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Hydrostatic Stokes equations with non-smooth data for mixed boundary conditions

Équations de Stokes hydrostatiques avec conditions aux limites mixtes à données peu régulières

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Abstract

The main subject of this work is to study the concept of very weak solution for the hydrostatic Stokes system with mixed boundary conditions (non-smooth Neumann conditions on the rigid surface and homogeneous Dirichlet conditions elsewhere on the boundary). In the Stokes framework, this concept has been studied by Conca [Rev. Mat. Apl. 10 (1989)] imposing non-smooth Dirichlet boundary conditions.

In this paper, we introduce the dual problem that turns out to be a hydrostatic Stokes system with non-free divergence condition. First, we obtain strong regularity for this dual problem (which can be viewed as a generalisation of the regularity results for the hydrostatic Stokes system with free divergence condition obtained by Ziane [Appl. Anal. 58 (1995)]). Afterwards, we prove existence and uniqueness of very weak solution for the (primal) problem.

As a consequence of this result, the existence of strong solution for the non-stationary hydrostatic Navier-Stokes equations is proved, weakening the hypothesis over the time derivative of the wind stress tensor imposed by Guillén-González, Masmoudi and Rodríguez-Bellido [Differential Integral Equations 50 (2001)].

Résumé

Le but principal de ce travail est d'étudier le concept de solution très faible pour le système de Stokes hydrostatique avec conditions aux limites mixtes (condition de Neumann non régulière sur la surface rigide et condition de Dirichlet homogène dans le reste). Dans le cas du problème de Stokes, ce sujet a été étudié par Conca [Rev. Mat. Apl. 10 (1989)] en imposant une condition aux limites de Dirichlet non homogène et peu régulière.

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Dans ce papier, on introduit le problème dual qui est aussi un système de Stokes hydrostatique mais avec une condition de divergence non nulle. D'abord, on obtient la régularité forte pour le problème dual (ce résultat peut être consideré comme une généralisation des résultats de régularité pour le système de Stokes hydrostatique avec la condition de divergence nulle, obtenus par Ziane [Appl. Anal. 58 (1995)]). On montre ensuite l'existence et unicité de solution très faible pour le problème primal.

Comme conséquence de ce résultat, on montre l'existence de solution forte pour le problème de Navier-Stokes hydrostatique non-stationnaire, avec une hypothèse sur la dérivée par rapport au temps du tenseur du vent plus faible que celle qui était imposée par Guillén-González, Masmoudi et Rodríguez-Bellido [Differential Integral Equations 50 (2001)].

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1. Introduction

The hydrostatic Navier–Stokes problem (also called Primitive Equations) is a model that appears in Geophysical fluid dynamics, in order to describe the general circulation in the Ocean or the Atmosphere [14]. This model has been extensively studied from a mathematical point of view by several authors [12,13,3,2,10,8,9], who in particular have established existence and uniqueness results for the stationary and non-stationary problems.

Let us recall that the Primitive Equations are variants of the Navier–Stokes system, where some simplifications have been made based on the analysis of physical scales (because the domain of study have a depth scale negligible in comparison to horizontal scales). Concretely, rigid-lid hypothesis and hydrostatic pressure are imposed [10]. These simplifications reduce the dimension of the system from a numerical point of view. However, it does not make any easier its mathematical analysis. For instance, this system is no longer parabolic for the vertical velocity, which depends upon derivatives for the horizontal velocity, loosing basically an order of regularity.

As far as we know, all the results concerning strong solutions for the Primitive Equations are based on Ziane's results for the hydrostatic Stokes problem [16].

In [9], Guillén-González, Masmoudi and Rodríguez-Bellido used Ziane's results in order to obtain existence of strong solutions (global in time for small data or local in time for any data) for the non-stationary hydrostatic Navier–Stokes problem. This fact forced to impose some regularity hypothesis on the data (more precisely, on the time derivative for the Neumann boundary condition) that we consider as not optimal.

In [6], Conca defines the very weak solution concept for the Stokes problem, analysing what kind of regularity can be obtained for a Stokes system when Dirichlet boundary data only belong to $L^2(\partial\Omega)$ (we recall that a weak solution has regularity $H^1(\Omega)$, therefore one has to impose Dirichlet data in $H^{1/2}(\partial\Omega)$). See also [1] for the non-hilbertian case.

In the present paper, we study a very weak solution concept for the hydrostatic Stokes problem, with mixed Dirichlet–Neumann boundary conditions. Moreover, we apply this research as an auxiliary problem in order to obtain strong regularity for the non-stationary case.

By sections, the main contributions of this paper are the following:

– In Section 2 we set up the formulation of the (stationary) hydrostatic Stokes problem (2), and we define the dual problem associated (3). Using a mixed formulation, we obtain, first, a weak solution for the dual problem (where the "divergence condition" does not vanish). Afterwards, we obtain strong solution for the dual problem. Finally, we define the very weak solution concept for the hydrostatic Stokes problem, by means of a transposition argument (using the strong regularity for the dual problem) and we prove the existence (and uniqueness) of very weak solutions.

– In Section 3 we give a differential interpretation of the very weak solution and a meaning for the boundary conditions; Dirichlet on the bottom and Neumann on the surface.
– In Section 4 we apply the results obtained in Section 2 in order to obtain strong solutions for the non-stationary hydrostatic Stokes Problem, weakening the hypothesis on the time derivative for the wind stress tensor imposed in [9] (here, a very weak solution will be used to lift the time derivative of the Neumann data). Finally, the application to the nonlinear non-stationary problem (or non-stationary Primitive Equations) is sketched, rewriting the arguments used in [9].

2. Existence of very weak solutions

2.1. Formulation of the problem

We consider an open, bounded and Lipschitz-continuous domain $\Omega \subseteq \mathbb{R}^3$ given by

$$\Omega = \big\{(\mathbf{x}, z) \in \mathbb{R}^3;\ \mathbf{x} = (x, y) \in S,\ -h(\mathbf{x}) < z < 0\big\}, \tag{1}$$

where S is an open bounded domain of $\mathbb{R}^2$ and $h : S \to \mathbb{R}_+$ is the depth function (which regularity assumptions will be precised later on). The boundary $\partial\Omega$ can be written as $\partial\Omega = \Gamma_s \cup \Gamma_b \cup \Gamma_l$ where:

$$\Gamma_s = \big\{(\mathbf{x}, 0) \mid \mathbf{x} \in S\big\},$$

$$\Gamma_b = \big\{\big(\mathbf{x}, -h(\mathbf{x})\big) \mid \mathbf{x} \in S\big\},$$

and

$$\Gamma_l = \big\{(\mathbf{x}, z) / \mathbf{x} \in \partial S,\ -h(\mathbf{x}) < z < 0\big\}.$$

In the main results of this work, we are going to impose the hypothesis of minimal depth strictly positive, i.e.

$$h \geqslant h_{\min} > 0 \quad \text{in } S.$$

Concretely, a vertical section of the domain can be viewed in Fig. 1.

We start from the hydrostatic Stokes problem:

$$\begin{cases} -\nu\Delta\mathbf{u} - \nu_3\partial^2_{zz}\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \langle\mathbf{u}\rangle = 0 & \text{in } S, \\ \nu_3\partial_z\mathbf{u} = \Upsilon & \text{on } \Gamma_s, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_b \cup \Gamma_l, \end{cases} \tag{2}$$

where $\langle\mathbf{u}\rangle(\mathbf{x}) = \int_{-h(\mathbf{x})}^0 \mathbf{u}(\mathbf{x}, z)\, dz$. The unknowns are $\mathbf{u} : \Omega \to \mathbb{R}^2$ the horizontal components of the velocity, and a potential $p : S \to \mathbb{R}$ representing the surface pressure stress (and the centripetal forces, see [14]). The data for (2) are the external forces, $\mathbf{f} : \Omega \to \mathbb{R}^2$, the wind tension stress on the surface, $\Upsilon : \Gamma_s \to \mathbb{R}^2$, and the (eddy) horizontal and vertical viscosities, $\nu > 0$ and $\nu_3 > 0$, respectively. Δ, ∇ and $\nabla\cdot$ denote the two-dimensional operators: $\partial^2_{xx} + \partial^2_{yy}$, $(\partial_x, \partial_y)^t$ and the horizontal divergence operator, respectively.

We define the following dual problem:

$$\begin{cases} -\nu\Delta\Phi - \nu_3\partial^2_{zz}\Phi + \nabla\pi = \mathbf{g} & \text{in } \Omega, \\ \nabla \cdot \langle\Phi\rangle = -\varphi & \text{in } S, \\ \nu_3\partial_z\Phi = \mathbf{0} & \text{on } \Gamma_s, \\ \Phi = \mathbf{0} & \text{on } \Gamma_b \cup \Gamma_l, \end{cases} \tag{3}$$

where (Φ, π) are the unknowns and $(\mathbf{g}, \varphi)$ the data.

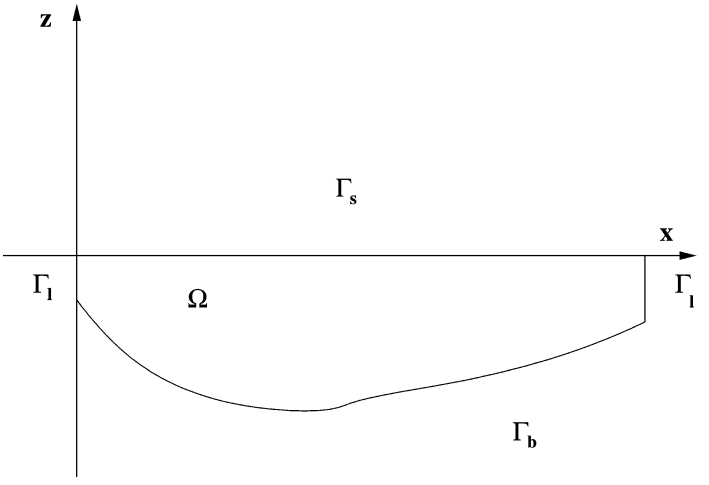

Fig. 1. A vertical section of the domain.

2.2. *Weak and strong regularity for the dual problem*

The following functional spaces will be used for the velocities:

$$H = \overline{\mathcal{V}}^{L^2} = \{\mathbf{v} \in L^2(\Omega)^2;\ \nabla \cdot \langle \mathbf{v}\rangle = 0 \text{ in } S, \langle \mathbf{v}\rangle \cdot \mathbf{n}_{\partial S} = 0\},$$
$$V = \overline{\mathcal{V}}^{H^1} = \{\mathbf{v} \in H^1(\Omega)^2;\ \nabla \cdot \langle \mathbf{v}\rangle = 0 \text{ in } S, \mathbf{v}|_{\Gamma_b \cup \Gamma_l} = \mathbf{0}\},$$

where $\mathbf{n}_{\partial S}$ denotes the outward normal vector to the boundary of S contained in the plane $z = 0$,

$$\mathcal{V} = \{\varphi \in C^\infty_{b,l}(\Omega)^2;\ \nabla \cdot \langle \varphi\rangle = 0 \text{ in } S\},$$

and

$$C^\infty_{b,l}(\Omega) = \{\varphi \in C^\infty(\Omega)^2;\ \operatorname{supp}(\varphi) \text{ is a compact set} \subseteq \overline{\Omega}\backslash(\Gamma_b \cup \Gamma_l)\}.$$

We denote $H^1_{b,l}(\Omega)$ the space of functions of $H^1(\Omega)$ vanishing on $\Gamma_b \cup \Gamma_l$ (i.e. $H^1_{b,l}(\Omega) = \overline{C^\infty_{b,l}(\Omega)}^{H^1}$).

Respect to spaces for the pressure, let us introduce:

$$L^2_0(S) = \left\{q \in L^2(S);\ \int_S q\,d\mathbf{x} = 0\right\},$$
$$\mathcal{H} = \left\{q \in H^1(S);\ \int_S q\,d\mathbf{x} = 0\right\} = H^1(S) \cap L^2_0(S).$$

In the following, by C we will denote different positive constants.

Now, we present the main result of this subsection:

Theorem 2.1 (Strong solution of (3)). *Suppose $S \subseteq \mathbb{R}^2$ with $\partial S \in C^3$ and $h \in C^3(\overline{S})$ with $h \geqslant h_{\min} > 0$ in S, and the corresponding domain Ω (defined in (1)). If $\mathbf{g} \in L^2(\Omega)^2$ and $\varphi \in \mathcal{H}$, then there exists a unique (strong) solution of (3) with $\Phi \in H^2(\Omega)^2 \cap H^1_{b,l}(\Omega)^2$, $\pi \in H^1(S)$. Moreover,*

$$\|\Phi\|^2_{H^2(\Omega)} + \|\pi\|^2_{H^1(S)} \leqslant C\{\|\mathbf{g}\|^2_{L^2(\Omega)} + \|\varphi\|^2_{H^1(S)}\}. \tag{4}$$

First of all, we will obtain a weak solution of (3) using a mixed formulation for the problem. We introduce the notation:

$$X = H^1_{b,l}(\Omega)^2, \qquad M = L^2_0(S),$$

$$a(\mathbf{u},\mathbf{v}) = \nu \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, d\Omega + \nu_3 \int_\Omega \partial_z\mathbf{u}\cdot\partial_z\mathbf{v}\, d\Omega, \quad \forall\mathbf{u},\ \mathbf{v}\in X,$$

$$b(\mathbf{u}, p) = -\int_S p\big(\nabla\cdot\langle\mathbf{u}\rangle\big)\, d\mathbf{x}, \quad \forall\mathbf{u}\in X,\ \forall p\in M,$$

$$\langle L, \mathbf{v}\rangle = \int_\Omega \mathbf{g}\cdot\mathbf{v}\, d\Omega, \quad \forall\mathbf{v}\in X,$$

$$\langle R, q\rangle = \int_S \varphi\, q\, d\mathbf{x}, \quad \forall q\in M.$$

It is easy to verify that $a(\cdot,\cdot)$ is a bilinear, symmetric, continuous and elliptic form on $X\times X$, $b(\cdot,\cdot)$ is a bilinear continuous form on $X\times M$, L is a linear continuous form on X and R is a linear continuous form on M.

Then, we consider the (abstract) mixed problem: Find $(\Phi,\pi)\in X\times M$ such that:

$$\begin{cases} a(\Phi,\mathbf{v}) + b(\mathbf{v},\pi) = \langle L,\mathbf{v}\rangle & \forall\mathbf{v}\in X,\\ b(\Phi,q) = \langle R,q\rangle & \forall q\in M. \end{cases} \tag{5}$$

Proposition 2.2 (Existence and uniqueness of solution for (5) [7]). *Suppose that*:

- $a(\cdot,\cdot)$ *is a bilinear continuous form on* $X\times X$, *satisfying the* V*-elliptic condition i.e. there exists* $a_0>0$ *such that*:

  $$a(\mathbf{v},\mathbf{v}) \geqslant a_0\|\mathbf{v}\|_X^2, \quad \forall\mathbf{v}\in V,$$

- $b(\cdot,\cdot)$ *is a bilinear continuous form on* $X\times M$ *satisfying the inf-sup condition, i.e. there exists* $\beta_0>0$ *such that*:

  $$\inf_{p\in M\setminus\{0\}} \sup_{\mathbf{v}\in X\setminus\{0\}} \frac{b(\mathbf{v},p)}{\|\mathbf{v}\|_X\|p\|_M} \geqslant \beta_0.$$

Then, for each pair $(L,R)\in X'\times M'$ (*the dual space of* $X\times M$) *the mixed problem* (5) *has a unique solution* $(\Phi,\pi)\in X\times M$. *Moreover, the following mapping is an isomorphism*:

$$(L,R)\in X'\times M' \to (\Phi,\pi)\in X\times M.$$

Therefore, in order to prove existence and uniqueness of weak solution of (3), $\Phi\in H^1_{b,l}(\Omega)^2$ and $\pi\in L^2_0(S)$, we only have to prove that the inf-sup condition holds. For this purpose, we use the following result:

Lemma 2.3 [7]. *Let* $\Omega\subset\mathbb{R}^3$ *be a Lipschitz-continuous domain. The three-dimensional divergence operator,* $\nabla_3\cdot : W^\perp \to L^2_0(\Omega)$ *is an isomorphism, where* $W=\{\mathbf{v}\in H^1_0(\Omega)^3;\ \nabla_3\cdot\mathbf{v}=0\}$, $W^\perp$ *is the orthogonal space of* W *respect to the* $H^1_0(\Omega)$*-norm, and* $L^2_0(\Omega)=\{g\in L^2(\Omega),\ \int_\Omega g\, d\Omega = 0\}$.

Lemma 2.4. *Assume Ω (defined in (1)) Lipschitz-continuous and $h \geqslant h_{\min} > 0$ in S. The following inf-sup condition holds*:

$$\sup_{\mathbf{v}\in H_0^1(\Omega)^2\setminus\{0\}} \frac{\int_S p\,\nabla\cdot\langle\mathbf{v}\rangle\,d\mathbf{x}}{\|\mathbf{v}\|_{H_0^1(\Omega)}} \geqslant C\sqrt{h_{\min}}\,\|p\|_{L^2(S)}, \quad \forall p \in L_0^2(S).$$

Proof. Basically, we follow here an argument introduced in [5]. Let $p \in L_0^2(S)$. Easily, we can deduce $\frac{1}{h(\mathbf{x})}p \in L_0^2(\Omega)$ (using that $h \geqslant h_{\min} \geqslant 0$) and $\partial_z p = 0$. Indeed, we have:

$$\left\|\frac{1}{h(\mathbf{x})}p\right\|_{L^2(\Omega)} \leqslant \left\|\frac{1}{\sqrt{h(\mathbf{x})}}p\right\|_{L^2(S)} \leqslant \frac{1}{\sqrt{h_{\min}}}\|p\|_{L^2(S)}.$$

Then, applying Lemma 2.3, there exists a function $\mathbf{U} = (\mathbf{u}, u_3) \in W^{\perp} \subset H_0^1(\Omega)^3$ such that $\nabla_3\cdot\mathbf{U} = \frac{1}{h(\mathbf{x})}p$ and (in particular)

$$\|\nabla_3\mathbf{U}\|_{L^2(\Omega)} \leqslant C\|\nabla_3\cdot\mathbf{U}\|_{L^2(\Omega)} \leqslant C\left\|\frac{1}{h(\mathbf{x})}p\right\|_{L^2(\Omega)} \leqslant C\frac{1}{\sqrt{h_{\min}}}\|p\|_{L^2(S)}. \tag{6}$$

Rewriting the bilinear form $b(\cdot,\cdot)$, one has that for all $\mathbf{v} \in H_0^1(\Omega)^2$

$$b(\mathbf{v}, p) = -\int_S p\big(\nabla\cdot\langle\mathbf{v}\rangle\big)\,d\mathbf{x} = -\int_S p\,\langle\nabla\cdot\mathbf{v}\rangle\,d\mathbf{x} = -\int_\Omega p\,\nabla\cdot\mathbf{v}\,d\Omega$$
$$(\text{as } \partial_z p = 0) = -\int_\Omega p\,\nabla_3\cdot(\mathbf{v}, v_3)\,d\Omega$$

where v_3 is any function belonging to $H_0^1(\Omega)$. Then, in particular

$$\big|b(\mathbf{u}, p)\big| = \left|-\int_\Omega p\,\nabla_3\cdot(\mathbf{u}, u_3)\,d\Omega\right| = \int_\Omega \frac{1}{h(\mathbf{x})}p^2 d\Omega = \int_S p^2 d\mathbf{x} = \|p\|^2_{L^2(S)}. \tag{7}$$

Therefore, using (6) and (7)

$$\frac{|b(\mathbf{u}, p)|}{\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|p\|_{L^2(S)}} = \frac{\|p\|_{L^2(S)}}{\|\nabla\mathbf{u}\|_{L^2(\Omega)}} \geqslant C\sqrt{h_{\min}}\,\frac{\|\nabla_3\mathbf{U}\|_{L^2(\Omega)}}{\|\nabla\mathbf{u}\|_{L^2(\Omega)}} \geqslant C\sqrt{h_{\min}},$$

hence

$$\sup_{\mathbf{v}\in H_0^1(\Omega)^2\setminus\{0\}} \frac{|b(\mathbf{v}, p)|}{\|\mathbf{v}\|_{H_0^1(\Omega)}} \geqslant C\sqrt{h_{\min}}\,\|p\|_{L_0^2(S)}, \quad \forall p \in L_0^2(S). \qquad \square$$

Applying Proposition 2.2 in our context, we deduce the following result:

Theorem 2.5 (Weak solution of (3)). *Suppose Ω Lipschitz-continuous and $h \geqslant h_{\min} > 0$ in S. If $\mathbf{g} \in H_{b,l}^{-1}(\Omega)^2$ and $\varphi \in L_0^2(S)'$, then there exists a unique weak solution of* (3), $(\Phi, \pi) \in H_{b,l}^1(\Omega)^2 \times L_0^2(S)$. *Moreover,*

$$\|\Phi\|_{H^1(\Omega)} + \|\pi\|_{L^2(S)} \leqslant C\big\{\|\mathbf{g}\|_{H_{b,l}^1(\Omega)'} + \|\varphi\|_{L_0^2(S)'}\big\}. \tag{8}$$

Remark 2.1. Taking into account that the space $L^2(S)/\mathbb{R}$ is isomorphic to $L_0^2(S)'$ (see [7] for instance), we can replace in (8) the norm $\|\varphi\|_{L_0^2(S)'}$ by $\|\varphi\|_{L^2(S)/\mathbb{R}}$.

In order to prove Theorem 2.1, we will use two auxiliary regularity results; a result for elliptic problems in Ω (see [16] and references therein cited) and Cattabriga's result for the Stokes problem in S (see [15] for instance), that we recall here:

Proposition 2.6 (Regularity for a elliptic problem in Ω, with mixed Neumann–Dirichlet boundary conditions). *Assume $h \in C^3(\bar{S})$ with $h \geqslant h_{\min} > 0$ in S and $\partial S \in C^3$, and the corresponding domain Ω. Let $\mathbf{u}$ be the unique solution of the problem*:

$$\begin{cases} -\nu\Delta\mathbf{w} - \nu_3\partial^2_{zz}\mathbf{w} = \mathbf{d} & \text{in } \Omega, \\ \nu_3\partial_z\mathbf{w} = \Upsilon & \text{on } \Gamma_s, \\ \mathbf{w} = \Psi_l \text{ (respectively } \Psi_b) & \text{on } \Gamma_l \text{ (respectively } \Gamma_b). \end{cases}$$

Suppose $\mathbf{d} \in L^2(\Omega)^2$. If $\Upsilon \in H_0^{s-3/2}(\Gamma_s)^2$, $\Psi_l \in H_0^{s-1/2}(\Gamma_l)^2$, $\Psi_b \in H_0^{s-1/2}(\Gamma_b)^2$ with $3/2 \leqslant s < 2$, then, $\mathbf{w} \in H^s(\Omega)^2$. Moreover, if $\Upsilon \in H_0^{1/2+\varepsilon}(\Gamma_s)^2$, $\Psi_l \in H_0^{3/2+\varepsilon}(\Gamma_l)^2$ and $\Psi_b \in H_0^{3/2+\varepsilon}(\Gamma_b)^2$, for some ε: $0 < \varepsilon < 1/2$, then $\mathbf{w} \in H^2(\Omega)^2$.

Proposition 2.7 (Regularity for the Stokes problem in S). *Let $S \subseteq \mathbb{R}^2$ be an open set with $\partial S \in C^3$. Let $(\mathbf{u}, p)$ be the solution for the Dirichlet–Stokes problem in S*:

$$\begin{cases} -\Delta\mathbf{v} + \nabla\pi = \mathbf{a} & \text{in } S, \\ \nabla\cdot\mathbf{v} = b & \text{in } S, \\ \mathbf{v} = \mathbf{c} & \text{on } \partial S. \end{cases}$$

If $\mathbf{a} \in L^\alpha(S)^2$, $b \in W^{s-1,\alpha}(S)$ and $\mathbf{c} \in W^{s-\frac{1}{\alpha},\alpha}(\partial S)^2$ with $1 < \alpha < +\infty$ and $1 \leqslant s \leqslant 2$ such that $s - \frac{1}{\alpha}$ is not an integer, and the compatibility condition $\int_S b\,d\mathbf{x} = \int_{\partial S} \mathbf{c}\cdot\mathbf{n}_{\partial S}\,ds$ holds, then:

$$\mathbf{v} \in W^{s,\alpha}(S)^2 \quad \text{and} \quad \pi \in W^{s-1,\alpha}(S).$$

This result was proved by Cattabriga [4], for any $s \geqslant 1$ integer, and can be generalized to $s \in \mathbb{R}$ using interpolation techniques, as in Lions and Magenes [11].

Proof of Theorem 2.1. Let us consider (Φ, π) the weak solution of the dual problem (3) (obtained in Theorem 2.5). Since $\pi \in L^2_0(S)$, in particular $\nabla\pi \in H^{-1}(S)$. Therefore, we might consider the auxiliary function $\mathbf{v} \in H^1_0(S)^2$ as the (unique) weak solution of the elliptic problem in S:

$$\begin{cases} \nu\Delta\mathbf{v} = \nabla\pi & \text{in } S, \\ \mathbf{v} = \mathbf{0} & \text{on } \partial S. \end{cases} \tag{9}$$

Now, we look at the elliptic problem verified by $\mathbf{w} = \Phi - \mathbf{v}$ (without pressure and free divergence restriction):

$$\begin{cases} -\nu\Delta\mathbf{w} - \nu_3\partial^2_{zz}\mathbf{w} = \mathbf{g} & \text{in } \Omega, \\ \nu_3\partial_z\mathbf{w} = \mathbf{0} & \text{on } \Gamma_s, \\ \mathbf{w} = \mathbf{0} & \text{on } \Gamma_l, \\ \mathbf{w} = -\mathbf{v} & \text{on } \Gamma_b. \end{cases} \tag{10}$$

Here, we have made the identification $\mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{x}, -h(\mathbf{x}))$ in order to consider $\mathbf{v}$ as a function defined in Γ_b.

Due to the regularity of the data ($\mathbf{g} \in L^2(\Omega)^2$, $\mathbf{v} \in H^1_0(\Gamma_b)^2$), we can apply Proposition 2.6 (for $s = 3/2$) and deduce that $\mathbf{w} \in H^{3/2}(\Omega)^2$. Then, using Lemma B.1 (see Appendix B), $\langle\mathbf{w}\rangle \in H^{3/2}(S)^2$. As $\mathbf{v}$ is independent of z, $\langle\mathbf{v}\rangle = h\mathbf{v}$. Then

$$-\nabla\cdot(h\,\mathbf{v}) = -\nabla\cdot\langle\mathbf{v}\rangle = \nabla\cdot\langle\mathbf{w}\rangle - \nabla\cdot\langle\Phi\rangle = \nabla\cdot\langle\mathbf{w}\rangle + \varphi \in H^{1/2}(S).$$

Now, as

$$\nabla\cdot\mathbf{v} = \frac{1}{h}\nabla\cdot(h\,\mathbf{v}) - \frac{\nabla h\cdot\mathbf{v}}{h},$$

then $\nabla\cdot\mathbf{v}\in H^{1/2}(S)$ (using that $h\geqslant h_{\min}>0$ in S). In particular, $\nabla\cdot\mathbf{v}\in H^{1/2-\varepsilon}(S)$ for all $\varepsilon>0$. Therefore, if we consider the Stokes problem in S that satisfies $(\mathbf{v},\pi)$:

$$\begin{cases} -\nu\Delta\mathbf{v}+\nabla\pi=\mathbf{0} & \text{in } S,\\ \nabla\cdot\mathbf{v}\in H^{1/2-\varepsilon}(S) & \text{in } S,\\ \mathbf{v}=\mathbf{0} & \text{on } \partial S,\end{cases}\tag{11}$$

from Proposition 2.7 (for $s=1/2-\varepsilon$ and $\alpha=2$) one has $\mathbf{v}\in H_0^{3/2-\varepsilon}(S)^2$ and $\pi\in H^{1/2-\varepsilon}(S)$, for all $\varepsilon>0$. Returning to system (10) (and (11)), using again Proposition 2.6 for $s=2-\varepsilon$ (and Proposition 2.7), one has $\mathbf{w}\in H^{2-\varepsilon}(\Omega)^2$ (and $\mathbf{v}\in H_0^{2-\varepsilon}(S)^2$ and $\pi\in H^{1-\varepsilon}(S)$).

Notice that, following with this “bootstrap” argument, it is not possible to obtain $\mathbf{w}\in H^2(\Omega)^2$ by means of Proposition 2.6, since $\mathbf{v}\notin H_0^{3/2+\varepsilon}(\Omega)^2$ because in general $\frac{\partial\mathbf{v}}{\partial\mathbf{n}}\neq 0$ on ∂S.

Therefore, we change the argument. In order to increase the regularity of the pressure to $\pi\in H^1(S)$, we integrate $(3)_1$ in the z-variable, arriving at:

$$\begin{cases} -\nu\Delta\langle\Phi\rangle+h(\mathbf{x})\nabla\pi=\mathbf{G} & \text{in } S,\\ \nabla\cdot\langle\Phi\rangle=-\varphi & \text{in } S,\\ \langle\Phi\rangle=\mathbf{0} & \text{on } \partial S,\end{cases}$$

where

$$\mathbf{G}=\langle\mathbf{g}\rangle+\nu_3\partial_z\Phi(x,0)-\nu_3\partial_z\Phi\big(x,-h(\mathbf{x})\big)+(\nabla\Phi)\big(\mathbf{x},-h(\mathbf{x})\big)\,\nabla h(\mathbf{x}).$$

The last term of $\mathbf{G}$ is the vector whose ith component is the scalar product $(\nabla\Phi_i)|_{\Gamma_b}\cdot\nabla h$, coming from the equality $\langle\Delta\Phi_i\rangle=\Delta\langle\Phi_i\rangle-(\nabla\Phi_i)|_{\Gamma_b}\cdot\nabla h$ $(i=1,2)$ (here, $\Phi|_{\Gamma_b}=\mathbf{0}$ is used). We will see that $\mathbf{G}\in L^2(S)^2$.

Since $\mathbf{w}\in H^{2-\varepsilon}(\Omega)^2$, in particular,

$$\partial_z\Phi=\partial_z\mathbf{w}\in H^{1-\varepsilon}(\Omega)^2.\tag{12}$$

From $\mathbf{g}\in L^2(\Omega)^2$ and (12), we have that $\langle\mathbf{g}\rangle+\nu_3\partial_z\Phi(\mathbf{x},0)-\nu_3\partial_z\Phi(\mathbf{x},-h(\mathbf{x}))\in L^2(S)^2$. Therefore, we focus our attention on the term $(\nabla\Phi_i)(\mathbf{x},-h(\mathbf{x}))\cdot\nabla h(\mathbf{x})$. Deriving with respect to the $\mathbf{x}$-variables the equality $\Phi|_{\Gamma_b}=\Phi(\mathbf{x},-h(\mathbf{x}))=\mathbf{0}$, we obtain:

$$(\nabla\Phi_i)|_{\Gamma_b}=(\partial_z\Phi_i)|_{\Gamma_b}\,\nabla h(\mathbf{x}).$$

Then,

$$(\nabla\Phi_i)|_{\Gamma_b}\cdot\nabla h=(\partial_z\Phi_i)|_{\Gamma_b}\,\big|\nabla h(\mathbf{x})\big|^2.$$

Therefore, it suffices to analyze the regularity of $|\nabla h(\mathbf{x})|^2\,(\partial_z\Phi)|_{\Gamma_b}$.

Since $h\in H^2(S)$, in particular $|\nabla h|^2\in L^p(S)$ for all $p>1$. From (12), $(\partial_z\Phi)|_{\Gamma_b}\in H^{1/2-\varepsilon}(\Gamma_b)^2\hookrightarrow L^{4/(1+2\varepsilon)}(\Gamma_b)^2$. In particular, $(\partial_z\Phi)|_{\Gamma_b}\in L^q(S)^2$ for a certain $q>2$. Then $|\nabla h(\mathbf{x})|^2\,(\partial_z\Phi)|_{\Gamma_b}\in L^2(S)^2$, hence we can conclude that $\mathbf{G}\in L^2(S)^2$.

Notice that, using the equality $h\,\nabla\pi=\nabla(h\,\pi)-\pi\,\nabla h$, the previous problem can be rewritten as the Stokes problem in S:

$$\begin{cases} -\nu\Delta\langle\Phi\rangle+\nabla(h\pi)=\mathbf{G}+\pi\,\nabla h & \text{in } S,\\ \nabla\cdot\langle\Phi\rangle=-\varphi & \text{in } S,\\ \langle\Phi\rangle=\mathbf{0} & \text{on } \partial S.\end{cases}$$

Since $\mathbf{G}+\pi\nabla h\in L^2(S)^2$ (recall that $\pi\in H^{1/2-\varepsilon}(S)$) and $\varphi\in H^1(S)$ such that $\int_S\varphi\,d\mathbf{x}=0$, by Proposition 2.7, $\langle\Phi\rangle\in H^2(S)^2$ and $h\pi\in H^1(S)$. In particular, $\pi\in H^1(S)$ (using once again that $h\geqslant h_{\min}>0$).

Now, we go back to the dual problem (3). Moving the pressure term to the right hand side, we consider the corresponding elliptic problem with the unknown Φ. Then, using the new regularity $\pi\in H^1(S)$ and applying Proposition 2.6, one has $\Phi\in H^2(\Omega)^2$. Finally, inequality (4) can be deduced by construction, thanks to the continuous dependence of the auxiliary problems (9), (10) and (11). □

2.3. Very weak regularity for the primal problem

Suppose the following regularity hypothesis for the data:

$$\mathbf{f} \in \big(H^2(\Omega)^2 \cap H^1_{b,l}(\Omega)^2\big)', \qquad \Upsilon \in H^{-3/2}(\Gamma_s). \tag{H}$$

Denote by $\langle\cdot,\cdot\rangle_S$ the duality between $H^1(S)'$ and $H^1(S)$, by $\langle\cdot,\cdot\rangle_\Omega$ the duality between $(H^2(\Omega)\cap H^1_{b,l}(\Omega))'$ and $H^2(\Omega)\cap H^1_{b,l}(\Omega)$ and by $\langle\cdot,\cdot\rangle_{\Gamma_s}$ the duality between $H^{-3/2}(\Gamma_s)$ and $H^{3/2}_0(\Gamma_s)$.

Definition 2.8. A pair $(\mathbf{u}, p)$ is said a *very weak solution of* (2) if $\mathbf{u} \in L^2(\Omega)^2$, $p \in H^1(S)'/\mathbb{R}$ and satisfies:

$$\begin{cases} \int_\Omega \mathbf{u}\cdot(-\nu\Delta\Phi - \nu_3\partial^2_{zz}\Phi + \nabla\pi)\,d\Omega - \langle p, \nabla\cdot\langle\Phi\rangle\rangle_S = \langle \mathbf{f},\Phi\rangle_\Omega + \langle\Upsilon,\Phi\rangle_{\Gamma_s} \\ \forall\Phi \in H^2(\Omega)^2\cap H^1_{b,l}(\Omega)^2 \quad \text{with} \quad \partial_z\Phi|_{\Gamma_s} = 0,\ \forall\pi\in H^1(S). \end{cases} \tag{13}$$

Remark 2.2. Notice that $p \in H^1(S)'/\mathbb{R}$ means that $p \in H^1(S)'$ is defined up to an additive constant. From $\Phi \in H^2(\Omega)^2 \cap H^1_{b,l}(\Omega)^2$ one has $\nabla\cdot\langle\Phi\rangle \in H^1(S)$ and $\int_S \nabla\cdot\langle\Phi\rangle = 0$ (therefore, $\langle p,\nabla\cdot\langle\Phi\rangle\rangle_S = \langle p+c,\nabla\cdot\langle\Phi\rangle\rangle_S$ for all $c\in\mathbb{R}$).

Remark 2.3. It is easy to see that if the data $(\mathbf{f},\Upsilon)$ are regular and $(\mathbf{u},p) \in H^1_{b,l}(\Omega)^2 \times L^2(\Omega)/\mathbb{R}$ is a weak solution of (2), then $(\mathbf{u},p)$ is also a very weak solution of (2) (i.e., the previous definition is a generalization of the variational formulation).

Let $l : L^2(\Omega)^2 \times \mathcal{H} \to \mathbb{R}$ defined by:

$$l(\mathbf{g},\varphi) = \langle\mathbf{f},\Phi\rangle_\Omega + \langle\Upsilon,\Phi\rangle_{\Gamma_s},$$

with (Φ,π) the strong solution for the dual problem (3) with data $(\mathbf{g},\varphi)$ (given in Theorem 2.1). It is easy to prove that l is a linear and continuous operator from $L^2(\Omega)^2\times\mathcal{H}$ into $\mathbb{R}$. Indeed, from (4) one has

$$\|l\|_{(L^2(\Omega)\times\mathcal{H})'} \leqslant C\big\{\|\mathbf{f}\|_{(H^2(\Omega)\cap H^1_{b,l}(\Omega))'} + \|\Upsilon\|_{H^{-3/2}(\Gamma_s)}\big\}.$$

Applying the classical Riesz' identification, one can easily prove the following:

Lemma 2.9. *Assuming* (H) *and the hypothesis of Theorem* 2.1*, there exists a unique pair* $(\mathbf{u},\tilde p) \in L^2(\Omega)^2 \times \mathcal{H}'$ *verifying*:

$$\int_\Omega \mathbf{u}\cdot\mathbf{g}\,d\Omega + \langle\tilde p,\varphi\rangle_{\mathcal{H}',\mathcal{H}} = l(\mathbf{g},\varphi), \quad \forall\mathbf{g}\in L^2(\Omega)^2,\ \forall\varphi\in\mathcal{H}. \tag{14}$$

Moreover, one has:

$$\|\mathbf{u}\|_{L^2(\Omega)} + \|\tilde p\|_{\mathcal{H}'} \leqslant C\big\{\|\mathbf{f}\|_{(H^2\cap H^1_{b,l})'} + \|\Upsilon\|_{H^{-3/2}(\Gamma_s)}\big\}. \tag{15}$$

In order to rewrite the linear form of (14) in terms of (Φ,π), one needs the following result

Proposition 2.10. *The space* $H^1(S)'/\mathbb{R}$ *is isomorphic to* $\mathcal{H}'$.

Proof. In Appendix A. □

Now, we are able to show the result about existence (and uniqueness) of very weak solutions.

Theorem 2.11. *Under conditions of Lemma* 2.9, *there exists a unique very weak solution* $(\mathbf{u}, p)$ *of* (2) *in* $L^2(\Omega)^2 \times (H^1(S))'/\mathbb{R}$. *Moreover, one has*:

$$\|\mathbf{u}\|_{L^2(\Omega)} + \|p\|_{H^1(S)'/\mathbb{R}} \leqslant C\big\{\|\mathbf{f}\|_{(H^2(\Omega)\cap H^1_{b,l}(\Omega))'} + \|\Upsilon\|_{H^{-3/2}(\Gamma_s)}\big\}. \tag{16}$$

Proof. From Lemma 2.9, there exists a unique pair $(\mathbf{u}, \tilde{p}) \in L^2(\Omega)^2 \times \mathcal{H}'$ verifying (14).

Using the isomorphism between $(H^1(S))'/\mathbb{R}$ and $\mathcal{H}'$, we can identify $\tilde{p}$ with a distribution p in $(H^1(S))'/\mathbb{R}$ such that $\langle \tilde{p}, \varphi\rangle_{\mathcal{H}',\mathcal{H}} = \langle p, \varphi\rangle_{H^1(S)',H^1(S)}$, $\forall \varphi \in \mathcal{H}$. Therefore, we conclude that $(\mathbf{u}, p)$ is a solution of (13), i.e. a very weak solution of (2). The uniqueness is deduced from the linearity of the problem. Finally, (16) is deduced from (15) and the isomorphism between $H^1(S)'/\mathbb{R}$ and $\mathcal{H}'$. □

3. Interpretation for the differential problem

3.1. Differential equation

From now on, we fix $\mathbf{f} \in L^2(\Omega)^2$ instead of $\mathbf{f} \in (H^2(\Omega) \cap H^1_{b,l})'$ (in order to have a space of distributions). Let us consider $(\mathbf{u}, p)$ the very weak solution of (2). Taking $\pi = 0$ and $\Phi \in \mathcal{D}(\Omega)$ in (13), one has:

$$\int_\Omega \mathbf{u} \cdot (-\nu\Delta\Phi - \nu_3\partial^2_{zz}\Phi)\, d\Omega - \big\langle p, \nabla \cdot \langle\Phi\rangle\big\rangle_S = \int_\Omega \mathbf{f} \cdot \Phi\, d\Omega,$$

hence we can deduce the system $(2)_1$ in the distributional sense. Taking $\Phi = \mathbf{0}$ and $\pi \in \mathcal{D}(S)$ in (13),

$$\int_\Omega \mathbf{u} \cdot \nabla\pi\, d\Omega = 0$$

hence we can deduce $\nabla \cdot \langle\mathbf{u}\rangle = 0$ in $\mathcal{D}'(S)$. Therefore, we have:

Proposition 3.1. *Assume* $\mathbf{f} \in L^2(\Omega)^2$. *Let* $(\mathbf{u}, p) \in L^2(\Omega)^2 \times (H^1(S))'/\mathbb{R}$ *be the unique very weak solution of* (2). *Then* $\mathbf{u}$ *and* p *verify Eqs.* $(2)_{1-2}$ *in the distributional sense in* Ω *and* S, *respectively.*

3.2. Sense for the boundary conditions

From $\mathbf{u} \in L^2(\Omega)^2$, we have that $\langle\mathbf{u}\rangle \in L^2(S)^2$. On the other hand, since $\nabla \cdot \langle\mathbf{u}\rangle = 0$, in particular $\nabla \cdot \langle\mathbf{u}\rangle \in L^2(S)$. Then we can conclude that $\langle\mathbf{u}\rangle \cdot \mathbf{n}_{\partial S} \in H^{-1/2}(S)$. Moreover, taking $\Phi = \mathbf{0}$ and $\pi \in H^1(S)$ in (13), we get:

$$\int_\Omega \mathbf{u} \cdot \nabla\pi\, d\Omega = 0,$$

hence

$$0 = \int_S \langle\mathbf{u}\rangle \cdot \nabla\pi\, dS = \big\langle\langle\mathbf{u}\rangle \cdot \mathbf{n}_{\partial S}, \pi\big\rangle_{\partial S}, \quad \forall\pi \in H^1(S)$$

where $\langle\cdot,\cdot\rangle_{\partial S}$ denotes the duality $H^{-1/2}(\partial S)$, $H^{1/2}(\partial S)$. Therefore,

$$\langle\mathbf{u}\rangle \cdot \mathbf{n}_{\partial S} = 0 \quad \text{in } H^{-1/2}(\partial S). \tag{17}$$

3.2.1. Dirichlet boundary conditions

In order to give a sense at the Dirichlet boundary conditions $(2)_4$ on $\Gamma_b \cup \Gamma_l$, we define the operator

$$\mathbf{D}_{(\mathbf{u},p)} : H^{1/2}(\Gamma_b \cup \Gamma_l)^2 \to \mathbb{R},$$
$$\mathbf{D}_{(\mathbf{u},p)}(\Psi) = \int_\Omega \mathbf{u} \cdot (-\nu \Delta \Phi - \nu_3 \partial^2_{zz} \Phi)\, d\Omega - \langle p, \nabla \cdot \langle \Phi \rangle \rangle_S - \langle \mathbf{f}, \Phi \rangle_\Omega, \quad \forall \Psi \in H^{1/2}(\Gamma_b \cup \Gamma_l)^2,$$

where $\Phi = \Phi(\Psi)$ is the unique weak solution, $\Phi \in H^2(\Omega)^2 \cap H^1_0(\Omega)^2$, of the problem:

$$\begin{cases} \Delta^2 \Phi = \mathbf{0} & \text{in } \Omega, \\ \Phi = \mathbf{0} & \text{on } \partial\Omega, \\ \frac{\partial \Phi}{\partial \mathbf{n}} = \Psi & \text{on } \Gamma_b \cup \Gamma_l, \quad \partial_z \Phi = \mathbf{0} \text{ on } \Gamma_s. \end{cases}$$

It is easy to show that $\mathbf{D}_{(\mathbf{u},p)}$ is a linear continuous operator, i.e.:

$$\mathbf{D}_{(\mathbf{u},p)} \in (H^{1/2}(\Gamma_b \cup \Gamma_l)^2)'.$$

Notice that if $(\mathbf{u}, p)$ is a weak solution of (2), then:

$$\mathbf{D}_{(\mathbf{u},p)}(\Psi) = \int_{\Gamma_b \cup \Gamma_l} \mathbf{u} \cdot \Psi \, d\sigma, \quad \forall \Psi,$$

therefore we can denote this operator as the *generalized trace over* $\Gamma_b \cup \Gamma_l$.

Replacing $\pi = 0$ and $\Phi = \Phi(\Psi)$ in (13), we obtain that:

$$\mathbf{D}_{(\mathbf{u},p)}(\Psi) = 0, \quad \forall \Psi \in H^{1/2}(\Gamma_b \cup \Gamma_l)^2.$$

Thus, it follows that $\mathbf{D}_{(\mathbf{u},p)} = 0$ in $(H^{1/2}(\Gamma_b \cup \Gamma_l)^2)'$.

3.2.2. Neumann boundary condition

In this case, in order to give a meaning to the Neumann boundary conditions $(2)_3$ on Γ_s, we define the operator

$$\mathbf{N}_{(\mathbf{u},p)} : H^{3/2}_0(S)^2 \to \mathbb{R},$$
$$\mathbf{N}_{(\mathbf{u},p)}(\Psi) = \int_\Omega \mathbf{u} \cdot (-\nu \Delta \Phi - \nu_3 \partial^2_{zz} \Phi)\, d\Omega - \langle p, \nabla \cdot \langle \Phi \rangle \rangle_S - \langle \mathbf{f}, \Phi \rangle_\Omega,$$

where $\Phi = \Phi(\Psi)$ is the unique weak solution, $\Phi \in H^2(\Omega)^2$, of the problem:

$$\begin{cases} \Delta^2 \Phi = \mathbf{0} & \text{in } \Omega, \\ \Phi = \Psi & \text{on } \Gamma_s, \quad \Phi = \mathbf{0} \text{ on } \Gamma_b \cup \Gamma_l, \\ \frac{\partial \Phi}{\partial \mathbf{n}} = \mathbf{0} & \text{on } \partial\Omega. \end{cases}$$

Now, assuming that $(\mathbf{u}, p)$ is a weak solution of (2), one has:

$$\mathbf{N}_{(\mathbf{u},p)}(\Psi) = \int_{\Gamma_s} \partial_z \mathbf{u} \cdot \Psi \, d\sigma, \quad \forall \Psi,$$

therefore we can denote this operator as *generalized normal trace over* Γ_s.

Replacing $\pi = 0$ and this new $\Phi = \Phi(\Psi)$ in (13), we obtain that:

$$\mathbf{N}_{(\mathbf{u},p)}(\Psi) = \langle \Upsilon, \Psi \rangle_{\Gamma_s}, \quad \forall \Psi \in H^{3/2}_0(S)^2$$

Thus, it follows that $\mathbf{N}_{(\mathbf{u},p)} = \Upsilon$ in $(H^{3/2}_0(S)^2)' \equiv H^{-3/2}(S)^2$.

4. Application to non-stationary problems

4.1. The linear case

In this subsection, we are going to apply the result obtained for the stationary case in order to get strong solutions for the linear non-stationary Primitive Equations (also called non-stationary hydrostatic Stokes problem):

$$\begin{cases} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} - \nu_3 \partial_{zz}^2 \mathbf{u} + \nabla p = \mathbf{F} & \text{in } (0,T) \times \Omega, \\ \nabla \cdot \langle \mathbf{u} \rangle = 0 & \text{in } (0,T) \times S, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \Omega, \\ \nu_3 \partial_z \mathbf{u} = \Upsilon & \text{on } (0,T) \times \Gamma_s, \\ \mathbf{u} = \mathbf{0} & \text{on } (0,T) \times (\Gamma_b \cup \Gamma_l). \end{cases} \tag{18}$$

The following result was given in [9]:

Theorem 4.1. *Suppose $S \subseteq \mathbb{R}^2$ with $\partial S \in C^3$ and $h \in C^3(\overline{S})$ with $h \geqslant h_{\min} > 0$ in S. If $\mathbf{F} \in L^2((0,T) \times \Omega)^2$, $\mathbf{u}_0 \in V$, $\Upsilon \in L^2(0,T; H_0^{1/2+\varepsilon}(\Gamma_s)^2)$ for some $\varepsilon > 0$, with $\partial_t \Upsilon \in L^2(0,T; H^{-1/2}(\Gamma_s)^2)$, then there exists a unique strong solution $\mathbf{u}$ of problem (18) in $(0,T)$. Moreover,*

$$\begin{aligned} &\|\mathbf{u}\|_{L^\infty(V)}^2 + \|\mathbf{u}\|_{L^2(H^2(\Omega))}^2 + \|\partial_t \mathbf{u}\|_{L^2(H)}^2 \\ &\quad \leqslant C\big\{\|\mathbf{u}_0\|_V^2 + \|\Upsilon(0)\|_{H^{-1/2}(\Gamma_s)}^2 + \|\mathbf{F}\|_{L^2(L^2(\Omega))}^2 + \|\Upsilon\|_{L^2(H_0^{1/2+\varepsilon}(\Gamma_s))}^2 + \|\partial_t \Upsilon\|_{L^2(H^{-1/2}(\Gamma_s))}^2\big\}. \end{aligned} \tag{19}$$

The proof of this theorem is based on the following two results about the stationary problem (2); a weak regularity result given by Lions, Temam and Wang in [13], and a strong regularity result due to M. Ziane [16], respectively:

Lemma 4.2. *Suppose $\Omega \subseteq \mathbb{R}^3$ be Lipschitz-continuous. If $\mathbf{f} \in H_{b,l}^{-1}(\Omega)^2$ and $\Upsilon \in H^{-1/2}(\Gamma_s)^2$, then the problem (2) has a unique weak solution $\mathbf{u} \in V$. Moreover,*

$$\|\mathbf{u}\|_V^2 \leqslant C\big\{\|\Upsilon\|_{H^{-1/2}(\Gamma_s)}^2 + \|\mathbf{f}\|_{H_{b,l}^{-1}(\Omega)}^2\big\}. \tag{20}$$

Lemma 4.3. *Suppose $S \subseteq \mathbb{R}^2$ with $\partial S \in C^3$ and $h \in C^3(\overline{S})$ with $h \geqslant h_{\min} > 0$ in S. If $\mathbf{f} \in L^2(\Omega)^2$ and $\Upsilon \in H_0^{1/2+\varepsilon}(\Gamma_s)^2$, for some $\varepsilon > 0$, the unique solution $\mathbf{u}$ of the problem (2) belongs to $H^2(\Omega)^2 \cap V$. Moreover,*

$$\|\mathbf{u}\|_{H^2(\Omega)}^2 \leqslant C\big\{\|\Upsilon\|_{H_0^{1/2+\varepsilon}(\Gamma_s)}^2 + \|\mathbf{f}\|_{L^2(\Omega)}^2\big\}. \tag{21}$$

In this section, using Theorem 2.11 instead of Lemma 4.2, we will obtain strong solutions of (19) imposing less regularity over $\partial_t \Upsilon$ (replacing $H^{-1/2}(\Gamma_s)$ by $H^{-3/2}(\Gamma_s)$). More precisely, the new result is:

Theorem 4.4. *Suppose $S \subseteq \mathbb{R}^2$ with $\partial S \in C^3$ domain and $h \in C^3(\overline{S})$ with $h \geqslant h_{\min} > 0$ in S. If $\mathbf{F} \in L^2((0,T) \times \Omega)^2$, $\mathbf{u}_0 \in V$, $\Upsilon \in L^2(0,T; H_0^{1/2+\varepsilon}(\Gamma_s)^2) \cap L^\infty(0,T; H^{-1/2}(\Gamma_s)^2)$ for some $\varepsilon > 0$ with $\partial_t \Upsilon \in L^2(0,T; H^{-3/2}(\Gamma_s)^2)$ and $\Upsilon(0) \in H^{-1/2}(\Gamma_s)^2$, then there exists a unique strong solution $\mathbf{u}$ of the problem (18) in $(0,T)$. Moreover,*

$$\begin{aligned} &\|\mathbf{u}\|_{L^\infty(V)}^2 + \|\mathbf{u}\|_{L^2(H^2(\Omega))}^2 + \|\partial_t \mathbf{u}\|_{L^2(H)}^2 \\ &\quad \leqslant C\big\{\|\mathbf{u}_0\|_V^2 + \|\Upsilon(0)\|_{H^{-1/2}(\Gamma_s)}^2 \\ &\qquad + \|\mathbf{F}\|_{L^2(L^2(\Omega))}^2 + \|\Upsilon\|_{L^2(H_0^{1/2+\varepsilon}(\Gamma_s)) \cap L^\infty(H^{-1/2}(\Gamma_s))}^2 + \|\partial_t \Upsilon\|_{L^2(H^{-3/2}(\Gamma_s))}^2\big\}. \end{aligned} \tag{22}$$

Remark 4.1. If S is smooth enough, the hypothesis $\Upsilon \in L^\infty(0,T;H^{-1/2}(\Gamma_s)^3)$ and $\Upsilon(0) \in H^{-1/2}(\Gamma_s)^3$ are not necessary, since from $\Upsilon \in L^2(0,T;H_0^{1/2+\varepsilon}(\Gamma_s)^3)$ and $\partial_t \Upsilon \in L^2(0,T;H^{-3/2}(\Gamma_s)^3)$ we can deduce $\Upsilon \in C([0,T];H^{-1/2}(\Gamma_s)^3)$ with continuous dependence (see Appendix C). In particular, we arrive again at (19).

Proof. The uniqueness is obtained thanks to the linearity of the problem. To get existence, we will separate the proof in the same steps of the proof done in [8,9]:

Step 1: *Existence of weak solution*. It can be obtained as the limit for the Galerkin approximate solutions $\mathbf{u}_m \in C^1([0,T];V_m)$ (being V_m a m-dimensional subspace of V) such that:

$$\begin{cases} \frac{d}{dt}\int_\Omega \mathbf{u}_m \cdot \varphi\, d\Omega + \nu \int_\Omega \nabla \mathbf{u}_m : \nabla\varphi\, d\Omega + \nu_3 \int_\Omega \partial_z \mathbf{u}_m \cdot \partial_z \varphi\, d\Omega = \int_\Omega \mathbf{F}_m \cdot \varphi\, d\Omega \\ \quad + \int_{\Gamma_s} \Upsilon_m \cdot \varphi|_{\Gamma_s}\, d\sigma \quad \forall \varphi \in V_m, \\ \mathbf{u}_m(0) \equiv \text{ projection of } \mathbf{u}_0 \text{ over } V_m, \end{cases}$$

where $\mathbf{F}_m \in C^0([0,T];H_{b,l}^{-1}(\Omega)^2)$ and $\Upsilon_m \in C^0([0,T];H^{-1/2}(\Gamma_s)^2)$ are smooth approximates of $\mathbf{F}$ and Υ, respectively.

Taking $\varphi = \mathbf{u}_m$, we can deduce that the sequence $\mathbf{u}_m$ is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$. Then, a standard limit process gives a weak solution $\mathbf{u}$ of (18).

Step 2: *Lifting of the Neumann boundary conditions*. We define $B:\Upsilon \to B\Upsilon = \mathbf{u}$ the solution of the stationary hydrostatic Stokes problem (2) with $\mathbf{f} = \mathbf{0}$.

Then, we can define the auxiliary function $\mathbf{e}(t) = B(\Upsilon(t))$ a.e. $t \in (0,T)$ that let us made an adequate lifting of the Neumann boundary condition data Υ. Using Lemmas 4.2 and 4.3, taking into account that $\Upsilon(t) \in H_0^{1/2+\varepsilon}(\Gamma_s)^2$ a.e. $t \in (0,T)$, we obtain that $\mathbf{e}(t) \in H^2(\Omega)^2 \cap V$ a.e. $t \in (0,T)$, and

$$\|\mathbf{e}(t)\|_V^2 \leqslant C\|\Upsilon(t)\|_{H^{-1/2}(\Gamma_s)}^2, \qquad \|\mathbf{e}(t)\|_{H^2(\Omega)}^2 \leqslant C\|\Upsilon(t)\|_{H_0^{1/2+\varepsilon}(\Gamma_s)}^2.$$

Therefore $\mathbf{e} \in L^2(0,T;H^2(\Omega)^2 \cap V) \cap L^\infty(0,T;V)$ and

$$\|\mathbf{e}\|_{L^\infty(V)}^2 \leqslant C\|\Upsilon\|_{L^\infty(H^{-1/2}(\Gamma_s))}^2, \tag{23}$$

$$\|\mathbf{e}\|_{L^2(H^2(\Omega))}^2 \leqslant C\|\Upsilon\|_{L^2(H_0^{1/2+\varepsilon}(\Gamma_s))}^2. \tag{24}$$

On the other hand, using Theorem 2.11 we have that, as $\partial_t \Upsilon(t) \in H^{-3/2}(\Gamma_s)$ a.e. $t \in (0,T)$, we can define $\tilde{\mathbf{e}}(t) = B(\partial_t \Upsilon(t))$ a.e. $t \in (0,T)$, with $\tilde{\mathbf{e}}(t) \in L^2(\Omega)$ and $\|\tilde{\mathbf{e}}(t)\|_{L^2(\Omega)} \leqslant C\|\partial_t \Upsilon\|_{H^{-3/2}(\Gamma_s)}$.

Now, let us see that $\tilde{\mathbf{e}}(t) = \partial_t \mathbf{e}(t)$; in fact, taking

$$\mathbf{u}_\delta(t) = \frac{\mathbf{e}(t+\delta) - \mathbf{e}(t)}{\delta} - \tilde{\mathbf{e}}(t) = B\left(\frac{\Upsilon(t+\delta) - \Upsilon(t)}{\delta} - \partial_t \Upsilon(t)\right),$$

as $\frac{\Upsilon(t+\delta)-\Upsilon(t)}{\delta} - \partial_t \Upsilon(t) \in H^{-3/2}(\Gamma_s)$, from Theorem 2.11, we deduce that:

$$\|\mathbf{u}_\delta(t)\|_{L^2(\Omega)} \leqslant C\left\|\frac{\Upsilon(t+\delta) - \Upsilon(t)}{\delta} - \partial_t \Upsilon(t)\right\|_{H^{-3/2}(\Gamma_s)} \to 0 \quad \text{as } \delta \to 0.$$

Therefore, we can conclude that $\widetilde{\mathbf{e}}(t) = \partial_t \mathbf{e}(t)$ in $L^2(\Omega)$ a.e. $t \in (0,T)$. Moreover, we obtain the bound:

$$\|\partial_t \mathbf{e}(t)\|_{L^2(\Omega)} \leqslant C\|\partial_t \Upsilon(t)\|_{H^{-3/2}(\Gamma_s)}$$

hence

$$\|\partial_t \mathbf{e}\|_{L^2(L^2(\Omega))} \leqslant C\|\partial_t \Upsilon\|_{L^2(H^{-3/2}(\Gamma_s))}. \tag{25}$$

Remark 4.2. Notice that this step 2 will be the fundamental step in the proof, because the fact of using estimate (16) (and Theorem 2.11) instead of estimate (20) (and Lemma 4.2) allows us to impose weaker hypothesis on $\partial_t \Upsilon$.

Step 3: *Strong solution for the homogeneous problem*. The function $\mathbf{y} = \mathbf{u} - \mathbf{e}$ (with a potential $q : S \to \mathbb{R}$ associated) verifies the following system:

$$\begin{cases} \partial_t \mathbf{y} - \nu \Delta \mathbf{y} - \nu_3 \partial_{zz}^2 \mathbf{y} + \nabla q = \mathbf{h} & \text{in } (0,T) \times \Omega, \\ \nabla \cdot \langle \mathbf{y} \rangle = 0 & \text{in } (0,T) \times S, \\ \mathbf{y}|_{t=0} = \mathbf{y}_0 & \text{in } S, \\ \nu_3 \partial_z \mathbf{y} = 0 & \text{on } (0,T) \times \Gamma_s, \\ \mathbf{y} = 0 & \text{on } (0,T) \times (\Gamma_b \cup \Gamma_l), \end{cases}$$

where $\mathbf{h} = \mathbf{F} - \partial_t \mathbf{e} \in L^2(0,T; L^2(\Omega)^d)$ and $\mathbf{y}_0 = \mathbf{u}_0 - \mathbf{e}(0) \in V$. Again, arguing by a Galerkin procedure, we denote by $\mathbf{y}_m : [0,T] \to V_m$ the Galerkin approximate functions, where V_m is the subspace of $V = \{\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^m, \ldots\}$ spanned by the eigenfunctions of the hydrostatic Stokes operator $A : V \to V'$ defined by:

$$\langle A\mathbf{u}, \mathbf{v} \rangle_{V',V} = \int_\Omega \big(\nu \nabla \mathbf{u} : \nabla \mathbf{v} + \nu_3 \partial_z \mathbf{u} \cdot \partial_z \mathbf{v} \big)\, d\Omega \quad \forall \mathbf{u}, \mathbf{v} \in V, \tag{26}$$

and associated to homogeneous boundary conditions (Neumann on the surface and Dirichlet on the bottom and the vertical sidewalls). These approximates solve the ordinary differential problem:

$$\begin{cases} \frac{d}{dt} \int_\Omega \mathbf{y}_m(t) \cdot \mathbf{v}_m \, d\Omega + \nu \int_\Omega \nabla \mathbf{y}_m : \nabla \mathbf{v}_m \, d\Omega + \nu_3 \int_\Omega \partial_z \mathbf{y}_m \cdot \mathbf{v}_m \, d\Omega = \int_\Omega \mathbf{h}_m \cdot \mathbf{v}_m \, d\Omega, \quad \forall \mathbf{v}_m \in V_m, \\ \mathbf{y}_m(0) = \mathbf{y}_{0m} = \sum_{j=1}^m (\int_\Omega \nabla \mathbf{y}_0 : \nabla \mathbf{w}^j + \partial_z \mathbf{y}_0 \cdot \partial_z \mathbf{w}^j) \mathbf{w}^j, \end{cases} \tag{27}$$

being $\mathbf{h}_m$ a smooth approximated function of $\mathbf{h}$. Let us now obtain strong estimates for $\mathbf{y}_m$. Taking $\mathbf{v}_m = A\mathbf{y}_m(t) \in V_m$ as test functions in (27), we deduce the inequality: $\forall t \in [0,T]$,

$$\frac{d}{dt} \big\| \mathbf{y}_m(t) \big\|_V^2 + \big\| A\mathbf{y}_m(t) \big\|_{L^2(\Omega)}^2 \leqslant \| \mathbf{h}_m \|_{L^2(\Omega)}^2.$$

Integrating in time, we get:

$$\big\| \mathbf{y}_m(t) \big\|_V^2 + \int_0^t \big\| A\mathbf{y}_m(s) \big\|_{L^2(\Omega)}^2 \, ds \leqslant \| \mathbf{y}_{0m} \|_V^2 + \int_0^t \big\| \mathbf{h}_m(s) \big\|_{L^2(\Omega)}^2 \, ds.$$

Therefore, the sequence $(\mathbf{y}_m)_m$ is bounded in $L^2(0,T; D(A)) \cap L^\infty(0,T; V)$, so there exists a limit function $\mathbf{y}$ that belongs to the same space and verifies the inequality:

$$\| \mathbf{y} \|_{L^\infty(V)}^2 + \| \mathbf{y} \|_{L^2(D(A))}^2 \leqslant \| \mathbf{y}_0 \|_V^2 + \| \mathbf{h} \|_{L^2(L^2(\Omega))}^2. \tag{28}$$

Now, taking $\partial_t \mathbf{y}_m \in V_m$ as test functions in (R) and integrating in time, we have:

$$\| \partial_t \mathbf{y}_m \|_{L^2(H)}^2 \leqslant \| \mathbf{y}_{0m} \|_V^2 + \| \mathbf{h}_m \|_{L^2(L^2\Omega))}^2,$$

and the limit $\partial_t \mathbf{y} \in L^2(H)$ and verifies the inequality:

$$\| \partial_t \mathbf{y} \|_{L^2(H)}^2 \leqslant \| \mathbf{y}_0 \|_V^2 + \| \mathbf{h} \|_{L^2(L^2\Omega))}^2. \tag{29}$$

Adding (28) and (29), and using that $\mathbf{y}_0 = \mathbf{u}_0 - \mathbf{e}(0)$ and $\mathbf{h} = \mathbf{F} - \partial_t \mathbf{e}$, we conclude that:

$$\| \mathbf{y} \|_{L^\infty(V)}^2 + \| \mathbf{y} \|_{L^2(D(A))}^2 + \| \partial_t \mathbf{y} \|_{L^2(H)}^2 \leqslant C \big\{ \| \mathbf{u}_0 \|_V^2 + \big\| \mathbf{e}(0) \big\|_V^2 + \| \mathbf{F} \|_{L^2(L^2(\Omega))}^2 + \| \partial_t \mathbf{e} \|_{L^2(L^2(\Omega))}^2 \big\}.$$

Finally, replacing estimates (20) for $\mathbf{e}(0)$ in V and estimates (25) for $\partial_t \mathbf{e}$ in $L^2(0,T; L^2(\Omega))$, we obtain (19). $\square$

4.2. Application to the non-stationary nonlinear Primitive Equations

The extension of the result from Theorem 4.4 to the nonlinear case follows similar arguments as in [8,9], replacing the use made there of Theorem 4.1 by Theorem 4.4. Therefore, we will get the same weaker hypothesis over $\partial_t \Upsilon$ as in the previous linear case.

We want to obtain the strong regularity result for the nonlinear problem of Primitive Equations:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + u_3\partial_z\mathbf{u} - \nu\Delta\mathbf{u} - \nu_3\partial_{zz}^2\mathbf{u} + \alpha\mathbf{u}^{\perp} + \nabla p_s = \mathbf{F} & \text{in } (0,T)\times\Omega,\\ \nabla\cdot\langle\mathbf{u}\rangle = 0 & \text{in } (0,T)\times S,\\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \Omega,\\ \nu_3\partial_z\mathbf{u}|_{\Gamma_s} = \Upsilon, \quad \mathbf{u}|_{\Gamma_b\cup\Gamma_l} = \mathbf{0} & \text{in } (0,T), \end{cases} \tag{30}$$

where $\alpha = 2f\sin(\lambda)$ is the Coriolis coefficient and the vertical velocity is computed by

$$u_3(t;\mathbf{x},z) = \int_z^0 \nabla\cdot\mathbf{u}(t;\mathbf{x},s)\,ds. \tag{31}$$

We give here an schedule of the result (see [9] for the details): *Under the hypothesis of Theorem* 4.4*, there exists a unique solution* $\mathbf{u}$ *of problem* (30)*, either defined in* $(0,T^{\star})$ *for a small time* $T^{\star}\in(0,T)$*, or in all the time interval* $(0,T)$ *under smallness assumptions for the data.*

For the proof, we use $(\mathbf{e}, q_s)$ the solution of (18) in order to lift the boundary conditions. Therefore, we only have to study the homogeneous problem that verifies $(\mathbf{w},\pi_s) = (\mathbf{u}-\mathbf{e}, p_s - q_s)$:

$$\begin{cases} \partial_t\mathbf{w} - \nu\Delta\mathbf{w} - \nu_3\partial_{zz}^2\mathbf{w} + \nabla\pi_s + (\mathbf{w}+\mathbf{e})\cdot\nabla(\mathbf{w}+\mathbf{e}) + (w_3+e_3)\partial_z(\mathbf{w}+\mathbf{e}) = \mathbf{0} & \text{in } (0,T)\times\Omega\ ,\\ \nabla\cdot\langle\mathbf{w}\rangle = 0 \quad \text{in } (0,T)\times S, \quad \mathbf{w}|_{t=0} = \mathbf{0} & \text{in } \Omega,\\ \nu_3\partial_z\mathbf{w}|_{\Gamma_s} = \mathbf{0}, \quad \mathbf{w}|_{\Gamma_b\cup\Gamma_l} = \mathbf{0} & \text{in } (0,T), \end{cases} \tag{32}$$

with $w_3 = \int_z^0 \nabla\cdot\mathbf{w}\,ds$ and the same form for e_3.

We approximate the functions $\mathbf{w}$ by the Galerkin functions $\mathbf{w}_m$ in the m-dimensional spaces V_m, which are the orthonormal basis of dimension m (in V) of eigenfunctions of the hydrostatic operator. In order to obtain estimates in the $H^2(\Omega)$-norm, we take $A\mathbf{w}_m(t)\in V_m$ as test functions, obtaining:

$$\begin{aligned} \frac{d}{dt}\|\mathbf{w}_m\|_V^2 + \|A\mathbf{w}_m\|_{L^2(\Omega)}^2 \leqslant C\int_\Omega \big|\big((\mathbf{w}_m+\mathbf{e}_m)\cdot\nabla\big)\,(\mathbf{w}_m+\mathbf{e}_m)\big|^2\,d\Omega\\ + C\int_\Omega \big|\big((w_3)_m+(e_3)_m\big)\partial_z(\mathbf{w}_m+\mathbf{e}_m)\big|^2\,d\Omega + C\|\mathbf{F}\|_{L^2(\Omega)}^2 \equiv \sum_{i=1}^3 I_i. \end{aligned} \tag{33}$$

We bound the I_i-terms using the estimates in strong norms of $\mathbf{e}_m$ and the left hand side of (33). The biggest difficulty is to bound the nonlinear terms:

$$\int_\Omega \big|(\mathbf{w}_m\cdot\nabla)\mathbf{w}_m\big|^2\,d\Omega \quad \text{and} \quad \int_\Omega \big|(w_m)_3\,\partial_z\mathbf{w}_m\big|^2\,d\Omega.$$

Remark that I_2 is less regular than I_1 due to the anisotropic regularity for the vertical velocity, since $\partial_z w_3 = -\nabla\cdot\mathbf{w}\in L^2(\Omega)$, and therefore (using a Poincaré inequality) $w_3\in L^2(\Omega)$. However, $\nabla w_3\notin L^2(\Omega)$ in general. If we use the usual estimates in the Sobolev spaces, as for the Navier–Stokes equation, this lack of regularity of the vertical velocity does not allow to obtain strong regularity for (32). Searching for an alternative method, we separate the regularity in $\mathbf{x}$ and z, considering the anisotropic spaces and estimates that were introduced in [9] and that we overview here:

Definition 4.5. Given $p, q \in [1, +\infty]$, we say that a function $\mathbf{u}$ belongs to $L^q_z L^p_{\mathbf{x}}(\Omega)$ if:

$$\mathbf{u}(\cdot, z) \in L^q(S_z) \quad \text{and} \quad \big\|\mathbf{u}(\cdot, z)\big\|_{L^q(S_z)} \in L^p(-h_{\max}, 0),$$

where $S_z = \{\mathbf{x} \in S \,/\, (\mathbf{x}, z) \in \Omega\}$ for a fixed $z \in (-h_{\max}, 0)$, and its norm is given by the expression:

$$\big\| \|\mathbf{u}(\cdot, z)\|_{L^q(S_z)} \big\|_{L^p(-h_{\max},0)}.$$

Lemma 4.6 (Interpolation inequalities [9]).

(a) *Let $v \in L^2(\Omega)$ be a function such that $\partial_z v \in L^2(\Omega)$ and $(v n_z)|_{\Gamma_b} = 0$. Then, $v \in L^\infty_z L^2_{\mathbf{x}}(\Omega)$ and satisfy the estimate*:

$$\|v\|^2_{L^\infty_z L^2_{\mathbf{x}}} \leqslant 2\, \|v\|_{L^2(\Omega)} \|\partial_z v\|_{L^2(\Omega)}. \tag{34}$$

More generally, if $v \in H^1(\Omega)$ then $v \in L^\infty_z L^2_{\mathbf{x}}(\Omega)$, and there exists a constant $C = C(\Omega) > 0$ such that:

$$\|v\|^2_{L^\infty_z L^2_{\mathbf{x}}} \leqslant C(\Omega) \|v\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega). \tag{35}$$

(b) *Let $v \in L^2(\Omega)$ be a function such that $\nabla v \in L^2(\Omega)^2$ and $(v n_{x_i})|_{\Gamma_b \cup \Gamma_l} = 0$ $(i = 1, 2)$. Then, $v \in L^2_z L^4_{\mathbf{x}}(\Omega)$ and verifies the estimate*:

$$\|v\|^2_{L^2_z L^4_{\mathbf{x}}} \leqslant 4\, \|v\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}. \tag{36}$$

More generally, if $v \in H^1(\Omega)$ then $v \in L^2_z L^4_{\mathbf{x}}$, and there exists a constant $C = C(\Omega) > 0$ such that:

$$\|v\|^2_{L^2_z L^4_{\mathbf{x}}} \leqslant C(\Omega) \|v\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}. \tag{37}$$

(c) *Let $\mathbf{v} \in L^2(\Omega)^2$ be a function such that $\nabla \cdot \mathbf{v} \in H^1(\Omega)$. Then, if we consider v_3 defined as in* (31), *we have that $v_3 \in L^\infty_z L^4_{\mathbf{x}}(\Omega)$ and satisfies the following estimate*:

$$\|v_3\|_{L^\infty_z L^4_{\mathbf{x}}} \leqslant C(\Omega) \|\nabla \cdot \mathbf{v}\|^{1/2}_{L^2(\Omega)} \|\nabla \cdot \mathbf{v}\|^{1/2}_{H^1(\Omega)}. \tag{38}$$

By using the above inequalities, we can bound the terms I_1 and I_2 as follows:

$$\begin{aligned} I_2 &\leqslant C \big\|(w_3)_m + (e_3)_m\big\|^2_{L^\infty_z L^4_{\mathbf{x}}} \|\partial_z \mathbf{w}_m + \partial_z \mathbf{e}_m\|^2_{L^2_z L^4_{\mathbf{x}}} \\ &\leqslant \frac{C}{\nu^{3/2}} \|A\mathbf{w}_m\|^2_{L^2(\Omega)} \|\mathbf{w}_m\|_V + \frac{C}{\nu^3} \|\mathbf{w}_m\|^2_V \|\mathbf{e}_m\|^2_{H^1(\Omega)} \|\mathbf{e}_m\|^2_{H^2(\Omega)} + C \|\mathbf{e}_m\|^2_{H^1(\Omega)} \|\mathbf{e}_m\|^2_{H^2(\Omega)} \end{aligned}$$

for a constant $C = C(\Omega) > 0$. For the I_1-term, we bound in the usual (isotropic) form:

$$\begin{aligned} I_1 &\leqslant C \|\mathbf{w}_m + \mathbf{e}_m\|^2_{L^4(\Omega)} \|\nabla \mathbf{w}_m + \nabla \mathbf{e}_m\|^2_{L^4(\Omega)} \\ &\leqslant C\big(\|\nabla \mathbf{w}_m\|^{3/2}_{L^2(\Omega)} \|\mathbf{w}_m\|^{1/2}_{L^2(\Omega)} + \|\mathbf{e}_m\|^{3/2}_{H^1(\Omega)} \|\mathbf{e}_m\|^{1/2}_{L^2(\Omega)}\big) \\ &\quad \times \big(\|\nabla \mathbf{w}_m\|^{3/2}_{H^1(\Omega)} \|\nabla \mathbf{w}_m\|^{1/2}_{L^2(\Omega)} + \|\mathbf{e}_m\|^{3/2}_{H^2(\Omega)} \|\mathbf{e}_m\|^{1/2}_{H^1(\Omega)}\big) \\ &\leqslant C\left(\frac{1}{\nu^{11}} \|\mathbf{w}_m\|^8_V + \frac{1}{\nu^7} \|\mathbf{e}_m\|^8_{H^1(\Omega)} + \frac{1}{\nu} \|\mathbf{e}_m\|^{1/2}_{H^1(\Omega)} \|\mathbf{e}_m\|^{3/2}_{H^2(\Omega)}\right) \|\mathbf{w}_m\|^2_V \\ &\quad + \frac{1}{2} \|A\mathbf{w}_m\|^2_{L^2(\Omega)} + C \|\mathbf{e}_m\|^{5/2}_{H^1(\Omega)} \|\mathbf{e}_m\|^{3/2}_{H^2(\Omega)}. \end{aligned}$$

Plugging the above estimates in (33), we have:

$$\frac{d}{dt}\|\mathbf{w}_m\|_V^2 + \|A\mathbf{w}_m\|_{L^2(\Omega)}^2 \leqslant \frac{C}{\nu^{3/2}}\|A\mathbf{w}_m\|_{L^2(\Omega)}^2\|\mathbf{w}_m\|_V + \frac{C}{\nu^{11}}\|\mathbf{w}_m\|_V^{10} + a(t)\|\mathbf{w}_m\|_V^2 + b(t), \tag{39}$$

where $a(t)$ and $b(t)$ are bounded functions in $L^1(0,T)$, depending on ν and the data.

From here, we can argue in two ways: supposing small data (to obtain global in time strong regularity) or for any data (to obtain local in time strong regularity).

In order to obtain *global in time strong solution for small data*, we reason as follows: First, supposing the estimate:

$$\left\|\mathbf{w}_m(t)\right\|_V < \gamma\nu^{3/2} \quad \text{for } \gamma \text{ small enough}, \tag{40}$$

we can control globally in time the terms $\|A\mathbf{w}_m\|_{L^2(\Omega)}^2\|\mathbf{w}_m\|_V$ and $\frac{C}{\nu^{11}}\|\mathbf{w}_m\|_V^{10}$ appearing on the right hand side of (39), and finish the proof in a standard way. Secondly, we have to prove the estimate (40), which is obtained using the homogeneous initial data and adequate smallness hypothesis on the data (see [9] for more details).

In order to obtain *local in time strong solution for any data*, using that $\mathbf{w}_m(0) = \mathbf{0}$ and $\mathbf{w}_m : [0,T] \to H^1(\Omega)^2$ is a continuous function, we can chose a time T_m such that:

$$\left\|\mathbf{w}_m(t)\right\|_V \leqslant \frac{\nu^{3/2}}{2C}, \quad \forall t \in [0, T_m]. \tag{41}$$

Then, we can prove that T_m is bounded from below by a time $T_* \geqslant 0$ independent on m. Indeed, integrating (39) between 0 and t, $t \in [0, T_m]$, and using (41) we obtain ([9]):

$$\left\|\mathbf{w}_m(t)\right\|_V^2 + \int_0^t \left\|A\mathbf{w}_m(s)\right\|_{L^2(\Omega)}^2 ds \leqslant K(\nu)\,t + C\nu^2\int_0^t \left\|\mathbf{e}_m(s)\right\|_{H^2(\Omega)}^2 ds.$$

Therefore, choosing now $T_\star$ such that:

$$K(\nu)T_\star + C\nu^2\|\mathbf{e}_m\|_{L^2(0,T_\star;H^2(\Omega))}^2 < \frac{\nu^3}{4C^2},$$

we verify that T_m can be chosen equal to $T_\star$ for each m. Now, the proof of the existence of strong solution in $(0, T_\star)$ follows in a standard manner. □

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Appendix A. The isomorphism

Proposition A.1 (before cited as Proposition 2.10). *The space $H^1(S)'/\mathbb{R}$ is isomorphic to $\mathcal{H}'$.*

Proof. Recall that $\mathcal{H} = \{\varphi/\varphi \in H^1(S),\ \int_S \varphi\, d\mathbf{x} = 0\}$.

(a) $H^1(S) = \mathcal{H} \oplus \mathcal{H}^\perp$ (identifying $\mathcal{H}^\perp$ with the space spanned by the constant functions over S). In fact, any constant function $\omega = \alpha$ in S ($\alpha \in \mathbb{R}$), verifies:

$$\big((\omega,\varphi)\big)_1 = \int_S \nabla\omega\cdot\nabla\varphi\, d\mathbf{x} + \int_S \omega\varphi\, d\mathbf{x} = \alpha\int_S \varphi\, d\mathbf{x} = 0 \quad \forall\varphi \in \mathcal{H},$$

where $((\cdot,\cdot))_1$ denotes the inner product in $H^1(S)$. Therefore, $\omega=\alpha\in\mathcal{H}^\perp$.

On the other hand, every function $v\in H^1(S)$ can be written in a unique manner as:

$$v=\varphi+\alpha,$$

with $\alpha=\frac{1}{|S|}\int_S v\,d\mathbf{x}\in\mathbb{R}$ and $\varphi=v-\alpha\in\mathcal{H}$.

(b) *The isomorphism*: We define the operator

$$T: H^1(S)'/\mathbb{R}\to\mathcal{H}',$$
$$q\mapsto Tq$$

as

$$\langle Tq,\varphi\rangle_{\mathcal{H}',\mathcal{H}}=\langle q,\varphi\rangle_{H^1(S)',H^1(S)},\quad \forall\varphi\in\mathcal{H}.$$

This operator is *well-defined* because if α is a constant, $T(q+\alpha)=Tq$. Indeed, $\langle T(q+\alpha),\varphi\rangle=\langle q+\alpha,\varphi\rangle=\langle q,\varphi\rangle$, because of $\int_S\varphi\,d\mathbf{x}$ for all $\varphi\in\mathcal{H}$. Therefore, it suffices to prove that T is a continuous bijection.

T *is one-to-one*: Suppose that $Tq=0$, i.e. $\langle Tq,\varphi\rangle=0$ for all $\varphi\in\mathcal{H}$. Let $v\in H^1(S)$, then $v=\varphi+\alpha$ with $\alpha=\frac{1}{|S|}\int_S v\,d\mathbf{x}$ and $\varphi\in\mathcal{H}$. Then,

$$\begin{aligned}\langle q,v\rangle_{H^1(S)',H^1(S)}&=\langle q,\varphi+\alpha\rangle_{H^1(S)',H^1(S)}=\langle Tq,\varphi\rangle_{\mathcal{H}',\mathcal{H}}+\frac{1}{|S|}\langle q,1\rangle_{H^1(S)',H^1(S))}\int_S v\,d\mathbf{x}\\&=\frac{1}{|S|}\langle q,1\rangle_{H^1(S)',H^1(S)}\int_S v\,d\mathbf{x},\end{aligned}$$

therefore, defining $\beta=\frac{1}{|S|}\langle q,1\rangle_{H^1(S)',H^1(S)}\in\mathbb{R}$, one has $\langle q-\beta,v\rangle_{H^1(S)',H^1(S)}=0$ for all $v\in H^1(S)$, hence $q-\beta$ belongs to the zero equivalent class in $H^1(S)'/\mathbb{R}$, and therefore $q=0$ in $H^1(S)'/\mathbb{R}$.

T *is onto*: For any $l\in\mathcal{H}'$, we have to prove that there exists an element $q\in H^1(S)'/\mathbb{R}$ such that $Tq=l$, i.e., $\langle q,\varphi\rangle_{H^1(S)',H^1(S)}=\langle l,\varphi\rangle_{\mathcal{H}',\mathcal{H}}$ $\forall\varphi\in\mathcal{H}$. Indeed, it suffices to define $\langle q,v\rangle_{H^1(S)',H^1(S)}=\langle l,\varphi\rangle_{\mathcal{H}',\mathcal{H}}$ if $v=\varphi+\alpha$.

T *is continuous*: Using the standard norm definitions,

$$\begin{aligned}\|q\|_{H^1(S)'/\mathbb{R}}&=\inf_{c\in\mathbb{R}}\|q+c\|_{H^1(S)'}=\inf_{c\in\mathbb{R}}\sup_{v\in H^1(S)}\frac{\langle q+c,v\rangle_{H^1(S)',H^1(S)}}{\|v\|_{H^1(S)}}\geqslant\inf_{c\in\mathbb{R}}\sup_{\varphi\in\mathcal{H}}\frac{\langle q+c,\varphi\rangle_{H^1(S)',H^1(S)}}{\|\varphi\|_{H^1(S)}}\\&=\sup_{\varphi\in\mathcal{H}}\frac{\langle q,\varphi\rangle_{H^1(S)',H^1(S)}}{\|\varphi\|_{H^1(S)}}=\sup_{\varphi\in\mathcal{H}}\frac{\langle Tq,\varphi\rangle_{\mathcal{H}',\mathcal{H}}}{\|\varphi\|_{H^1(S)}}=\|Tq\|_{\mathcal{H}'}.\qquad\square\end{aligned}$$

Appendix B. Regularity in S

Lemma B.1. *Suppose $h\in H^2(S)$. If $w\in H^s(\Omega)$ for any s: $1\leqslant s\leqslant 2$, then $\langle w\rangle\in H^s(S)$.*

Proof. If $w\in L^2(\Omega)$, then integrating in $(-h(\mathbf{x}),0)$ we obtain:

$$\left|\langle w\rangle\right|^2=\left|\int_{-h(\mathbf{x})}^0 w(\mathbf{x},z)\,dz\right|^2\leqslant\left(\int_{-h(\mathbf{x})}^0\left|w(\mathbf{x},z)\right|^2dz\right)h(\mathbf{x}).$$

Integrating now in S, we obtain:

$$\int_S \left|\langle w\rangle\right|^2 d\mathbf{x} \leqslant \int_S h(\mathbf{x}) \left(\int_{-h(\mathbf{x})}^0 \left|w(\mathbf{x},z)\right|^2 ds \right) d\mathbf{x} \leqslant \|h\|_{L^\infty(S)} \int_\Omega \left|w(\mathbf{x},z)\right|^2 d\Omega = \|h\|_{L^\infty(S)} \|w\|^2_{L^2(\Omega)},$$

that implies:

$$\left\|\langle w\rangle\right\|_{L^2(S)} \leqslant \|h\|^{1/2}_{L^\infty(S)} \|w\|_{L^2(\Omega)}. \tag{42}$$

Deriving $\langle w\rangle$, we get that $\nabla\langle w\rangle = \langle\nabla w\rangle + w|_{\Gamma_b}\nabla h$. Taking the $L^2(S)$-norm and using (42), we obtain:

$$\left\|\nabla\langle w\rangle\right\|_{L^2(S)} \leqslant \left\|\langle\nabla w\rangle\right\|_{L^2(S)} + \|w|_{\Gamma_b}\nabla h\|_{L^2(S)} \leqslant \|h\|^{1/2}_{L^\infty(S)} \|\nabla w\|_{L^2(\Omega)} + \|w|_{\Gamma_b}\|^{1/2}_{L^4(S)} \|\nabla h\|^{1/2}_{L^4(S)}.$$

Let us focus our attention on the terms of $L^4(S)$-type:

$$\begin{aligned}\|w|_{\Gamma_b}\|_{L^4(S)} &= \left(\int_S \left|w\big(\mathbf{x},-h(\mathbf{x})\big)\right|^4 d\mathbf{x} \right)^{1/4} = \left(\int_{\Gamma_b} \left|w\big(\mathbf{x},-h(\mathbf{x})\big)\right|^4 \big(1+\left|\nabla h(\mathbf{x})\right|^2\big)^{-1/2} d\sigma \right)^{1/4} \\ &\leqslant \|w\|_{L^4(\Gamma_b)} \leqslant C\,\|w\|_{H^1(\Omega)},\end{aligned}$$

where we have used the continuity of the trace function from $H^1(\Omega)$ in $L^4(\partial\Omega)$. On the other hand, using the Sobolev embedding $H^2(S) \hookrightarrow W^{1,4}(S)$, one has:

$$\|\nabla h\|_{L^4(S)} \leqslant C\,\|h\|_{H^2(S)}.$$

Therefore,

$$\left\|\nabla\langle w\rangle\right\|_{L^2(S)} \leqslant \|h\|^{1/2}_{L^\infty(S)} \|\nabla w\|_{L^2(\Omega)} + C\,\|w\|^{1/2}_{H^1(\Omega)} \|h\|^{1/2}_{H^2(\Omega)}. \tag{43}$$

Estimates (42) and (43) let us deduce that if $w \in H^1(\Omega)$ and $h \in H^2(S)$, then $\langle w\rangle \in H^1(S)$.

Now, we study the case where $w \in H^2(\Omega)$ and $h \in H^2(S)$. From the previous estimates, we only need to estimate second order derivatives for w. Without loss of generality, we will only reason for $\partial^2_{xx}\langle w\rangle$. We have:

$$\partial^2_{xx}\langle w\rangle = \langle\partial^2_{xx} w\rangle + 2(\partial_x w)|_{\Gamma_b}\,\partial_x h(\mathbf{x}) - (\partial_z w)|_{\Gamma_b}\left|\partial_x h(\mathbf{x})\right|^2 + w|_{\Gamma_b}\,\partial^2_{xx} h(\mathbf{x}).$$

Therefore, using the same arguments as before:

$$\begin{aligned}\left\|\partial^2_{xx}\langle w\rangle\right\|_{L^2(S)} &\leqslant \|h\|^{1/2}_{L^\infty(S)} \|\partial^2_{xx} w\|_{L^2(\Omega)} \\ &\quad + C\big(\|\partial_x w\|^{1/2}_{H^1(\Omega)} \|h\|^{1/2}_{H^2(S)} + \|\partial_z w\|^{1/2}_{H^1(\Omega)} \|h\|_{H^2(S)} + \|w|_{\Gamma_b}\|^{1/2}_{L^\infty(S)} \|h\|^{1/2}_{H^2(S)}\big) \\ &\leqslant \|h\|^{1/2}_{L^\infty(S)} \|\partial^2_{xx} w\|_{L^2(\Omega)} + C\|w\|^{1/2}_{H^2(\Omega)} \|h\|^{1/2}_{H^2(S)} \big(1+\|h\|^{1/2}_{H^2(S)}\big),\end{aligned} \tag{44}$$

where we have used in the last term that if $w \in H^2(\Omega)$, then $w|_{\partial\Omega} \in H^{3/2}(\partial\Omega) \hookrightarrow L^\infty(\partial\Omega)$. Expression (44) together with (42) and (43), let us deduce that if $w \in H^2(\Omega)$ and $h \in H^2(S)$, then $\langle w\rangle \in H^2(S)$.

We have just proved that, if $w \in H^1(\Omega)$ then $\langle w\rangle \in H^1(S)$, and if $w \in H^2(\Omega)$ then $\langle w\rangle \in H^2(S)$. In the case $H^s(\Omega)$ for $s \in (1,2)$, interpolation results [11] let us see $H^s(\Omega) = [H^1(\Omega), H^2(\Omega)]_\theta$ and $H^s(S) = [H^1(S), H^2(S)]_\theta$ with $\theta = s-1$, hence we can deduce that if $w \in H^s(\Omega)$, then $\langle w\rangle \in H^s(S)$. □

Appendix C. Interpolation results

Lemma C.1. *Let $S \subseteq \mathbb{R}^2$ be a bounded open set with $\partial S \in C^\infty$. If $\Upsilon \in L^2(0,T; H_0^{1/2+\varepsilon}(\Gamma_s))$ for some $\varepsilon > 0$ and $\partial_t\Upsilon \in L^2(0,T; H^{-3/2}(\Gamma_s))$, then $\Upsilon \in C^0([0,T]; H^{-1/2}(\Gamma_s))$.*

In order to prove this lemma, we will use some interpolation results appearing in Lions and Magenes [11]:

Theorem C.2 [11, p. 79]. *Suppose that $\Gamma_s \in C^\infty$. Let s_1, $s_2 \geqslant 0$ such that $s_i \neq \lambda + 1/2$ (λ integer, $i = 1, 2$). Let $\theta \in [0, 1]$ such that*:

$$(1-\theta)s_1 - \theta s_2 \neq \mu + 1/2 \quad \textit{and} \quad \neq -\mu - 1/2 \quad (\mu \textit{ integer} \geqslant 0). \tag{45}$$

Then,

$$\left[H_0^{s_1}(\Gamma_s), H^{-s_2}(\Gamma_s)\right]_\theta = \begin{cases} H_0^{(1-\theta)s_1-\theta s_2}(\Gamma_s) & \textit{if } (1-\theta)s_1 - \theta s_2 \geqslant 0, \\ H^{(1-\theta)s_1-\theta s_2}(\Gamma_s) & \textit{if } (1-\theta)s_1 - \theta s_2 \leqslant 0. \end{cases} \tag{46}$$

Proposition C.3 [11, p. 53]. *Let X and Y two separable Hilbert spaces such that $X \subset Y$, X dense in Y with continuous embedding. Then*:

$$\left[H^{t_1}(\Omega; X), H^{t_2}(\Omega; Y)\right]_\theta = H^{(1-\theta)t_1+\theta t_2}\big(\Omega; [X, Y]_\theta\big).$$

Proof of Lemma C.1. In our case, taking in Theorem C.2, $\mu = 0$, $s_1 = 1/2 + \varepsilon$, $s_2 = 3/2 + \varepsilon'$ with $\varepsilon' < \varepsilon$ one has

$$\left[H_0^{1/2+\varepsilon}(\Gamma_s), H^{-3/2-\varepsilon'}(\Gamma_s)\right]_\theta = H^{-1/2+\delta}(\Gamma_s),$$

if one imposes that $(1-\theta)s_1 - \theta s_2 = -1/2 + \delta < 0$ with δ is small enough, then:

$$\theta = \frac{1 + \varepsilon - \delta}{2 + \varepsilon + \varepsilon'} > \frac{1}{2} \quad (\text{taking } \varepsilon > \varepsilon' + 2\delta).$$

From hypothesis $\Upsilon \in H^1(0, T; H^{-3/2}(\Gamma_s)) \cap H^0(0, T; H_0^{1/2+\varepsilon}(\Gamma_s))$. Then, we use Proposition C.3 for $t_1 = 0$, $t_2 = 1$, $X = H_0^{1/2+\varepsilon}(\Gamma_s)$, $Y = H^{-3/2-\varepsilon'}(\Gamma_s)$ and $\Omega = (0, T)$, obtaining $\Upsilon \in H^\theta(0, T; H^{-1/2+\delta}(\Gamma_s))$ (with $\theta > 1/2$), hence in particular one has $\Upsilon \in C^0([0, T]; H^{-1/2}(\Gamma_s))$. $\square$

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