cdot L$ on $X \times X^*$ by

$$(\lambda \cdot L)(x, p) = \lambda^2 L\left(\frac{x}{\lambda}, \frac{p}{\lambda}\right).$$

Addition: If $L, M \in \mathcal{L}(X)$, define the sum $L + M$ on $X \times X^*$ by

$$(L \oplus M)(x, p) = \inf\{L(x, r) + M(x, p - r); r \in X^*\}.$$

Convolution: If $L, M \in \mathcal{L}(X)$, define the convolution $L \star M$ on $X \times X^*$ by

$$(L \star M)(x, p) = \inf\{L(z, p) + M(x - z, p); z \in X\}.$$

Right operator shift: If $L \in \mathcal{L}(X)$ and $\Lambda : X \to X^*$ is a bounded linear operator, define the Lagrangian L_Λ on $X \times X^*$ by

$$L_\Lambda(x, p) := L(x, \Lambda x + p).$$

Left operator shift: If $L \in \mathcal{L}(X)$ and if $\Lambda : X \to X^*$ is an invertible operator, define the Lagrangian ${}_\Lambda L$ on $X \times X^*$ by

$${}_\Lambda L(x, p) := L\big(x + \Lambda^{-1} p, \Lambda x\big).$$

Free product: If $\{L_i;\ i \in I\}$ is a finite family of Lagrangians on reflexive Banach spaces $\{X_i;\ i \in I\}$, define the Lagrangian $L := \sum_{i \in I} L_i$ on $(\prod_{i \in I} X_i) \times (\prod_{i \in I} X_i^*)$ by

$$L\big((x_i)_i, (p_i)_i\big) = \sum_{i \in I} L_i(x_i, p_i).$$

Twisted product: If $L \in \mathcal{L}(X)$ and $M \in \mathcal{L}(Y)$ where X and Y are two reflexive spaces, then for any bounded linear operator $A : X \to Y^*$, define the Lagrangian $L \oplus_A M$ on $(X \times Y) \times (X^* \times Y^*)$ by

$$(L \oplus_A M)\big((x, y), (p.q)\big) := L(x, A^* y + p) + M(y, -Ax + q).$$

Anti-dualisation: If φ is any convex function on $X \times Y$ and A is any bounded linear operator $A : X \to Y^*$, define the Lagrangian $L \oplus_{\mathrm{as}} A$ on $(X \times Y) \times (X^* \times Y^*)$ by

$$\varphi \oplus_{\mathrm{as}} A\big((x, y), (p.q)\big) = \varphi(x, y) + \varphi^*(-A^* y - p, Ax - q).$$

The above defined convolution operation should not be confused with the standard convolution for L and M as convex functions in both variables. It is easy to see that in the case where $L(x, p) = \varphi(x) + \varphi^*(-p)$ and $M(x, p) = \psi(x) + \psi^*(-p)$, addition corresponds to taking

$$(L \oplus M)(x, p) = (\varphi + \psi)(x) + \varphi^* \star \psi^*(-p)$$

while convolution reduces to:

$$(L \star M)(x, p) = (\varphi \star \psi)(x) + (\varphi^* + \psi^*)(-p),$$

which also means that they are dual operations. We do not know whether this is true in general, but for the sequel we shall only need the following:

Lemma 3.1. *Let X be a reflexive Banach space and consider two Lagrangians L and M in $\mathcal{L}(X)$. Then the following hold*:

(1) $(L \oplus M)^* \leqslant L^* \star M^*$ *and* $(L \star M)^* \leqslant L^* \oplus M^*$.
(2) *If L or M is a basic ASD Lagrangian, then* $(L \oplus M)^* = L^* \star M^*$ *and* $(L \star M)^* = L^* \oplus M^*$.
(3) *If L and M are in $\mathcal{L}_{\mathrm{AD}}(X)$, then $L^* \oplus M^*(q, y) = L \star M(-y, -q)$ for every $(y, q) \in X \times X^*$.*

Proof. To prove (1) fix $(q, y) \in X^* \times X$ and write:

$$\begin{aligned}
&(L\star M)^*(q,y)\\
&\quad=\sup\{\langle q,x\rangle+\langle y,p\rangle-L(z,p)-M(x-z,p);\,(z,x,p)\in X\times X\times X^*\}\\
&\quad=\sup\{\langle q,v+z\rangle+\langle y,p\rangle-L(z,p)-M(v,p);\,(z,v,p)\in X\times X\times X^*\}\\
&\quad=\sup\{\langle q,v+z\rangle+\sup\{\langle y,p\rangle-L(z,p)-M(v,p);\,p\in X^*\};\,(z,v)\in X\times X\}\\
&\quad=\sup_{(z,v)\in X\times X}\Big\{\langle q,v+z\rangle+\inf_{w\in X}\Big\{\sup_{p_1\in X^*}\big(\langle w,p_1\rangle-L(z,p_1)\big)+\sup_{p_2\in X^*}\big(\langle y-w,p_2\rangle-M(v,p_2)\big)\Big\}\Big\}\\
&\quad\leqslant\inf_{w\in X}\Big\{\sup_{(z,p_1)\in X\times X^*}\{\langle q,z\rangle+\langle w,p_1\rangle-L(z,p_1)\}+\sup_{(v,p_2)\in X\times X^*}\{\langle q,v\rangle+\langle y-w,p_2\rangle-M(v,p_2)\}\Big\}\\
&\quad=\inf_{w\in X}\{L^*(q,w)+M^*(q,y-w)\}\\
&\quad=(L^*\oplus M^*)(q,y).
\end{aligned}$$

For (2) assume that $M(x,p)=\varphi(x)+\varphi^*(-p)$ where φ is a convex lower semi-continuous function. Fix $(q,y)\in X^*\times X$ and write:

$$\begin{aligned}
(L\star M)^*(q,y)&=\sup\{\langle q,x\rangle+\langle y,p\rangle-L(z,p)-M(x-z,p);\,(z,x,p)\in X\times X\times X^*\}\\
&=\sup\{\langle q,v+z\rangle+\langle y,p\rangle-L(z,p)-M(v,p);\,(z,v,p)\in X\times X\times X^*\}\\
&=\sup_{p\in X^*}\Big\{\langle y,p\rangle+\sup_{(z,v)\in X\times X}\{\langle q,v+z\rangle-L(z,p)-\varphi(v)\}-\varphi^*(-p)\Big\}\\
&=\sup_{p\in X^*}\Big\{\langle y,p\rangle+\sup_{z\in X}\{\langle q,z\rangle-L(z,p)\}+\sup_{v\in X}\{\langle q,v\rangle-\varphi(v)\}-\varphi^*(-p)\Big\}\\
&=\sup_{p\in X^*}\Big\{\langle y,p\rangle+\sup_{z\in X}\{\langle q,z\rangle-L(z,p)\}+\varphi^*(q)-\varphi^*(-p)\Big\}\\
&=\sup_{p\in X^*}\sup_{z\in X}\{\langle y,p\rangle+\langle q,z\rangle-L(z,p)-\varphi^*(-p)\}+\varphi^*(q)\\
&=(L+T)^*(q,y)+\varphi^*(q),
\end{aligned}$$

where $T(z,p):=\varphi^*(-p)$ for all $(z,p)\in X\times X^*$. Note now that

$$T^*(q,y)=\sup_{z,p}\{\langle q,z\rangle+\langle y,p\rangle-\varphi^*(-p)\}=\begin{cases}+\infty & \text{if } q\neq 0,\\ \varphi(-y) & \text{if } q=0\end{cases}$$

in such a way that by using the duality between sums and convolutions in both variables, we get

$$\begin{aligned}
(L+T)^*(q,y)&=\operatorname{conv}(L^*,T^*)(q,y)\\
&=\inf_{r\in X^*,\,z\in X}\{L^*(r,z)+T^*(-r+q,-z+y)\}\\
&=\inf_{z\in X}\{L^*(q,z)+\varphi(z-y)\}
\end{aligned}$$

and finally

$$\begin{aligned}
(L\star M)^*(q,y)&=(L+T)^*(q,y)+\varphi^*(q)\\
&=\inf_{z\in X}\{L^*(q,z)+\varphi(z-y)\}+\varphi^*(q)\\
&=\inf_{z\in X}\{L^*(q,z)+\varphi^*(q)+\varphi(z-y)\}\\
&=(L^*\oplus M^*)(q,y).
\end{aligned}$$

The rest follows in the same way. For (3) write

$$\begin{aligned}
(L^*\oplus M^*)(q,y)&=\inf_{w\in X}\{L^*(q,w)+M^*(q,y-w)\}\\
&=\inf_{w\in X}\{L(-w,-q)+M(w-y,-q)\}\\
&=(L\star M)(-y,-q).\qquad\square
\end{aligned}$$

The following proposition summarizes some of the remarkable permanence properties of ASD Lagrangians.

Proposition 3.1. *Let X be a reflexive Banach space, then the following holds*:

(1) *If L is in $\mathcal{L}_{\mathrm{AD}}(X)$, then $L^* \in \mathcal{L}_{\mathrm{AD}}(X^*)$, and if $\lambda > 0$, then $\lambda \cdot L$ also belong to $\mathcal{L}_{\mathrm{AD}}(X)$.*
(2) *If L is in $\mathcal{L}_{\mathrm{AD}}(X)$, then for any $y \in X$ and $q \in X^*$, the translated Lagrangians M_y and N_p defined respectively by $M_y(x,p) = L(x+y,p) + \langle y,p\rangle$ and $N_q(x,p) = L(x,p+q) + \langle x,q\rangle$ are also in $\mathcal{L}_{\mathrm{AD}}(X)$.*
(3) *If L and M are in $\mathcal{L}_{\mathrm{AD}}(X)$ and one of them is basic, then the Lagrangians $L \oplus M$, and $L \star M$ also belong to $\mathcal{L}_{\mathrm{AD}}(X)$.*
(4) *If $L_i \in \mathcal{L}_{\mathrm{AD}}(X_i)$ where X_i is a reflexive Banach space for each $i \in I$, then $\sum_{i\in I} L_i$ is in $\mathcal{L}_{\mathrm{AD}}(\prod_{i\in I} X_i)$.*
(5) *If $L \in \mathcal{L}_{\mathrm{AD}}(X)$ and $\Lambda : X \to X^*$ is a skew-adjoint bounded linear operator (i.e., $\Lambda^* = -\Lambda$), then the Lagrangian L_Λ is also in $\mathcal{L}_{\mathrm{AD}}(X)$.*
(6) *If $L \in \mathcal{L}_{\mathrm{AD}}(X)$ and if $\Lambda : X \to X^*$ is an invertible skew-adjoint operator, then the Lagrangian ${}_\Lambda L$ is also in $\mathcal{L}_{\mathrm{AD}}(X)$.*
(7) *If $L \in \mathcal{L}_{\mathrm{AD}}(X)$ and $M \in \mathcal{L}_{\mathrm{AD}}(Y)$, then for any bounded linear operator $A : X \to Y^*$, the Lagrangian $L \oplus_A M$ belongs to $\mathcal{L}_{\mathrm{AD}}(X \times Y)$.*
(8) *If φ is a proper convex lower semi-continuous function on $X \times Y$ and A is any bounded linear operator $A : X \to Y^*$, then $\varphi \oplus_{\mathrm{as}} A$ belongs to $\mathcal{L}_{\mathrm{AD}}(X \times Y)$.*
(9) *If L is in $\mathcal{L}_{\mathrm{AD}}(X)$ and if U is a unitary operator ($U^{-1} = U^*$), then $M(x,p) = L(Ux, Up)$ also belongs to $\mathcal{L}_{\mathrm{AD}}(X)$.*

Proof. (1) and (2) are straightforward, while (3) follows from the above lemma and (4) is obvious. (5) has been shown in the proof of Theorem 2.3. For (6) let $r = x - \Lambda^{-1}p$ and $s = \Lambda x$ and write

$$
\begin{aligned}
{}_\Lambda L^*(q,y) &= \sup\bigl\{\langle q,x\rangle + \langle y,p\rangle - L\bigl(x - \Lambda^{-1}p, \Lambda x\bigr); (x,p) \in X \times X^*\bigr\} \\
&= \sup\bigl\{\bigl\langle q, \Lambda^{-1}s\bigr\rangle + \langle y, s - \Lambda r\rangle - L(r,s); (r,s) \in X \times X^*\bigr\} \\
&= \sup\bigl\{\bigl\langle -\Lambda^{-1}q + y, s\bigr\rangle + \langle \Lambda y, r\rangle - L(r,s); (r,s) \in X \times X^*\bigr\} \\
&= L^*\bigl(\Lambda y, -\Lambda^{-1}q + y\bigr) = L\bigl(-y + \Lambda^{-1}q, -\Lambda y\bigr) \\
&= {}_\Lambda L(-y,-q).
\end{aligned}
$$

For (7), it is enough to notice that for $(\tilde{x}, \tilde{p}) \in (X \times Y) \times (X^* \times Y^*)$, we can write

$$L \oplus_A M(\tilde{x}, \tilde{p}) = (L + M)(\tilde{x}, \tilde{A}\tilde{x} + \tilde{p}),$$

where $\tilde{A} : X \times Y \to X^* \times Y^*$ is the skew-adjoint operator defined by $\tilde{A}(\tilde{x}) = \tilde{A}((x,y)) = (A^*y, -Ax)$. Assertion (8) follows again from (5) since

$$\varphi \oplus_{\mathrm{as}} A\bigl((x,y),(p,q)\bigr) = \varphi(x,y) + \varphi^*(-A^*y - p, Ax - q) = L_{\tilde{A}}\bigl((x,y),(p,q)\bigr)$$

where $L((x,y),(p.q)) := \varphi(x,y) + \varphi^*(-p,-q)$ is obviously in $\mathcal{L}_{\mathrm{AD}}(X \times Y)$ and where $\tilde{A} : X \times Y \to X^* \times Y^*$ is again the skew-adjoint operator defined by $\tilde{A}((x,y)) = (A^*y, -Ax)$. □

Remark 3.2.

(i) The proofs of (5) and (6) above clearly show that L_Λ (resp., ${}_\Lambda L$) is partially anti-self-dual if and only if L is anti-self-dual on the graph of Λ.
(ii) An important use of the above proposition is when $M_\lambda(x,p) = \|x\|^2/(2\lambda^2) + \lambda^2\|p\|^2/2$, then $L_\lambda = L \star M_\lambda$ is a λ-regularization of the Lagrangian L, which is reminiscent of the Yosida theory for operators and for convex functions. This will be most useful in [10] and [17].

3.2. Anti-self-dual vector fields

Definition 3.3. Let X be a reflexive Banach space and let $L : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semi-continuous function, that is not identically equal to $+\infty$.

(1) The partial domains of L are defined as

$$\mathrm{Dom}_1(L) = \{x \in X; L(x,p) < +\infty \text{ for some } p \in X^*\} \quad \text{and}$$
$$\mathrm{Dom}_2(L) = \{p \in X^*; L(x,p) < +\infty \text{ for some } x \in X\}.$$

(2) The *derived ASD vector fields of* L at $x \in X$ (resp., $p \in X^*$) are the – possibly empty – sets

$$\bar{\partial} L(x) := \{p \in X^*;\ L(x,p) + \langle x,p\rangle = 0\} \quad \text{resp.,}\ \tilde{\partial} L(p) := \{x \in X;\ L(x,p) + \langle x,p\rangle = 0\}.$$

(3) The domains of the *ASD* vector field $\bar{\partial} L$ are the sets

$$\mathrm{Dom}(\bar{\partial} L) = \{x \in X; \bar{\partial} L(x) \neq \emptyset\} \quad \text{resp.,} \quad \mathrm{Dom}(\tilde{\partial} L) = \{p \in X^*; \tilde{\partial} L(p) \neq \emptyset\}.$$

The above defined potentials should not be confused with the subdifferential ∂L of L as a convex function on $X \times X^*$. It is easy to see that if L is an anti-self-dual Lagrangian, then we have

$$p \in \bar{\partial} L(x) \quad \text{iff} \quad x \in \tilde{\partial} L(p) \quad \text{iff} \quad (-p,-x) \in \partial L(x,p) \quad \text{iff} \quad 0 \in \bar{\partial} L_p(\bar{x}),$$

where $L_p(x,q) = L(x,p+q) + \langle x,p\rangle$. For a basic ASD Lagrangian $L(x,p) = \varphi(x) + \varphi^*(-p)$, it is clear that

$$\bar{\partial} L(x) = -\partial\varphi(x) \quad \text{while} \quad \tilde{\partial} L(p) = \partial\varphi^*(-p).$$

Recalling that an ASD Lagrangian L satisfies $L(x,p) + \langle x,p\rangle \geqslant 0$ for every $(x,p) \in X \times X^*$, the problem of proving that $\tilde{\partial} L(p)$ is non-empty for a given $p \in X^*$ amounts to showing that the infimum of the functional $I_p(x) = L(x,p) + \langle x,p\rangle$ over $x \in X$ is equal to zero and is attained. Proposition 2.1 applied to translated Lagrangians then give the following simple but far-reaching proposition.

Proposition 3.2. *Let L be an anti-self-dual Lagrangian L on a reflexive Banach space $X \times X^*$. The following assertions hold*:

(1) *If $q \mapsto L(0,q)$ is bounded on a ball of X^*, then for each $p \in X^*$ the infimum of $I_p(x) = L(x,p) + \langle x,p\rangle$ over X is equal to zero and is attained at some point $\bar{x} \in X$ so that $p \in \bar{\partial} L(\bar{x})$.*
(2) *The ASD vector field $x \to -\bar{\partial} L(x)$ is monotone in the sense that $\langle x-y, p-q\rangle \geqslant 0$ for $p \in -\bar{\partial} L(x)$ and $q \in -\bar{\partial} L(y)$.*
(3) *If L is strictly convex in the second variable, then the ASD vector field $x \to \bar{\partial} L(x)$ is single-valued.*
(4) *If L is uniformly convex in the second variable (i.e., if $L(x,p) - \epsilon\|p\|^2/2$ is convex in p for some $\epsilon > 0$) then the ASD vector field $x \to \bar{\partial} L(x)$ is a Lipschitz maximal monotone map.*

Corresponding dual statements about $p \to \tilde{\partial} L(p)$ also hold. The proof is straightforward provided one applies Theorem 2.1 with the automorphism $R(x) = -x$ and the translated Lagrangians M_x and N_p which are also anti-self-dual by Proposition 3.1. The details are left to the interested reader. For the theory of maximal monotone operators see [5].

As noted above, non-trivial examples of anti-self-dual Lagrangians are of the form

$$L(x,p) = \varphi(x) + \varphi^*(-\Lambda x - p), \tag{26}$$

where φ is a convex and lower semi-continuous function on X, φ^* is its Legendre conjugate on X^* and where $\Lambda : X \to X^*$ is skew-symmetric. In this case, it is easy to see that

$$\bar{\partial} L(x) = -\Lambda x - \partial\varphi(x) \quad \text{while} \quad \tilde{\partial} L(p) = (-\Lambda - \partial\varphi)^{-1}(p).$$

This suggests that ASD vector fields are natural extensions of operators of the form $A + \partial\varphi$, where A is positive and φ is convex. This is an important subclass of maximal monotone operators which can now be resolved variationally. Indeed, by considering the cone $\mathcal{C}(X)$ of bounded below, proper convex lower semi-continuous functions on X, and $\mathcal{A}(X)$ the cone of positive bounded linear operators from X into X^* (i.e., $\langle Ax,x\rangle \geqslant 0$ for all $x \in X$).

Proposition 3.3. *One can associate to any pair $(\varphi, A) \in \mathcal{C}(X) \times \mathcal{A}(X)$, a Lagrangian $L := L_{(\varphi,A)} \in \mathcal{L}_{\mathrm{AD}}(X)$ such that for $p \in X^*$, the following are equivalent*:

(1) *The equation* $Ax + \partial\varphi(x) = -p$ *has a solution* $\bar{x} \in X$.
(2) *The functional* $I_p(x) = L(x, p) + \langle x, p\rangle$ *attains its infimum at* $\bar{x} \in X$.
(3) $p \in \bar{\partial} L(\bar{x})$.

Proof. Indeed, one can associate to each pair $(\varphi, A) \in \mathcal{C}(X) \times \mathcal{A}(X)$, the anti-self-dual Lagrangian

$$L_{(\varphi,A)}(x, p) = \psi(x) + \psi^*\big(-A^a x - p\big) \quad \text{for any } (x, p) \in X \times X^*,$$

where $\psi(x) = \frac{1}{2}\langle Ax, x\rangle + \varphi(x)$, $A^a = \frac{1}{2}(A - A^*)$ is the anti-symmetric part of A, and $A^s = \frac{1}{2}(A + A^*)$ is its symmetric part. The fact that the minimum of $I(x) = \psi(x) + \psi^*(-A^a x - p) + \langle x, p\rangle$ is equal to 0 and is attained at some $\bar{x} \in X$ means that $\psi(\bar{x}) + \psi^*(-A^a\bar{x} - p) = 0 = -\langle A^a\bar{x} + p, \bar{x}\rangle$ which yields, in view of Legendre–Fenchel duality that $-A^a\bar{x} - p \in \partial\psi(\bar{x}) = A^s\bar{x} + \partial\varphi(\bar{x})$, hence $\bar{x}$ satisfies $-Ax - p \in \partial\varphi(x)$. □

4. Self-dual variational principles for homogeneous boundary value problems

An immediate corollary of Theorem 2.4 in the special case of ASD Lagrangians is the following result which will be used repeatedly in the sequel.

4.1. A non-linear Lax–Milgram theorem for homogeneous equations

Theorem 4.1. *Let* $\Lambda : X \to X^*$ *be a bounded linear skew-adjoint operator on a reflexive Banach space* X*, and let* L *be an anti-self-dual Lagrangian on the graph of* Λ*. Assume one of the following hypothesis*:

(A) $\lim_{\|x\|\to\infty} \frac{L(x,\Lambda x)}{\|x\|} = +\infty$, *or*
(B) *The operator* Λ *is invertible and the map* $x \to L(x, 0)$ *is bounded above on the ball of* X.

Then there exists $\bar{x} \in X$*, such that*:

$$\begin{cases} L(\bar{x}, \Lambda\bar{x}) = \inf_{x\in X} L(x, \Lambda x) = 0, \\ \Lambda\bar{x} \in \bar{\partial} L(\bar{x}). \end{cases} \tag{27}$$

Proof. It suffices to apply Theorem 2.3 with $R(x) = -x$. In the case where Λ is also invertible, then we directly apply Proposition 2.1 to the Lagrangian ${}_\Lambda L(x, p) = L(x + \Lambda^{-1} p, \Lambda x)$ which is partially anti-self-dual. □

Now we can a variational resolution to the following non-linear Lax–Milgram type result.

Corollary 4.2. *Assume one of the following conditions on a pair* $(\varphi, A) \in \mathcal{C}(X) \times \mathcal{A}(X)$:

(A) $\lim_{\|x\|\to\infty} \|x\|^{-1}(\varphi(x) + \frac{1}{2}\langle Ax, x\rangle) = +\infty$, *or*
(B) *The operator* $A^a = \frac{1}{2}(A - A^*) : X \to X^*$ *is onto and* φ *is bounded above on the ball of* X.

Then, there exists for any $f \in X^*$*, a solution* $\bar{x} \in X$ *to the equation* $-Ax + f \in \partial\varphi(x)$ *that can be obtained as a minimizer of the problem*:

$$\inf_{x\in X} \big\{\psi(x) + \psi^*\big(-A^a x\big)\big\} = 0, \tag{28}$$

where ψ *is the convex functional* $\psi(x) = \frac{1}{2}\langle Ax, x\rangle + \varphi(x) - \langle f, x\rangle$.

Note that all what is needed in the above proposition is that the function $\psi(x) = \frac{1}{2}\langle Ax, x\rangle + \varphi(x)$ is convex and lower semi-continuous.

Example 1 *(A variational formulation for the Lax–Milgram theorem).* Given a bilinear continuous functional a on a Banach space X, and assuming that a is coercive: i.e., for some $\lambda > 0$, we have that $a(v, v) \geqslant \lambda\|v\|^2$ for every $v \in X$.

It is well known that if a is symmetric, then for any $f \in X^*$, we can use a variational approach to find $u \in X$, such that for every $v \in X$, we have $a(u,v) = \langle v, f\rangle$. The procedure amounts to minimizing on H the convex functional $\psi(u) = \frac{1}{2}a(u,u) - \langle u, f\rangle$.

The theorem of Lax–Milgram deals with the case when a is not symmetric, for which the above variational argument does not work. Theorem 4.1 however yields the following variational formulation and proof of the original Lax–Milgram theorem.

Corollary 4.3. *Let a be a coercive continuous bilinear form on $X \times X$. For any $f \in X^*$, consider the functional $I(v) = \psi(v) + \psi^*(-\Lambda v)$ where $\psi(v) = \frac{1}{2}a(v,v) - \langle v, f\rangle$, ψ^* its Legendre conjugate and where $\Lambda : X \to X^*$ is the skew-adjoint operator defined by $\langle \Lambda v, w\rangle = \frac{1}{2}(a(v,w) - a(w,v))$. Then there exists $u \in X$ such that*

$$I(u) = \inf_{v\in H} I(v) = 0 \quad \textit{and} \quad a(u,v) = \langle v, f\rangle \quad \textit{for every } v \in X.$$

Proof. Consider the ASD Lagrangian $L(x,p) = \psi(x) + \psi^*(-p)$ and apply Theorem 4.1 to $I(u) = L(u, \Lambda u)$. Note that $L(u, \Lambda u) = 0$ if and only if $\psi(u) + \psi^*(-\Lambda u) = 0 = -\langle \Lambda u, u\rangle$ which means that $-\Lambda u \in \partial\psi(u)$. In other words, we have for every $v \in X$

$$-\frac{1}{2}\big(a(u,v) - a(v,u)\big) = \frac{1}{2}\big(a(u,v) + a(v,u)\big) - \langle v, f\rangle,$$

which yields our claim. □

Example 2 *(Inverting variationally a non-self-adjoint matrix).* An immediate finite dimensional application of the above corollary is the following variational solution for the linear equation $Ax = y$ where A is an $n \times n$-matrix and $y \in \mathbb{R}^n$. It then suffices to minimize

$$I(x) = \frac{1}{2}\langle Ax, x\rangle + \frac{1}{2}\big\langle A_s^{-1}(y - A_a x), y - A_a x\big\rangle - \langle y, x\rangle,$$

on $\mathbb{R}^n$, where A_a is the anti-symmetric part of A and A_s^{-1} is the inverse of the symmetric part. If A is coercive, i.e., $\langle Ax, x\rangle \geqslant c|x|^2$ for all $x \in \mathbb{R}^n$, then there is a solution $\bar{x} \in \mathbb{R}^n$ to the equation obtained as $I(\bar{x}) = \inf_{x\in\mathbb{R}^n} I(x) = 0$.

Example 3 *(A variational principle for a non-symmetric Dirichlet problem).* Let $\mathbf{a} : \Omega \to \mathbb{R}^n$ be a smooth function on a bounded domain Ω of $\mathbb{R}^n$, and consider the first order linear operator $Av = \mathbf{a}\cdot\nabla v = \sum_{i=1}^n a_i \frac{\partial v}{\partial x_i}$. Assume that the vector field $\sum_{i=1}^n a_i \frac{\partial v}{\partial x_i}$ is actually the restriction of a smooth vector field $\sum_{i=1}^n \bar{a}_i \frac{\partial v}{\partial x_i}$ defined on an open neighborhood X of $\overline{\Omega}$ and that each $\bar{a}_i$ is a $C^{1,1}$ function on X (see [4]). Consider the Dirichlet problem:

$$\begin{cases} \Delta u + \mathbf{a}\cdot\nabla u = |u|^{p-2}u + f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{29}$$

If $\mathbf{a} = 0$, then to find a solution, it is sufficient to minimize the functional

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{1}{p}\int_\Omega |u|^p\,\mathrm{d}x + \int_\Omega f u\,\mathrm{d}x$$

and get the solution as a critical point $\partial\Phi(u) = 0$. However, if the non-self-adjoint term $\mathbf{a}$ is not zero, we can use the above approach to get

Theorem 4.4. *Assume* $\mathrm{div}(\mathbf{a}) \geqslant 0$ *on Ω, $1 < p \leqslant \frac{2n}{n-2}$ and consider on $H_0^1(\Omega)$, the functional*

$$I(u) = \Psi(u) + \Psi^*\left(\mathbf{a}\cdot\nabla u + \frac{1}{2}\mathrm{div}(\mathbf{a}),\ u\right),$$

where $\Psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{1}{p}\int_\Omega |u|^p\,\mathrm{d}x + \int_\Omega f u\,\mathrm{d}x + \frac{1}{4}\int_\Omega \mathrm{div}(\mathbf{a})|u|^2\,\mathrm{d}x$ and Ψ^* is its Legendre transform. Then there exists $\bar{u} \in H_0^1(\Omega)$ such that $I(\bar{u}) = \inf\{I(u);\ u \in H_0^1(\Omega)\} = 0$, and $\bar{u}$ is a solution of (29).

Proof. Indeed, Ψ is clearly convex and lower semi-continuous on $H_0^1(\Omega)$ while the operator $\Lambda u = -\mathbf{a}\cdot\nabla u - \frac{1}{2}\operatorname{div}(\mathbf{a})u$ is skew-adjoint by Green's formula. Again the functional $I(u) = \Psi(u) + \Psi^*(\mathbf{a}\cdot\nabla u + \frac{1}{2}\operatorname{div}(\mathbf{a})u)$ is of the form $L(u, \Lambda u)$ where $L(u,v) = \Psi(u) + \Psi^*(-v)$ is an ASD Lagrangian on $H_0^1(\Omega)\times H^{-1}(\Omega)$. The existence follows from Theorem 2.3, since Ψ is clearly coercive. Note that $\bar{u}$ then satisfies

$$\mathbf{a}\cdot\nabla\bar{u} + \frac{1}{2}\operatorname{div}(\mathbf{a})\bar{u} = \partial\Psi(\bar{u}) = -\Delta\bar{u} + \bar{u}^{p-1} + f + \frac{1}{2}\operatorname{div}(\mathbf{a})\bar{u}$$

and therefore $\bar{u}$ is a solution for (29). □

Example 4 *(A variational solution for variational inequalities).* Given again a bilinear continuous functional a on $X\times X$, and $\varphi : X \to \mathbb{R}$ a convex l.s.c, then solving the corresponding variational inequality amounts to constructing for any $f \in X^*$, a point $y \in X$ such that for all $z \in X$,

$$a(y, y-z) + \varphi(y) - \varphi(z) \leqslant \langle y-z, f\rangle. \tag{30}$$

It is well known that this problem can be rewritten as

$$f \in Ay + \partial\varphi(y),$$

where A is the bounded linear operator from X into X^* defined by $a(u,v) = \langle Au, v\rangle$. This means that the variational inequality (30) can be rewritten and solved using our self-dual variational principle. For example, we can solve variationally the following "obstacle" problem.

Corollary 4.5. *Let a be bilinear continuous functional a on a reflexive Banach space $X\times X$ so that $a(v,v)\geqslant \lambda\|v\|^2$, and let K be a convex closed subset of X. Then, for any $f\in X^*$, there is $\bar{x}\in K$ such that*

$$a(\bar{x}, \bar{x}-z) \leqslant \langle \bar{x}-z, f\rangle \quad \textit{for all } z\in K. \tag{31}$$

The point $\bar{x}$ can be obtained as a minimizer of the following problem:

$$\inf_{x\in X}\bigl\{\varphi(x) + (\varphi+\psi_K)^*(-\Lambda x)\bigr\} = 0,$$

where $\varphi(u) = \frac{1}{2}a(u,u) - \langle f, x\rangle$, $\Lambda : X \to X^*$ is the skew-adjoint operator defined by

$$\langle \Lambda u, v\rangle = \frac{1}{2}\bigl(a(u,v) - a(v,u)\bigr)$$

and where $\psi_K(x) = 0$ on K and $+\infty$ elsewhere.

4.2. ASD Lagrangians associated to differential systems

The next proposition shows that the theory of ASD-Lagrangians is well suited for "anti-Hamiltonian" systems of the form

$$(-A^*y, Ax) \in \partial\varphi(x,y),$$

where A is any given bounded linear operator.

Proposition 4.1. *Let φ be a proper and coercive convex lower semi-continuous function on $X\times Y$ with $(0,0)\in \operatorname{dom}(\varphi)$, and let $A : X\to Y^*$ be any bounded linear operator. Assume $B_1 : X\to X^*$ (resp., $B_2 : Y \to Y^*$) to be skew-adjoint operators, then there exists $(\bar{x},\bar{y})\in X\times Y$ such that*

$$(-A^*\bar{y} + B_1\bar{x}, A\bar{x} + B_2\bar{y}) \in \partial\varphi(\bar{x},\bar{y}). \tag{32}$$

The solution is obtained as a minimizer on $X\times Y$ of the functional

$$I(x,y) = \varphi(x,y) + \varphi^*(-A^*y + B_1x, Ax + B_2y).$$

Proof. It is enough to apply Theorem 4.1 to the ASD Lagrangian

$$L\big((x,y),(p,q)\big)=\varphi(x,y)+\varphi^*(-A^*y+B_1x-p,Ax+B_2y-q),$$

obtained by shifting to the right the ASD Lagrangian $\varphi\oplus_{\mathrm{as}}A$ by the skew-adjoint operator $(-B_1,-B_2)$. This yields that $I(x,y)=L((x,y),(0,0))$ attains its minimum at some $(\bar{x},\bar{y})\in X\times Y$ and that the minimum is actually 0. In other words,

$$\begin{aligned}0=I(\bar{x},\bar{y})&=\varphi(\bar{x},\bar{y})+\varphi^*(-A^*\bar{y}+B_1\bar{x},A\bar{x}+B_2\bar{y})\\&=\varphi(\bar{x},\bar{y})+\varphi^*(-A^*\bar{y}+B_1\bar{x},A\bar{x}+B_2\bar{y})-\big\langle(\bar{x},\bar{y}),(-A^*\bar{y}+B_1\bar{x},A\bar{x}+B_2\bar{y})\big\rangle\end{aligned}$$

from which the equation follows. □

Corollary 4.6. *Given positive operators $B_1:X\to X^*$, $B_2:Y\to Y^*$ and convex functions φ_1 in $\mathcal{C}(X)$ and φ_2 in $\mathcal{C}(Y)$ having 0 in their respective domains, we consider the convex functionals $\psi_1(x)=\frac{1}{2}\langle B_1x,x\rangle+\varphi_1(x)$ and $\psi_2(x)=\frac{1}{2}\langle B_2x,x\rangle+\varphi_2(x)$. Assume*

$$\lim_{\|x\|+\|y\|\to\infty}\frac{\psi_1(x)+\psi_2(y)}{\|x\|+\|y\|}=+\infty.$$

Then, for any $(f,g)\in X^*\times Y^*$ and any $c\in\mathbb{R}$, and any bounded linear operator $A:X\to Y^*$, there exists a solution $(\bar{x},\bar{y})\in X\times Y$ to the system of equations

$$\begin{cases}-A^*y-B_1x+f\in\partial\varphi_1(x),\\ c^2Ax-B_2y+g\in\partial\varphi_2(y).\end{cases}\tag{33}$$

It can be obtained as a minimizer of the problem:

$$\inf_{x,y\in X\times Y}\big\{\chi_1(x)+\chi_1^*\big(-B_1^ax-A^*y\big)+\chi_2(y)+\chi_2^*\big(-B_2^ay+c^2Ax\big)\big\}=0,\tag{34}$$

where B_1^a (resp., B_2^a) are the skew-symmetric parts of B_1 and B_2 and where $\chi_1(x)=\psi_1(x)-\langle f,x\rangle$ and $\chi_2(x)=\psi_2(x)-\langle g,x\rangle$

Proof. This follows by applying the above proposition to the convex function $\varphi(x,y)=\chi_1(x)+\chi_2(y)$ and the skew-symmetric operators $-B_1^a$ and $-B_2^a$. Note that the operator $\tilde{A}:X\times Y\to X^*\times Y^*$ defined by $\tilde{A}(x,y)=(A^*y,-c^2Ax)$ is skew adjoint once we equip $X\times Y$ with the scalar product

$$\big\langle(x,y),(p,q)\big\rangle=\langle x,p\rangle+c^{-2}\langle y,q\rangle.$$

We then get

$$\begin{cases}-A^*y-B_1^ax+f\in\partial\varphi_1(x)+B_1^s(x),\\ c^2Ax-B_2^ay+g\in\partial\varphi_2(y)+B_2^s(y)\end{cases}\tag{35}$$

which gives the result. □

Example 5 *(A variational principle for coupled equations).* Let $\mathbf{b}_1:\Omega\to\mathbb{R}^n$ and $\mathbf{b}_2:\Omega\to\mathbb{R}^n$ be two smooth vector fields on a neighborhood of a bounded domain Ω of $\mathbb{R}^n$, verifying the conditions in Example 3. Consider the system:

$$\begin{cases}\Delta(v+u)+\mathbf{b}_1\cdot\nabla u=|u|^{p-2}u+f & \text{on }\Omega,\\ \Delta(v-c^2u)+\mathbf{b}_2\cdot\nabla v=|v|^{q-2}v+g & \text{on }\Omega,\\ u=v=0 & \text{on }\partial\Omega.\end{cases}\tag{36}$$

We can use the above to get the following result.

Theorem 4.7. *Assume $\operatorname{div}(\mathbf{b}_1)\geqslant 0$ and $\operatorname{div}(\mathbf{b}_2)\geqslant 0$ on Ω, $1<p,q\leqslant\frac{n+2}{n-2}$ and consider on $H_0^1(\Omega)\times H_0^1(\Omega)$ the functional*

$$I(u,v)=\Psi(u)+\Psi^*\left(\mathbf{b}_1\cdot\nabla u+\frac{1}{2}\operatorname{div}(\mathbf{b}_1)u+\Delta v\right)+\Phi(v)+\Phi^*\left(\mathbf{b}_2\cdot\nabla v+\frac{1}{2}\operatorname{div}(\mathbf{b}_2)\,v-c^2\Delta u\right),$$

where

$$\Psi(u)=\frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x+\frac{1}{p}\int_\Omega |u|^p\,\mathrm{d}x+\int_\Omega fu\,\mathrm{d}x+\frac{1}{4}\int_\Omega \mathrm{div}(\mathbf{b}_1)|u|^2\,\mathrm{d}x,$$
$$\Phi(v)=\frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x+\frac{1}{q}\int_\Omega |v|^q\,\mathrm{d}x+\int_\Omega gv\,\mathrm{d}x+\frac{1}{4}\int_\Omega \mathrm{div}(\mathbf{b}_2)|v|^2\,\mathrm{d}x$$

and Ψ^* and Φ^* are their Legendre transforms. Then there exists $(\bar u,\bar v)\in H_0^1(\Omega)\times H_0^1(\Omega)$ such that:

$$I(\bar u,\bar v)=\inf\big\{I(u,v);\,(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)\big\}=0,$$

and $(\bar u,\bar v)$ is a solution of (36).

We can also reduce general minimization problems of functionals of the form $I(x)=\varphi(x)+\psi(Ax)$ to the much easier problem of minimizing ASD Lagrangians. Indeed we have

Proposition 4.2. *Let φ (resp., ψ) be a convex lower semi-continuous function on a reflexive Banach space X (resp. Y^*) and let $A: X\to Y^*$ be a bounded linear operator. To minimize the functional $I(x)=\varphi(x)+\psi(Ax)$ on X, we consider on $X\times Y$ the functional*

$$I(x,y)=\varphi(x)+\psi^*(y)+\varphi^*(-A^*y)+\psi(Ax).$$

Assuming $\lim_{\|z\|\to\infty} I(z)=+\infty$, then the infimum of I is zero and is attained at a point $(\bar x,\bar y)$ which determines the extremals of the min–max problem:

$$\sup\big\{-\psi^*(y)-\varphi^*(-A^*y);\,y\in Y\big\}=\inf\big\{\varphi(x)+\psi(Ax);\,x\in X\big\}.$$

The pair $(\bar x,\bar y)$ also satisfies the system:

$$\begin{cases}-A^*y\in\partial\varphi(x),\\ Ax\in\partial\psi^*(y).\end{cases}\tag{37}$$

Proof. It is sufficient to note that $I(x,y)=L((x,y),(0,0)$ where L is an anti-self-dual Lagrangian defined on $X\times Y$ by:

$$L\big((x,y),(p,q)\big)=\varphi(x)+\psi^*(y)+\varphi^*(-A^*y-p)+\psi(Ax-q).\qquad\square$$

By considering more general twisted sum Lagrangians, we obtain the following application

Corollary 4.8. *Let X and Y be two reflexive Banach spaces and let $A: X\to Y^*$ be any bounded linear operator. Assume $L\in\mathcal{L}_{\mathrm{AD}}(X)$ and $M\in\mathcal{L}_{\mathrm{AD}}(Y)$ are such that*

$$\lim_{\|x\|+\|y\|\to\infty}\frac{L(x,A^*y)+M(y,-Ax)}{\|x\|+\|y\|}=+\infty.$$

Then there exists $(\bar x,\bar y)\in X\times Y$, such that:

$$L(\bar x,A^*\bar y)+M(\bar y,-A\bar x)=\inf_{(x,y)\in(X\times Y)}L(x,A^*y)+M(y,-Ax)=0.\tag{38}$$

Moreover, we have

$$\begin{cases}L(\bar x,A^*\bar y)+\langle\bar x,A^*\bar y\rangle=0,\\ M(\bar y,-A\bar x)+\langle\bar y,-Ax\rangle=0,\\ (-A^*\bar y,-\bar x)\in\partial L(\bar x,A^*\bar x),\\ (A\bar x,-\bar y)\in\partial M(\bar y,-A\bar x).\end{cases}\tag{39}$$

Proof. It is sufficient to apply Proposition 2.1 to the ASD Lagrangian $L\oplus_A M$. $\square$

5. Self-dual variational principles with boundary constraints

For problems involving boundaries, we may start with an ASD Lagrangian L, but the operator Λ may be skew-adjoint modulo a term involving a boundary operator $\mathcal{B}$ from X into some Hilbert space H. We can then try to recover anti-selfduality by adding a correcting term via a boundary Lagrangian ℓ on H, in such a way that a new Lagrangian

$$M(x,p)=L(x,\Lambda x+p)+\ell(\mathcal{B}x)$$

becomes anti-self-dual on $X\times X^*$. As will be shown in [11], this procedure can be applied to a large number of linear and non-linear boundary value problems. In this section, we shall only consider the case where the boundary operator is of the form $\mathcal{B}:=(b_1,b_2)$ from its domain in X into a product Hilbert space $H:=H_1\times H_2$, and where the anti-symmetry of Λ modulo $\mathcal{B}$ corresponds to the simplest "Green's formula":

$$\langle\Lambda x,y\rangle_{(X,X^*)}+\langle\Lambda y,x\rangle_{(X,X^*)}=\big\langle b_2(x),b_2(y)\big\rangle_{H_2}-\big\langle b_1(x),b_1(y)\big\rangle_{H_1}. \tag{40}$$

5.1. Skew-symmetry and anti-self-dual Lagrangians

Definition 5.1. Let Λ be a linear map from its domain $D(\Lambda)$ in a reflexive Banach space X into X^* and consider (b_1,b_2) to be a linear map from its domain $D(b_1,b_2)$ in X into the product of two Hilbert spaces $H_1\times H_2$. Let $S:=D(\Lambda)\cap D(b_1,b_2)$ and associate the set

$$D^*(\Lambda,b_1,b_2)=\left\{y\in X;\ \sup\left\{\langle y,\Lambda x\rangle-\frac{1}{2}\big(\|b_1(x)\|^2_{H_1}+\|b_2(x)\|^2_{H_2}\big);\ x\in S,\ \|x\|_X<1\right\}<\infty\right\}.$$

- Say that Λ is *skew-adjoint modulo the boundary operators* (b_1,b_2) if the following properties are satisfied:

(1) The space $X_0:=\operatorname{Ker}(b_1,b_2)\cap D(\Lambda)$ is dense in X.
(2) The image of S by (b_1,b_2) is dense in $H_1\times H_2$.
(3) For every $x,y\in S$, we have $\langle y,\Lambda x\rangle+\langle\Lambda y,x\rangle=\langle b_2(x),b_2(y)\rangle_{H_2}-\langle b_1(x),b_1(y)\rangle_{H_1}$.
(4) $D^*(\Lambda,b_1,b_2)=D(\Lambda)\cap D(b_1,b_2)$.

Definition 5.2. We say that $\ell:H_1\times H_2\to\mathbb{R}\cup\{+\infty\}$ is a *compatible boundary Lagrangian* if

$$\ell^*(-h_1,h_2)=\ell(h_1,h_2)\quad\text{for all }(h_1,h_2)\in H_1\times H_2. \tag{41}$$

It is easy to see that such a boundary Lagrangian will always satisfy the inequality

$$\ell(r,s)\geqslant\frac{1}{2}\big(\|s\|^2-\|r\|^2\big)\quad\text{for all }(r,s)\in H_1\times H_2. \tag{42}$$

The basic example of a compatible boundary Lagrangian is given by a function ℓ on $H_1\times H_2$, of the form $\ell(x,p)=\psi_1(x)+\psi_2(p)$, with $\psi_1^*(x)=\psi_1(-x)$ and $\psi_2^*(p)=\psi_2(p)$. Here the choices for ψ_1 and ψ_2 are rather limited and the typical sample is:

$$\psi_1(x)=\frac{1}{2}\|x\|^2-2\langle a,x\rangle+\|a\|^2,\quad\text{and}\quad\psi_2(p)=\frac{1}{2}\|p\|^2,$$

where a is given in H_1. Boundary operators and compatible boundary Lagrangians allow us to build new ASD Lagrangians.

Proposition 5.1. *Let $L:X\times X^*\to\mathbb{R}\cup\{+\infty\}$ be an ASD Lagrangian on a reflexive Banach space X and let $\ell:H_1\times H_2\to\mathbb{R}$ be a compatible boundary Lagrangian on the product of two Hilbert spaces $H_1\times H_2$. Consider $\Lambda:D(\Lambda)\subset X\to X^*$ to be a skew-adjoint operator modulo a boundary operator $(b_1,b_2):D(b_1,b_2)\subset X\to H_1\times H_2$. Assume the following conditions*:

(i) *For every $p\in X^*$, the function $x\to L(x,p)$ is continuous on X.*
(ii) *The function $x\to L(x,0)$ is bounded on a neighborhood of* 0 *in X.*
(iii) *For all $(r,s)\in H_1\times H_2$, $\ell(r,s)\leqslant C(1+\|r\|^2+\|s\|^2)$.*

Then the Lagrangian defined by

$$L_{\Lambda,\ell}(x,p)=\begin{cases}L(x,\Lambda x+p)+\ell(b_1(x),b_2(x)) & \text{if } x\in D(\Lambda)\cap D(b_1,b_2),\\ +\infty & \text{if } x\notin D(\Lambda)\cap D(b_1,b_2)\end{cases}$$

is anti-self-dual on X.

Proof. First take $(q,y)\in X^*\times S$, and write

$$L^*_{\Lambda,\ell}(q,y)=\sup\bigl\{\langle y,p\rangle+\langle x,q\rangle-L(x,\Lambda x+p)-\ell\bigl(b_1(x),b_2(x)\bigr);\,x\in S,\,p\in X^*\bigr\}.$$

Setting $r=\Lambda x+p$, and using that $\langle y,\Lambda x\rangle=-\langle x,\Lambda y\rangle-\langle b_1(x),b_1(y)\rangle+\langle b_2(x),b_2(y)\rangle$ for $y\in S$, we obtain

$$\begin{aligned}L^*_{\Lambda,\ell}(q,y)&=\sup_{\substack{x\in S\\ r\in X^*}}\bigl\{\langle y,r-\Lambda x\rangle+\langle x,q\rangle-L(x,r)-\ell\bigl(b_1(x),b_2(x)\bigr)\bigr\}\\
&=\sup_{\substack{x\in S\\ r\in X^*}}\bigl\{\langle x,\Lambda y\rangle+\bigl\langle b_1(x),b_1(y)\bigr\rangle-\bigl\langle b_2(x),b_2(y)\bigr\rangle+\langle y,r\rangle+\langle x,q\rangle-L(x,r)-\ell\bigl(b_1(x),b_2(x)\bigr)\bigr\}\\
&=\sup\bigl\{\langle x,\Lambda y+q\rangle+\bigl\langle b_1(x+x_0),b_1(y)\bigr\rangle-\bigl\langle b_2(x+x_0),b_2(y)\bigr\rangle+\langle y,r\rangle-L(x,r)\\
&\quad-\ell\bigl(b_1(x+x_0),b_2(x+x_0)\bigr);\,x\in S,\,r\in X^*,\,x_0\in\operatorname{Ker}(b_1,b_2)\cap D(\Lambda)\bigr\}.\end{aligned}$$

Since S is a linear space, we may set $w=x+x_0$ and write

$$\begin{aligned}L^*_{\Lambda,\ell}(q,y)&=\sup\bigl\{\langle w-x_0,\Lambda y+q\rangle+\bigl\langle b_1(w),b_1(y)\bigr\rangle-\bigl\langle b_2(w),b_2(y)\bigr\rangle+\langle y,r\rangle-L(w-x_0,r)\\
&\quad-\ell\bigl(b_1(w),b_2(w)\bigr);\ w\in S,\,r\in X^*,\,x_0\in\operatorname{Ker}(b_1,b_2)\cap D(\Lambda)\bigr\}.\end{aligned}$$

Now for each fixed $w\in S$ and $r\in X^*$, the supremum over $x_0\in\operatorname{Ker}(b_1,b_2)\cap D(\Lambda)$ can be taken as a supremum over $x_0\in X$ since $\operatorname{Ker}(b_1,b_2)\cap D(\Lambda)$ is dense in X and all terms involving x_0 are continuous in that variable. Furthermore, for each fixed $w\in S$ and $r\in X^*$, the supremum over $x_0\in X$ of the terms $w-x_0$ can be written as supremum over $v\in X$ where $v=w-x_0$. So setting $v=w-x_0$ we get

$$\begin{aligned}L^*_{\Lambda,\ell}(q,y)&=\sup\bigl\{\langle v,\Lambda y+q\rangle+\bigl\langle b_1(w),b_1(y)\bigr\rangle-\bigl\langle b_2(w),b_2(y)\bigr\rangle+\langle y,r\rangle-L(v,r)\\
&\quad-\ell\bigl(b_1(w),b_2(w)\bigr);\,v\in X,\,r\in X^*,\,w\in S\bigr\}\\
&=\sup_{v\in X}\sup_{r\in X^*}\bigl\{\langle v,\Lambda y+q\rangle+\langle y,r\rangle-L(v,r)\bigr\}\\
&\quad+\sup_{w\in S}\bigl\{\bigl\langle b_1(w),b_1(y)\bigr\rangle+\bigl\langle b_2(w),-b_2(y)\bigr\rangle-\ell\bigl(b_1(w),b_2(w)\bigr)\bigr\}.\end{aligned}$$

Since the range of $(b_1,b_2):S\to H_1\times H_2$ is dense in the $H_1\times H_2$ topology, the boundary term can be written as

$$\sup_{a\in H_1}\sup_{b\in H_2}\bigl\{\bigl\langle a,b_1(y)\bigr\rangle+\bigl\langle b,-b_2(y)\bigr\rangle-\ell(a,b)\bigr\}=\ell^*\bigl(b_1(y),-b_2(y)\bigr)=\ell\bigl(-b_1(y),-b_2(y)\bigr)$$

while the main term is clearly equal to $L^*(\Lambda y+q,y)=L(-y,-\Lambda y-q)$ in such a way that $L^*_{\Lambda,\ell}(q,y)=L_{\Lambda,\ell}(-y,-q)$ if $y\in D(\Lambda)\cap D(b_1,b_2)$.

Now assume $y\notin S=D(\Lambda)\cap D(b_1,b_2)$, then $-y\notin S$ and we may write

$$\begin{aligned}L^*_{\Lambda,\ell}(q,y)&=\sup_{\substack{x\in S\\ r\in X^*}}\left\{\langle y,r-\Lambda x\rangle+\langle x,q\rangle-L(x,r)-C\left(\frac{\|b_1(x)\|^2_{H_1}}{2}+\frac{\|b_2(x)\|^2_{H_2}}{2}\right)\right\}\\
&\geqslant\sup_{\substack{x\in S\\ \|x\|_X<1}}\left\{\langle -y,\Lambda x\rangle+\langle x,q\rangle-L(x,0)-C\left(\frac{\|b_1(x)\|^2_{H_1}}{2}+\frac{\|b_2(x)\|^2_{H_2}}{2}\right)\right\}.\end{aligned}$$

Since by assumption $L(x,0)<K$ whenever $\|x\|_X<1$, we finally obtain that

$$\begin{aligned}L^*_{\Lambda,\ell}(q,y)&\geqslant\sup_{\substack{x\in S\\ \|x\|_X<1}}\left\{\langle -y,\Lambda x\rangle+\langle x,q\rangle-K-C-C\left(\frac{\|b(x)\|^2_{H_2}}{2}+\frac{\|b_2(x)\|^2_{H_2}}{2}\right)\right\}\\
&=+\infty=L_{\Lambda,\ell}(-y,-q)\end{aligned}$$

since $-Cy \notin S$ as soon as $y \notin S$. Therefore $L^*_{\Lambda,\ell}(q,y) = L_{\Lambda,\ell}(-y,-q)$ for all $(y,q) \in X \times X^*$ and $L_{\Lambda,\ell}$ is an anti-self-dual Lagrangian. □

5.2. A Lax–Milgram type result with boundary constraints

One can now deduce the following

Theorem 5.3. *Let $L : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ be an ASD Lagrangian on a reflexive Banach space X and let $\ell : H_1 \times H_2 \to \mathbb{R}$ be a compatible boundary Lagrangian on the product of two Hilbert spaces $H_1 \times H_2$. Consider $\Lambda : D(\Lambda) \subset X \to X^*$ to be a skew-adjoint operator modulo a boundary operator $(b_1, b_2) : D(b_1, b_2) \subset X \to H_1 \times H_2$. Assume the following hypothesis*:

(i) *For every $p \in X^*$, the function $x \to L(x,p)$ is continuous on X.*
(ii) *The function $x \to L(x,0)$ is bounded on a neighborhood of* 0 *in X.*
(iii) *The function $p \to L(0,p)$ is bounded on a neighborhood of* 0 *X^*.*
(iv) *For all $(r,s) \in H_1 \times H_2$, $\ell(r,s) \leqslant C(1 + \|r\|^2 + \|s\|^2)$.*

Then, there exists $\bar{x} \in X$ such that:

$$L(\bar{x}, \Lambda\bar{x}) + \ell(b_1\bar{x}, b_2\bar{x}) = \inf_{x\in X}\big\{L(x, \Lambda x) + \ell(b_1x, b_2x)\big\} = 0. \tag{43}$$

Moreover, we have

$$\begin{cases} L(\bar{x}, \Lambda\bar{x}) + \langle \bar{x}, \Lambda\bar{x}\rangle = 0, \\ (-\Lambda\bar{x}, -\bar{x}) \in \partial L(\bar{x}, \Lambda\bar{x}), \\ \ell(b_1(\bar{x}), b_2(\bar{x})) = \frac{1}{2}(\|b_2\bar{x}\|^2 - \|b_1\bar{x}\|^2). \end{cases} \tag{44}$$

In particular, for any $a \in H_1$ there exists $\bar{x} \in X$ such that $b_1(\bar{x}) = a$ while satisfying (44)*. It is obtained as a minimizer on X of the functional*

$$I(x) = L(x, \Lambda x) + \frac{1}{2}\big\|b_1(x)\big\|^2 - 2\big\langle a, b_1(x)\big\rangle + \|a\|^2 + \frac{1}{2}\big\|b_2(x)\big\|^2.$$

Proof. Conditions (i), (ii) and (iv) allow us to use Proposition 5.1 to get that the Lagrangian

$$M(x,p) = L(x, \Lambda x + p) + \ell\big(b_1(x), b_2(x)\big)$$

is anti-self-dual on $X \times X^*$. Condition (iii) implies that $p \to M(0,p)$ is bounded above on the bounded sets of X^*, which means that we can apply Theorem 2.1 to obtain $\bar{x} \in X$ such that (43) is satisfied.

To establish (44), write

$$\begin{aligned} L(x, \Lambda x) + \ell(b_1x, b_2x) &= L(x, \Lambda x) + \langle x, \Lambda x\rangle - \langle x, \Lambda x\rangle + \ell(b_1x, b_2x) \\ &= L(x, \Lambda x) + \langle x, \Lambda x\rangle - \frac{1}{2}\big(\|b_2x\|^2 - \|b_1x\|^2\big) + \ell(b_1x, b_2x). \end{aligned}$$

Since $L(x,p) \geqslant -\langle x,p\rangle$ and $\ell(r,s) \geqslant \frac{1}{2}(\|s\|^2 - \|r\|^2)$, we immediately obtain (44).

For $a \in H_1$ we consider the compatible boundary Lagrangian $\ell(r,s) = \frac{1}{2}\|r\|^2 - 2\langle a, r\rangle + \|a\|^2 + \frac{1}{2}\|s\|^2$, to obtain

$$\begin{aligned} L(x, \Lambda x) + \ell(b_1x, b_2x) &= L(x, \Lambda x) + \langle x, \Lambda x\rangle - \frac{1}{2}\big(\|b_2x\|^2 - \|b_1x\|^2\big) + \ell(b_1x, b_2x) \\ &= L(x, \Lambda x) + \langle x, \Lambda x\rangle + \big\|b_1(x) - a\big\|^2. \end{aligned}$$

In other words, $\bar{x}$ is a solution of $\inf_{x\in X}\{L(x, \Lambda x) + \langle x, \Lambda x\rangle + \|b_1(x) - a\|^2\} = 0$, and since $L(x,p) \geqslant -\langle x,p\rangle$, we obtain:

$$\begin{cases} L(\bar{x}, \Lambda\bar{x}) + \langle \bar{x}, \Lambda\bar{x}\rangle = 0, \\ b_1(\bar{x}) = a. \end{cases} \qquad \Box \tag{45}$$

Theorem 5.4. *Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be proper convex and lower semi-continuous on a reflexive Banach space X and let $\Lambda_1 : D(\Lambda_1) \subset X \to X^*$ and $\Lambda_2 : D(\Lambda_2) \subset X \to X^*$ be two linear operators such that:*

(i) *Λ_1 is a positive operator and $D(\varphi) \subset D(\Lambda_1)$.*
(ii) *Λ_2 is skew-adjoint operator modulo a boundary operator $(b_1, b_2) : D(b_1, b_2) \to H_1 \times H_2$ where H_1, H_2 are two Hilbert spaces.*

Suppose there exist $C > 0$, $p_1, p_2 > 0$ such that for every $x \in X$,

$$\frac{1}{C}\big(\|x\|_X^{p_1} - 1\big) \leqslant \varphi(x) \leqslant C\big(\|x\|_X^{p_2} + 1\big). \tag{46}$$

Then for any $a \in H_1$ and any $f \in X^*$, the equation

$$\begin{cases} \Lambda_1 x + \Lambda_2 x \in -\partial\varphi(x) + f, \\ b_1(x) = a \end{cases} \tag{47}$$

has a solution $\bar{x} \in X$ that is a minimizer of the functional

$$I(x) = \psi(x) + \psi^*(-\Lambda_1^a x - \Lambda_2 x) + \frac{1}{2}\big\|b_1(x)\big\|^2 - 2\big\langle a, b_1(x)\big\rangle + \|a\|^2 + \frac{1}{2}\big\|b_2(x)\big\|^2,$$

where $\psi(x) = \varphi(x) + \frac{1}{2}\langle \Lambda_1 x, x\rangle - \langle f, x\rangle$ on $D(\varphi)$ and $+\infty$ elsewhere, and $\Lambda_1^a = \frac{1}{2}(\Lambda_1 - \Lambda_1^*)$.

Proof. The Lagrangian $M(x, p) : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$M(x, p) = \psi(x) + \psi^*(-\Lambda_1^a x - \Lambda_2 x - p) + \frac{1}{2}\big\|b_1(x)\big\|^2 - 2\big\langle a, b_1(x)\big\rangle + \|a\|^2 + \frac{1}{2}\big\|b_2(x)\big\|^2$$
$$\text{if } x \in D(\Lambda) \cap D(b_1, b_2)$$

and $+\infty$ otherwise is anti-self-dual on $X \times X^*$. Indeed, the ASD Lagrangian $L(x, p) := \psi(x) + \psi^*(-p)$ and the boundary Lagrangian $\ell(r, s) = \frac{1}{2}\|r\|^2 - 2\langle a, r\rangle + \|a\|^2 + \frac{1}{2}\|s\|^2$ verify all the properties of Theorem 5.3.

Note that

$$I(x) = M(x, 0) = \psi(x) + \psi^*(-\Lambda_1^a x - \Lambda_2 x) + \langle x, \Lambda_1^a x + \Lambda_2 x\rangle + \big\|b_1(x) - a\big\|^2.$$

The fact that the minimum is attained at some $\bar{x}$ and is equal to 0, implies that $b_1(\bar{x}) = a$ and that $\psi(\bar{x}) + \psi^*(-\Lambda_1^a \bar{x} - \Lambda_2 \bar{x}) = 0$ which means that

$$-\Lambda_1^a(\bar{x}) - \Lambda_2(\bar{x}) \in \partial\psi(\bar{x}) = \partial\varphi(\bar{x}) + \Lambda_1^s(\bar{x}) - f$$

and therefore $\bar{x}$ satisfies (47). □

Example 6 *(A variational principle for non-linear transport equations with no diffusion).* Consider the equation

$$\begin{cases} \mathbf{a} \cdot \nabla u - a_0 u - Bu = u|u|^{p-2} + f \quad \text{on } \Omega \subset \mathbb{R}^n, \\ u = u_0 \quad \text{on } \Sigma_+, \end{cases} \tag{48}$$

where $p > 1$, $B : L^p(\Omega) \to L^q(\Omega)$ is a bounded linear operator ($\frac{1}{p} + \frac{1}{q} = 1$), $\mathbf{a}$ is a smooth vector field defined on a neighborhood of a C^∞ bounded open set Ω in $\mathbb{R}^n$, and $\Sigma_\pm = \{x \in \partial\Omega;\ \pm\mathbf{a}(x) \cdot \hat{n}(x) \geqslant 0\}$ are the entrance and exit sets of the transport operator $\mathbf{a} \cdot \nabla$.

Let $X = L^p(\Omega)$ and consider the Hilbert spaces: $H_1 = L^2(\Sigma_+; |\mathbf{a} \cdot \hat{n}|\,\mathrm{d}\sigma)$, $H_2 = L^2(\Sigma_-; |\mathbf{a} \cdot \hat{n}|\,\mathrm{d}\sigma)$ as well as the boundary operators $(b_1 u, b_2 u) = (u|_{\Sigma_+}, u|_{\Sigma_-})$ whose domain is $D(b_1, b_2) = \{u \in L^p(\Omega);\ (u|_{\Sigma_+}, u|_{\Sigma_-}) \in H_1 \times H_2\}$.

The operator $\Lambda u = \mathbf{a} \cdot \nabla u + \frac{1}{2}\operatorname{div}(\mathbf{a})u$ with domain $D(\Lambda) = \{u \in L^p(\Omega);\ \mathbf{a} \cdot \vec{\nabla} u + \frac{1}{2}\operatorname{div}(\mathbf{a})u \in L^q(\Omega)\}$ into $L^q(\Omega)$ is then skew-adjoint modulo the boundary (b_1, b_2) on $L^p(\Omega)$. Indeed, observe that $D(\Lambda)$ is a Banach space – denoted $H^{1,p}_{0,A}(\Omega)$ under the norm $\|u\|_{D(\Lambda)} = \|u\|_p + \|\mathbf{a} \cdot \nabla u\|_q$ and that $S := D(\Lambda) \cap D(b_1, b_2)$ is also a Banach space – denoted $H^{1,p}_A(\Omega)$ – under the norm $\|u\|_S = \|u\|_p + \|\mathbf{a} \cdot \nabla u\|_q + \|u|_{\Sigma_+}\|_{L^2(\Sigma_+; |\mathbf{a}\cdot\hat{n}|\,\mathrm{d}\sigma)}$.

The space $C^\infty(\overline{\Omega})$ is dense in both spaces (see [4]), as well as in $X = L^p(\Omega)$. Similarly $C_0^\infty(\Omega) \subset \mathrm{Ker}(b_1, b_2) \cap D(\Lambda)$, in such a way that $\mathrm{Ker}(b_1, b_2) \cap D(\Lambda)$ is dense in X. Moreover, by Green's theorem we have

$$\int_\Omega (\mathbf{a}\cdot\nabla u)v\,\mathrm{d}x + \int_\Omega (\mathbf{a}\cdot\nabla v)u\,\mathrm{d}x + \int_\Omega \mathrm{div}(\mathbf{a})uv\,\mathrm{d}x = \int_{\partial\Omega} uv\mathbf{n}\cdot\mathbf{a}\,\mathrm{d}\sigma$$

for all $u, v \in C^\infty(\overline{\Omega})$ and the identity on S follows since $C^\infty(\overline{\Omega})$ is dense in S for the norm $\|u\|_S$. Moreover, the embedding of $C_0^\infty(\Sigma_\pm) \subset L^2(\Sigma_\pm; |\vec{a}\cdot\hat{n}|\,\mathrm{d}\sigma)$ is dense, and therefore the image of $C^\infty(\overline{\Omega})$ under (b_1, b_2) is dense in $H_1 \times H_2$.

The following now follows immediately from Theorem 5.4

Theorem 5.5. *Assume $2 \leqslant p < \infty$, that $a_0 + \frac{1}{2}\mathrm{div}(\mathbf{a}) \geqslant 0$ on Ω and that $\int_\Omega u(x)\cdot Bu(x)\,\mathrm{d}x \geqslant 0$ for all $u \in L^p(\Omega)$. For $f \in L^q(\Omega)$ where $\frac{1}{p}+\frac{1}{q}=1$, consider on $L^p(\Omega)$ the convex continuous functional*:

$$\varphi(u) := \frac{1}{2}\int_\Omega \left(\frac{1}{2}\mathrm{div}\,\mathbf{a} + a_0\right)|u|^2\,\mathrm{d}x + \frac{1}{p}\int_\Omega |u|^p\,\mathrm{d}x + \int_\Omega u(x)\cdot Bu(x)\,\mathrm{d}x + \int_\Omega uf\,\mathrm{d}x$$

and let φ^* be its Legendre transform. For any $u_0 \in L^2(\Sigma_+; |\mathbf{a}\cdot\hat{n}|\,\mathrm{d}\sigma)$, let $B_a = \frac{1}{2}(B - B^*)$ be the anti-symmetric part of B, and consider the following functional on the space $L^p(\Omega)$

$$I(u) = \varphi(u) + \varphi^*\left(\mathbf{a}\cdot\nabla u + \frac{1}{2}\mathrm{div}(\mathbf{a})u - B^a u\right) + \frac{1}{2}\int_{\Sigma_+} \left|u(x)\right|^2 \mathbf{n}(x)\cdot\mathbf{a}(x)\,\mathrm{d}\sigma - \frac{1}{2}\int_{\Sigma_-} \left|u(x)\right|^2 \mathbf{n}(x)\cdot\mathbf{a}(x)\,\mathrm{d}\sigma$$
$$- 2\int_{\Sigma_+} u(x)u_0(x)\mathbf{n}(x)\cdot\mathbf{a}(x)\,\mathrm{d}\sigma + \int_{\Sigma_+} \left|u_0(x)\right|^2 \mathbf{n}(x)\cdot\mathbf{a}(x)\,\mathrm{d}\sigma$$

if $u \in H_A^{1,p}(\Omega)$ and $+\infty$ elsewhere. Then I attains its minimum on $L^p(\Omega)$ at a point $\bar{u}$ such that

$$I(\bar{u}) = \inf\left\{I(u) \mid u \in L^p(\Omega)\right\} = 0,$$

which satisfies Eq. (48).

5.3. Self-dual coupled equations with prescribed boundaries

Assume $L \in \mathcal{L}_{\mathrm{AD}}(X)$ and $M \in \mathcal{L}_{\mathrm{AD}}(Y)$ where X and Y are two reflexive Banach spaces and let $A : D(A) \subset X \to Y^*$ be any bounded linear operator. Let $\Lambda_1 : D(\Lambda_1) \subset X \to X^*$ (resp., $\Lambda_2 : D(\Lambda_2) \subset Y \to Y^*$) be skew-adjoint operators modulo a boundary operator $(b_1, b_2) : X \to H_1 \times H_2$ (resp., $(c_1, c_2) : Y \to K_1 \times K_2$), and let ℓ (resp., m be a self dual boundary Lagrangian on $H_1 \times H_2$ (resp., $K_1 \times K_2$), in such a way that the Lagrangians

$$L_{\Lambda_1}(x, p) = L(x, \Lambda_1 x + p) + \ell\big(b_1(x), b_2(x)\big) \quad \text{and} \quad M_{\Lambda_2}(y, q) = L(y, \Lambda_2 y + q) + \ell\big(c_1(y), b_2(y)\big)$$

are ASD on $X \times X^*$ and $Y \times Y^*$ respectively. Note also that the operator $\tilde{A}(x, y) = (A^* y, -Ax)$ from $D(A) \times D(A^*) \subset X \times Y$ to $X^* \times Y^*$ is skew-symmetric (modulo zero-boundary operators!). It follows that the Lagrangian

$$L_{\Lambda_1} \oplus_A M_{\Lambda_2}\big((x, y), (p, q)\big) = L(x, A^* y + \Lambda_1 x + p) + M(y, -Ax + \Lambda_2 y + q) + \ell(b_1 x, b_2 x) + m(c_1 y, c_2 y)$$

is also anti-self-dual. Consider the functional $I(x, y) = L_\Lambda \oplus_A M_\Gamma((x, y), (0, 0))$, that is

$$I(x, y) := L(x, A^* y + \Lambda_1 x) + M(y, -Ax + \Lambda_2 y) + \ell(b_1 x, b_2 x) + m(c_1 y, c_2 y).$$

Assuming the appropriate boundedness and coercivity conditions, we can deduce the existence of $(\bar{x}, \bar{y}) \in X \times Y$ such that

$$I(\bar{x}, \bar{y}) = \inf_{(x,y)\in X\times Y} I(x, y) = 0. \tag{49}$$

In particular, for any $a \in H_1$ and $b \in K_1$, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that:

$$\begin{cases} L(\bar{x}, A^*\bar{y} + \Lambda_1\bar{x}) + \langle \bar{x}, A^*\bar{y} + \Lambda_1\bar{x}\rangle = 0, \\ M(\bar{y}, -A\bar{x} + \Lambda_2\bar{y}) + \langle \bar{y}, -A\bar{x} + \Lambda_2\bar{y}\rangle = 0, \\ (-A^*\bar{y} - \Lambda_1\bar{x}, -\bar{x}) \in \partial L(\bar{x}, A^*\bar{x} + \Lambda_1\bar{x}), \\ (A\bar{x} - \Lambda_2\bar{y}, -\bar{y}) \in \partial M(\bar{y}, -A\bar{x} + \Lambda_2\bar{y}), \\ b_1(\bar{x}) = a, \\ c_1(\bar{y}) = b. \end{cases} \tag{50}$$

It is obtained as a minimizer on $X \times Y$ of the functional

$$I(x, y) = L(x, A^*y + \Lambda_1 x) + \frac{1}{2}\big\|b_1(x)\big\|^2 - 2\big\langle a, b_1(x)\big\rangle + \|a\|^2 + \frac{1}{2}\big\|b_2(x)\big\|^2$$
$$+ M(y, -Ax + \Lambda_2 y) + \frac{1}{2}\big\|c_1(y)\big\|^2 - 2\big\langle b, c_1(y)\big\rangle + \|b\|^2 + \frac{1}{2}\big\|c_2(y)\big\|^2.$$

Note that we can rewrite

$$I(x, y) = L(x, A^*y + \Lambda_1 x) + \langle x, A^*y + \Lambda_1 x\rangle + M(y, -Ax + \Lambda_2 y) + \langle y, -Ax + \Lambda_2 y\rangle$$
$$+ \big\|b_1(x) - a\big\|^2 + \big\|c_1(x) - b\big\|^2$$

in such a way that if $I(\bar{x}, \bar{y}) = 0$, then the fact that the sum of each two consecutive terms constituting I above is non-negative, prove our claim (50).

Corollary 5.6. *Let $\Lambda_1 : D(\Lambda_1) \subset X \to X^*$ (resp., $\Lambda_2 : D(\Lambda_2) \subset Y \to Y^*$) be skew-adjoint operators modulo a boundary operator $(b_1, b_2) : X \to H_1 \times H_2$ (resp., $(c_1, c_2) : Y \to K_1 \times K_2$). Let φ_1 (resp. φ_2) be a convex lower semi-continuous function on X (resp. on Y) and let $A : D(A) \subset X \to Y^*$ be a linear operator. Assume that*

$$\frac{1}{C}\big(1 + \|x\| + \|y\|\big)^{p_1} \leqslant \varphi_1(x) + \varphi_2(y) \leqslant C\big(1 + \|x\| + \|y\|\big)^{p_2}.$$

Then, for any $(a, b) \in H_1 \times K_1$, any $(f, g) \in X^* \times Y^*$ and any $\alpha \in R$, there exists a solution $(\bar{x}, \bar{y}) \in X \times Y$ to the system of equations

$$\begin{cases} -A^*y - \Lambda_1 x + f \in \partial\varphi_1(x), \\ \alpha^2 Ax - \Lambda_2 y + g \in \partial\varphi_2(y), \\ b_1(\bar{x}) = a, \\ b_2(\bar{y}) = b. \end{cases} \tag{51}$$

It is obtained as a minimizer on $X \times Y$ of the functional

$$I(x, y) = \psi_1(x) + \psi_1^*(-\Lambda_1 x - A^*y) + \frac{1}{2}\big(\|b_1(x)\|^2 + \|b_2(x)\|^2\big) - 2\big\langle a, b_1(x)\big\rangle + \|a\|^2$$
$$+ \psi_2(y) + \psi_2^*\big(-\Lambda_2 y + \alpha^2 Ax\big) + \frac{1}{2}\big(\|c_1(y)\|^2 + \|c_2(y)\|^2\big) - 2\langle b, c_1(y)\rangle + \|b\|^2,$$

where $\psi_1(x) = \varphi_1(x) - \langle f, x\rangle$ and $\varphi_2(y) = \psi_2(y) - \langle g, y\rangle$.

Proof. Associate the following anti-self-dual Lagrangians on $X \times X^*$ and $Y \times Y^*$ respectively,

$$L(x, p) = \psi_1(x) + \psi_1^*(-\Lambda_1 x - A^*y - p) + \frac{1}{2}\big(\|b_1(x)\|^2 + \|b_2(x)\|^2\big) - 2\big\langle a, b_1(x)\big\rangle + \|a\|^2,$$

$$M(y, q) = \psi_2(y) + \psi_2^*(-\Lambda_2 y + \alpha^2 Ax - q) + \frac{1}{2}\big(\|c_1(y)\|^2 + \|c_2(y)\|^2\big) - 2\big\langle b, c_1(y)\big\rangle + \|b\|^2.$$

Now apply the preceding observation to these two ASD Lagrangians, to the operator $\tilde{A}(x, y) = (-A^*y, \alpha^2 Ax)$ and to the product $X \times Y$ equipped with the scalar product $\langle (x, y), (p, q)\rangle = \langle x, p\rangle + \alpha^{-2}\langle y, q\rangle$, to get the result. □

Example 7 *(A variational principle for a coupled system with prescribed boundary conditions).* Let $\mathbf{b}_1 : \Omega \to \mathbb{R}^n$ and $\mathbf{b}_2 : \Omega \to \mathbb{R}^n$ be two smooth vector fields on a neighborhood of a bounded domain Ω of $\mathbb{R}^n$, verifying the conditions in Example 6, and consider their corresponding first order linear operator $B_1 u = \mathbf{b}_1 \cdot \nabla u$ and $B_2 v = \mathbf{b}_2 \cdot \nabla v$. Let

$$\Sigma_-^1 = \{x \in \partial\Omega; \mathbf{b}_1 \cdot \mathbf{n}(x) < 0\} \quad \text{and} \quad \Sigma_-^2 = \{x \in \partial\Omega; \mathbf{b}_2 \cdot \mathbf{n}(x) < 0\}.$$

For $u_0 \in L^2_{B_1}(\Sigma_-^1)$ and $v_0 \in L^2_{B_2}(\Sigma_-^2)$, consider the Dirichlet problem:

$$\begin{cases} \Delta v - \mathbf{b}_1 \cdot \nabla u - b_1 u = |u|^{p-2}u + f & \text{on } \Omega, \\ -\alpha^2 \Delta u - \mathbf{b}_2 \cdot \nabla v - b_2 v = |v|^{q-2}v + g & \text{on } \Omega, \\ u = u_0 & \text{on } \Sigma_-^1, \\ v = v_0 & \text{on } \Sigma_-^2. \end{cases} \tag{52}$$

We can use the above to get

Theorem 5.7. *Assume $b_1(x) - \frac{1}{2}\operatorname{div}\mathbf{b}_1(x) \geqslant \alpha > 0$ and $b_2(x) - \frac{1}{2}\operatorname{div}\mathbf{b}_2(x) \geqslant \alpha > 0$ on Ω, $2 < p, q \leqslant \frac{2n}{n-2}$. For any $f, g \in L^2(\Omega)$ and $(u_0, v_0) \in L^2_{B_1}(\Sigma_-^1) \times L^2_{B_2}(\Sigma_-^2)$, consider on $L^p(\Omega) \times L^q(\Omega)$ the functional*

$$\begin{aligned} I(u,v) = {} & \Psi(u) + \Psi^*\Big(-\mathbf{b}_1 \cdot \nabla u - \frac{1}{2}\operatorname{div}(\mathbf{b}_1)u + \Delta v\Big) \\ & + \frac{1}{2}\int_{\Sigma_+^1} |u(x)|^2 \mathbf{n}(x) \cdot \mathbf{b}_1(x)\,d\sigma - \frac{1}{2}\int_{\Sigma_-^1} |u(x)|^2 \mathbf{n}(x) \cdot \mathbf{b}_1(x)\,d\sigma \\ & + 2\int_{\Sigma_-^1} u(x)u_0(x)\mathbf{n}(x) \cdot \mathbf{b}_1(x)\,d\sigma - \int_{\Sigma_-^1} |u_0(x)|^2 \mathbf{n}(x) \cdot \mathbf{b}_1(x)\,d\sigma \\ & + \Phi(v) + \Phi^*\Big(-\mathbf{b}_2 \cdot \nabla v - \frac{1}{2}\operatorname{div}(\mathbf{b}_2)v - \alpha^2 \Delta u\Big) \\ & + \frac{1}{2}\int_{\Sigma_+^2} |v(x)|^2 \mathbf{n}(x) \cdot \mathbf{b}_2(x)\,d\sigma - \frac{1}{2}\int_{\Sigma_-^2} |v(x)|^2 \mathbf{n}(x) \cdot \mathbf{b}_2(x)\,d\sigma \\ & + 2\int_{\Sigma_-^2} v(x)u_0(x)\mathbf{n}(x) \cdot \mathbf{b}_2(x)\,d\sigma - \int_{\Sigma_-^2} |v_0(x)|^2 \mathbf{n}(x) \cdot \mathbf{b}_2(x)\,d\sigma, \end{aligned} \tag{53}$$

where

$$\Psi(u) = \frac{1}{p}\int_\Omega |u|^p\,dx + \int_\Omega fu\,dx + \frac{1}{2}\int_\Omega \Big(b_1 - \frac{1}{2}\operatorname{div}(\mathbf{b}_1)\Big)|u|^2\,dx,$$

$$\Phi(v) = \frac{1}{q}\int_\Omega |v|^q\,dx + \int_\Omega gv\,dx + \frac{1}{2}\int_\Omega \Big(b_2 - \frac{1}{2}\operatorname{div}(\mathbf{b}_2)\Big)|v|^2\,dx,$$

and Ψ^* and Φ^* are their Legendre transforms. The infimum is zero and there exists a minimizer $(\bar{u}, \bar{v}) \in H^1_{B_1}(\Omega) \times H^1_{B_2}(\Omega)$ that is a solution of (52).

The conditions on $\mathbf{b}_1$ and $\mathbf{b}_2$ insure that the first order linear operators $B_1 u := \mathbf{b}_1 \cdot \nabla u + b_1 u$ (resp., $B_2 v := \mathbf{b_2} \cdot \nabla v + b_2 v$) are positive modulo the boundary operators $u \to (u_{|_{\Sigma_-^1}}, u_{|_{\Sigma_+^1}}) \in L^2_{B_1}(\Sigma_-^1) \times L^2_{B_1}(\Sigma_+^1)$ (resp., $v \to (v_{|_{\Sigma_-^2}}, v_{|_{\Sigma_+^2}}) \in L^2_{B_2}(\Sigma_-^2) \times L^2_{B_2}(\Sigma_+^2)$). Apply now the above with $A = \Delta$.

6. Time dependent anti-self-dual Lagrangians

In the next two sections we develop further the variational theory for dissipative evolution equations via the theory of ASD Lagrangians. The goal is to extend the variational theory of gradient flows [16] and [12] so as to include evolutions of the form

$$-\dot{x}(t) \in \partial\varphi\big(x(t)\big) + Ax(t) + \omega x(t) \quad \text{for all } t \in [0, T], \tag{54}$$

where A is a positive – possibly unbounded – operator on a Hilbert space H and ω is any real number. The framework proposed above for the stationary case, leads to the formulation of (54) as

$$\dot{x}(t) \in \bar{\partial} L\big(t, x(t)\big) \quad \text{for all } t \in [0, T], \tag{55}$$

where the anti-self-dual Lagrangians $L(t, \cdot, \cdot)$ on the state space are associated to the convex functional φ, the operator A, and the scalar ω in the following way:

$$L(t, x, p) = \mathrm{e}^{2wt}\big\{\varphi\big(\mathrm{e}^{-wt}x\big) + \varphi^*\big(-\mathrm{e}^{-wt}(Ax + p)\big)\big\}. \tag{56}$$

Consider now the path space $A^2_H = \{u\colon\ [0, T] \to H;\ \dot{u} \in L^2_H\}$ consisting of all absolutely continuous arcs $u : [0, T] \to H$, equipped with the norm $\|u\|_{A^2_H} = (\|u(0)\|^2_H + \int_0^T \|\dot{u}\|^2 \,\mathrm{d}t)^{1/2}$.

The main step is based on the fact – established below – stating that under appropriate boundedness conditions, a (time-dependent) anti-self dual Lagrangian $L : [0, T] \times H \times H \to \mathbb{R}$ on a Hilbert space H, "lifts" to a partially anti-self-dual Lagrangian $\mathcal{L}$ on path space $A^2_H = \{u : [0, T] \to H;\ \dot{u} \in L^2_H\}$ via the formula

$$\mathcal{L}(x, p) = \int_0^T L\big(t, x(t) + p(t), \dot{x}(t)\big)\,\mathrm{d}t + \ell\big(x(0) + a, x(T)\big), \tag{57}$$

where ℓ is an appropriate time-boundary Lagrangian and where $(p(t), a) \in L^2_H \times H$ which happens to be a convenient representation for the dual of A^2_H. Eq. (54) can then be formulated as a stationary equation on path space of the form

$$0 \in \bar{\partial}\mathcal{L}(x) \tag{58}$$

hence reducing the dynamic problem to the stationary case already considered above. We now formalize the following concept.

Definition 6.1. Let $L : [0, T] \times H \times H \to \mathbb{R} \cup \{+\infty\}$ be measurable with respect to the σ-field generated by the products of Lebesgue sets in $[0, T]$ and Borel sets in $H \times H$. We say that L is an *anti-self-dual Lagrangian* (*ASD*) on $[0, T] \times H \times H$ if for any $t \in [0, T]$, the map $L_t : (x, p) \to L(t, x, p)$ is in $\mathcal{L}_{\mathrm{AD}}(H)$: that is if

$$L^*(t, p, x) = L(t, -x, -p) \quad \text{for all } (x, p) \in H \times H,$$

where here L^* is the Legendre transform in the last two variables.

The most basic time-dependent *ASD* Lagrangians are again of the form $L(t, x, p) = \varphi(t, x) + \varphi^*(t, -p)$ where for each t, the function $x \to \varphi(t, x)$ is convex and lower semi-continuous. We now show how this property naturally "lifts" to path space.

6.1. ASD Lagrangians on path spaces

Proposition 6.1. *Suppose that L is an anti-self-dual Lagrangian on $[0, T] \times H \times H$, then for each $\omega \in \mathbb{R}$, the Lagrangian $M(u, p) := \int_0^T \mathrm{e}^{2wt} L(t, \mathrm{e}^{-wt}u(t), \mathrm{e}^{-wt}p(t))\,\mathrm{d}t$ is anti-self-dual on L^2_H.*

Proof. This follows from the following standard fact: for any Lagrangian $L(t, x, p)$, we have:

$$\int_0^T L^*\big(t, p(t), s(t)\big)\,\mathrm{d}t = \sup\left\{\int_0^T \big(\big\langle p(t), u(t)\big\rangle + \big\langle s(t), v(t)\big\rangle - L\big(t, u(t), v(t)\big)\big)\,\mathrm{d}t;\ (u, v) \in L^2_H \times L^2_H\right\}. \qquad \square$$

6.1.1. A representation of the space A^2_H

One way to represent the space A^2_H is to identify it with the product space $H \times L^2_H$, in such a way that its dual $(A^2_H)^*$ can also be identified with $H \times L^2_H$ via the formula:

$$\big\langle u, (p_1, p_0)\big\rangle_{A^2_H, H\times L^2_H} = \big\langle u(0), p_1\big\rangle_H + \int_0^T \big\langle \dot u(t), p_0(t)\big\rangle\,\mathrm{d}t,$$

where $u \in A^2_H$ and $(p_1, p_0) \in H \times L^2_H$.

Proposition 6.2. *Suppose L is an anti-self-dual Lagrangian on $[0,T] \times H \times H$ and that ℓ is a compatible boundary Lagrangian on $H \times H$, then the Lagrangian defined on $A^2_H \times (A^2_H)^* = A^2_H \times (H \times L^2_H)$ by*

$$N(u,p) = \int_0^T L\big(t, u(t) + p_0(t), \dot u(t)\big)\,\mathrm{d}t + \ell\big(u(0) + p_1, u(T)\big)$$

is anti-self-dual on $A^2_H \times (L^2_H \times \{0\})$.

Proof. For $(v,q) \in A^2_H \times (A^2_H)^*$ with q represented by $(q_0(t), 0)$ write:

$$N^*(q,v) = \sup_{p_1\in H}\ \sup_{p_0\in L^2_H}\ \sup_{u\in A^2_H} \Bigg\{\big\langle p_1, v(0)\big\rangle + \int_0^T \Big[\big\langle p_0(t), \dot v(t)\big\rangle + \big\langle q_0(t), \dot u(t)\big\rangle - L\big(t, u(t) + p_0(t), \dot u(t)\big)\Big]\,\mathrm{d}t \\ - \ell\big(u(0) + p_1, u(T)\big)\Bigg\}.$$

Making a substitution $u(0) + p_1 = a \in H$ and $u(t) + p_0(t) = y(t) \in L^2_H$, we obtain

$$N^*(q,v) = \sup_{a\in H}\ \sup_{y\in L^2_H}\ \sup_{u\in A^2_H} \Bigg\{\big\langle a - u(0), v(0)\big\rangle - \ell\big(a, u(T)\big) \\ + \int_0^T \Big[\big\langle y(t) - u(t), \dot v(t)\big\rangle + \big\langle q_0(t), \dot u(t)\big\rangle - L\big(t, y(t), \dot u(t)\big)\Big]\Bigg\}\,\mathrm{d}t.$$

Since $\dot u$ and $\dot v \in L^2_H$, we have $\int_0^T \langle u, \dot v\rangle = -\int_0^T \langle \dot u, v\rangle + \langle v(T), u(T)\rangle - \langle v(0), u(0)\rangle$, which implies

$$N^*(q,v) = \sup_{a\in H}\ \sup_{y\in L^2_H}\ \sup_{u\in A^2_H} \Bigg\{\big\langle a, v(0)\big\rangle - \big\langle v(T), u(T)\big\rangle - \ell\big(a, u(T)\big) \\ + \int_0^T \Big[\big\langle y(t), \dot v(t)\big\rangle + \big\langle v(t) + q_0(t), \dot u(t)\big\rangle - L\big(t, y(t), \dot u(t)\big)\Big]\,\mathrm{d}t\Bigg\}.$$

Identify now A^2_H with $H \times L^2_H$ via the correspondence:

$$(b, r) \in H \times L^2_H \mapsto b + \int_t^T r(s)\,\mathrm{d}s \in A^2_H,$$

$$u \in A^2_H \mapsto \big(u(T), -\dot u(t)\big) \in H \times L^2_H.$$

We finally obtain

$$
\begin{aligned}
N^*(q,v) &= \sup_{a\in H}\sup_{b\in H}\Bigg\{\langle a, v(0)\rangle - \langle v(T), b\rangle - \ell(a,b) \\
&\qquad + \sup_{y\in L^2_H}\sup_{r\in L^2_H}\int_0^T \big[\langle y(t),\dot v(t)\rangle + \langle v(t)+q_0(t), r(t)\rangle - L\big(t,y(t),r(t)\big)\big]\,\mathrm{d}t\Bigg\} \\
&= \int_0^T L^*\big(t,\dot v(t), v(t)+q_0(t)\big)\,\mathrm{d}t + \ell^*\big(v(0),-v(T)\big) \\
&= \int_0^T L\big(t,-v(t)-q_0(t),-\dot v(t)\big)\,\mathrm{d}t + \ell\big(-v(0),-v(T)\big) \\
&= N(-v,-q). \qquad \square
\end{aligned}
$$

6.2. ASD Lagrangians in the calculus of variations

Theorem 6.2. *Suppose L is an anti-self-dual Lagrangian on $[0,T]\times H\times H$, ℓ is a compatible boundary Lagrangian on $H\times H$, and consider the following functional*

$$
I_{\ell,L}(u) = \int_0^T L\big(t,u(t),\dot u(t)\big)\,\mathrm{d}t + \ell\big(u(0),u(T)\big).
$$

Suppose there exists $C>0$ such that

$$
\int_0^T L\big(t,x(t),0\big)\,\mathrm{d}t \leqslant C\big(1+\|x\|^2_{L^2_H}\big) \quad \textit{for all } x\in L^2_H. \tag{59}
$$

Then there exists $v\in A^2_H$ such that $(v(t),\dot v(t))\in \mathrm{Dom}(L)$ for almost all $t\in[0,T]$ and

$$
I_{\ell,L}(v) = \inf_{u\in A^2_H} I_{\ell,L}(u) = 0.
$$

In particular, for every $v_0\in H$ the following functional

$$
I_{\ell,L}(u) = \int_0^T L\big(t,u(t),\dot u(t)\big)\,\mathrm{d}t + \frac{1}{2}\big\|u(0)\big\|^2 - 2\big\langle v_0, u(0)\big\rangle + \|v_0\|^2 + \frac{1}{2}\big\|u(T)\big\|^2
$$

has minimum equal to zero on A^2_H. It is attained at a unique path v which then satisfies:

$$
v(0)=v_0 \quad \textit{and} \quad \big(v(t),\dot v(t)\big)\in\mathrm{Dom}(L) \quad \textit{for almost all } t\in[0,T], \tag{60}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\partial_p L\big(t,v(t),\dot v(t)\big) = \partial_x L\big(t,v(t),\dot v(t)\big), \tag{61}
$$

$$
\big(-\dot v(t),-v(t)\big)\in\partial L\big(t,v(t),\dot v(t)\big), \tag{62}
$$

$$
\big\|v(t)\big\|^2_H = \|v_0\|^2 - 2\int_0^t L\big(s,v(s),\dot v(s)\big)\,\mathrm{d}s \quad \textit{for every } t\in[0,T]. \tag{63}
$$

If L is autonomous and $v\in C^1([0,T],H)$, then we have:

$$
\big\|\dot v(t)\big\| \leqslant \big\|\dot v(0)\big\| \quad \textit{for all } t\in[0,T]. \tag{64}
$$

Proof. Apply Proposition 6.2 to get that

$$N(u,p)=\int_0^T L\big(t,u(t)+p_0(t),\dot u(t)\big)\,\mathrm{d}t+\ell\big(u(0)+p_1,u(T)\big)$$

is partially anti-self-dual on A_H^2. It now suffices to apply Theorem 2.1 since in this case

$$N(0,p)=\int_0^T L\big(t,p_0(t),0\big)\,\mathrm{d}t+\ell(p_1,0)\leqslant C_2\big(1+\|p_0\|^2_{L^2_H}\big)+\|p_1\|^2_H,$$

which means that $N(0,p)$ is bounded on the bounded sets of $(A_H^2)^*$.

Given $v_0\in H$, use the compatible boundary Lagrangian $\ell(r,s)=\frac12\|r\|^2-2\langle v_0,r\rangle+\|v_0\|^2+\frac12\|s\|^2$ to get that

$$I_{\ell,L}(u)=\int_0^T\big[L\big(t,u(t),\dot u(t)\big)+\big\langle u(t),\dot u(t)\big\rangle\big]\,\mathrm{d}t+\big\|u(0)-v_0\big\|^2.$$

Since $L(t,x,p)\geqslant-\langle x,p\rangle$ for all $(t,x,p)\in[0,T]\times H\times H$, the fact that $I_{\ell,L}(v)=\inf_{u\in A_H^2}I_{\ell,L}(u)=0$, then yields $v(0)=v_0$ and that

$$L\big(s,v(s),\dot v(s)\big)+\big\langle v(s),\dot v(s)\big\rangle=0\quad\text{for almost all } s\in[0,T]. \tag{65}$$

This clearly yields (63), since then $\mathrm{d}(|v(s)|^2)/\mathrm{d}s=-2L(s,v(s),\dot v(s))$. To prove now (62), use (65) and the fact that L is anti-self-dual to write:

$$L\big(s,v(s),\dot v(s)\big)+L^*\big(s,-\dot v(s),-v(s)\big)+\big\langle\big(v(s),\dot v(s)\big),\big(\dot v(s),v(s)\big)\big\rangle=0.$$

Now apply Legendre–Fenchel duality in the space $H\times H$. The uniqueness and (64) follow from the following observation.

Lemma 6.3. *Suppose $L(t,\cdot,\cdot)$ is convex on $H\times H$ for each $t\in[0,T]$, and that $x(t)$ and $v(t)$ are two paths in $C^1([0,T],H)$ satisfying $x(0)=x_0$, $v(0)=v_0$, $-(\dot x,x)\in\partial L(t,x,\dot x)$ and $-(\dot v,v)\in\partial L(t,v,\dot v)$. Then $\|x(t)-v(t)\|\leqslant\|x(0)-v(0)\|$ for each $t\in[0,T]$.*

Proof. Estimate $\alpha(t)=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\|x(t)-v(t)\|^2}{2}$ as follows:

$$\begin{aligned}
\alpha(t)&=\big\langle v(t)-x(t),\dot v(t)-\dot x(t)\big\rangle\\
&=\frac12\big\langle v(t)-x(t),\dot v(t)-\dot x(t)\big\rangle+\frac12\big\langle\dot v(t)-\dot x(t),v(t)-x(t)\big\rangle\\
&=\frac12\big(\big\langle\big(v(t)-x(t),\dot v(t)-\dot x(t)\big),\big(\dot v(t)-\dot x(t),v(t)-x(t)\big)\big\rangle_{H\times H}\big)\\
&=\frac12\big(\big\langle\big(v(t)-x(t),\dot v(t)-\dot x(t)\big),\big(\partial_x L\big(t,x(t),\dot x(t)\big)\\
&\quad-\partial_x L\big(t,v(t),\dot v(t)\big),\partial_p L\big(t,x(t),\dot x(t)\big)-\partial_p L\big(t,v(t),\dot v(t)\big)\big)\big\rangle\big)\\
&=\frac12\big(\big\langle\big(v(t),\dot v(t)\big)-\big(x(t),\dot x(t)\big),\big(\partial_x L\big(t,x(t),\dot x(t)\big),\partial_p L\big(t,x(t),\dot x(t)\big)\big)\\
&\quad-\big(\partial_x L\big(t,v(t),\dot v(t)\big),\partial_p L\big(t,v(t),\dot v(t)\big)\big)\big\rangle\big)\\
&=\frac12\big(\big\langle\big(v(t),\dot v(t)\big)-\big(x(t),\dot x(t)\big),\partial L\big(t,x(t),\dot x(t)\big)-\partial L\big(t,v(t),\dot v(t)\big)\big\rangle\big)\\
&\leqslant 0
\end{aligned}$$

in view of the convexity of L. □

It then follows that $\|x(t)-v(t)\|\leqslant\|x(0)-v(0)\|$ for all $t>0$. Now if L is autonomous, $v(t)$ and $x(t)=v(t+h)$ are solutions for any $h>0$, so that (64) follows from the above. □

6.2.1. ASD Lagrangians associated to gradient flows

The most basic example of a self-dual Lagrangian already provides a variational formulation and proof of existence for gradient flows. The following extends some of the results in [16].

Corollary 6.4. *Let $\varphi:[0,T]\times H\to\mathbb{R}\cup\{+\infty\}$ be a measurable function with respect to the σ-field in $[0,T]\times H$ generated by the products of Lebesgue sets in $[0,T]$ and Borel sets in H. Assume that for every $t\in[0,T]$, the function $\varphi(t,\cdot)$ is convex and lower semicontinuous on H, and A_t is a bounded linear positive operator on H such that for some positive functions $\gamma,\beta^{-1}\in L^\infty[0,T]$, we have*

$$\beta(t)\|x\|^p\leqslant\varphi(t,x)+\frac{1}{2}\langle A_tx,x\rangle\leqslant\gamma(t)\|x\|^q. \tag{66}$$

Then, for any $u_0\in H$, the functional

$$I(u)=\frac{1}{2}\big(\big|u(0)\big|^2+\big|u(T)\big|^2\big)-2\big\langle u(0),u_0\big\rangle+|u_0|^2+\int_0^T\big[\psi\big(t,u(t)\big)+\psi^*\big(t,-A_t^au(t)-\dot u(t)\big)\big]\,\mathrm{d}t, \tag{67}$$

where ψ is the convex functional $\psi(t,x)=\varphi(t,x)+\frac{1}{2}\langle A_tx,x\rangle$ has a unique minimizer v in A_H^2 such that:

$$I(v)=\inf_{u\in A_H^2}I(u)=0. \tag{68}$$

Among the paths in A_H^2, v is the unique solution to

$$\begin{cases}-A_tu(t)-\dot v(t)\in\partial\varphi(t,v(t)) & \text{a.e. on } [0,T],\\ v(0)=u_0.\end{cases} \tag{69}$$

Proof. This follows directly from Theorem 6.2 applied to the anti-self-dual Lagrangian $L(t,x,p)=\psi(t,x)+\psi^*(t,-A_t^ax-p)$. Note that the conditions in (66) yield that $\int_0^T L(t,x(t),0)\,\mathrm{d}t=\int_0^T\psi(t,x(t))+\psi^*(t,A_t^ax(t))\,\mathrm{d}t$ is bounded on the bounded sets of L_H^2. □

6.2.2. Variational resolution for parabolic-elliptic variational inequalities

Consider for each time t, a bilinear continuous functional a_t on a Hilbert space $H\times H$ and a convex l.s.c function $\varphi(t,\cdot):H\to\mathbb{R}\cup\{+\infty\}$. Solving the corresponding parabolic variational inequality amounts to constructing for a given $f\in L^2([0,T];H)$ and $x_0\in H$, a path $x(t)\in A_H^2([0,T])$ such that for all $z\in H$,

$$\big\langle\dot x(t),x(t)-z\big\rangle+a_t\big(x(t),x(t)-z\big)+\varphi\big(t,x(t)\big)-\varphi(t,z)\leqslant\big\langle x(t)-z,f(t)\big\rangle \tag{70}$$

for almost all $t\in[0,T]$. This problem can be rewritten as: $f(t)\in\dot y(t)+A_ty(t)+\partial\varphi(t,y)$, where A_t is the bounded linear operator on H defined by $a_t(u,v)=\langle A_tu,v\rangle$. This means that the variational inequality (70) can be rewritten and solved using the variational principle in Theorem 6.5 For example, one can then solve variationally the following "obstacle" problem.

Corollary 6.5. *Let $(a_t)_t$ be bilinear continuous functionals on $H\times H$ satisfying:*

- *For some $\lambda>0$, $a_t(v,v)\geqslant\lambda\|v\|^2$ on H for every $t\in[0,T]$.*
- *The map $u\to\int_0^T a_t(u(t),u(t))\,\mathrm{d}t$ is continuous on L_H^2.*

If K is a convex closed subset of H, then for any $f\in L^2([0,T];H)$ and any $x_0\in K$, there exists a path $x\in A_H^2([0,T])$ such that $x(0)=x_0$, $x(t)\in K$ for almost all $t\in[0,T]$ and

$$\big\langle\dot x(t),x(t)-z\big\rangle+a_t\big(x(t),x(t)-z\big)\leqslant\big\langle x(t)-z,f\big\rangle\quad\textit{for all } z\in K.$$

The path $x(t)$ is obtained as a minimizer of the following functional on $A_H^2([0,T])$:

$$I(y)=\int_0^T\big\{\varphi\big(t,y(t)\big)+\big(\varphi(t,\cdot)+\psi_K\big)^*\big(-\dot y(t)-A_ty(t)\big)\big\}\,\mathrm{d}t+\frac{1}{2}\big(\big|y(0)\big|^2+\big|y(T)\big|^2\big)-2\big\langle y(0),x_0\big\rangle+|x_0|^2.$$

Here $\varphi(t,y)=\frac{1}{2}a_t(y,y)-\langle f(t),y\rangle$ *and* $\psi_K(y)=0$ *on* K *and* $+\infty$ *elsewhere, while* $\Lambda_t: H\to H$ *is the skew-adjoint operator defined by* $\langle \Lambda_t u, v\rangle = \frac{1}{2}(a_t(u,v)-a_t(v,u))$.

Theorem 6.2 is not directly applicable to this situation, since the Lagrangian is not bounded, however one can replace ψ_K by its λ-regularization ψ_K^λ, apply the above variational principle to the function $\varphi_\lambda(t,\cdot)=\varphi(t,\cdot)+\psi_K^\lambda$, then let $\lambda\to 0$ to conclude.

7. Semi-groups associated to autonomous anti-self-dual Lagrangians

Under appropriate boundedness conditions, Theorem 6.2 naturally associates to any time-dependent anti-self-dual Lagrangian $L(t,\cdot,\cdot)$ a family of maps $T_t: H\to H$, where $T_t x_0 = x(t)$ which is the solution at time t of the equation

$$\dot{x}(t)\in\bar{\partial}L\big(t,x(t)\big)\quad\text{for all } t\in[0,T]\quad\text{and}\quad x(0)=x_0. \tag{71}$$

When the Lagrangian $L(x,p)$ is autonomous, the situation is much nicer since first $(T_t)_t$ becomes a semi-group, and secondly one can then associate a flow without stringent boundedness or coercivity conditions on the Lagrangian L. Indeed, we can then use a Yosida-type regularization of ASD Lagrangian reminiscent of the standard theory for operators and for convex functions. We then obtain the following result.

Theorem 7.1. *Let L be an anti-self-dual Lagrangian on a Hilbert space H that is uniformly convex in the first variable. Assuming* $\mathrm{Dom}(\bar{\partial}L)$ *is non-empty, then there exists a semi-group of* 1*-Lipschitz maps* $(T_t)_{t\in\mathbb{R}^+}$ *on* $\mathrm{Dom}(\bar{\partial}L)$ *such that* $T_0=\mathrm{Id}$ *and for any* $x_0\in\mathrm{Dom}(\bar{\partial}L)$*, the path* $x(t)=T_t x_0$ *satisfies the following*:

$$\dot{x}(t)\in\bar{\partial}L\big(t,x(t)\big)\quad\textit{for all } t\in[0,T] \tag{72}$$

and

$$\big\|x(t)\big\|_H^2=\|x_0\|^2-2\int_0^t L\big(x(s),\dot{x}(s)\big)\,\mathrm{d}s\quad\textit{for every } t\in[0,T]. \tag{73}$$

The path $(x(t))_t=(T_t x)_t$ *is obtained as a minimizer on* A_H^2 *of the functional*

$$I(u)=\int_0^T L\big(u(t),\dot{u}(t)\big)\,\mathrm{d}t+\frac{1}{2}\big\|u(0)\big\|^2-2\big\langle x_0,u(0)\big\rangle+\|x\|^2+\frac{1}{2}\big\|u(T)\big\|^2,$$

whose infimum is equal to zero.

As mentioned above, we can associate to the Lagrangian $L(x,p)$ its λ-regularization by considering $L_\lambda=L\star T_\lambda$ where

$$T_\lambda(x,p)=\frac{\|x\|^2}{2\lambda^2}+\frac{\lambda^2\|p\|^2}{2}.$$

Then L_λ satisfies the hypothesis of Theorem 6.2, and we can then find for each initial point $v\in H$, a path $v_\lambda\in A_H^2$, with $v_\lambda(0)=v$, which verify the above properties.

The uniform convexity of L in the first variable insures that the regularization L_λ is uniformly convex in both variables which then yield C^1-solutions. The 1-Lipschitz property follows from Lemma 6.3. The rest of the argument amounts to analyzing what happens when $\lambda\to 0$. The details will be given in [17] where the semi-convex case is also considered. The λ-regularization procedure is however made more complicated by the presence of the linear term ωx which prevents the Lagrangian from being autonomous. This factor will however allow – among other things – to relax the convexity assumptions on φ. We then obtain the following result.

Theorem 7.2. *Let L be an autonomous anti-self-dual Lagrangian on a Hilbert space $H\times H$ that is uniformly convex in the first variable. Assuming* $\mathrm{Dom}(\bar{\partial}L)$ *is non-empty, then for any* $\omega\in\mathbb{R}$ *there exists a semi-group of maps* $(T_t)_{t\in\mathbb{R}^+}$ *defined on* $\mathrm{Dom}(\bar{\partial}L)$ *such that*:

(1) *$T_0x = x$ and $\|T_tx - T_ty\| \leqslant e^{-\omega t}\|x - y\|$ for any $x, y \in \mathrm{Dom}(\bar\partial L)$.*
(2) *The semi-group is defined for any $x_0 \in \mathrm{Dom}(\bar\partial L)$ by $T_tx_0 = x(t)$ where $x(t)$ is the unique path that minimizes on A_H^2 the functional*

$$I(u) = \int_0^T e^{2\omega t} L\big(u(t), \omega u(t) + \dot u(t)\big)\,dt + \frac{1}{2}\big\|u(0)\big\|^2 - 2\big\langle x_0, u(0)\big\rangle + \|x_0\|^2 + \frac{1}{2}\big\|e^{\omega T}u(T)\big\|^2$$

in such a way that $I(x) = \inf_{u\in A_H^2} I(u) = 0$.
(3) *For any $x_0 \in \mathrm{Dom}(\bar\partial L)$ the path $x(t) = T_tx_0$ satisfies the following*:

$$-\big(\dot x(t) + \omega x(t), -x(t)\big) \in \partial L\big(x(t), \dot x(t) + \omega x(t)\big), \qquad (74)$$
$$x(0) = x_0.$$

Proof. We associate to L, the anti-self-dual Lagrangian

$$L_\omega(t, x, p) := \big(e^{\omega t} \cdot L\big)(x, p) = e^{2\omega t} L\big(e^{-\omega t}x, e^{-\omega t}p\big).$$

Note that if $y(t)$ satisfies:

$$\big(-\dot y(t), -y(t)\big) \in \partial L_\omega\big(t, y(t), \dot y(t)\big) \qquad (75)$$

then $x(t) = e^{-\omega t}y(t)$ satisfies

$$-\big(\dot x(t) + \omega x(t), x(t)\big) \in \partial L\big(x(t), \dot x(t) + \omega x(t)\big). \qquad (76)$$

However, we cannot apply Theorem 7.1 directly to the Lagrangian L_ω because the latter is not autonomous. However, we shall see in [17] that the Yosida regularization argument still works in this case. □

Now we can deduce the following which was established in [16] in the case of gradient flows of convex potentials (i.e., when $A = 0$ and $\omega = 0$), and in [12] in the case of gradient flows of semi-convex functions (i.e., when $A = 0$ and $\omega > 0$).

Theorem 7.3. *Let φ be a bounded below, proper convex lower semi-continuous functional on H and let A be a positive bounded linear operator on H. For any $\omega \in \mathbb{R}$ and $x_0 \in \mathrm{Dom}(\partial\varphi)$, consider the following functional on A_H^2:*

$$I(u) = \int_0^T e^{2\omega t}\big\{\psi\big(u(t)\big) + \psi^*\big(-A^a u(t) - \omega u(t) - \dot u(t)\big)\big\}\,dt + \frac{1}{2}\big\|u(0)\big\|^2$$
$$- 2\big\langle x_0, u(0)\big\rangle + \|x_0\|^2 + \frac{1}{2}\big\|e^{\omega T}u(T)\big\|^2,$$

where A^a is the anti-symmetric part of A, and $\psi(u) = \varphi(u) + \frac{1}{2}\langle Au, u\rangle$. The minimum of I is then zero and is attained at a path $x(t)$ which is a solution of

$$\begin{cases} -Ax(t) - \omega x(t) - \dot x(t) \in \partial\varphi(x(t)) & \text{a.e. } t \in [0, T], \\ x(0) = x_0. \end{cases} \qquad (77)$$

7.1. Variational resolution for nonlinear initial-value problems

Example 8 *(Quasi-linear parabolic equations).* Let Ω be a smooth bounded domain in $\mathbb{R}^n$. For $p \geqslant \frac{n-2}{n+2}$, the Sobolev space $W_0^{1,p+1}(\Omega) \subset H := L^2(\Omega)$, and so we define on $L^2(\Omega)$ the functional

$$\varphi(u) = \begin{cases} \frac{1}{p+1}\int_\Omega |\nabla u|^{p+1} & \text{on } W_0^{1,p+1}(\Omega), \\ +\infty & \text{elsewhere} \end{cases} \qquad (78)$$

and we let φ^* be its Legendre conjugate. For any $\omega \in \mathbb{R}$, any $u_0 \in W_0^{1,p+1}(\Omega)$ and any $f \in W^{-1,\frac{p+1}{p}}(\Omega)$, the infimum of the functional

$$I(u) = \frac{1}{p+1}\int_0^T \mathrm{e}^{2\omega t}\int_\Omega \left(\left|\nabla u(t,x)\right|^{p+1} - (p+1)f(x)u(x,t)\right)\mathrm{d}x\,\mathrm{d}t + \int_0^T \mathrm{e}^{2\omega t}\varphi^*\left(f - \omega u(t,\cdot) - \frac{\partial u}{\partial t}(t,\cdot)\right)\mathrm{d}t$$
$$- 2\int_\Omega u(0,x)u_0(x)\,\mathrm{d}x + \int_\Omega \left|u_0(x)\right|^2\mathrm{d}x + \frac{1}{2}\int_\Omega \left(\left|u(0,x)\right|^2 + \mathrm{e}^{2T}\left|u(T,x)\right|^2\right)\mathrm{d}x$$

on the space A_H^2 is equal to zero and is attained uniquely at an $W_0^{1,p+1}(\Omega)$-valued path u such that $\int_0^T \|\dot u(t)\|_2^2\,\mathrm{d}t < +\infty$ and which is a solution of the equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_p u + \omega u + f & \text{on } \Omega \times [0,T], \\ u(0,x) = u_0 & \text{on } \Omega, \\ u(t,0) = 0 & \text{on } \partial\Omega. \end{cases} \tag{79}$$

Similarly, we can deal with the equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta_p u - Au + \omega u(t,x) + f & \text{on } \Omega \times [0,T], \\ u(0,x) = u_0 & \text{on } \Omega, \\ u(t,0) = 0 & \text{on } \partial\Omega \end{cases} \tag{80}$$

whenever A is a positive operator on $L^2(\Omega)$.

Example 9 *(Porous media equations).* Let $H = H^{-1}(\Omega)$ equipped with the norm induced by the scalar product $\langle u, v\rangle_{H^{-1}(\Omega)} = \int_\Omega u(-\Delta)^{-1}v\,\mathrm{d}x$. For $m \geqslant \frac{n-2}{n+2}$, we have $L^{m+1}(\Omega) \subset H^{-1}$, and so we can consider the functional

$$\varphi(u) = \begin{cases} \frac{1}{m+1}\int_\Omega |u|^{m+1} & \text{on } L^{m+1}(\Omega), \\ +\infty & \text{elsewhere} \end{cases} \tag{81}$$

and its conjugate

$$\varphi^*(v) = \frac{m}{m+1}\int_\Omega \left|\Delta^{-1}v\right|^{\frac{m+1}{m}}\mathrm{d}x. \tag{82}$$

Then, for any $\omega \in \mathbb{R}$, $u_0 \in H^{-1}(\Omega)$ and $f \in L^2(\Omega)$, the infimum of the functional

$$I(u) = \frac{1}{m+1}\int_0^T \mathrm{e}^{2\omega t}\int_\Omega \left(\left|u(t,x)\right|^{m+1}\mathrm{d}x + m\left|(-\Delta)^{-1}\left(f(x) - \omega u(t,x) - \frac{\partial u}{\partial t}(t,x)\right)\right|^{\frac{m+1}{m}}\right)\mathrm{d}x\,\mathrm{d}t$$
$$- \int_0^T \mathrm{e}^{2\omega t}\int_\Omega u(x,t)(-\Delta)^{-1}f(x)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_\Omega \left|\nabla(-\Delta)^{-1}u_0(x)\right|^2\mathrm{d}x - 2\int_\Omega u_0(x)(-\Delta)^{-1}u(0,x)\,\mathrm{d}x + \frac{1}{2}\left(\left\|u(0)\right\|_{H^{-1}}^2 + \mathrm{e}^{2wT}\left\|u(T)\right\|_{H^{-1}}^2\right)$$

on the space A_H^2 is equal to zero and is attained uniquely at an $L^{m+1}(\Omega)$-valued path u such that $\int_0^T \|\dot u(t)\|_H^2\,\mathrm{d}t < +\infty$ and which is a solution of the equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u^m + \omega u(t,x) + f & \text{on } \Omega \times [0,T], \\ u(0,x) = u_0 & \text{on } \Omega. \end{cases} \tag{83}$$

7.2. Variational resolution for coupled flows and wave-type equations

Again, ASD Lagrangians are suited to treat variationally coupled evolution equations.

Proposition 7.1. *Let φ be a proper convex lower semi-continuous function on $X \times Y$ and let $A : X \to Y^*$ be any bounded linear operator. Assume $B_1 : X \to X$ (resp., $B_2 : Y \to Y$) are positive operators, then for any $(x_0, y_0) \in \mathrm{dom}(\partial\varphi)$ and any $(f, g) \in X^* \times Y^*$, there exists a path $(x(t), y(t)) \in A^2_X \times A^2_Y$ such that*

$$\begin{aligned}
&-\dot{x}(t) - A^* y(t) - B_1 x(t) + f \in \partial_1 \varphi\big(x(t), y(t)\big),\\
&-\dot{y}(t) + Ax(t) - B_2 y(t) + g \in \partial_2 \varphi\big(x(t), y(t)\big),\\
&x(0) = x_0,\\
&y(0) = y_0.
\end{aligned}$$

The solution is obtained as a minimizer on $A^2_X \times A^2_Y$ of the following functional

$$\begin{aligned}
I(x, y) = {}&\int_0^T \big\{\psi\big(x(t), y(t)\big) + \psi^*\big(-A^* y(t) - B_1^a x(t) - \dot{x}(t), Ax(t) - B_2^a y(t) - \dot{y}(t)\big)\big\}\,\mathrm{d}t\\
&+ \frac{1}{2}\big\|x(0)\big\|^2 - 2\big\langle x_0, x(0)\big\rangle + \|x_0\|^2 + \frac{1}{2}\big\|x(T)\big\|^2 + \frac{1}{2}\big\|y(0)\big\|^2 - 2\big\langle y_0, y(0)\big\rangle + \|y_0\|^2 + \frac{1}{2}\big\|y(T)\big\|^2,
\end{aligned}$$

whose infimum is zero. Here B_1^a (resp., B_2^a) are the skew-symmetric parts of B_1 and B_2 and

$$\psi(x, y) = \varphi(x, y) + \frac{1}{2}\langle B_1 x, x\rangle - \langle f, x\rangle + \frac{1}{2}\langle B_2 y, y\rangle - \langle g, x\rangle.$$

Proof. It suffices to apply Theorem 7.1 to the ASD Lagrangian

$$L\big((x, y), (p, q)\big) = \psi(x, y) + \psi^*\big(-A^* y - B_1^a x - p, Ax - B_2^a y - q\big).$$

If $(\bar{x}(t), \bar{y}(t))$ is where the infimum is attained, then we get

$$\begin{aligned}
0 &= I(\bar{x}, \bar{y})\\
&= \int_0^T \big\{\psi\big(\bar{x}(t), \bar{y}(t)\big) + \psi^*\big(-A^* \bar{y}(t) - B_1^a \bar{x}(t) - \dot{\bar{x}}(t), A\bar{x}(t) - B_2^a \bar{y}(t) - \dot{\bar{y}}(t)\big)\\
&\quad - \big\langle\big(\bar{x}(t), \bar{y}(t)\big), \big(-A^* \bar{y}(t) - B_1^a \bar{x}(t) - \dot{\bar{x}}(t), A\bar{x}(t) - B_2^a \bar{y}(t) - \dot{\bar{y}}(t)\big)\big\rangle\big\}\,\mathrm{d}t\\
&\quad + \big\|x(0) - x_0\big\|^2 + \big\|y(0) - y_0\big\|^2.
\end{aligned}$$

It follows that $\bar{x}(0) = x_0$, $\bar{y}(0) = 0$ and the integrand is zero for almost all t which yields

$$\begin{aligned}
&-\dot{x}(t) - A^* y(t) - B_1^a x(t) \in \partial_1 \psi\big(x(t), y(t)\big) = \partial_1 \varphi\big(x(t), y(t)\big) + B_1^s x(t) - f,\\
&-\dot{y}(t) + Ax(t) - B_2^a y(t) \in \partial_2 \psi\big(x(t), y(t)\big) = \partial_2 \varphi\big(x(t), y(t)\big) + B_2^s y(t) - g,\\
&x(0) = x_0,\\
&y(0) = y_0. \qquad \square
\end{aligned}$$

Consider now two convex lower semi-continuous φ_1 and φ_2 on Hilbert spaces X and Y respectively, as well as two positive operators B_1 on X and B_2 on Y. For any $(f, g) \in X \times Y$, consider the convex functionals $\psi_1(x) = \frac{1}{2}\langle B_1 x, x\rangle + \varphi_1(x)$ and $\psi_2(x) = \frac{1}{2}\langle B_2 x, x\rangle + \varphi_2(x)$, and the anti-self-dual Lagrangians

$$\begin{aligned}
L(x, p) &= \psi_1(x) - \langle f, x\rangle + \psi_1^*\big(-B_1^a x + f - p\big), \quad \text{for } (x, p) \in X \times X,\\
M(y, q) &= \psi_2(y) - \langle g, y\rangle + \psi_2^*\big(-B_2^a y + g - q\big), \quad \text{for } (y, q) \in Y \times Y.
\end{aligned}$$

For $w, w' \in \mathbb{R}$, we associate the following time-dependent ASD Lagrangian:

$$L_\omega(t,x,p) := \mathrm{e}^{-2wt} L\big(\mathrm{e}^{wt}x, \mathrm{e}^{wt}p\big) \quad \text{and} \quad M_{\omega'}(t,y,q) = \mathrm{e}^{-2w't} M\big(\mathrm{e}^{w't}y, \mathrm{e}^{w't}q\big).$$

Let $A : X \to Y$ be any bounded linear operator and consider for any $c \in \mathbb{R}$ the following twisted ASD Lagrangian on $X \times Y$

$$(L_\omega \oplus_{c^2 A} M_{\omega'})\big(t,(x,y),(p.q)\big) := L_\omega(t,x,A^*y+p) + M_{\omega'}\big(t,y,-c^2Ax+q\big),$$

where the duality in $X \times Y$ is given by $\langle (x,y),(p,q)\rangle = \langle x,p\rangle + c^{-2}\langle y,q\rangle$. Applying Theorem 7.2, we obtain

Proposition 7.2. *Assume* $0 \in \mathrm{Dom}(\partial\varphi_1)$ *and* $0 \in \mathrm{Dom}(\partial\varphi_2)$*, and consider on* $A_X^2 \times A_Y^2$ *the functional*:

$$\begin{aligned} I(u,v) = {} & \int_0^T \mathrm{e}^{-2\omega t}\big\{\psi_1\big(\mathrm{e}^{\omega t}u(t)\big) + \psi_1^*\big(\mathrm{e}^{\omega t}\big(-A^*v(t) - B_1^a u(t) - \dot u(t)\big)\big)\big\}\,\mathrm{d}t \\ & + \int_0^T \mathrm{e}^{-2\omega' t}\big\{\psi_2\big(\mathrm{e}^{\omega' t}v(t)\big) + \psi_2^*\big(\mathrm{e}^{\omega' t}\big(c^2Au(t) - B_2^a v(t) - \dot v(t)\big)\big)\big\}\,\mathrm{d}t \\ & + \frac12\big\|u(0)\big\|^2 - 2\big\langle x_0, u(0)\big\rangle + \|x_0\|^2 + \frac12\big\|u(T)\big\|^2 + \frac12\big\|v(0)\big\|^2 - 2\big\langle y_0, v(0)\big\rangle + \|y_0\|^2 + \frac12\big\|v(T)\big\|^2. \end{aligned}$$

The minimum of I is then zero and is attained at a path $(\bar x(t), \bar y(t))$, in such a way that $x(t) = \mathrm{e}^{\omega t}\bar x(t)$ and $y(t) = \mathrm{e}^{\omega' t}\bar y(t)$ form a solution of the system of equations

$$\begin{cases} -\dot x(t) + \omega x(t) - A^*y(t) - B_1x(t) + f \in \partial\varphi_1(x(t)), \\ -\dot y(t) + \omega' y(t) + c^2Ax(t) - B_2y(t) + g \in \partial\varphi_2(y(t)), \\ x(0) = x_0, \\ y(0) = y_0. \end{cases} \tag{84}$$

Example 10 *(A variational principle for coupled equations).* Let $\mathbf{b}_1 : \Omega \to \mathbb{R}^n$ and $\mathbf{b}_2 : \Omega \to \mathbb{R}^n$ be two smooth vector fields on a neighborhood of a bounded domain Ω of $\mathbb{R}^n$, verifying the conditions in Example 3, and consider the system of evolution equations:

$$\begin{cases} -\frac{\partial u}{\partial t} - \Delta(v-u) + \mathbf{b}_1\cdot\nabla u = |u|^{p-2}u + f & \text{on } (0,T]\times\Omega, \\ -\frac{\partial v}{\partial t} + \Delta(v + c^2u) + \mathbf{b}_2\cdot\nabla v = |v|^{q-2}v + g & \text{on } (0,T]\times\Omega, \\ u(t,x) = v(t,x) = 0 & \text{on } (0,T]\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{for } x\in\Omega, \\ v(0,x) = v_0(x) & \text{for } x\in\Omega. \end{cases} \tag{85}$$

We then have the following result.

Theorem 7.4. *Assume* $\mathrm{div}(\mathbf{b}_1) \geqslant 0$ *and* $\mathrm{div}(\mathbf{b}_2) \geqslant 0$ *on* Ω*,* $1 < p, q \leqslant \frac{n+2}{n-2}$ *and consider on* $A^2_{H_0^1(\Omega))} \times A^2_{H_0^1(\Omega)}$ *the functional*

$$\begin{aligned} I(u,v) = {} & \int_0^T \Big\{\Psi\big(u(t)\big) + \Psi^*\Big(\mathbf{b}_1\cdot\nabla u(t) + \frac12\mathrm{div}(\mathbf{b}_1)\,u(t) - \Delta v(t) - \dot u(t)\Big)\Big\}\,\mathrm{d}t \\ & + \int_0^T \Big\{\Phi\big(v(t)\big) + \Phi^*\Big(\mathbf{b}_2\cdot\nabla v(t) + \frac12\mathrm{div}(\mathbf{b}_2)v(t) + c^2\Delta u(t) - \dot v(t)\Big)\Big\}\,\mathrm{d}t \\ & + \int_\Omega \Big\{\frac12\big(\big|u(0,x)\big|^2 + \big|u(T,x)\big|^2\big) - 2u(0,x)u_0(x) + \big|u_0(x)\big|^2\Big\}\,\mathrm{d}x \\ & + \int_\Omega \Big\{\frac12\big(\big|v(0,x)\big|^2 + \big|v(T,x)\big|^2\big) - 2v(0,x)v_0(x) + \big|v_0(x)\big|^2\Big\}\,\mathrm{d}x, \end{aligned}$$

where

$$\Psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{1}{p}\int_\Omega |u|^p\,\mathrm{d}x + \int_\Omega f u\,\mathrm{d}x + \frac{1}{4}\int_\Omega \operatorname{div}(\mathbf{b}_1)|u|^2\,\mathrm{d}x,$$

$$\Phi(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x + \frac{1}{q}\int_\Omega |v|^q\,\mathrm{d}x + \int_\Omega g v\,\mathrm{d}x + \frac{1}{4}\int_\Omega \operatorname{div}(\mathbf{b}_2)|v|^2\,\mathrm{d}x$$

and Ψ^* and Φ^* are their Legendre transforms. Then there exists $(\bar u, \bar v) \in A^2_{H^1_0(\Omega))} \times A^2_{H^1_0(\Omega)}$ such that:

$$I(\bar u, \bar v) = \inf\bigl\{I(u, v);\ (u, v) \in A^2_{H^1_0(\Omega))} \times A^2_{H^1_0(\Omega)}\bigr\} = 0,$$

and $(\bar u, \bar v)$ is a solution of (85).

Example 11 *(Pressureless gaz of sticky particles).* Motivated by the recent work of Brenier [5] we consider equations of the form

$$\partial_{tt} X = c^2 \partial_{yy} X - \partial_t \partial_a \mu, \quad \partial_a X \geqslant 0, \quad \mu \geqslant 0, \tag{86}$$

where here $X(t) := X(t, a, y)$ is a function on $K = [0, 1] \times \mathbb{R}/\mathbb{Z}$, and $\mu(t, a, y)$ is a non-negative measure that plays the role of a Lagrange multiplier for the constraint $\partial_a X \geqslant 0$. Following Brenier, we reformulate the problem with the following system:

$$\begin{cases} -\dot X(t) - \frac{\partial U}{\partial y}(t) \in \partial\varphi_1(X(t)), \\ -\dot U(t) + \frac{\partial X}{\partial y}(t) = 0, \\ X(0) = X_0, \\ U(0) = U_0, \end{cases} \tag{87}$$

where φ_1 is the convex function defined on $L^2(K)$ by

$$\varphi_1(X) = \begin{cases} 0 & \text{if } \partial_a X \geqslant 0, \\ +\infty & \text{elsewhere.} \end{cases} \tag{88}$$

We can solve this system with the above method by first setting $\varphi_2(U) = 0$ for every $U \in L^2(K)$ and by considering the Hilbert spaces $X = Y = H^2_{\mathrm{per}}(K)$ to be the subspace of A^2_K consisting of functions that are periodic in y. Define on this space the operator $AX = \frac{\partial X}{\partial y}$ in such a way that $A^* = -A$. We consider now the functional

$$\begin{aligned} I(X, U) = {} & \int_0^T \Bigl\{\varphi_1\bigl(X(t)\bigr) + \varphi_1^*\Bigl(-\frac{\partial U}{\partial y}(t) - \dot X(t)\Bigr)\Bigr\}\,\mathrm{d}t + \int_0^T \Bigl\{\varphi_2^*\Bigl(\frac{\partial X}{\partial y}(t) - \dot U(t)\Bigr)\Bigr\}\,\mathrm{d}t \\ & + \frac{1}{2}\bigl\|X(0)\bigr\|^2 - 2\bigl\langle X_0, X(0)\bigr\rangle + \|X_0\|^2 + \frac{1}{2}\bigl\|X(T)\bigr\|^2 + \frac{1}{2}\bigl\|U(0)\bigr\|^2 - 2\bigl\langle Y_0, U(0)\bigr\rangle \\ & + \|Y_0\|^2 + \frac{1}{2}\bigl\|U(T)\bigr\|^2. \end{aligned}$$

It follows from Theorem 7.1 that if (X_0, U_0) are such that $\partial_a X_0 \geqslant 0$, then the minimum of I is then zero and is attained at a path $(\overline{X}(t), \overline{U}(t)$ which solves the above system of equations.

References

[1] G. Auchmuty, Saddle points and existence–uniqueness for evolution equations, Differential Integral Equations 6 (1993) 1161–1171.

[2] G. Auchmuty, Variational principles for operator equations and initial value problems, Nonlinear Analysis, Theory, Methods and Applications 12 (5) (1988) 531–564.

[3] V. Barbu, Optimal Control of Variational Inequalities, Research Notes in Mathematics, vol. 100, Pitman, 1984.

[4] C. Bardos, Problèmes aux limites pour les equations aux dérivées partielles du premier ordre a coefficients réels ; Théorèmes d'approximation ; Application à l'équation de transport, Ann. Sci. École Norm. Sup. (4) 3 (1970) 185–233.

[5] Y. Brenier, Order preserving vibrating strings and applications to electrodynamics and magnetohydrodynamics, Preprint, 2004.

[6] H. Brezis, Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland, Amsterdam–London, 1973.
[7] H. Brezis, I. Ekeland, Un principe variationnel associé à certaines equations paraboliques. Le cas independant du temps, C. R. Acad. Sci. Paris Sér. A 282 (1976) 971–974.
[8] H. Brezis, I. Ekeland, Un principe variationnel associé à certaines equations paraboliques. Le cas dependant du temps, C. R. Acad. Sci. Paris Sér. A 282 (1976) 1197–1198.
[9] N. Ghoussoub, A variational principle for non-linear transport equations, Comm. Pure Appl. Anal. 4 (4) (2005) 735–742.
[10] N. Ghoussoub, Anti-selfdual Hamiltonians: Variational resolution for Navier–Stokes equations and other nonlinear evolutions, Comm. Pure Applied Math. (2005) 25 pp., in press.
[11] N. Ghoussoub, A class of selfdual partial differential equations and its variational principles (2005), in preparation.
[12] N. Ghoussoub, R. McCann, A least action principle for steepest descent in a non-convex landscape, Contemp. Math. 362 (2004) 177–187.
[13] N. Ghoussoub, A. Moameni, On the existence of Hamiltonian paths connecting Lagrangian submanifolds (2005), submitted for publication.
[14] N. Ghoussoub, A. Moameni, Selfdual variational principles for periodic solutions of Hamiltonian and other dynamical systems, Comm. Partial Differential Equations (2006), in press.
[15] N. Ghoussoub, A. Moameni, Selfduality and periodic solutions of certain Schrödinger equations and infinite dimensional Hamiltonian systems (2006), in preparation.
[16] N. Ghoussoub, L. Tzou, A variational principle for gradient flows, Math. Ann. 30 (3) (2004) 519–549.
[17] N. Ghoussoub, L. Tzou, Anti-selfdual Lagrangians II: Unbounded non self-adjoint operators and evolution equations, Ann. Mat. Pura Appl. (2005) 30 pp.
[18] N. Ghoussoub, L. Tzou, Iterations of anti-selfdual Lagrangians and applications to Hamiltonian systems and multiparameter gradient flows, Calc. Var. Partial Differential Equations (2006) 28 pp., in press.
[19] J. Jost, Riemannian Geometry and Geometric Analysis, Springer University Text, 2002.