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A compactness theorem of n-harmonic maps

Un théorème de compacité pour applications n-harmoniques

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Abstract

For $n \geqslant 3$, let $\Omega \subset \mathbf{R}^n$ be a bounded domain and $N \subset \mathbf{R}^L$ be a compact smooth Riemannian submanifold without boundary. Suppose that $\{u_n\} \subset W^{1,n}(\Omega, N)$ are weak solutions to the (perturbed) n-harmonic map equation (1.2), satisfying (1.3), and $u_k \to u$ weakly in $W^{1,n}(\Omega, N)$. Then u is an n-harmonic map. In particular, the space of n-harmonic maps is sequentially compact for the weak-$W^{1,n}$ topology.

Résumé

Pour $n \geqslant 3$, soit $\Omega \subset \mathbf{R}^n$ un domaine borné et soit $N \subset \mathbf{R}^L$ une sous-variété compacte sans bord. Soient $\{u_n\} \subset W^{1,n}(\Omega, N)$ des solutions de l'équation (perturbée) (1.2) pour les applications n-harmoniques, telles que $u_k \to u$ faiblement dans $W^{1,n}(\Omega, N)$. Alors u est une application n-harmonique. En particulier, l'espace des applications n-harmoniques est sequentiellement compact dans la topologie $W^{1,n}$ faible.

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1. Introduction

For $n \geqslant 2$, let $\Omega \subset \mathbf{R}^n$ be a bounded domain, and $N \subset \mathbf{R}^L$ be a compact smooth Riemannian manifold without boundary, isometrically embedded into an Euclidean space $\mathbf{R}^L$ for some $L \geqslant 1$. For $2 \leqslant p \leqslant n$, the Sobolev space $W^{1,p}(\Omega, N)$ is defined by

$$W^{1,p}(\Omega, N) := \left\{u = (u^1, \ldots, u^L) \in W^{1,p}(M, \mathbf{R}^L) \mid u(x) \in N \text{ for a.e. } x \in \Omega\right\}.$$

The Dirichlet p-energy functional $E_p : W^{1,p}(\Omega, N) \to \mathbf{R}$ is defined by

$$E_p(u) = \int_\Omega |\nabla u|^p \,\mathrm{d}x = \int_\Omega \left(\sum_{\alpha=1}^n \left\langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\alpha}\right\rangle\right)^{p/2} \mathrm{d}x$$

where $\langle \cdot, \cdot \rangle$ is the scalar product of $\mathbf{R}^L$.

Recall that a map $u \in W^{1,p}(\Omega, N)$ is a p-harmonic map, if u is a critical point of E_p on the space $W^{1,p}(\Omega, N)$, i.e. u satisfies the p-harmonic map equation:

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |\nabla u|^{p-2} A(u)(\nabla u, \nabla u) \tag{1.1}$$

in the sense of distributions, where div is the divergence operator on $\mathbf{R}^n$ and $A(\cdot)(\cdot, \cdot)$ is the second fundamental form of $N \subset \mathbf{R}^L$.

Since the p-harmonic map equation (1.1) is a degenerate elliptic system with critical nonlinearity in the gradients, the analysis of both the regularity problem and the weak compactness for p-harmonic maps are extremely challenging.

This paper is motivated by the problem:

Question A. For $n \geqslant 3$ and $2 \leqslant p \leqslant n$, is any weak limit u in $W^{1,p}(\Omega, N)$ of a sequence of p-harmonic maps $\{u_k\} \subset W^{1,p}(\Omega, N)$ a p-harmonic map?

For $p = n = 2$, the answer to question A is affirmative, due to Hélein's celebrated regularity theorem [12]: *any 2-harmonic map from a Riemannian surface into any compact Riemannian manifold is smooth*.

Question A remains open for $n \geqslant 3$, although a lot of efforts have been made. We would like to mention some known results in the direction. Schoen–Uhlenbeck [24] ($p = 2$), Hardt–Lin [15] and Luckhaus [21] ($p \neq 2$) have shown that *any weak limit $u \in W^{1,p}$ of a sequence of minimizing p-harmonic maps is a strong limit and a minimizing p-harmonic map*. Question A is true for target manifolds N with symmetry, such as $N = S^{L-1}$ is the unit sphere in $\mathbf{R}^L$ (cf. Chen [3], Shatah [22], Evans [6] §5, and Hélein [13] §2) or $N = \mathbf{G}/\mathbf{H}$ is a compact Riemannian homogeneous manifold (cf. Toro–Wang [26]). Here the symmetry guarantees the existence of Killing tangent vector fields on N, under which the nonlinearity of the p-harmonic map equation (1.1) can be reduced to a form with Jacobian structure.

For manifolds N without symmetries, the idea of Coulomb moving frames, due to Hélein [12] (see also [13]), has played extremely important roles on the study of regularity of stationary 2-harmonic maps by Hélein [12] ($n = 2$) and Bethuel [2] ($n \geqslant 3$) (see also Evans [5]). The idea in [12] is that one first assumes that N is parallelizable and then uses the variational method to obtain a harmonic moving frame $\{e_\alpha\}$. It turns out that the nonlinearity of 2-harmonic map equation via a harmonic moving frame contains Jacobian structure. However, it is known that the harmonic moving frame by [12] is insufficient for the compactness of 2-harmonic maps. On the other hand, in the study on existence of wave maps in $\mathbf{R}^{2+1}$, Freire–Müller–Struwe [9,10] have discovered that for wave maps enjoying the energy monotonicity inequalities in $\mathbf{R}^{2+1}$, the concentration compactness method of Lions [19,20], in combination with the idea of Coulomb moving frames for wave maps and some end-point analytic estimates, can yield the weak compactness of wave maps enjoying energy monotonicity inequalities in $\mathbf{R}^{2+1}$. We would like to point out that Strzelecki, Zatorska-Goldstein [25] have used these ideas from [9,10] and [19,20] to show the weak compactness of weak solutions of higher dimensional H-systems.

There is a main difficulty that one encounters for p-harmonic maps for $p \neq 2$, namely the appropriate construction of Coulomb moving frames. Notice that neither minimizers of $\int |\langle \mathrm{d}e_\alpha, e_\beta\rangle|^p$ nor minimizers of $\int |\nabla u|^{p-2}|\langle \mathrm{d}e_\alpha, e_\beta\rangle|^2$ seem to work here. Instead, we observe that for $p = n$ case Uhlenbeck's construction of Coulomb gauges for Yang–Mills fields [27] can be adopted to obtain Coulomb moving frames along u^*TN under the smallness of $E_n(u)$. This kind of observation has been utilized by Wang [29,30] in the context of biharmonic maps. With such a Coulomb moving frame along u^*TN, we can modify the analytic techniques by [10] to show the weak compactness of a Palais–Smale sequence of the Dirichlet n-energy functional E_n on $W^{1,n}(\Omega, N)$.

We first recall

Definition. A sequence of maps $\{u_k\} \subset W^{1,n}(\Omega, N)$ is a Palais–Smale sequence for the Dirichlet n-energy functional E_n, if (a) $u_k \to u$ weakly in $W^{1,n}(\Omega, N)$, and (b) $E_n'(u_k) \to 0$ in $(W^{1,n}(\Omega, N))^*$. Here $(W^{1,n}(\Omega, N))^*$ is the dual of $W^{1,n}(\Omega, N)$.

Notice that (b) is equivalent to that u_k satisfies the perturbed n-harmonic map equation:

$$-\operatorname{div}\big(|\nabla u_k|^{n-2}\nabla u_k\big) = |\nabla u_k|^{n-2}A(u_k)(\nabla u_k, \nabla u_k) + \Phi_k, \tag{1.2}$$

in the sense of distributions, and

$$\lim_{k\to\infty} \|\Phi_k\|_{(W^{1,n}(\Omega,N))^*} = 0. \tag{1.3}$$

The question is whether any weak limit u of a Palais–Smale sequence is an n-harmonic map. This is highly nontrivial. Since E_n is conformally invariant and the conformal group is non-compact, E_n does not satisfy the Palais–Smale condition (cf. [23]). Our main result is

Theorem B. *For $n \geqslant 3$, assume that $\{u_k\} \subset W^{1,n}(\Omega, N)$ satisfy Eqs.* (1.2), (1.3), *and converge weakly to u in $W^{1,n}(\Omega, N)$, then $u \in W^{1,n}(\Omega, N)$ is an n-harmonic map.*

We would like to remark that for $n = 2$, Theorem B has first been proven by Bethuel [1], later reproved by Freire–Müller–Struwe [10], and also by Wang [28]. For $n \geqslant 3$, Hungerbhler [14] has obtained the existence of global weak solutions to the n-harmonic map flow. Theorem B is applicable to the n-harmonic map flow by [14] at time infinity.

As a corollary, we answer Question A in the affirmative for $p = n \geqslant 3$.

Corollary C. *For $n \geqslant 3$, assume that $\{u_k\} \subset W^{1,n}(\Omega, N)$ are a sequence of n-harmonic maps converging weakly to u in $W^{1,n}(\Omega, N)$, then u is an n-harmonic map.*

The paper is written as follows. In Section 2, we outline the construction of Coulomb moving frames. In Section 3, we first recall $\mathcal{H}^1(\mathbf{R}^n)$-estimate for functions with Jacobian structure by [4], the duality between $\mathcal{H}^1(\mathbf{R}^n)$ and $\mathrm{BMO}(\mathbf{R}^n)$ by [11], and then give a proof of Theorem B.

In this paper, we will use the following notations. For a ball $B = B_r(x) \subset \mathbf{R}^n$, denote $\alpha B = B_{\alpha r}(x)$ for any $\alpha > 0$. For $1 \leqslant i \leqslant n$, denote $\wedge^i(\mathbf{R}^n)$ as the ith wedge product of $\mathbf{R}^n$, $C^\infty(\mathbf{R}^n, \wedge^i(\mathbf{R}^n))$ as the space of smooth ith forms on $\mathbf{R}^n$, and $W^{m,p}(\mathbf{R}^n, \wedge^i(\mathbf{R}^n))$ as the space of ith forms on $\mathbf{R}^n$ with $W^{m,p}(\mathbf{R}^n)$ coefficients, for nonnegative integers m and $1 < p < \infty$. Denote by $\mathcal{D}'(\Omega)$ the dual of $C_0^\infty(\Omega)$. Denote d as the exterior differential operator on $\mathbf{R}^n$ and δ as the adjoint operator of d.

2. The construction of Coulomb moving frames

This section is devoted to the construction of Coulomb moving frames along u^*TN, under the smallness condition on $E_n(u)$.

For any open set $U \subset \mathbf{R}^n$ and $u \in W^{1,n}(U,N)$, denote u^*TN as the pull-back bundle of TN by u over U. For $l = \dim(N)$, we say that $\{e_\alpha\}_{\alpha=1}^l$ is a moving frame along u^*TN, if $\{e_\alpha(x)\}_{\alpha=1}^l$ is an orthonormal base of $T_{u(x)}N$, the tangent space of N at the point $u(x)$, for a.e. $x \in U$.

We now express the perturbed n-harmonic map equation, via a moving frame, as follows.

Lemma 2.1. *For $n \geqslant 3$ and $u \in W^{1,n}(\Omega,N)$, let $\{e_\alpha\}_{\alpha=1}^l$ be a moving frame along u^*TN. Then u is a weak solution to the perturbed n-harmonic map equation*:

$$-\operatorname{div}\big(|\nabla u|^{n-2}\nabla u\big) = |\nabla u|^{n-2}A(u)(\nabla u,\nabla u) + \Phi \tag{2.1}$$

if and only if for any $1 \leqslant \alpha \leqslant l$, the following equation

$$-\operatorname{div}\big(\big\langle|\nabla u|^{n-2}\nabla u, e_\alpha\big\rangle\big) = \sum_{\beta=1}^{l}\big\langle|\nabla u|^{n-2}\nabla u, e_\beta\big\rangle\langle\nabla e_\alpha, e_\beta\rangle + \langle\Phi, e_\alpha\rangle \tag{2.2}$$

holds in the sense of distributions. Here $\Phi \in (W^{1,n}(\Omega,N))^*$.

Proof. Observe that for a.e. $x \in \Omega$, we have

$$\big\langle e_\alpha(x), A\big(u(x)\big)\big(\nabla u(x), \nabla u(x)\big)\big\rangle = 0, \quad 1 \leqslant \alpha \leqslant l,$$

for $e_\alpha(x) \in T_{u(x)}N$ and $A(u(x))(\nabla u(x),\nabla u(x)) \perp T_{u(x)}N$. Then straightforward calculations deduce the equivalence between (2.2) and (2.1). □

We now state the construction of a Coulomb moving frame along u^*TN with estimates on its connection form. It is inspired by an earlier result of Wang [29,30] in the context of biharmonic maps and Uhlenbeck's Coulomb gauge construction for Yang–Mills fields [27].

Proposition 2.2. *For $n \geqslant 3$ and any ball $B \subset \mathbf{R}^n$, there exists an $\epsilon_0 > 0$ such that if $u \in W^{1,n}(2B,N)$ satisfies*

$$\|\nabla u\|_{L^n(2B)} \leqslant \epsilon_0 \tag{2.3}$$

then there exists a Coulomb moving frame $\{e_\alpha\}_{\alpha=1}^l$ along u^*TN in $W^{1,n}(B,\mathbf{R}^L)$ such that its connection form $A = (\langle de_\alpha, e_\beta\rangle)$ satisfies

$$\delta A = 0 \quad \text{in } B; \qquad x\cdot A = 0 \quad \text{on } \partial B \tag{2.4}$$

and

$$\|A\|_{L^n(B)} + \|\nabla A\|_{L^{n/2}(B)} \leqslant C\|\nabla u\|_{L^n(B)}^2. \tag{2.5}$$

Proof. Since the argument is very similar to that of [30] Proposition 3.2, we only sketch it briefly. First, it is well-known (cf. [24]) that the standard mollification process and the nearest point projection map yield that if $\epsilon_0 > 0$ in (2.3) is chosen sufficiently small, then there exist a sequence of smooth maps $\{u_k\} \subset C^\infty(B,N)$ such that $u_k \to u$ strongly in $W^{1,n}(B,N)$. In particular, there exists a $k_0 \geqslant 1$ such that

$$\sup_{k\geqslant k_0}\|\nabla u_k\|_{W^{1,n}(B)} \leqslant 2\epsilon_0. \tag{2.6}$$

Next, since $u_k^*TN|_B$ are trivial smooth vector bundles, there exist smooth moving frames $\{e_\alpha^k\}_{\alpha=1}^l$ along u_k^*TN on B. Let $A_k = (\langle \mathrm{d}e_\alpha^k, e_\beta^k\rangle)_{1\leqslant\alpha,\beta\leqslant l}$ and $F(A_k)$ be the connection form and curvature form of u_k^*TN with respect to the frame $\{e_\alpha^k\}_{\alpha=1}^l$ respectively. Then the same computation as in [30] Proposition 3.2 implies that

$$\big|F(A_k)\big|(x) \leqslant C|\nabla u_k|^2(x), \quad \forall x \in B. \tag{2.7}$$

This, combined with (2.6), implies

$$\sup_{k\geqslant k_0} \big\|F(A_k)\big\|_{L^{n/2}(B)} \leqslant C \sup_{k\geqslant k_0} \|\nabla u_k\|_{L^n(B)}^2 \leqslant C\epsilon_0^2. \tag{2.8}$$

Hence, for $k \geqslant k_0$, Uhlenbeck's theorem [27] implies that there are gauge transformation maps $\{R_k\} \subset W^{1,n}(B, \mathbf{SO}(l))$ such that the connection forms $\overline{A_k} = (\langle \mathrm{d}\overline{e_\alpha^k}, \overline{e_\beta^k}\rangle)_{1\leqslant\alpha,\beta\leqslant l}$ and the curvature forms $F(\overline{A_k})$ of the new moving frames $\overline{e_\alpha^k} = \sum_{\beta=1}^l R_k^{\alpha\beta} e_\beta^k$, $1 \leqslant \alpha \leqslant l$, satisfy

$$\delta\overline{A_k} = 0 \quad \text{in } B, \qquad x \cdot \overline{A_k} = 0, \quad \text{on } \partial B, \tag{2.9}$$

$$\|\overline{A_k}\|_{L^n(B)} + \|\nabla\overline{A_k}\|_{L^{n/2}(B)} \leqslant C\|F(A_k)\|_{L^{n/2}(B)} \leqslant C\|\nabla u_k\|_{L^n(B)}^2 \leqslant C\epsilon_0. \tag{2.10}$$

Finally, we want to take limit $k \to \infty$. For this, we need to estimate $\|\nabla\overline{e_\alpha^k}\|_{L^n(B)}$ for $1 \leqslant \alpha \leqslant l$.

For $y \in N$, let $P^\perp(y): \mathbf{R}^L \to (T_yN)^\perp$ denote the orthogonal projection from map $\mathbf{R}^L$ to the normal space $(T_yN)^\perp$. Then we have

$$\nabla\overline{e_\alpha^k} = \sum_{\beta=1}^l \langle\nabla\overline{e_\alpha^k}, \overline{e_\beta^k}\rangle\overline{e_\beta^k} + P^\perp(u_k)(\nabla\overline{e_\alpha^k}) = \sum_{\beta=1}^l \langle\nabla\overline{e_\alpha^k}, \overline{e_\beta^k}\rangle\overline{e_\beta^k} - A(u_k)(\overline{e_\alpha^k}, \nabla u_k) \tag{2.11}$$

where we have used

$$P^\perp(u_k)(\nabla\overline{e_\alpha^k}) = -\nabla\big(P^\perp(u_k)\big)(\overline{e_\alpha^k}) = -A(u_k)(\overline{e_\alpha^k}, \nabla u_k)$$

for $P^\perp(u_k)(\overline{e_\alpha^k}) = 0$. Therefore we have, for $k \geqslant k_0$,

$$|\nabla\overline{e_\alpha^k}|(x) \leqslant C\big(|A_k| + |\nabla u_k|\big)(x), \quad \text{for a.e. } x \in B. \tag{2.12}$$

This, combined with (2.6) and (2.10), yields

$$\sum_{\alpha=1}^l \|\nabla\overline{e_\alpha^k}\|_{L^n(B)} \leqslant C\big(\|A_k\|_{L^n(B)} + \|\nabla u_k\|_{L^n(B)}\big) \leqslant C\epsilon_0. \tag{2.13}$$

Therefore, after taking subsequences, we can assume that $\overline{e_\alpha^k} \to e_\alpha$ weakly in $W^{1,n}(B)$, strongly in $L^n(B)$, and a.e. in B. Since $u_k \to u$ strongly in $W^{1,n}(B)$, we have that $\{e_\alpha\}_{\alpha=1}^l \subset W^{1,n}(B)$ is a moving frame along u^*TN on B. Moreover, (2.10) implies that $A_k \to A \equiv (\langle \mathrm{d}e_\alpha, e_\beta\rangle)$, the connection form of $\{e_\alpha\}_{\alpha=1}^l$, weakly in $W^{1,n/2}(B)$. Hence (2.9) and (2.10) imply that A satisfies (2.4) and (2.5). The proof of Proposition 2.2 is complete. □

3. Proof of Theorem B

This section is devoted to the proof of Theorem B. First we recall some basic facts on the Hardy space $\mathcal{H}^1(\mathbf{R}^n)$ and the BMO space $\mathrm{BMO}(\mathbf{R}^n)$.

Recall that $f \in L^1(\mathbf{R}^n)$ belongs to the Hardy space $\mathcal{H}^1(\mathbf{R}^n)$ if

$$f_* := \sup_{\epsilon>0} |\phi_\epsilon * f| \in L^1(\mathbf{R}^n)$$

where $\phi_\epsilon(x) := \epsilon^{-n}\phi(\frac{x}{\epsilon})$ for a fixed nonnegative $\phi \in C_0^\infty(\mathbf{R}^n)$ with $\int_{\mathbf{R}^n}\phi\,\mathrm{d}y = 1$. Note that $\mathcal{H}^1(\mathbf{R}^n)$ is a Banach space with the norm

$$\|f\|_{\mathcal{H}^1(\mathbf{R}^n)} := \|f\|_{L^1(\mathbf{R}^n)} + \|f_*\|_{L^1(\mathbf{R}^n)}.$$

An important property of $f \in \mathcal{H}^1(\mathbf{R}^n)$ is the cancellation identity $\int_{\mathbf{R}^n} f\,\mathrm{d}y = 0$ (cf. [11]).

Recall also that $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ belongs to the BMO space $\mathrm{BMO}(\mathbf{R}^n)$ (cf. John–Nirenberg [18]), if

$$\|f\|_{\mathrm{BMO}(\mathbf{R}^n)} := \sup\left\{\frac{1}{|B|}\int_B |f - f_B|\,\mathrm{d}y\colon \text{ any ball } B \subset \mathbf{R}^n\right\} < \infty$$

where $f_B = \frac{1}{|B|}\int_B f\,\mathrm{d}y$ is the average of f over B. By the Poincaré inequality we have $W^{1,n}(\mathbf{R}^n) \subset \mathrm{BMO}(\mathbf{R}^n)$ and

$$\|f\|_{\mathrm{BMO}(\mathbf{R}^n)} \leqslant C\|\nabla f\|_{L^n(\mathbf{R}^n)}. \tag{3.1}$$

The celebrated theorem of Fefferman–Stein [11] says that the dual of $\mathcal{H}^1(\mathbf{R}^n)$ is $\mathrm{BMO}(\mathbf{R}^n)$. Moreover

$$\left|\int_{\mathbf{R}^n} fg\,\mathrm{d}y\right| \leqslant C\|f\|_{\mathcal{H}^1(\mathbf{R}^n)}\|g\|_{\mathrm{BMO}(\mathbf{R}^n)}. \tag{3.2}$$

Now we recall an important result of Coifman–Lions–Meyer–Semmes [4], see also [5].

Proposition 3.1 [4]. *For any $1 < p < \infty$, denote $p' = \frac{p}{p-1}$. Let $f \in W^{1,p}(\mathbf{R}^n)$, $g \in W^{1,p'}(\mathbf{R}^n, \wedge^1(\mathbf{R}^n))$, and $h \in W^{1,n}(\mathbf{R}^n)$. Then $\mathrm{d}f \cdot \delta g \in \mathcal{H}^1(\mathbf{R}^n)$ and*

$$\|\mathrm{d}f \cdot \delta g\|_{\mathcal{H}^1(\mathbf{R}^n)} \leqslant C\|\nabla f\|_{L^p(\mathbf{R}^n)}\|\nabla g\|_{L^{p'}(\mathbf{R}^n)}. \tag{3.3}$$

In particular, we have

$$\left|\int_{\mathbf{R}^n}\langle \mathrm{d}f \cdot \delta g, h\rangle\,\mathrm{d}y\right| \leqslant C\|\nabla f\|_{L^p(\mathbf{R}^n)}\|\nabla g\|_{L^{p'}(\mathbf{R}^n)}\|\nabla h\|_{L^n(\mathbf{R}^n)}. \tag{3.4}$$

We also recall the following pointwise convergence result, which is essentially due to Hardt–Lin–Mou [16] (see also [8]).

Lemma 3.2 [16]. *Suppose that $\{u_k\} \subset W^{1,n}(\Omega, \mathbf{R}^L)$ are weak solutions to*

$$-\operatorname{div}\big(|\nabla u_k|^{n-2}\nabla u_k\big) = f_k + \Phi_k, \tag{3.5}$$

where $f_k \to 0$ in $L^1(\Omega, \mathbf{R}^L)$, and $\Phi_k \to 0$ in $(W^{1,n}(\Omega, \mathbf{R}^L))^*$. Assume that $u_k \to u$ weakly in $W^{1,n}(\Omega, \mathbf{R}^L)$. Then, after taking possible subsequences, we have $\nabla u_k \to \nabla u$ a.e. in Ω. In particular, $\nabla u_k \to \nabla u$ strongly in $L^q(\Omega, \mathbf{R}^L)$ for any $1 \leqslant q < n$.

After these preparations, we are ready to give a proof of Theorem B. It turns out the crucial step is to show the following weak compactness under the smallness condition on E_n.

Lemma 3.3 (ϵ-weak compactness). *For any $n \geqslant 3$, there exists an $\epsilon_1 > 0$ such that if $\{u_k\} \subset W^{1,n}(2B, N)$ satisfy both Eq.* (1.2) *and the condition* (1.3) *with Ω replaced by $2B$, $u_k \to u$ weakly in $W^{1,n}(2B, N)$, and satisfy*

$$\int_{2B} |\nabla u_k|^n\,\mathrm{d}x \leqslant \epsilon_1^n, \quad \forall k \geqslant 1. \tag{3.6}$$

Then $u \in W^{1,n}(B, N)$ is an n-harmonic map.

Proof. For the convenience, we will write both equation (1.1) and (1.2) by using d and δ from now on.

Let $\epsilon_1 > 0$ be the same constant as in Proposition 2.2. Then we have that for any $k \geqslant 1$ there is a Coulomb moving frame $\{e_\alpha^k\}_{\alpha=1}^l$ along $u_k^* TN$ such that the connection form $A_k = (\langle \mathrm{d}e_\alpha^k, e_\beta^k \rangle)$ satisfies

$$\delta A_k = 0 \quad \text{in } B; \qquad x \cdot A_k = 0 \quad \text{on } \partial B \tag{3.7}$$

and

$$\|A_k\|_{L^n(B)} + \|\nabla A_k\|_{L^{n/2}(B)} \leqslant C\|\nabla u_k\|^2_{L^n(B)}. \tag{3.8}$$

Moreover, similar to (2.19), we have

$$\max_{\alpha=1}^{l} \|\nabla e_\alpha^k\|_{L^n(B)} \leqslant C\|\nabla u_k\|_{L^n(B)} \leqslant C\epsilon_1, \quad \forall k \geqslant 1. \tag{3.9}$$

Therefore we may assume, after passing to subsequences, that $e_\alpha^k \to e_\alpha$ weakly in $W^{1,n}(B, \mathbf{R}^L)$ and strongly in $L^n(B, \mathbf{R}^L)$, $A_k \to A$ weakly in $W^{1,n/2}(B)$ and strongly in $L^{n/2}(B)$. It is easy to see that $\{e_\alpha\}_{\alpha=1}^l$ is a moving frame along u^*TN, and $A = (\langle \mathrm{d}e_\alpha, e_\beta \rangle)$ satisfies

$$\delta A = 0 \quad \text{in } B; \qquad x \cdot A = 0 \quad \text{on } \partial B, \tag{3.10}$$

and

$$\|A\|_{L^n(B)} + \|\nabla A\|_{L^{n/2}(B)} \leqslant C \liminf_k \|\nabla u_k\|^2_{L^n(B)} \leqslant C\epsilon_1^2. \tag{3.11}$$

Using these moving frames, Lemma 2.1 yields that for any $1 \leqslant \alpha \leqslant l$

$$-\delta\big(\big\langle |\mathrm{d}u_k|^{n-2}\,\mathrm{d}u_k, e_\alpha^k\big\rangle\big) = \sum_{\beta=1}^{l} \big\langle |\mathrm{d}u_k|^{n-2}\,\mathrm{d}u_k, e_\beta^k\big\rangle \cdot \langle \mathrm{d}e_\alpha^k, e_\beta^k\rangle + \langle \Phi_k, e_\alpha^k\rangle. \tag{3.12}$$

It follows from Lemma 3.2 that we can assume that $\nabla u_k \to \nabla u$ strongly in $L^q(\Omega)$ for any $1 \leqslant q < n$. Therefore we have

$$|\mathrm{d}u_k|^{n-2}\,\mathrm{d}u_k \to |\mathrm{d}u|^{n-2}\,\mathrm{d}u, \quad \text{weakly in } L^{n/(n-1)}(2B). \tag{3.13}$$

This implies

$$-\delta\big(\big\langle |du_k|^{n-2}\,\mathrm{d}u_k, e_\alpha^k\big\rangle\big) \to -\delta\big(\big\langle |\mathrm{d}u|^{n-2}\,\mathrm{d}u, e_\alpha\big\rangle\big), \quad \text{in } \mathcal{D}'(B) \tag{3.14}$$

as $k \to \infty$, for all $1 \leqslant \alpha \leqslant l$.

It is readily seen that for any $\phi \in C_0^\infty(B)$ we have

$$\big|\langle \Phi_k, e_\alpha^k \phi\rangle_{\{(W^{1,n})^*, W^{1,n}\}}\big| \leqslant \|\Phi_k\|_{(W^{1,n}(B,N))^*} \|e_\alpha^k \phi\|_{W^{1,n}(B)} \to 0, \quad \text{as } k \to \infty. \tag{3.15}$$

In order to prove that u is an n-harmonic map, it suffices to prove that for any $1 \leqslant \alpha, \beta \leqslant l$

$$\big\langle |\mathrm{d}u_k|^{n-2}\,\mathrm{d}u_k, e_\beta^k\big\rangle \cdot \langle \mathrm{d}e_\alpha^k, e_\beta^k\rangle \to \big\langle |\mathrm{d}u|^{n-2}\,\mathrm{d}u, e_\beta\big\rangle \langle \mathrm{d}e_\alpha, e_\beta\rangle, \quad \text{in } \mathcal{D}'(B). \tag{3.16}$$

To prove (3.16), we first let $\bar{u}_k \in W^{1,n}(\mathbf{R}^n, \mathbf{R}^L)$ and $\overline{e_\alpha^k} \in W^{1,n}(\mathbf{R}^n, \mathbf{R}^L)$ be the extensions of u_k and e_α^k from B respectively such that

$$\|\nabla \bar{u}_k\|_{L^n(\mathbf{R}^n)} \leqslant C\|\nabla u_k\|_{L^n(B)}, \qquad \big\|\nabla(\overline{e_\alpha^k})\big\|_{L^n(\mathbf{R}^n)} \leqslant C\|\nabla e_\alpha^k\|_{L^n(B)}. \tag{3.17}$$

For $\langle |\mathrm{d}\bar{u}_k|^{n-2}\,\mathrm{d}\bar{u}_k, \overline{e_\beta^k}\rangle \in L^{n/(n-1)}(\mathbf{R}^n, \wedge^1(\mathbf{R}^n))$, the Hodge decomposition theorem (cf. Iwaniec–Martin [17]) implies that there are $f_\beta^k \in W^{1,n/(n-1)}(\mathbf{R}^n)$ and $g_\beta^k \in W^{1,n/(n-1)}(\mathbf{R}^n, \wedge^2(R^n))$ such that $\mathrm{d}g_\beta^k = 0$,

$$\big\langle |\mathrm{d}\bar{u}_k|^{n-2}\,\mathrm{d}\bar{u}_k, \overline{e_\beta^k}\big\rangle = \mathrm{d}f_\beta^k + \delta g_\beta^k, \tag{3.18}$$

and

$$\|\nabla f_\beta^k\|_{L^{n/(n-1)}(\mathbf{R}^n)} + \|\nabla g_\beta^k\|_{L^{n/(n-1)}(\mathbf{R}^n)} \leqslant C\|\nabla u_k\|^{n-1}_{L^n(B)}. \tag{3.19}$$

It follows from (3.19) that we may assume $f_\beta^k \to f_\beta$, $g_\beta^k \to g_\beta$ weakly in $W^{1,n/(n-1)}_{\mathrm{loc}}(\mathbf{R}^n)$. Therefore, by taking k to infinity, (3.18) implies

$$\langle |\mathrm{d}u|^{n-2}\,\mathrm{d}u, e_\beta\rangle = \mathrm{d}f_\beta + \delta g_\beta; \quad \mathrm{d}g_\beta = 0, \quad \text{in } B. \tag{3.20}$$

Moreover, (3.18) gives

$$\langle |\mathrm{d}u_k|^{n-2}\,\mathrm{d}u_k, e_\beta^k\rangle \cdot \langle \mathrm{d}e_\alpha^k, e_\beta^k\rangle = \mathrm{d}f_\beta^k \cdot \langle \mathrm{d}e_\alpha^k, e_\beta^k\rangle + \delta g_\beta^k \cdot \langle \mathrm{d}e_\alpha^k, e_\beta^k\rangle, \quad \text{in } B. \tag{3.21}$$

Since $\mathrm{d}f_\beta^k \to \mathrm{d}f_\beta$ weakly in $L^{n/(n-1)}(B)$, $\langle \mathrm{d}e_\alpha^k, e_\beta^k\rangle \to \langle \mathrm{d}e_\alpha, e_\beta\rangle$ weakly in $L^n(B)$, and $\delta\langle \mathrm{d}e_\alpha^k, e_\beta^k\rangle = 0$ in B, we can apply the Div–Curl lemma (cf. [6] page 53) to conclude

$$\mathrm{d}f_\beta^k \cdot \langle \mathrm{d}e_\alpha^k, e_\beta^k\rangle \to \mathrm{d}f_\beta \cdot \langle \mathrm{d}e_\alpha, e_\beta\rangle, \quad \text{in } \mathcal{D}'(B). \tag{3.22}$$

In fact, (3.22) follows directly from the integrations by parts: for any $\phi \in C_0^\infty(B)$,

$$\int_{\mathbf{R}^n} \mathrm{d}f_\beta^k \cdot \langle \mathrm{d}e_\alpha^k, e_\beta^k\rangle \phi\,\mathrm{d}x = -\int_{\mathbf{R}^n} f_\beta^k \langle \mathrm{d}e_\alpha^k, e_\beta^k\rangle \cdot \mathrm{d}\phi\,\mathrm{d}x$$
$$\to -\int_{\mathbf{R}^n} f_\beta \langle \mathrm{d}e_\alpha, e_\beta\rangle \cdot \mathrm{d}\phi\,\mathrm{d}x = \int_{\mathbf{R}^n} \mathrm{d}f_\beta \cdot \langle \mathrm{d}e_\alpha, e_\beta\rangle\phi$$

as $k \to \infty$. Here we have used both (3.7) and (3.10), i.e. $\delta\langle \mathrm{d}e_\alpha^k, e_\beta^k\rangle = \delta\langle \mathrm{d}e_\alpha, e_\beta\rangle = 0$, in B.

Now we need the compensated compactness result (cf. Lions [19,20]), which was developed by Freire–Müller–Struwe [9,10] in the context of wave maps on $\mathbf{R}^{2+1}$.

Lemma 3.4. *Under the same notations. After taking possible subsequences, we have*

$$\delta g_\beta^k \cdot \langle \mathrm{d}e_\alpha^k, e_\beta^k\rangle \to \delta g_\beta \cdot \langle \mathrm{d}e_\alpha, e_\beta\rangle + \nu, \quad \text{in } B \tag{3.23}$$

where ν is a signed Radon measure given by

$$\nu = \sum_{j\in J} a_j \delta_{x_j} \tag{3.24}$$

where J is at most countable, $a_j \in \mathbf{R}$, $x_j \in B$, and $\sum_{j\in J} |a_j| < +\infty$.

Proof. For the simplicity, we only outline a proof based on suitable modifications of [10].

First we observe that

$$\begin{aligned}
&\delta g_\beta^k \cdot \langle \mathrm{d}e_\alpha^k, e_\beta^k\rangle - \delta g_\beta \cdot \langle \mathrm{d}e_\alpha, e_\beta\rangle \\
&\quad = \delta(g_\beta^k - g_\beta) \cdot \langle \mathrm{d}(e_\alpha^k - e_\alpha), e_\beta^k\rangle + \delta g_\beta \cdot \langle \mathrm{d}(e_\alpha^k - e_\alpha), e_\beta^k\rangle + \left(\delta g_\beta^k \cdot \langle \mathrm{d}e_\alpha, e_\beta^k\rangle - \delta g_\beta \cdot \langle \mathrm{d}e_\alpha, e_\beta\rangle\right) \\
&\quad = \delta(g_\beta^k - g_\beta) \cdot \langle \mathrm{d}(e_\alpha^k - e_\alpha), e_\beta^k\rangle + I_k + II_k.
\end{aligned}$$

The dominated convergence theorem implies

$$I_k, II_k \to 0, \quad \text{in } L^1(B), \text{ as } k \to \infty.$$

Therefore (3.23) and (3.24) is equivalent to

$$\delta(g_\beta^k - g_\beta) \cdot \langle \mathrm{d}(e_\alpha^k - e_\alpha), e_\beta^k\rangle \to \nu \tag{3.25}$$

where ν is the Radon measure given by (3.24).

Since $|\nabla(e_\alpha^k - e_\alpha)|^n$, $|\nabla(g_\beta^k - g_\beta)|^{n/(n-1)}$ are bounded in $L^1(B)$, we may assume, after taking subsequences, that there is a nonnegative Radon measure μ on B such that

$$\left(\sum_{\alpha=1}^{l}\left|\nabla(e_\alpha^k - e_\alpha)\right|^n + \sum_{\beta=1}^{l}\left|\nabla(g_\beta^k - g_\beta)\right|^{n/(n-1)}\right)\mathrm{d}x \to \mu$$

as convergence of Radon measures on B.

Let $\mathcal{S} = \{x \in B\colon\ \mu(\{x\}) \equiv \lim_{r\to 0}\mu(B_r(x)) > 0\}$. Then it follows from $\mu(B) < +\infty$ that $\mathcal{S}$ is at most a countable set. Now we want to show

$$\operatorname{supp}(\nu) \subset \mathcal{S}. \tag{3.26}$$

It is easy to see that (3.26) yields (3.24) and hence the conclusion of Lemma 3.4.

To see (3.26), we proceed as follows. For $\phi \in C_0^\infty(B)$, we have

$$\begin{aligned}\langle \nu, \phi^3\rangle &= \lim_{k\to\infty}\int_{\mathbf{R}^n} \phi\delta(g_\beta^k - g_\beta)\cdot\left\langle \phi \mathrm{d}(e_\alpha^k - e_\alpha), \phi e_\beta^k\right\rangle \mathrm{d}x \\ &= \lim_{k\to\infty}\int_{\mathbf{R}^n}\left[\delta\big(\phi(g_\beta^k - g_\beta)\big) - \mathrm{d}\phi\cdot(g_\beta^k - g_\beta)\right]\cdot\left\langle\left[\mathrm{d}(\phi(e_\alpha^k - e_\alpha)\big) - (e_\alpha^k - e_\alpha)\,\mathrm{d}\phi\right], \phi e_\beta^k\right\rangle \mathrm{d}x \\ &= \lim_{k\to\infty}\int_{\mathbf{R}^n}\delta\big(\phi(g_\beta^k - g_\beta)\big)\cdot\left\langle \mathrm{d}\big(\phi(e_\alpha^k - e_\alpha)\big), \phi e_\beta^k\right\rangle \mathrm{d}x \end{aligned} \tag{3.27}$$

where we have used

$$\lim_{k\to\infty}\int_{\mathbf{R}^n}\left[(g_\beta^k - g_\beta)\,\mathrm{d}\phi\cdot\left\langle \phi d(e_\alpha^k - e_\alpha), \phi e_\beta^k\right\rangle - \delta\big(\phi(g_\beta^k - g_\beta)\big)\cdot\left\langle (e_\alpha^k - e_\alpha)\,\mathrm{d}\phi, \phi e_\beta^k\right\rangle\right]\mathrm{d}x = 0.$$

Note that Proposition 3.1 implies $H_k \equiv \delta(\phi(g_\beta^k - g_\beta))\cdot \mathrm{d}(\phi(e_\alpha^k - e_\alpha))$ is bounded in $\mathcal{H}^1(\mathbf{R}^n)$, and (3.22) implies $H_k \to 0$ in $\mathcal{D}'(\mathbf{R}^n)$. Therefore we have that $H_k \to 0$ weak* in $\mathcal{H}^1(\mathbf{R}^n)$. On the other hand, since $\phi e_\beta \in W^{1,n}(\mathbf{R}^n)$, we have $\phi e_\beta \in \mathrm{VMO}(\mathbf{R}^n)$, where $\mathrm{VMO}(\mathbf{R}^n) \subset \mathrm{BMO}(\mathbf{R}^n)$ is the closure of $C_0^\infty(\mathbf{R}^n)$ in the BMO norm. It is well-known [11] that the dual of $\mathrm{VMO}(\mathbf{R}^n)$ is $\mathcal{H}^1(\mathbf{R}^n)$. Hence we have

$$\lim_{k\to\infty}\int_{\mathbf{R}^n}\delta\big(\phi(g_\beta^k - g_\beta)\big)\cdot\left\langle \mathrm{d}\big(\phi(e_\alpha^k - e_\alpha)\big), \phi e_\beta\right\rangle \mathrm{d}x = 0. \tag{3.28}$$

Putting (3.28) together with (3.27) and applying (3.4), we have

$$\begin{aligned}\left|\langle \nu, \phi^3\rangle\right| &\leqslant C\lim_{k\to\infty}\left\|\nabla\big(\phi(e_\beta^k - e_\beta)\big)\right\|_{L^n(\mathbf{R}^n)}\left\|\nabla\big(\phi(e_\alpha^k - e_\alpha)\big)\right\|_{L^n(\mathbf{R}^n)}\left\|\nabla\big(\phi(g_\beta^k - g_\beta)\big)\right\|_{L^{n/(n-1)}(\mathbf{R}^n)} \\ &\leqslant C\lim_{k\to\infty}\Big\{\left[\left\|\phi\nabla(e_\beta^k - e_\beta)\right\|_{L^n(\mathbf{R}^n)} + \|\nabla\phi\|_{L^\infty}\|e_\beta^k - e_\beta\|_{L^n(B)}\right] \\ &\quad\times\left[\left\|\phi\nabla(e_\alpha^k - e_\alpha)\right\|_{L^n(\mathbf{R}^n)} + \|\nabla\phi\|_{L^\infty}\|e_\alpha^k - e_\alpha\|_{L^n(B)}\right] \\ &\quad\times\left[\left\|\phi\nabla(g_\beta^k - g_\beta)\right\|_{L^{n/(n-1)}(\mathbf{R}^n)} + \|\nabla\phi\|_{L^\infty}\|g_\beta^k - g_\beta\|_{L^{n/(n-1)}(B)}\right]\Big\} \\ &\leqslant C\big(\langle\mu, \phi^n\rangle\big)^{1/n}\big(\langle\mu, \phi^n\rangle\big)^{1/n}\big(\langle\mu, \phi^{n/(n-1)}\rangle\big)^{(n-1)/n}\end{aligned} \tag{3.29}$$

where we have used

$$\lim_{k\to\infty}\big(\|e_\alpha^k - e_\alpha\|_{L^n(B)} + \|g_\beta^k - g_\beta\|_{L^{n/(n-1)}(B)}\big) = 0.$$

By choosing $\phi_i \in C_0^\infty(B)$ such that $\phi_i \to \lambda_{B_r(y)}$, the characteristic function of a ball $B_r(y)$, we then have

$$\nu\big(B_r(y)\big) \leqslant C\mu\big(B_r(y)\big)^{(n+1)/n}. \tag{3.30}$$

Therefore ν is absolutely continuous with respect to μ. Moreover, for any $y \notin \mathcal{S}$, the Radon–Nikodyn derivative

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(y) = \lim_{r\to 0}\frac{\nu(B_r(y))}{\mu(B_r(y))} \leqslant C\lim_{r\to 0}\mu\big(B_r(y)\big)^{1/n} = 0.$$

Therefore the support of ν is contained in $\mathcal{S}$. This proves (3.26) and hence (3.24). The proof of Lemma 3.4 is complete. □

Now we return to the proof of Lemma 3.3. By putting (3.14), (3.20), (3.22), and (3.23) together, we have, for any $1 \leqslant \alpha \leqslant l$,

$$-\delta\big(\big\langle|\mathrm{d}u|^{n-2}\,\mathrm{d}u, e_\alpha\big\rangle\big) = \sum_{\alpha=1}^{l}\big\langle|\mathrm{d}u|^{n-2}\,\mathrm{d}u, e_\beta\big\rangle\cdot\langle \mathrm{d}e_\alpha, e_\beta\rangle + \sum_{j\in J} a_j\delta_{x_j} \tag{3.31}$$

where J is at most countable, $a_j \in \mathbf{R}$, $x_j \in B$, and $\sum_{j\in J}|a_j| < +\infty$.

In order to conclude that u is an n-harmonic map, one has to show that $a_j = 0$ for all $j \in J$. In fact, (3.31) implies that $\sum_{j\in J} a_j\delta_{x_j} \in W^{-1,n}(B) + L^1(B)$. One the other hand, it is well-known that $\delta_x \notin W^{-1,n}(B) + L^1(B)$ for any $x \in B$. Hence $a_j = 0$ for $j \in J$. The proof of Lemma 3.3 is complete. □

Based on Lemma 3.3, we can give a proof of Theorem B as follows.

Proof of Theorem B. Since $|\nabla u_k|^n$ is bounded in $L^1(\Omega)$, we may assume, after passing to subsequences, that there is a nonnegative Radon measure μ on Ω such that

$$|\nabla u_k|^n\,\mathrm{d}x \to \mu$$

as convergence of Radon measures. Let $\epsilon_1 > 0$ be the same constant as in Lemma 3.3 and define $\Sigma \subset \Omega$ by

$$\Sigma = \big\{x \in \Omega\colon\ \mu\big(\{x\}\big) \geqslant \epsilon_1^n\big\}.$$

Then Σ is a finite subset and

$$|\Sigma| \leqslant C\epsilon_1^{-n}, \quad C \equiv \limsup_{k\to\infty}\int_\Omega |\nabla u_k|^n\,\mathrm{d}x < +\infty.$$

For any $x_0 \in \Omega \setminus \Sigma$, there exists an $r_0 > 0$ such that $\mu(B_{4r_0}(x_0)) < \epsilon_1^n$. Since

$$\limsup_{k\to\infty}\int_{B_{2r_0}(x_0)} |\nabla u_k|^n\,\mathrm{d}x \leqslant \mu\big(B_{4r_0}(x_0)\big),$$

we can assume that there exists $k_0 \geqslant 1$ such that $\int_{B_{2r_0}(x_0)}|\nabla u_k|^2\,\mathrm{d}x \leqslant \epsilon_1^n,\ \forall k \geqslant k_0$. Therefore Lemma 3.3 implies that u is an n-harmonic map in $B_{r_0}(x_0)$. Since $x_0 \in \Omega \setminus \Sigma$ is arbitrary, we conclude that u is an n-harmonic map in $\Omega \setminus \Sigma$. Since Σ is finite, it is standard to show that u is also an n-harmonic map in Ω (cf. [7,26]). □

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