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On global smooth solutions to the 3D Vlasov–Nordström system

Sur les solutions régulières du système de Vlasov–Nordström tridimensionnel

Christophe Pallard

D.M.A., École Normale Supérieure, 45, rue d'Ulm, 75230 Paris cedex 05, France

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Abstract

The Vlasov–Nordström system is a relativistic model describing the motion of a self-gravitating collisionless gas. A conditional existence result for global smooth solutions was obtained in [Comm. Partial Differential Equations 28 (2003) 1863–1885]. We give a new proof for this result.

Résumé

Le système de Vlasov–Nordström est un modèle relativiste décrivant l'évolution d'un ensemble de particules massives soumises au champ gravitationnel qu'elles génèrent collectivement. Un théorème d'existence conditionnelle a été démontré dans [Comm. Partial Differential Equations 28 (2003) 1863–1885]. Nous donnons ici une nouvelle preuve de ce résultat.

MSC: 85A05; 82C22

1. Introduction

1.1. The Vlasov–Nordström system

This is a relativistic kinetic model describing the behaviour of a collisionless set of particles interacting through gravitational forces. It may be thought of as a relativistic generalization of the Vlasov–Poisson system, the latter being obtained as its Newtonian limit [5]. Using the framework of Nordström's theory [11], whereby gravitational

E-mail address: pallard@dma.ens.fr.

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effects are mediated by a scalar field, the Vlasov–Nordström system is a much simpler model than the Vlasov–Einstein system. Nevertheless, as it couples Vlasov equation with a hyperbolic equation, it remains less well understood than the standard Vlasov–Poisson system. For more background and references, we refer to [4], where a thorough derivation of the Vlasov–Nordström system can be found. See also [1,6–8,14]. We shall consider the following formulation. The unknowns are functions $f \equiv f(t,x,\xi) \geqslant 0$ and $\phi \equiv \phi(t,x)$ with $(t,x,\xi) \in \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3$, satisfying Vlasov equation

$$Tf = \nabla_\xi \cdot \left[\left((T\phi)\xi + \frac{\nabla_x \phi}{\sqrt{1+|\xi|^2}}\right) f\right] + fT\phi, \tag{1.1}$$

T being the streaming operator $T = \partial_t + v(\xi)\cdot\nabla_x$ and v the relativistic velocity of a particle of momentum ξ:

$$v(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}}.$$

The scalar field ϕ is supposed to solve the wave equation

$$\Box_{t,x}\phi = -\mu, \tag{1.2}$$

with

$$\mu = \int \frac{f\,\mathrm{d}\xi}{\sqrt{1+|\xi|^2}}. \tag{1.3}$$

The Cauchy problem for the Vlasov–Nordström system (VN) consists in Eqs. (1.1), (1.2) and (1.3) together with initial data

$$f_{|t=0} = f_I, \quad \phi_{|t=0} = \phi_I, \quad \partial_t\phi_{|t=0} = \phi'_I. \tag{1.4}$$

In these equations, all physical constants have been set equal to unity. The interpretation of a solution (f,ϕ) is the following: the space-time is a Lorentzian manifold with a conformally flat metric given in coordinates (t,x) by

$$g_{\mu\nu} = \mathrm{e}^{2\phi}\,\mathrm{diag}(-1,1,1,1)$$

and the particle density on the mass shell in this metric is $\mathrm{e}^{-4\phi} f(t,x,\mathrm{e}^{\phi}\xi)$.

This system should be compared to another kinetic model arising in plasma physics, the relativistic Vlasov–Maxwell system (RVM), which describes the behaviour of a collisionless set of charged particles interacting through a self-generated electromagnetic field. In particular, it is known since Glassey and Strauss [10]—and reproved in [3,13]—that smooth solutions to (RVM) do not develop singularities as long as the momentum of particles remains bounded. The corresponding result for (VN) was shown in [6,7] by similar means. Defining the size of the momentum support as

$$R(t) = \sup\bigl\{|\xi|:\ \exists x \in \mathbf{R}^3\ f(t,x,\xi) \neq 0\bigr\}, \tag{1.5}$$

we have the following theorem, established in [6,7].

Theorem 1.1. *Let $\tau > 0$. Let $f \in \mathcal{C}^1([0,\tau) \times \mathbf{R}^3 \times \mathbf{R}^3)$ and $\phi \in \mathcal{C}^2([0,\tau)\times\mathbf{R}^3)$ be a solution of* (VN) *with initial data $f_I \in \mathcal{C}^1_c(\mathbf{R}^3\times\mathbf{R}^3)$, $\phi_I \in \mathcal{C}^3_c(\mathbf{R}^3)$ and $\phi'_I \in \mathcal{C}^2_c(\mathbf{R}^3)$. Then for any $t \in [0,\tau]$ we have*

$$\sup_{s\in[0,t)} R(s) < +\infty \quad \Longrightarrow \quad \|f\|_{W^{1,\infty}([0,t)\times\mathbf{R}^6)} + \|\phi\|_{W^{2,\infty}([0,t)\times\mathbf{R}^3)} < +\infty. \tag{1.6}$$

A corollary of this result is that if a smooth solution blows up in finite time then R becomes infinite. For if it were not the case, the estimates (1.6) would allow to extend the solution as described in [6], p. 1881. The proof of theorem 1.1 in [6] relies essentially on the same procedures than those found in [10]. In this paper, we give a new proof by handling the fields and their derivatives using a method similar to [3], where an alternative derivation of the Glassey–Strauss' theorem is performed.

1.2. Kinetic formulation

The starting point in [3] is an adequate 'kinetic formulation' of the system, which was introduced in [2]. Let us show why this approach is relevant in the context of the Vlasov–Nordström system. Introduce a scalar potential $u \equiv u(t,x,\xi)$ solving the wave equation

$$\Box_{t,x} u = f, \quad u_{|t=0} = 0, \quad \partial_t u_{|t=0} = 0. \tag{1.7}$$

Let ϕ^0 be the solution to

$$\Box_{t,x} \phi^0 = 0, \quad \phi^0_{|t=0} = \phi_I, \quad \partial_t \phi^0_{|t=0} = \phi'_I. \tag{1.8}$$

And define

$$\phi_u = \phi^0 - \int \frac{u \, \mathrm{d}\xi}{\sqrt{1+|\xi|^2}}, \tag{1.9}$$

$$K_u = (T\phi_u)\xi + \frac{\nabla_x \phi_u}{\sqrt{1+|\xi|^2}}. \tag{1.10}$$

Then the Vlasov–Nordström system (VN) is equivalent to

$$\Box_{t,x} u = f, \tag{1.11}$$

$$Tf = \nabla_\xi \cdot (f K_u) + f T\phi_u, \tag{1.12}$$

with initial data

$$f_{|t=0} = f_0, \quad u_{|t=0} = 0, \quad \partial_t u_{|t=0} = 0. \tag{1.13}$$

This representation of the scalar field ϕ_u as a ξ average of u allows a treatment similar to [3]. That is, we derive suitable expressions of the derivatives of ϕ_u by working on the fundamental solution of the wave operator. The benefits of this approach are a unified treatment for all derivatives as well as a natural explanation for a key point in both the present paper and [6], namely the vanishing average of some particular coefficients. We also mention that this method extends to the two-dimensional case studied in [14], see the remarks in [3] on this question. In the next section we recall the so-called division lemma, on which we shall rely heavily. Section 3 is devoted to establishing estimates on f, ϕ_u and their derivatives leading to the proof of Theorem 1.1. We use standard notations. In inequalities, constants that depend on some parameters $\lambda_1, \ldots, \lambda_k$ are denoted by $C(\lambda_1, \ldots, \lambda_k)$ and may change from line to line.

2. A division lemma

Let $Y \in \mathcal{D}'(\mathbf{R}^4)$ be the forward fundamental solution of the wave operator:

$$Y(t,x) = \frac{\mathbf{1}_{t>0}}{4\pi t}\delta\big(|x| - t\big). \tag{2.1}$$

Notice that the distribution Y is homogeneous of degree -2 in $\mathbf{R}^4$. Let $\mathcal{M}_m$ be the space of C^∞ homogeneous functions of degree m on $\mathbf{R}^4 \setminus 0$. Below, we use the notation

$$x_0 := t, \quad \text{and} \quad \partial_j := \partial_{x_j}, \quad j = 0, \ldots, 3. \tag{2.2}$$

The following lemma can be found almost verbatim in [3].

Lemma 2.1 (Division lemma). *For each $\xi \in \mathbf{R}^3$,*

- *there exists functions* $a_i^k \equiv a_i^k(t,x)$ *where* $i = 0,\dots,3$ *and* $k = 0,1$*, such that* $a_i^k \in \mathcal{M}_{-k}$ *and*

$$\partial_i Y = T(a_i^0 Y) + a_i^1 Y, \quad i = 0,\dots,3; \tag{2.3}$$

- *there exists functions* $b_{ij}^k \equiv b_{ij}^k(t,x)$ *with* $i,j = 0,\dots,3$, $k = 0,1,2$*, such that* $b_{ij}^k \in \mathcal{M}_{-k}$ *and*

$$\partial_{ij}^2 Y = T^2(b_{ij}^0 Y) + T(b_{ij}^1 Y) + b_{ij}^2 Y, \quad i,j = 0,\dots,3; \tag{2.4}$$

- *moreover, the functions* b_{ij}^2 *satisfy the conditions*

$$\int_{\mathbf{S}^2} b_{ij}^2(1,y)\,\mathrm{d}\sigma(y) = 0, \quad i,j = 0,\dots,3, \tag{2.5}$$

  where $\mathrm{d}\sigma(y)$ *is the rotation invariant surface element on the unit sphere* $\mathbf{S}^2$ *of* $\mathbf{R}^3$*. In both formulas* (2.3) *and* (2.4)*,* $a_i^0 Y$, $a_i^1 Y$, $b_{ij}^0 Y$ *and* $b_{ij}^1 Y$ *designate, for each* $i,j = 0,\dots,3$*, the unique extensions as homogeneous distributions on* $\mathbf{R}^4$ *of those same expressions—which are a priori only defined on* $\mathbf{R}^4 \setminus 0$*. Likewise,* $b_{ij}^2 Y$ *designates, for* $i,j = 0,\dots,3$ *the unique extension as a homogeneous distribution of degree* -4 *on* $\mathbf{R}^4$ *of that same expressions for which the relation* (2.4) *holds in the sense of distributions on* $\mathbf{R}^4$*.*

Remarks.

1. The proof of Lemma 2 is in [3]. It is based on the commutation properties of the wave operator with the Lorentz boosts.
2. We refer the reader to the reference for the expressions of coefficients $a_i^k(t,x,\xi)$ and $b_{ij}^k(t,x,\xi)$. In the sequel, all we shall need are the following two properties: $a_i^k, b_{ij}^k \in \mathcal{C}^\infty(\mathbf{R}^4 \setminus 0 \times \mathbf{R}^3)$ and for any $\xi \in \mathbf{R}^3$ and $\alpha \in \mathbf{N}^3$ we have $\partial_\xi^\alpha a_i^k(\cdot,\cdot,\xi) \in \mathcal{M}_{-k}$ and $\partial_\xi^\alpha b_{ij}^k(\cdot,\cdot,\xi) \in \mathcal{M}_{-k}$.
3. We recall here some facts about homogeneous distributions. Any homogeneous distribution of degree $k > -3$ on $\mathbf{R}^4 \setminus 0$ has a unique extension on $\mathbf{R}^4$ that is also homogeneous of degree k. A homogeneous distribution of degree -4 on $\mathbf{R}^4 \setminus 0$ may not be extendable on $\mathbf{R}^4$. If such a homogeneous extension exists, then it is not unique: two extensions may differ by a multiple of $\delta_{x=0}$. For more details, see the appendix of [3] and references therein [9,12].

3. Proof of Theorem 1.1

3.1. Estimates on f

We begin by showing that the needed estimates on f and its first derivatives will follow from estimates on ϕ_u. This is done by working on the transport equation satisfied by f. Following [6], we thus rewrite (1.12) as

$$\begin{aligned}
T(\mathrm{e}^{-4\phi_u} f) &= -4\mathrm{e}^{-4\phi_u} f T\phi_u + \mathrm{e}^{-4\phi_u} Tf \\
&= -4\mathrm{e}^{-4\phi_u} f T\phi_u + \mathrm{e}^{-4\phi_u}\big(\nabla_\xi \cdot (fK_u) + fT\phi_u\big) \\
&= -3\mathrm{e}^{-4\phi_u} f T\phi_u + K_u \cdot \nabla_\xi(\mathrm{e}^{-4\phi_u} f) + \mathrm{e}^{-4\phi_u} f \nabla_\xi \cdot K_u.
\end{aligned}$$

The expression of K_u gives

$$\begin{aligned}
\nabla_\xi \cdot K_u &= \nabla_\xi \cdot \left(T\phi_u \xi + \frac{\nabla_x \phi_u}{\sqrt{1+|\xi|^2}}\right) \\
&= (\xi \cdot \nabla_\xi)(v \cdot \nabla_x \phi_u) + 3T\phi_u + (\nabla_x \phi_u) \cdot \nabla_\xi \left(\frac{1}{\sqrt{1+|\xi|^2}}\right).
\end{aligned}$$

A short computation shows that

$$(\xi \cdot \nabla_\xi)(v \cdot \nabla_x \phi_u) = \frac{v \cdot \nabla_x \phi_u}{1+|\xi|^2},$$

and

$$(\nabla_x \phi_u) \cdot \nabla_\xi \left(\frac{1}{\sqrt{1+|\xi|^2}} \right) = -\frac{v \cdot \nabla_x \phi_u}{1+|\xi|^2}.$$

So that we find

$$T(\mathrm{e}^{-4\phi_u} f) - \left(T\phi_u \xi + \frac{\nabla_x \phi_u}{\sqrt{1+|\xi|^2}} \right) \cdot \nabla_\xi (\mathrm{e}^{-4\phi_u} f) = 0. \tag{3.1}$$

The characteristic curves of this equation remain the same as those derived from (1.12). These are curves $t \mapsto (X(t), \Xi(t))$ satisfying

$$X'(t) = v\bigl(\Xi(t)\bigr),$$
$$\Xi'(t) = -(T\phi_u)\bigl(t, X(t), \Xi(t)\bigr)\Xi(t) - \frac{(\nabla_x \phi_u)(t, X(t), \Xi(t))}{\sqrt{1+|\Xi(t)|^2}},$$

with initial data $X(0) = x_0$ and $\Xi(0) = \xi_0$. We infer from (3.1) that $\mathrm{e}^{-4\phi_u} f$ is constant along these curves and we get equality (2.7) of [6]:

$$f\bigl(t, X(t), \Xi(t)\bigr) = f_I(x_0, \xi_0) \exp\bigl(4\phi_u\bigl(t, X(t)\bigr) - 4\phi_I(x_0)\bigr). \tag{3.2}$$

As was observed in [7], u solves the wave equation (1.7) with a right-hand side $f \geqslant 0$ and vanishing initial data, so that $u \geqslant 0$. From (1.9), it comes $\phi_u \leqslant \phi^0$ and we recover proposition 1 of [7]:

$$\bigl\| f(t,\cdot,\cdot) \bigr\|_{L^\infty} \leqslant C(f_I, \phi_I, \phi_I', \tau). \tag{3.3}$$

A look at (3.2) shows that since f_I is compactly supported, the momentum support of $f(t,\cdot,\cdot)$ remains bounded for any $t < \tau$. From now on, we assume

$$\sup_{t \in [0,\tau)} R(t) = r^* < +\infty. \tag{3.4}$$

Differentiating equality (1.12) in x or ξ, we find

$$T(Df) - \nabla_\xi \cdot \bigl((Df)K_u\bigr) = [T, D]f + \nabla_\xi \cdot (f DK_u) + D(fT\phi_u),$$

where D denotes ∂_{x_i} or ∂_{ξ_i}. Therefore with (3.3),

$$\bigl\| f(t,\cdot,\cdot) \bigr\|_{W^{1,\infty}}$$
$$\leqslant C(f_I, \phi_I, \phi_I', \tau, r^*) \left(1 + \int_0^t \bigl\| f(s,\cdot,\cdot) \bigr\|_{W^{1,\infty}} \bigl(1 + \bigl\| \phi_u(s,\cdot) \bigr\|_{W^{2,\infty}} + \bigl\| \partial_t \phi_u(s,\cdot) \bigr\|_{W^{1,\infty}} \bigr) \,\mathrm{d}s \right). \tag{3.5}$$

The next three subsections are devoted to estimating ϕ_u, its first and second derivatives. Note that we aim at using inequality (3.5) with Gronwall's lemma. This requires bounds that do not grow too fast with respect to the quantity $\| f(t,\cdot,\cdot) \|_{W^{1,\infty}}$.

3.2. Bound on ϕ_u

The easiest one. We have to estimate

$$\phi_u = \phi^0 - \int \frac{u\,\mathrm{d}\xi}{\sqrt{1+|\xi|^2}}. \tag{3.6}$$

We recall the following elementary inequalities for the wave equation

$$\|\phi^0\|_{W^{k,\infty}([0,t]\times\mathbf{R}^3)} \leqslant (1+t)\|\phi_I\|_{W^{k+1,\infty}} + t\|\phi_I'\|_{W^{k,\infty}}. \tag{3.7}$$

Thus the first term in (3.6) can be estimated by

$$\left\|\phi^0(t,\cdot)\right\|_{L^\infty} \leqslant (1+t)\|\phi_I\|_{W^{1,\infty}} + t\|\phi_I'\|_{L^\infty}.$$

Let $\chi \in \mathcal{C}_c^\infty(\mathbf{R}^3)$ be a cut-off function such that $\chi(\xi)=1$ when $|\xi| \leqslant r^*$ and vanishing when $|\xi| > 2r^*$. Define

$$m(\xi) = \frac{1}{\sqrt{1+|\xi|^2}}\chi(\xi).$$

From relation (1.7), we know that the momentum support of u and f are equal. Therefore the second term in (3.6) satisfy

$$\int \frac{u(t,x,\xi)\,\mathrm{d}\xi}{\sqrt{1+|\xi|^2}} = \int m(\xi)u(t,x,\xi)\,\mathrm{d}\xi.$$

The function u solves the wave equation (1.7), so that[1]

$$u = Y \star (f\mathbf{1}_{t>0}). \tag{3.8}$$

And since $Y(t,\cdot)$ is a positive measure of total mass t, it comes

$$\left\|\int m(\xi)u(t,\cdot,\xi)\,\mathrm{d}\xi\right\|_{L^\infty} \leqslant \frac{4}{3}\pi r^{*3}\int_0^t (t-s)\left\|f(s,\cdot,\cdot)\right\|_{L^\infty}\mathrm{d}s.$$

With (3.3), we find

$$\left\|\phi_u(t,\cdot)\right\|_{L^\infty} \leqslant C(f_I,\phi_I,\phi_I',\tau,r^*). \tag{3.9}$$

3.3. Bounds on first derivatives of ϕ_u

We intend here to estimate

$$I(t) = \sup_{i=0,\dots,3}\left\|\partial_i\phi_u(t,\cdot)\right\|_{L^\infty}.$$

Derivating (3.6), we find

$$\partial_i\phi_u(t,x) = \partial_i\phi^0(t,x) - \partial_i\int m(\xi)u(t,x,\xi)\,\mathrm{d}\xi,$$

for $i=0,\dots,3$. The first term is estimated with (3.7). It comes

$$\left\|\partial_i\phi^0(t,\cdot)\right\|_{L^\infty} \leqslant C(\phi_I,\phi_I',t).$$

[1] In the sequel, $\star$ denotes convolution in the space and time variables, while $\star_x$ denotes convolution in the space variable only.

Consider now the second term. In view of the remark following (3.5), straightforward estimates on $\partial_i u = Y \star \partial_i(f\mathbf{1}_{t>0})$ would not lead to interesting bounds. Instead, we use (3.8) with Lemma 2.1 to get

$$\partial_i u = (a_i^1 Y) \star (f\mathbf{1}_{t>0}) + (a_i^0 Y) \star T(f\mathbf{1}_{t>0}). \tag{3.10}$$

Besides, we infer from equation (1.12)

$$T(f\mathbf{1}_{t>0}) = (Tf)\mathbf{1}_{t>0} + f_I\delta_{t=0} = \nabla_\xi \cdot (fK_u)\mathbf{1}_{t>0} + f(T\phi_u)\mathbf{1}_{t>0} + f_I\delta_{t=0}.$$

It only remains to get rid of derivatives in the ξ variable by integrating by parts, leading eventually to the expression:

$$\begin{aligned}\partial_i \int m(\xi)u(t,x,\xi)\,\mathrm{d}\xi = &\int m(\xi)\big((a_i^1 Y) \star (f\mathbf{1}_{t>0})\big)(t,x,\xi)\,\mathrm{d}\xi \\ &+ \int m(\xi)\big(\big(a_i^0 Y(t,\cdot)\big) \star_x f_I\big)(x,\xi)\,\mathrm{d}\xi \\ &+ \int \big(\big(-\nabla_\xi(ma_i^0)Y\big) \star (f\mathbf{1}_{t>0}K_u)\big)(t,x,\xi)\,\mathrm{d}\xi \\ &+ \int \big((ma_i^0 Y) \star (f\mathbf{1}_{t>0}T\phi_u)\big)(t,x,\xi)\,\mathrm{d}\xi.\end{aligned}$$

The interest of Lemma 2.1 is now obvious: we don't need to differentiate f in the previous decomposition. Repeatedly using the fact that $Y(t,\cdot)$ is a positive measure of total mass t, we get

$$\begin{aligned}I(t) \leqslant C(\phi_I,\phi_I',t) + \frac{4}{3}\pi r^{*3}\Bigg(&\|mta_i^1\|_{L^\infty}\int_0^t \big\|f(s,\cdot,\cdot)\big\|_{L^\infty}\,\mathrm{d}s + \|ma_i^0\|_{L^\infty} t\|f_I\|_{L^\infty} \\ &+ \|ma_i^0\|_{L^\infty_{t,x}(W^{1,\infty}_\xi)}\int_0^t (t-s)\big\|fK_u(s,\cdot,\cdot)\big\|_{L^\infty}\,\mathrm{d}s + \|ma_i^0\|_{L^\infty}\int_0^t (t-s)\big\|fT\phi_u(s,\cdot,\cdot)\big\|_{L^\infty}\,\mathrm{d}s\Bigg).\end{aligned}$$

It follows from expression (1.10) that

$$\big\|K_u(s,\cdot,\cdot)\big\|_{L^\infty(\mathbf{R}^3\times B(0,r^*))} \leqslant C(r^*)I(s). \tag{3.11}$$

With inequality (3.3) and expression (1.9), we find

$$I(t) \leqslant C(f_I,\phi_I,\phi_I',\tau,r^*)\left(1 + \int_0^t I(s)\,\mathrm{d}s\right). \tag{3.12}$$

Applying Gronwall's lemma to inequality (3.12), it comes

$$\sup_{t\in[0,\tau)} I(t) \leqslant C(f_I,\phi_I,\phi_I',\tau,r^*). \tag{3.13}$$

3.4. Bounds on second derivatives of ϕ_u

We define

$$J(t) = \sup_{i,j=0,\ldots,3}\big\|\partial_{ij}\phi_u(t,\cdot)\big\|_{L^\infty}.$$

Differentiating (3.6) twice,

$$\partial_{ij}\phi_u(t,x) = \partial_{ij}\phi^0(t,x) + \partial_{ij}\int m(\xi)u(t,x,\xi)\,\mathrm{d}\xi,$$

for any $i, j = 0, \dots, 3$. From (3.7), it comes

$$\|\partial_{ij}\phi^0(t,\cdot)\|_{L^\infty} \leqslant C(\phi_I, \phi_I', t). \tag{3.14}$$

Using (3.8) and Lemma 2.1,

$$\partial_{ij}\int m(\xi)u(t,x,\xi)\,\mathrm{d}\xi = \int m(\xi)\big((b_{ij}^2 Y)\star(f\mathbf{1}_{t>0})\big)(t,x,\xi)\,\mathrm{d}\xi + \int m(\xi)\big((b_{ij}^1 Y)\star T(f\mathbf{1}_{t>0})\big)(t,x,\xi)\,\mathrm{d}\xi$$
$$+\int m(\xi)\big((b_{ij}^0 Y)\star T^2(f\mathbf{1}_{t>0})\big)(t,x,\xi)\,\mathrm{d}\xi = S_0 + S_1 + S_2.$$

Estimates for S_0. The key point here is the fact that the average of the coefficients b_{ij}^2 vanishes, which allows us to obtain sharp estimates for S_0. As will be seen below, the contribution of this term to $J(t)$ is crucial. First, let us determine a homogeneous extension of $b_{ij}^2 Y$ on $\mathbf{R}^4$. Let $\phi \in \mathcal{C}_c^\infty(\mathbf{R}^4 \setminus 0)$ be a test function and consider

$$\langle b_{ij}^2 Y, \phi\rangle = \int_0^\infty \int_{|y|=1} b_{ij}^2(1,y,\xi)\phi(t,ty)\frac{\mathrm{d}S_y}{4\pi t}\,\mathrm{d}t,$$

where we used the homogeneity of $b_{ij}^2(\cdot,\cdot,\xi) \in \mathcal{M}_{-2}$ for any ξ. Since b_{ij}^2 satisfy (2.5), the following equality holds for any $\theta \geqslant 0$:

$$\langle b_{ij}^2 Y, \phi\rangle = \int_0^\theta \int_{|y|=1} b_{ij}^2(1,y,\xi)\big(\phi(t,ty)-\phi(t,0)\big)\frac{\mathrm{d}S_y}{4\pi t}\,\mathrm{d}t + \int_\theta^\infty \int_{|y|=1} b_{ij}^2(1,y,\xi)\phi(t,ty)\frac{\mathrm{d}S_y}{4\pi t}\,\mathrm{d}t. \tag{3.15}$$

But the right-hand side of (3.15) still makes sense for test functions on $\mathbf{R}^4$. Denote by $\mathrm{p.v.}(b_{ij}^2 Y)$ the distribution defined by this expression.[2] This is a homogeneous distribution of degree -4 on $\mathbf{R}^4$ that extends $b_{ij}^2 Y$. It follows from the third remark in Section 2 the relation

$$b_{ij}^2 Y - \mathrm{p.v.}(b_{ij}^2 Y) = c(\xi)\delta_{(t,x)=(0,0)},$$

where $c_{ij} \in \mathcal{C}^\infty(\mathbf{R}^3)$; indeed, the left-hand side of this equality is smooth as a function of ξ – see the second remark below the lemma. Thus, for θ_t to be chosen later,

$$S_0 - \int m(\xi)c_{ij}(\xi)f(t,x,\xi)\,\mathrm{d}\xi = \int m(\xi)\big(\mathrm{p.v.}(b_{ij}^2 Y)\star(f\mathbf{1}_{t>0})\big)(t,x,\xi)\,\mathrm{d}\xi$$
$$= \int m(\xi)\int_0^{\theta_t}\int_{|y|=1} b_{ij}^2(1,y,\xi)\big(f(t-s,x-sy,\xi)-f(t-s,x,\xi)\big)\frac{\mathrm{d}S_y}{4\pi s}\,\mathrm{d}s\,\mathrm{d}\xi$$
$$+\int m(\xi)\int_{\theta_t}^{t}\int_{|y|=1} b_{ij}^2(1,y,\xi)f(t-s,x-sy,\xi)\frac{\mathrm{d}S_y}{4\pi s}\,\mathrm{d}s\,\mathrm{d}\xi.$$

For the first term in the right-hand side, we write

$$\left|\int_0^{\theta_t}\int_{|y|=1} b_{ij}^2(1,y,\xi)\big(f(t-s,x-sy,\xi)-f(t-s,x,\xi)\big)\frac{\mathrm{d}S_y}{4\pi s}\,\mathrm{d}s\right|$$
$$\leqslant \theta_t\big\|b_{ij}^2(1,\cdot,\xi)\big\|_{L^\infty(\mathbf{S}^2)}\|\nabla_x f\|_{L^\infty([0,t)\times\mathbf{R}^6)}.$$

[2] p.v. stands for principal value.

For the second term, we have

$$\left| \int_{\theta_t}^{t} \int_{|y|=1} b_{ij}^2(1,y,\xi) f(t-s,x-sy,\xi) \frac{\mathrm{d}S_y}{4\pi s}\,\mathrm{d}s \right| \leqslant \ln\left(\frac{t}{\theta_t}\right) \left\| b_{ij}^2(1,\cdot,\xi)\right\|_{L^\infty(\mathbf{S}^2)} \|f\|_{L^\infty([0,t]\times\mathbf{R}^6)}.$$

Thus if we choose

$$\theta_t = \inf\left(\frac{1}{\|\nabla_x f\|_{L^\infty([0,t]\times\mathbf{R}^6)}}, t\right)$$

we get

$$\begin{aligned} |S_0| \leqslant{}& Cr^{*3}\|m\|_{L^\infty}\big[\|c_{ij}\|_{L^\infty(B(0,r^{*3}))}\|f\|_{L^\infty([0,t]\times\mathbf{R}^6)} + \|b_{ij}^2\|_{L^\infty(\mathbf{S}^2\times\mathbf{R}^3)} \\ &\times \big(1+\|f\|_{L^\infty([0,t]\times\mathbf{R}^6)}\ln\big(1+t\|\nabla_x f\|_{L^\infty([0,t]\times\mathbf{R}^6)}\big)\big)\big]. \end{aligned}$$

In view of (3.3), this gives

$$|S_0| \leqslant C(f_I,\phi_I,\phi_I',\tau,r^*)\big(1+\ln\big(1+t\|\nabla_x f\|_{L^\infty([0,t]\times\mathbf{R}^6)}\big)\big). \tag{3.16}$$

Estimates for S_1. This term is very similar to the one arising from the second part of the right-hand side of (3.10). We find

$$\begin{aligned} S_1 ={}& \int m(\xi)\big(\big(b_{ij}^1 Y(t,\cdot)\big) \star_x f_I\big)(x,\xi)\,\mathrm{d}\xi + \int \big(\big(-\nabla_\xi(mb_{ij}^1)Y\big)\star(f\mathbf{1}_{t>0}K_u)\big)(t,x,\xi)\,\mathrm{d}\xi \\ &+ \int \big((mb_{ij}^1 Y)\star(f\mathbf{1}_{t>0}T\phi_u)\big)(t,x,\xi)\,\mathrm{d}\xi. \end{aligned}$$

The only difference with the estimates following (3.10) is the fact that $b_{ij}^1 \in \mathcal{M}_{-1}$ whereas $a_i^0 \in \mathcal{M}_0$. Consequently,

$$\begin{aligned} |S_1| \leqslant{}& \frac{4}{3}\pi r^{*3}\Big(\|mtb_{ij}^1\|_{L^\infty}\|f_I\|_{L^\infty} + \|mtb_{ij}^1\|_{L^\infty_{t,x}(W^{1,\infty}_\xi)} \int_0^t \left\|fK_u(s,\cdot,\cdot)\right\|_{L^\infty}\mathrm{d}s \\ &+ \|mtb_{ij}^1\|_{L^\infty} \int_0^t \left\|fT\phi_u(s,\cdot,\cdot)\right\|_{L^\infty}\mathrm{d}s\Big). \end{aligned}$$

With (3.3), (3.11) and (3.13), we infer that S_1 is bounded by a constant:

$$|S_1| \leqslant C(f_I,\phi_I,\phi_I',\tau,r^*). \tag{3.17}$$

Estimates for S_2. This last term requires lengthy computations but the strategy remains the same as above: our goal is to avoid differentiating f by using Eq. (1.12). Let us start with

$$\begin{aligned} T^2(f\mathbf{1}_{t>0}) &= T(\delta_{t=0}f_I) + T\big(\mathbf{1}_{t>0}\big(\nabla_\xi\cdot(fK_u)+fT\phi_u\big)\big) \\ &= \delta_{t=0}'f_I + \delta_{t=0}\big(v\cdot\nabla_x f_I + \nabla_\xi\cdot(f_I K_u^I) + f_I\phi_I' + f_I v\cdot\nabla_x\phi_I\big) \\ &\quad + \mathbf{1}_{t>0}T\big(\nabla_\xi\cdot(fK_u)\big) + \mathbf{1}_{t>0}T(fT\phi_u). \end{aligned}$$

Working on the last two terms, we find:

$$\begin{aligned} T\big(\nabla_\xi\cdot(fK_u)\big) &= \nabla_\xi\cdot\big(fTK_u + (\nabla_\xi\cdot(fK_u)+fT\phi_u)K_u\big) + [T,\nabla_\xi\cdot](fK_u) \\ &= \nabla_\xi\cdot\big(fTK_u + f(T\phi_u)K_u\big) + \nabla_\xi^{\otimes 2}: fK_u^{\otimes 2} - (\nabla_\xi v)^T : \nabla_x(fK_u). \end{aligned}$$

Note that the last term, which arises from the commutator, will require further computations. Besides,

$$\begin{aligned}T(fT\phi_u) &= (Tf)T\phi_u + fT^2\phi_u\\ &= \nabla_\xi\cdot(fK_u)T\phi_u + f(T\phi_u)^2 + fT^2\phi_u\\ &= \nabla_\xi\cdot\big(f(T\phi_u)K_u\big) - (fK_u)\cdot\nabla_\xi(T\phi_u) + f(T\phi_u)^2 + fT^2\phi_u\\ &= \nabla_\xi\cdot\big(f(T\phi_u)K_u\big) - \big((fK_u)\cdot\nabla_\xi v\big)\cdot\nabla_x\phi_u + f(T\phi_u)^2 + fT^2\phi_u.\end{aligned}$$

This leads to the following decomposition:

$$\begin{aligned}T^2(f\mathbf{1}_{t>0}) = {}&\delta'_{t=0}f_I + \delta_{t=0}\big(v\cdot\nabla_x f_I + \nabla_\xi\cdot(f_IK_u^I) + f_I\phi'_I + f_Iv\cdot\nabla_x\phi_I\big)\\ &+ \mathbf{1}_{t>0}\nabla_\xi\cdot\big(fTK_u + 2f(T\phi_u)K_u\big) + \mathbf{1}_{t>0}\nabla_\xi^{\otimes 2} : fK_u^{\otimes 2}\\ &- (\nabla_\xi v)^T : \nabla_x(f\mathbf{1}_{t>0}K_u) - f\mathbf{1}_{t>0}(K_u\cdot\nabla_\xi v)\cdot\nabla_x\phi_u + f\mathbf{1}_{t>0}\big(T^2\phi_u + (T\phi_u)^2\big).\end{aligned}$$

We are now ready to integrate in the ξ variable. The corresponding derivatives are removed by integrating by parts. Thus S_2 can be written as a sum $S'_{20} + S_{20} + S_{21} + S_{22} + S_{23} + S_{24} + S_{25}$ with

$$\begin{aligned}S'_{20} &= \int m(\xi)(b^0_{ij}Y)\star\big(\delta'_{t=0}f_I\big)\,\mathrm{d}\xi,\\ S_{20} &= \int m(\xi)(b^0_{ij}Y)\star\big(\delta_{t=0}\big(v\cdot\nabla_x f_I + \nabla_\xi\cdot(f_IK_u^I) + f_I\phi'_I + f_Iv\cdot\nabla_x\phi_I\big)\big)\,\mathrm{d}\xi,\\ S_{21} &= \int\big(-\nabla_\xi(mb^0_{ij})Y\big)\star\big(f\mathbf{1}_{t>0}\big(TK_u + 2(T\phi_u)K_u\big)\big)(t,x,\xi)\,\mathrm{d}\xi,\\ S_{22} &= \int\big(\nabla_\xi^{\otimes 2}(mb^0_{ij}Y)\star(f\mathbf{1}_{t>0}K_u^{\otimes 2})\big)(t,x,\xi)\,\mathrm{d}\xi,\\ S_{23} &= \int m(\xi)\big(\big(\nabla_\xi v\cdot\nabla_x(b^0_{ij}Y)\big)\star(f\mathbf{1}_{t>0}K_u)\big)(t,x,\xi)\,\mathrm{d}\xi,\\ S_{24} &= \int m(\xi)\big((b^0_{ij}Y)\star\big(f\mathbf{1}_{t>0}(K_u\cdot\nabla_\xi v)\cdot\nabla_x\phi_u\big)\big)(t,x,\xi)\,\mathrm{d}\xi,\\ S_{25} &= \int m(\xi)\big((b^0_{ij}Y)\star\big(f\mathbf{1}_{t>0}(T^2\phi_u + (T\phi_u)^2)\big)\big)(t,x,\xi)\,\mathrm{d}\xi.\end{aligned}$$

The first two terms only involve initial data. They are estimated by

$$\begin{aligned}|S'_{20} + S_{20}| \leqslant {}&\frac{4}{3}\pi r^{*3}\|mb^0_{ij}\|_{L^\infty_x(W^{1,\infty}_{t,\xi})}(1+t)^2\|f_I\|_{W^{1,\infty}}\\ &\times\big(1 + \|K_u^I\|_{L^\infty(R^3\times B(0,r^*))} + \|\phi_I\|_{W^{1,\infty}} + \|\phi'_I\|_{L^\infty}\big).\end{aligned}$$

The third, fourth, sixth and last terms are estimated in a familiar way:

$$|S_{21}| \leqslant \frac{4}{3}\pi r^{*3}\|mb^0_{ij}\|_{L^\infty_{t,x}(W^{1,\infty}_\xi)}\int_0^t(t-s)\big\|f\big(TK_u + 2(T\phi_u)K_u\big)(s,\cdot,\cdot)\big\|_{L^\infty}\,\mathrm{d}s,$$

$$|S_{22}| \leqslant \frac{4}{3}\pi r^{*3}\|mb^0_{ij}\|_{L^\infty_{t,x}(W^{2,\infty}_\xi)}\int_0^t(t-s)\big\|fK_u^{\otimes 2}(s,\cdot,\cdot)\big\|_{L^\infty}\,\mathrm{d}s,$$

$$|S_{24}| \leqslant \frac{4}{3}\pi r^{*3}\|mb^0_{ij}\|_{L^\infty}\int_0^t(t-s)\big\|f(K_u\cdot\nabla_\xi v)\cdot\nabla_x\phi_u(s,\cdot,\cdot)\big\|_{L^\infty}\,\mathrm{d}s,$$

$$|S_{25}| \leqslant \frac{4}{3}\pi r^{*3}\|mb_{ij}^0\|_{L^\infty}\int_0^t (t-s)\big\|f\big(T^2\phi_u+(T\phi_u)^2\big)(s,\cdot,\cdot)\big\|_{L^\infty}\,\mathrm{d}s.$$

Expression (1.10) shows that

$$\big\|TK_u(s,\cdot,\cdot)\big\|_{L^\infty(\mathbf{R}^3\times B(0,r^*)))} \leqslant C(r^*)J(s).$$

Using estimates (3.3) and (3.13), it comes then

$$|S_{21}+S_{22}+S_{24}+S_{25}| \leqslant C(f_I,\phi_I,\phi_I',\tau,r^*)\left(1+\int_0^t J(s)\,\mathrm{d}s\right).$$

As said above, the remaining term S_{23} requires an additional step. We brought the derivatives to the left side of the convolution in order to use Lemma 2.1 one more time. We have

$$\partial_k(b_{ij}^0Y) = T(b_{ij}^0a_k^0Y) + \big(b_{ij}^0a_k^1 - a_k^0T(b_{ij}^0) + \partial_k b_{ij}^0\big)Y,$$

which yields

$$\nabla_\xi v\cdot\nabla_x(b_{ij}^0Y) = T(c_{ij}^0Y) + c_{ij}^1Y,$$

where we set

$$c_{ij}^0 = b_{ij}^0\nabla_\xi v\cdot a^0,$$
$$c_{ij}^1 = b_{ij}^0\nabla_\xi v\cdot a^1 - (\nabla_\xi v\cdot a^0)Tb_{ij}^0 + \nabla_\xi v\cdot\nabla_x b_{ij}^0.$$

Therefore S_{23} can be written as

$$S_{23} = \int m(\xi)\big((c_{ij}^0Y)\star T(f\mathbf{1}_{t>0}K_u)\big)(t,x,\xi)\,\mathrm{d}\xi + \int m(\xi)\big((c_{ij}^1Y)\star(f\mathbf{1}_{t>0}K_u)\big)(t,x,\xi)\,\mathrm{d}\xi.$$

Using another time the transport equation,

$$T(f\mathbf{1}_{t>0}K_u) = f_IK_u^I\delta_{t=0} + \mathbf{1}_{t>0}fTK_u + \mathbf{1}_{t>0}\nabla_\xi\cdot(fK^{\otimes 2}) - \mathbf{1}_{t>0}f(K_u\cdot\nabla_\xi)K_u + f(T\phi_u)K_u,$$

it is now routine work to see that

$$|S_{23}| \leqslant C(f_I,\phi_I,\phi_I',\tau,r^*)\left(1+\int_0^t J(s)\,\mathrm{d}s\right).$$

Using (3.13) and gathering the inequalities above, we infer that

$$|S_2| \leqslant C(f_I,\phi_I,\phi_I',\tau,r^*)\left(1+\int_0^t J(s)\,\mathrm{d}s\right). \tag{3.18}$$

Collecting estimates (3.14), (3.16), (3.17) and (3.18),

$$J(t) \leqslant C(f_I,\phi_I,\phi_I',\tau,r^*)\left(1+\ln(1+t\|\nabla_x f\|_{L^\infty([0,t]\times\mathbf{R}^6)}) + \int_0^t J(s)\,\mathrm{d}s\right)$$

for any $0<t<\tau$. Applying Gronwall's lemma, we get for $0<t<\tau$,

$$J(t) \leqslant C(f_I,\phi_I,\phi_I',\tau,r^*)\ln\big(1+t\|\nabla_x f\|_{L^\infty([0,t]\times\mathbf{R}^6)}\big). \tag{3.19}$$

Note that the behaviour of this bound is governed by the contribution from the most singular term, namely S_0.

3.5. Proof of Theorem 1.1

With (3.9) and (3.13), (3.19) yields

$$\|\phi_u\|_{W^{2,\infty}([0,t]\times\mathbf{R}^3)} \leqslant C(f_I,\phi_I,\phi_I',\tau,r^*)\bigl(1+\ln\bigl(1+\|f\|_{W^{1,\infty}([0,t]\times\mathbf{R}^6)}\bigr)\bigr). \tag{3.20}$$

Using this in (3.5) gives

$$\bigl\|f(t,\cdot,\cdot)\bigr\|_{W^{1,\infty}} \leqslant C(f_I,\phi_I,\phi_I',\tau,r^*)\left(1+\int_0^t \bigl\|f(s,\cdot,\cdot)\bigr\|_{W^{1,\infty}}\bigl(1+\ln\bigl(1+\|f\|_{W^{1,\infty}([0,s]\times\mathbf{R}^6)}\bigr)\bigr)\,\mathrm{d}s\right).$$

The growth rate in this estimate is decisive and allows the use of a logarithmic Gronwall's lemma, showing that

$$\|f\|_{W^{1,\infty}([0,\tau)\times\mathbf{R}^6)} \leqslant C(f_I,\phi_I,\phi_I',\tau,r^*).$$

We eventually infer from (3.20) the expected estimate

$$\|\phi_u\|_{W^{2,\infty}([0,\tau)\times\mathbf{R}^6)} \leqslant C(f_I,\phi_I,\phi_I',\tau,r^*).$$

This ends the proof of Theorem 1.1.

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