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Semilinear equations with exponential nonlinearity and measure data

Équations semi linéaires avec non linéarité exponentielle et données mesures

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Abstract

We study the existence and non-existence of solutions of the problem

$$\begin{cases} -\Delta u + \mathrm{e}^u - 1 = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geqslant 3$, and μ is a Radon measure. We prove that if $\mu \leqslant 4\pi\mathcal{H}^{N-2}$, then (0.1) has a unique solution. We also show that the constant 4π in this condition cannot be improved.

Résumé

Nous étudions l'existence et la non existence des solutions de l'équation

$$\begin{cases} -\Delta u + \mathrm{e}^u - 1 = \mu & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases} \tag{0.2}$$

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où Ω est un domaine borné dans $\mathbb{R}^N$, $N \geqslant 3$, et μ est une mesure de Radon. Nous démontrons que si μ vérifie $\mu \leqslant 4\pi \mathcal{H}^{N-2}$, alors le problème (0.2) admet une unique solution. Nous montrons que la constante 4π dans cette condition ne peut pas être améliorée.

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Keywords: Nonlinear elliptic equations; Exponential nonlinearity; Measure data

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a bounded domain with smooth boundary. We consider the problem

$$\begin{cases} -\Delta u + \mathrm{e}^u - 1 = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mu \in \mathcal{M}(\Omega)$, the space of bounded Radon measures in Ω. We say that a function u is a solution of (1.1) if $u \in L^1(\Omega)$, $\mathrm{e}^u \in L^1(\Omega)$ and the following holds:

$$-\int_\Omega u\Delta\zeta + \int_\Omega (\mathrm{e}^u - 1)\zeta = \int_\Omega \zeta\,\mathrm{d}\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}). \tag{1.2}$$

Here $C_0^2(\overline{\Omega})$ denotes the set of functions $\zeta \in C^2(\overline{\Omega})$ such that $\zeta = 0$ on $\partial\Omega$. A measure μ is a *good measure* for problem (1.1) if (1.1) has a solution. We shall denote by $\mathcal{G}$ the set of good measures. Problem (1.1) has been recently studied by Brezis, Marcus and Ponce in [1], where the general case of a continuous nondecreasing nonlinearity $g(u)$, with $g(0) = 0$, is dealt with. Applying Theorem 1 of [1] to $g(u) = \mathrm{e}^u - 1$, it follows that for every $\mu \in \mathcal{M}(\Omega)$ there exists a largest good measure $\leqslant \mu$ for (1.1), which we shall denote by μ^*.

In the case $N = 2$, the set of good measures for problem (1.1) has been characterized by Vázquez in [9]. More precisely, a measure μ is a good measure if and only if $\mu(\{x\}) \leqslant 4\pi$ for every x in Ω. Note that any $\mu \in \mathcal{M}(\Omega)$ can be decomposed as

$$\mu = \mu_0 + \sum_{i=1}^{\infty} \alpha_i \delta_{x_i},$$

with $\mu_0(\{x\}) = 0$ for every x in Ω, and δ_{x_i} is the Dirac mass concentrated at x_i. Using Vázquez's result, it is not difficult to check that (see [1, Example 5])

$$\mu^* = \mu_0 + \sum_{i=1}^{\infty} \min\{4\pi, \alpha_i\}\,\delta_{x_i}.$$

This paper is devoted to the study of problem (1.1) in the case $N \geqslant 3$. First of all, let us recall that if μ is a good measure, then (1.1) has a unique solution u (see [1, Corollary B.1]). This solution can be either obtained as the limit of the sequence (u_n) of solutions of

$$\begin{cases} -\Delta u_n + \min\{\mathrm{e}^{u_n} - 1, n\} = \mu & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

or as the limit of a sequence (v_n) of solutions of

$$\begin{cases} -\Delta v_n + \mathrm{e}^{v_n} - 1 = \mu_n & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\mu_n = \rho_n * \mu$, where (ρ_n) is a sequence of mollifiers. If μ is not a good measure, then both sequences (u_n) and (v_n) converge to the solution u^* of problem (1.1) with datum μ^* (see [1]). It has also been proved in [1] that the set $\mathcal{G}$ of good measures is convex and closed with respect to the strong topology in $\mathcal{M}(\Omega)$. Moreover, it is easy to see that if $\nu \leqslant \mu$ and $\mu \in \mathcal{G}$, then $\nu \in \mathcal{G}$.

Before stating our results, let us briefly recall the definitions of Hausdorff measure and Hausdorff dimension of a set. Let $s \geqslant 0$, and let $A \subset \mathbb{R}^N$ be a Borel set. Given $\delta > 0$, let

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_i \omega_s r_i^s\colon\ K \subset \bigcup_i B_{r_i} \text{ with } r_i < \delta,\ \forall i\right\},$$

where the infimum is taken over all coverings of A with open balls B_{r_i} of radius $r_i < \delta$, and $\omega_s = \pi^{s/2}/\Gamma(s/2+1)$. We define the (spherical) s-dimensional Hausdorff measure in $\mathbb{R}^N$ as

$$\mathcal{H}^s(A) = \lim_{\delta\downarrow 0} \mathcal{H}^s_\delta(A),$$

and the Hausdorff dimension of A as

$$\dim_{\mathcal{H}}(A) = \inf\{s \geqslant 0\colon\ \mathcal{H}^s(A) = 0\}.$$

Given a measure μ in $\mathcal{M}(\Omega)$, we say that it is concentrated on a Borel set $E \subset \Omega$ if $\mu(A) = \mu(E \cap A)$ for every Borel set $A \subset \Omega$. Given a measure μ in $\mathcal{M}(\Omega)$, and a Borel set $E \subset \Omega$, the measure $\mu \llcorner E$ is defined by $\mu \llcorner E(A) = \mu(E \cap A)$ for every Borel set $A \subset \Omega$.

One of our main results is the following

Theorem 1. *Let $\mu \in \mathcal{M}(\Omega)$. If $\mu \leqslant 4\pi\mathcal{H}^{N-2}$, that is, if $\mu(A) \leqslant 4\pi\mathcal{H}^{N-2}(A)$ for every Borel set $A \subset \Omega$ such that $\mathcal{H}^{N-2}(A) < \infty$, then there exists a unique solution u of* (1.1).

As a corollary of Theorem 1, we have

Corollary 1. *Let $\mu \in \mathcal{M}(\Omega)$. If $\mu \leqslant 4\pi\mathcal{H}^{N-2}$, then $\mu^* = \mu$.*

The proof of Theorem 1 relies on a decomposition lemma for Radon measures (see Section 3 below) and on the following sharp estimate concerning the exponential summability for solutions of the Laplace equation. We denote by $\mathrm{M}^{N/2}(\Omega)$ the Morrey space with exponent $\frac{N}{2}$ equipped with the norm $\|\cdot\|_{N/2}$ (see Definition 1 below).

Theorem 2. *Let f be a function in $\mathrm{M}^{N/2}(\Omega)$, and let u be the solution of*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Then, for every $0 < \alpha < 2N\omega_N$, it holds

$$\int_\Omega \mathrm{e}^{((2N\omega_N - \alpha)/\|f\|_{N/2})|u|} \leqslant \frac{(N\omega_N)^2}{\alpha} \operatorname{diam}(\Omega)^N. \tag{1.4}$$

This theorem is the counterpart in the case $N \geqslant 3$ of a result proved, for $N = 2$ and $f \in L^1(\Omega)$, by Brezis and Merle in [2]. Note that, for $N = 2$, the space $\mathrm{M}^{N/2}(\Omega)$ coincides with $L^1(\Omega)$.

As a consequence of Theorem 1, we have that the set of good measures $\mathcal{G}$ contains all measures μ which satisfy $\mu \leqslant 4\pi\,\mathcal{H}^{N-2}$. If $N = 2$, then the result of Vázquez states that the converse is also true. In our case, that is $N \geqslant 3$, this is *false*. After this work was completed, A.C. Ponce found explicit examples of good measures which are not

$\leqslant 4\pi\mathcal{H}^{N-2}$ (see [7, Theorems 2 and 3]). The existence of such measures was conjectured by L. Véron in a personal communication.

We now present some necessary conditions a measure $\mu \in \mathcal{G}$ has to satisfy. We start with the following

Theorem 3. *Let $\mu \in \mathcal{M}(\Omega)$. If $\mu(A) > 0$ for some Borel set $A \subset \Omega$ such that $\dim_{\mathcal{H}}(A) < N-2$, then* (1.1) *has no solution.*

Observe that in the case of dimension $N = 2$, no measure μ satisfies the assumptions of Theorem 3.

As a consequence of Theorem 3 we have

Corollary 2. *Let $\mu \in \mathcal{M}(\Omega)$. If μ^+ is concentrated on a Borel set $A \subset \Omega$ with $\dim_{\mathcal{H}}(A) < N-2$, then $\mu^* = -\mu^-$.*

The next theorem, which is one of the main results of this paper, states that there exists no solution of (1.1) if μ is strictly larger than $4\pi\,\mathcal{H}^{N-2}$ on an $(N-2)$-rectifiable set.

Theorem 4. *Let $\mu \in \mathcal{M}(\Omega)$. Assume there exist $\varepsilon > 0$ and an $(N-2)$-rectifiable set $E \subset \Omega$, with $\mathcal{H}^{N-2}(E) > 0$, such that $\mu \llcorner E \geqslant (4\pi + \varepsilon)\,\mathcal{H}^{N-2} \llcorner E$. Then,* (1.1) *has no solution.*

Corollary 3. *Assume $\mu = \alpha(x)\mathcal{H}^{N-2} \llcorner E$, where $E \subset \Omega$ is $(N-2)$-rectifiable and α is $\mathcal{H}^{N-2} \llcorner E$-integrable. Then, $\mu^* = \min\{4\pi, \alpha(x)\}\mathcal{H}^{N-2} \llcorner E$.*

In Theorem 4 (and also in Corollary 3), the assumption that E is $(N-2)$-rectifiable is important. In fact, one can find $(N-2)$-unrectifiable sets $F \subset \Omega$, with $0 < \mathcal{H}^{N-2}(F) < \infty$, such that $\nu = \alpha\mathcal{H}^{N-2} \llcorner F$ is a good measure for *every* $\alpha > 0$ (see [7]).

As a consequence of the previous results, we can derive some information on μ^*. To this extent, let $\mu \in \mathcal{M}(\Omega)$. Since $e^u - 1$ is bounded for $u < 0$, μ^- will play no role in the existence-nonexistence theory for (1.1). Therefore, we only have to deal with μ^+, which we recall can be uniquely decomposed as

$$\mu^+ = \mu_1 + \mu_2 + \mu_3, \tag{1.5}$$

where

$$\mu_1(A) = 0 \quad \text{for every Borel set } A \subset \Omega \text{ such that } \mathcal{H}^{N-2}(A) < \infty, \tag{1.6}$$

$$\mu_2 = \alpha(x)\mathcal{H}^{N-2} \llcorner E \quad \text{for some Borel set } E \subset \Omega \text{, and some } \mathcal{H}^{N-2}\text{-measurable } \alpha, \tag{1.7}$$

$$\mu_3(\Omega \backslash F) = 0 \quad \text{for some Borel set } F \subset \Omega \text{ with } \mathcal{H}^{N-2}(F) = 0. \tag{1.8}$$

By a result of Federer (see [4] and also [6, Theorem 15.6]), the set E can be uniquely decomposed as a disjoint union $E = E_1 \cup E_2$, where E_1 is $(N-2)$-rectifiable and E_2 is purely $(N-2)$-unrectifiable. In particular,

$$\mu_2 = \alpha(x)\mathcal{H}^{N-2} \llcorner E_1 + \alpha(x)\mathcal{H}^{N-2} \llcorner E_2. \tag{1.9}$$

Combining Corollaries 1–3, we establish the following

Theorem 5. *Given $\mu \in \mathcal{M}(\Omega)$, decompose μ^+ as in* (1.5)–(1.9). *Then,*

$$\mu^* = (\mu_1)^* + (\mu_2)^* + (\mu_3)^* - \mu^-. \tag{1.10}$$

In addition,

$$(\mu_1)^* = \mu_1, \tag{1.11}$$
$$(\mu_2)^* = \big(\alpha(x)\mathcal{H}^{N-2} \llcorner E_1\big)^* + \big(\alpha(x)\mathcal{H}^{N-2} \llcorner E_2\big)^*, \tag{1.12}$$
$$\big(\alpha(x)\mathcal{H}^{N-2} \llcorner E_1\big)^* = \min\{4\pi, \alpha(x)\}\mathcal{H}^{N-2} \llcorner E_1, \tag{1.13}$$
$$\big(\alpha(x)\mathcal{H}^{N-2} \llcorner E_2\big)^* \geqslant \min\{4\pi, \alpha(x)\}\mathcal{H}^{N-2} \llcorner E_2, \tag{1.14}$$
$$(\mu_3)^*(A) = 0 \quad \textit{for every Borel set } A \subset \Omega \textit{ with } \dim_{\mathcal{H}}(A) < N-2. \tag{1.15}$$

In view of the examples presented in [7], one can find measures $\mu \geqslant 0$ for which equality in (1.14) fails and such that $(\mu_3)^*(F) > 0$ for some Borel set $F \subset \Omega$, with $\mathcal{H}^{N-2}(F) = 0$.

The plan of the paper is as follows. In the next section we will prove Theorem 2. In Section 3 we will present a decomposition result for Radon measures. Theorem 1 will then be proved in Section 4. Theorems 3 and 4 will be established in Section 5. The last section will be devoted to the proof of Theorem 5 and Corollaries 1–3.

2. Proof of Theorem 2

We first recall the definition of the Morrey space $\mathrm{M}^p(\Omega)$; see [5].

Definition 1. Let $p \geqslant 1$ be a real number. We say that a function $f \in L^1(\Omega)$ belongs to the Morrey space $\mathrm{M}^p(\Omega)$ if

$$\|f\|_p = \sup_{B_r} \frac{1}{r^{N(1-1/p)}} \int_{\Omega \cap B_r} \big|f(y)\big|\,\mathrm{d}y < +\infty,$$

where the supremum is taken over all open balls $B_r \subset \mathbb{R}^N$.

The following theorem is well-known (for the proof, see for example [5, Section 7.9]).

Theorem 6. *Let $f \in \mathrm{M}^p(\Omega)$ for some $p \geqslant \frac{N}{2}$, and let u be the solution of*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If $p > \frac{N}{2}$, then u belongs to $L^\infty(\Omega)$. If $p = \frac{N}{2}$, then $\mathrm{e}^{\beta|u|}$ is uniformly bounded in $L^1(\Omega)$ norm, for every $\beta < \beta_0 = 2N\omega_N/(\mathrm{e}\,\|f\|_{N/2})$.

Theorem 2 in the Introduction improves the upper bound β_0 given in [5]. It turns out that the constant $\frac{2N\omega_N}{\|f\|_{N/2}}$ is sharp. Indeed we have the following

Example 1. Let $E = \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N\colon\ x_1 = x_2 = 0\}$, and let $\mu = 4\pi\mathcal{H}^{N-2} \llcorner E$. Define $\mu_n = \rho_n * \mu$, where (ρ_n) is a sequence of mollifiers, and let u_n be the solution of

$$\begin{cases} -\Delta u_n = \mu_n & \text{in } B_2(0), \\ u_n = 0 & \text{on } \partial B_2(0). \end{cases}$$

By standard elliptic estimates, $u_n \to u$ in $W_0^{1,q}(B_2(0))$, for every $q < \frac{N}{N-1}$ and a.e., where u is the solution of

$$\begin{cases} -\Delta u = 4\pi\mathcal{H}^{N-2} \llcorner E & \text{in } B_2(0), \\ u = 0 & \text{on } \partial B_2(0). \end{cases}$$

Using the Green representation formula, and setting $\rho(x) = \operatorname{dist}(x, E)$, one can prove that $u(x)$ behaves as $-2\ln\rho(x)$, for any x in a suitable neighborhood of $E \cap B_1(0)$. Moreover, it is easy to verify that

$$\|\mu_n\|_{N/2} \to 2N\omega_N \quad \text{as } n \to \infty.$$

Then, by Fatou's lemma

$$\liminf_{n\to+\infty} \int\limits_{B_2(0)} \mathrm{e}^{(2N\omega_N/\|\mu_n\|_{N/2})u_n} \geqslant \int\limits_{B_2(0)} \mathrm{e}^{u} = +\infty.$$

We now turn to the proof of Theorem 2. We start with the following well-known

Lemma 1. *Let $f : [0, d] \to \mathbb{R}^+$ be a C^1-function, and*

$$g(r) = \sup_{t\in[0,r]} f(t).$$

Then, g is absolutely continuous on $[0, d]$, and its derivative satisfies the following inequality:

$$0 \leqslant g'(r) \leqslant \left[f'(r)\right]^+ \quad \textit{a.e.}, \tag{2.1}$$

where $s^+ = \max\{s, 0\}$ is the positive part of $s \in \mathbb{R}$.

Proof. First of all, observe that since f is continuous, then so is g. We now prove that, for every $x < y$ in $[0, d]$, there exist $\tilde{x} \leqslant \tilde{y}$ in $[x, y]$ such that

$$0 \leqslant g(y) - g(x) \leqslant \left[f(\tilde{y}) - f(\tilde{x})\right]^+. \tag{2.2}$$

Indeed, if $g(y) = g(x)$, then it is enough to choose $\tilde{x} = x$ and $\tilde{y} = y$. If $g(y) > g(x)$, then let us define

$$\tilde{x} = \max\bigl\{z \geqslant x\colon\ g(z) = g(x)\bigr\} \quad \text{and} \quad \tilde{y} = \min\bigl\{z \leqslant y\colon\ g(z) = g(y)\bigr\}.$$

Clearly, since g is nondecreasing, we have $\tilde{x} \leqslant \tilde{y}$. In order to prove (2.2), simply observe that $f(\tilde{x}) = g(x)$ and $f(\tilde{y}) = g(y)$. Indeed, if for example $f(\tilde{x}) \neq g(x)$, then it must be $f(\tilde{x}) < g(x)$, and this implies that $g(z) = g(x)$ for some $z > x$, thus contradicting the definition of $\tilde{x}$.

Since f is absolutely continuous, (2.2) implies that g is absolutely continuous, as required, so that $g'(r)$ exists for almost every r. We now establish (2.1). Starting from (2.2), and applying the mean value problem to f, we have that there exists $\tilde{\xi} \in [\tilde{x}, \tilde{y}]$ such that

$$0 \leqslant g(y) - g(x) \leqslant \left[f(\tilde{y}) - f(\tilde{x})\right]^+ = \left[f'(\tilde{\xi})\right]^+(\tilde{y} - \tilde{x}) \leqslant \left[f'(\tilde{\xi})\right]^+(y - x).$$

Dividing by $y - x$, and letting $y \to x$, the result follows. □

Proof of Theorem 2. We split the proof into two steps:

Step 1. Given $f \in C_c^\infty(\Omega)$, $f \geqslant 0$, let

$$v(x) = \frac{1}{N(N-2)\omega_N} \int\limits_{\Omega} \left(\frac{1}{|x-y|^{N-2}} - \frac{1}{d^{N-2}}\right) f(y)\,\mathrm{d}y \quad \forall x \in \Omega, \tag{2.3}$$

where d is the diameter of Ω. Then, for every $0 < \alpha < 2N\omega_N$, it holds

$$\int\limits_{\Omega} \mathrm{e}^{((2N\omega_N-\alpha)/\|f\|_{N/2})v(x)}\,\mathrm{d}x \leqslant \frac{(N\omega_N)^2}{\alpha} d^N. \tag{2.4}$$

Let us set

$$\nu(x,r)=\int\limits_{B_r(x)} f(y)\,\mathrm{d}y \quad \forall x\in\Omega.$$

In particular,

$$\nu(x,r)\leqslant \omega_N r^N\|f\|_{L^\infty} \quad \text{and} \quad \nu'(x,r)=\int\limits_{\partial B_r(x)} f(y)\,\mathrm{d}\sigma(y)\leqslant N\omega_N r^{N-1}\|f\|_{L^\infty}, \tag{2.5}$$

where $'$ denotes the derivative with respect to r and $d\sigma$ is the $(N-1)$-dimensional measure on $\partial B_r(x)$. Then,

$$\begin{aligned} v(x)&=\frac{1}{N(N-2)\omega_N}\int\limits_0^d\left(\frac{1}{r^{N-2}}-\frac{1}{d^{N-2}}\right)\left(\int\limits_{\partial B_r(x)} f(y)\,\mathrm{d}\sigma(y)\right)\mathrm{d}r\\ &=\frac{1}{N(N-2)\omega_N}\int\limits_0^d\left(\frac{1}{r^{N-2}}-\frac{1}{d^{N-2}}\right)\nu'(x,r)\,\mathrm{d}r. \end{aligned}$$

Integrating by parts, we have

$$v(x)=\frac{1}{N(N-2)\omega_N}\left(\frac{1}{r^{N-2}}-\frac{1}{d^{N-2}}\right)\nu(x,r)\bigg|_0^d+\frac{1}{N\omega_N}\int\limits_0^d\frac{\nu(x,r)}{r^{N-1}}\,\mathrm{d}r.$$

By (2.5),

$$\lim_{r\to 0}\frac{\nu(x,r)}{r^{N-2}}=0,$$

and so

$$v(x)=\frac{1}{N\omega_N}\int\limits_0^d\frac{\nu(x,r)}{r^{N-1}}\,\mathrm{d}r.$$

Define now

$$\psi(x,r)=\sup_{t\in[0,r]}\frac{\nu(x,t)}{t^{N-2}}.$$

It follows from Lemma 1 that $\psi(x,\cdot)$ is absolutely continuous. Then, integrating by parts,

$$\begin{aligned} v(x)&\leqslant\frac{1}{N\omega_N}\int\limits_0^d\frac{\psi(x,r)}{r}\,\mathrm{d}r=-\frac{1}{N\omega_N}\int\limits_0^d\left(\ln\left(\frac{d}{r}\right)\right)'\psi(x,r)\,\mathrm{d}r\\ &=-\frac{1}{N\omega_N}\psi(x,r)\ln\left(\frac{d}{r}\right)\bigg|_0^d+\frac{1}{N\omega_N}\int\limits_0^d\ln\left(\frac{d}{r}\right)\psi'(x,r)\,\mathrm{d}r. \end{aligned}$$

By (2.5),

$$\lim_{r\to 0}\psi(x,r)\ln\left(\frac{d}{r}\right)=0,$$

and then, observing that $\psi(x,d) \geqslant \nu(x,d)/d^{N-2} = \|f\|_{L^1}/d^{N-2} > 0$,

$$v(x) \leqslant \frac{1}{N\omega_N}\int_0^d \ln\left(\frac{d}{r}\right)\psi'(x,r)\,\mathrm{d}r = \int_0^d \frac{\psi(x,d)}{N\omega_N}\ln\left(\frac{d}{r}\right)\frac{\psi'(x,r)}{\psi(x,d)}\,\mathrm{d}r.$$

Therefore, for any $0 < \alpha < 2N\omega_N$,

$$\mathrm{e}^{((2N\omega_N-\alpha)/\|f\|_{N/2})v(x)} \leqslant \exp\left(\int_0^d \frac{2N\omega_N-\alpha}{\|f\|_{N/2}}\frac{\psi(x,d)}{N\omega_N}\ln\left(\frac{d}{r}\right)\frac{\psi'(x,r)}{\psi(x,d)}\,\mathrm{d}r\right).$$

Since $\frac{\psi'(x,r)}{\psi(x,d)}\,\mathrm{d}r$ is a probability measure on $(0,d)$, Jensen's inequality implies

$$\mathrm{e}^{((2N\omega_N-\alpha)/\|f\|_{N/2})v(x)} \leqslant \int_0^d \left(\frac{d}{r}\right)^{((2N\omega_N-\alpha)/\|f\|_{N/2})(\psi(x,d)/N\omega_N)}\frac{\psi'(x,r)}{\psi(x,d)}\,\mathrm{d}r.$$

Clearly,

$$\psi(x,d) \leqslant \sup_{y\in\Omega}\psi(y,d) = \|f\|_{N/2} \quad \text{and} \quad \psi(x,d) \geqslant \frac{\|f\|_{L^1}}{d^{N-2}}.$$

Thus,

$$\mathrm{e}^{((2N\omega_N-\alpha)/\|f\|_{N/2})v(x)} \leqslant \frac{d^{N-\alpha/N\omega_N}}{\|f\|_{L^1}}\int_0^d \frac{\psi'(x,r)}{r^{2-\alpha/N\omega_N}}\,\mathrm{d}r. \tag{2.6}$$

Now, by (2.1) we have

$$\psi'(x,r) \leqslant \left[\left(\frac{\nu(x,r)}{r^{N-2}}\right)'\right]^+ \leqslant \frac{\nu'(x,r)}{r^{N-2}},$$

so that

$$\int_\Omega \psi'(x,r)\,\mathrm{d}x \leqslant \frac{1}{r^{N-2}}\int_\Omega\left(\int_{\partial B_r(x)} f(y)\,\mathrm{d}\sigma(y)\right)\mathrm{d}x = \frac{1}{r^{N-2}}\int_\Omega\left(\int_{\partial B_r(0)} f(y+x)\,\mathrm{d}\sigma(y)\right)\mathrm{d}x$$
$$= \frac{1}{r^{N-2}}\int_{\partial B_r(0)}\left(\int_\Omega f(y+x)\,\mathrm{d}x\right)\mathrm{d}\sigma(y) \leqslant N\omega_N r\|f\|_{L^1}.$$

Hence, from (2.6),

$$\int_\Omega \mathrm{e}^{((2N\omega_N-\alpha)/\|f\|_{N/2})v(x)}\,\mathrm{d}x \leqslant N\omega_N d^{N-\alpha/N\omega_N}\int_0^d \frac{dr}{r^{1-\alpha/N\omega_N}} = \frac{(N\omega_N)^2}{\alpha}d^N$$

which is (2.4). This concludes the proof of Step 1.

Step 2. Proof of Theorem 2 completed.

Let $f \in \mathrm{M}^{N/2}(\Omega)$. Clearly, it suffices to prove the theorem for $f \geqslant 0$. By extending f to be identically zero outside Ω, we have

$$\int_{B_r} f(y)\,\mathrm{d}y \leqslant \|f\|_{N/2}r^{N-2} \quad \text{for every ball } B_r \subset \mathbb{R}^N. \tag{2.7}$$

Let $(\rho_n) \subset C_c^\infty(B_1)$, $\rho_n \geqslant 0$, be a sequence of mollifiers. Take $(\zeta_n) \subset C_c^\infty(\Omega)$ to be such that $0 \leqslant \zeta_n \leqslant 1$ in Ω, and $\zeta_n(x) = 1$ if $d(x, \partial\Omega) \geqslant \frac{1}{n}$. Set $f_n = \zeta_n(\rho_n * f)$. We claim that

$$\|f_n\|_{N/2} \leqslant \|f\|_{N/2} \quad \forall n \geqslant 1. \tag{2.8}$$

In fact, given any ball $B_r(z) \subset \mathbb{R}^N$, we have

$$\int_{B_r(z)} f_n(x)\,\mathrm{d}x \leqslant \int_{B_r(z)} (\rho_n * f)(x)\,\mathrm{d}x = \int_{B_r(z)} \biggl(\int_{\mathbb{R}^N} \rho_n(x-y) f(y)\,\mathrm{d}y\biggr)\mathrm{d}x = \int_{\mathbb{R}^N} \biggl(\int_{B_r(z-t)} f(y)\,\mathrm{d}y\biggr)\rho_n(t)\,\mathrm{d}t.$$

Since (2.7) holds, we get

$$\int_{B_r(z)} f_n(x)\,\mathrm{d}x \leqslant \|f\|_{N/2} r^{N-2} \int_{\mathbb{R}^N} \rho_n(t)\,\mathrm{d}t = \|f\|_{N/2} r^{N-2},$$

which is precisely (2.8).

Let u_n be the unique solution of

$$\begin{cases} -\Delta u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

We shall denote by v_n the function given by (2.3), with f replaced by f_n. Note that, by the standard maximum principle, $0 \leqslant u_n \leqslant v_n$ in Ω, $\forall n \geqslant 1$. Given $0 < \alpha < 2N\omega_N$, it follows from (2.8) and the previous step that

$$\int_\Omega \mathrm{e}^{((2N\omega_N-\alpha)/\|f\|_{N/2})u_n(x)}\,\mathrm{d}x \leqslant \int_\Omega \mathrm{e}^{((2N\omega_N-\alpha)/\|f_n\|_{N/2})v_n(x)}\,\mathrm{d}x \leqslant \frac{(N\omega_N)^2}{\alpha} d^N \quad \forall n \geqslant 1. \tag{2.9}$$

Since $f_n \to f$ in $L^1(\Omega)$, standard elliptic estimates imply that $u_n \to u$ in $L^1(\Omega)$ and a.e. Thus, as $n \to \infty$ in (2.9), it follows from Fatou's lemma that $\mathrm{e}^{((2N\omega_N-\alpha)/\|f\|_{N/2})u} \in L^1(\Omega)$ and

$$\int_\Omega \mathrm{e}^{((2N\omega_N-\alpha)/\|f\|_{N/2})u(x)}\,dx \leqslant \frac{(N\omega_N)^2}{\alpha} d^N.$$

This concludes the proof of the theorem. □

3. A useful decomposition result

Our goal in this section is to establish the following:

Lemma 2. *Let $\mu \in \mathcal{M}(\mathbb{R}^N)$, $\mu \geqslant 0$. Given $\delta > 0$, there exists an open set $A \subset \mathbb{R}^N$ such that*

(a) *$\mu(B_r \backslash A) \leqslant 2N\omega_N r^{N-2}$ for every ball $B_r \subset \mathbb{R}^N$ with $0 < r < \delta$;*
(b) *for every compact set $K \subset A$,*

$$\mu\bigl(N_{2\delta}(K)\bigr) \geqslant 4\pi \mathcal{H}_\delta^{N-2}(K),$$

where $N_{2\delta}(K)$ denotes the neighborhood of K of radius 2δ.

Proof. Given a sequence of open sets $(A_k)_{k\geqslant 0}$, for each $k \geqslant 1$ we let

$$R_k = \sup\bigl\{r \in [0,\delta)\colon\ \mu(B_r \backslash A_{k-1}) \geqslant 2N\omega_N r^{N-2} \text{ for some ball } B_r \subset \mathbb{R}^N\bigr\}. \tag{3.1}$$

We now construct the sequence (A_k) inductively as follows. Let $A_0 = \phi$. We have two possibilities. If $R_1 = 0$, then we take $A_k = \phi$ for every $k \geqslant 1$. Otherwise, $R_1 > 0$ and there exists $r_1 \in (\frac{R_1}{2}, R_1]$ and $x_1 \in \mathbb{R}^N$ such that

$$\mu\big(B_{r_1}(x_1)\big) \geqslant 2N\omega_N r_1^{N-2}.$$

Let $A_1 = B_{r_1}(x_1)$. If $R_2 = 0$, then we let $A_k = \phi$ for every $k \geqslant 2$. Assume $R_2 > 0$. In this case, we may find $r_2 \in (\frac{R_2}{2}, R_2]$ and $x_2 \in \mathbb{R}^N$ such that

$$\mu\big(B_{r_2}(x_2)\backslash A_1\big) \geqslant 2N\omega_N r_2^{N-2}.$$

Proceeding by induction, we obtain a sequence of balls $B_{r_1}(x_1), B_{r_2}(x_2), \ldots$ and open sets

$$A_k = B_{r_1}(x_1) \cup \cdots \cup B_{r_k}(x_k) \tag{3.2}$$

such that

$$\frac{R_k}{2} < r_k \leqslant R_k \tag{3.3}$$

and

$$\mu\big(B_{r_k}(x_k)\backslash A_{k-1}\big) \geqslant 2N\omega_N r_k^{N-2} \quad \forall k \geqslant 1. \tag{3.4}$$

Note that $R_k \to 0$ as $k \to \infty$. In fact, by (3.3) and (3.4) we have

$$\frac{N\omega_N}{2^{N-3}} \sum_{k=1}^{\infty} R_k^{N-2} \leqslant 2N\omega_N \sum_{k=1}^{\infty} r_k^{N-2} \leqslant \sum_{k=1}^{\infty} \mu\big(B_{r_k}(x_k)\backslash A_{k-1}\big) = \mu\bigg(\bigcup_k B_{r_k}(x_k)\bigg) \leqslant \|\mu\|_{\mathcal{M}}.$$

In particular, $\sum_k R_k^{N-2} < \infty$, which implies the desired result.

Let

$$A = \bigcup_{j=1}^{\infty} A_j = \bigcup_{k=1}^{\infty} B_{r_k}(x_k).$$

We claim that A satisfies (a) and (b).

Proof of (a). Given $B_r \subset \mathbb{R}^N$ such that $0 < r < \delta$, let $k \geqslant 1$ be sufficiently large so that $R_k < r$. By the definition of R_k, we have $\mu(B_r\backslash A_k) \leqslant 2N\omega_N r^{N-2}$. Since $A_k \subset A$, we have $B_r\backslash A \subset B_r\backslash A_k$ and the result follows.

Proof of (b). Given a compact set $K \subset A$, let

$$J = \big\{j \geqslant 1\colon\ B_{r_j}(x_j) \cap K \neq \phi\big\}.$$

In particular,

$$K \subset \bigcup_{j\in J} B_{r_j}(x_j).$$

Moreover, since $r_j < \delta$, we have $B_{r_j}(x_j) \subset N_{2\delta}(K)$ for every $j \in J$. Thus,

$$\begin{aligned}\mu\big(N_{2\delta}(K)\big) &\geqslant \mu\bigg(\bigcup_{j\in J} B_{r_j}(x_j)\bigg) \geqslant \mu\bigg(\bigcup_{j\in J}\big[B_{r_j}(x_j)\backslash A_{j-1}\big]\bigg)\\ &= \sum_{j\in J} \mu\big(B_{r_j}(x_j)\backslash A_{j-1}\big) \geqslant 2N\omega_N \sum_{j\in J} r_j^{N-2} \geqslant \frac{2N\omega_N}{\omega_{N-2}} \mathcal{H}_\delta^{N-2}(K).\end{aligned}$$

Since $2N\omega_N/\omega_{N-2} = 4\pi$, we get

$$\mu\big(N_{2\delta}(K)\big) \geqslant 4\pi \mathcal{H}_\delta^{N-2}(K).$$

This concludes the proof of Lemma 2. □

4. Proof of Theorem 1

We first observe that, as a consequence of Theorem 2, we have the following

Proposition 1. *Let $\mu \in \mathcal{M}(\Omega)$ be such that*

$$\mu^+(\Omega \cap B_r) \leqslant 2N\omega_N r^{N-2} \quad \textit{for every ball } B_r \subset \mathbb{R}^N.$$

Then, μ is a good measure for (1.1).

Proof. Since $\mu \leqslant \mu^+$, it is enough to show that μ^+ is a good measure. Thus, without loss of generality, we may assume that $\mu \geqslant 0$. Moreover, extending μ to be identically zero outside Ω, we may also assume that $\mu \in \mathcal{M}(\mathbb{R}^N)$ and

$$\mu(B_r) \leqslant 2N\omega_N r^{N-2} \quad \text{for every ball } B_r \subset \mathbb{R}^N.$$

We shall split the proof of Proposition 1 into two steps:

Step 1. Assume there exists $\varepsilon > 0$ such that

$$\mu(B_r) \leqslant 2N\omega_N (1-\varepsilon) r^{N-2} \quad \text{for every ball } B_r \subset \mathbb{R}^N.$$

Then, μ is a good measure.

Let $(\rho_n) \subset C_c^\infty(B_1)$, $\rho_n \geqslant 0$, be a sequence of mollifiers. Set $\mu_n = \rho_n * \mu$. Proceeding as in the proof of Theorem 2, Step 2, we have

$$\|\mu_n\|_{N/2} \leqslant 2N\omega_N(1-\varepsilon) \quad \forall n \geqslant 1.$$

Let v_n be the unique solution of

$$\begin{cases} -\Delta v_n = \mu_n & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Applying Theorem 2 to $\alpha = 2N\omega_N - \|\mu_n\|_{N/2} \geqslant 2N\omega_N \varepsilon > 0$, we conclude that

$$\int_\Omega \mathrm{e}^{v_n} \leqslant C \quad \forall n \geqslant 1, \tag{4.1}$$

for some constant $C > 0$ independent of n. By standard elliptic estimates $v_n \to v$ a.e., where v is a solution for

$$\begin{cases} -\Delta v = \mu & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, by Fatou's lemma and (4.1), it follows that $\mathrm{e}^v \in L^1(\Omega)$. Since

$$-\Delta v + \mathrm{e}^v - 1 = \mu + \mathrm{e}^v - 1 \quad \text{in } \Omega,$$

$\mu + \mathrm{e}^v - 1$ is a good measure. In particular, $\mu \leqslant \mu + \mathrm{e}^v - 1$ and $v \geqslant 0$, imply that μ is a good measure as well.

Step 2. Proof of the proposition completed.

Let $\alpha_n \uparrow 1$. For every $n \geqslant 1$, the measure $\alpha_n\mu$ satisfies the assumptions of Step 1. Thus, $\alpha_n\mu \in \mathcal{G}$, $\forall n \geqslant 1$. Since $\alpha_n\mu \to \mu$ strongly in $\mathcal{M}(\Omega)$ and $\mathcal{G}$ is closed in $\mathcal{M}(\Omega)$, we have $\mu \in \mathcal{G}$. □

We recall the following result:

Lemma 3. *If $\mu_1, \ldots, \mu_k \in \mathcal{M}(\Omega)$ are good measures for* (1.1), *then so is* $\sup_i \mu_i$.

Proof. If $k = 2$, this is precisely [1, Corollary 4]. The general case easily follows by induction on k. $\square$

We then have a slightly improved version of Proposition 1:

Proposition 2. *Let $\mu \in \mathcal{M}(\Omega)$. Assume there exists $\delta > 0$ such that*

$$\mu^+(\Omega \cap B_r) \leqslant 2N\omega_N r^{N-2} \quad \textit{for every ball } B_r \subset \mathbb{R}^N \textit{ with } r \in (0, \delta).$$

Then, μ is a good measure for (1.1).

Proof. Let $B_\delta(x_1), \ldots, B_\delta(x_k)$ be a finite covering of Ω. For each $i = 1, \ldots, k$, let $\mu_i = \mu \llcorner B_\delta(x_i) \in \mathcal{M}(\Omega)$. It is easy to see that μ_i satisfies the assumptions of Proposition 1, so that each μ_i is a good measure for (1.1). Thus, by the previous lemma, $\sup_i \mu_i \in \mathcal{G}$. Since $\mu \leqslant \sup_i \mu_i$, we conclude that μ is also a good measure for (1.1). $\square$

We can now present the

Proof of Theorem 1. As above, since $\mu \leqslant \mu^+$, it suffices to show that μ^+ is a good measure. In particular, we may assume that $\mu \geqslant 0$. Moreover, it suffices to establish the theorem for a measure μ such that $\mu \leqslant (4\pi - \varepsilon)\mathcal{H}^{N-2}$ for some $\varepsilon > 0$. The general case follows as in Step 2 of Proposition 1.

We first extend μ to be identically zero outside Ω. By Lemma 2, there exists an open set $\hat{A}_1 \subset \mathbb{R}^N$ such that (a) and (b) hold with $\delta = 1$ and $A = \hat{A}_1$. By induction, given an open set $\hat{A}_{k-1} \subset \mathbb{R}^N$, we apply Lemma 2 to $\mu \llcorner \hat{A}_{k-1}$ and $\delta_k = \frac{1}{k}$ to obtain an open set $\hat{A}_k \subset \hat{A}_{k-1}$ such that

(a_k) $\mu \llcorner \hat{A}_{k-1}(B_r \backslash \hat{A}_k) \leqslant 2N\omega_N r^{N-2}$ for every ball $B_r \subset \mathbb{R}^N$ with $0 < r < \frac{1}{k}$;

(b_k) for every compact set $K \subset \hat{A}_k$,

$$\mu\big(N_{2/k}(K)\big) \geqslant \mu \llcorner \hat{A}_{k-1}\big(N_{2/k}(K)\big) \geqslant 4\pi \mathcal{H}^{N-2}_{1/k}(K).$$

By Proposition 2, each measure $\mu \llcorner \Omega \backslash \hat{A}_1$, $\mu \llcorner \hat{A}_1 \backslash \hat{A}_2, \ldots, \mu \llcorner \hat{A}_{k-1} \backslash \hat{A}_k$ is good. We now invoke Lemma 3 to conclude that

$$\mu \llcorner \Omega \backslash \hat{A}_k = \sup\{\mu \llcorner \Omega \backslash \hat{A}_1, \mu \llcorner \hat{A}_1 \backslash \hat{A}_2, \ldots, \mu \llcorner \hat{A}_{k-1} \backslash \hat{A}_k\}$$

is a good measure for every $k \geqslant 1$. Let $\hat{A} = \bigcap_k \hat{A}_k$. Since $\mu \llcorner \Omega \backslash \hat{A}_k \to \mu \llcorner \Omega \backslash \hat{A}$ strongly in $\mathcal{M}(\Omega)$ and the set $\mathcal{G}$ of good measures is closed with respect to the strong topology, we conclude that $\mu \llcorner \Omega \backslash \hat{A}$ is also a good measure for (1.1).

We now claim that $\mu(\hat{A}) = 0$. In fact, let $K \subset \hat{A}$ be a compact set. In particular, $K \subset \hat{A}_k$. By (b_k), we have

$$\mu\big(N_{2/k}(K)\big) \geqslant 4\pi \mathcal{H}^{N-2}_{1/k}(K) \quad \forall k \geqslant 1.$$

As $k \to \infty$, we conclude that

$$\mu(K) \geqslant 4\pi \mathcal{H}^{N-2}(K). \tag{4.2}$$

In particular, $\mathcal{H}^{N-2}(K) < \infty$. Recall that, by assumption,

$$\mu(K) \leqslant 4\pi(1 - \varepsilon)\mathcal{H}^{N-2}(K). \tag{4.3}$$

Combining (4.2) and (4.3), we get $\mu(K) = 0$. Since $K \subset \hat{A}$ is arbitrary, we conclude that $\mu(\hat{A}) = 0$. Therefore, $\mu = \mu \llcorner \Omega \backslash \hat{A}$ and so μ is a good measure. This concludes the proof of Theorem 1. $\square$

5. Proofs of Theorems 3 and 4

In this section we derive some necessary conditions for a measure to be good for problem (1.1). Let us start with a regularity property for solutions of elliptic equations with measure data.

Lemma 4. *Let $\nu \in \mathcal{M}(\Omega)$ and let u be the solution of the Dirichlet problem*

$$\begin{cases} -\Delta u = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

If $\mathrm{e}^u \in L^1(\Omega)$, then u^+ belongs to $W_0^{1,p}(\Omega)$ for every $p < 2$, and

$$\|u^+\|_{W_0^{1,p}} \leqslant C\big(p, \operatorname{meas}\Omega, \|\nu\|_{\mathcal{M}}, \|\mathrm{e}^u\|_{L^1}\big) \quad \forall p < 2. \tag{5.2}$$

Proof. Let $\nu_n = \rho_n * \nu$, where (ρ_n) is a sequence of mollifiers, and let u_n be the solution of

$$\begin{cases} -\Delta u_n = \nu_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

Then it is well-known that the sequence (u_n) converges to u in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$ (see [8]).

Using $T_k(u_n^+) = \min\{k, \max\{u_n, 0\}\}$ as a test function in (5.3), we have

$$\int_\Omega \big|\nabla T_k(u_n^+)\big|^2 \,\mathrm{d}x = \int_\Omega T_k(u_n^+)\, \nu_n \,\mathrm{d}x \leqslant k\|\nu_n\|_{L^1} \leqslant k\|\nu\|_{\mathcal{M}}.$$

Letting $n \to \infty$, by weak lower semicontinuity we obtain

$$\int_\Omega \big|\nabla T_k(u^+)\big|^2 \,\mathrm{d}x \leqslant k\|\nu\|_{\mathcal{M}}. \tag{5.4}$$

On the other hand, assumption $\mathrm{e}^u \in L^1(\Omega)$ implies, for every $k > 0$,

$$\mathrm{e}^k \operatorname{meas}\{u > k\} \leqslant \int_{\{u>k\}} \mathrm{e}^u \,\mathrm{d}x \leqslant \|\mathrm{e}^u\|_{L^1},$$

and so

$$\operatorname{meas}\{u > k\} \leqslant \mathrm{e}^{-k}\|\mathrm{e}^u\|_{L^1}. \tag{5.5}$$

For every $\eta > 1$ we have

$$\big\{|\nabla u^+| > \eta\big\} = \left\{ \begin{array}{c} |\nabla u| > \eta \\ u > k \end{array} \right\} \cup \left\{ \begin{array}{c} |\nabla u| > \eta \\ 0 \leqslant u \leqslant k \end{array} \right\},$$

so that, by (5.4) and (5.5),

$$\begin{aligned} \operatorname{meas}\big\{|\nabla u^+| > \eta\big\} &\leqslant \operatorname{meas}\{u > k\} + \operatorname{meas}\left\{ \begin{array}{c} |\nabla u| > \eta \\ 0 \leqslant u \leqslant k \end{array} \right\} \\ &\leqslant \mathrm{e}^{-k}\|\mathrm{e}^u\|_{L^1} + \frac{1}{\eta^2}\int_\Omega \big|\nabla T_k(u^+)\big|^2 \,\mathrm{d}x \leqslant C\left(\mathrm{e}^{-k} + \frac{k}{\eta^2}\right), \end{aligned}$$

where $C = \max\{\|\mathrm{e}^u\|_{L^1}, \|\nu\|_{\mathcal{M}}\}$. Minimizing on k, we find

$$\operatorname{meas}\big\{|\nabla u^+| > \eta\big\} \leqslant C\,\frac{1 + 2\ln\eta}{\eta^2}.$$

Therefore, $|\nabla u^+|$ belongs to the Marcinkiewicz space of exponent p, for every $p<2$. Since Ω is bounded, it follows that $|\nabla u^+| \in L^p(\Omega)$, for every $p<2$, and that (5.2) holds. □

Theorem 3 can now be obtained as a consequence of the above results.

Proof of Theorem 3. By inner regularity, it is enough to prove that if $\mu \in \mathcal{M}(\Omega)$ is a good measure for problem (1.1), then $\mu(K) \leqslant 0$ for every compact set $K \subset \Omega$ with $\dim_{\mathcal{H}}(K) < N-2$.

By Lemma 3, if μ is a good measure, then so is $\mu^+ = \sup\{\mu, 0\}$. Let $v \geqslant 0$ be the solution of problem (1.1) with datum μ^+. In particular, v satisfies

$$\int_\Omega \nabla v\, \nabla\zeta + \int_\Omega (\mathrm{e}^v - 1)\zeta = \int_\Omega \zeta\, \mathrm{d}\mu^+ \quad \forall \zeta \in C_c^\infty(\Omega). \tag{5.6}$$

Take now a compact set $K \subset \Omega$ with $\dim_{\mathcal{H}}(K) < N-2$, and let q be such that $2<q<N-\dim_{\mathcal{H}}(K)$. Then the q-capacity of K is zero (see e.g. [3]), and there exists a sequence of smooth functions $\zeta_n \in C_c^\infty(\Omega)$ such that

$$0 \leqslant \zeta_n \leqslant 1 \quad \text{in } \Omega, \qquad \zeta_n = 1 \quad \text{in } K, \qquad \zeta_n \to 0 \quad \text{in } W_0^{1,q}(\Omega) \text{ and a.e.} \tag{5.7}$$

Using ζ_n as test function in (5.6) yields

$$0 \leqslant \mu^+(K) \leqslant \int_\Omega \zeta_n\, \mathrm{d}\mu^+ = \int_\Omega \nabla v\, \nabla\zeta_n + \int_\Omega (\mathrm{e}^v - 1)\,\zeta_n.$$

Since, by Lemma 4, $v \in W_0^{1,q'}(\Omega)$, the right-hand side tends to 0 as $n \to \infty$. Hence, $\mu^+(K) = 0$, which implies $\mu(K) \leqslant 0$, as desired. □

Before presenting the proof of Theorem 4, we need some preliminary lemmas. The first one is well-known (see e.g. [3]).

Lemma 5. *If $f \in L^1(\mathbb{R}^N)$, then, for every $0 \leqslant s < N$,*

$$\lim_{r\to 0} \frac{1}{r^s} \int_{B_r(x)} \bigl|f(y)\bigr|\,\mathrm{d}y = 0 \quad \mathcal{H}^s\text{-a.e. in } \mathbb{R}^N.$$

In the following, we will denote the angular mean of a function $w \in L^1(\mathbb{R}^N)$ on the sphere centered at $x \in \mathbb{R}^N$ with radius $r>0$ by

$$\overline{w}(x,r) = \fint_{\partial B_r(x)} w\,\mathrm{d}\sigma = \frac{1}{N\omega_N r^{N-1}} \int_{\partial B_r(x)} w\,\mathrm{d}\sigma. \tag{5.8}$$

The next result provides an estimate of the asymptotic behavior, as $r \to 0$, of the angular mean of a function in terms of its Laplacian.

Lemma 6. *Let $w \in L^1(\mathbb{R}^N)$ be such that $\Delta w \in \mathcal{M}(\mathbb{R}^N)$. Set $\mu = -\Delta w$. Then,*

$$\frac{1}{N\omega_N} \liminf_{r\to 0} \frac{\mu(B_r(x))}{r^{N-2}} \leqslant \liminf_{r\to 0} \frac{\overline{w}(x,r)}{\ln(1/r)} \leqslant \limsup_{r\to 0} \frac{\overline{w}(x,r)}{\ln(1/r)} \leqslant \frac{1}{N\omega_N} \limsup_{r\to 0} \frac{\mu(B_r(x))}{r^{N-2}}.$$

Proof. We claim that, for every $0<r<s<1$, we have

$$\overline{w}(x,r) - \overline{w}(x,s) = \frac{1}{N\omega_N} \int_r^s \frac{\mu(B_\rho(x))}{\rho^{N-1}}\,\mathrm{d}\rho. \tag{5.9}$$

Indeed, if $\mu \in L^1(\mathbb{R}^N)$, then, integrating by parts, we have

$$\int_{B_\rho(x)} \mu(y)\,\mathrm{d}y = -N\omega_N \rho^{N-1}\overline{w}'(x,\rho), \tag{5.10}$$

where $'$ denotes the derivative with respect to ρ. Integrating (5.10) from r to s we have

$$\overline{w}(x,r) - \overline{w}(x,s) = \frac{1}{N\omega_N}\int_r^s \frac{1}{\rho^{N-1}}\left(\int_{B_\rho(x)} \mu(y)\,\mathrm{d}y\right)\mathrm{d}\rho,$$

which is precisely (5.9) if $\mu \in L^1(\mathbb{R}^N)$. The general case then follows by regularizing via convolution and taking the limit. Thus, from (5.9) we have

$$\frac{1}{N\omega_N}\inf_{0<\rho<s}\left(\frac{\mu(B_\rho(x))}{\rho^{N-2}}\right)\ln\left(\frac{s}{r}\right) \leqslant \overline{w}(x,r) - \overline{w}(x,s) \leqslant \frac{1}{N\omega_N}\sup_{0<\rho<s}\left(\frac{\mu(B_\rho(x))}{\rho^{N-2}}\right)\ln\left(\frac{s}{r}\right).$$

Dividing by $\ln(1/r)$ and letting $r \to 0$ yields

$$\frac{1}{N\omega_N}\inf_{0<\rho<s}\left(\frac{\mu(B_\rho(x))}{\rho^{N-2}}\right) \leqslant \liminf_{r\to 0}\frac{\overline{w}(x,r)}{\ln(1/r)} \leqslant \limsup_{r\to 0}\frac{\overline{w}(x,r)}{\ln(1/r)} \leqslant \frac{1}{N\omega_N}\sup_{0<\rho<s}\left(\frac{\mu(B_\rho(x))}{\rho^{N-2}}\right),$$

and the conclusion follows by letting $s \to 0$. □

An immediate consequence of Lemmas 5 and 6 is the following

Corollary 4. *Let $w \in L^1(\mathbb{R}^N)$ be such that $\Delta w \in L^1(\mathbb{R}^N)$. Then,*

$$\lim_{r\to 0}\frac{\overline{w}(x,r)}{\ln(1/r)} = 0 \quad \text{for } \mathcal{H}^{N-2}\text{-a.e. } x \in \mathbb{R}^N.$$

We can now prove Theorem 4.

Proof of Theorem 4. By contradiction, assume that μ is a good measure for problem (1.1), so that $(4\pi+\varepsilon)\mathcal{H}^{N-2} \llcorner E$ is also a good measure. Let u be the solution of (1.1) with datum $(4\pi+\varepsilon)\mathcal{H}^{N-2} \llcorner E$ and let v the solution of

$$\begin{cases} -\Delta v = (4\pi+\varepsilon)\mathcal{H}^{N-2} \llcorner E & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Since E is $(N-2)$-rectifiable, then (see [6])

$$\lim_{r\to 0}\frac{\mathcal{H}^{N-2}(E\cap B_r(x))}{r^{N-2}} = \omega_{N-2} \quad \text{for } \mathcal{H}^{N-2}\text{-a.e. } x \in E.$$

Thus, from Lemma 6 we obtain

$$\lim_{r\to 0}\frac{\overline{v}(x,r)}{\ln(1/r)} = \frac{(4\pi+\varepsilon)\omega_{N-2}}{N\omega_N} = \frac{4\pi+\varepsilon}{2\pi} \quad \text{for } \mathcal{H}^{N-2}\text{-a.e. } x \in E. \tag{5.11}$$

On the other hand, the function $w = v - u$ satisfies $-\Delta w = \mathrm{e}^u - 1 \in L^1(\Omega)$, so that, by Corollary 4,

$$\lim_{r\to 0}\frac{\overline{w}(x,r)}{\ln(1/r)} = \lim_{r\to 0}\frac{\overline{v}(x,r) - \overline{u}(x,r)}{\ln(1/r)} = 0 \quad \text{for } \mathcal{H}^{N-2}\text{-a.e. } x \in \Omega. \tag{5.12}$$

Combining (5.11) and (5.12) we deduce

$$\lim_{r\to 0}\frac{\overline{u}(x,r)}{\ln(1/r)}=\frac{4\pi+\varepsilon}{2\pi}>2\quad \text{for } \mathcal{H}^{N-2}\text{-a.e. } x\in E.$$

Thus, for $\mathcal{H}^{N-2}$-a.e. $x\in E$, there exists $\delta=\delta(x)>0$ such that

$$\frac{\overline{u}(x,r)}{\ln(1/r)}>2\quad \forall r\in(0,\delta). \tag{5.13}$$

Since

$$\int_{B_\delta(x)} \mathrm{e}^{u(y)}\,\mathrm{d}y=\int_0^\delta\biggl(\int_{\partial B_r(x)}\mathrm{e}^{u}\,\mathrm{d}\sigma\biggr)\mathrm{d}r=N\omega_N\int_0^\delta r^{N-1}\biggl(\fint_{\partial B_r(x)}\mathrm{e}^{u}\,\mathrm{d}\sigma\biggr)\mathrm{d}r,$$

by Jensen's inequality and (5.13), it follows that

$$\int_{B_\delta(x)} \mathrm{e}^{u(y)}\,\mathrm{d}y\geqslant N\omega_N\int_0^\delta r^{N-1}\mathrm{e}^{\overline{u}(x,r)}\,\mathrm{d}r\geqslant N\omega_N\int_0^\delta r^{N-3}\,\mathrm{d}r=\frac{N\omega_N}{N-2}\delta^{N-2}.$$

Consequently, as $\delta\to 0$, we obtain

$$\liminf_{\delta\to 0}\frac{1}{\delta^{N-2}}\int_{B_\delta(x)}\mathrm{e}^{u(y)}\,\mathrm{d}y>0\quad \text{for } \mathcal{H}^{N-2}\text{-a.e. } x\in E,$$

which contradicts Lemma 5 being $\mathcal{H}^{N-2}(E)>0$. □

6. Proof of Theorem 5

We first establish Corollaries 1–3.

Proof of Corollary 1. Let $\mu\in\mathcal{M}(\Omega)$ be such that $\mu\leqslant 4\pi\mathcal{H}^{N-2}$. It follows from Theorem 1 that μ is a good measure. Since μ^* is the largest good measure $\leqslant\mu$, we must have $\mu=\mu^*$. □

Proof of Corollary 2. By Corollary 10 in [1], for every $\mu\in\mathcal{M}(\Omega)$ we have

$$\mu^*=(\mu^+)^*+(-\mu^-)^*=(\mu^+)^*-\mu^-. \tag{6.1}$$

Assume that there exists a Borel set $A\subset\Omega$, with $\dim_{\mathcal{H}}(A)<N-2$, such that $\mu^+=\mu^+\llcorner A$. We claim that $(\mu^+)^*=0$.

By contradiction, suppose that $(\mu^+)^*\neq 0$. Since $0\leqslant(\mu^+)^*\leqslant\mu^+$, the measure $(\mu^+)^*$ is also concentrated on A. In addition, $(\mu^+)^*\neq 0$ implies $(\mu^+)^*(A)>0$. Applying Theorem 3, we conclude that $(\mu^+)^*$ is not a good measure, which is a contradiction. Thus, $(\mu^+)^*=0$. It then follows from (6.1) that $\mu^*=-\mu^-$. □

Proof of Corollary 3. Without loss of generality we can assume that $\alpha(x)\geqslant 0$ for $\mathcal{H}^{N-2}$-a.e. in $x\in E$. Let $\nu=\min\{4\pi,\alpha(x)\}\mathcal{H}^{N-2}\llcorner E$. Since $\nu\leqslant 4\pi\mathcal{H}^{N-2}$, Theorem 1 implies that ν is a good measure. Clearly, $\nu\leqslant\mu$; thus, $\nu\leqslant\mu^*$. Since $\mu^*\leqslant\mu=\alpha(x)\mathcal{H}^{N-2}\llcorner E$, there exists an $\mathcal{H}^{N-2}$-measurable function β, such that $\mu^*=\beta(x)\mathcal{H}^{N-2}\llcorner E$. Assume by contradiction that $\beta\neq\min\{4\pi,\alpha\}$. Since

$$\min\{4\pi,\alpha\}\leqslant\beta\leqslant\alpha,$$

we conclude that there exists $\varepsilon > 0$ and a Borel set $F \subset E$, with $\mathcal{H}^{N-2}(F) > 0$, such that

$$(4\pi + \varepsilon) \leqslant \beta \quad \mathcal{H}^{N-2}\text{-a.e. on } F.$$

Since E is $(N-2)$-rectifiable and $F \subset E$, then F is also $(N-2)$-rectifiable (see e.g. [6, Lemma 15.5]). Moreover,

$$(4\pi + \varepsilon)\mathcal{H}^{N-2} \llcorner F \leqslant \beta \mathcal{H}^{N-2} \llcorner F \leqslant \mu^*.$$

Thus, $(4\pi + \varepsilon)\,\mathcal{H}^{N-2} \llcorner F$ is a good measure. But this contradicts Theorem 4. Therefore, $\beta = \min\{4\pi, \alpha\}$ and so $\mu^* = \nu$. □

We now present the

Proof of Theorem 5. Clearly, the measures μ_1, μ_2, μ_3 and $-\mu^-$ are singular with respect to each other; (1.10) then follows from Theorem 8 in [1]. For the same reason, (1.12) holds. Next, Corollaries 1–3 imply (1.11), (1.13) and (1.15). Finally, since $\min\{4\pi, \alpha\}\,\mathcal{H}^{N-2} \llcorner E_2$ is a good measure by Theorem 1, we have (1.14). □

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